MAT.495UB / SS 2021
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Institute for Mathematics
and Scientific Computing

Calculus of Variations

Problem Sheet 1

Due date: 12.03.2021,8am

Problem 1.1 (30 pts). Let (X, d) be a metric space.

a) Let $\{x_n\} \subset X$ be a sequence such that $x_n \to x$ as $n \to \infty$, for some $x \in X$. Show that $\{x_n\}$ is a Cauchy sequence.

b) Suppose in addition that X is a real vector space and that the distance d satisfies:

   i) $d(x + a, y + a) = d(x, y)$ for all $a, x, y \in X$ (translation invariance),
   ii) $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x, y \in X$, $\lambda \in \mathbb{R}$ (one-homogeneity).

   Prove that $\|x\| := d(x, 0)$ defines a norm over X.

c) Define a metric on a real vector space X which does not satisfy either (i) and/or (ii).

Problem 1.2 (15 pts). Let X be a real vector space, such that $X \neq \{0\}$. Show that there exists at least one norm on X.

Hint: every real vector space X has an algebraic basis, that is, there exists $B = \{e_i,\, i \in I\} \subset X$ such that every $x \in X$ with $x \neq 0$ can be uniquely written as $x = \sum_{j=1}^{n} \lambda_{i_j} e_{i_j}$ for some $n \in \mathbb{N}$, $\lambda_{i_j} \in \mathbb{R} \setminus \{0\}$ and $i_j \in I$ pairwise distinct for $j = 1, \dots, n$. Use this fact to define a norm over X.

Remember: For a metric space (X, d) the collection of sets

$$\tau := \{A \subset X : \forall x \in X, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subset A\}$$

is called the topology induced by d over X, where $B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}$. The sets $A \in \tau$ are called open. A set $C \subset X$ is closed if $C^c := X \setminus C$ is open.

Problem 1.3 (30 pts). The aim of this exercise is to show the difference between metrics and norms.

a) Let X be a real vector space. Suppose that $\|\cdot\|_1, \|\cdot\|_2 : X \to \mathbb{R}$ are norms on X which induce the same topology τ. Prove that $(X, \|\cdot\|_1)$ is complete if and only if $(X, \|\cdot\|_2)$ is complete.

   Hint: Consider the identity map $I\colon (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$. Is it a linear and bounded operator?

b) Let $d_1, d_2 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that $d_1(x, y) := |x - y|$, $d_2(x, y) := |\varphi(x) - \varphi(y)|$, with $\varphi(x) := x/(1 + |x|)$. We know that $(\mathbb{R}, d_1)$ is a complete metric space. Prove that:

   i) d_2 is a metric over $\mathbb{R}$;
   ii) d_1 and d_2 induce the same topology τ over $\mathbb{R}$;
   iii) $(\mathbb{R}, d_2)$ is not complete.
   iv) Does there exist a norm $\|\cdot\|_2$ on $\mathbb{R}$ such that $\|x - y\|_2 = d_2(x, y)$ for all $x, y \in \mathbb{R}$?

Problem 1.4 (25 pts). Let (X, d_X), (Y, d_Y) be metric spaces and denote by τ_X, τ_Y the respective induced topologies. Recall that a map $F \colon X \to Y$ is continuous if $F^{-1}(A) \in \tau_X$ for all $A \in \tau_Y$, where $F^{-1}(A) := \{x \in X : \exists\, y \in A \text{ s.t. } F(x) = y\}$. Show that they are equivalent:

i) F is continuous,

ii) For all $x_0 \in X$ it holds: for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(F(x), F(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$ (here δ depends on x_0),

iii) F is sequentially continuous, that is, for all $x_0 \in X$ and $\{x_n\} \subset X$ such that $d_X(x_n, x_0) \to 0$, it holds $d_Y(F(x_n), F(x_0)) \to 0$.

Hint: It may be easier to show that (i) is equivalent to (ii), and (ii) is equivalent to (iii).

