Geometric Patterns and Microstructures in the study of Material Defects and Composites

Silvio Fanzon

supervised by

Mariapia Palombaro

University of Sussex

Department of Mathematics

UNIVERSITY OF SUSSEX

Presentation Plan

1. **Geometric Patterns of Dislocations**

2. **Microstructures in Composites**

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   - Dislocations

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   - Semi-coherent interfaces (Chapter 3)

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   - Proof of the main theorem

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   - Semi-coherent interfaces
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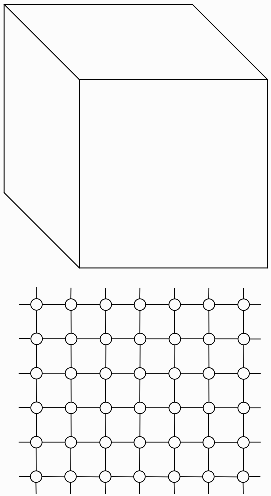
2. **Microgeometries in Composites**
   - Critical lower integrability
   - Convex integration
   - Proof of our main result

Edge dislocations

Dislocations: topological defects in the otherwise periodic structure of a crystal.



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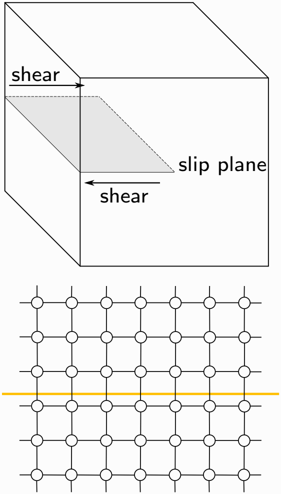

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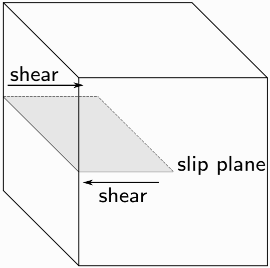

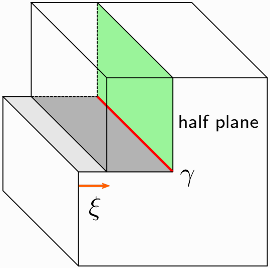

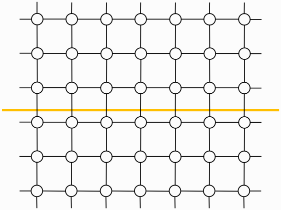

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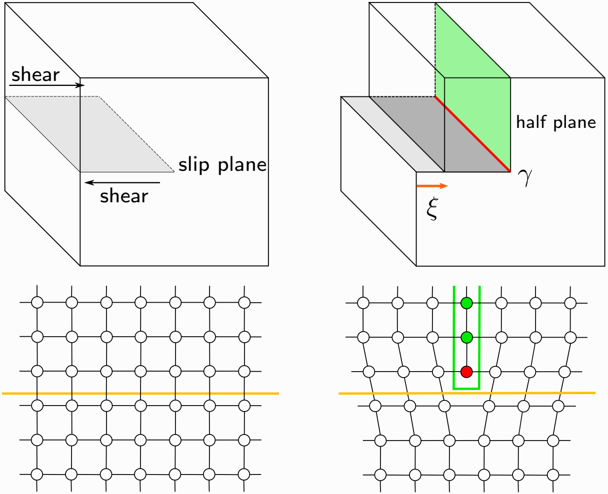

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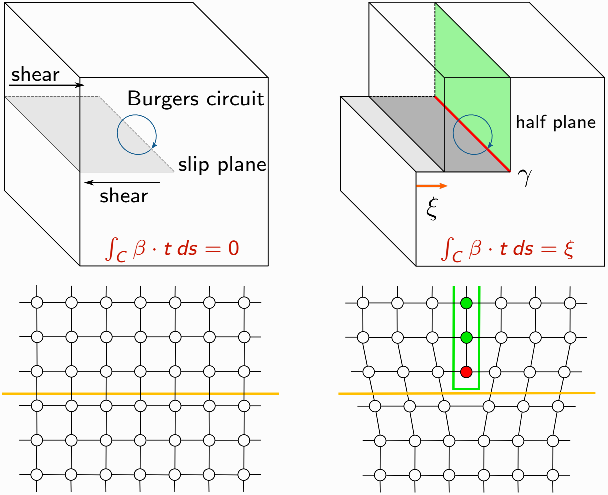

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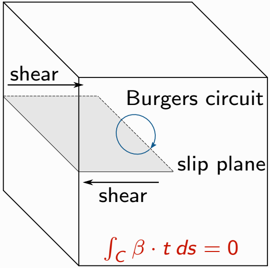

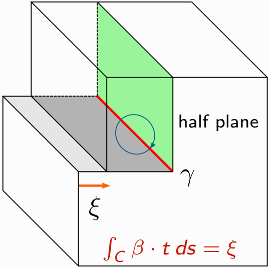

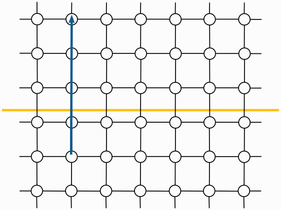

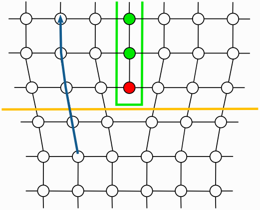

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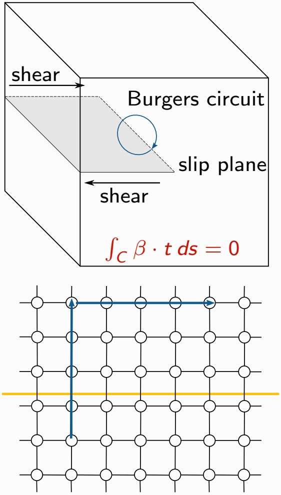

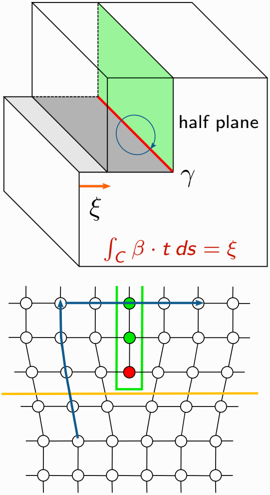

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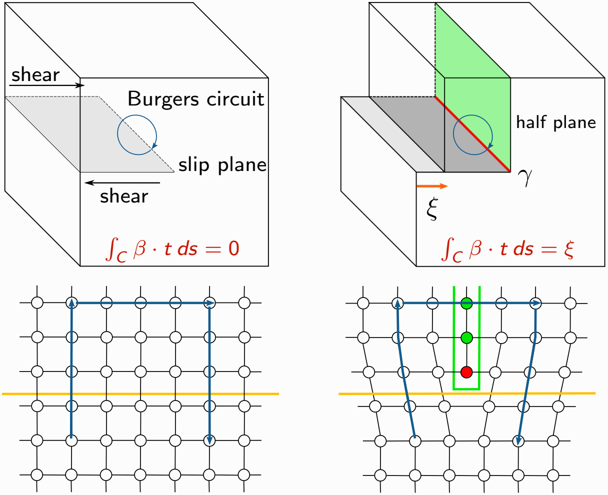

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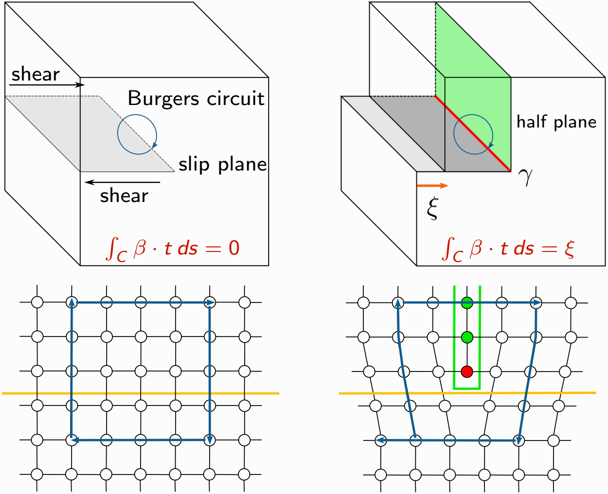

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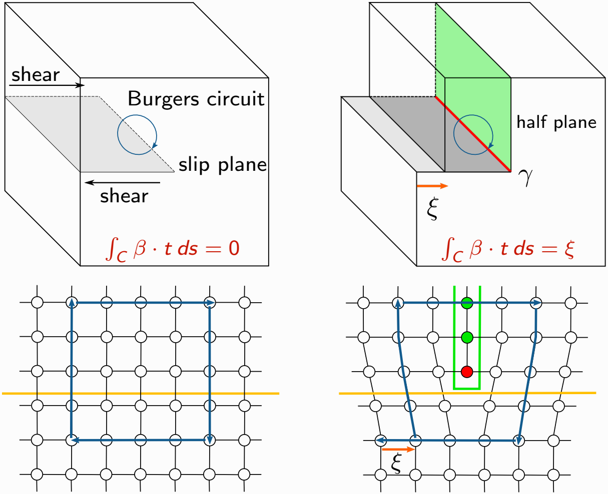

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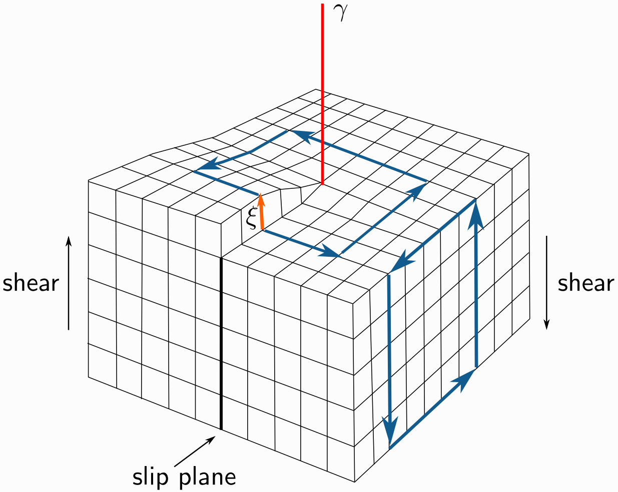

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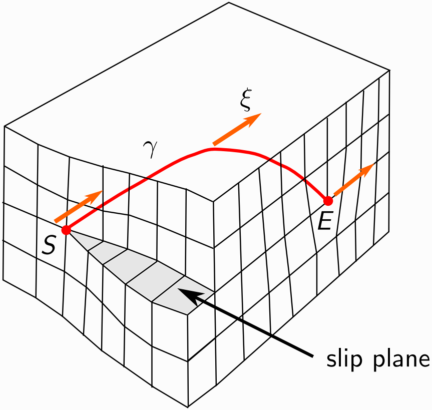
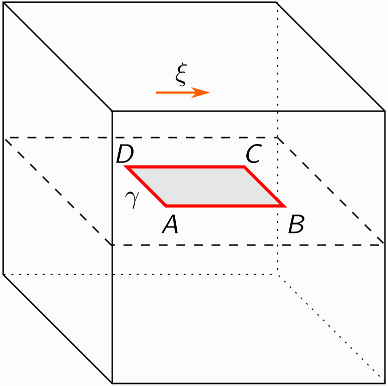
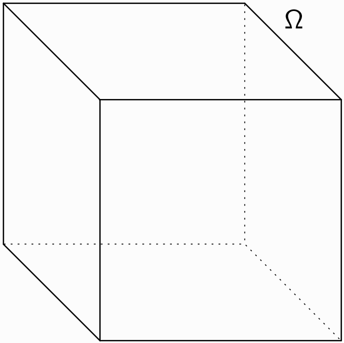




Nonlinear Elasticity

Reference configuration: $\Omega \subset \mathbb{R}^3$ open bounded



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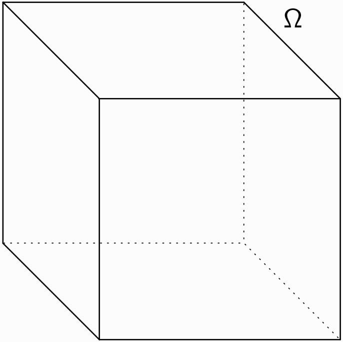

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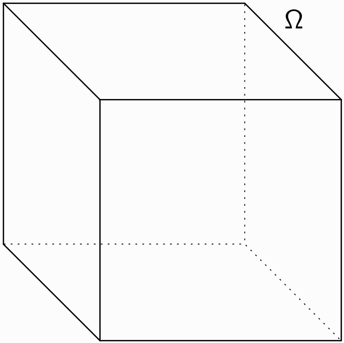

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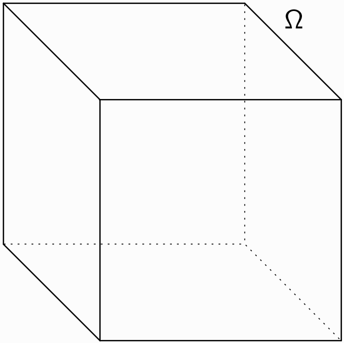

