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Analysis 3 - Exercise Sheet 1

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We start with some revision exercises on Analysis 2 topics

Exercise 1.1 (20 pts) Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := f(x+2y) + f(7y-3x)\,,$$

for all $x, y \in \mathbb{R}$. Is F differentiable? In that case, compute ∇F.

Exercise 1.2 (20 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting $F(x,y) := \sqrt{|xy|}$. Is F differentiable in $(0,0)$? Justify your answer.

Recall: Let (X,d) be a non-empty complete metric space and $F\colon X \to X$. We say that F is a *contraction* if there exists a constant $C \in [0,1)$ such that

$$d(F(z_1), F(z_2)) \le C d(z_1, z_2)$$

for all $z_1, z_2 \in X$. We say that z^* is a *fixed point* for F if $F(z^*) = z^*$. The Banach fixed point theorem states that if F is a contraction, then F admits a unique fixed point. Recall that $\mathbb{R}^n$ is a complete metric space with the Euclidean distance.

Exercise 1.3 (20 pts) Let (X,d) be a non-empty complete metric space. Prove the Banach fixed point theorem stated above.

Exercise 1.4 (20 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$F(x,y) := (x + y/2, x/2 + y + 1)\,.$$

Define the map $G(x,y) = (x,y) - F(x,y)$. Using the Banach fixed point theorem on G, prove that F admits a unique zero, i.e., there exists a unique $(x^*, y^*) \in \mathbb{R}^2$ such that $F(x^*, y^*) = (0,0)$.

Exercise 1.5 (20 pts) Define $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x,y,z) := (x+y+z, xy+yz+zx, xyz)\,.$$

Determine all the points in $\mathbb{R}^3$ in which F is locally invertible.

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Analysis 3 - Exercise Sheet 2

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Recall: Let $A \subset \mathbb{R}^n$ be an open set. Suppose $f : A \to \mathbb{R}$ is differentiable at $z^* \in \mathbb{R}^n$. We say that z^* is a critical point of f if $\nabla f(z^*) = 0$. Recall that a local minimizer or maximinizer of f is always a critical point. Suppose now $n = 2$ and that f is C^2. The Hessian of f is defined by

$$Hf(x, y) := \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx} f_{yy} - (f_{xy})^2 .$$

Suppose (x^*, y^*) is a critical point of f. If

$$Hf(x^*, y^*) > 0 , \quad f_{xx}(x^*, y^*) > 0 ,$$

then (x^*, y^*) is a local mimizer for f. If

$$Hf(x^*, y^*) > 0 , \quad f_{xx}(x^*, y^*) < 0 ,$$

then (x^*, y^*) is a local maximizer for f. If

$$Hf(x^*, y^*) < 0 ,$$

then (x^*, y^*) is a saddle point for f.

Exercise 2.1 (25 pts) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$f(x, y) := \frac{xy}{1 + x^2 + y^2} .$$

Find all the critical points of f and classify them into local maximizers, local minimizers and saddle points.

Exercise 2.2 (25 pts) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$f(x, y) := 2(x^4 + y^4 + 1) - (x + y)^2 .$$

Find all the critical points of f and classify them into local maximizers, local minimizers and saddle points.

Hint: if at some critical point (x^*, y^*) one has $Hf(x^*, y^*) = 0$, then nothing can be concluded about the nature of (x^*, y^*). In such case, one has to proceed manually, for example by considering the restriction of f to the line through the origin $\{y = mx\}$ for $m \in \mathbb{R}$, and analyze the resulting function of one variable.

Exercise 2.3 (25 pts) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = xy^2$ and consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : \; y \geq 0, \, y \leq 1 + x, \, y \leq 1 - x\} .$$

Find the global maximizers and minimizers of f restricted to the set A.

Hint: Draw A and consider the separate cases of internal points and boundary points.

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Exercise 2.4 (25 pts)

(a) Suppose $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x^* \in \mathring{A}$, with $\mathring{A}$ denoting the interior of A. Show that if x^* is a local minimizer or maximizer for f, then $\nabla f(x^*) = 0$.

(b) (Rolle's Theorem in $\mathbb{R}^n$) Suppose that $A \subset \mathbb{R}^n$ is compact with $\mathring{A} \neq \emptyset$. Let $f\colon A \to \mathbb{R}$ be continuous in A, differentiable in $\mathring{A}$, and constant on ∂A. Prove that there exists $x^* \in \mathring{A}$ such that $\nabla f(x^*) = 0$.

Hint: Use Weierstrass' Theorem.

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Analysis 3 - Exercise Sheet 3

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Remark: The following exercise addresses a question that a few of you asked in the previous class. Enjoy!

Exercise 3.1 (25 pts)

(a) Suppose that $z = 0$ is a local minimizer for a given function $F : \mathbb{R}^n \to \mathbb{R}$. Let $v \in \mathbb{R}^n \smallsetminus \{0\}$ and consider the restriction of F along the line of direction v, that is, the function $g_v(t) := F(tv)$ for $t \in \mathbb{R}$. Prove that $t = 0$ is a local minimizer for g_v.

(b) We now show that the converse of point (a) does not hold, even if F is smooth. To this end, consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, y) := (y - x^2)(y - 2x^2) \,.$$

Prove the following statements:

(i) $(0, 0)$ is the only critical point of F.

(ii) The Hessian of F vanishes in $(0, 0)$.

