Institute for Mathematics
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Advanced Functional Analysis

Problem Sheet 1

Due date: October 14, 2019

I will refer to the book H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 2011, Springer-Verlag New York, which you can download from the university library website.

Problem 1.1 (30 pts). Assume that $\Omega \subset \mathbb{R}^n$ is an open set with $0 < |\Omega| < \infty$ and fix some p with $1 < p < \infty$. Recall that the dual of $L^p(\Omega)$ is given by $L^q(\Omega)$ with $1/p + 1/q = 1$ (Theorem 4.11 Brezis). Let $f_n \in L^p(\Omega)$ be a sequence such that

i) $f_n(x) \to f(x)$ for a.e. $x \in \Omega$ as $n \to \infty$,

ii) $\sup_n \|f_n\|_{L^p} \leq C$ for some constant $C > 0$.

The aim is to show, by means of Real Analysis tools, that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ as $n \to \infty$. Follow the strategy below:

a) Show that $f \in L^p(\Omega)$ by using Fatou's Lemma (Lemma 4.1 Brezis).

b) Prove absolute continuity of the Lebesgue integral: namely, assume that $f \in L^1(\Omega)$. Then show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |f| \, dx < \varepsilon$ if $|E| < \delta$.

   Hint: by definition of Lebesgue integrability, there exists a step function $s = \sum_{j=1}^N s_j \chi_{A_j}$, with $s_j \in \mathbb{R}$, $\{A_j\}$ measurable partition of Ω, such that $0 \leq s \leq |f|$ and $\int_\Omega (|f| - s) \, dx < \varepsilon/2$.

c) Show that $f_n \rightharpoonup f$ by directly estimating the quantity

$$\int_\Omega f_n g \, dx - \int_\Omega f g \, dx \,,$$

   for some fixed $g \in L^q(\Omega)$.

   Hint: use point (b) in combination with Egoroff's Theorem (Theorem 4.29 Brezis) and Hölder's inequality (Theorem 4.6 Brezis).

Banach-Saks Property: a Banach space X is said to have the *Banach-Saks property* if for any sequence $\{x_n\} \subset X$ such that $x_n \rightharpoonup x$ weakly as $n \to \infty$, there exists a subsequence x_{n_k} of x_n such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N x_{n_k} = x$$

strongly in X. The Banach-Saks property holds when X is uniformly convex: in particular reflexive spaces, such as $X = L^p$ for $1 < p < \infty$, are uniformly convex.

Problem 1.2 (20 pts). Assume that $\Omega \subset \mathbb{R}^n$ is an open set with $0 < |\Omega| < \infty$ and fix some p with $1 < p < \infty$. Let $f_n, f \in L^p(\Omega)$ be such that (i)-(ii) from Problem 1.1. hold.

a) Show that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ by employing the Banach-Saks property.

   Hint: you can assume as given the following facts: Banach-Alaoglu's Theorem in $L^p(\Omega)$ (Theorem 3.16 Brezis for the general statement), uniqueness of the pointwise a.e. limit, Theorem 4.9 in Brezis, and the elementary fact that if a sequence of real numbers x_n is such that $\lim_n x_n = x$, then also $\lim_N \frac{1}{N} \sum_{n=1}^N x_n = x$.

b) Show that the above claim fails for $p = 1$, for example considering the case of $L^1(0,1)$: namely, construct a sequence $f_n \in L^1(0,1)$ such that (i)-(ii) from Problem 1.1 hold, but f_n does not have any weak accumulation point.

Recall: a Banach space X is said to be reflexive if $X^{**} = j(X)$, where $j\colon X \to X^{**}$ is the canonical immersion. Also recall that the dual of L^1 is given by L^∞.

Problem 1.3 (20 pts). Show that $L^1(0,1)$ is not reflexive.
Hint: assume by contradiction that $L^1(0,1)$ is reflexive (hence Banach-Alaoglu holds) and examine the sequence $f_n(x) = n\chi_{(0,\frac{1}{n})}(x)$ to derive a contradiction. It might be useful to employ dominated convergence (Theorem 4.2 Brezis).

Recall: let H be a real Hilbert space, with scalar product denoted by $\langle \cdot, \cdot \rangle$ and induced norm $\|x\| := \sqrt{\langle x, x\rangle}$ and distance $d(x,y) := \|x - y\|$. A map $a\colon H \times H \to \mathbb{R}$ is said to be a *bilinear form* if

$$a(\lambda x + \mu y, z) = \lambda\, a(x,z) + \mu\, a(y,z)\,, \qquad a(x, \lambda y + \mu z) = \lambda\, a(x,y) + \mu\, a(x,z)$$

for all $x, y, z \in H$, $\lambda, \mu \in \mathbb{R}$. A bilinear form $a\colon H \times H \to \mathbb{R}$ is *continuous* if there exists $C \geq 0$ such that

$$|a(x,y)| \leq C\|x\|\,\|y\| \quad \text{for all} \quad x, y \in H\,,$$

and *coercive* if there exists $\alpha > 0$ such that

$$a(x,x) \geq \alpha\|x\|^2 \quad \text{for all} \quad x \in H\,.$$

Lax-Milgram's Theorem: Let H be a real Hilbert space, $a\colon H \times H \to \mathbb{R}$ a continuous and coercive bilinear form and $T\colon H \to \mathbb{R}$ a linear continuous operator. Then there exists a unique $\hat{x} \in H$ such that

$$a(\hat{x}, y) = T(y) \quad \text{for all} \quad y \in H\,. \tag{1}$$

Problem 1.4 (30 pts). Prove Lax-Milgram's Theorem by following the strategy below:

a) Show that there exists a linear continuous operator $A\colon H \to H$ such that

$$a(x,y) = \langle A(x), y\rangle \quad \text{for all} \quad x, y \in H\,.$$

Hint: for $x \in H$ fixed, the map $y \in H \mapsto a(x,y) \in \mathbb{R}$ is linear and continuous. Use Riesz Theorem (Theorem 5.5 Brezis).

b) Fix $\lambda > 0$. Show that solving (1) is equivalent to finding a fixed point $\hat{x} \in H$ for the map $S_\lambda\colon H \to H$ defined by

$$S_\lambda(x) := x - \lambda A(x) + \lambda z$$

where $z \in H$ is uniquely determined and only depends on T (we recall that a fixed point is such that $S_\lambda(\hat{x}) = \hat{x}$).

c) Show that for some $\lambda > 0$ the map S_λ is a contraction, that is, prove the existence of some $0 \leq \theta < 1$ such that

$$d(S_\lambda(x), S_\lambda(y)) \leq \theta\, d(x,y) \quad \text{for all} \quad x, y \in H\,.$$

Then conclude the proof by Theorem 5.7 Brezis.

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Problem Sheet 2

Due date: October 28, 2019

I will refer to the book H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 2011, Springer-Verlag New York, which you can download from the university library website.

Notation: If $\Omega \subset \mathbb{R}^N$ is an open set, we define the spaces of locally integral functions

$$L^p_{\mathrm{loc}}(\Omega) := \{f \colon \Omega \to \mathbb{R} : \, f\chi_K \in L^p(\Omega) \ \text{ for all compact sets } \ K \subset \Omega\}$$

and of compactly supported continuous functions

$$C_c(\Omega) := \{f \in C(\Omega) : \ \exists \text{ a compact set } \ K \subset \Omega \ \text{ such that } \ f \equiv 0 \ \text{ in } \ \Omega \smallsetminus K\} \, .$$

For $k \geq 1$, we define the set of k-times continuously differentiable functions with compact support by $C^k_c(\Omega) := C^k(\Omega) \cap C_c(\Omega)$. For $f \in C^1(\Omega)$ we denote the gradient of f by $\nabla f := (\partial_{x_1} f, \dots, \partial_{x_n} f)$ where $\partial_{x_i} f := \frac{\partial f}{\partial x_1}$ is the i-th partial derivative of f. For $f \in C^k$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$ with $|\alpha| := \alpha_1 + \ldots + \alpha_N \leq k$ we denote

$$D^\alpha f := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_1^{\alpha_N}} \, f \, .$$

Finally the set of arbitrarily differentiable functions with compact support is denoted by $C^\infty_c(\Omega) := C^\infty(\Omega) \cap C_c(\Omega)$ where $C^\infty(\Omega) := \cap_{k=1}^\infty C^k(\Omega)$.