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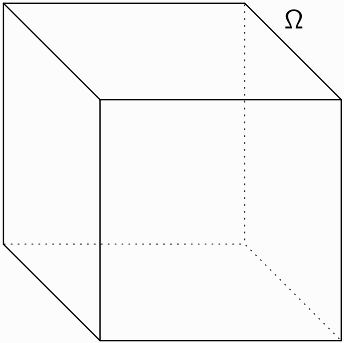

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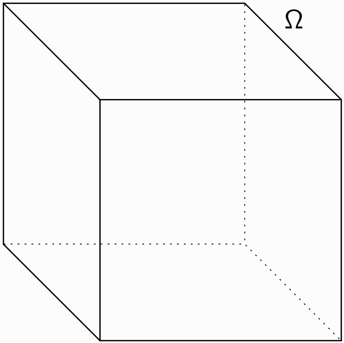

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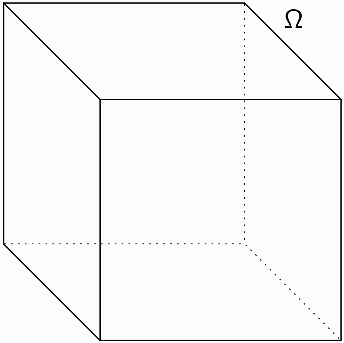

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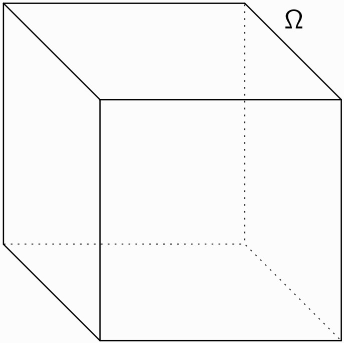
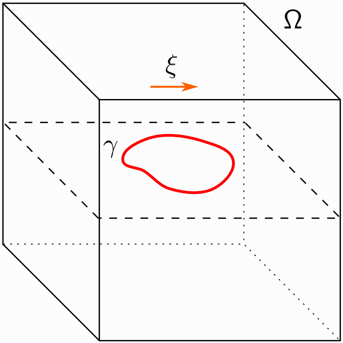


Semi-discrete model for dislocations

Dislocation lines: Lipschitz curves $\gamma \subset \Omega$ such that $\Omega \setminus \gamma$ is not simply connected



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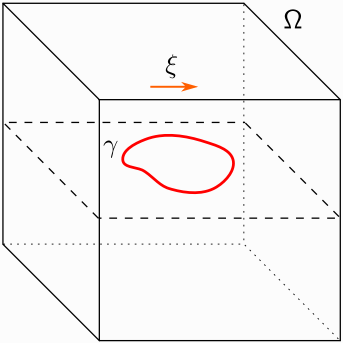

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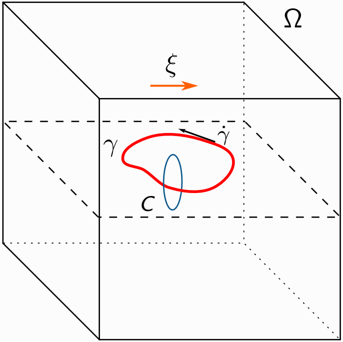
Strain generating (γ, ξ): map $\beta \colon \Omega \to \mathbb{M}^{3\times 3}$ s.t.

$$\operatorname{Curl} \beta = -\xi \otimes \dot{\gamma}\, \mathcal{H}^1 \llcorner \gamma \iff \int_C \beta \cdot t \, d\mathcal{H}^1 = \xi \, .$$



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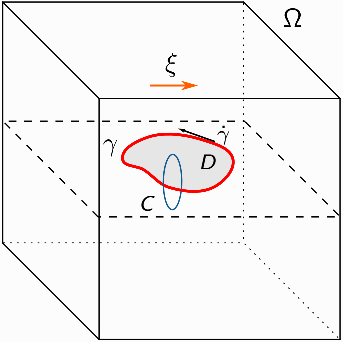


Geometric interpretation: if D encloses γ, there exists a deformation $v \in SBV(\Omega; \mathbb{R}^3)$ s.t.

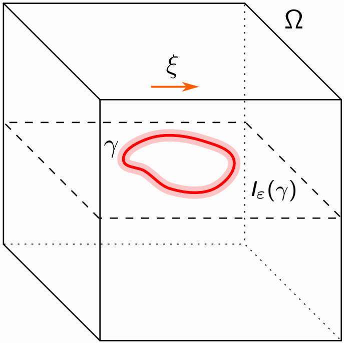
$$Dv = \nabla v \, dx + \xi \otimes n \, \mathcal{H}^2 \llcorner D \, , \quad \beta = \nabla v \, .$$

v has constant jump ξ across the slip region D.

Strains are not L^2

Let β generate (γ, ξ). Consider $\varepsilon > 0$ and

$$I_\varepsilon(\gamma) := \{x \in \mathbb{R}^3 : \ \mathrm{dist}(x, \gamma) < \varepsilon\} .$$



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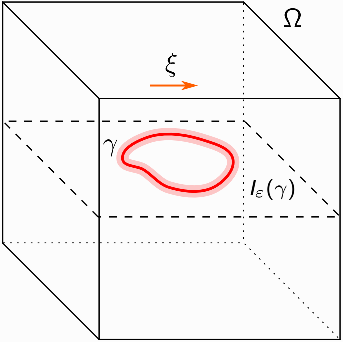

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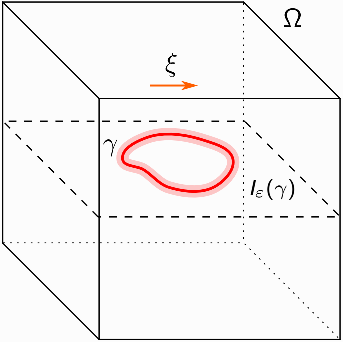
Proof: let $\sigma > \varepsilon$ and $L := \mathrm{length}(\gamma)$



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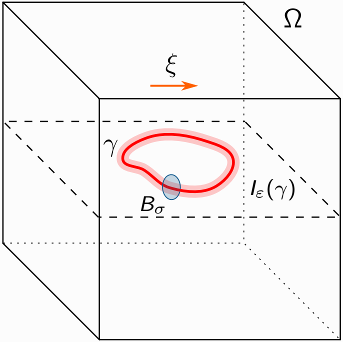

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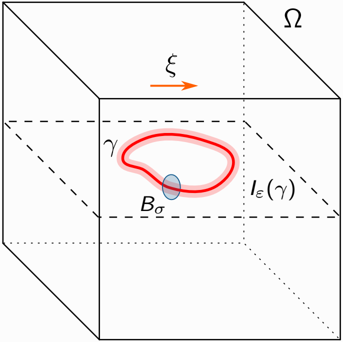

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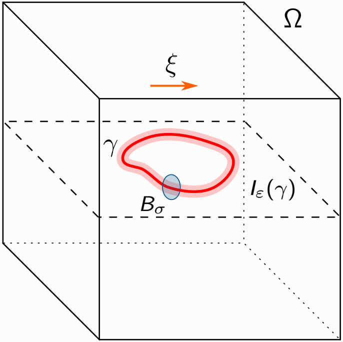
$$= L \frac{|\xi|^2}{2\pi} \log \frac{\sigma}{\varepsilon} \to \infty \ \text{ as } \ \varepsilon \to 0$$



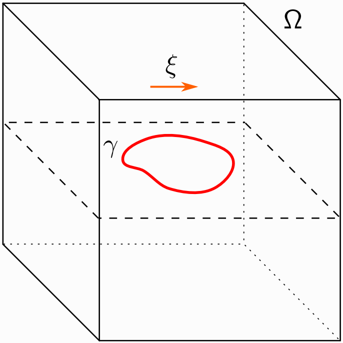
Regularise the problem

Energy Truncation. Fix $p \in (1,2)$ and assume

$$W(F) \sim \mathrm{dist}(F, SO(3))^2 \wedge (|F|^p + 1)\,.$$

Strains are maps $\beta \in L^2(\Omega; \mathbb{M}^{3\times 3})$ such that

$$\mathrm{Curl}\,\beta = -\xi \otimes \dot{\gamma}\, \mathcal{H}^1 \llcorner \gamma\,.$$



Regularise the problem

Energy Truncation. Fix $p \in (1,2)$ and assume

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Core Radius Approach. Assume

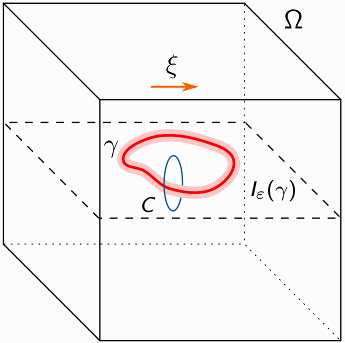
$$W(F) \sim \operatorname{dist}(F, SO(3))^2\,.$$

Let $\varepsilon > 0$ ($\propto$ atomic distance) and consider

$$\Omega_\varepsilon(\gamma) := \Omega \setminus I_\varepsilon(\gamma)\,.$$

Strains are maps $\beta \in L^2(\Omega_\varepsilon(\gamma); \mathbb{M}^{3\times 3})$ such that

$$\operatorname{Curl} \beta \llcorner \Omega_\varepsilon(\gamma) = 0\,, \qquad \int_C \beta \cdot t \, d\mathcal{H}^1 = \xi\,.$$



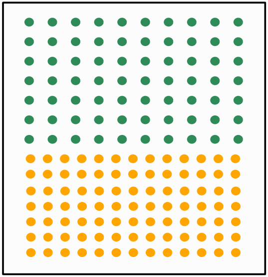
Presentation Plan

1. **Geometric Patterns of Dislocations**
   - Dislocations
   - Semi-coherent interfaces
   - Linearised polycrystals

2. **Microgeometries in Composites**
   - Critical lower integrability
   - Convex integration
   - Proof of our main result

Semi-coherent interfaces

Two different crystalline materials joined at a flat interface:



Λ^+

Λ^-

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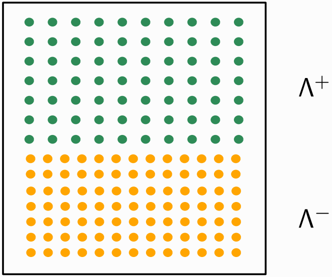

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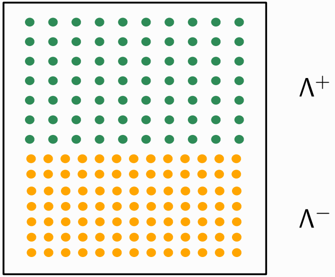

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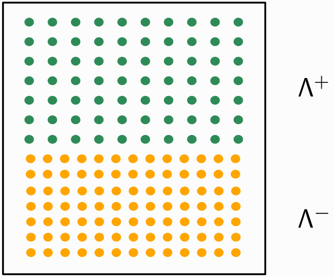

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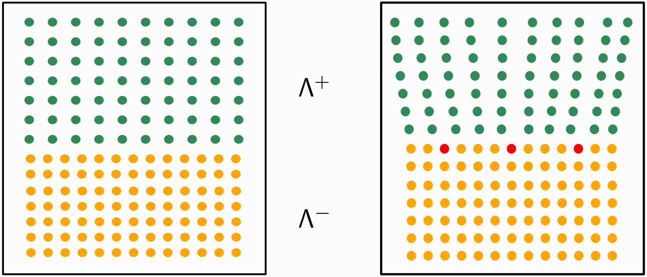
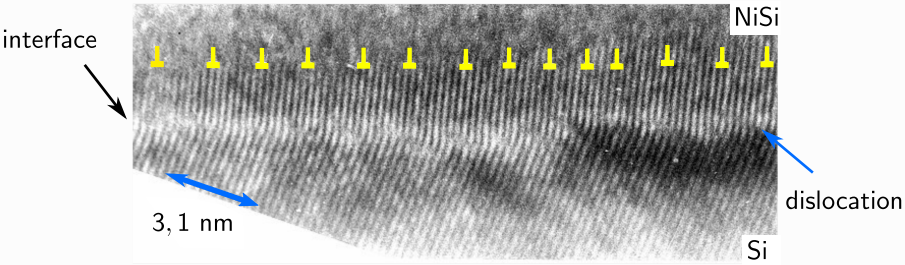
Equilibrium: Λ^+ has lower density than Λ^- $\implies$ **edge dislocations** at interface.