(iii) Consider the restriction of F along the lines through the origin:

$$g_m(x) := \begin{cases} F(x, mx) & \text{if } \ m \in \mathbb{R} \,, \\ F(0, x) & \text{if } \ m = \infty \,. \end{cases}$$

Show that for all $m \in \mathbb{R} \cup \{\infty\}$ the point $x = 0$ is a local minimizer for g_m.

(iv) Show that $(0, 0)$ is a saddle point for F.

Hint: To understand what is happening, it might be helpful to draw the sets in $\mathbb{R}^2$ where F is positive, negative and zero.

Implicit Function: Let $A \subset \mathbb{R}^2$ be open, and $F : A \to \mathbb{R}$. Define the set

$$Z := \{(x, y) \in A : \ F(x, y) = 0\} \,.$$

Assume $(x_0, y_0) \in Z$, i.e., $F(x_0, y_0) = 0$. We say that the equation $F = 0$ defines an implicit function $y = f(x)$ at the point (x_0, y_0) if the set Z coincides with the graph of f around (x_0, y_0), that is, if there exist $\varepsilon, \delta > 0$ and $f : I_\varepsilon(x_0) \to I_\delta(y_0)$, with $I_\varepsilon(x_0) := [x_0 - \varepsilon, x_0 + \varepsilon], I_\delta(y_0) := [y_0 - \delta, y_0 + \delta]$ such that

$$\{(x, y) \in I_\varepsilon(x_0) \times I_\delta(y_0) : \ F(x, y) = 0\} = \{(x, f(x)) : \ x \in I_\varepsilon(x_0)\} \,.$$

Clearly, we say that $F = 0$ defines an implicit function $x = f(y)$ at the point (x_0, y_0) if the set Z coincides with the graph of f around (x_0, y_0)

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Implicit Function Theorem: Let $A \subset \mathbb{R}^2$ be open, and $F\colon A \to \mathbb{R}$ with $F \in C^1(A)$. Assume there exists a point $(x_0, y_0) \in A$ such that

$$F(x_0, y_0) = 0\,, \quad F_y(x_0, y_0) \neq 0\,.$$

Then there exist $\varepsilon, \delta > 0$ and a unique implicit function $f\colon I_\varepsilon(x_0) \to I_\delta(y_0)$, i.e., f satisfies $f(x_0) = y_0$ and

$$F(x, f(x)) = 0\,, \quad \text{for all } x \in I_\varepsilon(x_0)\,.$$

Moreover $f \in C^1(I_\varepsilon(x_0), I_\delta(y_0))$ and

$$f'(x) = -\frac{F_x(x, f(x))}{F_y(x, f(x))}\,.$$

Clearly, the analogous statement holds if

$$F(x_0, y_0) = 0\,, \quad F_x(x_0, y_0) \neq 0\,,$$

and in this case $F = 0$ defines an implicit function $x = f(y)$.

Tangent line to a set: Let $A \subset \mathbb{R}^2$ be open, and $F\colon A \to \mathbb{R}$ with $F \in C^1(A)$. Define the set

$$Z := \{(x, y) \in A : \ F(x, y) = 0\}\,.$$

Suppose that the point $(x_0, y_0) \in Z$ is such that either $F_x(x_0, y_0) \neq 0$ or $F_y(x_0, y_0) \neq 0$. Then the equation of the tangent line to Z at (x_0, y_0) is given by

$$F_x(x_0, y_0)\,(x - x_0) + F_y(x_0, y_0)\,(y - y_0) = 0\,.$$

Exercise 3.2 (25 pts) Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + y^3 - 3xy\,.$$

Find the points $(x_0, y_0) \in \mathbb{R}^2$ such that $F = 0$ defines implicitly a map $y = f(x)$.

Exercise 3.3 (25 pts) Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = 2y^3 + 4x^2y - 3x^4 + x + 6y\,.$$

Prove that the equation $F = 0$ defines an implicit function $y = f(x)$ for all $(x_0, y_0) \in \mathbb{R}^2$.

Exercise 3.4 (25 pts) Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + y^3 - 4x^2y + 2\,.$$

(a) Show that the equation $F = 0$ defines an implicit function $y = f(x)$ around the point $(1, 1)$.

(b) Compute $f'(1)$.

(c) Compute the equation of the line tangent to the set

$$Z = \{(x, y) \in \mathbb{R}^2 : \ F(x, y) = 0\}$$

at the point $(1, 1)$.

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Analysis 3 - Exercise Sheet 4

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Consider the following statement you saw in class.

Theorem 1. Let $A \subset \mathbb{R}^n$ be open and $F: A \to \mathbb{R}$. Suppose that there exist $z_0 \in A$ and a neighbourhood $U \subset A$ of z_0 such that ∇F exists and is continuous in U. Then F is differentiable in z_0.

The converse of Theorem 1 does not hold, as shown in the next exercise.

Exercise 4.1 (25 pts) Define $F: \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} 0 & \text{if } y = 0\,, \\ y^2 \cos\left(\dfrac{1}{y}\right) & \text{if } y \neq 0\,. \end{cases}$$

(a) Compute F_x and F_y. Prove that F_y is not continuous in $(x,0)$ for all $x \in \mathbb{R}$.

(b) Prove that F is differentiable in $(x,0)$ for all $x \in \mathbb{R}$.

In class you saw the following theorem.

