WEAK LIMIT THEOREMS FOR DENSE MULTIGRAPHON-VALUED STOCHASTIC PROCESSES WITH APPLICATIONS TO CONFIGURATION MODELS

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Abstract

Dynamics of random graph models have been applied widely in various fields. Although the mathematical theory of simple graphs has achieved a rapid development in recent years, the study of multigraphs is still in its early stage. A graph is called a multigraph if loops and multiple edges are involved. The multigraphon is an important tool in building the limiting theory for dense multigraphs. In this paper, we establish a weak limit theorem for multigraphon-valued stochastic process. As an application, we prove a weak limiting theorem for the edge-reconnection model.

1 INTRODUCTION

The understanding of large graphs and their limits has seen a rapid development in recent years, starting with the seminal paper of Lovász and Szegedy (2006) on dense graph sequences and their limits, called graphons, which are symmetric functions defined on the unit square and hence surprisingly concrete objects. A thorough mathematical treatment was given by Borgs, Chayes, Lovász, Sós and Vesztergombi (2008, 2012) and a book-length discussion by Lovász (2012). Diaconis and Janson (2008) made the connection between this new theory with earlier work of Aldous (1981) and Hoover (1989) on exchangeable arrays, which allowed further extensions to sparse graph sequences via an analogous result of Kallenberg and exchangeable random measures; see Caron and Fox (2017) and Kallenberg (2006).

Much of this literature is concerned with simple graphs, but sometimes graphs with multiple edges and self-looped graphs appear in practice. Kolossváry and Ráth (2011) developed a generalization of the theory of Lovász and Szegedy (2006) to multigraphs, and they identified the resulting limiting objects, called *multigraphons*.

Another line of research that has become increasingly important is that of dynamics of networks, since only rarely are networks static over time. However, the mathematical treatment of network dynamics is still not well developed, despite a rather large literature on such models. Erdős and Rényi (1960) analysed growing random graphs, Holland and Leinhardt (1977) looked at the evolution of social networks, and accounts of subsequent developments are given by Snijders (2001) and Snijders, Koskinen and Schweinberger (2010) with a more statistical perspective. Recently, results have been appearing more frequently in mathematical literature, too, such as those of Basak, Durrett and Zhang (2015) and Basu and Sly (2017) to name a few. In the context of dense graph limit theory, Crane (2016) was the first to develop a cohesive stochastic-process point of view, and he introduced and studied graphon-valued processes through the lens of the theory of Aldous and Hoover. Another, more direct approach was taken by Athreya, den Hollander and Röllin (2021+), who established a weak limit theory for graph-valued stochastic processes with graphon-valued limits. Many questions remain open, such as how to describe generators of graphon-valued Markov processes.

The aim of the present article is to develop a weak limiting theory for multigraphon-valued stochastic processes analogous to that of Athreya et al. (2021+), which is essentially defining the Skorohod topology on the space of càdlàg multigraphon-valued paths. We also construct and study a class of multigraph-valued processes which give rise to these limits, and our workhorse will be the configuration random graph model with dynamics defined through by flipping, deleting and adding edges.

The rest of this paper is organized as follows. In Section 2, we give a brief overview of multigraph-valued spaces and establish the weak convergence theorem for multigraphon-valued stochastic processes. Since Skorohod's approach to stochastic processes works best on complete and separable metric spaces, one major contribution is to define a suitable metric on the space of multigraphons that makes it complete and separable. Unlike in the case of graphons, the quotient space induced by measure-preserving mappings of multigraphons is not compact for the simple reason that the number of edges between any two vertices is unbounded. In Section 3, a weak limit of the configuration random graph processes is obtained.

2 MULTIGRAPHON-VALUED STOCHASTIC PROCESSES

{sec1a}

2.1 Multigraphs and multigraphons

{sub1}

In this article, by *multigraph*, we mean a graph G on a vertex set $V(G)$, where we allow for multiple edges and multiple loops. We loosely follow the setup of Kolossváry and Ráth (2011), and represent a multigraph G by its adjacency matrix $(z_{ij})_{i,j\in[n]}$, where z_{ij} equals the number of edges connecting the vertices labelled by i and j if $i \neq j$, and where it equals two times the number of loops of vertex i if $i = j$. Let $v(G)$ be the number of vertices, let $e(G) = \sum_{1\leqslant i<j\leqslant v(G)} z_{ij}$ be the number of non-loop edges, and let $l(G) = \sum_{i=1}^{v(G)} z_{ii}/2$ be the number of loops in G. For $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the set of multigraphs on $[n]$ and let $\mathcal{M} = \cup_{n=1}^{\infty}\mathcal{M}_n$. If $G_1 \in \mathcal{M}_n$ and $G_2 \in \mathcal{M}_n$, we denote by $G_1 + G_2$ the multigraph on $[n]$ whose adjacency matrix is the sum of adjacency matrices of G_1 and G_2.

In order to define the distance between two multigraphs, we follow the paper of Kolossváry and Ráth (2011), and define the subgraph density functionals as follows. Let $k \geqslant 1$ and $n \geqslant 1$, and let $F = (a_{ij})_{i,j\in[k]} \in \mathcal{M}_k$ and

$G = (z_{ij})_{i,j\in[n]} \in \mathcal{M}_n$; then, define the *homomorphism density of F in G* as

$$t_F(G) = \frac{1}{n^k} \sum_{\sigma\colon [k]\to[n]} \mathbb{I}\big[\forall i,j \in [k] : a_{ij} \leqslant z_{\sigma(i)\sigma(j)}\big],$$

where the summation $\sum_{\sigma\colon [k]\to[n]}$ ranges over all maps σ from $[k]$ to $[n]$. For finite multigraphs, it is more convenient to work with injective homomorphism densities and induced homomorphism densities, which are both equivalent forms of homomorphism densities. Let $F = (a_{ij})_{i,j\in[k]} \in \mathcal{M}_k$ and $G = (z_{ij})_{i,j\in[n]} \in \mathcal{M}_n$, define

$$t_F^{\text{inj}}(G) = \frac{1}{(n)_k} \sum_{\sigma\colon [k]\hookrightarrow[n]} \mathbb{I}\big\{\forall i,j \in [k] : a_{ij} \leqslant z_{\sigma(i)\sigma(j)}\big\}$$

if $k \leqslant n$, and $t_F^{\text{inj}}(G) = 0$ otherwise; here, the summation is over all injective maps σ from $[k]$ to $[n]$ and where $(n)_k = n(n-1)\cdots(n-k+1)$ is the falling factorial. Similarly, define

{{eq2-13}}
$$t_F^{\text{ind}}(G) = \frac{1}{(n)_k} \sum_{\sigma\colon [k]\hookrightarrow[n]} \mathbb{I}\big\{\forall i,j \in [k] : a_{ij} = z_{\sigma(i)\sigma(j)}\big\} \tag{2.1}$$

if $k \leqslant n$ and $t_F^{\text{ind}}(G) = 0$ otherwise. By a standard inclusion–exclusion argument,

$$\big|t_F^{\text{inj}}(G) - t_F(G)\big| \leqslant \frac{1}{v(G)}\binom{v(F)}{2}.$$

Note that $\mathcal{M}$ is countable. In order to define an appropriate distance between multigraphs, consider the map $\tau : \mathcal{M} \to [0,1]^{\mathcal{M}}$ defined as

$$\tau(G) := \big(t_F(G)\big)_{F\in\mathcal{M}} \in [0,1]^{\mathcal{M}}.$$

Since $[0,1]^{\mathcal{M}}$ is a compact space (equipped with the canonical metric), it would be tempting to take closure of the image of $\tau(\mathcal{M})$, which would then also be compact; see discussion of Diaconis and Janson (2008, p. 7). However, there is no guarantee that the closure has a nice representation, as happens to be the case for simple graphons. Indeed, if K_n denotes a graph on n vertices with n edges between every pair of vertices, we have $t_F(K_n) \to 1$ as $n \to \infty$ for every $F \in \mathcal{M}$, but the limiting element $(1)_{F\in\mathcal{M}} \in [0,1]^{\mathcal{M}}$ does not have a multigraphon representation (see Definition 2.1 below). However, we do not need compactness of the underlying metric space — completeness and separability will suffice to develop a suitable theory.

To this end, we define the *multisubgraph distance* d_{ms} between two multigraphs $G_1, G_2 \in \mathcal{M}$ as

{{eq2-1}}
$$\begin{aligned} d_{\text{ms}}(G_1,G_2) = &\sum_{i=1}^{\infty} 2^{-i}\big|t_{F_i^*}(G_1) - t_{F_i^*}(G_2)\big| \\ &+ \sum_{r\geqslant 0} |t_{K_{2,r}}^{\text{ind}}(G) - t_{K_{2,r}}^{\text{ind}}(G)| \\ &+ \sum_{r\geqslant 0} |t_{L_r}^{\text{ind}}(G) - t_{L_r}^{\text{ind}}(G)|, \end{aligned} \tag{2.2}$$

where $F_1^*, F_2^*, \ldots$ is some enumeration of all multigraphons, where $K_{2,r}$ is the graph on two vertices with r edges connecting them, and where L_r is the graph on one vertex with r loops. Note that for different orderings of $F_1^*, F_2^*, \ldots$, the subgraph distances are equivalent

In order to define the completion of $\mathcal{M}$ with respect to the distance d_{ms}, we introduce multigraphons. For $j = 1, 2$, let $L_1([0,1]^j)$ be a space of Lesbegue integrable functions $\varphi : [0,1]^j \to \mathbb{R}$, where functions which agree almost everywhere with respect to the j-dimensional Lebesgue measure are identified as one object.

{def1}

Definition 2.1. We say $h : \mathbb{N}_0 \times [0,1]^2 \to [0,1]$ is a *multigraphon* if

(i) for each $r \geqslant 0$, the function $(x,y) \mapsto h(r; x, y)$ belongs to $L_1([0,1]^2)$ and the function $x \mapsto h(r; x, x)$ belongs to $L_1([0,1])$;

(ii) for any $r \geqslant 0$ and for $(x,y) \in [0,1]^2$,

{{111}}
$$h(r; x, y) = h(r; y, x), \quad \sum_{r=0}^{\infty} h(r; x, y) = 1, \tag{2.3}$$

and for $x \in [0,1]$,

{{112}}
$$h(2r+1; x, x) = 0. \tag{2.4}$$

For any two multigraphons h_1 and h_2, we write $h_1 \equiv h_2$ if for all $r \geqslant 0$,

$$\int_{[0,1]^2} |h_1(r; x, y) - h_2(r; x, y)| dx dy = 0,$$
$$\int_{[0,1]} |h_1(r; x, x) - h_2(r; x, x)| dx = 0.$$

Let $\mathcal{H}$ be the class of all equivalent classes of multigraphons with respect to "$\equiv$". Let $h \in \mathcal{H}$; while strictly speaking, h is an equivalence class of multigraphons, we will always interpret h as a *representative* of the corresponding equivalence class, that is, as an actual multigraphon, without making a notational distinction between the two. But the reader needs to keep in mind that statements about $\mathcal{H}$ are to be understood as statements about the respective equivalence classes.

For each $h \in \mathcal{H}$ and $F = (a_{ij})_{i,j \in [k]} \in \mathcal{M}_k$, define the *homomorphism density of F in h* as

{{eq2-2}}
$$t_F(h) = \int_{[0,1]^k} \prod_{1 \leqslant i \leqslant j \leqslant k} \sum_{r=a_{ij}}^{\infty} h(r; x_i, x_j)\, dx_1 \ldots dx_k. \tag{2.5}$$

Similarly, define the *induced homomorphism density of F in h* as

$$t_F^{\mathrm{ind}}(h) = \int_{[0,1]^k} \prod_{1 \leqslant i \leqslant j \leqslant k} h(a_{ij}; x_i, x_j)\, dx_1 \ldots dx_k.$$

Alternatively, if $U_1, \ldots, U_k$ are independent random variables, distributed uniformly on $[0,1]$, we can write

{{eq2-4}}

$$\begin{aligned} t_F(h) &= \mathbb{E}\Big\{ \prod_{1\leqslant i\leqslant j\leqslant k} \sum_{r=a_{ij}}^{\infty} h(r; U_i, U_j) \Big\}, \\ t_F^{\mathrm{ind}}(h) &= \mathbb{E}\Big\{ \prod_{1\leqslant i\leqslant j\leqslant k} h(a_{ij}; U_i, U_j) \Big\}. \end{aligned} \tag{2.6}$$

Moreover, if F_1 is isomorphic to F_2, then $t_{F_1}(h) = t_{F_2}(h)$ and $t_{F_1}^{\mathrm{ind}}(h) = t_{F_2}^{\mathrm{ind}}(h)$. Similarly as for multigraphs, we define d_{ms} for multigraphons as

{{eq2-6}}

$$\begin{aligned} d_{\mathrm{ms}}(h, h') = \sum_{i\geqslant 1} 2^{-i} \big| t_{F_i^*}(h) - t_{F_i^*}(h') \big| \\ + \sum_{r\geqslant 0} \big| t_{K_{2,r}}^{\mathrm{ind}}(h) - t_{K_{2,r}}^{\mathrm{ind}}(h') \big| \\ + \sum_{r\geqslant 0} \big| t_{L_r}^{\mathrm{ind}}(h) - t_{L_r}^{\mathrm{ind}}(h') \big|, \end{aligned} \tag{2.7}$$

Note that the second and third sums in (2.7) are always finite due to the requirement that $\sum_{r\geqslant 0} h(r; x, y) = 1$.

We can embed the space of multigraphs in the space of multigraphons in the usual manner: For any multigraph $G = (z_{ij})_{i,j\in[n]} \in \mathcal{M}_n$, let the corresponding multigraphon h^G be defined as

$$h^G(r; x, y) = \mathbb{I}\big[z_{\lceil nx\rceil\lceil ny\rceil} = r\big], \quad k \geqslant 0.$$

Kolossváry and Ráth (2011) showed that $t_F(G) = t_F(h^G)$ for any $F \in \mathcal{M}$; this justifies defining d_{ms} between a multigraph and a multigraphon as

$$d_{\mathrm{ms}}(G, h) = d_{\mathrm{ms}}(h^G, h).$$

Note that d_{ms} is only a pseudo-metric; that is, $d_{\mathrm{ms}}(h, h')$ may be zero, even though h and h' are not equal almost everywhere. This happens if h and h' are related via measure-preserving transformations, which is analogous to the graphon case. We will discuss this later.

The distance d_{ms} is novel in two ways. First, although multigraphon and its subgraph density functionals were introduced by Kolossváry and Ráth (2011) and further discussed by Ráth and Szakács (2012), distances on the multigraphon space have not yet been defined and analysed to the best of our knowledge. Second, the metric d_{ms} is not a naive generalization of the subgraph distance and cut distance for simple graphon space (c.f. Lovász and Szegedy (2006)), because compared to the subgraph distance for simple graphons, there are two additional terms involved in d_{ms}, which is what ensures the completeness property of the space $(\mathcal{H}, d_{\mathrm{ms}})$.

{lem0}

Lemma 2.2. *The pseudo-metric space $(\mathcal{H}, d_{\mathrm{ms}})$ is complete and separable.*

Proof. We first prove that $(\mathcal{H}, d_{\mathrm{ms}})$ is complete. To this end, let $h_1, h_2, \ldots$ be a Cauchy sequence in $(\mathcal{H}, d_{\mathrm{ms}})$. By the first sum in the definition of

d_{ms}, it follows that, for any $F \in \mathcal{M}$, $(t_F(h_n))_{n\geqslant 1}$ is also a Cauchy sequence. Hence, $\lim_{n\to\infty} t_F(h_n)$ exists. Define the function $f : \mathcal{M} \to [0,1]$ as $f(F) = \lim_{n\to\infty} t_F(h_n)$. We proceed in two steps: We first prove that there exists a multigraphon $h \in \mathcal{H}$ such that $t_F(h) = f(F)$ for all $F \in \mathcal{M}$; then, we prove that $d_{\mathrm{ms}}(h_n, h) \to 0$ as $n \to \infty$.

For the first step, we need to prove that f is non-defective; that is, we need to show that for any $k \geqslant 1$ and any sequence $F_1, F_2, \ldots \in \mathcal{M}_k$ with $\lim_{j\to\infty}(e(F_j) + l(F_j)) = \infty$, it follows that $\lim_{j\to\infty} f(F_j) = 0$.

Recall that $K_{2,j}$ denotes the multigraph on two vertices with j multiple edges and that L_j denotes the multigraph on one vertex with j loops. As $(h_n)_{n\geqslant 1}$ is a Cauchy sequence in $(\mathcal{H}, d_{\mathrm{ms}})$, we have that for any $\varepsilon > 0$, there exists $n_0 := n_0(\varepsilon)$ such that

{{eqbb}}
$$\sum_{r\geqslant 0}\big(|t^{\mathrm{ind}}_{K_{2,r}}(h_n) - t^{\mathrm{ind}}_{K_{2,r}}(h_{n_0})| + |t^{\mathrm{ind}}_{L_r}(h_n) - t^{\mathrm{ind}}_{L_r}(h_{n_0})|\big) \leqslant \varepsilon/2 \qquad \text{for all } n \geqslant n_0. \tag{2.8}$$

For this n_0, as $h_{n_0} \in \mathcal{H}$, by (2.3), there exists $r_0 := r_0(n_0, \varepsilon)$ such that

{{eqaa}}
$$\sum_{r\geqslant r_0}\big(t^{\mathrm{ind}}_{K_{2,r}}(h_{n_0}) + t^{\mathrm{ind}}_{L_r}(h_{n_0})\big) \leqslant \varepsilon/2, \tag{2.9}$$

By (2.8) and (2.9), we have for all $n \geqslant n_0$,

{{eqcc}}
$$\sum_{r\geqslant r_0}\big(t^{\mathrm{ind}}_{K_{2,r}}(h_n) + t^{\mathrm{ind}}_{L_r}(h_n)\big) \leqslant \varepsilon. \tag{2.10}$$

Since $\lim_{j\to\infty}(e(F_j) + l(F_j)) \to \infty$, there exists $j_0 := j_0(r_0, k) > 1$ such that $e(F_j) + l(F_j) \geqslant k^2 r_0$ for all $j \geqslant j_0$. Now, for each $j \geqslant j_0$, at least one of the following two statements must be true:

(a) F_j contains a vertex with r_0 loops;

(b) F_j containts a pair of vertices with r_0 multiple edges between them.