Network of dislocations

Experimentally observed phenomena:



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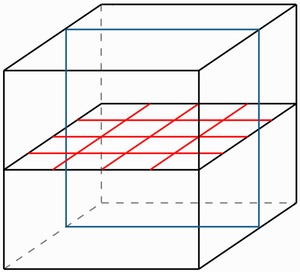

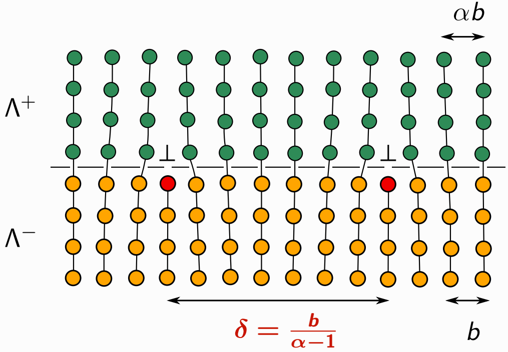

D.A. Porter, K.E. Easterling. *Phase transformations in metals and alloys.* CRC Press (2009)

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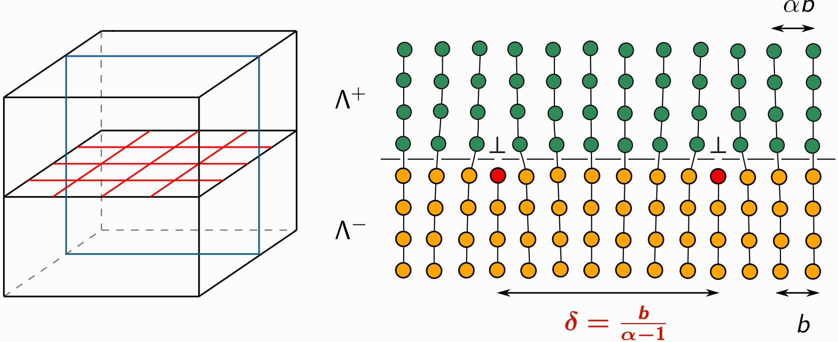

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- derive the simplified (dislocation density) **continuum model**.

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Semi-discrete line defect model

Reference configuration: $\Omega_r := \Omega_r^- \cup S_r \cup \Omega_r^+, \;\; r > 0,$

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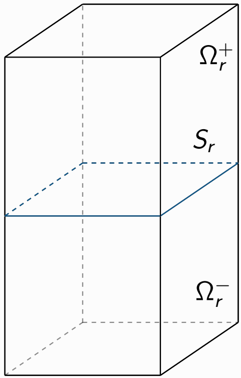

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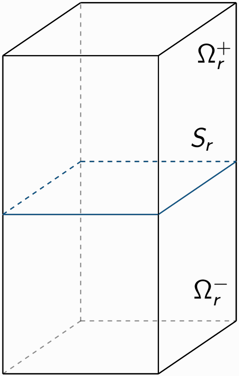

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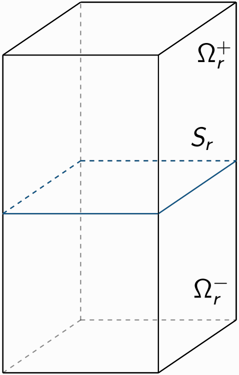

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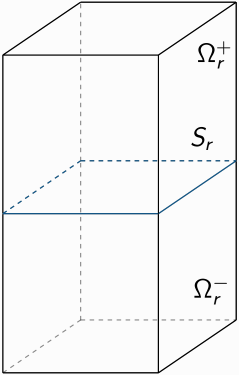

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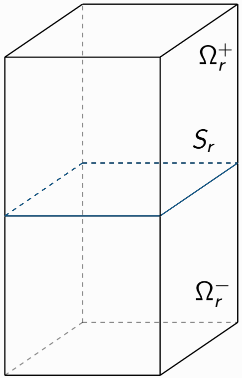

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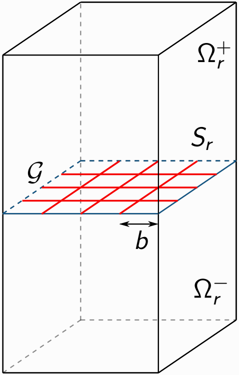

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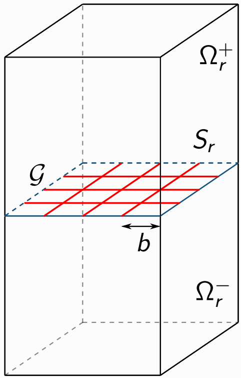

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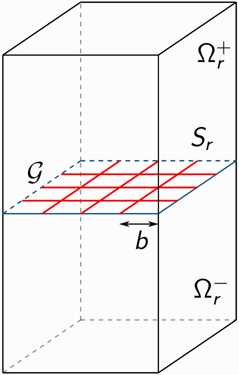

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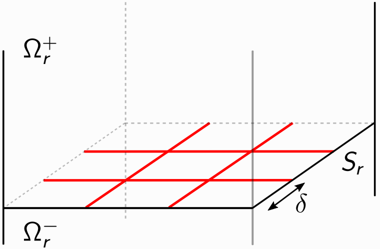

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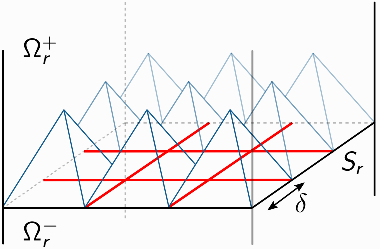

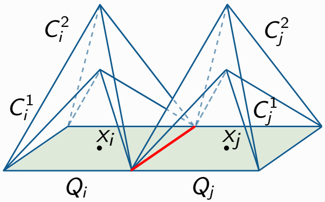

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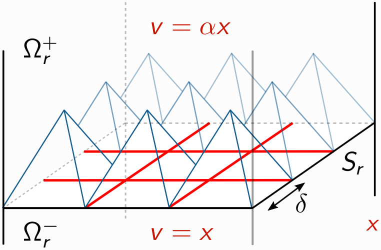

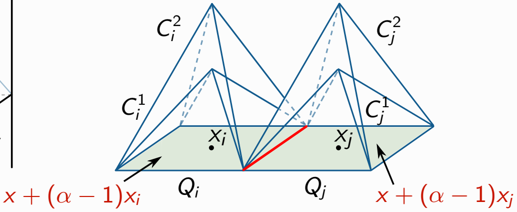

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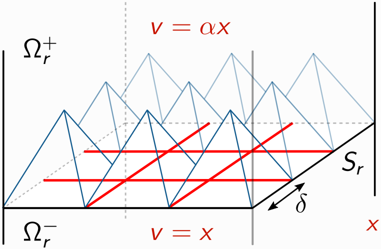

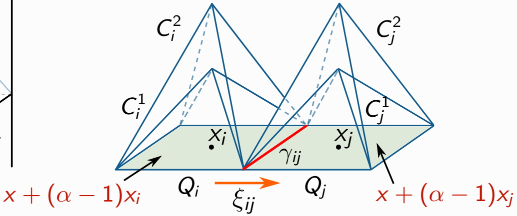

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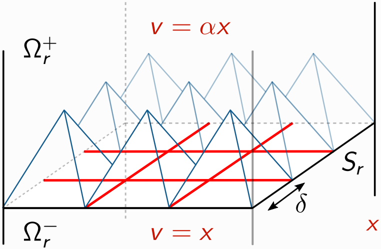
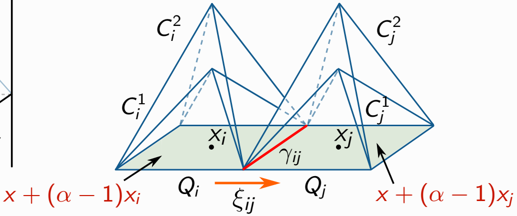
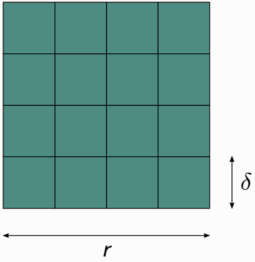
Energy: in each pyramid is $c = c(\alpha, b, p) \implies E_{\alpha,r} \leq c \dfrac{r^2}{\delta^2}$ (as $W(\alpha I) = 0$).





Remarks on the semi-discrete model

Deformed configuration: $v(S_R)$ with v from the upper bound construction



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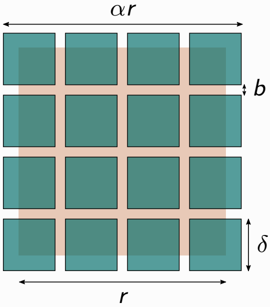

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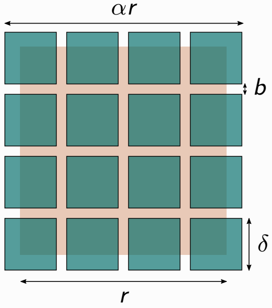

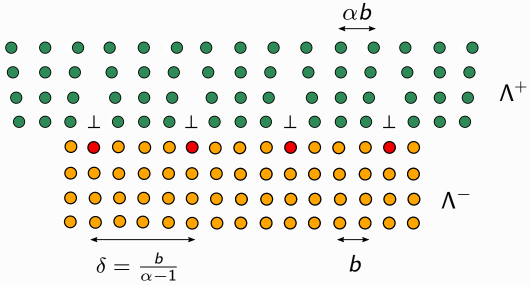

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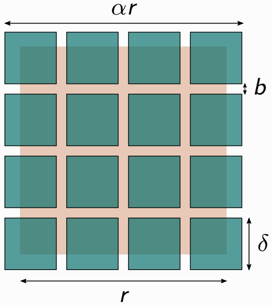

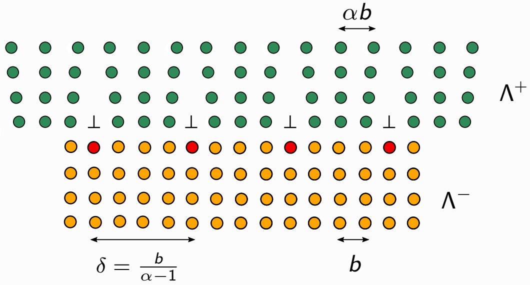

Limitations of the considered model:

- $v(S_r)$ does not match $S_r \implies$ not appropriate for semi-coherent interfaces,

Remarks on the semi-discrete model

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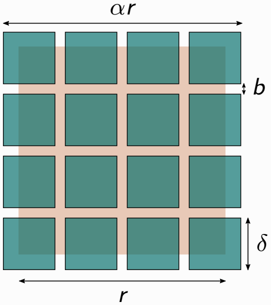

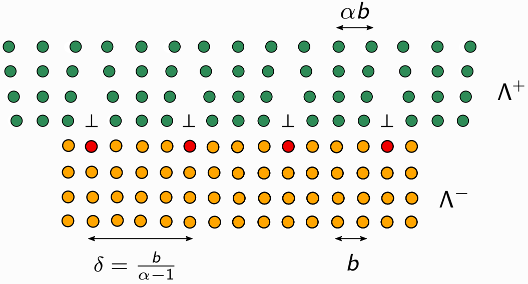

Limitations of the considered model:

- $v(S_r)$ does not match $S_r \implies$ not appropriate for semi-coherent interfaces,
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Remarks on the semi-discrete model

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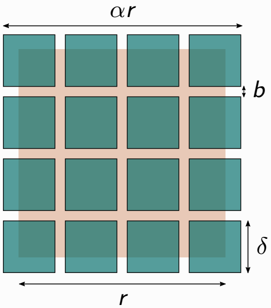

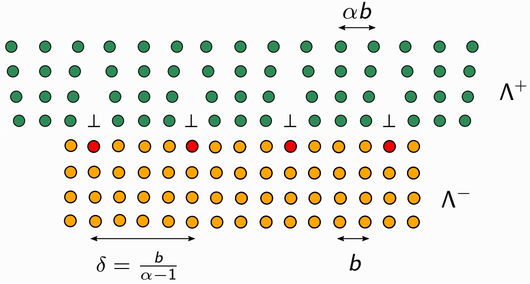

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Remarks on the semi-discrete model

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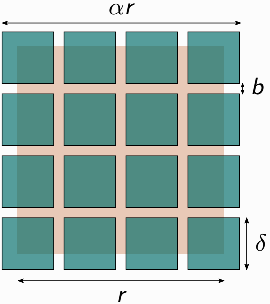

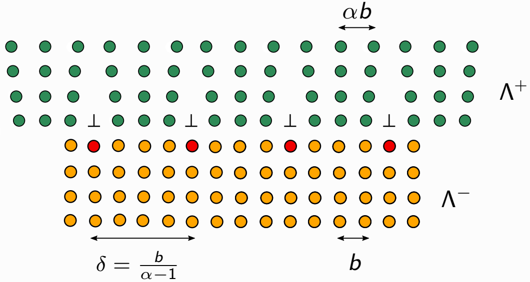

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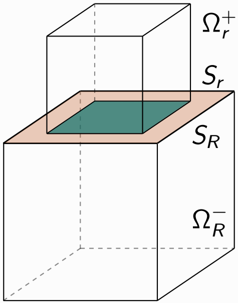
What we do now:

- take a smaller overlayer and enforce match at the interface,
- introduce a simplified continuum (dislocation density) model to better describe true minimisers.