Theorem 2. Let $A \subset \mathbb{R}^n$ be open and $F: A \to \mathbb{R}$. Suppose that there exist $z_0 \in A$ and a neighbourhood $U \subset A$ of z_0 such that $\nabla^2 F$ exists and is continuous in U. Then $F_{x_i x_j}(z_0) = F_{x_j x_i}(z_0)$ for all i,j in $\{1,\dots,n\}$.

The aim of the next exercise is to prove that the assumption of $\nabla^2 F$ being continuous cannot be removed.

Exercise 4.2 (25 pts) Define $F: \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} 0 & \text{if } (x,y) = (0,0)\,, \\ \dfrac{x^3 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,. \end{cases}$$

(a) Prove that F is continuous in $\mathbb{R}^2$.

(b) Compute $\nabla F = (F_x, F_y)$ and prove that F is differentiable in $\mathbb{R}^2$.

(c) Prove that F_{xy} and F_{yx} exist in $\mathbb{R}^2$ and that

$$F_{xy}(0,0) \neq F_{yx}(0,0)\,.$$

(d) Check that F_{xy} and F_{yx} are not continuous in $(0,0)$.

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For the next exercise it will be useful to recall the following Taylor formula in one dimension.

Theorem 3. Let $a, b \in \mathbb{R}$ and $g \in C^2(I)$ with $I = [a, b] \subset \mathbb{R}$. Let $t \in I$ and $s > 0$ be such that $t + s \in I$. Then there exists $\xi \in (0, 1)$ such that

$$g(t+s) = g(t) + g'(t)s + \frac{1}{2} g''(t + \xi s)s^2 .$$

Exercise 4.3 (25 pts) Let $A \subset \mathbb{R}^2$ be open and $F \in C^2(A)$.

(a) Let $(x, y), (h, k) \in \mathbb{R}^2$ be such that $P_t := (x + th, y + tk) \in A$ for all $t \in [0, 1]$. Using Theorem 3, show that there exists $\xi \in (0, 1)$ such that

$$F(x+h, y+k) = F(x, y) + F_x(x, y)h + F_y(x, y)k + \frac{1}{2} \left\{ F_{xx}(P_\xi)h^2 + 2F_{xy}(P_\xi)hk + F_{yy}(P_\xi)k^2 \right\} .$$

(b) The second order Taylor polynomial of F in $(0, 0)$ is defined by

$$P_2(x, y) := F(0, 0) + F_x(0, 0)x + F_y(0, 0)y + \frac{1}{2} \left\{ F_{xx}(0, 0)x^2 + 2F_{xy}(0, 0)xy + F_{yy}(0, 0)y^2 \right\} .$$

Compute P_2 for $F \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) := (2x + y)e^{x^2 - y^2}$.

Defintion. Define $\mathbb{S}^n := \{v \in \mathbb{R}^{n+1} : \ \|v\| = 1\}$. Let $A \subset \mathbb{R}^{n+1}$ be open, $F \colon A \to \mathbb{R}$, and $v \in \mathbb{S}^n$. The directional derivative of F at $z_0 \in A$ in direction v is defined by

$$F_v(z_0) := \lim_{t \to 0} \frac{F(z_0 + tv) - F(z_0)}{t} ,$$

whenever the limit exists.

Theorem 4. Let $A \subset \mathbb{R}^{n+1}$ be open, $F \colon A \to \mathbb{R}$. If F is differentiable in $z_0 \in A$, then F admits all the directional derivatives in z_0 and

$$F_v(z_0) = \nabla F(z_0) \cdot v = \sum_{i=1}^{n+1} F_{x_i}(z_0) v_i , \qquad \text{for all } \ v \in \mathbb{S}^n .$$

The next exercise shows that the converse of Theorem 4 does not hold, i.e., there exists F which admits all the directional derivatives at some point z_0, but is not differentiable at z_0.

Exercise 4.4 (25 pts) Define $F \colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x, y) := \begin{cases} 0 & \text{if } \ (x, y) = (0, 0) , \\ \dfrac{x^3 y}{x^4 + y^2} + y & \text{if } \ (x, y) \neq (0, 0) . \end{cases}$$

(a) Prove that $F_v(0, 0)$ exists for all $v \in \mathbb{S}^1$ and compute it.

(b) Prove that F is not differentiable in $(0, 0)$.

(c) Prove that for all $v \in \mathbb{S}^1$ it holds

$$F_v(0, 0) = \nabla F(0, 0) \cdot v .$$

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Analysis 3 - Exercise Sheet 5

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Exercise 5.1 (25 pts) Let $g\colon \mathbb{R} \to \mathbb{R}$ be differentiable in $t = 0$. Moreover suppose g is bounded, that is, there exists $M \geq 0$ such that $|g(t)| \leq M$ for all $t \in \mathbb{R}$. Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} x^2\, g\left(\dfrac{y}{x}\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Prove that $F_{xy}(0,0) = F_{yx}(0,0)$ if and only if $g'(0) = 0$.

Define $\mathbb{S}^{n-1} := \{v \in \mathbb{R}^n : \ \|v\| = 1\}$.

Theorem 1. Let $A \subset \mathbb{R}^n$ be open, $F\colon A \to \mathbb{R}$. If F is differentiable in $z_0 \in A$, then F admits all the directional derivatives in z_0 and

$$F_v(z_0) = \nabla F(z_0) \cdot v = \sum_{i=1}^{n} F_{x_i}(z_0) v_i\,, \qquad \text{for all } \ v \in \mathbb{S}^{n-1}\,. \tag{1}$$

The next exercise shows that, in general, formula (1) does not hold if F is not differentiable.