By (2.10), for any $n \geqslant n_0$ and $j \geqslant j_0$ (both n_0 and j_0 depend only on ε),

{{eq2-16}}
$$\begin{aligned} t_{F_j}(h_n) &\leqslant \max\{t_{K_{2,r_0}}(h_n), t_{L_{r_0}}(h_n)\} \\ &\leqslant \sum_{r\geqslant r_0}\big(t^{\mathrm{ind}}_{K_{2,r}}(h_n) + t^{\mathrm{ind}}_{L_r}(h_n)\big) \leqslant \varepsilon. \end{aligned} \tag{2.11}$$

Letting $n \to \infty$ in (2.11), we have

$$f(F_j) = \lim_{n\to\infty} t_{F_j}(h_n) \leqslant \varepsilon \quad \text{for all } j \geqslant j_0.$$

Noting that $f(F_j) \geqslant 0$ for all $j \geqslant 1$, we then conclude that $f(F_j) \to 0$ as $j \to \infty$, which implies by definition that f is non-defective. By Kolossváry and Ráth (2011, Theorem 1), we conclude that there exists a multigraphon $h \in \mathcal{H}$ such that

$$f(F) = t_F(h) \text{ for all } F \in \mathcal{M}.$$

This concludes the first step, and it remains to show that $d_{\mathrm{ms}}(h_n, h) \to 0$ as $n \to \infty$ as a second step. As $t_F(h_n) \to t_F(h)$ for all $F \in \mathcal{M}$, by Kolossváry and Ráth (2011, Lemma 1), we have $t_F^{\mathrm{ind}}(h_n) \to t_F^{\mathrm{ind}}(h)$ for all $F \in \{K_{2,r}, L_r : r = 0, 1, \dots\}$. Thus, it follows that for all $r \geqslant 0$,

{{eq2-20}}

$$t^{\mathrm{ind}}_{K_{2,r}}(h_n) \to t^{\mathrm{ind}}_{K_{2,r}}(h), \qquad t^{\mathrm{ind}}_{L_r}(h_n) \to t^{\mathrm{ind}}_{L_r}(h). \tag{2.12}$$

Recalling that $h_n, h \in \mathcal{H}$, and hence, by (2.3), we have

$$\sum_{r\geqslant 0} t^{\mathrm{ind}}_{K_{2,r}}(h_n) = \sum_{r\geqslant 0} t^{\mathrm{ind}}_{K_{2,r}}(h) = \sum_{r\geqslant 0} t^{\mathrm{ind}}_{L_r}(h_n) = \sum_{r\geqslant 0} t^{\mathrm{ind}}_{L_r}(h_n) = 1.$$

By (2.12) and the dominated convergence theorem, we have as $n \to \infty$,

{{eq2-21}}

$$\sum_{r\geqslant 0} |t^{\mathrm{ind}}_{K_{2,r}}(h_n) - t^{\mathrm{ind}}_{K_{2,r}}(h)| \to 0, \qquad \sum_{r\geqslant 0} |t^{\mathrm{ind}}_{L_r}(h_n) - t^{\mathrm{ind}}_{L_r}(h)| \to 0. \tag{2.13}$$

Recalling the fact that $t_F(h_n) \to t_F(h)$ for every $F \in \mathcal{M}$ together with (2.13), it is now routine to conclude that $d_{\mathrm{ms}}(h_n, h) \to 0$.

Now, we move to prove the separability of $(\mathcal{H}, d_{\mathrm{ms}})$ by showing that there exists a countable subset $\mathcal{H}^{\mathrm{sep}} \subset \mathcal{H}$ with the property that, for every $h \in \mathcal{H}$, there is a sequence $h_1, h_2, \dots \in \mathcal{H}^{\mathrm{sep}}$ such that $d_{\mathrm{ms}}(h_n, h) \to 0$. The latter is implied if we can show that $t_F(h_n) \to t_F(h)$ (which in particular implies that $t^{\mathrm{ind}}_{K_{2,r}}(h_n) \to t^{\mathrm{ind}}_{K_{2,r}}(h)$ and $t^{\mathrm{ind}}_{L_r}(h_n) \to t^{\mathrm{ind}}_{L_r}(h)$).

We first introduce some notation. Recall that for $j = 1, 2$, $L_1([0,1]^j)$ is the space of functions $\varphi : [0,1]^j \to \mathbb{R}$ such that $|\varphi|$ is Lebesgue integrable, where functions which agree almost everywhere are identified. For $\varphi \in L_1([0,1]^2)$, let $\varphi_{\mathrm{dg}}(x) = \varphi(x, x)$. Let

$$\mathcal{G} = \{\varphi \in L_1([0,1]^2) : \varphi \geqslant 0, \varphi_{\mathrm{dg}} \in L_1([0,1]) \text{ and } \varphi(x,y) = \varphi(y,x)\}.$$

For $\varphi, \varphi' \in \mathcal{G}$, we introduce the metric

$$d_1(\varphi, \varphi') = d_{\mathrm{sq}}(\varphi, \varphi') + d_{\mathrm{dg}}(\varphi, \varphi'),$$

where

{{eq:box}}

$$\begin{aligned} d_{\mathrm{sq}}(\varphi, \varphi') &= \int_{[0,1]^2} \big|\varphi(x,y) - \varphi'(x,y)\big| dxdy, \\ d_{\mathrm{dg}}(\varphi, \varphi') &= \int_{[0,1]} |\varphi(x,x) - \varphi'(x,x)| dx. \end{aligned} \tag{2.14}$$

It is routine to show that $(\mathcal{G}, d_1)$ is a metric space.

Next, we prove that $(\mathcal{G}, d_1)$ is separable. Recall that the metric space $L_1([0,1]^2)$ is separable. As $(\mathcal{G}, d_{\mathrm{sq}})$ is a subspace of $L_1([0,1]^2)$, it is also separable, since every subspace of a separable metric space is again separable. Let $\mathcal{U}_0$ be a countable and dense subset of $(\mathcal{G}, d_{\mathrm{sq}})$. Let $\mathcal{G}_{\mathrm{dg}} = \{f \in L_1([0,1]) : f \geqslant 0\}$. By a similar argument, we have the space $\mathcal{G}_{\mathrm{dg}}$ contains a dense countable subset $\mathcal{V}_0$. Let

$$\begin{aligned} \mathcal{G}^{\mathrm{sep}} = \{\varphi \in \mathcal{G} : \exists\, U \in \mathcal{U}_0 \text{ and } f \in \mathcal{V}_0 \text{ such that} \\ \varphi(x,y) = U(x,y) \text{ a.e. for } (x,y) \in [0,1]^2 \text{ with } x \neq y, \end{aligned}$$

and $\varphi(x,x)=f(x)$ a.e. for $x\in[0,1]\}$.

Since $\mathcal{U}_0\times\mathcal{V}_0$ is countable, it follows that $\mathcal{G}_{\text{sep}}$ is also countable. Moreover, by the definition of d_1 and by the dense properties of $\mathcal{U}_0$ and $\mathcal{V}_0$, we have $\mathcal{G}^{\text{sep}}$ is also dense in $\mathcal{G}$ with respect to d_1. This implies that $(\mathcal{G},d_1)$ is separable.

For $m\geqslant 0$, let

$$\mathcal{G}_m=\{g=(g(0),g(1),\dots)\in\mathcal{G}^{\mathbb{N}}: \\ g(r)\in\mathcal{G}\text{ for }0\leqslant r\leqslant m\text{ and }g(r)\equiv 0\text{ for }r>m\}.$$

For any $g\in\mathcal{G}^{\mathbb{N}}$ and $r\geqslant 0$, let $g^{\geqslant r}:[0,1]^2\to[0,\infty)$ be defined as

{{eq-hsqr}}
$$g^{\geqslant r}(x,y)=\sum_{s=r}^{\infty}g(s;x,y). \tag{2.15}$$

Note that $g^{\geqslant r}\in\mathcal{G}$. For any $m\geqslant 1$, we equip the space $\mathcal{G}_m$ with the distance

{{eq-dotimes}}
$$d_2(g_1,g_2)=\sup_{r\geqslant 0}d_1(g_1^{\geqslant r},g_2^{\geqslant r})\quad\text{for }g_1,g_2\in\mathcal{G}_m. \tag{2.16}$$

Again, we have for each $m\geqslant 0$, $(\mathcal{G}_m,d_2)$ is a metric space.

We then move on to prove separability of $(\mathcal{G}_m,d_2)$ for every finite $m\geqslant 0$. To this end, let

$$\mathcal{G}_m^{\text{sep}}=\{g\in\mathcal{G}_m:g(r)\in\mathcal{G}^{\text{sep}}\text{ for }0\leqslant r\leqslant m\text{ and }g(r)\equiv 0\text{ for }r>m\}.$$

Thus, $\mathcal{G}_m^{\text{sep}}$ is countable. Now, we prove $\mathcal{G}_m^{\text{sep}}$ is dense in $(\mathcal{G}_m,d_2)$. For any $g\in\mathcal{G}_m$, we have $g(r)\in\mathcal{G}$ for $0\leqslant r\leqslant m$ and $g(r)=0$ for $r>m$. By separability of $(\mathcal{G},d_1)$, for any $n\geqslant 1$ and $r\geqslant 0$, there exists a sequence $(\psi_{r,M})_{M\geqslant 1}\subset\mathcal{G}^{\text{sep}}$ that converges to $g(r)$. Then, there exists a number $M(r,n,m)$ such that

$$d_1(\psi_{r,M(r,n)},g(r))<\frac{1}{2^{r+1}(m+1)n}.$$

Let $g_n\in\mathcal{G}^{\mathbb{N}}$ be defined as $g_n(r)=\psi_{r,M(r,n)}$ for $0\leqslant r\leqslant m$ and $g_n(r)=0$ for $r>m$. Then, we have $g_n\in\mathcal{G}_m^{\text{sep}}$ and

$$\begin{aligned}d_2(g_n,g)=\sup_{r\geqslant 0}d_1(g_n^{\geqslant r},g^{\geqslant r})&\leqslant\sum_{r=0}^{\infty}\sum_{s=r}^{m}d_1(g_n(s),g(s))\\&=\sum_{r=0}^{\infty}\sum_{s=r}^{m}d_1(\psi_{s,M(s,n)},g(s))\leqslant\frac{1}{n}.\end{aligned}$$

Thus, $(g_n)_{n\geqslant 1}$ converges to g with respect to d_2. This shows that $\mathcal{G}_m^{\text{sep}}$ is dense in $(\mathcal{G}_m,d_2)$, and hence, $(\mathcal{G}_m,d_2)$ is separable.

We are now ready to construct $\mathcal{H}^{\text{sep}}$. For $m\geqslant 0$, let

$$\mathcal{H}_m=\Big\{h\in\mathcal{G}_m:\sum_{r\geqslant 0}h(r)\equiv 1,\,h_{\text{dg}}(2r+1)\equiv 0\text{ for all }r\geqslant 0\Big\}.$$

For each $m \geqslant 0$, we have $(\mathcal{H}_m, d_2)$ is a subspace of the metric space $(\mathcal{G}_m, d_2)$, and thus, is also separable. Let $\mathcal{H}_m^{\mathrm{sep}}$ be a countable and dense subset of $(\mathcal{H}_m, d_2)$, and $\mathcal{H}^{\mathrm{sep}} = \cup_{m\geqslant 0}\mathcal{H}_m^{\mathrm{sep}}$. Thus, $\mathcal{H}^{\mathrm{sep}}$ is countable. Moreover, we have $\mathcal{H}^{\mathrm{sep}} \subset \mathcal{H}$.

We finish this proof by showing that for any $h \in \mathcal{H}$, there exists a sequence $(h_n)_{n\geqslant 1} \subset \mathcal{H}^{\mathrm{sep}}$ such that for any $F \in \mathcal{M}$, $|t_F(h_n) - t_F(h)| \to 0$ as $n \to \infty$. To this end, fix $h \in \mathcal{H}$.

For each $n \geqslant 1$, there exists $m(n) > 0$ such that

{{eq2-hbound}}

$$\sum_{r\geqslant m(n)} \int_{[0,1]^2} h(r; x, y)\, dxdy + \sum_{r\geqslant m(n)} \int_{[0,1]} h(r; x, x) dx < \frac{1}{n}. \tag{2.17}$$

Let $\bar{h}_n \in \mathcal{H}$ be such that

{{eq-hbarn}}

$$\bar{h}_n(r) = \begin{cases} h(r) & \text{if } 0 \leqslant r < m(n), \\ \sum_{s\geqslant m(n)} h(s) & \text{if } r = m(n), \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Thus, we have $\bar{h}_n \in \mathcal{H}_{m(n)}$ and $\bar{h}_n^{\geqslant r} = h^{\geqslant r}$ for $0 \leqslant r \leqslant m(n)$. By the separability of $\mathcal{H}_{m(n)}$, there exists a sequence $(h_M^{\mathrm{sep}})_{M\geqslant 1} \subset \mathcal{H}_{m(n)}^{\mathrm{sep}} \subset \mathcal{H}^{\mathrm{sep}}$ such that $d_2(h_M^{\mathrm{sep}}, \bar{h}_n) \to 0$ as $M \to \infty$. Therefore, there exists an $M(n) > 0$ such that

{{eq23}}

$$d_2(h_{M(n)}^{\mathrm{sep}}, \bar{h}_n) \leqslant 1/n. \tag{2.19}$$

Choose $h_n = h_{M(n)}^{\mathrm{sep}}$. Now, it suffices to show that $|t_F(h_n) - t_F(h)| \to 0$ for all $F \in \mathcal{M}$ as $n \to \infty$. Let $k \geqslant 1$ and $F = (a_{ij})_{i,j\in[k]} \in \mathcal{M}_k$ be arbitrary. If $\max_{i,j} a_{ij} > m(n)$, by (2.17) and by definition of the multigraph parameter t_F, it is easy to see that $t_F(h_n) = 0$ and $t_F(h) \leqslant k^2/n$, and hence that

{{eq-tF0}}

$$|t_F(h_n) - t_F(h)| \leqslant k^2/n. \tag{2.20}$$

Moreover, by (2.16), (2.18) and (2.19), it follows that

{{eq2-24}}

$$\sup_{r\geqslant 0} d_1(h_n^{\geqslant r}, h^{\geqslant r}) = \sup_{r\geqslant 0} d_1(h_n^{\geqslant r}, \bar{h}_n^{\geqslant r}) = d_2(h_{M(n)}^{\mathrm{sep}}, \bar{h}_n) \leqslant \frac{1}{n} \qquad \text{for } 0 \leqslant r \leqslant m(n). \tag{2.21}$$

If $\max_{i,j} a_{ij} \leqslant m(n)$, we apply Lemma 2.3 (see below) and (2.21), and obtain

$$\begin{aligned} |t_F(h_n) - t_F(h)| &\leqslant \sum_{1\leqslant i<j\leqslant k} d_{\mathrm{sq}}(h_n^{[a_{ij}]}, h^{[a_{ij}]}) + \sum_{1\leqslant i\leqslant k} d_{\mathrm{dg}}(h_n^{[a_{ii}]}, h^{[a_{ii}]}) \\ &\leqslant \frac{k(k-1)}{n} + \frac{k}{n}. \end{aligned}$$

Hence, $|t_F(h_n) - t_F(h)| \to 0$ as required. □

{lem:10}

Lemma 2.3. *Let $h_1, h_2 \in \mathcal{H}$, let $k \geqslant 1$, and let $F \in \mathcal{M}_k$. Let $h^{\geqslant r}$ be defined as in* (2.15)*. Then*

$$\big|t_F(h_1) - t_F(h_2)\big| \leqslant \sum_{1\leqslant i<j\leqslant k} d_{\mathrm{sq}}(h_1^{[a_{ij}]}, h_2^{[a_{ij}]}) + \sum_{1\leqslant i\leqslant k} d_{\mathrm{dg}}(h_1^{[a_{ii}]}, h_2^{[a_{ii}]}).$$

Proof. Let

$$\theta(u) = \int_{[0,1]^k} \prod_{1\leqslant i\leqslant j\leqslant k} \Big(u \sum_{r\geqslant a_{ij}} h_1(r; x_i, x_j) \\ + (1-u) \sum_{r\geqslant a_{ij}} h_2(r; x_i, x_j) \Big) dx_1 \dots dx_k,$$

and thus

$$\theta'(u) = \int_{[0,1]^k} \sum_{1\leqslant i\leqslant j\leqslant k} b_{ij}(u) \Big(\sum_{r\geqslant a_{ij}} \big(h_1(r; x_i, x_j) - h_2(r; x_i, x_j) \big) \Big) dx_1 \dots dx_k.$$

where

$$b_{ij}(u) = \begin{cases} \prod_{\substack{1\leqslant i'\leqslant j'\leqslant k \\ (i',j')\neq(i,j)}} c_{i'j'}(u) & \text{if } \sum_{r\geqslant a_{ij}} (h_1(r; x_i, x_j) \neq \sum_{r\geqslant a_{ij}} h_2(r; x_i, x_j)), \\ 0 & \text{otherwise,} \end{cases}$$

and where

$$c_{ij}(u) = u \sum_{r\geqslant a_{ij}} h_1(r; x_i, x_j) + (1-u) \sum_{r\geqslant a_{ij}} h_2(r; x_i, x_j).$$

It follows that $0 \leqslant b_{ij}(u) \leqslant 1$ for all $0 \leqslant i \leqslant j \leqslant k$ and $u \in [0,1]$. Hence, for all $u \in [0,1]$,

$$|\theta'(u)| \leqslant \sum_{1\leqslant i<j\leqslant k} \int_{[0,1]^2} \Big| \sum_{r\geqslant a_{ij}} \big(h_1(r; x, y) - h_2(r; x, y) \big) \Big| dxdy \\ + \sum_{i=1}^{k} \int_{[0,1]} \Big| \sum_{r\geqslant a_{ii}} \big(h_1(r; x, x) - h_2(r; x, x) \big) \Big| dx.$$

The claim now easily follows. □

The collection of maps $(t_{F_i^*})_{i\geqslant 1}$ is not injective, since the values $(t_{F_i^*}(h))_{i\geqslant 1}$ determine $h \in \mathcal{H}$ only up to a measure-preserving transformation. Hence, we proceed to define an equivalence relation "$\cong$" in the canonical way. Let $h_1, h_2 \in \mathcal{H}$; we say h_1 and h_2 are equivalent and write $h_1 \cong h_2$ if $t_F(h_1) = t_F(h_2)$ for all $F \in \mathcal{M}$. Observing that

$$t^{\mathrm{ind}}_{K_{2,r}}(h) = t_{K_{2,r}}(h) - t_{K_{2,r+1}}(h), \quad t^{\mathrm{ind}}_{L_r}(h) = t_{L_r}(h) - t_{L_{r+1}}(h),$$

and using (2.3) and (2.4) to represent the second and third sum of in (2.7), it easily follows that $h_1 \cong h_2$ if and only if $d_{\mathrm{ms}}(h_1, h_2) = 0$. As an immediate consequence of Lemma 2.2, we have the following result.