Convolutions: Let $N \geq 1$. Assume that $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$. We define the convolution between f and g as

$$(f \star g)(x) := \int_{\mathbb{R}^N} f(x-y) g(y) \, dy \qquad \text{for a.e. } \ x \in \mathbb{R}^N \, .$$

It should be well-known (otherwise see Theorem 4.15 Brezis) that $f \star g \in L^p(\mathbb{R}^N)$. Moreover (Proposition 4.18 Brezis)

$$\operatorname{supp}(f \star g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g} \, . \tag{1}$$

Mollifiers: let $N \geq 1$ and denote by B_r the N-dimensional ball of radius $r > 0$ centered at the origin, that is $B_r := \{x \in \mathbb{R}^N : \, |x| < r\}$. A sequence of *mollifiers* is any sequence of functions $\rho_n \colon \mathbb{R}^N \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$,

$$\rho_n \in C^\infty_c(\mathbb{R}^N) \, , \quad \operatorname{supp} \rho_n \subset \overline{B_{1/n}} \, , \quad \rho_n \geq 0 \ \text{ on } \ \mathbb{R}^N \, , \quad \int_{\mathbb{R}^n} \rho_n(x) \, dx = 1 \, . \tag{2}$$

Assume given some function $\rho \in C^\infty_c(\mathbb{R}^N)$ such that

$$\operatorname{supp} \rho \subset \overline{B_1} \, , \quad \rho \geq 0 \ \text{ on } \ \mathbb{R}^N \, , \quad \int_{\mathbb{R}^N} \rho(x) \, dx = 1 \, .$$

It is immediate to see that the sequence $\rho_n(x) := n^N \rho(nx)$ defines a sequence of mollifiers. One standard choice for ρ is given by the map

$$\rho(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } \ |x| < 1 \, , \\ 0 & \text{if } \ |x| \geq 1 \, , \end{cases}$$

where the constant $C > 0$ is chosen so that $\int_{\mathbb{R}^N} \rho(x)\,dx = 1$.

Sobolev spaces: let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be an open set. If $f \in L^1(\Omega)$, we say that $g \in L^1(\Omega)$ is the i-th weak partial derivative of f, with $i \in \{1, \dots, N\}$, if

$$\int_\Omega f(x)\, \partial_{x_i}\varphi(x)\,dx = -\int_\Omega g(x)\varphi(x)\,dx \quad \text{for all} \quad \varphi \in C_c^1(\Omega)\,.$$

If the i-th partial weak derivative of f exists, then it is unique (up to sets of zero Lebesgue measure). We denote such weak derivative by $g := \partial_{x_i} f$. For $1 \leq p \leq \infty$ we define the Sobolev space $W^{1,p}(\Omega)$ as

$$W^{1,p}(\Omega) := \{f \in L^p(\Omega) : \, \partial_{x_i} f \in L^p(\Omega) \text{ for all } i = 1, \dots, N\}\,.$$

Problem 2.1 (20 pts).

a) Let $f \in C_c(\mathbb{R}^N)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Show that $f \star g$ is well defined and $f \star g \in C(\mathbb{R}^N)$.

   Hint: let $x_n \to x$. Since f is compactly supported, there exists some compact set K such that $(x_n - \operatorname{supp} f) \subset K$ for all $n \in \mathbb{N}$.

b) Let $k \geq 1$, $f \in C_c^k(\mathbb{R}^N)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Show that $f \star g \in C^k(\mathbb{R}^N)$ with

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

   for each multi-index α with $|\alpha| \leq k$. In particular if $f \in C_c^\infty(\mathbb{R}^N)$ then $f \star g \in C^\infty(\mathbb{R}^N)$.

   Hint: by induction it is sufficient to check the statement for $k = 1$. You can directly check, using the definition of differentiability, that $\nabla(f \star g) = (\nabla f) \star g$. Notice that ∇f is uniformly continuous in $\mathbb{R}^N$, since $\operatorname{supp} \nabla f \subset \operatorname{supp} f$, and $\operatorname{supp} f$ is compact. Moreover it may be useful to recall the fundamental theorem of calculus, namely, $f(x+h) - f(x) = \int_0^1 \nabla f(x + hs) \cdot h\,ds$.

Problem 2.2 (40 pts). Let $\rho_n \colon \mathbb{R}^n \to \mathbb{R}$ be a sequence of mollifiers, so that (2) holds. Let $1 \leq p < \infty$.

a) Let $f \in C(\mathbb{R}^N)$. Show that, as $n \to \infty$, $(\rho_n \star f) \to f$ uniformly on each compact $K \subset \mathbb{R}^N$.

   Hint: fix K compact. Then f is uniformly continuous on K (why?). Hence for $\varepsilon > 0$, there exists some $\delta > 0$ (depending on ε and K) such that $|f(x-y) - f(x)| < \varepsilon$ for $x \in K$, $y \in B_\delta$.

b) Let $f \in L^p(\mathbb{R}^N)$. Show that, as $n \to \infty$, $(\rho_n \star f) \to f$ strongly in $L^p(\mathbb{R}^N)$, by following the strategy below:

   i) Show that, if $f \in C_c(\mathbb{R}^N)$ then $(\rho_n \star f) \to f$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$.
      Hint: use point (a) and (1).

   ii) Show that, for $f \in L^p(\mathbb{R}^N)$, it holds that $\|\rho_n \star f\|_{L^p} \leq \|f\|_{L^p}$ for all $n \in \mathbb{N}$.
      Hint: note that $\rho_n = \rho_n^p \rho_n^{1-1/p}$ and use Hölder's inequality.

   iii) Using that $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ (Thm 4.12 Brezis), and (i)-(ii), conclude (b).

c) (Fundamental Lemma of Calculus of Variations) Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume that $f \in L^1(\Omega)$ is such that

$$\int_\Omega f(x)\,\varphi(x)\,dx = 0 \quad \text{for all} \quad \varphi \in C_c(\Omega)\,.$$

   Show that $f = 0$ a.e. on Ω.

   Hint: first show that if $g \in L^\infty(\mathbb{R}^N)$ is such that $\operatorname{supp} g$ is compact and contained in Ω, then $\int_\Omega f\,g = 0$. This can be done by considering $g_n := \rho_n \star g$ and by using point (b), Problem 2.1 (with Ω instead of $\mathbb{R}^N$) and dominated convergence. Then apply what you just proved to some particular $L^\infty(\mathbb{R}^N)$ function in order to conclude.

The goal of the next exercise it to prove the following characterization theorem for one dimensional Sobolev functions, in the case when $\Omega = (a, b)$ is a bounded interval.

Theorem 1. Let $1 \leq p < \infty$ and $I \subset \mathbb{R}$ interval (bounded or unbounded). Let $f \in W^{1,p}(I)$. Then there exists $\tilde{f} \in C(\overline{I})$ such that $f = \tilde{f}$ a.e. in I and

$$\tilde{f}(y) - \tilde{f}(x) = \int_x^y f'(t)\, dt \quad \text{for all} \quad x, y \in \overline{I}\,.$$

The function $\tilde{f}$ is called the *continuous representative of* f.

Problem 2.3 (40 pts). Let $I = (a, b) \subset \mathbb{R}$ be a bounded interval.

a) Let $f \in L^1(I)$ be such that

$$\int_I f(x)\, \varphi'(x)\, dx = 0 \quad \text{for all} \quad \varphi \in C^1_c(I)\,.$$

Show that f is constant, i.e., $f = c$ a.e. on I for some $c \in \mathbb{R}$.

Hint: Fix $\Psi \in C_c(I)$ such that $\int_I \Psi = 1$. Then for all $w \in C_c(I)$ the map $h := w - \left(\int_I w\right) \Psi$ admits a unique continuous primitive (why?). Apply point (c) from Problem 2.2.

b) Let $g \in L^1(I)$ and define the function

$$f(x) := \int_a^x g(t)\, dt \quad \text{for} \quad x \in I\,.$$

Show that $f \in C(I)$ with $f' = g$ in the weak sense, that is,

$$\int_I f(x)\, \varphi'(x)\, dx = -\int_I g(x)\, \varphi(x)\, dx \quad \text{for all} \quad \varphi \in C^1_c(I)\,.$$

Hint: for the continuity use dominated convergence. The remaining part of the statement can be checked by employing Fubini (Theorem 4.5 Brezis).

c) Prove the statement of Theorem 1 for I bounded, with the help of (a)-(b).

Hint: study the behavior of $\bar{f}(x) := \int_a^x f'(t)\, dt$.

MAT.405 / WS 2019/2020
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Advanced Functional Analysis

Problem Sheet 3

Due date: November 11, 2019

I will refer to the book H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 2011, Springer-Verlag New York, which you can download from the university library website.

Problem 3.1 (30 pts).

a) Let $f(x) := |x|$ for $x \in [-1,1]$. Show that $f \in W^{1,p}(-1,1)$ for all $1 \leq p \leq \infty$.

b) Let $f(x) := -\chi_{(-1,0)}(x) + \chi_{(0,1)}(x) = \operatorname{sign} x$. Show that $f \notin W^{1,p}(-1,1)$ for any $1 \leq p \leq \infty$, **without** using the characterization theorem for 1D Sobolev functions (Theorem 1 in Worksheet 2).

Hint: assume by contradiction that f has a weak derivative in $L^1(-1,1)$. Then derive a contradiction by means of a sequence of functions localized around the jump, i.e. a sequence $\psi_n \in C_c^\infty(-1,1)$ such that $0 \leq \psi_j \leq 1$, $\psi_j(1) = 1$ and $\lim_j \psi_j(x) = 0$ for all $x \neq 0$.