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Problem Sheet 2

Due date: 26.03.2021

Problem 2.1 (15 pts). Let (X, d) be a metric space.

a) Let $C \subset X$. Show that C is closed if and only if

$$\{x_n\} \subset C,\, x_n \to x_0 \text{ implies } x_0 \in C\,.$$

b) Let $A \subset X$. Recall that the closure of A is defined by $\overline{A} := \cap\{C \subset X : A \subset C,\ C \text{ closed }\}$. Define the set $L := \{x \in X : \exists\, \{x_n\} \subset A \text{ s.t. } x_n \to x\}$. Prove that $\overline{A} = L$.

Hint: Show that L is closed by means of a diagonal argument.

Problem 2.2 (15 pts). Let (X, d_X), (Y, d_Y) be metric spaces and assume that X is compact.

a) Let $\{x_n\} \subset X$ be a sequence having the property that every convergent subsequence converges to the same point x_0. Prove that $x_n \to x_0$ as $n \to \infty$.

b) Suppose that $F\colon X \to Y$ is continuous. Show that $F(X)$ is compact in Y.

c) Suppose that $F\colon X \to Y$ is continuous. Show that F is also uniformly continuous, that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(F(x_1), F(x_2)) < \varepsilon$, for all $x_1, x_2 \in X$ satisfying $d_X(x_1, x_2) < \delta$.

Problem 2.3 (15 pts). Let (X, d) be a compact metric space and $F\colon X \to \mathbb{R}$ be lower semicontinuous. Prove that:

a) F is bounded from below, that is, there exists $m \in \mathbb{R}$ such that $F(x) \geq m$ for all $x \in X$,

b) F admits minimum, that is, there exists $\hat{x} \in X$ such that $F(\hat{x}) = \inf_{x \in X} F(x)$.

Hint: Consider a minimizing sequence, i.e., $\{x_n\} \subset X$ such that $F(x_n) \to \inf_{X \in X} F(x)$ (by definition of infimum it always exists). What happens to x_n to the limit?

Problem 2.4 (25 pts). Let X be a real normed space, and $\Lambda_n, \Lambda \in X^*$, $x_n, x \in X$ for $n \in \mathbb{N}$.

a) Show the following implications between convergences in X^*:

   i) if $\Lambda_n \to \Lambda$ strongly then $\Lambda_n \rightharpoonup \Lambda$ weakly,

   ii) if $\Lambda_n \rightharpoonup \Lambda$ weakly then $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly*.

   Hint: Use the canonical embedding $J\colon X \to X^{**}$.

b) Assume in addition that X is reflexive. Prove that in X^* we have that $\Lambda_n \rightharpoonup \Lambda$ weakly if and only if $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly*.

Hint: In this case, by definition, the canonical embedding is surjective.

c) Prove that weak* limits in X^* and weak limits in X are unique.

Hint: Use one of the corollaries of the Hahn-Banach Theorem (Conway III.6.8, Pag 79).

d) Assume that $x_n \rightharpoonup x$ weakly in X. Show that x_n is norm bounded, that is, $\sup_n \|x_n\|_X < \infty$.

Hint: Use the canonical embedding and the Principle of Uniform Boundedness (Conway III.14.1, Pag 95).

e) Assume in addition that X is a Banach space. Show that if $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ then $\sup_n \|\Lambda_n\|_{X^*} < \infty$.

Hint: Use the Principle of Uniform Boundedness.

Problem 2.5 (30 pts). Let H be a real Hilbert space, $x_n, x, y_n, y \in H$ for $n \in \mathbb{N}$.

