

Perfect set theorems for closed and analytic sets

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Perfect set theorems are theorems of the following kind:

Theorem Template. *Let X be some kind of set, then one of the following will hold:*

1. *X has a perfect subset (and then possibly something about this perfect subset)*
2. *X is tractable in some way*

This note contains the perfect set theorems for closed and analytic (Σ_1^1) sets in the Baire space.

1 Perfect set theorem for closed sets

Definition 1.1. A closed set $X \subseteq \omega^\omega$ is the body $[T]$ of a tree $T \subseteq \omega^{<\omega}$.

The proof of the perfect set theorem demonstrates a common technique that has many different variants for later use. The technique involves eliminating isolated elements, which will remind the analysts of the Cantor-Bendixson derivative. Let us first define an increasing sequence of isolated nodes in a tree.

Definition 1.2. Let T be a tree $\subseteq \omega^{<\omega}$. We say a node s is *isolated in T* iff it is an element of T and doesn't split below (i.e., there are no extensions $\sigma_0, \sigma_1 \succ s$ in T that are incompatible).

Definition 1.3. Given a tree $T \subseteq \omega^{<\omega}$, define an increasing sequence of sets T_α as follows:

$$\begin{aligned} T_0 &= \emptyset \\ T_\lambda &= \bigcup_{\beta < \lambda} T_\beta \text{ if } \lambda \text{ is a limit ordinal} \\ T_{\alpha+1} &= T_\alpha \cup \{s \in T \mid s \text{ is isolated in } T \setminus T_\alpha\} \end{aligned}$$

Lemma 1.4. *If $T \subseteq \omega^{<\omega}$ is a tree, then there is some countable ordinal δ when $T_\delta = T_{\delta+1}$.*

Proof. This is because, if $T_0 \subsetneq T_1 \subsetneq T_2 \subseteq \dots \subsetneq T_\alpha \subsetneq T_{\alpha+1} \subsetneq \dots$ lasts $\geq \omega_1$ many steps, we would have collected an uncountable subset of T , which has at most countably many elements. \square

So for any tree T , there will be a least ordinal countable ordinal δ for which $T_\delta = T_{\delta+1}$.

Theorem 1.5 (The case where $T \neq T_\delta$). *For any T and δ as above, if $T \neq T_\delta$, then there is an injection from 2^ω into $[T]$.*

Proof. Almost by definition, $T \setminus T_\delta$ is the set of nodes in T that will always split. So we can map $2^{<\omega}$ into T by mapping splits to splits recursively. More concretely, set $f(\langle \rangle) = \langle \rangle$, and if $f(s) \in T \setminus T_\delta$ is defined, then by assumption there are at least two incompatible extensions $f(s)^\frown m, f(s)^\frown n$ below it. Pick the least such pair $m < n$ and map $s^\frown 0$ to $f(s)^\frown m$ and $s^\frown 1$ to $f(s)^\frown n$. (Visually, we are “stretching” the infinite binary tree to “fit” its splits to those on $T \setminus T_\delta$). Finally for $x \in 2^\omega$ set $F(x) = \bigcup_{n \in \omega} f(x \upharpoonright n)$. \square

On the other hand, if $T = T_\delta$, then this provides a fertile ground for an effective analysis of T . First, notice that this would imply that each node in T becomes isolated at some point, and then gets collected at the next stage. This is just another way of saying every node of T is isolated in some $T \setminus T_\alpha$ for $\alpha < \delta$, and then it gets in $T_{\alpha+1}$.

Definition 1.6. For each node $s \in T$, call the unique ordinal α where this happens its isolation rank (i.e., least α such that $s \in T_{\alpha+1} \setminus T_\alpha$), written $\rho(s)$.

A few things to notice:

- Nodes with smaller isolation ranks get picked up by T_α earlier in the process.
- The empty sequence always has the maximum isolation rank, because it always gets picked up last.
- If $s \prec t$, then $\rho(s) \geq \rho(t)$.

Observation 1.7. If $x \in [T]$, then the isolation ranks of $x \upharpoonright 0, x \upharpoonright 1, x \upharpoonright 2, \dots$ is a non-increasing sequence of ordinals. So this sequence of ordinals must be eventually constant. Since it won't cause any confusion, we also call this eventual ordinal constant the isolation rank ρ_x of x .