Corollary 2.4. *The metric space* $(\mathcal{H} \backslash \cong, d_{\mathrm{ms}})$ *is complete and separable.*

Remark 2.5. Since functions that are continuous on the pseudo-metric space $(\mathcal{H}, d_{\mathrm{ms}})$ are also continuous on the induced metric space $(\mathcal{H} \backslash \cong, d_{\mathrm{ms}})$ and vice versa, there is no need to distinguish the two spaces when it comes to weak convergence, since weak convergence is determined by continuous and bounded functions. Therefore, in what follows, there is no need to make a distinction between the pseudo-metric space $(\mathcal{H}, d_{\mathrm{ms}})$ and the metric space $(\mathcal{H} \backslash \cong, d_{\mathrm{ms}})$, and so we will simply use the notation $(\mathcal{H}, d_{\mathrm{ms}})$ throughout.

2.2 Simple graphons

We now discuss some relations between multigraphons and simple graphons. First, let h be a multigraphon, and for fixed $r \geqslant 0$, we note that $h(r;\cdot,\cdot) : [0,1]^2 \to [0,1]$ is a simple graphon. Second, a simple graphon is a special case of multigraphon — for any graphon $\widehat{h}$, we can define its corresponding multigraphon as

$$h(0;x,y) = 1 - \widehat{h}(x,y), \quad h(1;x,y) = \widehat{h}(x,y), \quad h(r;x,y) = 0 \text{ for } r \geqslant 2. \tag{2.22}$$

{{eq2-9}}

Recall now that the homomorphism density of any simple graph F on k vertices in a simple graphon $\widehat{h}$ is defined as

$$t_F^{\mathrm{sim}}(\widehat{h}) = \int_{[0,1]^k} \prod_{ij\in F} \widehat{h}(x_i,x_j)dx_1\dots dx_k,$$

where $ij \in F$ indicates that (i,j) is an edge in F; see Lovász and Szegedy (2006). It is easy to see that $t_F^{\mathrm{sim}}(\widehat{h}) = t_F(h)$, where h is defined as in (2.22).

2.3 Weak convergence for multigraphon-valued random elements

{sub2.2}

In what follows, we use "$\longrightarrow$" to denote the convergence with respect to the underlying (pseudo)metric space, and we use "$\Longrightarrow$" to denote weak convergence, defined in the usual way. Specifically, in the metric space $(\mathcal{H}, d_{\mathrm{ms}})$, we say that a sequence of equivalence classes of multigraphons $(h_n)_{n\geqslant 1}$ of $\mathcal{H}$-valued random element converges weakly to $h \in \mathcal{H}$ as $n \to \infty$, written as "$h_n \Longrightarrow h$ in $\mathcal{H}$", if $\lim_{n\to\infty} \mathbb{E}f(h_n) = \mathbb{E}f(h)$ for any continuous and bounded function $f : (\mathcal{H}, d_{\mathrm{ms}}) \to \mathbb{R}$.

Although multigraphons have been introduced by Kolossváry and Ráth (2011), the characterisation of weak convergence for multigraphon sequences has not been discussed in the literature to the best of our knowledge. The following theorem provides some equivalent conditions of the weak convergence of multigraphon sequences, which is a generalization of Theorem 3.1 in Diaconis and Janson (2008).

{thm0}

Theorem 2.6. *Let $h, h_1, h_2, \dots \in (\mathcal{H}, d_{\mathrm{ms}})$ be a sequence of random multigraphons. Then the following are equivalent:*

(i) $h_n \Longrightarrow h$ *in* $(\mathcal{H}, d_{\mathrm{ms}})$ *as* $n \to \infty$*;*

(ii) for every $F \in \mathcal{M}$*, we have* $t_F(h_n) \Longrightarrow t_F(h)$ *in* $\mathbb{R}$ *as* $n \to \infty$*;*

(iii) for every $F \in \mathcal{M}$*, we have* $\lim_{n\to\infty} \mathbb{E}\{t_F(h_n)\} = \mathbb{E}\{t_F(h)\}$*;*

Proof. (i) $\Longrightarrow$ (ii). By the definition of d_{ms}, it follows that for any nonrandom $F \in \mathcal{M}$, the map $t_F(\cdot) : (\mathcal{H}, d_{\mathrm{ms}}) \to \mathbb{R}$ is continuous. By the continuous mapping theorem, we have *(i)* implies *(ii)*.

(ii) $\Longrightarrow$ (iii). This is a consequence of the bounded convergence theorem.

(iii) $\Longrightarrow$ (i). For any $F_1, F_2 \in \mathcal{M}$ with $v(F_1) = k_1$ and $v(F_2) = k_2$, and denoting by $A_1 = (a_{1;i,j})_{1\leqslant i,j\leqslant k_1}$ and $A_2 = (a_{2;i,j})_{1\leqslant i,j\leqslant k_2}$ by their adjacency

matrices, respectively. We have by definition that

$$\begin{aligned} t_{F_1}(h)t_{F_2}(h) &= \Big(\int_{[0,1]^{k_1}} \prod_{1\leqslant i\leqslant j\leqslant k_1} \sum_{r\geqslant a_{1;i,j}} h(r;x_i,x_j)dx_1\dots dx_{k_1}\Big) \\ &\quad\times \Big(\int_{[0,1]^{k_2}} \prod_{1\leqslant i'\leqslant j'\leqslant k_2} \sum_{r'\geqslant a_{2;i',j'}} h(r';y_{i'},y_{j'})dy_1\dots dy_{k_2}\Big) \\ &= t_{F_1\uplus F_2}(h), \end{aligned}$$

where $F_1 \uplus F_2$ is the disjoint union of F_1 and F_2. As $F_1 \uplus F_2 \in \mathcal{M}$, it follows that the class $\{t_F : F \in \mathcal{M}\}$ forms an algebra. Noting that $(\mathcal{H}, d_{\mathrm{ms}})$ is a complete and separable metric space, and by Lemma 2.8 below and Ethier and Kurtz (1986, Theorem 4.5(b), p. 113), we have $\{t_F, F \in \mathcal{M}\} \subset \mathcal{C}_b(\mathcal{H})$ is convergence determining, where $\mathcal{C}_b(\mathcal{H})$ is the class of bounded and continuous functions from $(\mathcal{H}, d_{\mathrm{ms}})$ to $\mathbb{R}$. Moreover, by *(iv)*, we have $\lim_{n\to\infty} \mathbb{E}\{t_F(h_n)\} = \mathbb{E}\{t_F(h)\}$ for all $F \in \mathcal{M}$. By Ethier and Kurtz (1986, Eq. (4.4), p. 112), we conclude that $h_n \Longrightarrow h$. □

Remark 2.7. It is tempting to interpret subgraph densities as the "moments" of random graphons. It may then come somewhat as a surprise that the family of functions $(t_F)_{F\in\mathcal{M}}$ is convergence determining even though the space $\mathcal{H}$ is not compact. In analogy to real-valued random variables, moments are convergence determining for probability measures on compact subsets of $\mathbb{R}$, but they are in general not convergence determining for measures on the whole real line. The reason is in essence that polynomials are bounded functions on compact sets and rich enough to be convergence determining, but they are unbounded when seen as functions on the whole real line, and so do not fall within the usual framework of weak convergence. This is in contrast to subgraph densities, which are always bounded functions, and so interpreting subgraph densities simply as the equivalent of moments of random variables does not fully capture the role they play in the theory of graphons and multigraphons.

The following lemma, used in the proof of Theorem 2.6, ensures that the family $\{t_F(\cdot), F \in \mathcal{M}\}$ strongly separates points in $(\mathcal{H}, d_{\mathrm{ms}})$.

{lem4}

Lemma 2.8. *The family of functions $\{t_F(\cdot), F \in \mathcal{M}\}$ strongly separates points in $(\mathcal{H}, d_{\mathrm{ms}})$.*

Proof. We need to show that for each $h \in \mathcal{H}$ and each $\varepsilon > 0$ there exists $m \geqslant 1$ such that

{{eq2-32}}

$$\inf_{h':d_{\mathrm{ms}}(h,h')\geqslant\varepsilon} \max_{1\leqslant i\leqslant m} \big|t_{F_i^*}(h) - t_{F_i^*}(h')\big| > 0, \tag{2.23}$$

where $(F_i^*)_{i\geqslant 1}$ is the enumeration of all multigraphs that generates the distance d_{ms}.

Now, fix $h \in \mathcal{H}$ and $\varepsilon > 0$. Recall that $K_{2,r}$ is the graph on two vertices with r edges connecting them, and let L_r is the graph on one vertex with r loops. Let

$$d_{\mathrm{sub}}(h,h') = \sum_{i\geqslant 1} 2^{-i}|t_{F_i^*}(h) - t_{F_i^*}(h')|. \tag{2.24}$$

By (2.13), it follows that $d_{\mathrm{ms}}(h,h') \to 0$ as $d_{\mathrm{sub}}(h,h') \to 0$. Then, there exists $\delta := \delta(\varepsilon)$ such that $d_{\mathrm{sub}}(h,h') < \delta$ implies $d_{\mathrm{ms}}(h,h') < \varepsilon$. Therefore, $\{h' : d_{\mathrm{ms}}(h,h') \geqslant \varepsilon\} \subset \{h' : d_{\mathrm{sub}}(h,h') \geqslant \delta\}$. Then, to show (2.23), it suffices to prove that there exists $m \geqslant 1$ such that

{{eq2-33}}
$$\inf_{h' : d_{\mathrm{sub}}(h,h') \geqslant \delta} \max_{1 \leqslant i \leqslant m} \bigl| t_{F_i^*}(h) - t_{F_i^*}(h') \bigr| > 0. \tag{2.25}$$

To this end, letting $m := m(\delta)$ be the smallest integer such that $\sum_{i>m} 2^{-i} < \delta/2$, we claim that for all h' satisfying $d_{\mathrm{sub}}(h,h') \geqslant \delta$,

{{eq2-34}}
$$\max_{1 \leqslant i \leqslant m} \bigl| t_{F_i^*}(h) - t_{F_i^*}(h') \bigr| \geqslant \frac{\delta}{2m}, \tag{2.26}$$

which implies (2.25) and hence (2.23).

We prove (2.26) by contradiction. If (2.26) does not hold, then

$$d_{\mathrm{sub}}(h,h') \leqslant \sum_{i=1}^{m} \bigl| t_{F_i^*}(h) - t_{F_i^*}(h') \bigr| + \sum_{i>m} 2^{-i} < \delta,$$

which contradicts $d_{\mathrm{sub}}(h,h') \leqslant \delta$. □

The definition of weak convergence extends naturally to multigraph sequences $G_1, G_2, \ldots$ through their multigraphon representation $h^{G_1}, h^{G_2}, \ldots$, and we simply write $G_n \Longrightarrow h$ if $h^{G_n} \Longrightarrow h$. As $v(G) \to \infty$, $t_F(G)$, $t_F^{\mathrm{inj}}(G)$ and $t_F^{\mathrm{ind}}(G)$ are equivalent. These equivalence relations, together with Theorem 2.6, yields the following corollary.

{cor0}

Corollary 2.9. *Let $G_1, G_2, \ldots \in \mathcal{M}$ be a sequence of random multigraphs defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $v(G_n) \to \infty$ $\mathbb{P}$-a.s. $(n \to \infty)$, and let $h \in \mathcal{H}$ be a random multigraphon. Then the following are equivalent:*

(i) $G_n \Longrightarrow h$ in $(\mathcal{H}, d_{\mathrm{ms}})$ as $n \to \infty$;

(ii) for every $F \in \mathcal{M}$, we have $t_F^{\mathrm{inj}}(G_n) \Longrightarrow t_F(h)$ in $\mathbb{R}$ as $n \to \infty$;

(iii) for every $F \in \mathcal{M}$, we have $t_F^{\mathrm{ind}}(G_n) \Longrightarrow t_F^{\mathrm{ind}}(h)$ in $\mathbb{R}$ as $n \to \infty$;

(iv) for every $F \in \mathcal{M}$, we have $\lim_{n\to\infty} \mathbb{E}\bigl\{t_F^{\mathrm{inj}}(G_n)\bigr\} \to \mathbb{E}\{t_F(h)\}$;

(v) for every $F \in \mathcal{M}$, we have $\lim_{n\to\infty} \mathbb{E}\bigl\{t_F^{\mathrm{ind}}(G_n)\bigr\} \to \mathbb{E}\bigl\{t_F^{\mathrm{ind}}(h)\bigr\}$.

2.4 Multigraphon-valued stochastic processes

{sub2.3}

Let $\mathcal{D} := \mathcal{D}([0,\infty), \mathcal{H})$, the càdlàg paths in $(\mathcal{H}, d_{\mathrm{ms}})$. Let κ be a $\mathcal{H}$-valued stochastic process. We write $\kappa(s)$ to denote the value of the process at time $s \geqslant 0$, which is an element of $\mathcal{H}$. For any $\kappa \in \mathcal{D}$ and $F \in \mathcal{M}$, we denote by $t_F(\kappa)$ the induced stochastic process defined as $t_F(\kappa)(s) = t_F(\kappa(s))$. By definition, it follows that $t_F(\kappa)$ takes values in $\mathcal{D}([0,\infty),[0,1])$.

We proceed to define the Skorohod topology on $\mathcal{D}$ in the usual way. Let

$$\Lambda = \bigl\{\lambda : [0,\infty) \to [0,\infty) :$$

$$\lambda \text{ is onto and increasing satisfying that } \gamma(\lambda) < \infty \big\}, \quad (2.27)$$

where

$$\gamma(\lambda) \coloneqq \sup_{0<s_1<s_2} \left| \log \frac{\lambda(s_2) - \lambda(s_1)}{s_2 - s_1} \right|.$$

We equip the space $\mathcal{D}$ with the distance d° defined as

{{eq-dcir}}

$$d^\circ(\kappa_1, \kappa_2) = \inf_{\lambda \in \Lambda} \left\{ \gamma(\lambda) \vee \int_0^\infty e^{-u} (\sup_{s \geqslant 0} d_{\mathrm{ms}}(\kappa_1(s \wedge u), \kappa_2(\lambda(s) \wedge u)) \wedge 1) du \right\}. \tag{2.28}$$

Again, we use "$\Longrightarrow$" to denote weak convergence with respect to the underlying (pseudo)metric space.

We have the following characterization of weak convergence in terms of subgraph densities.

{thm1}

Theorem 2.10. *Let $\kappa, \kappa_1, \kappa_2, \dots$ be random elements in $\mathcal{D}$. Then the following are equivalent:*

(i) $\kappa_n \Longrightarrow \kappa$ *in* $(\mathcal{D}, d^\circ)$ *as* $n \to \infty$;

(ii) for every $q \geqslant 1$ *and every* $F_1, \dots, F_q \in \mathcal{M}$, *we have*

$$\big(t_{F_1}(\kappa_n), \dots, t_{F_q}(\kappa_n)\big) \Longrightarrow \big(t_{F_1}(\kappa), \dots, t_{F_q}(\kappa)\big) \text{ in } \mathcal{D}([0,\infty), \mathbb{R}^q)$$

as $n \to \infty$;

(iii) for every $F \in \mathcal{M}$, *the sequence* $(t_F(\kappa_n))_{n \geqslant 1}$ *is tight, and for every* $q \geqslant 1$, *all real numbers* $0 \leqslant s_1 < \dots < s_q < \infty$ *where* κ *is continuous almost surely, and every* $F_1, \dots, F_q \in \mathcal{M}$, *we have*

$$\lim_{n \to \infty} \mathbb{E}\{t_{F_1}(\kappa_n(s_1)) \dots t_{F_q}(\kappa_n(s_q))\} = \mathbb{E}\{t_{F_1}(\kappa(s_1)) \dots t_{F_q}(\kappa(s_q))\}.$$

Proof. We apply several results from Ethier and Kurtz (1986), and use Lemmas 2.2 and 2.8.

{par2} *(i)* $\Longrightarrow$ *(ii).* By the definition of d_{ms} in (2.2) and (2.7), it follows that the homomorphism map t_F is continuous from $\mathcal{D}$ to $\mathcal{D}([0,\infty], \mathbb{R})$. By the continuous mapping theorem (c.f. Ethier and Kurtz (1986, Problem 13, p. 151), we have *(i)* implies *(ii)*.

{par3} *(ii)* $\Longrightarrow$ *(iii).* For $q = 1$, it follows from (ii) that $t_F(\kappa_n) \Longrightarrow t_F(\kappa)$, which implies that $(t_F(\kappa_n))_{n \geqslant 1}$ is tight. By the definition of weak convergence, for the points of almost sure continuity of κ, the finite dimensional convergence in *(iii)* follows from *(ii)*.