Problem 3.2 (30 pts). Let $N \geq 3$. Assume that $\Omega \subset \mathbb{R}^N$ is an open connected set with regular boundary $\partial\Omega$. Assume that $A \colon \Omega \to \mathbb{R}^{N\times N}$ is a Lebesgue measurable matrix field such that there exist constants $C \geq 0, \alpha > 0$ satisfying

$$|A(x)| \leq C\,, \quad A(x)\xi \cdot \xi \geq \alpha|\xi|^2 \ \text{ for all } \ \xi \in \mathbb{R}^N\,, \quad \text{for a.e. } x \in \Omega\,.$$

Let $K > 0$ be the Poincaré constant of Ω, that is, a constant such that

$$\|u\|^2_{L^2(\Omega)} \leq K\,\|\nabla u\|^2_{L^2(\Omega)} \qquad \text{for all} \qquad u \in H_0^1(\Omega)\,,$$

(see Corollary 9.19 Brezis). Fix $\lambda \in \mathbb{R}$ such that $\lambda + \alpha/K > 0$. Let $f \in L^{2_*}(\Omega)$, where 2_* is the Hölder conjugate of $2^* := \frac{2N}{N-2}$ (that is, $\frac{1}{2^*} + \frac{1}{2_*} = 1$). We say that $u \in H_0^1(\Omega)$ is a *weak solution* to the boundary value problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u(x)) + \lambda u(x) = f(x) & \text{if } \ x \in \Omega\,, \\ u(x) = 0 & \text{if } \ x \in \partial\Omega\,, \end{cases}$$

if u satisfies

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla v(x)\,dx + \lambda \int_\Omega u(x)v(x)\,dx = \int_\Omega v(x)f(x)\,dx \qquad \text{for all} \qquad v \in H_0^1(\Omega)\,. \tag{1}$$

Show that there exists a unique solution $u \in H_0^1(\Omega)$ to (1).

Hint: consider the Hilbert space $H := H_0^1(\Omega)$ and apply Lax-Milgram (see Worksheet 1). It is also useful to recall Sobolev embeddings and Poincaré's inequality.

The goal of the next exercise is to prove the following theorem:

Theorem 1 (Partitions of unity): Let $N, k \in \mathbb{N}$ with $N, k \geq 1$. Let $\Omega \subset \mathbb{R}^N$ be a bounded set such that $\Omega \subset\subset \cup_{j=1}^k U_j$, where $U_j \subset \mathbb{R}^N$ is open for each $j = 1, \dots, k$. Then there exist functions $\eta_j \in C_c^\infty(U_j)$ such that $0 \leq \eta_j \leq 1$, $\operatorname{supp} \eta_j \subset U_j$ for all $j = 1, \dots, k$, and

$$\sum_{j=1}^{k} \eta_j(x) = 1 \quad \text{for all} \quad x \in \Omega\,.$$

We recall that a family $\{\eta_j\}_{j=1}^k$ satisfying the properties of Theorem 1 is called a *partition of unity* subordinate to the open cover $\{U_j\}_{j=1}^k$ of Ω.

Problem 3.3 (40 pts). Let $N, k \in \mathbb{N}$, $N, k \geq 1$.

a) Let $K \subset \mathbb{R}^N$ be compact and $U \subset \mathbb{R}^N$ be open, such that $K \subset U$. Show that there exists $\psi \in C_c^\infty(\mathbb{R}^N)$ with the following properties:

- $0 \leq \psi \leq 1$ in $\mathbb{R}^N$,
- $\psi(x) = 1$ for all $x \in K$,
- $\operatorname{supp} \psi \subset U$.

Hint: For $\varepsilon > 0$ consider $K_\varepsilon := K + \overline{B_\varepsilon} = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) \leq \varepsilon\}$. Take $\varepsilon > 0$ small enough so that $K_{3\varepsilon} \subset U$ (you can do it since $K \subset\subset U$). Then use the standard mollifiers ρ_ε to construct ψ (recall the basic properties of mollifiers: $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$, $\rho_\varepsilon \geq 0$, $\operatorname{supp} \rho_\varepsilon \subset \overline{B_\varepsilon}$, $\int_{\mathbb{R}^N} \rho_\varepsilon(x)\, dx = 1$). Also recall the properties of convolutions from Worksheet 2.

b) (Refining the cover) Let $K \subset \mathbb{R}^N$ be compact and $U_j \subset \mathbb{R}^N$ be open for all $j = 1, \dots, k$. Assume that $K \subset \cup_{j=1}^k U_j$. Show that exists a family $\{K_j\}_{j=1}^k$ of compact sets of $\mathbb{R}^N$ such that $K_j \subset U_j$ for all $j = 1, \dots, k$ and $K \subset \cup_{j=1}^k K_j$.

Hint: since K is bounded, it is not restrictive to assume that U_j is bounded for all $j = 1, \dots, k$. It might be useful to consider the sets $U_{n,j} := \{x \in U_j : \operatorname{dist}(x, \partial U_j) > 1/n\}$ for $n \in \mathbb{N}$. Also recall the topological definition of compactness: from an arbitrary open cover of a compact set you can extract a finite subcover.

c) (Proof of Theorem 1) Let $\Omega \subset \mathbb{R}^N$ be bounded and assume that $\Omega \subset\subset \cup_{j=1}^k U_j$, with $U_j \subset \mathbb{R}^N$ open for each $j = 1, \dots, k$. Show that there exists a family $\{\eta_j\}_{j=1}^k$ with $\eta_j \in C_c^\infty(\mathbb{R}^N)$ satisfying the following properties:

- $0 \leq \eta_j \leq 1$ in $\mathbb{R}^N$, for all $j = 1, \dots, k$,
- $\operatorname{supp} \eta_j \subset U_j$ for all $j = 1, \dots, k$,
- $\sum_{j=1}^k \eta_j(x) = 1$ for all $x \in \Omega$.

Hint: use points (a) and (b). Notice that it is not restrictive to assume that the sets U_j are bounded. Also: if $\psi_1, \dots, \psi_k$ are real numbers and you define $\eta_l := \psi_l \prod_{j=1}^{l-1}(1 - \psi_j)$ for each $l = 1, \dots, k$ (with the understanding that the empty product is equal to 1), the following identity holds

$$1 - \sum_{j=1}^{k} \eta_j = \prod_{j=1}^{k} (1 - \psi_j)\,.$$

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Problem Sheet 4

Due date: November 25, 2019

The goal of Problems 4.1 and 4.2 is to prove Morrey's inequality and the consequent embedding.

Theorem 1 (Morrey's inequality): Let $N \in \mathbb{N}, N \geq 1$, $N < p \leq \infty$ and set $\alpha := 1 - \frac{N}{p}$. There exists a constant $C > 0$ depending only on N and p, such that

$$\|f\|_{C^{0,\alpha}(\mathbb{R}^N)} \leq C \|f\|_{W^{1,p}(\mathbb{R}^N)} \quad \text{for all} \quad f \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N). \tag{1}$$

Theorem 2 (Embedding): Let $N \in \mathbb{N}, N \geq 1$, $N < p \leq \infty$ and $\alpha := 1 - \frac{N}{p}$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^1 boundary. Then every function $f \in W^{1,p}(\Omega)$ coincides a.e. in Ω with a function $\tilde{f} \in C^{0,\alpha}(\overline{\Omega})$. Moreover there exists a constant $C > 0$ such that

$$\left\|\tilde{f}\right\|_{C^{0,\alpha}(\overline{\Omega})} \leq C \|f\|_{W^{1,p}(\Omega)} \quad \text{for all} \quad f \in W^{1,p}(\Omega).$$

Recall: For $0 < \alpha \leq 1$, $C^{0,\alpha}(\overline{\Omega})$ is the space of continuous bounded functions $f \colon \Omega \to \mathbb{R}$ normed by

$$\left\|\tilde{f}\right\|_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x \in \overline{\Omega}} |f(x)| + \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Problem 4.1 (30 pts). Let $N \in \mathbb{N}, N \geq 1$, $N < p \leq \infty$ and $\alpha := 1 - \frac{N}{p}$. Prove Theorem 1 by following the strategy below:

a) Let $B_r(x)$ be the open ball centred at $x \in \mathbb{R}^N$ with radius $r > 0$. Show that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \, dy \leq C \int_{B_r(x)} \frac{|\nabla f(y)|}{|x-y|^{N-1}} \, dy \tag{2}$$

for all $f \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ and all balls $B_r(x) \subset \mathbb{R}^N$, where $C > 0$ is a constant.

Hint: Recall that $\int_{B_r(x)} f(y) \, dy = \int_0^r \int_{\partial B_\rho(x)} f(w) \, dS(w) \, d\rho$, where dS is the surface element on $\partial B_\rho(x)$ (or the $(N-1)$-dimensional Hausdorff measure restricted to $\partial B_\rho(x)$). Also $|B_r(x)| = \tilde{C} r^N$, where $\tilde{C} > 0$ depends only on N.

b) Prove, by using (2), that there exists $C > 0$ (depending only on N and p) such that

$$\sup_{x \in \mathbb{R}^N} |f(x)| \leq C \|f\|_{W^{1,p}(\mathbb{R}^N)} \quad \text{for all} \quad f \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N).$$

c) Prove that there exists $C > 0$ (depending only on N and p) such that

$$\sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq C \|\nabla f\|_{L^p(\mathbb{R}^N)} \quad \text{for all} \quad f \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N).$$

Hint: Fix $x, y \in \mathbb{R}^N$ and define $r := |x-y|$, $W := B_r(x) \cap B_r(y)$. Use (2) to estimate $\frac{1}{|W|} \int_W |f(x) - f(z)| + |f(y) - f(z)| \, dz$.

Problem 4.2 (10 pts). Prove the embedding Theorem 2.

Hint: you can use (1) in conjunction with extension and density arguments.

We now pass to some exercises on topological vector spaces and on locally convex spaces. Let us establish some definitions first.