Exercise 5.2 (25 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

(a) Prove that $F_v(0,0)$ exists for all $v \in \mathbb{S}^1$ and compute it.

(b) Prove that (1) does not hold, i.e., that there exists some $v \in \mathbb{S}^1$ such that

$$F_v(0,0) \neq \nabla F(0,0) \cdot v\,.$$

(c) Can F be differentiable in $(0,0)$?

Definition. Consider a vector valued function $F = (F^1, \dots, F^n)\colon A \subset \mathbb{R}^n \to \mathbb{R}^n$. The Jacobian of F at $z \in A$ is defined as the $n \times n$ matrix of partial derivatives

$$J_F(z) := \left(F^i_{x_j}(z)\right)_{ij}\,.$$

Inverse Function Theorem. Let $A \subset \mathbb{R}^n$ be open. Let $F\colon A \to \mathbb{R}^n$ be a C^1 function and suppose that

$$\det J_F(z_0) \neq 0$$

for some $z_0 \in A$. Then F is *locally invertible* around z_0, that is, there exist $U \subset A$ neighbourhood of z_0, V neighbourhood of $F(z_0)$ and a C^1 function $G\colon V \to U$ such that $(F \circ G)(w) = w$ for all $w \in V$ and $(G \circ F)(z) = z$ for all $z \in U$. We denote $F^{-1} := G$. In particular for all $w \in V$ it holds

$$J_{F^{-1}}(w) = \left[J_F(F^{-1}(w))\right]^{-1}\,.$$

Exercise 5.3 (25 pts)

(a) Consider the map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x,y,z) = (xz, 2xy, 3yz)\,.$$

For which points of $\mathbb{R}^3$ is the map F locally invertible?

(b) Consider the map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = (e^x \cos y, e^x \sin y)\,.$$

Show that F is locally invertible for every point in $\mathbb{R}^2$. Is F globally invertible?

Exercise 5.4 (25 pts) Suppose $F \in C^2(\mathbb{R}^2)$ and that there exists $(x_0,y_0) \in \mathbb{R}^2$ such that

$$F(x_0,y_0) = F_x(x_0,y_0) = F_y(x_0,y_0) = 0\,.$$

Moreover assume that

$$F_{xx}(x_0,y_0)F_{yy}(x_0,y_0) > F_{xy}^2(x_0,y_0)\,.$$

Use the Inverse Function Theorem and the Minimality/Maximality Criterion from Exercise Sheet 2 to prove the existence of a neighbourhood U of (x_0,y_0) such that

$$F(x,y) \neq 0 \quad \text{for all} \quad (x,y) \in U \setminus \{(x_0,y_0)\}\,.$$

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Analysis 3 - Exercise Sheet 6

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Exercise 6.1 (20 pts) Define $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) := \left(\sin(xy) + x\cos y, e^{x+y} - \frac{1}{1+x^2+y^2}\, . \right)$$

(a) Show that F is locally invertible in $(0,0)$.

(b) Denote by $G = (G^1, G^2)$ the local inverse of F around $(0,0)$. Compute the first order Taylor approximation of G^i around $(0,0)$, that is,

$$G^i(x,y) = G^i(0,0) + G^i_x(0,0)x + G^i_y(0,0)y + o(\sqrt{x^2+y^2})\,,$$

for $i = 1,2$.

In the following we assume familiarity with the concepts of *topology*, *induced topology*, *continuity* for maps between topological spaces, and *homeomorphism* of topological spaces.

Definition. A topological space X is called *connected* if the only subsets that are both open and closed are $\emptyset$ and X. A non-connected topological space is called *disconnected*.

Exercise 6.2 (20 pts) Let X be a topological space. Prove that the following statements are equivalent:

(a) X is disconnected.

(b) $X = A_1 \cup A_2$ with A_1, A_2 open, disjoint and proper, i.e., $A_i \neq \emptyset$, $A_i \neq X$.

(c) $X = C_1 \cup C_2$ with C_1, C_2 closed, disjoint and proper, i.e., $C_i \neq \emptyset$, $C_i \neq X$.

Definition. The euclidean topology on $\mathbb{R}^n$ is defined as the collection of sets

$$\tau := \{A \subset \mathbb{R}^n : \forall\, x \in A, \exists\, \varepsilon > 0 \text{ such that } B_\varepsilon(x) \subset A\}\,,$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^n : \ \|y - x\| < \varepsilon\}$ and $\|\cdot\|$ is the euclidean norm.

Note. In the following $\mathbb{R}^n$ is always equipped with the euclidean topology. Any subset $A \subset \mathbb{R}^n$ is itself a topological space with the topology induced by the euclidean one.

Definition. A topological space X is called *path-connected* if for every pair $x, y \in X$ there exists a continuous map $\alpha \colon [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. The map α is a *path* between x and y.

Note. In the following you can assume this fact: $I \subset \mathbb{R}$ is connected if and only if it is an interval.

Exercise 6.3 (20 pts) Let X and Y be topological spaces.

(a) Assume that $f \colon X \to Y$ is continuous. Show that if X is connected then $f(X)$ is connected.

(b) Show that if X is path-connected then it is connected.

(c) Suppose that $A, B \subset X$ are path-connected and such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is path-connected.

(d) Assume that $f \colon X \to Y$ is continuous. Show that if X is path-connected then $f(X)$ is path-connected.