{par4} *(iii)* $\Longrightarrow$ *(i).* Let $\mathcal{C}_b(\mathcal{H})$ be the family of bounded and continuous functions that maps from $\mathcal{H}$ to $\mathbb{R}$ and let $\mathcal{F} \coloneqq \{t_F : F \in \mathcal{M}\}$; clearly, $\mathcal{F} \subset \mathcal{C}_b(\mathcal{H})$. By Lemma 2.8, we have that the family $\mathcal{F}$ strongly separates points in $(\mathcal{H}, d_{\mathrm{ms}})$. By the assumption of *(iii)*, we have that $(t_F(\kappa_n))_{n \geqslant 1}$ is tight for every $F \in \mathcal{M}$. Recall that $(\mathcal{H}, d_{\mathrm{ms}})$ is a complete and separable metric space. By Ethier and Kurtz (1986, p. 153, Problem 24), we have $\kappa_n \Longrightarrow \kappa$ follows from the convergence of finite dimensional distribution of κ_n to that of κ.

Now, it suffices to prove the convergence of finite-dimensional of κ_n. By Lemma 2.8 and by Ethier and Kurtz (1986, Theorem 4.5(b), p. 113), $\{t_F, F \in \mathcal{M}\}$ is convergence determining. By Ethier and Kurtz (1986, Proposition 4.6(b), p. 115), functions of the form $t_{F_1} \cdots t_{F_q}$ are convergence determining on the product space $(\mathcal{H})^q$ with the metric d_{ms}, and so convergence of finite dimensional distributions follows. This establishes *(i)*. $\square$

Let $(\boldsymbol{G}_n)_{n\geqslant 1} \subset \mathcal{D}([0,\infty),\mathcal{M})$ be a sequence of multigraph-valued processes; we denote by $G_n(s)$ the value of $\boldsymbol{G}_n$ at time s, which is a multigraph. We write $\boldsymbol{G}_n \Longrightarrow \kappa$ if the induced $\mathcal{H}$-valued process $\kappa^{\boldsymbol{G}_n}$ converges weakly to κ. For any $\boldsymbol{G} = (G(s))_{s\geqslant 0} \in \mathcal{D}([0,\infty),\mathcal{M})$ and $F \in \mathcal{M}$, we let $t_F(\boldsymbol{G})$, $t_F^{\mathrm{inj}}(\boldsymbol{G})$, $t_F^{\mathrm{ind}}(\boldsymbol{G})$ be the induced stochastic processes with paths in $\mathcal{D}([0,\infty),[0,1])$ defined as $t_F(\boldsymbol{G})(s) = t_F(G(s))$, $t_F^{\mathrm{inj}}(\boldsymbol{G})(s) = t_F^{\mathrm{inj}}(G(s))$ and $t_F^{\mathrm{ind}}(\boldsymbol{G})(s) = t_F^{\mathrm{ind}}(G(s))$. The following corollary provides some additional equivalent conditions for the weak convergence in terms of functionals t_F^{inj} and t_F^{ind}, which are direct consequences of Theorem 2.10.

{cor1}

Corollary 2.11. *Let $\boldsymbol{G}_1, \boldsymbol{G}_2, \ldots \in \mathcal{D}([0,\infty),\mathcal{M})$ be a sequence of multigraph-valued stochastic process such that*

$$\inf_{s\geqslant 0} v(G_n(s)) \to \infty \quad (n \to \infty),$$

where $v(G)$ is the number of vertices of G. Let κ be a random element in $\mathcal{D}$. Then the following are equivalent:

(i) $\boldsymbol{G}_n \Longrightarrow \kappa$ *in* $(\mathcal{D}, d^\circ)$ *as* $n \to \infty$*;*

(ii) $\big(t_{F_1}^{\mathrm{inj}}(\boldsymbol{G}_n), \ldots, t_{F_q}^{\mathrm{inj}}(\boldsymbol{G}_n)\big) \Longrightarrow \big(t_{F_1}(\kappa), \ldots, t_{F_q}(\kappa)\big)$ *in* $\mathcal{D}([0,\infty),\mathbb{R}^q)$ *as* $n \to \infty$ *for all* $q \geqslant 1$ *and all multigraphs* $F_1, \ldots, F_q \in \mathcal{M}$*;*

(iii) $\big(t_{F_1}^{\mathrm{ind}}(\boldsymbol{G}_n), \ldots, t_{F_q}^{\mathrm{ind}}(\boldsymbol{G}_n)\big) \Longrightarrow \big(t_{F_1}^{\mathrm{ind}}(\kappa), \ldots, t_{F_q}^{\mathrm{ind}}(\kappa)\big)$ *in* $\mathcal{D}([0,\infty),\mathbb{R}^q)$ *as* $n \to \infty$ *for all* $q \geqslant 1$ *and every* $F_1, \ldots, F_q \in \mathcal{M}$*;*

(iv) for every $F \in \mathcal{M}$, the sequence $\big(t_F^{\mathrm{inj}}(\boldsymbol{G}_n)\big)_{n\geqslant 1}$ is tight, and for every $q \geqslant 1$, all real numbers $0 \leqslant s_1 < \cdots < s_q < \infty$ where κ is continuous almost surely, and every $F_1, \ldots, F_q \in \mathcal{M}$, we have

$$\lim_{n\to\infty} \mathbb{E}\{t_{F_1}^{\mathrm{inj}}(G_n(s_1)) \ldots t_{F_q}^{\mathrm{inj}}(G_n(s_q))\} = \mathbb{E}\{t_{F_1}(\kappa(s_1)) \ldots t_{F_q}(\kappa(s_k))\}.$$

(v) for every $F \in \mathcal{M}$, the sequence $(t_F^{\mathrm{ind}}(\boldsymbol{G}_n))_{n\geqslant 1}$ is tight, and for all $q \geqslant 1$, all real numbers $0 \leqslant s_1 < \cdots < s_q < \infty$ where κ is continuous almost surely, and every $F_1, \ldots, F_q \in \mathcal{M}$, we have

$$\lim_{n\to\infty} \mathbb{E}\{t_{F_1}^{\mathrm{ind}}(G_n(s_1)) \ldots t_{F_q}^{\mathrm{ind}}(G_n(s_q))\} = \mathbb{E}\{t_{F_1}^{\mathrm{ind}}(\kappa(s_1)) \ldots t_{F_q}^{\mathrm{ind}}(\kappa(s_k))\}.$$

2.5 Erased graphs generated from multigraphs

{sub3}

In this subsection, we consider graphs that are simple graphs obtained from multigraphs by removing loops and merging multiple edges; we call these graphs *erased graphs* (see, e.g., van der Hofstad (2017, Chapter 7)). Specifically, let $G = (z_{ij})_{ij\in[n]} \in \mathcal{M}$ be a multigraph. The corresponding erased graph $\widehat{G} = (\widehat{z}_{ij})_{i,j\in[n]}$ of G is defined as

$$\widehat{z}_{ij} = \begin{cases} \mathbb{I}\{z_{ij} \geqslant 1\}, & i \neq j, \\ 0, & i = j. \end{cases}$$

The weak limiting behavior of simple graphon-valued stochastic process has been studied by Athreya, den Hollander and Röllin (2021+). We now introduce some notation. Let $\mathcal{W}$ be the space of graphons. We say $h_1, h_2 \in \mathcal{W}$ are equivalent if there exists two measure-preserving bijections σ_1 and σ_2 such that $h_1(\sigma_1 x, \sigma_1 y) = h_2(\sigma_2 x, \sigma_2 y)$. This equivalence relation yields the quotient space $\widetilde{\mathcal{W}}$. Let $\mathcal{D}([0,\infty), \widetilde{\mathcal{W}})$ be the set of càdlàg paths in $\widetilde{\mathcal{W}}$.

Let h be a multigraphon; we define its *erased graphon* $\widehat{h} : [0,1]^2 \to [0,1]$ by

$$\widehat{h}(x,y) = \sum_{r=1}^{\infty} h(r; x, y).$$

Similarly, for $\kappa \in \mathcal{D}$, we define the $\widetilde{\mathcal{W}}$-valued process $\widehat{\kappa}$ as at each $s \geqslant 0$, the element $\widehat{\kappa}(s) \in \widetilde{\mathcal{W}}$ is the equivalence class of the erased graphon of $\kappa(s)$.

{thm2}

Corollary 2.12. *Let $\widehat{\kappa}, \widehat{\kappa}_1, \widehat{\kappa}_2, \ldots \in \mathcal{D}([0,\infty), \widetilde{\mathcal{W}})$ be the corresponding erased process of $\kappa, \kappa_1, \kappa_2, \ldots$, respectively. If $\kappa_n \implies \kappa$ in $\mathcal{D}$, then $\widehat{\kappa}_n \implies \widehat{\kappa}$ in $\mathcal{D}([0,\infty), \widetilde{\mathcal{W}})$.*

Proof of Corollary 2.12. Let t^{sim} be the homomorphism density for simple graphons; that is, for any $h \in \mathcal{W}$, and any simple graph F with k vertices, let

$$t_F^{\mathrm{sim}}(h) = \int_{[0,1]^k} \prod_{ij\in F} h(x_i, x_j) dx_1 \ldots dx_k.$$

By Theorem 3.1 of Athreya, den Hollander and Röllin (2021+), it suffices to prove the following two conditions:

(i) Tightness. For every graph $F \in \mathcal{F}$, the sequence $(t_F^{\mathrm{sim}}(\widehat{\kappa}_n))_{n\geqslant 1}$ is tight.

(ii) Finite dimensional convergence. For all $q \geqslant 1$, all $0 \leqslant s_1 < \cdots < s_q < \infty$ and all $F_1, \ldots, F_q \in \mathcal{F}$,

$$\lim_{n\to\infty} \mathbb{E}\{t_{F_1}^{\mathrm{sim}}(\widehat{\kappa}_n(s_1)) \cdots t_{F_q}^{\mathrm{sim}}(\widehat{\kappa}_n(s_q))\} = \mathbb{E}\{t_{F_1}^{\mathrm{sim}}(\widehat{\kappa}(s_1)) \cdots t_{F_q}^{\mathrm{sim}}(\widehat{\kappa}(s_q))\}.$$

Now, for κ and $(\kappa_j)_{j\geqslant 1}$, consider the truncated multigraphon processes $\bar{\kappa}$ and $(\bar{\kappa}_j)_{j\geqslant 1}$, that are defined by, for $x \neq y$,

$$\bar{\kappa}(s;0;x,y) = \kappa(s;0;x,y), \quad \bar{\kappa}(s;1;x,y) = \sum_{r=1}^{\infty} \kappa(s;r;x,y),$$

$$\bar{\kappa}_j(s;0;x,y) = \kappa_j(s;0;x,y), \quad \bar{\kappa}_j(s;1;x,y) = \sum_{r=1}^{\infty} \kappa_j(s;r;x,y),$$

and

$$\bar{\kappa}(s;0;x,x)=1,\quad \bar{\kappa}(s;1;x,x)=1,\quad \bar{\kappa}_j(s;0;x,x)=1,\quad \bar{\kappa}_j(s;1;x,x)=0.$$

Then, it is easy to check that the map $\kappa \in \mathcal{D} \mapsto \bar{\kappa} \in \mathcal{D}$ is continuous.

By the construction of $\widehat{\kappa}$ and $(\widehat{\kappa}_j)_{j\geqslant 1}$, we have for any simple graph F,

$$t_F^{\mathrm{sim}}(\widehat{\kappa}) = t_F(\bar{\kappa}),\quad t_F^{\mathrm{sim}}(\widehat{\kappa}_j) = t_F(\bar{\kappa}_j).$$

Since $\kappa_n \Longrightarrow \kappa$ as $n \to \infty$, and by the continuous mapping theorem, we have $\bar{\kappa}_n \Longrightarrow \bar{\kappa}$ as $n \to \infty$. Thus, we have *(i)* and *(ii)* are satisfied, and hence the theorem is proved. □

3 DYNAMICS ON CONFIGURATION RANDOM GRAPHS

{sec1}

3.1 Configuration random graph

{sub6}

The configuration model was originally introduced by Bender and Canfield (1978) and Bollobás (1980), who considered a uniform simple d-regular graph on n nodes. This model was later generalized by Molloy and Reed (1995), who obtained conditions for the existence of a giant component; we refer to van der Hofstad (2017) for an in-depth discussion.

We proceed with the mathematical definition of the model. Let $n \geqslant 1$ be an integer and let $d_n = (d_{n,1}, \dots, d_{n,n})$ be a sequence of positive integers. Let $\ell_n \coloneqq \sum_{i=1}^n d_{n,i}$ be the sum of all degrees; we assume that ℓ_n is even. To construct a multigraph where vertex j has degree $d_{n,j}$ we start with n vertices, where vertex j has $d_{n,j}$ half-edges for $1 \leqslant j \leqslant n$. We further assume that the half-edges are numbered in an arbitrary order from 1 to ℓ_n. We construct the configuration random multigraph as follows. Connect the first half-edge with one of the $\ell_n - 1$ remaining ones, chosen uniformly at random. Continue the procedure for the remaining half-edges until all of them are connected. The distribution of the resulting multigraph on the set $\mathcal{M}_n$ is denoted by $\mathrm{CM}(d_n)$.

Let $D_n \coloneqq (D_{n,1}, \dots, D_{n,n})$ be a random degree sequence defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We proceed to prove that $G_n \sim \mathrm{CM}(D_n)$ converges in distribution to a random multigraphon, and the limiting multigraphon depends on the limiting behaviour of the random degree sequence D_n. To specify the limiting multigraphon, we need to introduce some assumptions. Let $L_n = \sum_{i=1}^n D_{n,i}$ and $Y_n = L_n/n^2$. For $k \geqslant 1$, let $Z_{n,1}, \dots, Z_{n,k}$ be a simple random sample from the set $\{nD_{n,1}/L_n, \dots, nD_{n,n}/L_n\}$, chosen uniformly and without replacement. Assume that for each $k \geqslant 1$, there exists a vector of random variables $(Z_1, \dots, Z_k, Y)$ such that, as $n \to \infty$,

{{eq-lim}}

$$(Z_{n,1}, \dots, Z_{n,k}, Y_n) \Longrightarrow (Z_1, \dots, Z_k, Y) \quad \text{in } \mathbb{R}^k \times \mathbb{R}, \tag{3.1}$$

where $Z_1, \dots, Z_k$ are conditionally independent given Y, and have a common distribution function Ψ. Here, Ψ may depend on Y. Define the generalised inverse of Ψ as

{{eq-barpsi}}

$$\bar{\Psi}(x) = \inf\{y : \Psi(y) \geqslant x\}, \quad x \in [0,1]. \tag{3.2}$$

Now, we are ready to define the limiting multigraphon. Let

{{eq19}}
$$h(r;x,y)=\begin{cases} p\big(r;Y\bar\Psi(x)\bar\Psi(y)\big) & \text{if } x\neq y,\\ p\Big(\dfrac{r}{2};\dfrac{Y\bar\Psi(x)^2}{2}\Big) & \text{if } x=y \text{ and if } r \text{ is even},\\ 0 & \text{otherwise,}\end{cases} \tag{3.3}$$

where $p(r;\lambda)=e^{-\lambda}\lambda^k/k!$ for $r\geqslant 0$ and where $\bar\Psi$ and Y are as in (3.1) and (3.2).

{t3.1}

Theorem 3.1. *Let $G_n\sim \mathrm{CM}(D_n)$, and let $L_n:=\sum_{i=1}^n D_{n,i}$. Assume that* (3.1) *holds and that*

{{eq-rho}}
$$L_n\geqslant n \quad \textit{and} \quad \max_{1\leqslant i\leqslant n} D_{n,i}/(L_n^{1/2}(\log n)^2)\to 0 \quad \mathbb{P}\text{-a.s.} \tag{3.4}$$

Then $G_n\Longrightarrow h$.

Before proving Theorem 3.1, we first prove a lemma.

{lem:prob}

Lemma 3.2. *For each $n\geqslant 1$, let d_n be a degree sequence, and let $\ell_n:=\sum_{i=1}^n d_{n,i}$. Assume that $\ell_n\geqslant n$ and that*

{{eq-conlp}}
$$\max_{1\leqslant i\leqslant n} d_{n,i}/(\ell_n^{1/2}(\log n)^2)\to 0 \quad \textit{as } n\to\infty. \tag{3.5}$$

Let $G_n=(G_{n,ij})_{i,j\in[n]}\sim\mathrm{CM}(d_n)$. Then, for any σ_n and any multigraph $F=(a_{ij})_{1\leqslant i\leqslant j\leqslant k}\in\mathcal{M}_k$, we have

{{100}}
$$\Big|\mathbb{P}[G_{n,\sigma_n}=F]-\prod_{1\leqslant i<j\leqslant k}p(a_{ij};y_nz_{n,\sigma_n(i)}z_{n,\sigma_n(j)})\prod_{i=1}^k p\Big(\frac{a_{ii}}{2};\frac{y_nz_{n,\sigma_n(i)}^2}{2}\Big)\Big| \leqslant C_Fn^{-1/4}, \tag{3.6}$$

where $\{G_{n,\sigma_n}=F\}$ is the event that $G_{n,\sigma_n(i),\sigma_n(j)}=a_{ij}$ for all $1\leqslant i\leqslant j\leqslant k$, and where $z_{n,j}=nd_{n,j}/\ell_n$ for $1\leqslant j\leqslant n$, $y_n=\ell_n/n^2$ and $C_F>0$ is a constant depending only on F.