Topological vector spaces: Let X be real vector space (that is, X is a vector space over $\mathbb{R}$). We say that X is a *topological vector space* (TVS) with the topology τ if the operations

$$\begin{aligned}(x, y) &\mapsto x + y\,, \quad X \times X \to X\,,\\ (\lambda, x) &\mapsto \lambda x\,, \quad \mathbb{R} \times X \to X\,,\end{aligned}$$

are continuous with respect to τ.

Subsets: Let X be a real vector space and $V \subset X$ be a subset. We say that V is *convex* if for all $x, y \in V$, $\lambda \in [0, 1]$ we have that $\lambda x + (1 - \lambda)y \in V$. We say that V is *balanced* if $\alpha x \in V$ whenever $x \in V$ and $|\alpha| \leq 1$. Notice that a non-empty balanced set always contains the origin. Finally, we say that V is *absorbing* at the point $a \in V$ if for every $x \in X$ there exists $\varepsilon > 0$ such that $a + tx \in V$ for all $0 \leq t < \varepsilon$.

Seminorms: Let X be a real vector space. A *seminorm* on X is a function $p \colon X \to [0, \infty)$ (notice that p is required to be non-negative and finite) satisfying the following properties:

i) p is positively homogenous, that is, $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{R}$, $x \in X$.

ii) p is subadditive, that is, $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Notice that $p(0) = 0$ as a consequence of (i). Recall that a norm is a seminorm with the additional property that if $p(x) = 0$ for some $x \in X$, then necessarily $x = 0$.

Minkowski functional: Let X be a real vector space and $V \subset X$ be a subset. The Minkowski functional associated to V is the function $p = p_V \colon X \to [0, \infty]$ defined by

$$p(x) := \inf \{t \geq 0 : \ x \in tV\}\ ,$$

where we define $p(x) := \infty$ if the set $\{t \geq 0 : \ x \in tV\}$ is empty.

Families of seminorms: Let X be a real vector space and $\mathcal{P}$ be a family of seminorms on X. Let $\tau_{\mathcal{P}}$ be the topology that has a subbase the sets

$$\{x \in X : \ p(x - x_0) < \varepsilon\}\,,$$

for some $x_0 \in X$, $p \in \mathcal{P}$, $\varepsilon > 0$. In other words, $A \subset X$ is open (with respect to $\tau_{\mathcal{P}}$) if and only if for every $x_0 \in A$ there exist $p_1, \ldots, p_n \in \mathcal{P}$, $\varepsilon_1, \ldots, \varepsilon_n > 0$, $n \in \mathbb{N}$, such that

$$\bigcap_{j=1}^{n} \{x \in X : \ p_j(x - x_0) < \varepsilon_j\} \subset A\,.$$

In particular, $\tau_{\mathcal{P}}$ is the coarsest topology for which all the elements of $\mathcal{P}$ are continuous. It is standard that X equipped with $\tau_{\mathcal{P}}$ is a TVS. We say that $\mathcal{P}$ is *separating* if

$$\bigcap_{p \in \mathcal{P}} \{x \in X : \ p(x) = 0\} = \{0\}\,.$$

Locally convex spaces: Let (X, τ) be a TVS. We say that X is a *locally convex space* (LCS) if $\tau = \tau_{\mathcal{P}}$ for some family $\mathcal{P}$ of seminorms on X.

Some topological definitions: Let X be a topological space and fix $x \in X$. We say that $U \subset X$ is a neighbourhood of x if there exists $A \subset X$ open, such that $x \in A$ and $A \subset U$. We say that X is *Hausdorff* if for all $x, y \in X$ with $x \neq y$, there exist $U, V \subset X$ neighbourhoods of x and y

respectively, such that $U \cap V = \emptyset$. If Y is another topological space, we say that a map $f\colon X \to Y$ is an *homeomorphism* if f is continuous, invertible, and the inverse of f is continuous. Finally, we say that a collection of open sets $\mathcal{N}$ is a *basis for the neighbourhood system* of a point $x \in X$ if the following properties are satisfied: $x \in N$ for all $N \in \mathcal{N}$; if $A \subset X$ is an open set such that $x \in A$, then there exists some $N \in \mathcal{N}$ such that $N \subset A$.

Problem 4.3 (20 pts). Let X be a TVS. Denote by τ its topology.

a) Fix $x_0 \in X$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Define the translation and dilation maps $T, D\colon X \to X$ by $T(x) := x + x_0$, $D(x) := \lambda x$ for all $x \in X$. Show that T and D are homeomorphisms.

b) Let $A \subset X$ be open. Show that A is absorbing at each of its points.

   Hint: Recall that the vector space operations on X are continuous by definition.

c) Let $n \in \mathbb{N}$ and $p_1, \dots, p_n$ be seminorms on X. Define

$$p(x) := \max_{j=1,\dots,n} p_j(x) \quad \text{for all} \quad x \in X\,.$$

   Show that p is a seminorm on X.

d) Assume that X is a LCS, that is, $\tau = \tau_{\mathcal{P}}$ for some family of seminorms $\mathcal{P}$. Show that (X, τ) is Hausdorff if the family $\mathcal{P}$ is separating.

In the next exercise we show that the topology of a LCS X can be equivalently described by a system of open, convex, balanced sets.

Problem 4.4 (40 pts). Let X be a real vector space.

a) Let $p\colon X \to [0, \infty)$ be a seminorm on X and define the set

$$V := \{x \in X : \, p(x) < 1\}\,. \tag{3}$$

   Show that V is convex, balanced and absorbing at each of its points.

b) Let $V \subset X$ be a subset. Assume that V is convex, balanced and absorbing at each of its points. Prove that the Minkowski functional p associated to V is a seminorm on X. Moreover show that p and V satisfy relation (3).

   Hint: if V is convex and $\alpha, \beta \geq 0$, then $\alpha V + \beta V \subset (\alpha + \beta) V$ (prove it). Also if $p(x) < \infty$, by definition of infimum, for each $\varepsilon > 0$ there exists $t \geq 0$ such that $t \leq p(x) + \varepsilon$ and $x \in tV$.

c) Assume that X is a TVS and define

$$\mathcal{U} := \{V \subset X : \, V \;\text{ open, convex and balanced }\}\,.$$

   Show that X is a LCS if and only if $\mathcal{U}$ is a basis for the neighbourhood system at 0.

   Hint: points (a), (b) from this exercise and points (b), (c) from Problem 4.3 are very useful!

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Problem Sheet 5

Due date: December 9, 2019

In the following we will adopt the definitions given in Worksheet 4.

Weak topologies: let X be a real LCS and denote by τ the topology induced by the family of seminorms $\mathcal{P}$. Consider the topological dual of X with respect to τ, defined by

$$X^* = (X, \tau)^* := \{x^* \colon X \to \mathbb{R} : \; x^* \; \text{ linear and } \tau\text{-continuous}\} .$$

Notice that X^* has a natural vector space structure, with

$$(x^* + \lambda y^*)(x) := x^*(x) + \lambda y^*(x) \quad \text{ for } \quad x^*, y^* \in X^*, \lambda \in \mathbb{R}, x \in X$$

For some $x^* \in X^*$ we define the map

$$p_{x^*} \colon \; X \to \mathbb{R} \quad \text{ by } \quad p_{x^*}(x) := |x^*(x)| \quad \text{ for all } \quad x \in X \, .$$

It is immediate to check that p_{x^*} is a seminorm on X. The *weak topology* on X (denoted by wk) is defined as the topology induced by the family of seminorms $\{p_{x^*} : \; x^* \in X^*\}$. Similarly, for $x \in X$ define the seminorm

$$p_x \colon X^* \to \mathbb{R} \quad \text{ by } \quad p_x(x^*) := |x^*(x)| \quad \text{ for all } \quad x^* \in X^* \, .$$

The *weak* topology* on X^* (denoted by wk^*) is the topology induced by the family of seminorms $\{p_x \colon \; x \in X\}$. Notice that both (X, wk) and (X^*, wk^*) are LCS. Therefore one has naturally a second topology on X, in addition to τ.

Problem 5.1 (30 pts).

a) Let X be a real vector space and $\varphi, \varphi_1, \dots, \varphi_n \colon X \to \mathbb{R}$ be linear functionals. Show that (i) and (ii) are equivalent, where

   i) There exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\varphi(x) = \sum_{j=1}^n \alpha_j \varphi_j(x)$ for all $x \in X$,

   ii) $(\cap_{j=1}^n \ker \varphi_j) \subset \ker \varphi$, where ker denotes the kernel.

   Hint: Consider the maps $\pi \colon X \to \mathbb{R}^n$ where $\pi(x) := (\varphi_1(x), \dots, \varphi_n(x))$ and $F \colon \pi(X) \to \mathbb{R}$ where $F(\pi(x)) := \varphi(x)$. Is F well defined?

b) Let X be a real LCS, τ its topology and X^* its dual. Prove that

   $$(X^*, wk^*)^* = X \, ,$$

   in the sense that if $\varphi \colon X^* \to \mathbb{R}$ is linear and weak* continuous, then there exists $x \in X$ (depending on φ) such that $\varphi(x^*) = x^*(x)$ for all $x^* \in X^*$.

   Hint: Recall the definition of open set with respect to a topology defined by seminorms (see Worksheet 4) and apply it to $\{x^* \in X^* \colon |\varphi(x^*)| < 1\}$. Then use (a).