Exercise 6.4 (20 pts)

(a) Define $X := \{x \in \mathbb{R} : \ x \neq 0\}$. Prove that X is disconnected.

(b) Let $a, b \in \mathbb{R}^n$ and define the line segment

$$[a, b] := \{ta + (1 - t)b\,,\, t \in [0, 1]\}\,.$$

Prove that $[a, b]$ is path-connected.

(c) Let $C \subset \mathbb{R}^n$ be convex. Show that C is path-connected.

Exercise 6.5 (20 pts)

(a) Let $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \ \|x\| = 1\}$. Show that $\mathbb{S}^n$ is path-connected.

(b) Show that $\mathbb{S}^1$ is not homeomorphic to $(0, 1)$.

(c) Prove that the intervals $(0, 1)$ and $(0, 1]$ are not homeomorphic.

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Analysis 3 - Exercise Sheet 7

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Note. We assume the definitions given in Exercise Sheet 6. Also, all the statements in Exercise Sheet 6 can be used to solve the problems below.

Exercise 7.1 (20 pts)

(a) Suppose that $f\colon [0,1] \to [0,\infty)$ is continuous and such that $f(1) = 0$. Show that there exists $\hat{x} \in [0,1]$ such that $f(\hat{x}) = \hat{x}$.

Hint: $A \subset \mathbb{R}$ is connected if and only if A is an interval.

(b) Let $n \geq 1$. Suppose that $f\colon \mathbb{S}^n \to \mathbb{R}$ is continuous. Show that there exists $\hat{x} \in \mathbb{S}^n$ such that $f(\hat{x}) = f(-\hat{x})$.

Hint: Recall that $\mathbb{S}^n$ is connected for all $n \geq 1$. Also it might be useful to consider $g(x) := f(x) - f(-x)$.

Definition. Consider the points $(z_1, \ldots, z_m)$ with $z_i \in \mathbb{R}^n$. Denote by S_k the line segment connecting z_k to z_{k+1}, that is, $S_k := [z_k, z_{k+1}]$. The set $P = \cup_{i=1}^{m-1} S_i$ is called *polygonal path* through $(z_1, \ldots, z_m)$. We also say that P connects z_1 to z_m.

Definition. A subset $A \subset \mathbb{R}^n$ is called *polygonally path-connected* if for every $x, y \in A$ there exists a polygonal path $P \subset A$ connecting x to y.

Exercise 7.2 (20 pts) Fix some integer $n \geq 2$ and let $A \subset \mathbb{R}^n$ be countable. Prove that $\mathbb{R}^n \setminus A$ is polygonally path-connected.

Exercise 7.3 (20 pts) Fix some integer $n \geq 2$ and Let $A \subset \mathbb{R}^n$ be convex and bounded. Prove that $\mathbb{R}^n \setminus A$ is path-connected.

Exercise 7.4 (20 pts) Let $A \subset \mathbb{R}^n$ be open. Prove that A is connected if and only if it is polygonally path-connected.

Let $\gamma \colon [a,b] \subset \mathbb{R} \to \mathbb{R}^n$ with $\gamma = (\gamma_1, \dots, \gamma_n)$ be a curve.

Definition. We say that γ is *regular* if $\gamma_i \in C^1([a,b])$ for all $i = 1, \dots, n$ and

$$\|\gamma'(t)\| = \sqrt{(\gamma_1'(t))^2 + \dots + (\gamma_n'(t))^2} > 0$$

for all $t \in (a,b)$. We say that γ is *piecewise regular* if there exist $a_1 = a < a_2 < \ldots < a_k = b$ such that γ is regular in each $[a_i, a_{i+1}]$.

Theorem. Let $\gamma \colon [a,b] \subset \mathbb{R} \to \mathbb{R}^n$ be piecewise regular. Then the length of γ is given by

$$\ell(\gamma) = \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} \sqrt{(\gamma_1')^2 + \dots + (\gamma_n')^2} \, dt \,.$$

Definition. Let $\gamma \colon [a,b] \subset \mathbb{R} \to \mathbb{R}^n$ be piecewise regular and $F \colon \gamma([a,b]) \subset \mathbb{R}^n \to \mathbb{R}$ continuous. The integral of F along γ is defined by

$$\int_\gamma F \, ds := \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} F(\gamma(t)) \sqrt{(\gamma_1')^2 + \dots + (\gamma_n')^2} \, dt \,.$$

Exercise 7.5 (20 pts) Let $\gamma \colon [0, \pi] \to \mathbb{R}^2$ defined by

$$\gamma(t) := (\cos^3 t, \sin^3 t) \,.$$

(a) Prove that γ is piecewise regular.

(b) Compute $\ell(\gamma)$.

(c) Compute $\int_\gamma F \, ds$ where $F \colon \mathbb{R}^2 \to \mathbb{R}$ is defined by $F(x,y) := \sqrt[3]{|xy|}$.

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Analysis 3 - Exercise Sheet 8

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Exercise 8.1 (20 pts) Consider the set of unitary matrices $O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^{-1} = A^T\} \subset \mathbb{R}n \times n$.

- Show that $O(n)$ is connected.
- Show that the tangent space of $O(n)$ at the identity matrix is the set of skew-symmetric matrices.

Exercise 8.2 (20 pts) Show that a topological space with a connected, dense subset is connected.

Exercise 8.3 (20 pts) Let A, B be subsapces of a topological space such that $A \cup B$ and $A \cap B$ are connected. Prove that if A, B are both closed or both open, A and B are connected.