Theorem 1.8 (The case where $T = T_\delta$). *For a tree T with $T = T_\delta$ (recall the notations above), if $x \in [T]$ has isolation rank $\rho_x < \delta$, as witnessed by $x \upharpoonright m$, then x is definable from $T, \rho_x, x \upharpoonright m$*

Proof. To say x has isolation rank ρ_x (witnessed by $x \upharpoonright m$) is to say that, in $T \setminus T_{\rho_x}$, the only extensions to $x \upharpoonright m$ are $x \upharpoonright (m+1), x \upharpoonright (m+2), x \upharpoonright (m+3), \dots$. But then it is now easy to define x : set $x(i) = j$ iff
$$\begin{cases} (x \upharpoonright m)(i) = j & i < m \\ (\exists q \in T \setminus T_{\rho_x})(x \upharpoonright m \prec q \wedge q(i) = j) & i \geq m \end{cases}$$

In other words, to “compute” $x(i)$ from T and ρ_x , we only need to build T_{ρ_x} and remove it from T , and then just brute-force search through finite sequences of natural numbers in that tree to see if any extends $x \upharpoonright m$. By the assumptions that $x \in [T]$ and $x \upharpoonright m$ is isolated in $T \setminus T_{\rho_x}$, for each length $k \geq m$ a unique extension exists. Moreover, all the extensions cohere (meaning their union is a real number, in this case x). \square

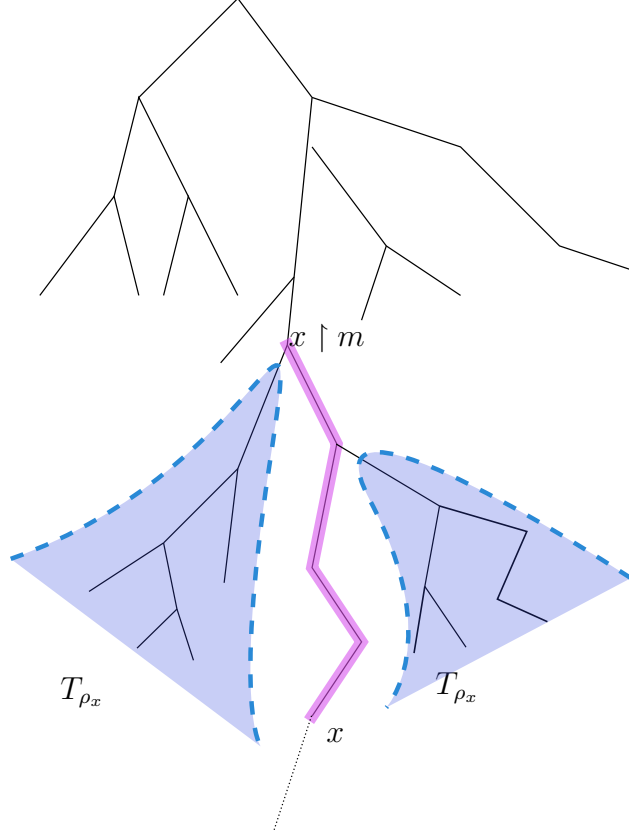


Figure 1: Once T_{ρ_x} is removed from T , $x \restriction m$ becomes isolated and hence x becomes definable by brute-force search through remaining nodes that extend it

Observation 1.9. The definition of x above is absolute to $L[T]$.

Proof. This is because the transfinite recursion constructing T_α 's from T , the definition of isolation ranks, and the definition of x (which is arithmetic) only involve Δ_0 formulas with parameters in $L[T]$. \square

Corollary 1.10 (Perfect set theorem for closed sets). *Let $X \subseteq \omega^\omega$ be the body of some tree T , then either X has a perfect subset, or $X \in L_{\omega_1}[T]$ (hence countable).*

Proof. We've proved almost everything, except that $X \in L_{\omega_1}[T]$. This is via absoluteness considerations. The construction process T_α and the arithmetical definition of x are both absolute between V and $L[T]$. And $\delta < \omega_1$. We've shown that $X \subseteq L_{\delta+\omega}[T]$, and so X can be defined as the paths through T in that level, which we know are all of them. \square

Remark. If you really think about it, all that there is to the proof is already captured in the lyrics to Lemon Tree by Fool's Garden. Isolation is good for you; if all that you can see is the lemon tree then you can define the lemon tree, etc etc.

