MAT.494UB / SS 2021
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and Scientific Computing

Calculus of Variations

Summary of Course Contents

Lesson 1 (3 March 2021)

Introduction to the course. Definition of minimization problem. Mentioned classes of methods to solve them: Indirect Methods, Direct Methods, Relaxation, Γ-convergence. Given elementary examples to illustrate each class.

Definition of Integral Functional with two examples (not analyzed in detail). Examples of variational problems which will not be treated in this course.

Started Functional Analysis Revision (no proofs): Metric Spaces, Normed Spaces, Banach Spaces. Space of Linear Continuous operators. Dual of a normed spaces. Weak and weak* topologies. Reflexivity. Compactness in infinite dimensions and Banach-Alaoglu Theorem. Weak and weak* convergence of sequences implies boundedness of the sequence. Definition of lower semicontinuity.

Lesson 2 (10 March 2021)

Continued Functional Analysis Revision (no proofs): Definition of Inner Product Space and Hilbert Space. Cauchy-Schwarz inequality. Basis of a Hilbert Space. Every separable Hilbert space admits a basis. Coordinates with respect to a basis. A separable Hilbert space is isometric to the space of square summable real sequences l_2. Riesz's Representation Theorem. Properties of weak convergence in Hilbert spaces.

Started Calculus in Normed Spaces: Fréchet derivative in normed space. C^1 functions. Examples in $\mathbb{R}^d$ and Hilbert. Fréchet differentiable implies continuous (with proof). Chain rule for Fréchet differentiable maps (no proof). Gâteaux derivative. Fréchet differentiable implies Gâteaux differentiable (with proof). Counterexample to show that the converse is not true. Example of Fréchet derivative for integral functional $F\colon C^1([0,1]) \to \mathbb{R}$. Mean Value Theorem (with proof). Gâteaux differentiable implies Fréchet differentiable under suitable assumptions (with proof).

Lesson 3 (17 March 2021)

Continued Calculus in Normed Spaces: Higher Order derivatives. The space of n-linear bounded operators. Generalization of Schwarz Theorem for twice Fréchet differentiable maps (no proof). Taylor's Formula for Fréchet differentiable maps (no proof).

Indirect Method: Definition of First Variation in a general set. Examples for the cases of $X = \mathbb{R}^d$ and X normed space. The first variation vanishes at global minimizers (with proof). Definition of Affine Space. First Variation in Affine Spaces. Abstract Euler-Lagrange Equation. Detailed Example: Minimization of three integral functionals over the space $C^1([a,b])$ having strictly convex, convex, and non-convex Lagrangians.

Lesson 4 (24 March 2021)

Fundamental Lemmas: Support of a functions, Compactly supported functions. Bump functions to construct smooth functions with arbitrary support. Existence of Cut Off function (no proof). Fundamental Lemma of Calculus of Variations (FLCV) for continuous functions with two proofs: one by contradiction, one by density. Generalization of FLCV to sets V such that the closure of their span (in a suitable sense) yields $C(0,1)$ (proof just sketched). Du Bois Reymond Lemma (DBR) for continuous functions with proof (based on FLCV). Alternative proof by density just sketched. Alternative formulation of DBR Lemma (with proof). Generalization of DBR Lemma to smaller classes of test functions (no proof).

Introduction to Boundary Conditions by Examples: considered the functional $F(u) = \int_0^1 u^2 + \dot{u}^2 \, dx$ over different subsets of $X = C^1(0,1)$ and derived boundary conditions of Dirichlet type, Neumann type, mixed Dirichlet and Neumann, Periodic, and finally a last example where the minimizer does not exist due to enforcing too many boundary conditions. The derivation was done by means of the Euler-Lagrange Equation and by the FLCV.

Lesson 5 (14 April 2021)

Euler-Lagrange Equation (ELE): First variation for general integral functionals defined over $C^1([a,b])$ (with proof). Dirichlet boundary conditions for general integral functionals: first integral form of the ELE, second integral form of the ELE, ELE in differential form (all with proof). Neumann boundary conditions for general integral functionals (with proof). ELE in Erdmann form (with proof). ELE for higher order integral functionals (no proof), ELE for integral functionals depending on more unknowns (no proof).