Heuristic for the continuum model



Reference configuration: $\Omega_{R,r} := \Omega_R^- \cup S_r \cup \Omega_r^+$, with $r := \theta R,\ \theta \in [\alpha^{-1}, 1]$

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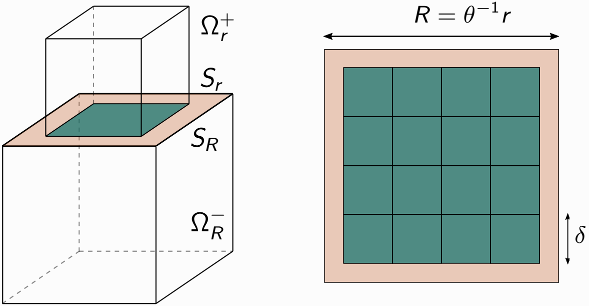

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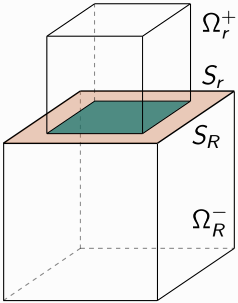

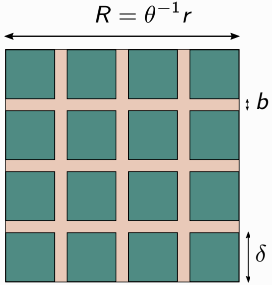

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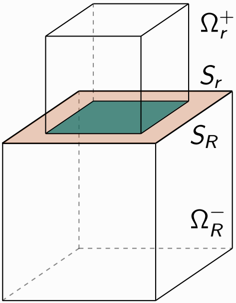

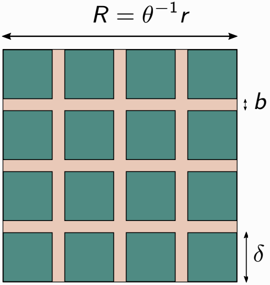

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$$L = 2R\frac{r}{\delta} = \frac{2r^2}{b}(\theta^{-2} - \theta^{-1}) \overset{(\theta^{-1}\approx 1)}{\approx} \frac{r^2}{b}(\theta^{-2} - 1) = \frac{1}{b}(R^2 - r^2) = \frac{1}{b}\,\text{Area Gap}$$

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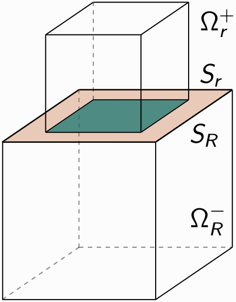

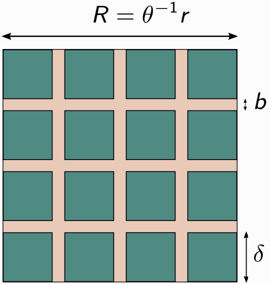

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$$\text{Dislocation Length} \approx \frac{1}{b}\,\text{Area Gap}$$

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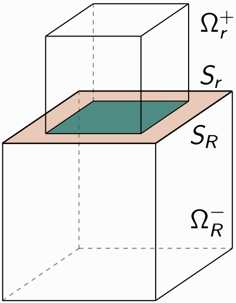

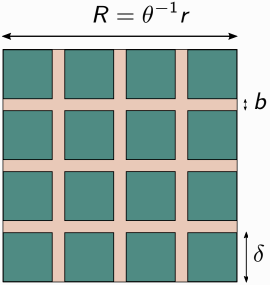

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$E_{\alpha,r} \approx r^2 E_\alpha$

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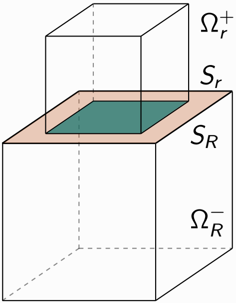

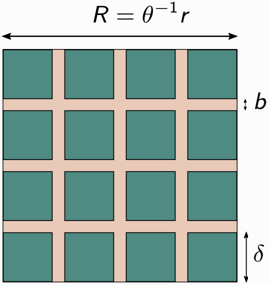

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$$E_{\alpha,r} \approx r^2 E_\alpha = \sigma\,\text{Area Gap} \quad \text{with} \quad \sigma := \frac{E_\alpha}{\theta^{-2}-1}$$

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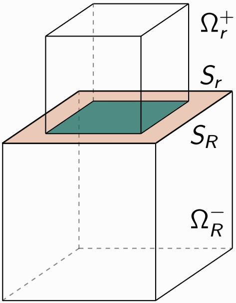

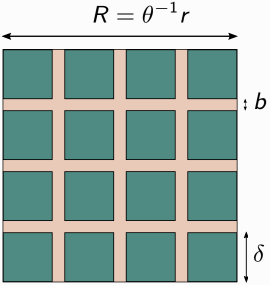

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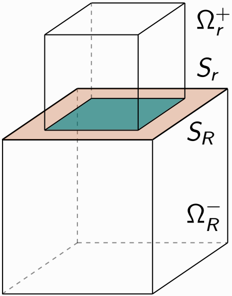
Hypothesis: Dislocation Energy $\propto$ Dislocation Length. Then optimise over θ.

Continuum model

Reference configuration: $\Omega_{R,r} := \Omega_R^- \cup S_r \cup \Omega_r^+$, with $r := \theta R\,,\ \ \theta \in [\alpha^{-1}, 1]$



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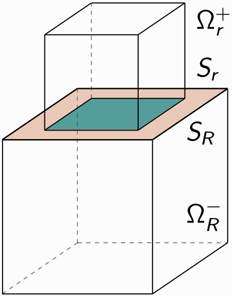
Deformations: $v \in W^{1,2}(\Omega_r^+; \mathbb{R}^3)$ such that $v = \dfrac{x}{\theta}$ on S_r

$\implies v(S_r) = S_R$ (interface match)



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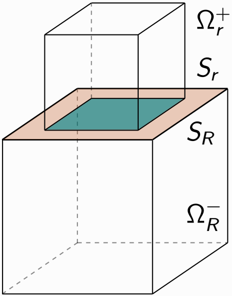

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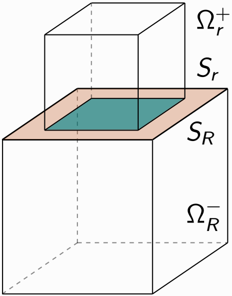

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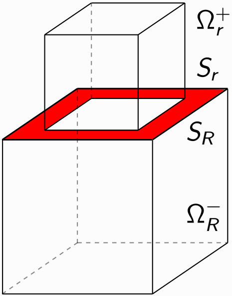

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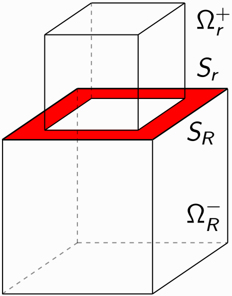
Total Energy: $E^{tot}_{\alpha,R}(\theta) := \min\limits_{\theta} \left(E^{el}_{\alpha,R}(\theta) + E^{pl}_R(\theta) \right)$



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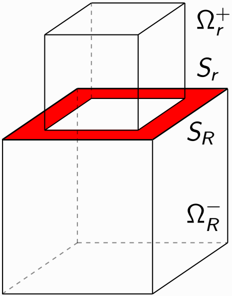

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Elastic: $E_{\alpha,R}^{el}(\theta) := \inf \left\{ \int_{\Omega_r^+} W(\nabla v)\, dx :\ v = x/\theta \text{ on } S_r \right\}$

Plastic: $E_R^{pl}(\theta) := \sigma\,\text{Area Gap} = \sigma R^2(1-\theta^2),\ \sigma > 0$

Total Energy: $E_{\alpha,R}^{tot}(\theta) := \min\limits_{\theta} \left(E_{\alpha,R}^{el}(\theta) + E_R^{pl}(\theta) \right)$

Question: behaviour of $E_{\alpha,R}^{tot}(\theta)$ as $R \to \infty$?

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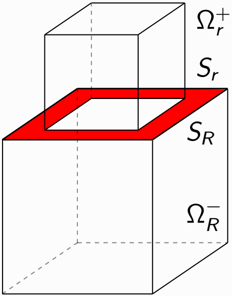

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Energy competition:

- $\theta = 1 \implies$ no dislocation energy

Continuum model

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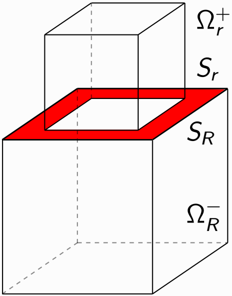

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- $\theta = 1 \implies$ no dislocation energy
- $\theta = \alpha^{-1} \implies$ no elastic energy $\qquad (v := \alpha x,\ W(\alpha I) = 0)$

Continuum model

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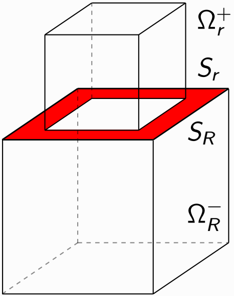

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Question: behaviour of $E^{tot}_{\alpha,R}(\theta)$ as $R \to \infty$?

Energy competition:

- $\theta = 1 \implies$ no dislocation energy
- $\theta = \alpha^{-1} \implies$ no elastic energy $\quad (v := \alpha x, \; W(\alpha I) = 0)$
- $\theta \in (\alpha^{-1}, 1) \implies$ both present

Asymptotic for $E^{tot}_{\alpha,R}$

Let $\theta_R \in [\alpha^{-1}, 1]$ be a minimiser for $E^{tot}_{\alpha,R}$ and define

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Step 4. Write the elastic energy as a polynomial

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where $k^{el}_R := C^{el} + \varepsilon_R > 0$ and $k^{el}_R \to C^{el}$.

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Step 5. The total energy computed along θ_R is equal to

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Step 8. Since $\theta_R - \theta^m_R \to 0$, by using (1.1), minimality, and computing $P_{R,k}(\theta^m_R)$, we have the thesis

$$E^{tot}_{\alpha,R}(\theta_R) = \underbrace{\frac{\sigma^2}{\alpha^3 C^{el}}\, R}_{\text{Elastic}} + \underbrace{\sigma R^2\left(1-\alpha^{-2}\right) - 2\frac{\sigma^2}{\alpha^3 C^{el}}\, R}_{\text{Plastic}} + O(R)\,.$$

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- Grain boundaries, the misfit between the crystal lattices are described by rotations rather than dilations.
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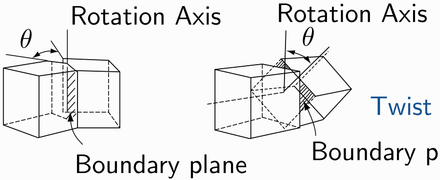
Tilt boundary



Twist boundary

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- Optimal geometry for the dislocation net (square vs hexagonal)
  Koslowski, Ortiz (2004)

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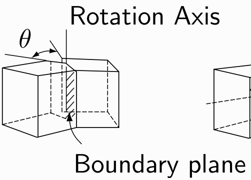

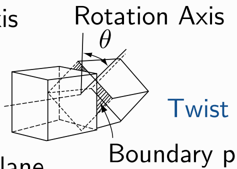

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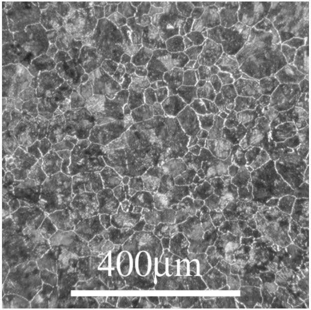
Presentation Plan

1. **Geometric Patterns of Dislocations**
   - Dislocations
   - Semi-coherent interfaces
   - Linearised polycrystals
2. **Microgeometries in Composites**
   - Critical lower integrability
   - Convex integration
   - Proof of our main result

Polycrystals

Polycrystal: formed by many grains, having the **same** lattice structure, mutually rotated $\implies$ interface misfit at **grain boundaries**.



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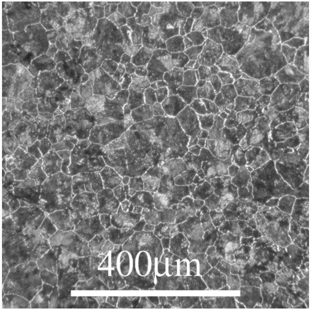
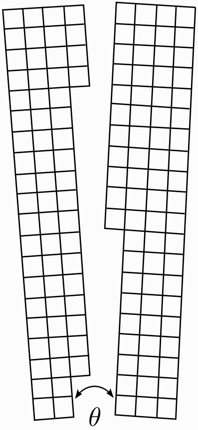


Goal: obtain polycrystalline structures as minimisers of some energy functional.