2 Perfect set theorem for analytic sets

Definition 2.1. An analytic set $X \subseteq \omega^\omega$ is the projection $p[T]$ of the body of a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$. Recall: $p[T] := \{x \mid (\exists y)(x, y) \in [T]\}$

Example 2.2. One of the first analytic sets is Luzin's set A of sequences containing a progressively divisible subsequence:

$$A(x) \Leftrightarrow \exists n_0 < n_1 < n_2 < \dots x(n_i) \text{ divides } x(n_{i+1})$$

It is the projection of the tree of attempts searching for such a sequence:

$$T := \{(s, t) \mid \text{the } t(0), t(1), \dots, t(n)\text{th places of } s \text{ are not a counterexample to progressive divisibility for any } n < \text{length}(t)\}$$

Furthermore, this set is Σ_1^1 -complete, meaning that every analytic set can be obtained as the continuous pre-image of this set.

Remark. Hopefully, the above example brings to mind the set of directed graphs containing a clique or a Hamiltonian path. These are of course classic examples of NP-complete sets. Indeed, in many aspects we have good reasons to think of the NP sets as finitary analogues of the analytic sets and vice versa. As a matter of fact, the sets

$$\begin{aligned} A &:= \{x \in \omega^\omega \mid x \text{ codes a countable graph with an infinite clique}\} \\ B &:= \{x \in \omega^\omega \mid x \text{ codes a countable graph with a Hamiltonian path}\} \end{aligned}$$

are both Σ_1^1 -complete.

The perfect set theorem for analytic sets is proved using a similar technique to closed sets. First we define the notion of isolation.¹

Definition 2.3. Given a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ and (u, v) , we say (u, v) is *isolated in* T iff $(u, v) \in T$ and it has no extensions in T that are incompatible in the first coordinate. In other words, every extension of (u, v) in T will be compatible in the first coordinate.

Given a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$, construct again a sequence of sets T_α as follows:

$$\begin{aligned} T_0 &= \emptyset \\ T_\lambda &= \bigcup_{\beta < \lambda} T_\beta \text{ if } \lambda \text{ is a limit ordinal} \\ T_{\alpha+1} &= T_\alpha \cup \{(u, v) \in T \mid (u, v) \text{ is isolated in } T \setminus T_\alpha\} \end{aligned}$$

Lemma 2.4 and Theorem 2.5 are proved in the same manner as before.

¹There's some slight clash of notation here. I've defined isolation and isolation rank for closed sets in the previous section. Technically the same definition generalizes to the closed set $[T] \subseteq \omega \times \omega$. But what is terminological consistency in the face of a convenient proof? Hence I've chosen to redefine what isolation means for this specific proof.

Lemma 2.4. *There is a least countable ordinal δ for which $T_\delta = T_{\delta+1}$.*

Theorem 2.5 (The case where $T \neq T_\delta$). *For any T and δ as above, if $T \neq T_\delta$, then there is an injection from 2^ω into $p[T]$.*

Now, if $T = T_\delta$, then every node in T becomes isolated at some point. That is, for every $(u, v) \in T$, there is a least ordinal α such that (u, v) is isolated in $T \setminus T_\alpha$ and then $(u, v) \in T_{\alpha+1}$. Let us call this ordinal α the isolation rank of (u, v) , written $\rho(u, v)$. Again, (\emptyset, \emptyset) has the highest isolation rank, and if $(u, v) \prec (u', v')$, then $\rho(u, v) \geq \rho(u', v')$. That is, as you descend down a path, the isolation ranks can't go up (it can only go down or remain the same).

Recall if $x \in p[T]$, then there is some y such that $(x, y) \in [T]$. That is, there is some y such that $(x \upharpoonright n, y \upharpoonright n) \in T$ for all $n \in \omega$.

Observation. So, if $x \in p[T]$, as witnessed by y , then the sequence

$$(\rho(\emptyset, \emptyset), \rho(x \upharpoonright 1, y \upharpoonright 1), \rho(x \upharpoonright 2, y \upharpoonright 2), \dots)$$

is a non-increasing sequence of ordinals. Hence it must be eventually constant.