a) Show that $x_n \to x$ strongly in H if and only if

$$x_n \rightharpoonup x \quad \text{weakly in} \quad H \quad \text{and} \quad \|x_n\| \to \|x\| \, .$$

b) Prove that the norm is weakly lower semicontinuous, that is, $x_n \rightharpoonup x$ implies

$$\|x\| \leq \liminf_{n \to +\infty} \|x_n\| \, .$$

c) Assume that $x_n \to x$ and $y_n \rightharpoonup y$. Prove that

$$\lim_{n \to +\infty} \langle x_n, y_n \rangle = \langle x, y \rangle \, .$$

d) Let $W \subset H$ be such that $\operatorname{span}(W)$ is dense in H with respect to the induced norm. Suppose that $\sup_n \|x_n\| \leq M$ for all $n \in \mathbb{N}$ and that there exists $x \in H$ such that

$$\lim_{n \to +\infty} \langle x_n, w \rangle = \langle x, w \rangle \qquad \text{for all} \quad w \in W \, . \tag{1}$$

Prove that $x_n \rightharpoonup x$.

e) Find a counterexample to prove that the boundedness assumption for the sequence $\{x_n\}$ in point (d) is necessary to have weak convergence, i.e., construct a sequence $\{x_n\}$ such that $\sup_n \|x_n\| = +\infty$, and that (1) holds for some $x \in H$, but $x_n \not\rightharpoonup x$.

Hint: Assume that H is separable and take $W = \{e_n\}$ basis. With it, construct a sequence satisfying the required properties.

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Problem Sheet 3

Due date: 23.04.2021

Throughout the exercise paper whenever we say differentiable, we mean Fréchet differentiable.

Problem 3.1 (15 pts). In this exercise we show that the Fréchet derivative is linear and that the chain rule and product rule hold.

a) (Linearity) Let X, Y be normed spaces, $U \subset X$ open, $\alpha \in \mathbb{R}$, and $F, G \colon U \to Y$ be differentiable at $u_0 \in U$. Prove that $\alpha F + G$ is differentiable at u_0 with $(\alpha F + G)'(u_0) \in \mathcal{L}(X, Y)$ given by

$$(\alpha F + G)'(u_0) = \alpha F'(u_0) + G'(u_0)\,.$$

b) (Chain Rule) Let X, Y, Z be normed spaces, $U \subset X$, $V \subset Y$ open sets. Let $F \colon U \to V$, $G \colon V \to Z$. Assume that F and G are differentiable at $u_0 \in U$ and at $F(u_0)$, respectively. Then $G \circ F \colon U \to Z$ is differentiable at u_0, with $(G \circ F)'(u_0) \in \mathcal{L}(X, Z)$ given by the composition of linear continuous operators

$$(G \circ F)'(u_0) = G'(F(u_0)) \circ F'(u_0)\,.$$

c) (Product Rule) Let X be a normed space, $U \subset X$ open and $F, G \colon U \to \mathbb{R}$ be differentiable at $u_0 \in U$. Show that the product function FG is differentiable at u_0 with $(FG)'(u_0) \in X^*$ given by

$$(FG)'(u_0) = G(u_0)F'(u_0) + F(u_0)G'(u_0)\,.$$

Problem 3.2 (15 pts). Let H be a Hilbert space with induced norm $\|\cdot\|$. Define $F, G \colon H \to \mathbb{R}$ by $F(x) := \|x\|^2$, $G(x) := \|x\|$.

a) Show that F is differentiable in H and that $F \in C^1(H)$.

b) Show that G is differentiable for all $x \in H \smallsetminus \{0\}$ but not differentiable at $x = 0$.

Hint: Chain rule for the first part of the statement. By contradiction for the second.

c) Find an example of a normed space X such that $G(x) := \|x\|$ is not differentiable in $X \smallsetminus \{0\}$.

Problem 3.3 (15 pts). Let H be a Hilbert space and $a \colon H \times H \to \mathbb{R}$ be bilinear, symmetric and continuous, that is, there exists $M > 0$ such that $|a(x, y)| \leq M \|x\| \|y\|$ for all $x, y \in H$. Let $b \in H$ and define the map $F \colon H \to \mathbb{R}$ by

$$F(u) := Q(u) + L(u), \quad Q(u) := \frac{1}{2}a(u, u), \quad L(u) := \langle b, u \rangle\,.$$

a) Prove that $F \in C^1(H)$, with derivative

$$F'(u)(v) = a(u, v) + L(v)\,.$$

b) Prove that $F \in C^2(H)$, with

$$F''(u)(v_1, v_2) = a(v_1, v_2)\,.$$

c) Suppose that a is only bilinear and continuous (not symmetric). Compute F', F'' in this case.

