# Optimal transport regularization of dynamic inverse problems

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ICCOPT 2019 Berlin

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**Proposed model:** optimal transport regularization for dynamic reconstruction

K. Bredies, S. Fanzon - An optimal transport approach for solving dynamic inverse problems in spaces of measures. Preprint 2019

# Optimal Transport - Static Formulation

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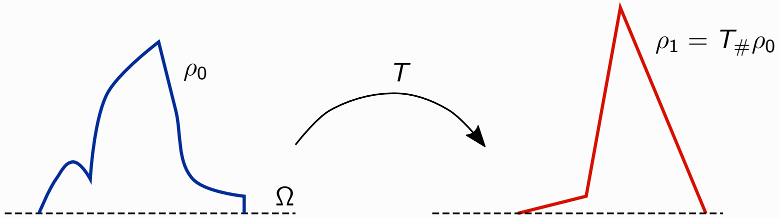

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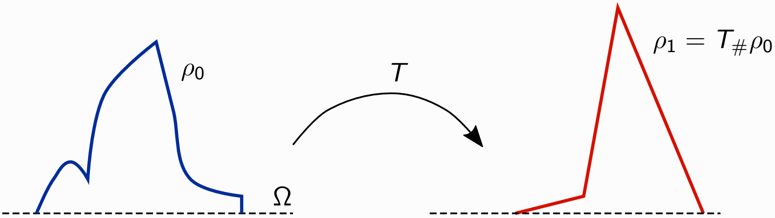


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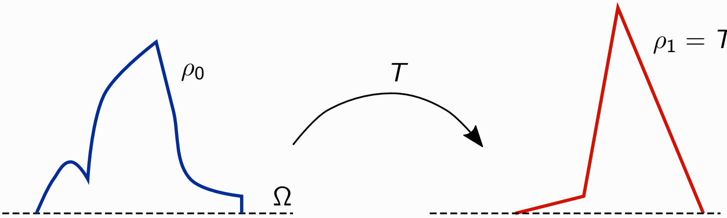


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**Optimal Transport:** a transport plan $T$ solving

$$\min \left\{ \int_\Omega |T(x)-x|^2 \, d\rho_0(x) : \; T\colon \Omega \to \Omega, \; T_{\#}\rho_0 = \rho_1 \right\}$$

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- $(\rho_t, v_t)$ solves the continuity equation with initial conditions

$$\begin{cases} \partial_t \rho_t + \operatorname{div}(\rho_t v_t) = 0 \\ \text{Initial data } \rho_0, \text{ final data } \rho_1 \end{cases} \tag{CE-IC}$$

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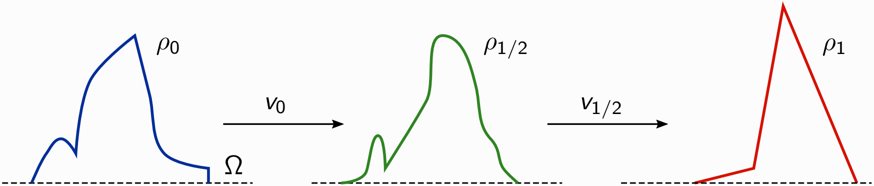

# Connection and Advantages

**Theorem (Benamou-Brenier '00)**

$$\min_{\substack{(\rho_t, v_t) \\ \textit{solving (CE-IC)}}} \int_0^1 \int_\Omega |v_t(x)|^2 \rho_t(x) dx\, dt = \min_{\substack{T : \Omega \to \Omega \\ T_\# \rho_0 = \rho_1}} \int_\Omega |T(x) - x|^2 \rho_0(x)\, dx$$

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**Advantages of Dynamic Formulation:**

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2. we know the full trajectory $\rho_t$ and can recover the velocity field $v_t$ from $m_t$

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**Problem**

*Given some data $\{f_t\}_{t \in [0,1]}$ with $f_t \in H_t$, find a curve of measures*

$$t \mapsto \rho_t \in \mathcal{M}(\overline{\Omega})$$

*such that they solve the dynamic inverse problem*

$$K_t^* \rho_t = f_t \qquad \text{for a.e.} \quad t \in [0,1]\,. \tag{P}$$

# Optimal transport regularization

Consider a triple $(\rho_t, v_t, g_t)$ with

- $v_t \colon (0,1) \times \overline{\Omega} \to \mathbb{R}^d$ velocity field
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- $v_t$ keeps track of motion
- $g_t$ allows the presence of a contrast agent
- continuity equation enforces “regular” motion

# Formal definition of the OT Energy

Set $X := (0,1) \times \overline{\Omega}$ and consider triples $(\rho, m, \mu) \in \mathcal{M}(X)^{d+2}$