Exercise 8.4 (20 pts) Determine the tangent space of the surface defined by $x^2 + y^2 - z^2 = 25$ in all points where $z = 0$.

Exercise 8.5 (20 pts) Where does the mapping $\mathbb{R}^2 \to \mathbb{R}^3$, $(u, v) \mapsto (x, y, z)$ via

$$x = u\ \cos(v), \quad y = u\ \sin(v), \quad z = u^2 + v^2$$

locally define a regular 2 dimensional hyperplane? Compute the tangent space in $(1, 0, 1) \in \mathbb{R}^3$.

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Analysis 3 - Exercise Sheet 9

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Exercise 9.1 (20 pts) Sei $f : U \to V$ stetig differenzierbar mit $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ beide offen. Sei $x_0 \in U$ und $Df(x_0)$ habe vollen Rang.

a) Sei $n < m$. Zeige, dass f lokal injektiv ist, d.h., es gibt U' offen in $\mathbb{R}^n$ mit $x_0 \in U'$, sodass $f : U' \to V$ injektiv ist.

b) Sei $n > m$. Zeige, dass f lokal surjektiv ist, d.h., es gibt V' offen in $\mathbb{R}^m$ mit $f(x_0) \in V'$, sodass $V' \subset f(U)$.

Exercise 9.2 (20 pts) Sei X ein endlich-dimensionaler Vektorraum und $f : X \to X$ ein Diffeomorphismus. Weiters sei $g : X \to X$ eine C^1 Abbildung, die außerhalb einer kompakten Teilmenge von X verschwindet. Zeige, dass es ein $\epsilon > 0$ derart gibt, dass for alle $|\lambda| < \epsilon$ die Abbildung $f + \lambda g : X \to X$ ein Diffeomorphismus ist.

Exercise 9.3 (20 pts) Sei $SL(n) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$. Finde eine Darstellung von $T_I SL(N)$, dem Tangentialraum zu $SL(n)$ in der Einheitsmatrix I. Zeige weiters, dass für $A \in \mathbb{R}^{n\times n}$ mit $tr(A) = 0$ ($tr(A)$ bezeichnet die Spur von A) gilt, dass $\gamma(t) = \exp(tA)$ eine Kurve in $SL(n)$ mit $\dot\gamma(0) = A$.

Exercise 9.4 (20 pts) Sei $M = f^{-1}(0)$ eine erklpf im $\mathbb{R}^2$ mit der Dimension 1, wobei $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ eine stetig differenzierbare Funktion mit 0 als regulärem Wert ist (d.h., $Df(x)$ hat vollen rang für alle x mit $f(x) = 0$) und $(0, 0) \notin M$. Zeige, dass

$$R = \{(x, y, z) \in \mathbb{R}^3 \mid f(\sqrt{x^2 + y^2}, z) = 0\}$$

eine 2-dimensionale erklpf des $\mathbb{R}^3$ ist. Veranschauliche dieses Resultat am Beispiel eines Torus.

Exercise 9.5 (20 pts) Sind die folgenden Mengen erklpf? Begründen Sie Ihre Antworten.

a) Die Ebene als Teilmenge des $\mathbb{R}^3$.

b) Die Sphäre S^n als Teilmenge des $\mathbb{R}^n$.

c) Die Sphäre S^n als Teilmenge des $\mathbb{R}^m$ für $m > n$.

d) Die linke Menge in Figure 1 im $\mathbb{R}^2$.

e) Die rechte Menge in Figure 1 im $\mathbb{R}^2$.

Figure 1: Erklpf?

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Analysis 3 - Exercise Sheet 10

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Wie besprochen, die zwei letzten Beispiele vom 8. Übungszettel noch mal.

Exercise 10.1 (20 pts) Sei $M = f^{-1}(0)$ eine erklpf im $\mathbb{R}^2$ mit der Dimension 1, wobei $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ eine stetig differenzierbare Funktion mit 0 als regulärem Wert ist (d.h., $Df(x)$ hat vollen rang für alle x mit $f(x) = 0$) und $(0, 0) \notin M$. Zeige, dass

$$R = \{(x, y, z) \in \mathbb{R}^3 \mid f(\sqrt{x^2 + y^2}, z) = 0\}$$

eine 2-dimensionale erklpf des $\mathbb{R}^3$ ist. Veranschauliche dieses Resultat am Beispiel eines Torus.

Exercise 10.2 (20 pts) Sind die folgenden Mengen erklpf? Begründen Sie Ihre Antworten.

a) Die Ebene als Teilmenge des $\mathbb{R}^3$.

b) Die Sphäre S^n als Teilmenge des $\mathbb{R}^{n+1}$.

c) Die n-dimensionale Sphäre als Teilmenge des $\mathbb{R}^m$ für $m > n+1$, das heißt, $S = \{x \in \mathbb{R}^m \mid (x_1, \ldots, x_{n+1}) \in S^n\}$.

d) Die linke Menge in Figure 1 im $\mathbb{R}^2$.

e) Die rechte Menge in Figure 1 im $\mathbb{R}^2$.

Figure 1: Erklpf?