Metrizable LCS: Let (X,τ) be a real LCS with the topology τ induced by the separating family of seminorms $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ (so in particular (X,τ) is Hausdorff). Assume that $d\colon X\times X\to\mathbb{R}$ is a translation invariant metric on X, that is, d is a metric on X and

$$d(x+a, y+a) = d(x,y) \quad \text{for all} \quad x,y,a\in X\,. \tag{1}$$

Denote by τ_d the topology induced by d on X. Since (1) holds, the topology τ_d is local, meaning that it is determined (up to translations) by the neighbourhood system at 0, given by $\mathcal{U} = \{U_\delta \colon \delta > 0\}$ where $U_\delta := \{x\in X : d(x,0) < \delta\}$. Recall that also τ is local, with the neighbourhoods of 0 given by $\mathcal{V} = \{V_{\alpha,\varepsilon} \colon \alpha\in A, \varepsilon>0\}$ where $V_{\alpha,\varepsilon} := \{x\in X \colon p_\alpha(x) < \varepsilon\}$.

We say that X is *metrized* by d if $\tau = \tau_d$, in the sense that for each $U_\delta\in\mathcal{U}$ there exists some $V_{\alpha,\varepsilon}\in\mathcal{V}$ such that $V_{\alpha,\varepsilon}\subset U_\delta$, and for each $V_{\alpha,\varepsilon}\in\mathcal{V}$ there exists some $U_\delta\in\mathcal{U}$ such that $U_\delta\subset V_{\alpha,\varepsilon}$. We say that (X,τ) is *metrizable* if there exists a translation invariant metric d such that $\tau=\tau_d$.

Problem 5.2 (30 pts). Let X be a real LCS whose topology τ is generated by a countable family of separating seminorms $\mathcal{P} := \{p_n\}_{n\in\mathbb{N}}$. Show that X is metrizable, by following the strategy below:

a) Define the map $d\colon X\times X\to\mathbb{R}$ by

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n}\,\frac{p_n(x-y)}{1+p_n(x-y)}\,.$$

Show that d is a translation invariant metric on X.

Hint: The scalar map $t\mapsto \frac{t}{1+t}$ is increasing for $t>-1$ and bounded by 1 from above.

b) Denote by τ_d the topology induced by d on X. Show that $\tau_d=\tau$ (in particular you showed that (X,τ) is metrizable).

Hint: You can assume that $p_n(x)\le p_{n+1}(x)$ for all $n\in\mathbb{N}$, $x\in X$ (since the family of seminorms $\{q_n\}$ defined by $q_n(x) := \max_{k\le n} p_k(x)$ induces the topology τ and $q_n\le q_{n+1}$). Also recall that $\sum_{n=1}^{k} 2^{-n} = 1-2^{-k}$ and $\sum_{n=k+1}^{\infty} 2^{-n} = 2^{-k}$.

Dual of continuous functions: Let (K,τ) be a compact Hausdorff topological space and consider the Banach space $C(K) := \{f\colon K\to\mathbb{R} : f \text{ is } \tau\text{-continuous}\}$ equipped with the supremum norm $\|f\|_\infty := \sup_{x\in K}|f(x)|$. We denote by $\mathcal{M}(K)$ the space of bounded Borel measures on K. If $\mu\in\mathcal{M}(K)$ we denote by $|\mu|$ its total variation measure. Recall that $\mathcal{M}(K)$ equipped with the norm $\|\mu\| := |\mu|(K)$ is a Banach space. The Riesz theorem states that the dual of $(C(K),\|\cdot\|_\infty)$ coincides with $\mathcal{M}(K)$. More precisely:

Theorem (Riesz): Let (K,τ) be a compact Hausdorff topological space. Let $\mu\in\mathcal{M}(K)$ and define the functional $\Lambda_\mu\colon C(K)\to\mathbb{R}$ by

$$\Lambda_\mu(f) := \int_K \varphi(x)\,d\mu(x) \quad \text{for all} \quad f\in C(K)\,.$$

Then Λ_μ is linear and continuous, and its operator norm satisfies $\|\Lambda_\mu\| = |\mu|(K)$. Conversely, let $\Lambda\colon C(K)\to\mathbb{R}$ be a linear and continuous functional. Then there exists a unique $\mu\in\mathcal{M}(K)$ such that $\Lambda=\Lambda_\mu$.

In view of the above theorem we have that a sequence $\{f_n\}_{n\in\mathbb{N}}\subset C(K)$ is *weakly* converging to some $f\in C(K)$ if and only if

$$\int_K f_n(x)\,d\mu(x) \to \int_K f(x)\,d\mu(x) \quad \text{as} \quad n\to\infty$$

for all $\mu\in\mathcal{M}(K)$ fixed. We also recall the dominated convergence theorem in higher generality:

Theorem (Dominated convergence): Let K be a compact Hausdorff space and fix $\mu\in\mathcal{M}(K)$. Let $f_n\colon K\to\mathbb{R}$ be a sequence of μ-measurable maps such that

i) $f_n(x)\to f(x)$ for μ-a.e. x in K, as $n\to\infty$,

ii) $\sup_n |f_n(x)| \le g(x)$ for μ-a.e. x in K, with $g\colon K\to\mathbb{R}$ is μ-measurable and $\int_K g(x)\,d\mu(x)<\infty$.

Then f is μ-measurable and $\int_K |f_n(x)-f(x)|\,d\mu(x)\to 0$ as $n\to\infty$.

Problem 5.3 (20 pts). Let K be a compact Hausdorff space and consider the Banach space $C(K)$ equipped with the supremum norm.

a) Let $f_n, f \in C(K)$ for $n \in \mathbb{N}$. Show that conditions (i) and (ii) given below are equivalent:

   i) f_n weakly converges to f as $n \to \infty$,

   ii) $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in K$ fixed.

   Hint: Use dominated convergence and Riesz theorem.

b) Let $\varphi \in C[0,1]$ be fixed and define the sequence $\{f_n\} \subset C[0,1]$ by setting

$$f_n(x) := \varphi(x^n) \quad \text{for} \quad x \in [0,1]\,.$$

   By using (a), prove that conditions (i') and (ii') given below are equivalent:

   i') f_n weakly converges to some $f \in C[0,1]$ as $n \to \infty$,

   ii') $\varphi(0) = \varphi(1)$.

Weak completeness: Let X be a real normed space. We say that a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is *weakly Cauchy* if for every $x^* \in X^*$ the sequence $\{x^*(x_n)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. We say that X is *weakly sequentially complete* if every weak Cauchy sequence is weakly convergent.

In the next exercise we will see that reflexivity is sufficient for weak sequential completeness, but not necessary.

Problem 5.4 (20 pts).

a) Let X be a reflexive Banach space. Show that X is weakly sequentially complete.

   Hint: Principle of uniform boundedness and Banach-Alaoglu might be useful.

b) Consider the Banach space $C[0,1]$ equipped with the supremum norm. Prove that $C[0,1]$ is not weakly sequentially complete.

   Hint: Examine the sequence f_n defined by $f_n(x) := 1 - nx$ if $0 \le x \le 1/n$ and $f_n(x) := 0$ if $1/n \le x \le 1$.

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Problem Sheet 6

Due date: January 20, 2020

You are required to present Problems 6.2, 6.4, 6.5, 6.6, 6.7, 6.9. The rest of the problems will not be marked, but I recommend doing them as a preparation for the final exam.

I will refer to the following books:

- Haim Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 2011, Springer-Verlag New York.
- John B. Conway, *A Course in Functional Analysis* (Second Edition), 1990, Springer.
- Walter Rudin, *Functional Analysis* (Second Edition), 1991, McGraw-Hill.

Locally convex spaces

Let $(X, \|\cdot\|_X)$ be a real normed space and let X^* be its dual, taken with respect to the norm of X. Then X^* is a normed space with $\|\Lambda\|_{X^*} := \sup\{|\Lambda x| : \ \|x\| \leq 1\}$ for $\Lambda \in X^*$. Recall that, since the field $\mathbb{R}$ is complete, X^* is a Banach space (Conway III.5.4). Let X^{**} be the dual of X^*, taken with respect to the operator norm of X^*. We equip X^{**} with the operator norm $\|T\|_{X^{**}} := \sup\{|T\Lambda| : \ \|T\|_{X^*} \leq 1\}$ for $T \in X^{**}$. Recall that the canonical embedding $J \colon X \to X^{**}$ is defined by $J(x)\Lambda := \Lambda x$ for $\Lambda \in \Lambda^*$. We have that J is an isometry, that is, $\|J(x)\|_{X^{**}} = \|x\|_X$ for all $x \in X$ (Conway III.6.7).

Convergences on normed spaces: Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence and $x \in X$. We say that $x_n \to x$ strongly if $\|x_n - x\|_X \to 0$ as $n \to \infty$. We say that $x_n \rightharpoonup x$ weakly if $\Lambda x_n \to \Lambda x$ as $n \to \infty$ for each $\Lambda \in X^*$. Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset X^*$ be a sequence and $\Lambda \in X^*$. We that $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly* if $\Lambda_n x \to \Lambda x$ as $n \to \infty$ for all $x \in X$. Notice that, since X^* is a normed space, we can also consider the strong convergence with respect to $\|\cdot\|_{X^*}$ and the weak convergence induced by X^{**}.