Problem 3.4 (15 pts). Let X be a normed space, $U \subset X$ be open. Let $F\colon U \to \mathbb{R}$. We say that $\hat{u} \in U$ is a local minumum for F if there exists a neighbourhood V of $\hat{u}$ such that

$$F(\hat{u}) \leq F(u) \quad \text{for all} \quad u \in V\,.$$

a) Suppose that F is differentiable at $\hat{u}$. Show that

$$F'(\hat{u}) = 0\,.$$

b) Suppose that F is differentiable in U and twice differentiable at $\hat{u}$. Prove that

$$F''(\hat{u})(v,v) \geq 0 \quad \text{for all} \quad v \in X\,.$$

Problem 3.5 (40 pts). Consider the functionals in $C^1([0,3])$

$$F(u) = \int_0^3 \dot{u}^2\,dx, \qquad G(u) = \int_0^3 (\dot{u}^2 + u^2)\,dx, \qquad H(u) = \int_0^3 (\dot{u}^2 - 6u)\,dx\,.$$

(A) For the above functionals:

(A1) Compute the first variation of F, G, H at $u \in C^1([0,3])$ in direction $v \in C^1([0,1])$.

(A2) Define

$$X = \{u \in C^1([0,3]) : u(0) = 2,\ u(3) = 6\}\,.$$

For $u \in C^2([0,3]) \cap X$, integrate by parts the first variation of F, G, H. After that, characterize all the stationary points of F, G, H in $C^2([0,3]) \cap X$.

(A3) Verify that the found stationary points are the unique minimizers for F, G, H.

(B) Study the minimum problem for F, G and H in the following sets

- $X_1 = \{u \in C^1([0,3]) : u(0) = 2\}$,
- $X_2 = \{u \in C^1([0,3])\}$,
- $X_3 = \{u \in C^1([0,3]) : u(0) = 2,\ u(3) = 6,\ \int_0^3 u\,dx = 1\}$.

Determine if the problem has a solution or not. If the minimum exists, characterize all minimizers (for G in the case of X_3, it is ok not to compute the exact coefficients).

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Problem Sheet 4

Due date: 07.05.2021

We start with a few definitions which are useful for the following exercises.

$\mathbf{L^p}$ spaces: Let $I \subset \mathbb{R}$ be open, and $1 \leq p < +\infty$. We define

$$L^p(I) := \left\{ u \colon \Omega \to \mathbb{R} : \, u \text{ Lebesgue measurable}, \int_I |u|^p \, dx < +\infty \right\} ,$$

where dx denotes the one-dimensional Lebesgue measure on $\mathbb{R}$. Recall that $L^p(I)$ is a Banach space with the norm

$$\|u\|_p := \left(\int_I |u|^p \, dx \right)^{1/p} .$$

Similarly, we define the space of locally integrable functions

$$L^p_{\text{loc}}(I) := \{ u \colon \Omega \to \mathbb{R} : \, u \chi_K \in L^p(I) \, \text{ for all compact sets } \, K \subset I \} ,$$

where χ_K is the characteristic function of K, i.e., $\chi_K(x) = 1$ if $x \in K$ and $\chi_K(x) = 0$ if $x \notin K$.

Convolutions: Given two measurable functions $u, v \colon \mathbb{R} \to \mathbb{R}$, their convolutions is defined as

$$(u \star v)(x) := \int_{\mathbb{R}} u(x-y) v(y) \, dy ,$$

for all $x \in \mathbb{R}$ such that the right-hand side is well-defined. Note that, as soon as the right-hand side is finite, we have $u \star v = v \star u$. The following result gives sufficient conditions for the convolution to be well-defined.