Exercise 10.3 (20 pts) Sei $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ und definiere

$$\iota_N : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad \iota_N(x) = N + \frac{2}{\|x - N\|_2^2}(x - N)$$

und die stereographische Projektion

$$\sigma_N : \mathbb{R}^n \to S^n \setminus \{N\}, \quad \sigma_N(x) = \iota_N(x, 0)$$

mit der n-dimensionalen Einheitssphäre $S^n \subset \mathbb{R}^{n+1}$. Zeigen Sie, dass σ_N eine Parametrisierung von $S^n \setminus \{N\}$ ist. Fertigen Sie Skizzen für n=1,2 an. Das heißt:

a) Zeigen Sie, dass σ_N surjektiv ist. Fertigen Sie eine Skizze an, um zu sehen, wie sie für $y \in S^n$ das passende x finden, sodass $\sigma_N(x) = y$.

b) Zeigen Sie, dass σ_N injektiv ist.

c) Zeigen Sie, dass die Jacobimatrix von σ_N injektiv ist. Die Jacobimatrix kann recht schön vereinfacht werden.

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Exercise 10.4 (20 pts) Bestimmen Sie die Tangentialräume von $S^n \setminus \{N\}$ mithilfe der Parametrisierung aus 10.3.

Exercise 10.5 (20 pts) Sei die Hyperfläche $S \subset \mathbb{R}^3$ der Graph einer Funktion $g : \mathbb{R}^2 \to \mathbb{R}$. Erstellen Sie eine Gleichung für die Punkte (x, y, z) der Tangentialebene in $(x0, y0, z0) \in S$ mittels der partiellen Ableitungen von g.

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Analysis 3 - Exercise Sheet 11

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Wie besprochen, die zwei letzten Beispiele vom 10. Übungszettel noch mal.

Exercise 11.1 (20 pts) Bestimmen Sie die Tangentialräume von $S^n \setminus \{N\}$ mithilfe der Parametrisierung aus 10.3. Zeigen Sie, dass der Tangentialraum $T_x S^n = \{x\}^\perp$, also dass orthogonale Komplement zum Vektor x ist. Zeigen sie gleiches auch über die implizite Darstellung der Sphäre als die Menge aller x für die gilt $\|x\|_2^2 = 1$.

Exercise 11.2 (20 pts) Sei die Hyperfläche $S \subset \mathbb{R}^3$ der Graph einer Funktion $g : \mathbb{R}^2 \to \mathbb{R}$. Erstellen Sie eine Gleichung für die Punkte (x, y, z) der Tangentialebene in $(x_0, y_0, z_0) \in S$ mittels der partiellen Ableitungen von g.

Exercise 11.3 (20 pts) In diesem Beispiel widmen wir uns Funktionen zwischen Flächen. Seien $M \subset \mathbb{R}^d$ und $N \subset \mathbb{R}^d$ zwei erklpf der Dimensionen m und n. Sei $F : M \to N$, $p \in M$ und $q = F(p) \in N$ und $f : U \to M$ eine lokale Parametrisierungen um p. Wir nennen F differenzierbar in p, falls $F \circ f$ differenzierbar in $x = f^{-1}(p)$ ist.

a) Warmup: Wozu ist diese umständliche Definition der Differenzierbarkeit überhaupt nötig? Warum betrachten wir die Differenzierbarkeit von F nicht einfach "direkt"?

b) Zeige, dass Differenzierbarkeit unabhängig von den konkreten Parametrisierung f ist. D.h., zeige, dass Differenzierbarkeit bezüglich der Parametrisierung f auch impliziert, dass F bezüglich einer beliebigen anderen Parametrisierung g differenzierbar ist.

Exercise 11.4 (20 pts) Sei $T \subset \mathbb{R}^3$ ein Torus mit beliebigen Radien r, R. Zeige, dass $F : T \to N$, $F(p) = \frac{1}{\|p\|}$ differenzierbar ist.

Exercise 11.5 (20 pts) Eine Abbildung $f : M \to N$ zwischen zei erklpf heißt Diffeomorphismus, wenn f bijektiv ist und f und f^{-1} stetig differenzierbar sind. Wir nennen M und N diffeomorph, falls ein Diffeomorphismus zwischen m und N existiert. Zeigen Sie, dass das Ellipsoid

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1\}$$

für $a, b, c > 0$ diffeomorph zur Sphäre S^2 ist.

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Analysis 3 - Exercise Sheet 12

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Exercise 12.1 (20 pts) Seien M, N eingebettete, reguläre, lokal parametrisierte Flächen und $F : M \to N$ differenzierbar. Wir betrachten die Ableitung von F als lineare Abbildung analog zur Ableitung im $\mathbb{R}^d$. Die Ableitung $DF(p) : T_pM \to T_qN$ für $p \in M$, $q = F(p)$ sei wie folgt definiert: Sei $w \in T_pM$ und γ Kurve in U mit $\gamma(0) = x$, sodass $w = \frac{d}{dt} f \circ \gamma(t)|_{t=0}$. Definiere $DF(p)(w) \coloneqq \frac{d}{dt} F \circ f \circ \gamma(t)|_{t=0}$. Beweise, dass $DF(x)$ wohldefiniert, insbesondere auch unabhängig von der Parametrisierung f und der Kurve γ, und linear ist.

Exercise 12.2 (20 pts) Beweisen Sie die Kettenregel für zwei hintereinander ausgeführte Abbildungen zwischen erklpf.

Exercise 12.3 (20 pts) Sei $F : M \to N$ eine Abbildung zwischen zwei erklpf.