Extremal points: Let X be a real vector space and let $K \subset X$ be a convex subset. We say that $a \in K$ is an *extremal point* of K if the following condition holds:

$$\text{if} \quad a = \lambda x_1 + (1-\lambda) x_2 \quad \text{for} \quad \lambda \in (0,1)\,, \ x_1, x_2 \in K \quad \text{then} \quad x_1 = x_2\,.$$

In other words, $a \in K$ is an extremal point if it does not lie in the interior of any open segment contained in K. We denote by ext(K) the set of extremal points of K. For an arbitrary set $E \subset X$ we define its *convex hull* by

$$\mathrm{co(E)} := \left\{ \sum_{j=1}^{n} \lambda_j x_j \ : \ n \in \mathbb{N}\,, \ x_1, \dots, x_n \in E\,, \ \sum_{j=1}^{n} \lambda_j = 1 \right\},$$

that is, the set of all convex combinations of points of E.

Theorem (Krein-Milman): Let (X, τ) be a LCS. Assume that $K \subset X$ is non-empty, convex and compact with respect to τ. Then

$$\mathrm{ext(K)} \neq \emptyset \quad \text{and} \quad \mathrm{K} = \overline{\mathrm{co(ext(K))}}$$

where the closure is taken with respect to τ.

Problem 6.1. Let X be a real normed space and let $\Lambda_n, \Lambda \in X^*$ and $x_n, x \in X$ for $n \in \mathbb{N}$.

a) Show the following implications between convergences in X^*:

   i) if $\Lambda_n \to \Lambda$ strongly then $\Lambda_n \rightharpoonup \Lambda$ weakly,

   ii) if $\Lambda_n \rightharpoonup \Lambda$ weakly then $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly*.

   Hint: Use the canonical embedding J.

b) Assume in addition that X is reflexive. Prove that in X^* we have that $\Lambda_n \rightharpoonup \Lambda$ weakly if and only if $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly*.

   Hint: In this case, by definition, the canonical embedding J is surjective.

c) Prove that weak* limits in X^* and weak limits in X are unique.

   Hint: Use one of the corollaries of the Hahn-Banach Theorem (Conway III.6.8, Pag 79).

d) Assume that $x_n \rightharpoonup x$ weakly in X. Show that x_n is norm bounded, that is, $\sup_n \|x_n\|_X < \infty$.
   Hint: Use J and the Principle of Uniform Boundedness (Conway III.14.1, Pag 95).

e) Assume in addition that X is a Banach space. Show that if $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ then $\sup_n \|\Lambda_n\|_{X^*} < \infty$.
   Hint: Use the Principle of Uniform Boundedness.

Problem 6.2 (15 pts).

a) Let X be a real normed space and $K := \{x \in X : \|x\| \leq 1\}$ its (convex) unit ball. Show that
$$\mathrm{ext(K)} \subset \{\mathrm{x} \in \mathrm{X} : \|\mathrm{x}\| = 1\}\,.$$

b) Let $X = L^1(0,1)$, $K := \{f \in L^1(0,1) : \|f\|_1 \leq 1\}$, where $\|f\|_1 := \int_0^1 |f(x)|\,dx$. Prove that
$$\mathrm{ext(K)} = \emptyset\,.$$

   Hint: If $f \in L^1(0,1)$ the function $\Psi\colon [0,1] \to \mathbb{R}$ defined by $\Psi(x) := \int_0^x |f(t)|\,dt$ is continuous and non-decreasing.

c) By using point (b) and Krein-Milman, prove that $L^1(0,1)$ is not the dual of a Banach space.
   Hint: Banach-Alaoglu (Conway V.3.1, Pag 130).

Distributions

Let $d \in \mathbb{N}, d \geq 1$ and $\Omega \subset \mathbb{R}^d$ be open and non-empty. We denote by $\mathcal{D}(\Omega)$ the set of $C^\infty(\Omega)$ functions with compact support. The space $\mathcal{D}(\Omega)$ is endowed with a topology τ which makes it into a complete LCS (Rudin 6.2). If $\phi_n, \varphi \in \mathcal{D}(\Omega)$, we have that $\phi_n \to \phi$ with respect to τ if and only if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp}\phi_n \subset K$ for all n and
$$D^\alpha \phi_n \to D^\alpha \phi \quad \text{uniformly on } K \text{ for all } \alpha \in \mathbb{N}_0^d\,,$$
where $D^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ (Rudin 6.5). We recall that the order of ∂^α is $|\alpha| := \sum_{i=1}^d \alpha_i$. The linear differential operator $D^\alpha\colon \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is continuous (Rudin 6.6). The space of distributions over Ω is denoted by $\mathcal{D}(\Omega)^*$ and it is defined as the set of linear operators $\Lambda\colon \mathcal{D}(\Omega) \to \mathbb{R}$ which are τ-continuous. For a linear operator $\Lambda\colon \mathcal{D}(\Omega) \to \mathbb{R}$ the following conditions are equivalent:

i) $\Lambda \in \mathcal{D}(\Omega)^*$,

ii) $\Lambda\phi_n \to 0$ whenever $\phi_n \to 0$ in $\mathcal{D}(\Omega)$,

iii) For each compact set $K \subset \Omega$ there exist constants $N \in \mathbb{N}_0$, $C > 0$ such that
$$|\Lambda\phi| \leq C\,\|\phi\|_{K,N} \qquad \text{for all} \qquad \phi \in \mathcal{D}_K\,, \tag{1}$$
   where $D_K := \{\phi \in C^\infty(\Omega) : \ \operatorname{supp}\phi \subset K\}$ and
$$\|\phi\|_{K,N} := \max\{|\partial^\alpha \phi(x)| : \ x \in K,\, \alpha \in \mathbb{N}_0^d,\, |\alpha| \leq N\}\,,$$

(Rudin 6.6 and 6.8). If the constant N in (1) is independent on K, we call the smallest N with such property the *order* of Λ.

For $f \in L^1_{\mathrm{loc}}(\Omega)$ we define the distribution Λ_f via

$$\Lambda_f(\phi) := \int_\Omega f(x)\,\phi(x)\,dx \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega)\,.$$

Also, for $p \in \Omega$ define the delta distribution at p by $\delta_p(\phi) := \phi(p)$ for each $\phi \in \mathcal{D}(\Omega)$.

Derivatives: If $\Lambda \in \mathcal{D}(\Omega)^*, \alpha \in \mathbb{N}_0^d$ the α-derivative of Λ is the linear operator $\partial^\alpha \Lambda \colon \mathcal{D}(\Omega) \to \mathbb{R}$ defined by

$$(\partial^\alpha \Lambda)(\phi) := (-1)^{|\alpha|}\Lambda(\partial^\alpha \phi) \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega)\,.$$

We have that $\partial^\alpha \Lambda \in \mathcal{D}(\Omega)^*$ (Rudin 6.12). Notice that $\partial^\alpha \Lambda_f = \Lambda_{\partial^\alpha f}$ whenever f is regular and $f, \partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$.

Supports: Let $\Lambda \in \mathcal{D}(\Omega)^*$. If $\omega \subset \Omega$ is an open set, we say that $\Lambda = 0$ in ω if $\Lambda\phi = 0$ for all $\phi \in \mathcal{D}(\omega)$. We define W as the union of all the open sets $\omega \subset \Omega$ such that $\Lambda = 0$ in ω. The support of Λ is then defined by

$$\operatorname{supp} \Lambda := \Omega \smallsetminus W\,.$$

If $\operatorname{supp} \Lambda$ is compact, then Λ has finite order and it extends in a unique way to a linear continuous functional on $C^\infty(\Omega)$ (Rudin 6.24). Also recall the following structure theorem (Rudin 6.25):

Theorem 1. Let $\Lambda \in \mathcal{D}(\Omega)^*$ be such that $\operatorname{supp} \Lambda \subset \{p\}$ for some $p \in \Omega$. Then there exist $N \in \mathbb{N}$ and coefficients $c_\alpha \in \mathbb{R}$ for each $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq N$ such that

$$\Lambda = \sum_{|\alpha| \leq N} c_\alpha\, \partial^\alpha \delta_p \quad \text{in} \;\; \mathcal{D}(\Omega)^*\,.$$

Multiplication: Let $f \in C^\infty(\Omega)$ and $\Lambda \in \mathcal{D}(\Omega)^*$. Their multiplication is the distribution $f\Lambda$ defined by

$$(f\Lambda)(\phi) := \Lambda(f\phi) \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega)\,.$$

Limits: For $\Lambda_n \in \mathcal{D}(\Omega)^*$ we say that $\Lambda_n \to \Lambda$ in the sense of distributions if for each $\phi \in \mathcal{D}(\Omega)$ the limit

$$\lim_n \Lambda_n \phi = \Lambda \phi \tag{2}$$

exists and is finite. Whenever (2) is satisfied for some linear functional Λ, then automatically $\Lambda \in \mathcal{D}(\Omega)^*$ (Rudin 6.17). If $f_n \in L^1_{\mathrm{loc}}(\Omega), \Lambda \in \mathcal{D}(\Omega)^*$, we write $f_n \to \Lambda$ in place of $\Lambda_{f_n} \to \Lambda$.