L^p Spaces Revision: Uploaded self-contained notes with essential information needed about measure theory and L^p spaces (this will not be an examination topic)

Lesson 6 (21 April 2021)

Sufficient Conditions for Minimality: Convexity. Definition of convex sets, convex functions and strictly convex functions. Characterization of convexity for C^1 and C^2 functions (no proof). Convex functions have non-empty subdifferential (no proof). Continuously differentiable solutions to the Integral ELE are minimizers if the Lagrangian is C^2 and convex in s, p (with proof). Uniqueness of minimizers for strictly convex Lagrangians (with proof). Example for Lagrangians depending only on p: the straight line always solve ELE.

Sufficient Conditions for Minimality: “Trivial Lemma”. Stated and proved the Trivial Lemma. Application of the Trivial Lemma to the double-well potential in one-dimension: proved that if $|\beta - \alpha| > 1$ the solution is the straight line, while if $|\beta - \alpha| \leq 1$ no minimizer exists. Summary of Indirect Method.

Convolutions in L^p: definition of convolution. Convolution is commutative and associative (no proof). Young's Inequality for convolutions (with proof).

Lesson 7 (28 April 2021)

Convolutions in L^p: Definition of support for arbitrary functions. Comparison with classical definition of support. Support of a convolution (no proof). Definition of locally integrable function. Theorem on smoothing by convolution (no proof). Definition of mollifiers and example of standard mollifiers. Proposition: if $u \in C_c(\mathbb{R})$ and ρ_n are mollifiers then $\rho_n \star u \to u$ uniformly on compact sets (no proof). Theorem: if $u \in L^p(\mathbb{R})$, ρ_n mollifiers, then $\rho_n \star u \to u$ strongly in L^p (with proof). Corollary: $C_c^\infty(I)$ is dense in $L^p(I)$ (no proof).

FLCV and DBR Lemma: FLCV for L^1_{loc} functions (with proof). Du Bois-Reymond Lemma for L^1_{loc} functions (with proof). Alternative formulation of DBR Lemma for L^1_{loc} functions (with proof).

Sobolev Spaces: Motivation of weak derivative for functions in C^1_{pw}. Definition of the Sobolev Space $W^{1,p}$. Proposition: the weak derivative, if it exists, is unique (with proof). Examples: C^1 functions and C^1_{pw} functions belong to $W^{1,p}$; functions with jumps do not belong to $W^{1,p}$. Definition of norms on $W^{1,p}$ and inner product on $H^1 := W^{1,2}$. Proposition: $W^{1,p}$ is Banach for $1 \leq p \leq +\infty$; $W^{1,p}$ is reflexive for $1 < p < +\infty$; $W^{1,p}$ is separable for $1 \leq p < +\infty$; H^1 is Hilbert separable (all with proof). Remark: if $\{u_n\} \subset W^{1,p}$ is such that $u_n \to u$ and $\dot{u}_n \to g$ strongly in L^p then $\dot{u} = g$ weakly and $u_n \to u$ strongly in $W^{1,p}$ (with proof). Remark: if $\{u_n\} \subset H^1$ is such that $u_n \rightharpoonup u$ and $\dot{u}_n \rightharpoonup g$ weakly in L^2 then $\dot{u} = g$ weakly and $u_n \rightharpoonup u$ weakly in H^1 (with proof).

Lesson 8 (5 May 2021)

Sobolev Spaces, regularity results: Theorem: Sobolev functions admit a continuous representative (with proof). Lemma: primitives of L^p functions are continuous and weakly differentiable (with proof). Proposition: Sobolev functions having continuous weak derivative are C^1 (with proof). Theorem: $W^{1,p}$ embeds in $C^{0,1-1/p}$ for $p > 1$ (with proof).

Sobolev Spaces, density results: Lemma: Extension result for Sobolev functions (with proof). Lemma: if $\rho \in L^1(\mathbb{R})$ and $u \in W^{1,p}(\mathbb{R})$ then $\rho \star u \in W^{1,p}(\mathbb{R})$ (with proof).