Theorem 2.6 (The case where $T = T_\delta$). *For any T and δ as above, such that $T = T_\delta$, if $x \in p[T]$ as witnessed by y , and $(\rho(\emptyset, \emptyset), \rho(x \upharpoonright 1, y \upharpoonright 1), \rho(x \upharpoonright 2, y \upharpoonright 2), \dots)$ is eventually constant starting at $\rho(x \upharpoonright m, y \upharpoonright m)$, then x is definable from $T, x \upharpoonright m, y \upharpoonright m, \rho(x \upharpoonright m, y \upharpoonright m)$.*

Proof. Again, the assumptions imply that $(x \upharpoonright m, y \upharpoonright m)$ is isolated in $T \setminus T_{\rho(x \upharpoonright m, y \upharpoonright m)}$. This means that every extension of the pair $(x \upharpoonright m, y \upharpoonright m)$ in $T \setminus T_{\rho(x \upharpoonright m, y \upharpoonright m)}$ must be compatible in the first coordinate. In other words, every extension of $(x \upharpoonright m, y \upharpoonright m)$ in $T \setminus T_{\rho(x \upharpoonright m, y \upharpoonright m)}$ will have the form $(x \upharpoonright k, q)$ for some $k \geq m$ and $q \in \omega^{<\omega}$.

But it is now easy to define x : set $x(i) = j$ iff

$$\begin{cases} (x \upharpoonright m)(i) = j & i < m \\ (\exists (p, q) \in T \setminus T_{\rho(x \upharpoonright m, y \upharpoonright m)})((x \upharpoonright m, y \upharpoonright m) \prec (p, q) \wedge p(i) = j) & i \geq m \end{cases}$$

In other words, to “compute” $x(i)$ from $T, x \upharpoonright m, y \upharpoonright m, \rho(x \upharpoonright m, y \upharpoonright m)$, we only need to build $T_{\rho(x \upharpoonright m, y \upharpoonright m)}$ and remove it from T , and then just brute-force search through pairs of finite sequences of natural numbers to see if any extends $(x \upharpoonright m, y \upharpoonright m)$. Since $(x \upharpoonright m, y \upharpoonright m)$ is isolated and for each length $k \geq m$, extensions of length k exist and all cohere in the first coordinate (recall the isolation rank of $(x \upharpoonright m, y \upharpoonright m)$ is that eventual ordinal constant), this definition defines $x \in p[T]$. \square

Corollary 2.7 (Perfect set theorem for analytic sets). *Let $X \subseteq \omega^\omega$ be the projection $p[T]$ the body of some tree T , then either X has a perfect subset, or $X \in L_{\omega_1}[T]$ (hence countable).*

Proof. Again, this is via absoluteness considerations by noticing that the construction process T_α and the arithmetical definition of x are both absolute between V and $L[T]$. \square

3 Some more examples of perfect set theorems

Theorem 3.1 (Harrison). *If A is Σ_1^1 , then either X has a perfect subset or there exists a computable ordinal α such that every element of A is computable by $\emptyset^{(\alpha)}$.*

Theorem 3.2 (Guaspari, Sacks, Kechris). *If X is Π_1^1 , then either X has a perfect subset, or $X \subseteq C_1 := \{x \in \omega^\omega \mid x \in L_{\omega_1^x}\}$*

The set C_1 is called the largest countable thin set.

Theorem 3.3. *If X is $\Sigma_2^1(a)$, then either A has a perfect subset, or $A \in L[a]$*

Under the assumption that $C_2(a) := \omega^\omega \cap L[a]$ is countable, $C_2(a)$ is called the largest countable $\Sigma_2^1(a)$ set. People have been somewhat obsessed with these sets since the 80s.

The above theorem is a consequence of the more general result.

Theorem 3.4 (Mansfield, Solovay). *If X is the projection $p[T]$ of the body of $T \subseteq \omega^{<\omega} \times Y^{<\omega}$ for some set Y , then either X has a perfect subset (moreover the tree corresponding to that perfect set is in $L[T]$), or $X \in L[T]$.*