Convolutions 1: Let $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$, $\phi \in \mathcal{D}(\mathbb{R}^d)$. Introduce the translation and reflection operators

$$(\tau_x \phi)(y) := f(y - x)\,, \quad \check{\phi}(y) := \phi(-y) \quad \text{for} \quad y \in \mathbb{R}^d\,,$$

where $x \in \mathbb{R}^d$ is fixed. The *convolution* between Λ and ϕ is the map $\Lambda \star \phi \colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$(\Lambda \star \phi)(x) := \Lambda(\tau_x \check{\phi}) \quad \text{for all} \;\; x \in \mathbb{R}^d\,. \tag{3}$$

It is important that ϕ is compactly supported for the definition to make sense. Notice that, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\Lambda = \Lambda_f$, then (5) coincides with the classical definition, since

$$(\Lambda_f \star \phi)(x) = \int_{\mathbb{R}^d} f(y)(\tau_x \check{\phi})(y)\,dy = \int_{\mathbb{R}^d} f(y)\phi(x - y)\,dy = (f \star \phi)(x)$$

Motivated by the equality

$$\int_{R^d} (\tau_x f)(y)\phi(y)\,dy = \int_{R^d} f(y)(\tau_{-x}\phi)(y)\,dy\,,$$

we also define the translation of Λ by $x \in \mathbb{R}^d$ as the linear functional $\tau_x \Lambda \colon \mathcal{D}(\mathbb{R}^d) \to \mathbb{R}$ such that

$$(\tau_x \Lambda)(\phi) := \Lambda(\tau_{-x}\phi) \quad \text{for all} \;\; \phi \in \mathcal{D}(\mathbb{R}^d)\,. \tag{4}$$

Convolutions 2: Assume that $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$ has compact support and that $\phi \in C^\infty(\Omega)$. Since Λ extends to a linear continuous functional on $C^\infty(\Omega)$, it makes sense to define $\Lambda \star \phi \colon \mathbb{R}^d \to \mathbb{R}$ by

$$(\Lambda \star \phi)(x) := \Lambda(\tau_x \check{\phi}) \quad \text{for all} \;\; x \in \mathbb{R}^d\,, \tag{5}$$

in the same way we did in (5).

Problem 6.3. Let $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$.

a) Show that $\tau_x \Lambda$ belongs to $\mathcal{D}(\mathbb{R}^d)^*$ for all $x \in \mathbb{R}^d$, where τ_x is defined at (4).

b) Prove that
$$\tau_x(\Lambda \star \phi) = (\tau_x \Lambda) \star \phi = \Lambda \star (\tau_x \phi)$$
for all $x \in \mathbb{R}^d$.

c) Show that $\Lambda \star \phi \in C^\infty(\mathbb{R}^d)$ and that for each $\alpha \in \mathbb{N}_0^d$
$$D^\alpha(\Lambda \star \phi) = (D^\alpha \Lambda) \star \phi = \Lambda \star (D^\alpha \phi)\,.$$

d) Let $\rho_n \in \mathcal{D}(\mathbb{R}^d)$ be a sequence of mollifiers, that is, $\rho_n \geq 0$, $\operatorname{supp} \rho_n \subset B_{1/n}(0)$, $\int_{\mathbb{R}^d} \rho_n \, dx = 1$. Prove that
$$\lim_n \Lambda \star \rho_n = \Lambda \quad \text{ in } \ \mathcal{D}(\mathbb{R}^d)^*\,.$$
Hint: Use that $\Lambda \star (\phi \star \psi) = (\Lambda \star \phi) \star \psi$ for all $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$, $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$.

e) Assume in addition that Λ is compactly supported and let $\psi \in C^\infty(\mathbb{R}^d)$. Prove that the claims in points (b) and (c) hold for $\Lambda \star \psi$.

f) Assume in addition that Λ is compactly supported. Prove that $\Lambda \star \phi \in \mathcal{D}(\mathbb{R}^d)$.

Problem 6.4 (15 pts).

a) Let $\delta \in \mathcal{D}(\mathbb{R})^*$ be the Dirac distribution at 0, that is, $\delta(\phi) := \phi(0)$ for all $\phi \in \mathcal{D}(\mathbb{R})$. For each $m \in \mathbb{N}$, $m \geq 0$, characterize the set
$$A_m = \{f \in C^\infty(\mathbb{R}) : \ f\delta^{(m)} = 0 \text{ in } \mathcal{D}(\mathbb{R})^*\}\,.$$

b) Give an example of $f \in C^\infty(\mathbb{R})$ and $\Lambda \in \mathcal{D}(\mathbb{R})^*$ such that $f = 0$ on $\operatorname{supp} \Lambda$ but $f\Lambda \neq 0$.

c) Fix $m \in \mathbb{N}$, $m \geq 1$. Show that they are equivalent:

   i) $x^m \Lambda = 0$ in $\mathcal{D}(\mathbb{R})^*$,

   ii) There exist $c_0, c_1, \ldots, c_{m-1} \in \mathbb{R}$ such that $\Lambda = \sum_{k=0}^{m-1} c_k \delta^{(k)}$.

   Hint: If $x^m \Lambda = 0$, first prove that $\operatorname{supp} \Lambda \subset \{0\}$. Then you can use Theorem 1. It is useful to notice that, since Λ is compactly supported, you can test against functions in $C^\infty(\mathbb{R})$.

Problem 6.5 (20 pts). Suppose that $f \in L^1((-\infty, -\varepsilon) \cup (\varepsilon, \infty))$ for all $\varepsilon > 0$. The principal value integral of f is defined by
$$\mathrm{PV} \int_{\mathbb{R}} f(x)\, dx := \lim_{\varepsilon \to 0} \int_{\{|x| \geq \varepsilon\}} f(x)\, dx$$
whenever the limits exists (finite). Here $\{|x| \geq \varepsilon\}$ is a shorthand for $\{x \in \mathbb{R} : \ x \geq \varepsilon \ \text{ or } \ x \leq -\varepsilon\}$. For $\phi \in \mathcal{D}(\mathbb{R})$ define
$$\left(\mathrm{PV}\, \frac{1}{x}\right)(\phi) := \mathrm{PV} \int_{\mathbb{R}} \frac{\phi(x)}{x}\, dx\,.$$

a) Prove that $\mathrm{PV}\,\frac{1}{x}$ is well defined, that it belongs to $\mathcal{D}(\mathbb{R})^*$ and that its order is at most 1.

   Hint: Notice that $1/x$ is anti-symmetric, therefore $\int_{\{|x| \geq \varepsilon\}} x^{-1}\, dx = 0$.

b) Prove that $\mathrm{PV}\,\frac{1}{x}$ is a distribution of order 1.

   Hint: We already know that the order is at most 1. Assume by contradiction that the order is 0, so that for any $K \subset \mathbb{R}$ compact there exists $C > 0$ such that $|\left(\mathrm{PV}\,\frac{1}{x}\right)(\phi)| \leq C\, \|\phi\|_{K,0}$ for all $\phi \in \mathcal{D}_K$. Take $K = [0, 1]$ and produce a sequence $\phi_n \in \mathcal{D}_K$ such such that $0 \leq \phi_n \leq 1$, which makes the previous estimate fail.

c) Show that, in the sense of distributions,

$$(\log |x|)' = \mathrm{PV}\,\frac{1}{x}\,.$$

d) Show that for all $\phi \in \mathcal{D}(\mathbb{R})$

$$\left(\mathrm{PV}\,\frac{1}{x}\right)'(\phi) = -\lim_{\varepsilon\to 0}\int_{\{|x|\geq\varepsilon\}} \frac{\phi(x)-\phi(0)}{x^2}\,dx\,.$$

Compact operators and spectral theory

For a Banach space X we denote by B_X its unit ball, that is, $B_X := \{x \in X\colon\ \|x\|_X \leq 1\}$. If Y is another Banach space, we denote by $\mathcal{L}(X,Y)$ the space of linear continuous operators $T\colon X \to Y$. Recall that $\mathcal{L}(X,Y)$ is a Banach space with the operator norm. We also denote $\mathcal{L}(X) := \mathcal{L}(X,X)$.

Compact operators: Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. We say that T is a *compact operator* if the closure of $T(B_X)$ is compact in Y. We denote the space of compact operators from X to Y by $\mathcal{K}(X,Y)$. Also we denote $\mathcal{K}(X) := \mathcal{K}(X,X)$.

Finite rank: Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. We say that T has *finite rank* if $T(X)$ is finite dimensional.

Adjoint: Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. The *adjoint* of T is the linear operator $T^*\colon Y^* \to X^*$ defined by

$$\langle T^*y^*, x\rangle_{X^*,X} = \langle y^*, Tx\rangle_{Y^*,Y} \quad \text{for all} \quad x \in X,\ y^* \in Y^*\,.$$

It is well-known that $T^* \in \mathcal{L}(Y^*, X^*)$, with $\|T\| = \|T^*\|$.

Theorem 2 (Brezis 6.1, 6.4): Let X, Y, Z be Banach spaces, $T \in \mathcal{L}(X,Y), S \in \mathcal{L}(Y,Z)$. Then:

i) $T \in \mathcal{K}(X,Y)$ if and only if $T^* \in \mathcal{K}(Y^*, X^*)$,

ii) If T has finite rank, then $T \in \mathcal{K}(X,Y)$,

iii) $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$: if $T_n \in \mathcal{K}(X,Y)$, $\|T_n - T\| \to 0$ then $T \in \mathcal{K}(X,Y)$,

iv) If $T \in \mathcal{K}(X,Y)$ or $S \in \mathcal{K}(X,Y)$, then $ST \in \mathcal{K}(X,Z)$.