Young's Theorem. Let $1 \leq p \leq +\infty$ and $u \in L^1(\mathbb{R})$, $v \in L^p(\mathbb{R})$. The for a.e. $x \in \mathbb{R}$ the map $y \mapsto u(x-y)v(y)$ belongs to $L^1(\mathbb{R})$, so that $(u \star v)(x)$ is finite. Moreover $u \star v \in L^p(\mathbb{R})$ and

$$\|u \star v\|_p \leq \|u\|_1 \, \|v\|_p .$$

Mollifiers: A family of mollifiers $\{\rho_\varepsilon\}_{\varepsilon > 0}$ is any family of functions $u_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ such that

$$\rho_\varepsilon \in C^\infty_c(\mathbb{R}) , \quad \operatorname{supp} \rho_\varepsilon \subset [-\varepsilon, \varepsilon] , \quad \rho_\varepsilon \geq 0 \, \text{ on } \, \mathbb{R} , \quad \int_{\mathbb{R}} \rho_\varepsilon(x) \, dx = 1 , \tag{1}$$

for all $\varepsilon > 0$. Sometimes it is more convenient to deal with a sequence, rather than a family indicized by ε: In this case the mollifiers are denoted by $\{\rho_n\}_{n \in \mathbb{N}}$, with ρ_n satisfying (1) with $\varepsilon = 1/n$.

Standard Mollifiers: Let $\rho \colon \mathbb{R} \to \mathbb{R}$ be given by

$$\rho(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1} \right) & \text{if } \, |x| < 1 , \\ 0 & \text{if } \, |x| \geq 1 , \end{cases}$$

where $C > 0 := 1/\int_{\mathbb{R}} \rho$. Notice that ρ satisfies

$$\rho \in C_c^\infty(\mathbb{R}), \quad \rho \geq 0\,, \quad \operatorname{supp} \rho \subset [-1,1]\,, \quad \int_{\mathbb{R}} \rho(x)\,dx = 1\,.$$

With ρ we can define the standard family of mollifiers

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon}\,\rho\left(\frac{x}{\varepsilon}\right)\,.$$

Notice that $\{\rho_\varepsilon\}_{\varepsilon>0}$ satisfies (1). If we want instead a sequence of standard mollifiers, just set

$$\rho_n(x) := n\,\rho(nx)\,.$$

Notation: Let $a < b$. Recall that the space $C^1([a,b])$ is normed by $\|u\| := \|u\|_\infty + \|u'\|_\infty$. Moreover we introduce the space of piecewise C^1 functions

$$C^1_{\text{pw}}([a,b]) := \big\{ u \in C([a,b]) : \exists\, N \in \mathbb{N},\ a = p_1 < p_2 < \ldots < p_N = b \ \text{ s.t.} \\ u \in C^1[p_i, p_{i+1}],\ \forall\, i = 1,\ldots,N-1 \big\}$$

Notice that the above partition can change depending on u. The map

$$\|u\|_{\text{pw}} := \|u\|_\infty + \max_{i=1,\ldots,N-1}\ \max_{x\in[p_1,p_{i+1}]} |u'(x)|$$

defines a norm over $C^1_{\text{pw}}([a,b])$.

Problem 4.1 (20 pts) - Smoothing by convolution.

a) Let $u \in L^1_{\text{loc}}(\mathbb{R})$ and $v \in C_c(\mathbb{R})$. Prove that $u \star v$ is well defined and $u \star v \in C(\mathbb{R})$.

b) Let $k \geq 1$, $u \in L^1_{\text{loc}}(\mathbb{R})$ and $v \in C^k_c(\mathbb{R})$. Prove that $u \star v \in C^k(\mathbb{R})$ with

$$\frac{d^k}{dx^k}(u \star v) = u \star v^{(k)}\,.$$

Hint: (a) Let $x_n \to x_0$. Since v is compactly supported, there exists some compact set $K \subset \mathbb{R}$ such that $(x_n - \operatorname{supp} v) \subset K$ for all $n \in \mathbb{N}$. (b) You can just show this for $k = 1$. The case $k > 1$ follows trivially by induction. Also notice that v' is uniformly continuous in $\mathbb{R}$, since $\operatorname{supp} v' \subset \operatorname{supp} v$, and $\operatorname{supp} v$ is compact. Moreover it may be useful to recall the Fundamental Theorem of Calculus, namely, $v(x+t) - v(x) = \int_0^1 v'(x+ts)\,t\,ds$.