Proof. Let $d_i(F)$ be the degree of node i in F and let $\ell(F)=\sum_{i=1}^k d_i(F)$. Let $c(F)=(\prod_{1\leqslant i<j\leqslant k}a_{ij}!\prod_{i=1}^k a_{ii}!!)^{-1}$. Rewriting the second term of the left hand side of (3.6) gives

$$\prod_{1\leqslant i<j\leqslant k}p(a_{ij};y_nz_{n,\sigma_n(i)}z_{n,\sigma_n(j)})\prod_{i=1}^k p\Big(\frac{a_{ii}}{2};\frac{y_nz_{n,\sigma_n(i)}^2}{2}\Big) = c(F)\exp\Big(-\frac12\Big(\sum_{j=1}^k\frac{d_{n,\sigma_n(j)}}{\ell_n^{1/2}}\Big)^2\Big)\prod_{i=1}^k\ell_n^{-d_i(F)/2}d_{n,\sigma_n(i)}^{d_i(F)}.$$

Thus, it suffices to prove that for large n,

$$\Big|\mathbb{P}[G_{n,\sigma_n}=F]-c(F)\exp\Big(-\frac12\Big(\sum_{j=1}^k\frac{d_{n,\sigma_n(j)}}{\ell_n^{1/2}}\Big)^2\Big)\prod_{i=1}^k\ell_n^{-d_i(F)/2}d_{n,\sigma_n(i)}^{d_i(F)}\Big|$$

{{p1-05}}
$$\leqslant C_F n^{-1/4}. \quad (3.7)$$

Let $\ell_{\sigma_n} = \sum_{i=1}^k d_{n,\sigma_n(i)}$. Now, by Ráth and Szakács (2012, Eqs. (49) and (50)), we have

{{p1-00}}
$$\begin{aligned}&\mathbb{P}[G_{n,\sigma_n} = F] \\ &= c(F)\frac{\prod_{i=1}^k d_{n,\sigma_n(i)}!}{\prod_{i=1}^k (d_{n,\sigma_n(i)} - d_i(F))!}\frac{(\ell_n/2)!2^{\ell_{\sigma_n}-\ell(F)/2}}{(\ell_n/2-\ell_{\sigma_n}+\ell(F)/2)!}\frac{(\ell_n-\ell_{\sigma_n})!}{\ell_n!}. \quad (3.8)\end{aligned}$$

The rest of the proof includes two steps.

Step 1. We show that there exists $n_1 > 1$ such that for all $n \geqslant n_1$,

{{eq-Step1}}
$$\left|\frac{(\ell_n/2)!2^{\ell_{\sigma_n}-\ell(F)/2}}{(\ell_n/2-\ell_{\sigma_n}+\ell(F)/2)!}\frac{(\ell_n-\ell_{\sigma_n})!}{\ell_n!} - (\ell_n)^{-\ell(F)/2}e^{-\ell_{\sigma_n}^2/(2\ell_n)}\right| \leqslant C_F n^{-1/3}. \quad (3.9)$$

To this end, we use the well-known Stirling's approximation to estimate the first term of the left hand side of (3.9):

{{111-0}}
$$\sqrt{2\pi}x^{x+1/2}e^{-x+1/(12x+1)} \leqslant \Gamma(x+1) \leqslant \sqrt{2\pi}x^{x+1/2}e^{-x+1/(12x)} \quad (3.10)$$

for $x > 0$, where Γ is the Gamma function. Rewriting the first term of (3.9) as

$$\frac{(\ell_n/2)!2^{\ell_{\sigma_n}-\ell(F)/2}}{(\ell_n/2-\ell_{\sigma_n}+\ell(F)/2)!}\frac{(\ell_n-\ell_{\sigma_n})!}{\ell_n!} = I_1 \times I_2,$$

where

$$I_1 = \frac{2^{-\ell(F)/2}(\ell_n/2-\ell_{\sigma_n})!}{(\ell_n/2-\ell_{\sigma_n}+\ell(F)/2)!}, \qquad I_2 = \frac{(\ell_n/2)!2^{\ell_{\sigma_n}}}{(\ell_n/2-\ell_{\sigma_n})!}\frac{(\ell_n-\ell_{\sigma_n})!}{\ell_n!}.$$

By (3.5), we have

{{eq-conlp2}}
$$\ell_{\sigma_n}/(\ell_n^{1/2}(\log n)^2) \leqslant k \max_{1\leqslant i\leqslant n} d_{n,i}/(\ell_n^{1/2}(\log n)^2) \to 0 \quad \text{as } n \to \infty. \quad (3.11)$$

Recalling the assumption that $\ell_n \geqslant n$, we have there exists $n_1 \geqslant 1$ such that for all $n \geqslant n_1$,

{{eq-111-ell}}
$$\ell_{\sigma_n}/\ell_n \leqslant n^{-1/3} \leqslant 0.1, \quad \ell_n - 2\ell_{\sigma_n} \geqslant \frac{1}{2}\ell_n. \quad (3.12)$$

By $\ell_n \geqslant n$ again, and by (3.10), we have $\ell_n/2 - \ell_{\sigma_n} \geqslant n/4$ for $n \geqslant n_1$, and

{{eq-111-I1}}
$$\begin{aligned}I_1 &\leqslant \frac{(2e)^{-\ell(F)/2}(\ell_n/2-\ell_{\sigma_n})^{\ell_n/2-\ell_{\sigma_n}+1/2}}{(\ell_n/2-\ell_{\sigma_n}+\ell(F)/2)^{\ell_n/2-\ell_{\sigma_n}+\ell(F)/2+1/2}}e^{1/(3n)} \\ &= \frac{e^{-\ell(F)/2}(\ell_n/2-\ell_{\sigma_n})^{\ell_n/2-\ell_{\sigma_n}+1/2}}{(\ell_n/2-\ell_{\sigma_n}+\ell(F)/2)^{\ell_n/2-\ell_{\sigma_n}+\ell(F)/2+1/2}}(\ell_n-2\ell_{\sigma_n})^{-\ell(F)}e^{1/(3n)}. \quad (3.13)\end{aligned}$$

Moreover, we have for $n \geqslant n_1$, the fraction term in the right hand side of (3.13) can be bounded by

$$\left|\frac{e^{-\ell(F)/2}(\ell_n/2-\ell_{\sigma_n})^{\ell_n/2-\ell_{\sigma_n}+\ell(F)/2+1/2}}{(\ell_n/2-\ell_{\sigma_n}+\ell(F)/2)^{\ell_n/2-\ell_{\sigma_n}+\ell(F)/2+1/2}} - 1\right| \leqslant C_F n^{-1}.$$

Therefore,

{{p1-02}}

$$\begin{aligned} I_1 &\leqslant \big(\ell_n - 2\ell_{\sigma_n}\big)^{-\ell(F)/2}(1+Q_1) \\ &\leqslant \ell_n^{-\ell(F)/2}(1+Q_2), \end{aligned} \tag{3.14}$$

for some $|Q_1| \leqslant C_F n^{-1}$ and $|Q_2| \leqslant C_F n^{-1/3}$, and we used (3.12) in the last line. Using a similar argument we obtain

{{p1-02'}}

$$I_1 \geqslant \ell_n^{-\ell(F)/2}(1-Q_2). \tag{3.15}$$

Now we consider I_2. Observe that for $n \geqslant n_1$,

{{eq-11l-39}}

$$\left|\frac{\ell_n}{\ell_n - 2\ell_{\sigma_n}}\frac{\ell_n - \ell_{\sigma_n}}{\ell_n} - 1\right| \leqslant \frac{2\ell_{\sigma_n}}{\ell_n} \leqslant 2n^{-1/3}. \tag{3.16}$$

By (3.10) and (3.16), and noting that $\ell_n \geqslant n$,

$$\begin{aligned} I_2 &\leqslant \sqrt{\frac{\ell_n}{\ell_n - 2\ell_{\sigma_n}}\frac{\ell_n - \ell_{\sigma_n}}{\ell_n}}\left(\frac{(\ell_n - \ell_{\sigma_n})^{2-2\ell_{\sigma_n}/\ell_n}}{\ell_n(\ell_n - 2\ell_{\sigma_n})^{1-2\ell_{\sigma_n}/\ell_n}}\right)^{\ell_n/2}\exp(C_F n^{-1/3}) \\ &= \left(\frac{(\ell_n - \ell_{\sigma_n})^{2-2\ell_{\sigma_n}/\ell_n}}{\ell_n(\ell_n - 2\ell_{\sigma_n})^{1-2\ell_{\sigma_n}/\ell_n}}\right)^{\ell_n/2}\exp(C_F n^{-1/3}) \\ &= \left(\frac{(1-x_n)^{2-2x_n}}{(1-2x_n)^{1-2x_n}}\right)^{\ell_n/2}\exp(C_F n^{-1/3}), \end{aligned}$$

where $x_n = \ell_{\sigma_n}/\ell_n$. Let $\psi(x) = (1-x)^{2-2x}/(1-2x)^{1-2x}$; by Taylor's expansion and recalling (3.12), we have for $n \geqslant n_1$,

$$\psi(x_n) = 1 - x_n^2 + \frac{\psi'''(\xi_n)}{6}x_n^3,$$

for some $|\xi_n| \leqslant 0.1$. A direct calculation implies $\sup_{|x|\leqslant 0.1}|\psi'''(x)| \leqslant 8$, and we have for $n \geqslant n_1$,

$$\psi(x_n) = 1 - x_n^2 + u_n x_n^3,$$

for some $|u_n| \leqslant 1.4$. Moreover, recalling (3.12), we have for $n \geqslant n_1$,

$$|(1 - x_n^2 + u_n x_n^3)^{\ell_n/2} - e^{-\ell_{\sigma_n}^2/(2\ell_n)}| \leqslant C_F n^{-1/3},$$

and therefore, for $n \geqslant n_1$,

{{p1-03}}

$$I_2 - e^{-\ell_{\sigma_n}^2/(2\ell_n)} \leqslant C_F n^{-1/3}. \tag{3.17}$$

Similarly, for $n \geqslant n_1$,

{{p1-03'}}

$$I_2 - e^{-\ell_{\sigma_n}^2/(2\ell_n)} \geqslant -C_F n^{-1/3}. \tag{3.18}$$

By (3.14), (3.15), (3.17) and (3.18), we obtain (3.9).

Step 2. Let $b_n = \min\{d_{n,\sigma_n(i)} : d_i(F) > 0\}$. In this step, we prove (3.7) for the two cases that $b_n < \ell_n^{1/4}$ and $b_n \geqslant \ell_n^{1/4}$ separately. Observe that

{{p1-04}}

$$\frac{\prod_{i=1}^k d_{n,\sigma_n(i)}!}{\prod_{i=1}^k (d_{n,\sigma_n(i)} - d_i(F))!} \leqslant \prod_{i=1}^k d_{n,\sigma_n(i)}^{d_i(F)}. \tag{3.19}$$

By (3.5), we have there exists $n_2 \geqslant 1$ such that for all $n \geqslant n_2$,

{{eq-l11-d1}}
$$\max_{1 \leqslant i \leqslant n} d_{n,i}/\ell_n \leqslant 1. \tag{3.20}$$

By (3.8), (3.9), (3.19) and (3.20), we have for $n \geqslant \max\{n_1, n_2\}$,

{{p1-06}}
$$\begin{aligned}
&\Big|\mathbb{P}[G_{n,\sigma_n} = F] - c(F)\ell_n^{-\ell(F)/2} \exp\Big(-\frac{\ell_{\sigma_n}^2}{2\ell_n}\Big) \prod_{i=1}^{k} d_{n,\sigma_n(i)}^{d_i(F)}\Big| \\
&\qquad \leqslant C_F n^{-1/3} \ell_n^{-\ell(F)/2} \prod_{i=1}^{k} d_{n,\sigma_n(i)}^{d_i(F)} \\
&\qquad \leqslant C_F n^{-1/3}.
\end{aligned} \tag{3.21}$$

Noting that if $b_n < \ell_n^{1/4}$, then there exists $j^* \in \{1, \ldots, k\}$ and $n_3 \geqslant 1$ such that $d_{j^*}(F) \geqslant 1$ and $d_{n,\sigma_n(j^*)}/\sqrt{\ell_n} \leqslant \ell_n^{-1/4} \leqslant n^{-1/4}$ for $n \geqslant n_3$. Then it follows that for $n \geqslant n_3$,

{{p1-07}}
$$\begin{aligned}
&\Big|\ell_n^{-\ell(F)/2} \exp\Big(-\frac{\ell_{\sigma_n}^2}{2\ell_n}\Big) \prod_{i=1}^{k} d_{n,\sigma_n(i)}^{d_i(F)}\Big| \\
&\qquad = \Big|\exp\Big(-\frac{1}{2}\Big(\sum_{i=1}^{k} \frac{d_{n,\sigma_n(i)}}{\ell_n^{1/2}}\Big)^2\Big) \prod_{i=1}^{k} \ell_n^{-d_i(F)/2} d_{n,\sigma_n(i)}^{d_i(F)}\Big| \\
&\qquad \leqslant C_F \exp\Big(-\frac{d_{n,\sigma_n(j^*)}}{2\ell_n^{1/2}}\Big) \ell_n^{-d_{j^*}(F)/2} d_{n,\sigma_n(j^*)}^{d_{j^*}(F)} \\
&\qquad \leqslant C_F n^{-1/4}.
\end{aligned} \tag{3.22}$$

Therefore, if $b_n < \ell_n^{1/4}$, by (3.21) and (3.22), we have for $n \geqslant \max\{n_1, n_2, n_3\}$,

{{p1-08}}
$$\mathbb{P}[G_{n,\sigma_n} = F] \leqslant C_F n^{-1/4}. \tag{3.23}$$

Combining (3.22) and (3.23) we have that (3.7) holds for $b_n < \ell_n^{1/4}$ and $n \geqslant \max\{n_1, n_2, n_3\}$.

If $b_n \geqslant \ell_n^{1/4}$, then it follows that $b_n \geqslant n^{1/4}$. By Stirling's formula (3.10), we have

{{p1-01}}
$$\frac{\prod_{i=1}^{k} d_{n,\sigma_n(i)}!}{\prod_{i=1}^{k} (d_{n,\sigma_n(i)} - d_i(F))!} = (1 + Q_3) \prod_{i=1}^{k} d_{n,\sigma_n(i)}^{d_i(F)}, \tag{3.24}$$

for some $|Q_3| \leqslant C_F n^{-1/4}$. Also, by (3.11), we have there exists $n_4 \geqslant 1$ such that $d_{n,\sigma_n(i)} \leqslant \ell_n^{2/3}$ for all $n \geqslant n_4$. Thus, for $n \geqslant n_4$,

$$\prod_{i=1}^{k} \Big(\ell_n^{-d_i(F)} d_{n,\sigma_n(i)}^{d_i(F)}\Big) \leqslant C_F n^{-1/3}.$$

Substituting (3.9) and (3.17) to (3.8), we have for $n \geqslant \max\{n_1, n_2, n_4\}$,

$$
\begin{aligned}
&\Big|\mathbb{P}[G_{n,\sigma_n} = F] - c(F)\exp\Big(-\frac{1}{2}\Big(\sum_{j=1}^{k}\frac{d_{n,\sigma_n(j)}}{\ell_n^{1/2}}\Big)^2\Big)\prod_{i=1}^{k}\ell_n^{-d_i(F)/2}d_{n,\sigma_n(i)}^{d_i(F)}\Big| \\
&\qquad \leqslant C_F n^{-1/4}\exp\Big(-\frac{1}{2}\Big(\sum_{j=1}^{k}\frac{d_{n,\sigma_n(j)}}{\ell_n^{1/2}}\Big)^2\Big)\prod_{i=1}^{k}\ell_n^{-d_i(F)/2}d_{n,\sigma_n(i)}^{d_i(F)} + C_F n^{-1/4} \\
&\qquad \leqslant C_F n^{-1/4}.
\end{aligned}
$$

Then, (3.7) also holds if $b_n \geqslant \ell_n^{1/4}$ and $n \geqslant \max\{n_1, n_2, n_4\}$. This proves (3.7) for $n \geqslant \max\{n_1, n_2, n_3, n_4\}$. □

Proof of Theorem 3.1. Denote by $\mathbb{E}_{D_n}$ and $\mathbb{P}_{D_n}$, respectively, the conditional expectation operator the conditional probability operator, respectively, given D_n. By (2.1) and Lemma 3.2, we have for any $k \geqslant 1$ and $F = (a_{ij})_{1\leqslant i\leqslant j\leqslant k} \in \mathcal{M}_k$,

{{eq-t31-1}}

$$
\begin{aligned}
&\Big|\mathbb{E}_{D_n}\{t_F^{\mathrm{ind}}(G_n)\} - \mathbb{E}_{D_n}\Big\{\prod_{1\leqslant i<j\leqslant k} p(a_{ij}; Y_n Z_{n,i} Z_{n,j})\prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{Y_n Z_{n,i}^2}{2}\Big)\Big\}\Big| \\
&\qquad\qquad \leqslant C_F n^{-1/4}, \qquad (3.25)
\end{aligned}
$$

where $Z_{n,1}, \ldots, Z_{n,k}$ are independently chosen with replacement from the set $\{nD_{n,1}/L_n, \ldots, nD_{n,n}/L_n\}$ and $Y_n = L_n/n^2$. By (3.1),

{{eq-t31-2}}

$$
\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\Big\{\prod_{1\leqslant i<j\leqslant k} p(a_{ij}; Y_n Z_{n,i} Z_{n,j})\prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{Y_n Z_{n,i}^2}{2}\Big)\Big\} \\
&\quad = \mathbb{E}\Big\{\prod_{1\leqslant i<j\leqslant k} p(a_{ij}; Y\bar{\Psi}(U_i)\bar{\Psi}(U_j))\prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{Y\bar{\Psi}(U_i)^2}{2}\Big)\Big\},
\end{aligned}
\tag{3.26}
$$

where $U_1, \ldots, U_k$ are independent random variables uniformly distributed on $[0, 1]$ and also independent of all others. By (2.6), (3.25) and (3.26) we obtain

{{eq18}}

$$
\lim_{n\to\infty}\mathbb{E}\{t_F^{\mathrm{ind}}(G_n)\} = \mathbb{E}\{t_F^{\mathrm{ind}}(h)\} \quad \text{for all } F \in \mathcal{M}. \tag{3.27}
$$

By *(iii)* of Corollary 2.9, we conclude that $G_n \Longrightarrow h$, which completes the proof. □

3.2 Edge reconnection model: A dynamic network model

{sec2}

In this subsection, we consider a dynamic network model, which we call the *edge reconnection model*. This dynamic model is based on a random multigraph growth process, and the latter was introduced by Pittel (2010) and further studied by Borgs, Chayes, Lovász, Sós and Vesztergombi (2011) and Ráth and Szakács (2012) for the graph limit behavior in the dense graphs.