Lesson 9 (12 May 2021)

Sobolev Spaces, density results: Theorem: if $1 \leq p < +\infty$ and $u \in W^{1,p}(a,b)$, there exists $\{u_n\}$ in $C^\infty_c(\mathbb{R})$ such that $u_n \to u$ strongly in $L^p(a,b)$ (with proof).

Sobolev embedding: Definition of embedding and compact embeddings. Proposition: compact operators transform weakly converging sequences into strong converging sequences (with proof). Theorem: Ascoli-Arzelà in metric space (no proof). Theorem: characterization of Sobolev function by continuity of translations (no proof). Theorem (Sobolev Embedding): $W^{1,p}(I) \hookrightarrow L^\infty(I)$ for all $1 \leq p \leq +\infty$, $I \subset \mathbb{R}$ open; for I bounded, the following are compact embedding: $W^{1,p}(I) \hookrightarrow C(\overline{I})$ for all $1 < p \leq +\infty$, $W^{1,1}(I) \hookrightarrow L^q(I)$ for all $1 \leq q < +\infty$, $W^{1,p}(I) \hookrightarrow L^p(I)$ for all $1 \leq p \leq +\infty$ (all with proof). Remark: considerations about embeddings not covered by the Theorem we saw. Corollary: if $u_n \rightharpoonup u$ in $W^{1,p}(I)$ with I bounded, then $u_n \to u$ in $L^p(I)$ (with proof).

Lesson 10 (19 May 2021)

Further Sobolev Spaces Topics: Definition of higher order Sobolev Spaces $W^{k,p}$ by induction. Remark: $u \in W^{k,p}$ iff u admits weak partial derivatives up to order k (no proof). Remark: $W^{k,p} \subset C^{k-1}$ (no proof).

Definition of $W^{1,p}_0$ by density. Proposition: $u \in W^{1,p}_0(I)$ if and only if $u = 0$ on ∂I (proof of the “only if” implication). Theorem: Poincaré inequality (with two proofs). Theorem: Generalized Poincaré inequality (no proof). Example of spaces in which the generalized Poincaré inequality holds.

Euler Lagrange Equation in Sobolev spaces: Recap on Theorems 4.5 and 5.4 on necessary and sufficient conditions for minimality for integral functionals defined on C^1. Definition: Carathéodory function. Proposition: composition between a Carathéodory function and a measurable function is measurable (no proof). Definition: variational problem for integral functionals on $W^{1,p}$ with Dirichlet boundary conditions. Theorem 8.4: minimizers solve the weak (or weaker) ELE, while they solve differential ELE if the Lagrangian and the minimizer are regular (with proof).

Lesson 11 (26 May 2021)

General boundary conditions: Definition: variational problem for integral functionals on $W^{1,p}$ with general boundary conditions. Theorem 8.5: Generalization of Theorem 8.4 to arbitrary boundary conditions (no proof).

Sufficient conditions for minimality: Theorem: solutions to weak ELE or differential ELE are minimizers if the Lagrangian is convex in (s, ξ) (no proof).

Direct Method: definition of Space with Notion of Convergence (SNC). Definition: compact sets in SNC. Definition: continuity and LSC in SNC. Theorem: Existence of minimizers in SNC under assumption of compactness of X and LSC of f (with proof). Theorem: Existence of minimizers in SNC under assumption of coercivity and LSC of f (with proof). Theorem: Existence of minimizers in SNC under assumption of compactness of sublevels and LSC of f (with proof).

Direct Method, Action Plan: strategy to solve minimization problems in 4 steps, which are Weak Formulation, Compactness, LSC and Regularity.

Lesson 12 (2 June 2021)

Direct Method, Action Plan: Example of minimization of $F(u) := \int_0^1 \dot{u}^2 + \sin(u^5)\, dx$ among functions $u \in C^1([0,1])$ such that $u(0) = 0$, $u(1) = 1$. Proved that a minimum exists for the extended problem in Sobolev (by the Direct Method), and that minimizers are $C^\infty([0,1])$.

Existence Result: General existence result via Direct Method for integral functionals in Sobolev spaces (see Theorem 9.9, with proof).