F., Palombaro, Ponsiglione. *Linearised Polycrystals from a 2D System of Edge Dislocations.* Preprint (2017)

Tilt grain boundaries

Tilt boundary: small angle rotation θ between grains



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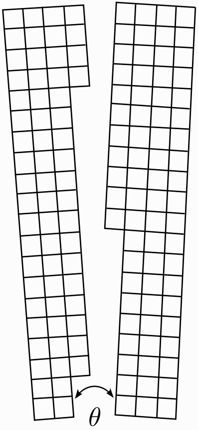

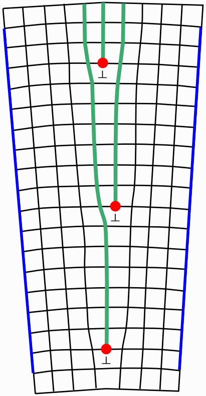

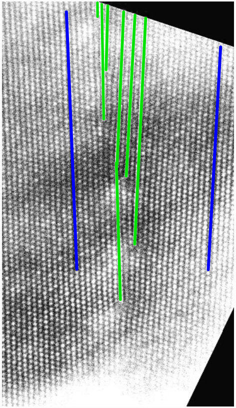

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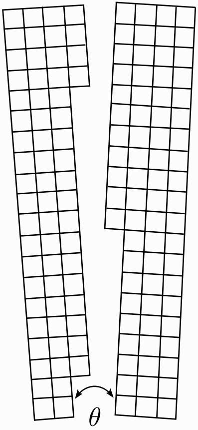
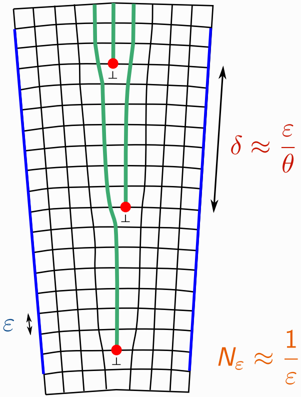
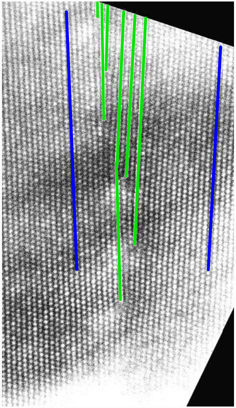
Boundary structure: periodic array of edge dislocations with spacing $\boldsymbol{\delta = \frac{\varepsilon}{\theta}}$.







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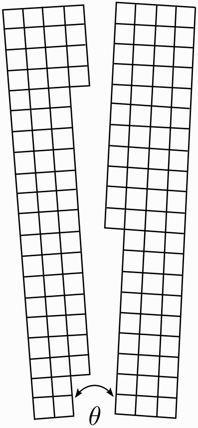

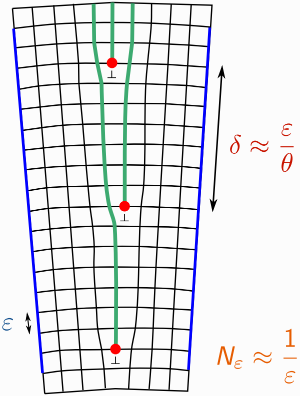

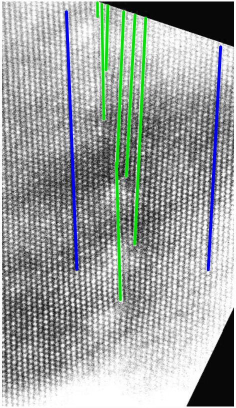

Porter, Easterling. CRC Press (2009) - Gottstein. Springer (2013)

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Setting: consider a 2D system of N_ε edge dislocations, where $\varepsilon > 0$ is the lattice spacing and

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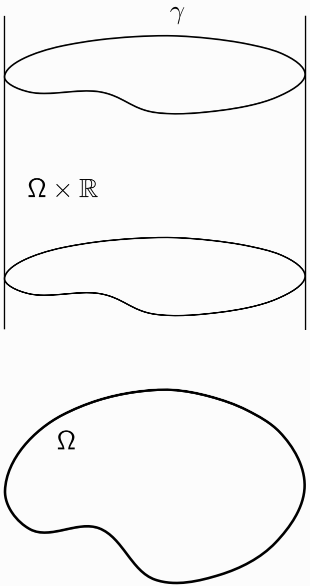
Linearised polycrystals: our energy regime will imply

$$N_\varepsilon \ll \frac{1}{\varepsilon}$$

$\implies$ we have less dislocations than tilt grain boundaries. However we still obtain polycrystalline minimisers, but with grains rotated by an infinitesimal angle $\theta \approx 0$.

Setting (linearised planar elasticity)

Reference configuration: $\Omega \subset \mathbb{R}^2$ open bounded.



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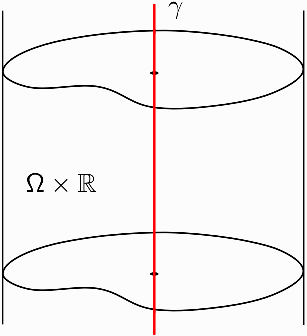

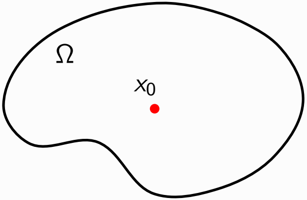

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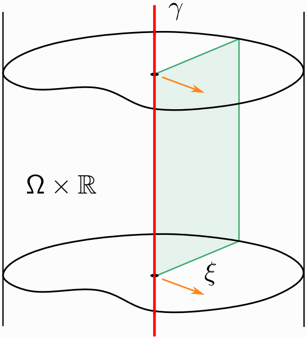

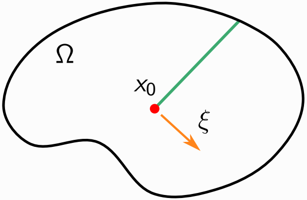

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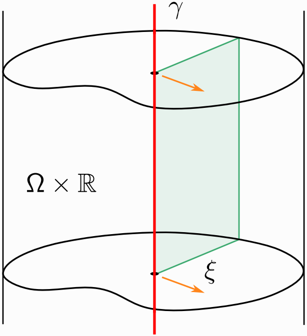

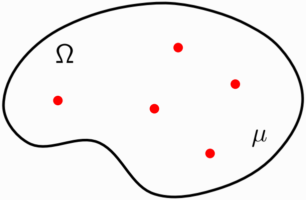

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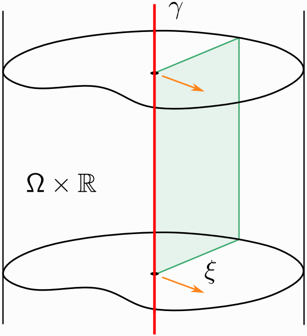

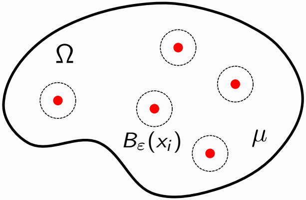

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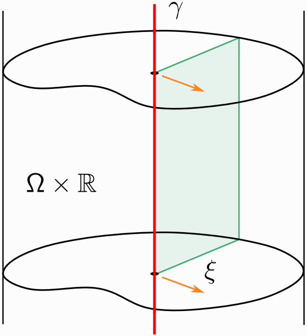
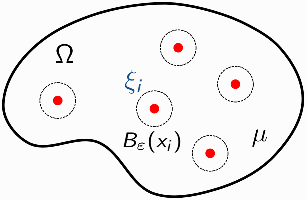
$$\operatorname{Curl} \beta \llcorner \Omega_\varepsilon(\mu) = 0 , \quad \int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \xi_i .$$





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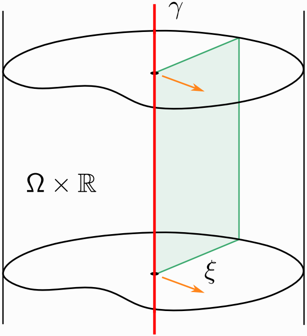

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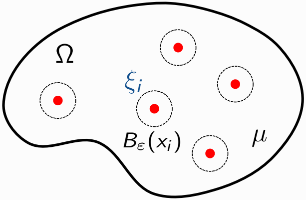
$$\operatorname{Curl} \beta \llcorner \Omega_\varepsilon(\mu) = 0 , \quad \int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \xi_i .$$



Linearised Energy: $\mathbb{C}F : F \sim |F^{\mathrm{sym}}|^2$, then

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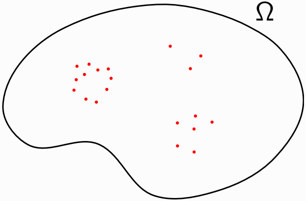

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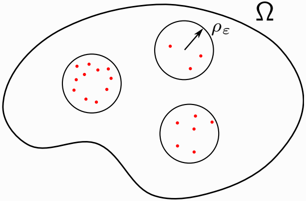

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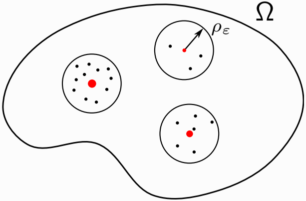

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with $\mathbb{S} := \mathrm{Span}_{\mathbb{Z}} \, \mathcal{S}$ set of multiple Burgers vectors, and

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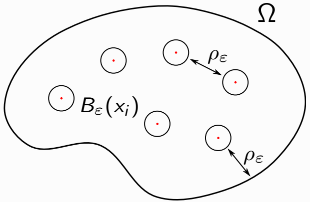

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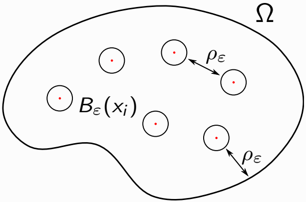

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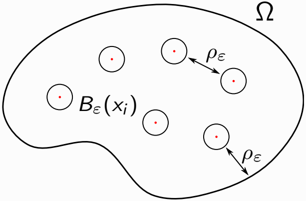

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Define $\varphi \colon \mathbb{R}^2 \to [0, \infty)$ as the relaxation of ψ (splitting multiple dislocations)

$$\varphi(\xi) := \min \left\{ \sum_{i=1}^M \lambda_i \psi(\xi_i) : \, \xi = \sum_{i=1}^M \lambda_i \xi_i, \, M \in \mathbb{N}, \, \lambda_i \geq 0, \, \xi_i \in \mathbb{S} \right\} .$$

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Compactness: *consider* $(\mu_\varepsilon, \beta_\varepsilon)$ *s.t.* "$\mathrm{Curl}\, \beta_\varepsilon = \mu_\varepsilon$" *and* $\mathcal{F}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \leq C \implies$

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- In the critical regime $N_\varepsilon \approx |\log \varepsilon|$ we have $S \approx A$ and $\operatorname{Curl}(S + A) = \mu$.

Compactness of the measures

Let $\mu_n := \sum_{i=1}^{M_n} \xi_{n,i} \delta_{x_{n,i}}$ and "$\operatorname{Curl} \beta_n = \mu_n$". We show that

$$\frac{1}{N_{\varepsilon_n}} |\mu_n|(\Omega) = \frac{1}{N_{\varepsilon_n}} \sum_{i=1}^{M_n} |\xi_{n,i}| \leq C \,, \tag{1.2}$$

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Dirichlet type BC: at level $\varepsilon > 0$ fix a boundary condition $g_\varepsilon : \Omega \to \mathbb{M}^{2\times 2}$ s.t.

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$$\mathcal{F}_{\mathrm{BC}}(\mu, S, A) := \int_\Omega \mathbb{C}S : S \, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu| + \int_{\partial\Omega} \varphi((g_A - A)\cdot t)\, ds \,,$$

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Minimising $\mathcal{F}_{\mathrm{BC}}$ with piecewise constant BC

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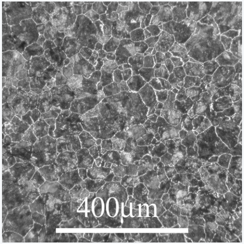
Polycrystals as energy minimisers

Theorem (F., Palombaro, Ponsiglione '17)

Given a piecewise constant boundary condition g_A, there exists a piecewise constant minimiser of $\mathcal{F}_{\mathrm{BC}}(\mu, 0, A)$

$$A = \sum_{k=1}^{M} A_k \chi_{E_k} ,$$

with $A_k \in \mathbb{M}^{2\times 2}_{\mathrm{skew}}$ and $\{E_k\}_{k=1}^{M}$ Caccioppoli partition of Ω. We interpret A as a linearised polycrystal.



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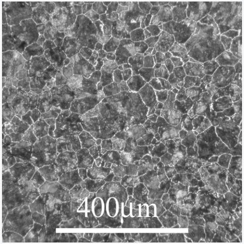

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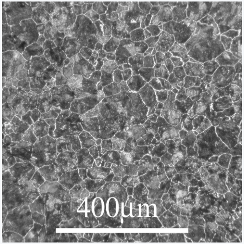

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Essential: that the boundary condition is piecewise affine on the whole $\partial\Omega$.

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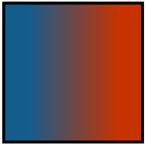
0  1

Idea of the proof

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Proof: let $\tilde{u}$ be a minimiser for (1.3). By anisotropic Coarea Formula

$$\int_\Omega \varphi \left(\frac{dD\tilde{u}}{d|D\tilde{u}|} \right) d|D\tilde{u}| = \int_{\mathbb{R}} Per_\varphi(\{x \in \Omega : \ \tilde{u}(x) > t\}) \, dt ,$$

we can select the levels with minimal perimeter. This defines the Caccioppoli partition.