The random multigraph growth model is defined as follows. Let $n \geqslant 1$ and let $\theta > 0$. Let $H_n(0)$ be the empty graph on the vertex set $[n]$. For

$m \geqslant 0$, and given $H_n(m)$ having the degree sequence $d_n = (d_{n,1}, \ldots, d_{n,n})$, we construct $H_n(m+1)$ by adding a new edge (i,j) with the following preferential-attachment-type probability:

{{eq-growth}}
$$\begin{cases} \dfrac{2(d_{n,i}+\theta)(d_{n,j}+\theta)}{(2m+n\theta)(2m+n\theta+1)} & \text{if } i \neq j, \\ \dfrac{(d_{n,i}+\theta)(d_{n,j}+\theta+1)}{(2m+n\theta)(2m+n\theta+1)} & \text{if } i = j. \end{cases} \tag{3.28}$$

Note that by this construction, both loops and multiple edges are allowed in $(H_n(m))_{m\geqslant 0}$, and for each $m \geqslant 0$, there are $2m$ half-edges in $H_n(m)$. For each $m \geqslant 0$, let $D_n^*(m) = (D_{n,1}^*(m), \ldots, D_{n,n}^*(m))$ be the degree sequence of $H_n(m)$.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, write $(x)_n = x(x-1)\cdots(x-n+1)$ as the falling factorial and write $x^{(n)} = x(x+1)\ldots(x+n-1)$ as the rising factorial; the value of each is taken to be 1 if $n = 0$. The following lemma states that, conditional on the degree sequence, the random multigraph $H_n(m)$ has distribution $\mathrm{CM}(d_n)$.

{lem7}

Lemma 3.3. *Let $d_n = (d_{n,1}, \ldots, d_{n,n})$ be a degree sequence satisfying that $\sum_{i=1}^n d_{n,i} = 2m$. Then, we have*

{{eq-proH}}
$$\mathscr{L}\big(H_n(m)\big|D_n^*(m) = d_n\big) = \mathrm{CM}(d_n). \tag{3.29}$$

Proof of Lemma 3.3. Let $G = (x_{ij})_{1\leqslant i\leqslant j\leqslant n} \in \mathcal{M}_n$ be a multigraph with the given degree sequence d_n. It follows from Pittel (2010, Eqs. (2.1) and (2.13)) that

{{lab40}}
$$\mathbb{P}\big[H_n(m) = G\big] = \frac{\prod_{i=1}^n \theta^{(d_{n,i})}}{(n\theta)^{(2m)}} \frac{(2m)!!}{\prod_{1\leqslant i<j\leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!}, \tag{3.30}$$

and

{{lab41}}
$$\mathbb{P}\big[D_n^*(m) = d_n\big] = \frac{(2m)!}{(n\theta)^{(2m)}} \prod_{i=1}^n \frac{\theta^{(d_{n,i})}}{d_{n,i}!}. \tag{3.31}$$

It can be shown (see, e.g., Lemma 1.6 of Bordenave (2006)) that

{{eq-proG}}
$$\mathrm{CM}(d_n)\{G\} = \frac{1}{(2m-1)!!} \frac{\prod_{i=1}^n d_{n,i}!}{\prod_{1\leqslant i<j\leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!}. \tag{3.32}$$

This completes the proof by combining (3.30)–(3.32). □

Now, we proceed to define an $\mathcal{M}$-valued stochastic process $(G_n(m))_{m\geqslant 0}$ which is built on $(H_n(m))_{m\geqslant 0}$. For each $m \geqslant 0$, let $D_n(m) = (D_{n,1}(m), \ldots, D_{n,n}(m))$ be the degree sequence of $G_n(m)$ and let $L_n(m) = \sum_{i=1}^n D_{n,i}(m)$. For each $m \geqslant 1$, we consider the following three types of updates:

(I) *Add one edge.* In this step, we choose two vertices at random and add an edge between them. Formally, given the graph $G_n(m-1)$, add one edge between i and j with probability

{{eq-growth2}}
$$\begin{cases} \dfrac{2(D_{n,i}(m-1)+\theta)(D_{n,j}(m-1)+\theta)}{(L_n(m-1)+n\theta)(L_n(m-1)+1+n\theta)}, & i \neq j, \\ \dfrac{(D_{n,i}(m-1)+\theta)(D_{n,i}(m-1)+\theta+1)}{(L_n(m-1)+n\theta)(L_n(m-1)+1+n\theta)}, & i = j. \end{cases} \tag{3.33}$$

We note that (3.33) is (3.28) with $d_{n,i}$ being replaced by the random variable $D_{n,i}(m-1)$ for each $1 \leqslant i \leqslant n$. In this step, if $i \neq j$, then the degrees of vertices i and j both increase by 1; if $i = j$, then the degree of the vertex i increases by 2.

(II) *Delete one edge or loop uniformly.* In this step, choose an edge (including loops) uniformly at random and remove it. If we remove the edge (i,j), then the degrees of vertices i and j both decrease by 1 and if we remove a loop on vertex i, then the degree of vertex i decreases by 2.

(III) *Move one half-edge.* In this step, we detach a uniformly chosen half-edge from its vertex and attach it back to another vertex according to a preferential attachment rule. Formally, choose a half-edge $j \in [L_n(m-1)]$ uniformly at random, and let $j' \in [L_n(m-1)]$ be the half-edge currently matched with j. Then, detach half-edge j' from its vertex and attach it to a new vertex i chosen with probability

$$\frac{D_{n,i}(m-1)+\theta}{L_n(m-1)+n\theta}.$$

If $i \neq j$, then the degree of i increases by 1 and that of j decreases by 1; if $i = j$, then $D_n(m) = D_n(m-1)$.

Assume that there exists a positive number $\rho_0 > 0$ such that $L_n(0)/n^2 \to \rho_0$ in probability as $n \to \infty$. Let a be a constant such that $0 < a \leqslant \rho_0 < \infty$ and let $p_1, p_2 \in [0,1]$ such that $1 - p_1 - p_2 \geqslant 0$. Let $(G_n(m))_{m\in\mathbb{N}_0}$ be defined by the following dynamics. Start with $G_n(0)$ having distribution $H(L_n(0)/2)$. For $m \geqslant 1$ and given the graph $G_n(m-1)$, do the following:

- If $L_n(m-1) > an^2+1$, generate $G_n(m)$ by Step (I) with probability p_1, via Step (II) with probability p_2 and via Step (III) with probability $1-p_1-p_2$;
- If $L_n(m-1) \leqslant an^2+1$, generate $G_n(m)$ via Step (I) with probability p_1+p_2 and via Step (III) with probability $1-p_1-p_2$.

Therefore, we obtain a sequence of multigraphs $(G_n(m))_{m\geqslant 0}$, which we call the *edge reconnection model.* The following lemma says that $(G_n(m))_{m\geqslant 0}$ is a multigraph-valued Markov chain with the property that, for each $m \geqslant 0$ and given $L_n(m) = \ell$, the multigraph $G_n(m)$ has the same distribution as $H_n(\ell/2)$.

{lem3.5}

Lemma 3.4. *For each $m \geqslant 0$ and any even integer ℓ, we have*

$$\mathscr{L}\big(G_n(m)|L_n(m)=\ell\big) = \mathscr{L}\big(H_n(\ell/2)\big).$$

Proof of Lemma 3.4. Let $G = (x_{ij})_{i,j\in[n]}$ be a nonrandom multigraph with degree sequence $d_n = (d_{n,1},\dots,d_{n,n})$ satisfying that $\sum_{i=1}^n d_{n,i} = \ell$. Recalling (3.30), it suffices to prove the identity

{{eq-l3.5-a}}

$$\mathbb{P}\big[G_n(m)=G\big|L_n(m)=\ell\big] = \frac{\prod_{i=1}^n \theta^{(d_{n,i})}}{(n\theta)^{(\ell)}} \frac{(\ell)!!}{\prod_{1\leqslant i<j\leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!}, \quad (3.34)$$

which we prove by induction. The identity is trivial for $m = 0$, which proves the base case. For $m \geqslant 1$, assume that (3.34) holds for $m-1$. Assume that $\ell > an^2 + 1$, the other case being similar. Denote by A_1, A_2 and A_3, respectively, the events that $G_n(m)$ is obtained from $G_n(m)$ via Steps (I), (II) and (III), respectively. By the construction of $G_n(m)$, we have

$$\begin{aligned}
&\mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell\big] \\
&\quad = p_1 \, \mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell, A_1\big] + p_2 \, \mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell, A_2\big] \\
&\qquad + (1 - p_1 - p_2) \, \mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell, A_3\big].
\end{aligned}$$

Now, given (i,j), let $G_-^{(i,j)}$ be the multigraph that is generated by replacing x_{ij} in G by $x_{ij} - 1$ for $i \neq j$ and by replacing x_{ij} by $x_{ij} - 2$ if $i = j$. We have

$$\begin{aligned}
&\mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell, A_1\big] \\
&\quad = \mathbb{P}\big[G_n(m) = G \big| L_n(m-1) = \ell - 2, A_1\big] \\
&\quad = \sum_{i \leqslant j} \mathbb{I}\{x_{ij} \geqslant 1\} \, \mathbb{P}\big[G_n(m) = G \big| G_n(m-1) = G_-^{(i,j)}, A_1\big] \\
&\qquad\qquad \times \mathbb{P}\big[G_n(m-1) = G_-^{(i,j)} \big| L_n(m-1) = \ell - 2, A_1\big].
\end{aligned}$$

Observe that for any (i,j) such that $x_{ij} \geqslant 1$,

$$\begin{aligned}
&\mathbb{P}\big[G_n(m) = G \big| G_n(m-1) = G_-^{(i,j)}, A_1\big] \\
&\quad = \begin{cases} \dfrac{2(d_{n,i} + \theta - 1)(d_{n,j} + \theta - 1)}{(\ell - 2 + n\theta)(\ell - 1 + n\theta)} & \text{if } i \neq j, \\[2ex] \dfrac{(d_{n,i} + \theta - 2)(d_{n,i} + \theta - 1)}{(\ell - 2 + n\theta)(\ell - 1 + n\theta)} & \text{if } i = j. \end{cases}
\end{aligned}$$

By induction assumption, noting that A_1 is independent of $(G_n(m-1), L_n(m-1))$, we obtain

$$\begin{aligned}
&\mathbb{P}\big[G_n(m-1) = G_-^{(i,j)} \big| L_n(m-1) = \ell - 2, A_1\big] \\
&\quad = \mathbb{P}\big[G_n(m-1) = G_-^{(i,j)} \big| L_n(m-1) = \ell - 2\big] \\
&\quad = \left(\frac{\prod_{i=1}^n \theta^{(d_{n,i})}}{(n\theta)^{(\ell)}} \frac{(\ell)!!}{\prod_{1 \leqslant i < j \leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!} \right) \times \frac{(n\theta + \ell - 2)(n\theta + \ell - 1)}{\ell} \\
&\qquad\qquad \times \begin{cases} \dfrac{\theta^{(d_{n,i}-1)} \theta^{(d_{n,j}-1)}}{\theta^{(d_{n,i})} \theta^{(d_{n,j})}} \dfrac{x_{ij}!}{(x_{ij} - 1)!} & \text{if } i \neq j, \\[2ex] \dfrac{\theta^{(d_{n,i}-2)}}{\theta^{(d_{n,i})}} \dfrac{x_{ii}!!}{(x_{ii} - 2)!!} & \text{if } i = j. \end{cases}
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
&\mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell, A_1\big] \\
&\quad = \frac{1}{\ell} \left(2 \sum_{i<j} x_{ij} + \sum_{i=1}^n x_{ii} \right) \times \left(\frac{\prod_{i=1}^n \theta^{(d_{n,i})}}{(n\theta)^{(\ell)}} \frac{(\ell)!!}{\prod_{1 \leqslant i < j \leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!} \right) \\
&\quad = \frac{\prod_{i=1}^n \theta^{(d_{n,i})}}{(n\theta)^{(\ell)}} \frac{(\ell)!!}{\prod_{1 \leqslant i < j \leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!}.
\end{aligned}$$

Given (i,j), let $G_+^{(i,j)}$ be the multigraph generated by replacing x_{ij} in G by $x_{ij}+1$ if $i \neq j$ and by replacing x_{ij} by $x_{ij}+2$ if $i=j$. Then,

$$\begin{aligned}
&\mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell, A_2\big] \\
&\quad = \sum_{i \leqslant j} \mathbb{P}\big[G_n(m) = G \big| G_n(m-1) = G_+^{(i,j)}, A_2\big] \\
&\qquad\qquad \times \mathbb{P}\big[G_n(m-1) = G_+^{(i,j)} \big| L_n(m-1) = \ell+2, A_2\big].
\end{aligned}$$

Now,

$$\mathbb{P}\big[G_n(m) = G \big| G_n(m-1) = G_+^{(i,j)}, A_2\big] = \begin{cases} \dfrac{2(x_{ij}+1)}{(\ell+2)} & \text{if } i \neq j, \\ \dfrac{(x_{ij}+2)}{(\ell+2)} & \text{if } i = j, \end{cases}$$

and

$$\begin{aligned}
&\mathbb{P}\big[G_n(m-1) = G_+^{(i,j)} \big| L_n(m-1) = \ell+2, A_2\big] \\
&\quad = \left(\frac{\prod_{i=1}^n \theta^{(d_{n,i})}}{(n\theta)^{(\ell)}} \frac{(\ell)!!}{\prod_{1 \leqslant i < j \leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!} \right) \times \frac{(\ell+2)}{(n\theta+\ell)(n\theta+\ell+1)} \\
&\qquad\qquad \times \begin{cases} \dfrac{\theta^{(d_{n,i}+1)}\theta^{(d_{n,j}+1)}}{\theta^{(d_{n,i})}\theta^{(d_{n,j})}} \dfrac{x_{ij}!}{(x_{ij}+1)!} & \text{if } i \neq j, \\ \dfrac{\theta^{(d_{n,i}+2)}}{\theta^{(d_{n,i})}} \dfrac{x_{ii}!!}{(x_{ii}+2)!!} & \text{if } i = j. \end{cases}
\end{aligned}$$

Then, it follows that

$$\mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell, A_2\big] = \left(\frac{\prod_{i=1}^n \theta^{(d_{n,i})}}{(n\theta)^{(\ell)}} \frac{(\ell)!!}{\prod_{1 \leqslant i < j \leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!} \right).$$

Using a similar argument, we have

$$\mathbb{P}\big[G_n(m) = G \big| L_n(m) = \ell, A_3\big] = \left(\frac{\prod_{i=1}^n \theta^{(d_{n,i})}}{(n\theta)^{(\ell)}} \frac{(\ell)!!}{\prod_{1 \leqslant i < j \leqslant n} x_{ij}! \prod_{i=1}^n x_{ii}!!} \right).$$

Combining the foregoing inequalities, we conclude that (3.34) also holds for m. This completes the proof by induction. □

The limiting behavior of the edge reconnection model was firstly studied by Ráth and Szakács (2012) who defined the dynamics only based on Step (II). In that case, the total number of the edges does not change over time. We remark that the model $(G_n(m))$ in the present paper is more general. Specially, if $p_1 = p_2 = 0$, then our model reduces to Ráth and Szakács (2012)'s model. In what follows, we proceed to prove the scaled multigraphon process converges in distribution to a non-trivial multigraphon-valued limiting process.

Let $\kappa_n = (\kappa_n(s))_{s \geqslant 0} \in \mathcal{D}$, where for each $s \geqslant 0$, multigraphon $\kappa_n(s)$ is the corresponding multigraphon generated by a scaled process $G_n(\lfloor n^4 p_1^{-1} s \rfloor)$. Let $Y_n(s) = L_n(\lfloor n^4 p_1^{-1} s \rfloor)/n^2$ and $Y_n = (Y_n(s))_{s \geqslant 0}$. In order to specify the limiting multigraphon process of κ_n, we need to introduce the limiting process of Y_n.

Let $Y = (Y(s))_{s \geqslant 0}$ be defined as

{{eq-limY}}
$$Y(s) = a + |2B(s) + \rho_0 - a| \quad \text{for } s \geqslant 0, \tag{3.35}$$

where $(B(s))_{s\geqslant 0}$ is a standard Brownian motion. Recalling that θ is given as in (3.28), let

{{eq-Psi}}
$$\Psi(x) = \begin{cases} \frac{\theta^\theta}{\Gamma(\theta)} \int_0^x z^{\theta-1} e^{-\theta z} dz & \text{if } x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.36}$$

{{eq-Psi-inv}}
$$\bar{\Psi}(x) = \inf\{y : \Psi(y) \geqslant x\}. \tag{3.37}$$

Then, it follows that $\bar{\Psi}(x)$ is the general inverse function of $\Psi(x)$ with respect to x. Let $\kappa = (\kappa(s))_{s\geqslant 0} \in \mathcal{D}$ be a multigraphon process such that for $s \geqslant 0$ and $r \in \mathbb{N}_0$,

{{eq17}}
$$\kappa(s; r; x, y) = \begin{cases} p\big(r; Y(s)\bar{\Psi}(x)\bar{\Psi}(y)\big) & \text{if } x \neq y, \\ p\left(\frac{r}{2}; \frac{Y(s)\bar{\Psi}(x)\bar{\Psi}(y)}{2}\right) & \text{if } x = y \text{ and if } r \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \tag{3.38}$$

where $p(r; \lambda) = \lambda^r e^{-\lambda}/r!$ as before. We have the following result.

{thm5}
Theorem 3.5. *Assume that $p_1 = p_2 > 0$. Then, $\kappa_n \Longrightarrow \kappa$ in $(\mathcal{D}, d^\circ)$.*

Remark 3.6. When $p_1 \neq p_2$, we need to use a different time-scaling for $G_n(m)$. If $p_1 > p_2$, we have by the law of large numbers that $L_n(\lfloor n^2 s \rfloor)/n^2$ diverges to ∞ in probability as both n and s tend to infinity. As a result, the multigraph $G_n(\lfloor n^2 s\rfloor)$ diverges to a multigraph with infinite edges and infinite loops as $n, s \to \infty$. If, on the other hand, $p_1 < p_2$, then $L_n(\lfloor n^2 s\rfloor)/n^2$ converges to a in probability as n and s go to infinity, where a is as in the generation of $(G_n(m))_{m\geqslant 1}$. Consequently, as n and s tend to infinity, by Theorem 3.1, the limiting multigraphon of $G_n(\lfloor n^2 s\rfloor)$ is a nonrandom multigraph given by

$$h(r; x, y) = \begin{cases} p(r; a\bar{\Psi}(x)\bar{\Psi}(y)) & \text{if } x \neq y, \\ p\left(\frac{r}{2}; \frac{a\bar{\Psi}(x)\bar{\Psi}(y)}{2}\right) & \text{if } x = y \text{ and } r \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Before giving the proof of Theorem 3.5, we introduce some notation and prove some auxiliary results. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a measurable function and we say f is symmetric if $f(x_1, \ldots, x_k) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ for any $(x_1, \ldots, x_k) \in \mathbb{R}^k$ and $\sigma : [k] \hookrightarrow [k]$. For any symmetric function f and $x = (x_1, \ldots, x_n)$, define the U-statistic

{{eq-Ufx}}
$$U_f(x) = \frac{1}{(n)_k} \sum_{\sigma:[k]\hookrightarrow[n]} f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}). \tag{3.39}$$

We have the following concentration inequality result.