Problem 4.2 (30 pts) - Cut-off. The goal of this exercise is to construct a cut-off function like the one introduced in Remark 3.3 in the lecture Notes. This is requested in point (b).

(a) Let $u \in L^1(\mathbb{R})$, $g \in L^p(\mathbb{R})$ for some $1 \leq p \leq +\infty$. Show that

$$\operatorname{supp}(u \star v) \subset \overline{\operatorname{supp} u + \operatorname{supp} v}\,.$$

(b) Let $K \subset \mathbb{R}$ be compact and $\varepsilon > 0$ fixed. Construct a function $\eta_\varepsilon \in C_c^\infty(\mathbb{R})$ such that:

i) $0 \leq \eta_\varepsilon \leq 1$ on $\mathbb{R}$,

ii) $\eta_\varepsilon(x) = 1$ for all $x \in K$,

iii) $\eta_\varepsilon(x) = 0$ for all $x \in \mathbb{R} \setminus K_\varepsilon$, with $K_\varepsilon := K + (-\varepsilon, \varepsilon)$,

iv) $|\eta_\varepsilon^{(k)}(x)| \leq C_k\,\varepsilon^{-k}$, for all $x \in \mathbb{R}$, $k \in \mathbb{N}$, with $C_k > 0$ constant depending only on k.

Hint: (a) $u \star v$ is well-defined by Young's Theorem stated above. (b) Take $\eta_\varepsilon := \chi_{K_{\varepsilon/2}} \star \rho_{\varepsilon/2}$, with ρ_ε the standard mollifier. Of course you can invoke Exercise 4.1 to prove smoothness and compute derivatives.

Problem 4.3 (25 pts) - Rounding corners. The goal of this exercise is to make rigorous the procedure of "rounding the corner" which has been used several times during the course.
Consider the functional $F\colon X \to \mathbb{R}$

$$F(u) := \int_0^1 \sqrt{|\dot{u}(x)|}\, dx\,, \qquad X := \{u \in C^1([0,1]) :\ u(0) = 2,\ u(1) = 0\}\,.$$

a) Let $a < x_0 < b$, $u \in C([a,b])$ such that $u \in C^1([a,x_0])$, $u \in C^1([x_0,b])$. Thus $u \in C^1_{\mathrm{pw}}([a,b])$. Let $\delta > 0$ be such that $I_\delta := (x_0 - \delta, x_0 + \delta) \subset (a,b)$. Using Exercise 4.2 point (b), construct $\tilde{u} \in C^1([a,b])$ such that

$$\tilde{u} = u \ \text{ in } \ [a,b] \smallsetminus I_\delta\,, \qquad \|\tilde{u}\| \leq \left(c_1 + c_2\,\frac{1}{\delta}\right) \|u\|_{\mathrm{pw}}\,,$$

where $c_1, c_2 > 0$ are constants not depending on δ and u.

b) Let $0 < \varepsilon < \sqrt{2} - 1$. Define $u_\varepsilon\colon [0,1] \to \mathbb{R}$ by

$$u_\varepsilon(x) := \begin{cases} 2 - \dfrac{x}{\varepsilon} & \text{if } \ 0 \leq x \leq 2\varepsilon\,, \\ 0 & \text{if } \ 2\varepsilon \leq x \leq 1\,. \end{cases}$$

Note that $u_\varepsilon \in C([0,1])$, $u_\varepsilon \in C^1([0,2\varepsilon])$ and $u_\varepsilon \in C^1([2\varepsilon,1])$. Let $\tilde{u}_\varepsilon$ be constructed by applying point (a) to u_ε with $a = 0$, $b = 1$, $x_0 := 2\varepsilon$ and $\delta := \varepsilon^2$. Prove that

i) $\tilde{u}_\varepsilon \in X$,

ii) $m = 0$, where $m := \inf\{F(u) :\ u \in X\}$,

iii) F admits no minimizer over X.