- Sei F die Einschränkung einer linearen Abbildung $\bar{F} : \mathbb{R}^d \to \mathbb{R}^d$, $\bar{F}(x) = Ax$, $A \in \mathbb{R}^{d\times d}$. Berechnen Sie $DF(p)w$ für beliebieges $p \in M$ und $w \in T_pM$.
- F^{-1} existiere und sei differenzierbar. Stelle $D(F^{-1})$ in geeigneter Weise durch F, DF dar.

Exercise 12.4 (20 pts) Seien $0 < r_1, r_2 < R$ und S_1, S_2 die Tori gegeben durch

$$f_i(\phi, \psi) = \begin{pmatrix} (R + r_i \cos(\phi)) \cos\psi \\ (R + r_i \cos(\phi)) \sin\psi \\ r_i \sin(\phi) \end{pmatrix}, \quad (\phi, \psi) \in [0, 2\pi)^2, \quad i = 1, 2.$$

Definiere $F : S_1 \to S_2$ über $f_1(\phi, \psi) \mapsto f_2(\phi, \psi)$. Bestimmen Sie für ein $x = f_1(\phi, \psi)$, $(\phi, \psi) \in (0, 2\pi)^2$ die Matrixdarstellung von DF bzgl. der Parametrisierungen f_i.

Exercise 12.5 (20 pts) Sei M das Ellipsoid

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1\}$$

für $a, b, c > 0$. Wir wissen vom letzten Übungszettel, dass $F : M \to S^2$, $F(x, y, z) = (\frac{x}{\sqrt{a}}, \frac{y}{\sqrt{b}}, \frac{z}{\sqrt{c}})$ ein Diffeomorphismus ist. Berechne die Ableitung von F.

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Analysis 3 - Exercise Sheet 13

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Theoretical preliminaries: Let $G \subset \mathbb{R}^d$ a bounded, open set and $F : \tilde{G} \to \mathbb{R}^d$, $f : \tilde{G} \to \mathbb{R}$ continuously differentiable with $\tilde{G}$ open and $\overline{G} \subset \tilde{G}$. We denote

$$\mathrm{div} F = \sum_{i=1}^{d} \partial_{x_i} F_i,$$

$$\Delta f = \sum_{i=1}^{d} \partial_{x_i}^2 f.$$

Further $\nu(x)$ denotes the unit normal vector on ∂G in $x \in \partial G$ which points outside of G. We denote the surface measure for integration on the boundary of G as σ.
Gauss' theorem It holds true that

$$\int_G \mathrm{div}(F)\; d(x,y) = \int_{\partial G} \langle F, \nu \rangle \; d\sigma.$$

Exercise 12.1 (20 pts) Show that

$$\int\limits_{\mathbb{R}} e^{-z^2}\; dz = \sqrt{\pi}$$

by computing

$$\int\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}\; d(x,y).$$

Hint: Polar coordinates.

Exercise 12.2 (20 pts) Compute the surface volume of a general torus with $0 < r < R$ by integrating the function $f \equiv 1$ over the surface.

Exercise 12.3 (20 pts) Let $G \subset \mathbb{R}^d$ a bounded, open set and $F : \tilde{G} \to \mathbb{R}^d$, $f, g : \tilde{G} \to \mathbb{R}$, f twice and F, g once continuously differentiable with $\tilde{G}$ open and $\overline{G} \subset \tilde{G}$. Use Gauss' theorem to prove that

a)

$$\int_G \langle \nabla f, F \rangle \; dx = \int_{\partial G} \langle fF, \nu \rangle \; d\sigma - \int_G f \mathrm{div} F \; dx.$$

b)

$$\int_G -\Delta f g = \int_G \langle \nabla f, \nabla g \rangle - \int_{\partial G} g \langle \nabla f, \nu \rangle \; \sigma.$$

We prove a version of Gauss' theorem Obviously, do not use Gauss' theorem in the following.
Let $a < 0$, $b > 1$ and $f : (a,b) \to (0,\infty)$ be a strictly increasing diffeomorphism. Denote the set $G = \{(x,y) \mid x \in (0,1),\ 0 < y < f(x)\}$. Let $F = (F_1, F_2)$ be a continuously differentiable vector field defined in a neighborhood of $\overline{G}$. We will show together that

$$\int_G \mathrm{div}(F)\; d(x,y) = \int_{\partial G} \langle F, n \rangle \; ds.$$

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Exercise 12.4 (20 pts) Denoting $\alpha = f(0)$, $\beta = f(1)$, use the fundamental theorem of calculus to show that

$$\int_G \mathrm{div} F \, d(x,y) = \int_0^1 -F_2(x,0)\,dx + \int_0^1 F_2(x,f(x))\,dx + \int_0^\beta F_1(1,y)\,dy + \int_0^\alpha -F_1(0,y)\,dy + \int_0^1 -F_1(x,f(x))f'(x)\,dx \tag{1}$$

Exercise 12.5 (20 pts) Compute the line integral $\int_{\partial G} \langle \nabla F, n\rangle \, ds$ by dividing it into 4 segments $\{(x,y) \in \delta G \mid x = 0,\ y \in (0,\alpha)\}$, $\{(x,y) \in \delta G \mid x = 1,\ y \in (0,\beta)\}$, $\{(x,y) \in \delta G \mid x \in (0,1),\ y = 0\}$, $\{(x,y) \in \delta G \mid x \in (0,1),\ y = f(x)\}$ to show that it agrees with the above from 12.4. Why can we neglect the corners $(x,y) = (0,0)$, $(x,y) = (1,0)$, etc.