{lem-11}

Lemma 3.7. *Let m be any positive integer satisfying that $n \leqslant m \leqslant n^3$, let $H(m)$ be defined as above and let $D^*(m) = (D_1^*(m), \dots, D_n^*(m))$ be its degree sequence. Let $Z = (Z_1, \dots, Z_n)$ be a vector of independent random variables with the common probability mass function*

$$\mathbb{P}[Z_1 = x] = \left(\frac{n\theta}{2m + n\theta}\right)^{\theta}\left(\frac{2m}{2m + k\theta}\right)^{x}\frac{\theta^{(x)}}{x!}, \quad x \in \mathbb{N}_0.$$

Then there exist positive constants C and C' that depend only on k and θ, such that, for any symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ with $0 \leqslant f \leqslant 1$ and any $\varepsilon > 0$, we have

{{l11-a}}
$$\mathbb{P}\big[\big|U_f(D^*(m)) - \mathbb{E}\{f(Z_1, \dots, Z_k)\}\big| \geqslant \varepsilon\big] \leqslant Cn^{5/2}e^{-C'n\varepsilon^2} \tag{3.40}$$

Proof of Lemma 3.7. Let $\mathbb{P}^*$ and $\mathbb{E}^*$ denote probability and expectation conditional on the event that $\sum_{i=1}^n Z_i = 2m$. Let $C_1, C_2, \dots$ denote positive constants depending only on k and θ. It has been shown that (see Pittel (2010, p. 624))

{{l11-c}}
$$\mathscr{L}(D_1^*(m), \dots, D_n^*(m)) = \mathscr{L}\bigg(Z_1, \dots, Z_n \,\bigg|\, \sum_{i=1}^n Z_i = 2m\bigg). \tag{3.41}$$

By definition, we have

{{l11-2}}
$$\begin{aligned}\mathbb{P}\bigg[\sum_{i=1}^n Z_i = 2m\bigg] &= \sum_{z_1+\dots+z_n=2m}\mathbb{P}\big[Z_1 = z_1, \dots, Z_n = z_n\big]\\ &= \left(\frac{n\theta}{2m + n\theta}\right)^{n\theta}\left(\frac{2m}{2m + n\theta}\right)^{2m}\frac{(n\theta)^{(2m)}}{(2m)!}\\ &= \frac{1}{2m}\left(\frac{n\theta}{2m + n\theta}\right)^{n\theta}\left(\frac{2m}{2m + n\theta}\right)^{2m}\frac{\Gamma(n\theta + 2m)}{\Gamma(n\theta)\Gamma(2m)}\\ &\geqslant \frac{(n\theta)^{1/2}}{2m + n\theta}\exp\Big(-\frac{1}{12n\theta} - \frac{1}{24m}\Big)\\ &\geqslant C_1 n^{-5/2},\end{aligned} \tag{3.42}$$

where we used (3.10) and the fact that $n \leqslant m(n) \leqslant n^3$ in the last two lines. By (3.41) and (3.42), the left hand side of (3.40) becomes

{{l11-1}}
$$\begin{aligned}\mathbb{P}^*\big[\big|U_f(Z) - \mathbb{E}\{U_f(Z)\}\big| \geqslant \varepsilon\big] &= \frac{\mathbb{P}\big[\big|U_f(Z) - \mathbb{E}\{U_f(Z)\}\big| \geqslant \varepsilon\big]}{\mathbb{P}[\sum_{i=1}^n Z_i = 2m]}\\ &\leqslant C_2 n^{5/2}\,\mathbb{P}\big[\big|U_f(Z) - \mathbb{E}\{U_f(Z)\}\big| \geqslant \varepsilon\big].\end{aligned} \tag{3.43}$$

As $0 \leqslant f \leqslant 1$, the value of $U_f(Z)$ changes by at most $(n-1)_{k-1}/(n)_k$ if the i-th variable Z_i changes. Recalling that $Z_1, \dots, Z_n$ are independent, by the McDiarmid inequality, we have

$$\mathbb{P}\big[\big|U_f(Z) - \mathbb{E}\{U_f(Z)\}\big| \geqslant \varepsilon\big] \leqslant 2\exp\big(-C_3 n\varepsilon^2\big).$$

This completes the proof together with (3.43). □

The following lemma provides a general concentration inequality for graph functionals. For any two multigraphs $G, G' \in \mathcal{M}_n$, we say G and G' differ from each other by a single switch of edges, if G' is a multigraph generated by choosing two edges or loops from G and reconnecting these four half-edges.

{lem5}

Lemma 3.8 (Remark 3.31 of Bordenave (2006))**.** *Let d_n be a degree sequence, let $G_n \sim \mathrm{CM}(d_n)$ and let $f : \mathcal{M}_n \to \mathbb{R}$ be a measurable function. Assume that there exists $c_1 > 0$ such that*

$$\big|f(G) - f(G')\big| \leqslant c_1$$

for any multigraphs $G, G' \in \mathcal{M}_n$ differing from each other by a single switch of edges. Then, for any $\varepsilon \geqslant 0$,

$$\mathbb{P}\big[\big|f(G_n) - \mathbb{E}f(G_n)\big| \geqslant \varepsilon\big] \leqslant 2\exp\bigg(-\frac{\varepsilon^2}{c_1^2\sum_{i=1}^n d_{n,i}}\bigg).$$

{lem6}

Lemma 3.9. *We have for each $F \in \mathcal{M}_k$, $m \geqslant 0$ and $\varepsilon \geqslant 0$,*

{{label34}}

$$\begin{aligned}\mathbb{P}\big[|t_F^{\mathrm{ind}}(G_n(m)) - \mathbb{E}\{t_F^{\mathrm{ind}}(G_n(m))|D_n(m)\}| \geqslant \varepsilon\big|L_n(m) \leqslant n^3\big]& \\ \leqslant 2\exp\big(-Cn\varepsilon^2\big),& \qquad (3.44)\end{aligned}$$

where $C > 0$ is a constant depending on k.

Proof. Let $G, G' \in \mathcal{M}_n$ be two multigraphs differ from each other by a single switch of edges or loops. By definition, for any $F \in \mathcal{M}_k$,

$$\big|t_F^{\mathrm{ind}}(G) - t_F^{\mathrm{ind}}(G')\big| \leqslant \frac{C_1}{n(n-1)},$$

where $C_1 > 0$ is a constant depending only on k. By Lemma 3.3, for each $m \geqslant 0$ and given $D_n(m) = d_n$, the random multigraph $G_n(m)$ has the same distribution as $\mathrm{CM}(d_n)$, and then Lemma 3.8 implies (3.44), as desired. $\square$

{lem-3.10}

Lemma 3.10. *Recall that θ is as defined in* (3.36) *and a is as in the construction of $G_n(m)$. Let $k \geqslant 1$, $y \geqslant a$ and let $Z_{n,1}, \ldots, Z_{n,k}$ be independent random variables with the common probability mass function*

{{eq-Zmass}}

$$\mathbb{P}[Z_{n,j} = r] = \left(\frac{\theta}{ny+\theta}\right)^{\theta}\left(\frac{ny}{ny+\theta}\right)^{r}\frac{\theta^{(r)}}{r!}, \quad 1 \leqslant j \leqslant k,\ r \in \mathbb{N}_0. \qquad (3.45)$$

Let $\zeta_{n,j} = Z_{n,j}/(ny)$ for every $1 \leqslant j \leqslant k$, and let $\zeta_1, \ldots, \zeta_k$ be independent random variables with the common distribution function (3.36)*. Let $\varphi : \mathbb{R}^k \to \mathbb{R}$ be a bounded measurable function satisfying that there exists $c > 0$ such that $\|\varphi\| \leqslant c$. We have*

{{eq-Ephi}}

$$|\,\mathbb{E}\varphi(\zeta_{n,1}, \ldots, \zeta_{n,k}) - \mathbb{E}\varphi(\zeta_1, \ldots, \zeta_k)| \leqslant Cn^{-1/2}, \qquad (3.46)$$

where $C > 0$ is a constant depending only on a, k, c and θ.

Remark 3.11. We remark that the random variables $Z_{n,j}$'s follow a negative binomial distribution $\mathrm{NB}(\theta, ny/(ny+\theta))$.

Proof. This proof includes two parts. In the first part, we prove an approximate representation of (3.45), and in the second part, we prove (3.46).

Denote by C a general constant depending only on a, k, c and θ, which might take different values in different places. Letting $u_n(r) = r/(ny)$, we have the probability mass function of $Z_{n,1}$ can be rewritten as

{{eq-Zmass1}}

$$\mathbb{P}[Z_{n,1} = r] = \theta^\theta (ny+\theta)^{-\theta} \big(1 + \theta/(ny)\big)^{-nyu_n(r)} \frac{\theta^{(r)}}{r!}. \tag{3.47}$$

For the second factor of the right hand side of (3.47), noting that $y \geqslant a$, we have

{{eq-13.10-0}}

$$(ny+\theta)^{-\theta} = (ny)^{-\theta}\left(1 + \frac{\theta}{ny}\right)^{-\theta} \leqslant (ny)^{-\theta}(1+Q_0) \tag{3.48}$$

for some $Q_0 \leqslant Cn^{-1}$. For the third factor of the right hand side of (3.47), noting that $y \geqslant a$, we have if $u_n(r) \geqslant n^{-1/2}$,

{{eq-13.10-1}}

$$\big(1 + \theta/(ny)\big)^{-nyu_n(r)} \leqslant e^{-\theta u_n(r)}(1+Q_1) \tag{3.49}$$

for some $|Q_1| \leqslant Cn^{-1/2}$. For the last factor of the right hand side of (3.47), we obtain if $u_n(r) \geqslant n^{-1/2}$, then $r = nyu_n(r) \geqslant n^{-1/2}a$, and therefore, by Stirling's formula (3.10) again,

{{eq-13.10-2}}

$$\frac{\theta^{(r)}}{r!} = \frac{\Gamma(r+\theta)}{\Gamma(\theta) r!} \leqslant \frac{(r+\theta)^{\theta-1}}{\Gamma(\theta)}(1+Q_2) \tag{3.50}$$

for some $|Q_2| \leqslant Cn^{-1/2}$. Moreover, if $u_n(r) \geqslant n^{-1/2}$, we have

{{eq-13.10-3}}

$$\begin{aligned}(r+\theta)^{\theta-1} &= (nyu_n(r)+\theta)^{\theta-1} \\ &= (ny)^{\theta-1}u_n(r)^{\theta-1}\left(1 + \frac{\theta}{nyu_n(r)}\right)^{\theta-1} \\ &\leqslant (ny)^{\theta-1}u_n(r)^{\theta-1}(1+Q_3),\end{aligned} \tag{3.51}$$

for some $|Q_3| \leqslant Cn^{-1/2}$. Substituting (3.48)–(3.51) to (3.47), we have if $u_n(r) \geqslant n^{-1/2}$,

$$\mathbb{P}[Z_{n,1} = r] \leqslant \frac{\theta^\theta}{ny\Gamma(\theta)} u_n(r)^{\theta-1} e^{-\theta u_n(r)} (1 + e^{Cn^{-1/2}}).$$

A similar lower bound still holds. Hence, it follows that if $u_n(r) \geqslant n^{-1/2}$, we have

{{eq-13.10-4}}

$$\mathbb{P}[Z_{n,1} = r] = \frac{\theta^\theta}{ny\Gamma(\theta)} u_n(r)^{\theta-1} e^{-\theta u_n(r)}(1+Q_3), \tag{3.52}$$

for some $|Q_3| \leqslant Cn^{-1/2}$. Similarly, if $0 \leqslant u_n(r) \leqslant n^{-1/2}$, we have

{{eq-13.10-5}}

$$\mathbb{P}[Z_{n,1} = r] \leqslant C(ny)^{-1}. \tag{3.53}$$

Now, we apply (3.52) and (3.53) to prove (3.46). Recalling that $\zeta_{n,j} = Z_{n,j}/(ny)$, denote by A_n the event that $\{\zeta_{n,j} \geqslant n^{-1/2}$ for $1 \leqslant j \leqslant k\}$ and by

B_n the event that $\{\zeta_j \geqslant n^{-1/2}$ for $1 \leqslant j \leqslant k\}$. By (3.52) and recalling again that $ny \geqslant na$, we have

{{eq-l3.10-6}}

$$\begin{aligned}
&\mathbb{E}\{\varphi(\zeta_{n,1}, \dots, \zeta_{n,k})\mathbb{I}(A_n)\} \\
&\quad = \frac{\theta^{k\theta}}{(ny)^k \Gamma(\theta)^k} \sum_{n^{1/2} y \leqslant r_1 \leqslant \infty} \cdots \sum_{n^{1/2} y \leqslant r_k \leqslant \infty} \\
&\qquad\qquad \times \left(\varphi\Big(\frac{r_1}{ny}, \dots, \frac{r_k}{ny}\Big) \mathbb{P}\big[Z_{n,1} = r_1, \dots, Z_{n,k} = r_k\big]\right) \qquad (3.54) \\
&\quad = \frac{(1+Q_4)\theta^{k\theta}}{\Gamma(\theta)^k} \int_{[n^{-1/2}, \infty]^k} \varphi(u_1, \dots, u_k) \prod_{j=1}^{k} (u_j^{\theta-1} e^{-\theta u_j} du_j) \\
&\quad = \mathbb{E}\{\varphi(\zeta_1, \dots, \zeta_k)\mathbb{I}(B_n)\}(1+Q_5),
\end{aligned}$$

for some $|Q_4| \leqslant Cn^{-1/2}$ and $|Q_5| \leqslant Cn^{-1/2}$. On the event A_n^c, we have

{{eq-l3.10-6'}}

$$\begin{aligned}
\big|\mathbb{E}\{\varphi(\zeta_{n,1}, \dots, \zeta_{n,k})\mathbb{I}(A_n^c)\}\big| &\leqslant c\,\mathbb{P}\Big[\min_{1 \leqslant j \leqslant k} \zeta_{n,j} < n^{-1/2}\Big] \\
&\leqslant c \sum_{j=1}^{k} \mathbb{P}\big[\zeta_{n,j} < n^{-1/2}\big] \qquad (3.55) \\
&\leqslant Cn^{-1/2},
\end{aligned}$$

where we used (3.53) in the last line. Since $\zeta_1, \dots, \zeta_k$ follow the common Gamma distribution $\Gamma(\theta, \theta)$, using a similar argument, we have

{{eq-l3.10-9}}

$$|\,\mathbb{E}\{\varphi(\zeta_1, \dots, \zeta_k)\mathbb{I}(B_n^c)\}| \leqslant Cn^{-1/2}. \qquad (3.56)$$

Combining (3.54)–(3.56), we complete the proof. □

We are now ready to give the proof of Theorem 3.5.

Proof of Theorem 3.5. We use *(v)* of Corollary 2.11 to prove this result. The proof is separated into three parts. We first show that Y_n is weakly convergent, then we verify the tightness property of $(t_F^{\mathrm{ind}}(\kappa_n))_{n \geqslant 1}$, and finally, we prove the finite dimensional convergence of $(t_{F_1}^{\mathrm{ind}}(\kappa_n(s_1)), \dots, t_{F_q}^{\mathrm{ind}}(\kappa_n(s_q)))$.

Step 1. Weak convergence of Y_n. Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables with common probability distribution

$$\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = p_1, \quad \mathbb{P}(\xi_1 = 0) = 1 - 2p_1.$$

Let $S(m) = \xi_1 + \dots + \xi_m$. Note that

$$L_n(m) \stackrel{d}{=} an^2 + |2S(m) + L_n(0) - an^2|,$$

and that $L_n(0)/n^2 \to \rho_0$ as $n \to \infty$, and then we have

$$Y_n(s) \stackrel{d}{=} a + |2n^{-2} S(\lfloor n^4 p_1^{-1} s \rfloor) + \rho_0 - a|.$$

Now, as $(n^{-2} S(\lfloor n^4 p_1^{-1} s \rfloor))_{s \geqslant 0} \Longrightarrow (B(s))_{s \geqslant 0}$ $(n \to \infty)$, where $(B(s))_{s \geqslant 0}$ is a standard Brownian motion, and by continuous mapping theorem, we have

$Y_n \Longrightarrow Y$ $(n \to \infty)$, where Y is as in (3.35). By Skorokhod's representation theorem, we may assume that Y_n and Y are constructed in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $Y_n \longrightarrow Y$ $\mathbb{P}$-a.s. as $n \to \infty$.

Moreover, for $n \geqslant 4(a+\rho_0)$ and for any $T \geqslant 0$, we have

$$\begin{aligned}\mathbb{P}\Big[\sup_{0\leqslant s\leqslant T} Y_n(s) \geqslant n\Big] &\leqslant \mathbb{P}\Big[\sup_{0\leqslant s\leqslant T} |S(\lfloor n^4 p_1^{-1} s\rfloor)| \geqslant \frac{n^3}{2} - (a+\rho_0)n^2\Big] \\ &\leqslant \mathbb{P}\Big[\sup_{0\leqslant s\leqslant T}\Big|\sum_{i=1}^{\lfloor n^4 p_1^{-1} s\rfloor} \xi_i\Big| \geqslant \frac{n^3}{4}\Big] \\ &\leqslant 2\,\mathbb{P}\Big[\Big|\sum_{i=1}^{\lfloor n^4 p_1^{-1} T\rfloor} \xi_i\Big| \geqslant \frac{n^3}{4}\Big],\end{aligned}$$

where we used Lévy's inequality since ξ_i's are symmetric. Applying Hoeffding's inequality, we have for $n \geqslant 4(a+\rho_0)$ and $T \geqslant 0$,

{{eq-expY}}

$$\mathbb{P}\Big[\sup_{0\leqslant s\leqslant T} Y_n(s) \geqslant n\Big] \leqslant 4\exp\Big(-\frac{8n^2}{p_1^{-1}(T+1)}\Big). \tag{3.57}$$

The inequality (3.57) will be used in *Step 2*.