Comparison with classical Read-Shockley formula

Read-Shockley formula: Elastic energy$= E_0\theta(1+|\log\theta|)$.

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- Supercritical regime analysis starting from a non-linear energy?
  Müller, Scardia, Zeppieri. *Geometric rigidity for incompatible fields and an application to strain-gradient plasticity.* Indiana University Mathematics Journal (2014).

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   - Dislocations
   - Semi-coherent interfaces
   - Linearised polycrystals

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   - Critical lower integrability
   - Convex integration
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2. **Microgeometries in Composites**
   - Critical lower integrability
   - Convex integration
   - Proof of our main result

Gradient integrability for solutions to elliptic equations

$\Omega \subset \mathbb{R}^2$ bounded open domain. A map $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2})$ is **uniformly elliptic** if

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Problem

Study the gradient integrability of distributional solutions $u \in W^{1,1}(\Omega)$ to

$$\operatorname{div}(\sigma \nabla u) = 0 , \tag{2.1}$$

when

$$\sigma = \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2} ,$$

with $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$ constant elliptic matrices, $\{E_1, E_2\}$ measurable partition of Ω.

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Application to composites:

- Ω is a section of a composite conductor obtained by mixing two materials with conductivities σ_1 and σ_2,
- the electric field ∇u solves (2.1),
- concentration of ∇u in relation to the geometry $\{E_1, E_2\}$.

Astala's Theorem

Theorem (Astala '94)

Let $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2})$ *be uniformly elliptic. There exists exponents* $1 < q < 2 < p$ *such that if* $u \in W^{1,q}(\Omega)$ *solves*

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Question

Are the exponents q *and* p *optimal among two-phase elliptic conductivities*

$$\sigma = \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2}\,?$$

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Astala's exponents for two-phase conductivities

For two-phase conductivities Astala's exponents $q = q_{\sigma_1,\sigma_2}$ and $p = p_{\sigma_1,\sigma_2}$ have been characterised.

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Remark: it is sufficient to prove optimality in the case

$$\sigma_1 = \begin{pmatrix} 1/K & 0 \\ 0 & 1/S_1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} K & 0 \\ 0 & S_2 \end{pmatrix},$$

where

$$K > 1 \qquad \text{and} \qquad \frac{1}{K} \leq S_j \leq K\,, \quad j = 1, 2\,.$$

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Upper exponent optimality

Theorem (Nesi, Palombaro, Ponsiglione '14)

Let $\sigma_1 = \mathrm{diag}(1/K, 1/S_1), \sigma_2 = \mathrm{diag}(K, S_2)$ *with* $K > 1$ *and* $S_1, S_2 \in [1/K, K]$.

(i) *If* $\sigma \in L^\infty(\Omega; \{\sigma_1, \sigma_2\})$ *and* $u \in W^{1,\frac{2K}{K+1}}(\Omega)$ *solves*

$$\mathrm{div}(\sigma \nabla u) = 0 \tag{2.2}$$

then $\nabla u \in L^{\frac{2K}{K-1}}_{\mathrm{weak}}(\Omega; \mathbb{R}^2)$.

(ii) *There exists* $\bar\sigma \in L^\infty(\Omega; \{\sigma_1, \sigma_2\})$ *and a weak solution* $\bar u \in W^{1,2}(\Omega)$ *to* (2.2) *with* $\sigma = \bar\sigma$, *satisfying affine boundary conditions and such that* $\nabla \bar u \notin L^{\frac{2K}{K-1}}(\Omega; \mathbb{R}^2)$.

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Question we address

Is the lower exponent $\frac{2K}{K+1}$ optimal?

Lower exponent optimality

$$1 \qquad p_n \longrightarrow \frac{2K}{K+1} \qquad 2 \qquad \frac{2K}{K-1}$$

Theorem (F., Palombaro '17)

Let $\sigma_1 = \mathrm{diag}(1/K, 1/S_1), \sigma_2 = \mathrm{diag}(K, S_2)$ *with* $K > 1$ *and* $S_1, S_2 \in [1/K, K]$. *There exist*

- *coefficients* $\sigma_n \in L^\infty(\Omega; \{\sigma_1; \sigma_2\})$,
- *exponents* $p_n \in \left[1, \frac{2K}{K+1}\right]$,
- *functions* $u_n \in W^{1,1}(\Omega)$ *such that* $u_n(x) = x_1$ *on* $\partial\Omega$,

such that

$$\mathrm{div}(\sigma_n \nabla u_n) = 0\,,$$

$$\nabla u_n \in L^{p_n}_{\mathrm{weak}}(\Omega; \mathbb{R}^2), \quad p_n \to \frac{2K}{K+1}, \quad \nabla u_n \notin L^{\frac{2K}{K+1}}(\Omega; \mathbb{R}^2)\,.$$

F., Palombaro. Calculus of Variations and Partial Differential Equations (2017)

Presentation Plan

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   - Dislocations
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Solving differential inclusions

Theorem (Approximate solutions for two phases)

Let $A, B \in \mathbb{M}^{2\times 2}$, $C := \lambda A + (1-\lambda)B$ *with* $\lambda \in [0,1]$, *and* $\delta > 0$. *Assume that*

$$B - A = a \otimes n \quad \text{for some} \quad a \in \mathbb{R}^2, n \in S^1 . \qquad \text{(Rank-one connection)}$$

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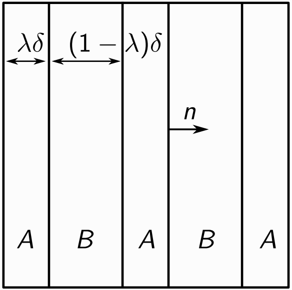

Müller. *Variational models for microstructure and phase transitions.*

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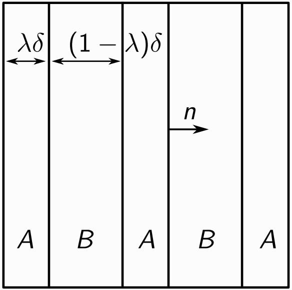

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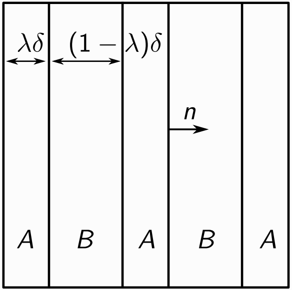

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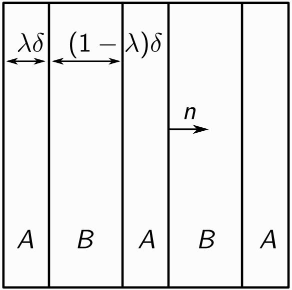

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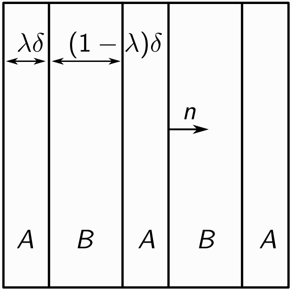
- this allows to recover boundary data C.



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$$\nu_\delta := (\nabla f_\delta)_\#(\mathcal{L}^2_\Omega) \overset{*}{\rightharpoonup} \nu := \lambda\delta_A + (1-\lambda)\delta_B \quad \text{in} \quad \mathcal{M}(\mathbb{M}^{2\times 2}).$$

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$$\frac{1}{|\Omega|}\int_\Omega |\nabla f_\delta|^p \, dx = \int_{\mathbb{M}^{2\times 2}} |M|^p \, d\nu_\delta(M).$$

Iterating the Proposition

Let $C = \lambda A + (1-\lambda)B$ with $\lambda \in [0,1]$ and $\operatorname{rank}(B-A) = 1$. Let $f : \Omega \to \mathbb{R}^2$ such that $f(x) = Cx$ on $\partial\Omega$,

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Let $\nu = \sum_{i=1}^{N} \lambda_i \delta_{A_i}$ *be a laminate of finite order, s.t.*

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Fix $\delta > 0$. $\exists$ *a* **piecewise affine Lipschitz** *map* $f : \Omega \to \mathbb{R}^2$ *s.t.* $\nabla f \sim \nu$, *that is,*

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Laminates of finite order: laminates obtained iteratively through the splitting procedure in the previous slide.

Proposition (Convex integration)

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Presentation Plan

1. **Geometric Patterns of Dislocations**
   - Dislocations
   - Semi-coherent interfaces
   - Linearised polycrystals

2. **Microgeometries in Composites**
   - Critical lower integrability
   - Convex integration
   - Proof of our main result

Strategy of the Proof

Strategy: explicit construction of u_n by **convex integration methods**.

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1. Rewrite the equation $\operatorname{div}(\sigma \nabla u) = 0$ as a differential inclusion

$$\nabla f(x) \in T\,, \qquad \text{for a.e.} \quad x \in \Omega \tag{2.3}$$

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In this way f solves (2.3) and
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These methods were developed for isotropic conductivities $\sigma \in L^\infty(\Omega; \{KI, \frac{1}{K}I\})$.
The adaptation to our case is non-trivial because of the lack of symmetry of the target set T, due to the anisotropy of σ_1 and σ_2.

Astala, Faraco, Székelyhidi. *Convex integration and the L^p theory of elliptic equations.*
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2008)

Rewriting the PDE as a differential inclusion

Let $K > 1$, $S_1, S_2 \in [1/K, K]$ and define

$$\sigma_1 := \operatorname{diag}(1/K, 1/S_1)\,, \quad \sigma_2 := \operatorname{diag}(K, S_2)\,, \qquad \sigma := \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2}\,,$$

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Targets in conformal coordinates

Conformal coordinates: Let $A \in \mathbb{M}^{2\times 2}$. Then $A = (a_+, a_-)$ for $a_+, a_- \in \mathbb{C}$, defined by

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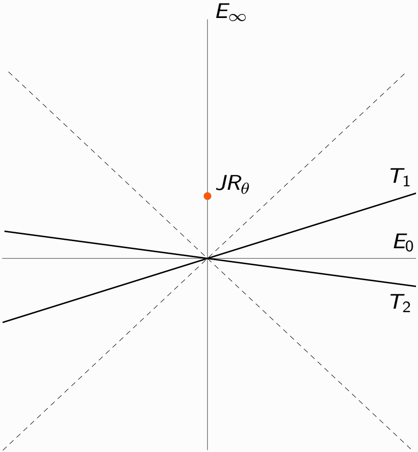

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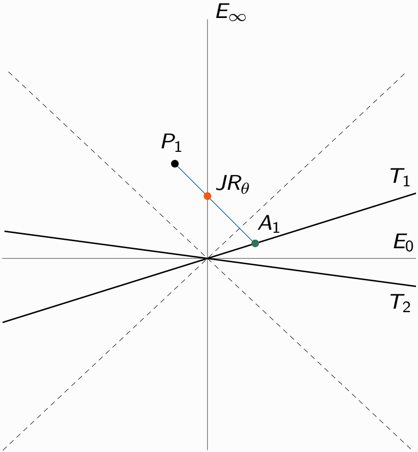

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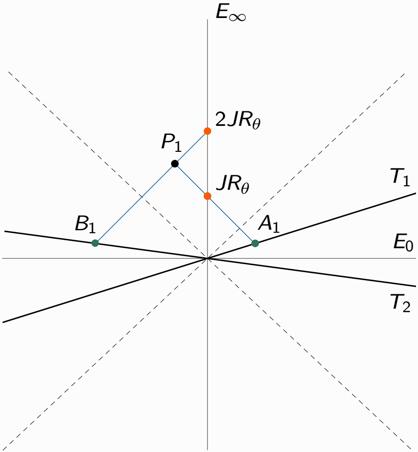

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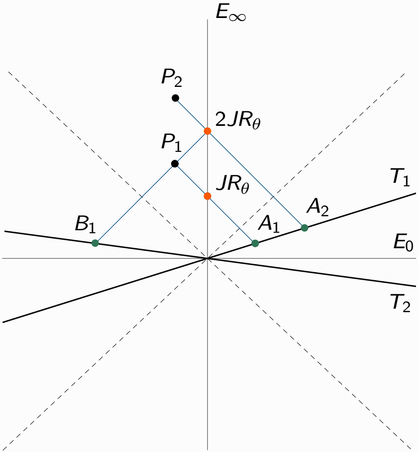

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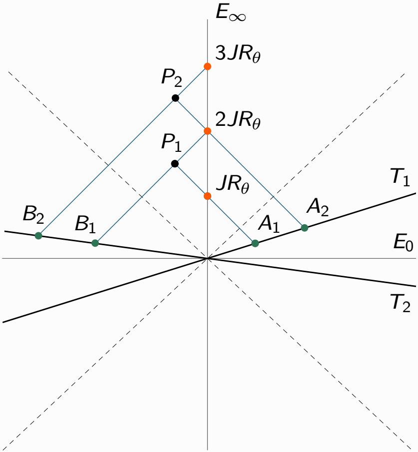