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$$B(\rho, m, \mu) := \int_X \Psi \left( \frac{d\rho}{d\lambda}, \frac{dm}{d\lambda}, \frac{d\mu}{d\lambda} \right) \, d\lambda$$

where $\lambda \in \mathcal{M}^+(X)$ is such that $\rho, m, \mu \ll \lambda$ and

$$\Psi(t, x, y) := \frac{x^2 + |y|^2}{2t} \quad \text{if} \quad t > 0, \;\; \Psi = +\infty \text{ else}$$

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## Proposition (Fanzon, Bredies '19)

*$B$ is weak* lower-semicontinuous. If $B(\rho, m, \mu) < +\infty$ and $\partial_t \rho + \operatorname{div} m = \mu$ then*

- $\rho = dt \otimes \rho_t$ *for a weak*-continuous curve* $t \mapsto \rho_t \in \mathcal{M}^+(\overline{\Omega})$
- $m = \rho v_t$ *for some velocity field* $v_t \colon (0,1) \times \overline{\Omega} \to \mathbb{R}^d$
- $\mu = \rho g_t$ *for some growth rate* $g_t \colon (0,1) \times \overline{\Omega} \to \mathbb{R}$

$$B(\rho, m, \mu) = \int_0^1 \int_{\overline{\Omega}} |v_t(x)|^2 + |g_t(x)|^2 \, d\rho_t(x) \, dt$$

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$f \colon [0,1] \to \cup_t H_t$ with $f_t \in H_t$ is strongly measurable if $\exists\ \varphi^n \colon [0,1] \to D$ step functions s.t.

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We then define the Hilbert space

$$L^2([0,1]; H) := \left\{ f \colon [0,1] \to \cup_t H_t : \ f \text{ strongly meas }, \ \int_0^1 \|f_t\|_{H_t}^2 \, dt < \infty \right\}$$

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## Definition (Regularization)

Let $f \in L^2([0,1]; H)$. For $(\rho, m, \mu) \in \mathcal{M}(X)^{d+2}$ set

$$J_{\alpha,\beta}(\rho, m, \mu) := \frac{1}{2} \int_0^1 \|K_t^* \rho_t - f_t\|_{H_t}^2 \, dt + \alpha \, B(\rho, m, \mu) + \beta \, \|\rho\|_{\mathcal{M}(X)}$$

if $\partial_t \rho + \operatorname{div} m = \mu$, and $J_{\alpha,\beta}(\rho, m, \mu) = +\infty$ else.

# Existence & Regularity

**Theorem (Fanzon, Bredies '19)**

*Assume (H)-(K) and let* $f \in L^2([0,1]; H)$*. Then*

$$\min_{(\rho,m,\mu)\in\mathcal{M}} J_{\alpha,\beta}(\rho, m, \mu) \tag{MIN}$$

*admits a solution. If* $K_t^*$ *is injective for a.e.* $t$*, then the solution is unique.*

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## Theorem (Fanzon, Bredies '19)

*Assume (H)-(K). Let $f^\dagger$ be exact data and $f^n$ be noisy data, with $f^n \to f^\dagger$ in $L^2$. Let $(\rho^n, m^n, \mu^n)$ be a minimizer of (MIN) with par. $\alpha_n, \beta_n \to 0$ and data $f^n$. Then*

$$(\rho^n, m^n, \mu^n) \overset{*}{\rightharpoonup} (\rho^\dagger, m^\dagger, \mu^\dagger)$$

$$K_t^* \rho_t^\dagger = f_t^\dagger \quad \textit{for all} \quad t \in [0,1]$$

$$(\rho^\dagger, m^\dagger, \mu^\dagger) \in \arg\min \alpha^* B(\rho,m,\mu) + \beta^* \|\rho\|_{\mathcal{M}(X)}$$

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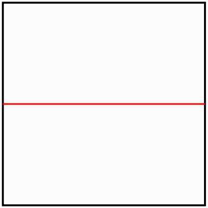

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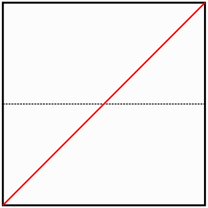

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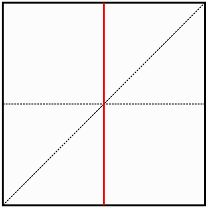

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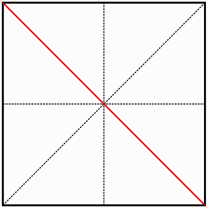

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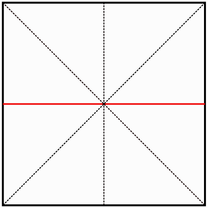

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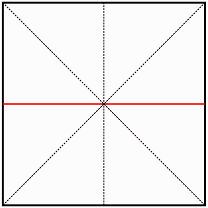

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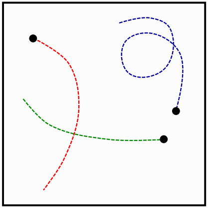