Step 2. Tightness of $(t_F^{\mathrm{ind}}(\kappa_n))_{n\geqslant 1}$. Fix $F = (a_{ij})_{i,j\in[k]}$. Let $(\zeta_1, \ldots, \zeta_k)$ be a vector of independent and identically distributed random variables (independent of all other random variables) having the common distribution function (3.36). Then, with (3.38), we have

{{eq-3.46}}

$$t_F^{\mathrm{ind}}(\kappa(s)) = \psi(Y(s)), \tag{3.58}$$

where

$$\psi(y) = \mathbb{E}\Big\{\prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{y\zeta_i}{2}\Big) \prod_{1\leqslant i<j\leqslant k} p(a_{ij}; y\zeta_i\zeta_j)\Big\},$$

which is a continuous function of y. Let $\psi(Y_n)$ be a stochastic process defined as $\psi(Y_n)(s) = \psi(Y_n(s))$ for each time s. As $Y_n \Longrightarrow Y$, we have $(Y_n)_{n\geqslant 1}$ is tight. Since compact sets remain compact under continuous mappings, it follows that $(\psi(Y_n))_{n\geqslant 1}$ is also tight. For every $s \geqslant 0$, given $Y_n(s)$, let $Z_{n,1}(s), \ldots, Z_{n,n}(s)$ be conditionally independent discrete random variables with the common probability mass function

$$\mathbb{P}[Z_{n,1}(s) = r | Y_n(s)] = \Big(\frac{\theta}{nY_n(s)+\theta}\Big)^{\theta}\Big(\frac{nY_n(s)}{nY_n(s)+\theta}\Big)^{r}\frac{\theta^{(r)}}{r!}, \quad r \in \mathbb{N}_0.$$

Then, for each $s \geqslant 0$, given $Y_n(s) = y$, the random variable $Z_{n,1}(s)$ has the same distribution as in $Z_{n,1}$ defined as in Lemma 3.10.

Let $\psi_n(Y_n)$ be a $D([0,\infty), \mathbb{R})$-valued random element defined as

$$\begin{aligned}&\psi_n(Y_n)(s) \\ &= \mathbb{E}\Big\{\prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{Y_n(s)\zeta_{n,i}(s)}{2}\Big) \prod_{1\leqslant i<j\leqslant k} p(a_{ij}; Y_n(s)\zeta_{n,i}(s)\zeta_{n,j}(s)) \,\Big|\, Y_n(s)\Big\}.\end{aligned}$$

where $\zeta_{n,i}(s) = Z_{n,i}(s)/(nY_n(s))$ for each $s \geqslant 0$. Noting that the function

$$\prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{yz_i}{2}\Big) \prod_{1\leqslant i<j\leqslant k} p(a_{ij}; yz_iz_j)$$

is a bounded function of $(z_1,\dots,z_k)$, and by Lemma 3.10, we have for every $y \geqslant a$, as $n\to\infty$,

$$\sup_{s\geqslant 0}\bigg|\mathbb{E}\Big\{\prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{y\zeta_{n,i}(s)}{2}\Big) \prod_{1\leqslant i<j\leqslant k} p(a_{ij}; y\zeta_{n,i}(s)\zeta_{n,j}(s)) \,\Big|\, Y_n(s) = y\Big\} \\ - \mathbb{E}\Big\{\prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{y\zeta_i}{2}\Big) \prod_{1\leqslant i<j\leqslant k} p(a_{ij}; y\zeta_i\zeta_j)\Big\}\bigg| \to 0,$$

which can be rewritten as

{{eq-3.48}} $$\sup_{s\geqslant 0}\big|\mathbb{E}\{\psi_n(Y_n)(s)\} - \psi(Y_n)(s)\big| \longrightarrow 0 \quad \mathbb{P}\text{-a.s. as } n\to\infty. \tag{3.59}$$

Hence, by Ethier and Kurtz (1986, Problem 18, p. 152) and recalling that $(\psi(Y_n))_{n\geqslant 1}$ is tight, we have the family of stochastic process $(\psi_n(Y_n))_{n\geqslant 1}$ is also tight.

Write $D'_n(s) = D_n(\lfloor n^4p_1^{-1}s\rfloor)$; that is, $D'_n(s)$ is the degree sequence of the multigraph $G_n(\lfloor n^4p_1^{-1}s\rfloor)$. Clearly, $D'_{n,1}(s)+\cdots+D'_{n,n}(s) = n^2Y_n(s)$. Let $\mathbb{E}'$ and $\mathbb{P}'$ be the expectation operator and probability operator conditional on $(D'_n(s))_{s\geqslant 0}$, and let $\mathbb{E}^*$ and $\mathbb{P}^*$ be the expectation operator and probability operator conditional on $(Y_n(s))_{s\geqslant 0}$. Recalling that $\sum_{i=1}^n D'_{n,i}(s) \geqslant an^2$ for all $s\geqslant 0$, we have the conditions in Lemma 3.2 are satisfied for every $s\geqslant 0$. By Lemmas 3.3 and 3.4, for each $s\geqslant 0$, the multigraph corresponding to $\kappa_n(s)$ has the same distribution as the configuration model with the degree sequence $D'_n(s)$. Let

$$f^*(x_1,\dots,x_k; y_n) = \prod_{1\leqslant i<j\leqslant k} p(a_{ij}; y_nx_ix_j) \prod_{i=1}^{k} p\Big(\frac{a_{ii}}{2}; \frac{y_nx_i^2}{2}\Big).$$

Let $\bar D_n(s) = (\bar D_{n,1}(s),\dots,\bar D_{n,n}(s))$ where $\bar D_{n,i}(s) = D'_{n,i}(s)/(nY_n(s))$ for $1\leqslant i\leqslant n$. Recalling (3.39), write

$$\begin{aligned} U_n^*(s) &:= U_{f^*(\cdot;Y_n)}(\bar D_n(s)) \\ &= \frac{1}{(n)_k}\sum_{\sigma\hookrightarrow[n]} f^*(\bar D_{n,\sigma(1)}(s),\dots,\bar D_{n,\sigma(k)}(s); Y_n(s)). \end{aligned}$$

Noting that $U_n^*(s)$ is $\sigma(D'_n(s))$-measurable, and observing that $|t_F^{\mathrm{ind}}(\kappa_n(s))| \leqslant 1$ and $|U_n^*(s)|\leqslant 1$, we obtain

$$\sup_{s\geqslant 0}\big|\mathbb{E}' t_F^{\mathrm{ind}}(\kappa_n(s)) - U_n^*(s)\big| \\ \leqslant \sup_{s\geqslant 0}\Big|\mathbb{E}'\Big\{t_F^{\mathrm{ind}}(\kappa_n(s)) - U_n^*(s) \,\Big|\, \max_{1\leqslant i\leqslant n} D'_{n,i}(s) \leqslant (nY_n^{1/2}(s)(\log n)^2)x\Big\}\Big|$$

{{eq-3.8-111}}

$$+ 2\sup_{s\geqslant 0}\mathbb{P}'\Big[\max_{1\leqslant i\leqslant n} D'_{n,i}(s) \geqslant (nY_n^{1/2}(s)(\log n)^2)x\Big]. \tag{3.60}$$

For the first term of the right hand side of (3.60), and recalling that $Y_n(s) \geqslant a$, we have $n^2Y_n(s) \geqslant n$ for $n \geqslant a^{-1}$. Choosing $x = 27a^{1/2}/(\theta \log n)$, we have if $\max_{1\leqslant i\leqslant n} D'_{n,i}(s) \leqslant (nY_n^{1/2}(s)(\log n)^2)x$, then the conditions in Lemma 3.2 are satisfied. Then, by Lemmas 3.2 and 3.3, and noting that the function p is continuous, we have with $x = 27a^{1/2}/(\theta \log n)$,

{{eq-3.8-112}}

$$\begin{aligned}\sup_{s\geqslant 0}\Big|\mathbb{E}'\Big\{t_F^{\mathrm{ind}}(\kappa_n(s)) - U_n^*(s) \,\Big|\, \max_{1\leqslant i\leqslant n} D'_{n,i}(s) \leqslant (nY_n^{1/2}(s)(\log n)^2)x\Big\}\Big| \\ \leqslant C_1 n^{-1/4},\end{aligned} \tag{3.61}$$

where $C_1 > 0$ is a constant depending on a, k, θ and the multigraph F.

For the second term of the right hand side of (3.60), for any $0 < u < \theta$,

{{eq-mg-zeta}}

$$\begin{aligned}\mathbb{E}^* e^{u\zeta_{n,i}(s)} &= \left(\frac{\theta}{nY_n(s)(1 - e^{u/(nY_n(s))}) + \theta}\right)^{\theta} \\ &\leqslant \left(1 - \frac{u}{\theta} - \frac{u^2}{2\theta nY_n(s)}\right)^{-\theta}.\end{aligned} \tag{3.62}$$

By (3.41) and (3.42) and the fact that $\inf_{s\geqslant 0} Y_n(s) \geqslant a$, taking $\lambda = \theta/(6nY_n(s))$ and $x = 27a^{1/2}/(\theta \log n)$, we have for any $s \geqslant 0$,

{{eq-3.8-113}}

$$\begin{aligned}&\mathbb{P}'\Big[\max_{1\leqslant i\leqslant n} D'_{n,i}(s) \geqslant (nY_n^{1/2}(s)(\log n)^2)x\Big] \\ &\quad\leqslant C_2 n^{5/2}\sum_{i=1}^n \mathbb{P}^*\big[Z_{n,i}(s) \geqslant (nY_n^{1/2}(s)(\log n)^2)x\big] \\ &\quad\leqslant C_2 n^{5/2}\sum_{i=1}^n e^{-\lambda(nY_n^{1/2}(s)(\log n)^2)x}\,\mathbb{E}^* e^{\lambda Z_{n,i}(s)} \\ &\quad\leqslant C_2 n^{5/2}\sum_{i=1}^n e^{-\lambda(nY_n^{1/2}(s)(\log n)^2)x}\,\mathbb{E}^* e^{\theta\zeta_{n,i}(s)/6} \\ &\quad\leqslant C_3 n^{7/2} e^{-\theta(\log n)^2 x/(6a^{1/2})} = C_3 n^{-1},\end{aligned} \tag{3.63}$$

where we used (3.62) in the last line, and C_2 and C_3 are positive constants depending only on θ. Substituting (3.61) and (3.63) to (3.60) yields

{{eq-3.8-1}}

$$\sup_{s\geqslant 0}|\mathbb{E}' t_F^{\mathrm{ind}}(\kappa_n(s)) - U_n^*(s)| \leqslant (C_1 + 2C_3)n^{-1/4}. \tag{3.64}$$

Now, we prove the tightness of $\mathbb{E}'\{t_F^{\mathrm{ind}}(\kappa_n)\}$ by showing that for any $\varepsilon_1 > 0$ and $T \geqslant 0$,

{{eq-3.8-2}}

$$\limsup_{n\to\infty}\mathbb{P}\Big[\sup_{0\leqslant s\leqslant T}\big|U_n^*(s) - \psi_n(Y_n(s))\}\big| \geqslant \varepsilon_1\Big] \leqslant \varepsilon_1. \tag{3.65}$$

Then, by Ethier and Kurtz (1986, Problem 18, p. 152) and (3.64), and recalling the tightness property of $\psi_n(Y_n(s))$, we have $\mathbb{E}'\{t_F^{\mathrm{ind}}(\kappa_n)\}$ is tight.

Now we prove (3.65). By Lemma 3.7 and (3.57) and noting that $\psi_n(Y_n(s)) = \mathbb{E}^*\{f(\zeta_{n,1},\ldots,\zeta_{n,k};Y_n(s))\}$, we have for $n \geqslant 4(a+\rho_0)$,

$$\begin{aligned}
&\mathbb{P}\left[\sup_{0\leqslant s\leqslant T}\left|U_n^*(s)-\psi_n(Y_n(s))\right| \geqslant \varepsilon_1\right]\\
&\qquad\leqslant \mathbb{P}\left[\sup_{0\leqslant s\leqslant T}\left|U_n^*(s)-\psi_n(Y_n(s))\right| \geqslant \varepsilon_1 \,\middle|\, \sup_{0\leqslant s\leqslant T} Y_n(s) \leqslant n\right]\\
&\qquad\quad + \mathbb{P}\left[\sup_{0\leqslant s\leqslant T} Y_n(s) > n\right]\\
&\qquad\leqslant C_4 n^{5/2} \sum_{m=0}^{\lfloor n^4 p_1^{-1} T\rfloor+1} e^{-C_5 n\varepsilon_1^2} + 4\exp\left(-\frac{8n^2}{p_1^{-1}(T+1)}\right)\\
&\qquad\leqslant C_4(1+T)n^{13/2}e^{-C_5 n\varepsilon_1^2} + 4\exp\left(-\frac{8n^2}{p_1^{-1}(T+1)}\right).
\end{aligned}$$

Then, there exists an $n_1 > 0$ depending on $\varepsilon_1, T, p_1, C_4$ and C_5 such that

$$\mathbb{P}\left[\sup_{0\leqslant s\leqslant T}\left|U_n^*(s)-\psi_n(Y_n(s))\right| \geqslant \varepsilon_1\right] \leqslant \varepsilon_1 \quad \text{for all } n \geqslant n_1.$$

This proves (3.65), and thus $\mathbb{E}' t_F^{\mathrm{ind}}(\kappa_n)$ is tight.

It suffices to prove the tightness of $t_F^{\mathrm{ind}}(\kappa_n)$. To this end, for any $\varepsilon_2 > 0$ and $T \geqslant 0$, we prove that

{{eq140}}

$$\limsup_{n\to\infty} \mathbb{P}\left[\sup_{0\leqslant s\leqslant T}\left|t_F^{\mathrm{ind}}(\kappa_n(s)) - \mathbb{E}' t_F^{\mathrm{ind}}(\kappa_n(s))\right| \geqslant \varepsilon_2\right] \leqslant \varepsilon_2. \tag{3.66}$$

Then, by Ethier and Kurtz (1986, Problem 18, p. 152), we conclude that $t_F^{\mathrm{ind}}(\kappa_n)$ is tight. Now, it suffices to prove (3.66). For any $\varepsilon_2 > 0$, by condition (C1) and Lemma 3.9, there exists $C_6 > 0$ depending on k such that for $n \geqslant 4(a+\rho_0)$,

$$\begin{aligned}
&\mathbb{P}\left[\sup_{0\leqslant s\leqslant T}\left|t_F^{\mathrm{ind}}(\kappa_n(s)) - \mathbb{E}'\{t_F^{\mathrm{ind}}(\kappa_n(s))\}\right| \geqslant \varepsilon_2\right]\\
&\leqslant \sum_{m=0}^{\lfloor n^4 p_1^{-1} T\rfloor+1} \mathbb{P}\left[\left|t_F^{\mathrm{ind}}(G_n(m)) - \mathbb{E}'\{t_F^{\mathrm{ind}}(G_n(m))\}\right| \geqslant \varepsilon_2 \,\middle|\, \sup_{0\leqslant s\leqslant T} Y_n(s) \leqslant n\right]\\
&\quad + \mathbb{P}\left[\sup_{0\leqslant s\leqslant T} Y_n(s) > n\right]\\
&\leqslant 2 \sum_{m=0}^{\lfloor n^4 p_1^{-1} T\rfloor+1} \exp\bigl(-C_6 n\varepsilon_2^2/(T+1)\bigr) + 4\exp\left(-\frac{8n^2}{p_1^{-1}(T+1)}\right)\\
&\leqslant 2\bigl(n^4+1\bigr)\bigl(T+1\bigr)\exp\bigl(-C_6 n\varepsilon_2^2/(T+1)\bigr) + 4\exp\left(-\frac{8n^2}{p_1^{-1}(T+1)}\right).
\end{aligned} \tag{3.67}$$

q:exp-inequality0}}

Then, there exists an $n_2 \geqslant 0$ depending on ε_2, T, p_1 and C_6 such that for all $n \geqslant n_2$, the right hand side of (3.67) can be bounded by ε_2. This proves (3.66) and hence the tightness of $t_F^{\mathrm{ind}}(\kappa_n)$.

Step 3. Finite dimensional convergence. By (3.58), (3.59) and (3.64)–(3.66), we have for any $F \in \mathcal{M}$ and $s \geqslant 0$,

$$t_F^{\mathrm{ind}}(\kappa_n(s)) \longrightarrow t_F^{\mathrm{ind}}(\kappa(s)) \quad \mathbb{P}\text{-a.s. as } n \to \infty.$$

Then, for any $F_1, \dots, F_q \in \mathcal{M}$ and $0 \leqslant s_1 < \cdots < s_q < \infty$, we have

$$\prod_{j=1}^{q} t_{F_j}^{\mathrm{ind}}(\kappa_n(s_j)) \longrightarrow \prod_{j=1}^{q} t_{F_j}^{\mathrm{ind}}(\kappa(s)) \quad \mathbb{P}\text{-a.s.} \quad \text{as } n \to \infty.$$

By the boundedness property of t_F^{ind} and bounded convergence theorem, we have

$$\lim_{n\to\infty} \mathbb{E}\left\{\prod_{j=1}^{q} t_{F_j}^{\mathrm{ind}}(\kappa_n(s_j))\right\} = \mathbb{E}\left\{\prod_{j=1}^{q} t_{F_j}^{\mathrm{ind}}(\kappa(s))\right\}.$$

This completes the proof. □

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