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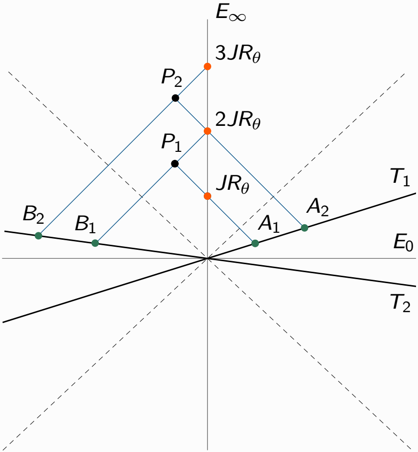

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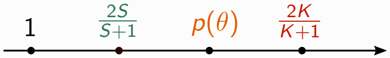

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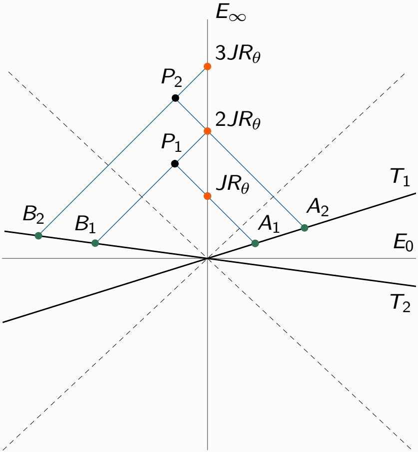

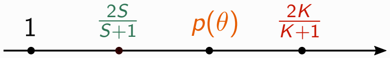

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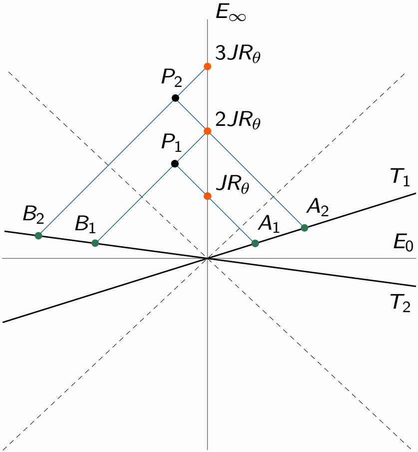

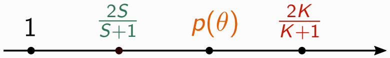

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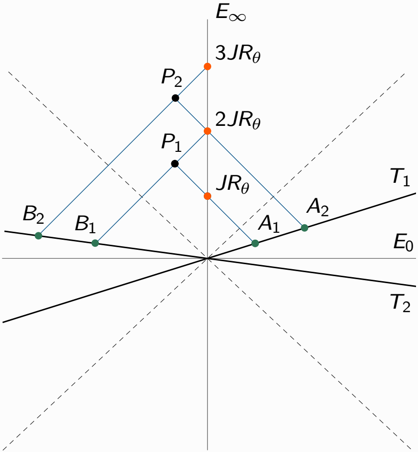

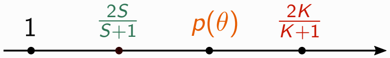

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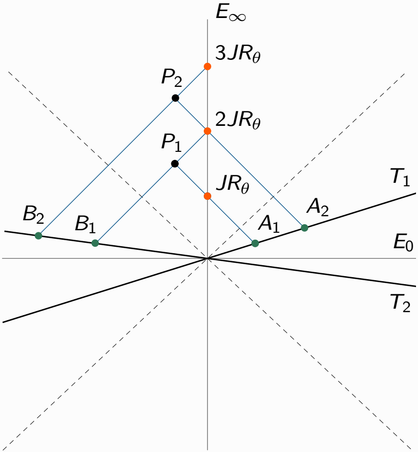

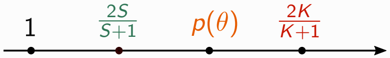

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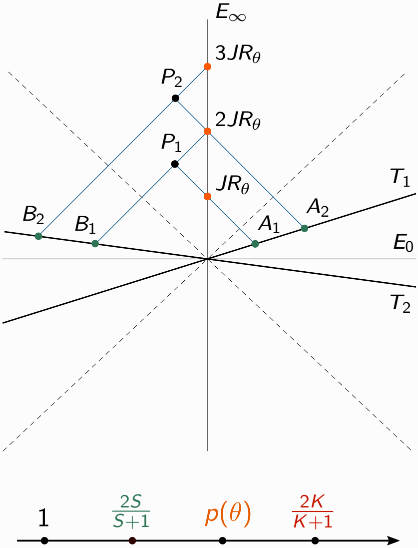

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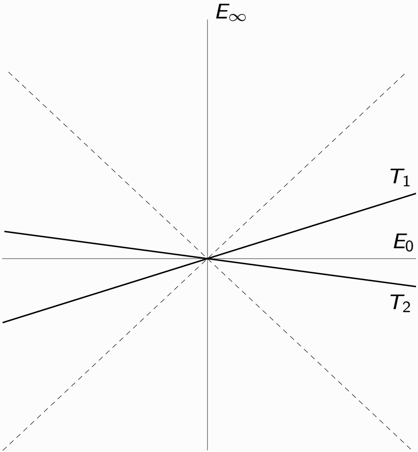
Remark: barycentre J gives the right growth.

Constructing approximate solutions

We want to construct $f\colon \Omega \to \mathbb{R}^2$ such that



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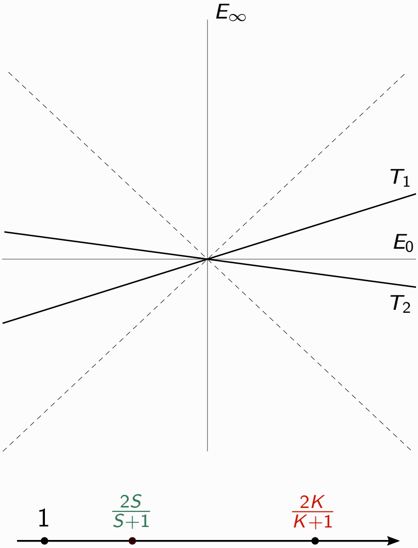

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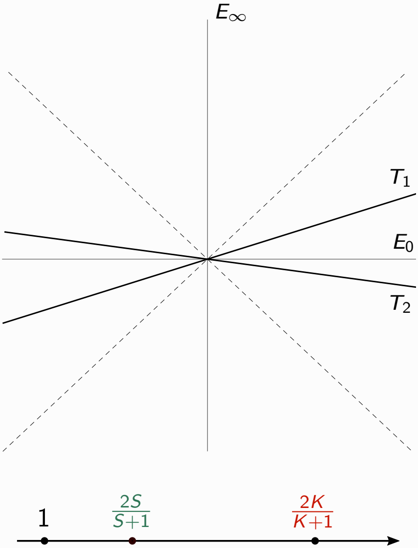

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We want to construct $f\colon \Omega \to \mathbb{R}^2$ such that

- $\operatorname{dist}(\nabla f, T_1 \cup T_2) < \varepsilon$ a.e. in Ω,
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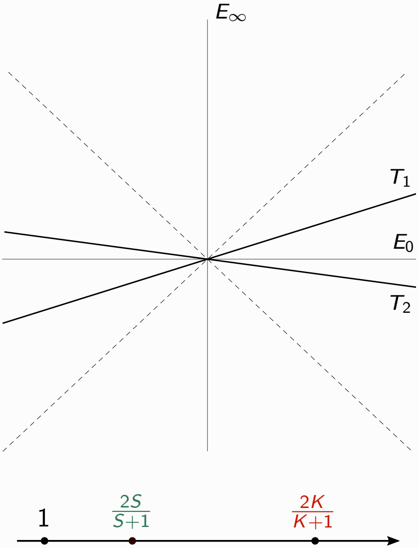

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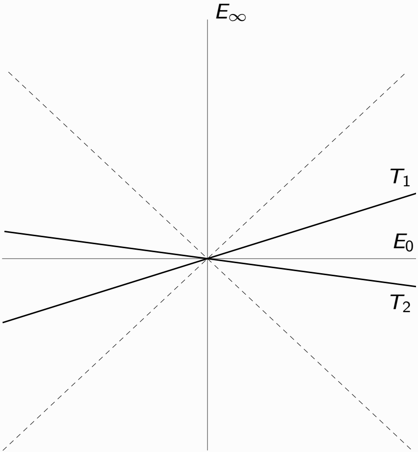

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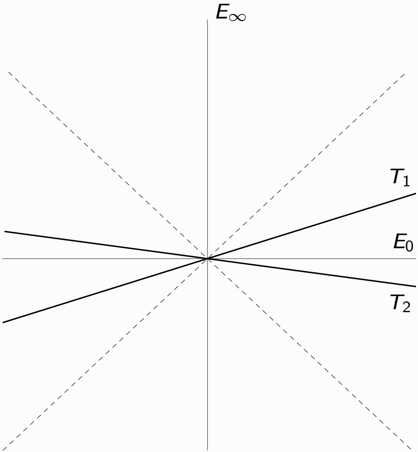
Idea: alternate one step of the staircase laminate with the convex integration Proposition.



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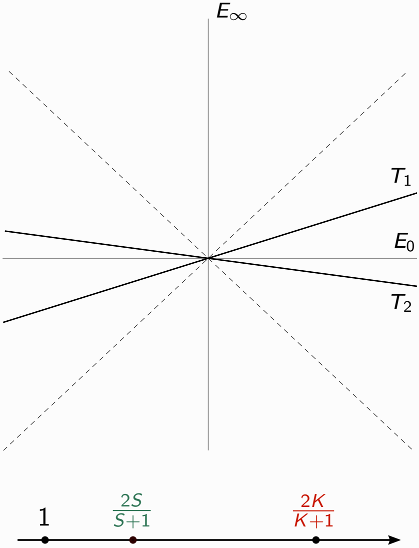

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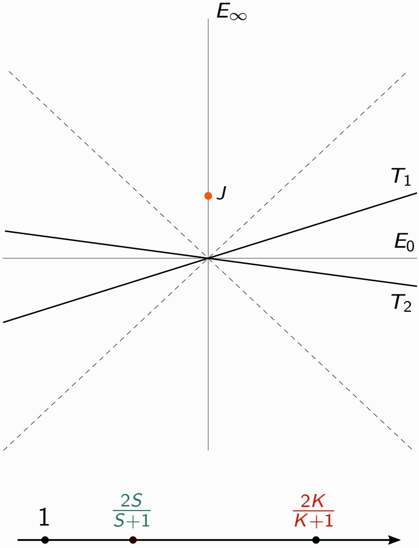

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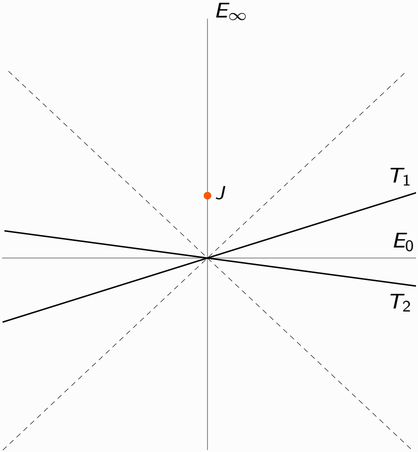

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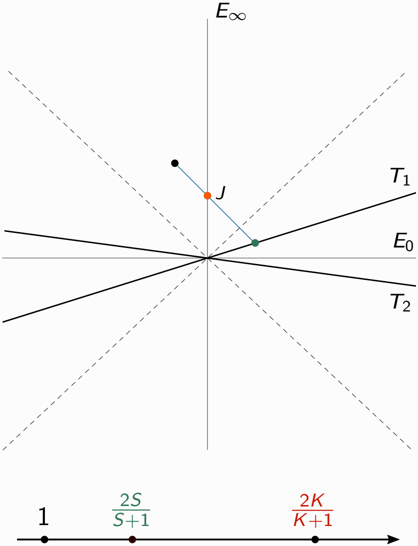

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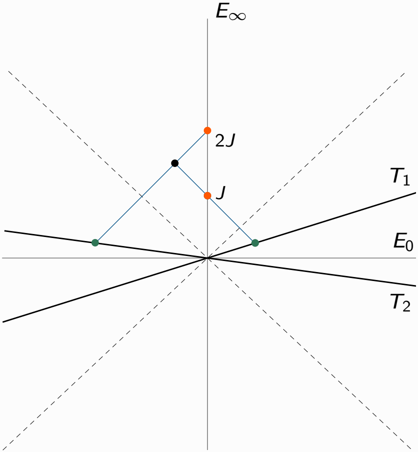

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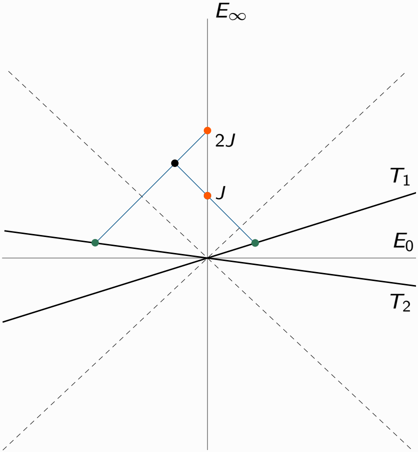