Lesson 13 (9 June 2021)

Lower Semicontinuous Envelope: Definition of LSC function on metric space. Proposition: Properties of LSC functions (no proof). Proposition: supremum of LSC functions is LSC (with proof). Remark: supremum of continuous functions is continuous in general, but only LSC (with counterexample). Definition: LSC Envelope.

Relaxation: Definition of Relaxation. Lemma 10.6: values of a function can be chosen close to the value of its Relaxation (with proof). Definition: Recovery Sequence for Relaxation. Lemma 10.8: existence of recovery sequence (with proof). Proposition: Relaxation and LSC Envelope are equivalent (with proof).

Properties: Proposition: $\inf_X f = \inf_X \bar{f}$ (with proof). Warning: If $A \subset X$ then in general $\inf_A f > \inf_A \bar{f}$ (given a counterexample). Proposition: if $A \subset X$ is open then $\inf_A f = \inf_A \bar{f}$ (with proof). Proposition: if f is coercive then $\bar{f}$ admits minimum and $\inf_X f = \min_X \bar{f}$ (with proof). Proposition 10.13 and Corollary 10.14 on behavior of infimizing sequences.

Computing the Relaxation: Proposition: Strategy 1 for computing relaxation (with proof). Definition: Energy Dense Subsets. Lemma 10.17: Inequalities on energy dense subsets can be extended to the whole space, under some assumptions (with proof). Proposition: Strategy 2 for computing relaxation (with proof).

Lesson 14 (16 June 2021)

Relaxation of integral functionals: Considered functional $F\colon C^1([a,b]) \to \overline{\mathbb{R}}$, $F(u) := \int_a^b \psi(\dot{u})\, dx$ with $\psi\colon \mathbb{R} \to \mathbb{R}$. Theorem 10.19: Extension of F by relaxation to $L^p(a,b)$ when ψ is convex and satisfies growth from below (with proof). Example of extension with convex Lagrangian. Definition: convex envelope. Proposition: properties of convex envelope (no proof). Theorem 10.22: Extension of F by relaxation to $L^p(a,b)$ when ψ only satisfies growth from below (no proof). Example of extension with non-convex Lagrangian.

Γ-convergence: Definition of Γ-convergence in metric space. Relationship with pointwise convergence: Proposition: if $f_n = f$ for all $n \in \mathbb{N}$ then $\Gamma - \lim f_n = \bar{f}$ (with proof). Proposition: if $f_n \to f$ uniformly on compact sets and f is LSC then $\Gamma - \lim f_n = f$ (with proof). Stability properties of Γ-convergence under sum (no proof). Definition of $\Gamma - \liminf$ and $\Gamma - \limsup$. Proposition: $\Gamma - \liminf f_n \leq \Gamma - \limsup f_n$ and $\Gamma - \liminf f_n = \Gamma - \limsup f_n$ if and only if $\Gamma - \lim f_n = f$ for some $f\colon X \to \overline{\mathbb{R}}$ (with proof).

Lesson 15 (23 June 2021)

Γ-convergence: Lemma: The Γ-limit is always LSC (with proof). Proposition 11.9: On the limit of infimums on open and compact sets (with proof). Definition: Equicoercivity in metric space. Theorem 11.12: Γ-convergence and equicoercivity imply convergence of minimization problems and compactness of almost-minimizers (with proof).

Example: Studied Γ-convergence in $L^2(0,1)$ for the sequence of functionals $F_n\colon C^1([0,1]) \to \mathbb{R}$ defined by $F_n(u) := \int_0^1 n\dot{u}^2 + (u - \arctan x)^2\,dx$.

Application: Homogenization problems. Considered the sequence of functionals $F_n\colon C^1([a,b]) \to \mathbb{R}$ defined by $F_n(u) := \int_a^b A_n \dot{u}^2\,dx$, where $A_n(x) := A(nx)$, $A(x) := \alpha$ in $[0,\lambda)$, $A(x) := \beta$ in $[\lambda, 1)$. Proved that $F_n \to F$ in the sense of Γ-convergence in $L^2(a,b)$, with $F(u) := c\int_a^b \dot{u}^2\,dx$ and $c := \alpha\beta/(\lambda\beta + (1-\lambda)\alpha)$.

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