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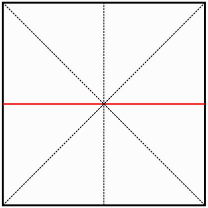

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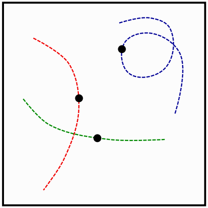

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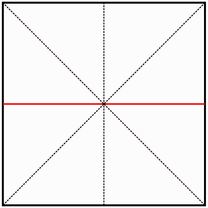

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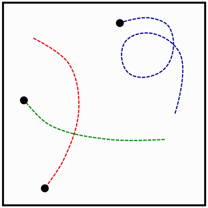

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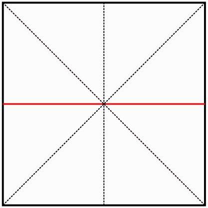

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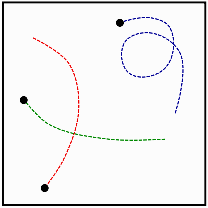

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- $K_t^* : \mathcal{M}(\overline{\Omega}) \to H_t$ masked Fourier transform

$$K_t^* \rho := (\mathfrak{F}(c_1 \rho), \ldots, \mathfrak{F}(c_N \rho))$$

with $c_j \in C_0(\mathbb{R}^2; \mathbb{C})$ coil sensitivities (accounting for phase inhomogeneities)

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## Theorem (Fanzon, Bredies '19)

*Assume (M1)-(M2). Let $\alpha, \beta, \delta > 0$, $f \in L^2([0,1]; H)$ and $c \in C_0(\mathbb{R}^2;\mathbb{C}^N)$. Then*

$$\min_{\substack{(\rho,m,\mu)\\ \partial_t \rho + \operatorname{div} m = \mu}} \frac{1}{2} \sum_{j=1}^{N} \int_0^1 \|\mathfrak{F}(c_j \rho_t) - f_t\|^2_{L^2_{\sigma_t}}\, dt + \alpha B_\delta(\rho, m, \mu) + \beta \|\rho\|$$

*admits a solution $(\rho, m, \mu)$ with*

- $\rho = dt \otimes \rho_t$ *with* $t \mapsto \rho_t$ *weak* continuous*
- $m = \rho v$ *for some velocity* $v\colon (0,1) \times \overline{\Omega} \to \mathbb{R}^2$
- $\mu = \rho g$ *for some growth rate* $g\colon (0,1) \times \overline{\Omega} \to \mathbb{R}^2$

# Extremal Points

Consider the regularizer for the homogenous case (no source): $(\rho, m) \in \mathcal{M}(X)^{d+1}$

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**Theorem (Fanzon, Bredies, Carioni, Romero '19)**

*Let* $C := \{(\rho, m) : \ R_{\alpha,\beta}(\rho, m) \leq 1\}$. *Then*

$$\mathrm{Ext}(C) = \{(0,0)\} \cup \mathcal{C}$$

*where*

$$\mathcal{C} := \left\{ (\rho_\gamma, m_\gamma) : \ \gamma \in \mathrm{AC}^2([0,1]; \overline{\Omega}) \right\}$$

$$\rho_\gamma := a_\gamma \, dt \otimes \delta_{\gamma(t)} \, , \ \ m_\gamma := \dot{\gamma} \, \rho_\gamma \, , \ \ a_\gamma^{-1} := \frac{\alpha}{2} \int_0^1 |\dot{\gamma}(t)|^2 \, dt + \beta$$

L. Ambrosio. Inventiones mathematicae, 158(2) '04

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**Regularization:** we regularize with

$$J_{\alpha,\beta}(\rho, m) := \frac{1}{2} \sum_{i=1}^{N} \|K_i \rho_{t_i} - f_i\|_{H_i}^2 + \alpha B(\rho, m) + \beta \|\rho\|_{\mathcal{M}(X)}$$

# Sparse minimizers

**Theorem (Fanzon, Bredies, Carioni, Romero '19)**

*The minimization problem*

$$\min_{(\rho,m)\in\mathcal{M}} \frac{1}{2}\sum_{i=1}^{N} \|K_i\rho_{t_i} - f_i\|_{H_i}^2 + \alpha B(\rho,m) + \beta\,\|\rho\|_{\mathcal{M}(X)}$$

*admits a sparse minimizer of the form*

$$(\rho^*, m^*) = \sum_{i=1}^{p} c_i\,(\rho_{\gamma_i}, m_\gamma)$$

*where* $c_i > 0$, $\gamma_i \in \mathrm{AC}^2([0,1];\overline{\Omega})$ *and* $p \leq \dim\mathcal{H}$.

K. Bredies, M. Carioni '18

C. Boyer, A. Chambolle, Y. De Castro, V. Duval, F. De Gournay, P. Weiss '18

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Thank You!