Theorem 3 (Riesz's Lemma, Brezis 6.1): Let X be a normed space, and $M \subset X$ a closed subspace with $M \neq X$. Then for each $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| = 1$ and $\operatorname{dist}(x, M) \geq 1 - \varepsilon$.

Spectral theory: Let X be a Banach space, $T \in \mathcal{L}(X)$. The *resolvent* set of T is defined by

$$\rho(T) := \{\lambda \in \mathbb{R} :\ T - \lambda I \ \text{ is bijective from } \ X \ \text{ onto } \ X\}\,,$$

where I denotes the identity operator from X into itself. The *spectrum* of T is

$$\sigma(T) := \mathbb{R} \smallsetminus \sigma(T)\,.$$

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of T if $\ker(T - \lambda I) \neq \{0\}$. We denote by $\mathrm{EV}(T)$ the set of eigenvalues of T. For $\lambda \in \mathrm{EV}(T)$, the corresponding *eigenspace* is $\ker(T - \lambda I)$. Notice that $\mathrm{EV}(T) \subset \sigma(T)$, but they are not equal in general.

Theorem 4 (Brezis 6.7, 6.8): Let X be a Banach space and $T \in \mathcal{L}(X)$. Then $\sigma(T)$ is a compact set, $\sigma(T) = \sigma(T^*)$ and

$$\sigma(T) \subset [-\|T\|, \|T\|]\,.$$

Assume in addition that X is infinite dimensional and $T \in \mathcal{K}(X)$. Then

i) $0 \in \sigma(T)$,

ii) $\sigma(T) \smallsetminus \{0\} = \mathrm{EV}(T) \smallsetminus \{0\}$,

iii) Either $\sigma(T) = \{0\}$, or $\sigma(T) \smallsetminus \{0\}$ is a finite set, or $\sigma(T) \smallsetminus \{0\} = \{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \to 0$.

Relative compactness in $C(X)$: Assume that (X,d) is a compact metric space. We denote by $C(X)$ the space of continuous functions $f\colon X \to \mathbb{R}$. Then $C(X)$ is a Banach space with the supremum norm $\|f\|_\infty := \sup_{x\in X}|f(x)|$. For a family $\mathcal{A} \subset C(X)$ we say that $\mathcal{A}$ is *uniformly bounded* if there exists a constant $M > 0$ such that

$$\sup_{x\in X}|f(x)| \le M \quad \text{for all} \quad f \in \mathcal{A}\,.$$

We say that $\mathcal{A}$ is *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε) with the following property:

$$\text{for all } \; x, y \in X \; \text{ such that } \; d(x,y) < \delta\,, \; \text{ it follows that } \; |f(x) - f(y)| < \varepsilon \; \text{ for all } \; f \in \mathcal{A}\,.$$

A characterization of relative compactness in $C(X)$ is given by the following:

Theorem 5 (Ascoli-Arzelà): Let (X,d) be a compact metric space. Let $\mathcal{A} \subset C(X)$. They are equivalent:

i) The closure of $\mathcal{A}$ is compact in $C(X)$ (with respect to the supremum norm);

ii) $\mathcal{A}$ is uniformly bounded and equicontinuous;

iii) each sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$ admits a subsequence converging uniformly.

Relative compactness in L^p: Let $d \in \mathbb{N}, d \ge 1$. For a map $f\colon \mathbb{R}^d \to \mathbb{R}$ we define its translation by $h \in \mathbb{R}^d$ as the new map $\tau_h f\colon \mathbb{R}^d \to \mathbb{R}$ defined by $(\tau_h f)(x) := (x - h)$. The following theorem is a version of Ascoli-Arzelà for L^p spaces.

Theorem 6 (Fréchet-Kolmogorov): Let $1 \le p < \infty$ and consider a family $\mathcal{A} \subset L^p(\mathbb{R}^d)$. Suppose that $\mathcal{A}$ is bounded, that is,

$$\sup_{f\in\mathcal{A}} \|f\|_{L^p(\mathbb{R}^d)} < +\infty\,.$$

Moreover assume that

$$\lim_{|h|\to 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^d)} = 0 \quad \text{uniformly in } \; f \in \mathcal{A}$$

that is, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\tau_h f - f\|_{L^p} < \varepsilon$ for all $f \in \mathcal{A}$, $h \in \mathbb{R}^d$ with $|h| < \delta$. Then the closure of $\mathcal{A}|_\Omega$ in $L^p(\Omega)$ is compact for any Lebesugue measurable set $\Omega \subset \mathbb{R}^d$ with $|\Omega| < \infty$.

In the above theorem we denote by $\mathcal{A}|_\Omega$ the restriction to Ω of functions in $\mathcal{A}$.

Problem 6.6 (10 pts).

a) Let X be a normed space. Show that the identity map $I\colon X \to X$ is compact if and only if $\dim X < +\infty$.

   Hint: Riesz Lemma (Theorem 3).

b) Consider $C^1[0,1]$ equipped with the norm $\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$ and $C[0,1]$ equipped with the supremum norm. Prove that the identity $I\colon C^1[0,1] \to C[0,1]$ is continuous and compact.

   Hint: Use Theorem 5.

Problem 6.7 (10 pts). Let H be a real Hilbert space and $T \in \mathcal{L}(H)$. Let $x_n, x \in H$ for $n \in \mathbb{N}$.

a) Show that $x_n \to x$ strongly in H if and only if

$$x_n \rightharpoonup x \;\text{ weakly in }\; H \;\text{ and }\; \|x_n\|_H \to \|x\|_H\,.$$

b) Show that $T \in \mathcal{K}(H)$ if and only if the following condition holds:

$$\text{If } \; x_n \rightharpoonup x \; \text{ weakly in } \; H\,, \; \text{ then } \; Tx_n \to Tx \; \text{ strongly in } \; H\,.$$

   Hint: Since H is reflexive, $T(B_H)$ is closed (see Problem 6.8 Point (c)).

Problem 6.8.

a) Let X, Y be normed spaces, $T \in \mathcal{L}(X, Y)$. Assume there exists a constant $c > 0$ such that

$$\|Tx\|_Y \geq c\,\|x\|_X \qquad \text{for all} \qquad x \in X\,.$$

Show that T is compact if and only if $\dim X < +\infty$.

b) Let X be a Banach space with $\dim X = +\infty$ and let $T \in \mathcal{K}(X)$. Show that T cannot be surjective, that is, there exists $y \in X$ such that the equation

$$Tx = y$$

has no solution in X.

c) Let X, Y be Banach spaces and assume that X is reflexive. Let $T \in \mathcal{L}(X, Y)$ and $M \subset X$ be closed, convex and bounded.

i) Show that $T(M)$ is closed in Y.

ii) In addition, assume that $T \in \mathcal{K}(X, Y)$. Show that $T(M)$ is compact.

d) Let H be a Hilbert space and $T \in \mathcal{K}(H)$. Show that T attains its norm, that is, there exists $\hat{x} \in H$ such that $\|\hat{x}\| \leq 1$ and $\|T\| = \|T\hat{x}\|$.

Problem 6.9 (30 pts). Consider the space $C[0,1]$ equipped with the supremum norm and let $1 \leq p \leq \infty$. Define the linear operator $T\colon L^p(0,1) \to L^p(0,1)$ by

$$(Tf)(x) := \int_0^x f(t)\,dt \qquad \text{for} \qquad x \in [0,1]\,.$$

Also consider the linear operator $S\colon C[0,1] \to C[0,1]$ defined by $Sf := Tf$ for $f \in C[0,1]$.

a) Prove that S is bounded and compute $\|S\|$.

b) Let $B := \{f \in C[0,1] : \ \|f\|_\infty \leq 1\}$. Prove that $S(B)$ is not closed.

Hint: Notice that $Sf \in C^1[0,1]$ for all $f \in C[0,1]$. Therefore construct a sequence $f_n \in B$ such that $Sf_n \to g$ uniformly but $g \notin C^1[0,1]$.

c) Prove that S is compact.

d) Prove that T is bounded for all $p \in [1,\infty]$ and compute its adjoint T^*.

e) Prove that T is compact for each $p \in [1,\infty]$.

Hint: For $1 < p \leq \infty$ use the fact that $Tf \in C[0,1]$. Therefore if you show compactness in $C[0,1]$ (by employing Theorem 5), you also have it in $L^p(0,1)$. For $p = 1$ you do not have compactness in $C[0,1]$ (see point (g)), but you can still prove compactness in $L^1(0,1)$ by means of Theorem 6.

f) Compute $\sigma(T)$, $\mathrm{EV}(T)$ and $\rho(T)$.

Hint: Try to compute $\mathrm{EV}(T)$ first. Remember that if $f \in L^p(0,1)$, then its primitive is Sobolev (see Problem 2.3). By a bootstrap argument you can infer regularity of the eigenvectors.

g) Show that $T\colon L^1(0,1) \to C[0,1]$ is not compact.

Hint: Consider $f_n(x) := n\chi_{(0,1/n)}(x)$.