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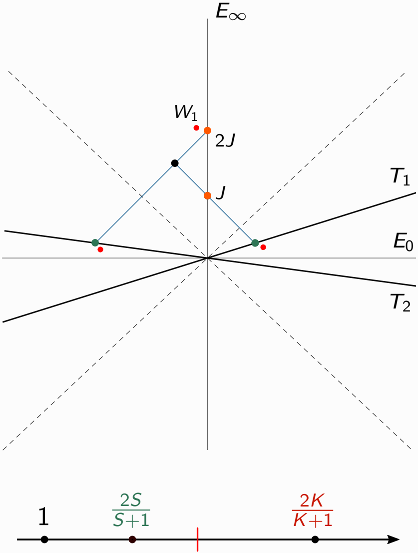

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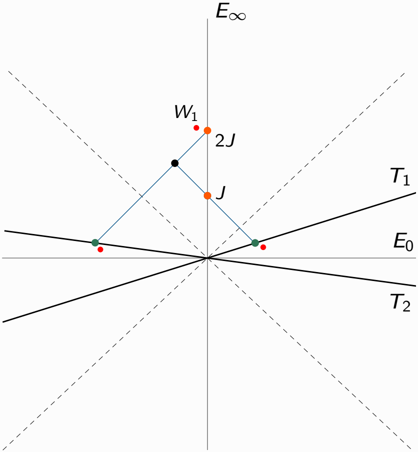

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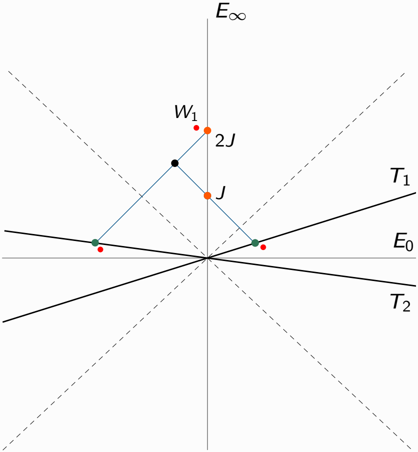

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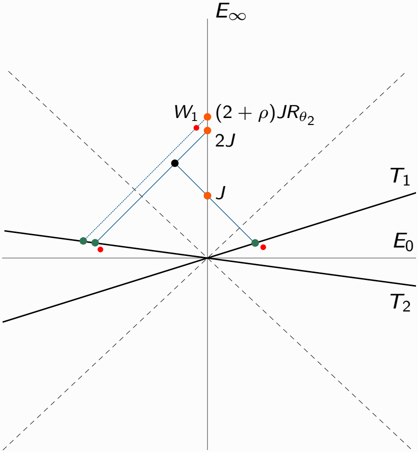

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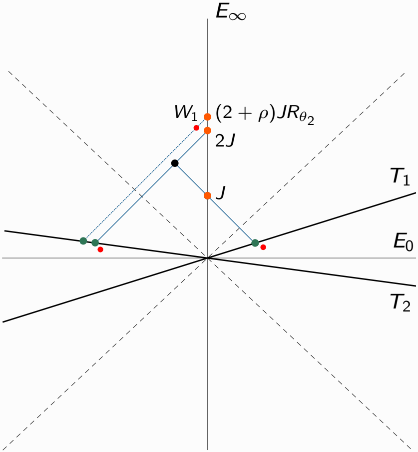

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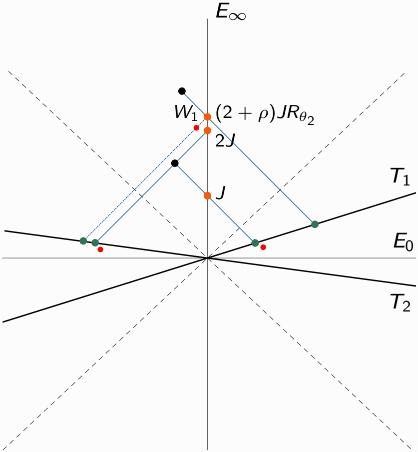

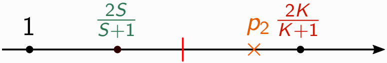

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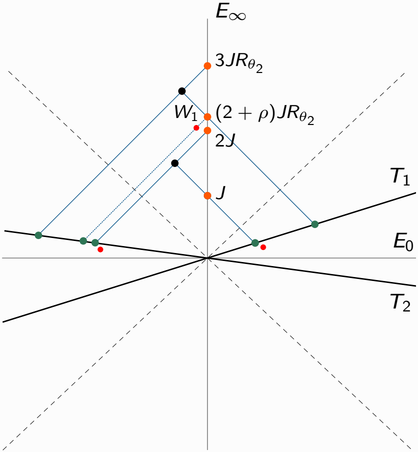

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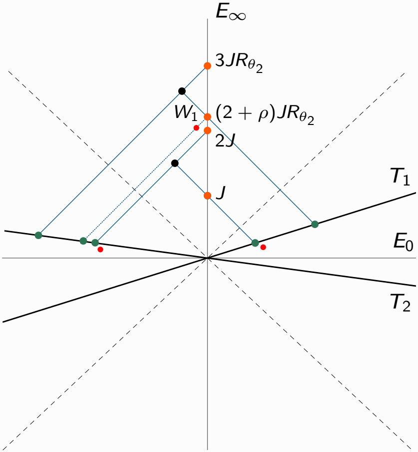

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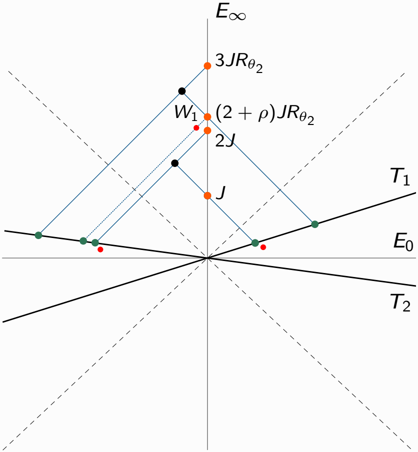

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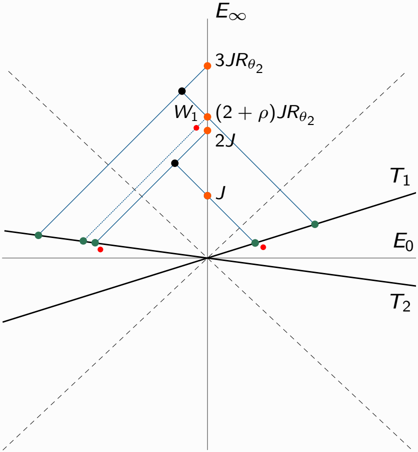

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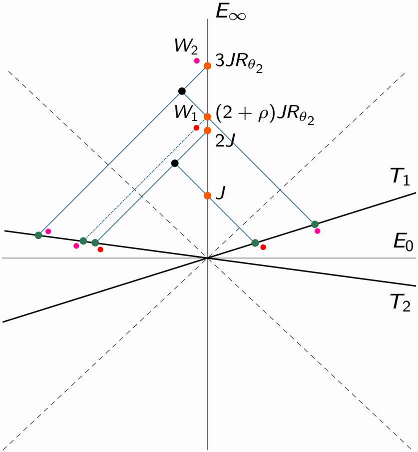

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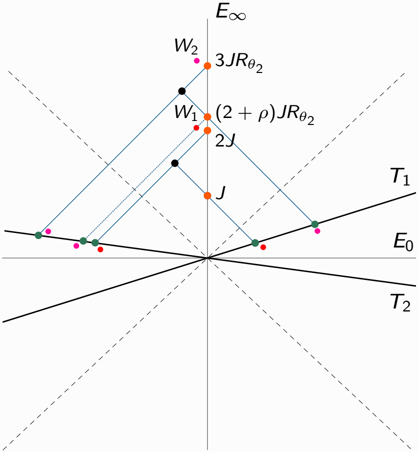

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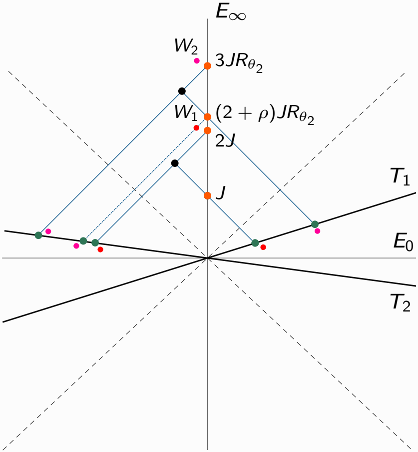

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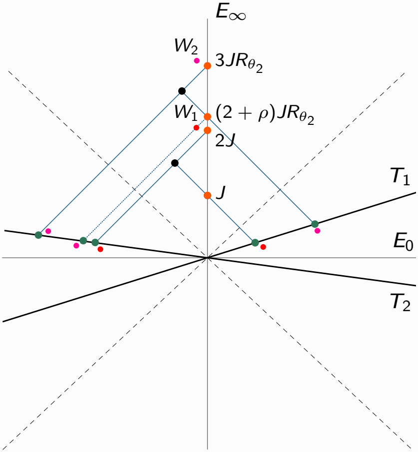
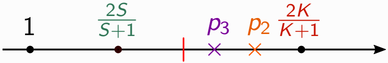




Constructing approximate solutions

Recall $I_\delta := \left(\frac{2K}{K+1} - \delta, \frac{2K}{K+1}\right]$.

Step A. Define $f_1(x) := Jx \implies \theta_1 = 0, p_1 = \frac{2K}{K+1}$

Step B. Laminate ν_1 from J to $2J \rightsquigarrow$ growth p_1

Step C. Proposition $\implies \exists$ map f_2 s.t. $f_2 = Jx$ on $\partial\Omega$ and $\nabla f_2 \sim \operatorname{supp} \nu_1 \implies \nabla f_2$ grows like p_1

This determines the exponent range I_δ

Step 1. One step of the staircase

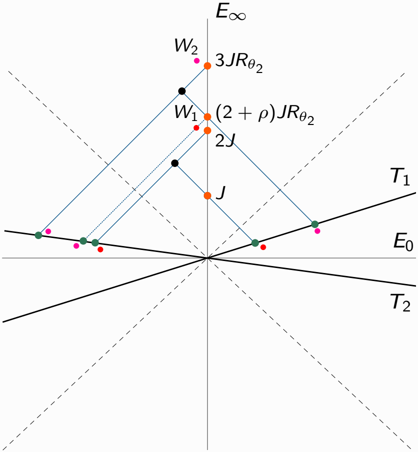
- Split W_1. Since $W_1 \sim 2J \implies$ point $(2+\rho)JR_{\theta_2}$ with θ_2, ρ small. $\implies p_2 \in I_\delta$
- Climb from $(2+\rho)JR_{\theta_2}$ to $3JR_{\theta_2}$
- $\rightsquigarrow$ Laminate ν_2 with $\overline{\nu}_2 = W_1$ and growth p_2

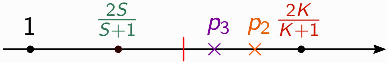
Step 2. Define map f_3 by modifying f_2

- Proposition $\implies \exists$ map g s.t. $g = W_1 x$ on $\partial\Omega$ and $\nabla g \sim \operatorname{supp} \nu_2 \implies \nabla g$ grows like p_2
- Set $f_3 := g$ in the set $\{\nabla f_2 \sim W_1\}$ and $f_3 := f_2$ otherwise $\implies \nabla f_3$ grows like p_2

Step 1. Split $W_2 \rightsquigarrow$ Laminate ν_3 with growth $p_3 \in I_\delta$

Iterating: $\rightsquigarrow f_n$ obtained by successive modifications on nested sets going to zero in measure $\implies f_n \to f$





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$$\operatorname{div}(\sigma \nabla u) = 0 \,, \quad \text{in } \Omega \,, \tag{2.4}$$

when $\sigma \in \{\sigma_1, \sigma_2\}$ for $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$ elliptic.

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- Stronger result for lower critical exponent: showing $\exists\ u \in W^{1,1}(\Omega)$ solution to (2.4) and s.t. $\nabla u \in L^{\frac{2K}{K+1}}_{\mathrm{weak}}(\Omega; \mathbb{R}^2)$ but $\nabla u \notin L^{\frac{2K}{K+1}}(B; \mathbb{R}^2)$, $\forall$ ball $B \subset \Omega$.
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- Extend these results to three-phase conductivities $\sigma \in \{\sigma_1, \sigma_2, \sigma_3\}$.
- Dimension $d \geq 3$? Even only in the isotropic case $\sigma \in \{KI, K^{-1}I\}$ for $K > 1$.
  Main difficulty: Astala's Theorem is missing in higher dimensions.