Problem 4.4 (15 pts) - Approximation Result. The goal of this exercise is to prove Remark 3.5 from the Lecture Notes (in the case $(a,b) = \mathbb{R}$).

a) Let ρ_n be a sequence of standard mollifiers and $u \in C(\mathbb{R})$. Prove that $\rho_n \star u \to u$ uniformly on compact sets, i.e., for all $K \subset \mathbb{R}$ compact it holds

$$\lim_{n\to+\infty} \max_{x\in K} |(\rho_n \star u)(x) - u(x)| = 0\,.$$

b) Let $u \in C_c(\mathbb{R})$. Use point (a) to construct a sequence $u_n \in C_c^\infty(\mathbb{R})$ such that

i) $u_n \to u$ uniformly on compact sets of $\mathbb{R}$,

ii) u_n is uniformly bounded, i.e.,

$$\sup_{n\in\mathbb{N}} \|u_n\|_\infty < +\infty\,.$$

Hint: You can use Exercise 4.1 and Exercise 4.2 point (b).

Problem 4.5 (10 pts). Consider the functional $F\colon X \to \mathbb{R}$

$$F(u) := \int_0^1 \exp(-\dot{u}^2)\, dx\,, \qquad X := \{u \in C^1([0,1]) : u(0) = u(1) = 0\}\,.$$

a) Prove that $u_0 \equiv 0$ is stationary for F.

b) Prove that $u_0 \equiv 0$ is a maximum point of F in X.

c) Let $m := \inf\{F(u) :\ u \in X\}$. Prove that $m = 0$ and that F admits no minimizer over X.

Hints: (a) Use Theorem 4.5 from the Lecture Notes to compute the first variation. (c) Consider $u_n(x) := n\,(x - 1/2)^2 - n/4$.

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Problem Sheet 5
Due date: xx.xx.2021

Problem 5.1 (20 pts).

(a) Let $-\infty < a < b < +\infty$ and $u \in C^1([a,b])$. Prove that $u \in W^{1,p}(a,b)$ for all $1 \leq p \leq +\infty$.

(b) Define $u(x) := |x| + 1$. Prove that $u \in W^{1,p}(-1,1)$ for all $1 \leq p \leq +\infty$.

(c) Define

$$u(x) := \begin{cases} 1 & \text{if } x > 0\,, \\ -1 & \text{if } x < 0\,. \end{cases}$$

Prove that $u \notin W^{1,p}(-1,1)$ for any $1 \leq p \leq +\infty$.

Hint: (c) Argue by contradiction and consider a sequence of Cut-off functions around the origin with supports converging to $\{0\}$.

Problem 5.2 (30 pts) - Characterization of Sobolev functions.

(a) Let $I \subset \mathbb{R}$ be open and $1 < p \leq +\infty$. Assume that $u \in L^p(I)$. Prove that they are equivalent:

(i) $u \in W^{1,p}(I)$,

(ii) There exists a constant $C > 0$ such that

$$\left| \int_I u\varphi' \, dx \right| \leq C \, \|\varphi\|_{L^{p'}} \qquad \text{for all} \quad \varphi \in C^1_c(I)\,,$$

where $p' = p/(p-1)$.

(b) Let $I \subset \mathbb{R}$ be open and $p = 1$. Does the equivalence between (i) and (ii) in point (a) hold?

Hint: (a) For the implication (ii) $\implies$ (i), consider the functional $T\colon C^1_c(I) \to \mathbb{R}$ defined by $T(\varphi) := \int_I u\varphi' \, dx$. Can it be extended to $L^{p'}(I)$?

Problem 5.3 (30 pts) - Another characterization of Sobolev functions.
Let $1 < p < +\infty$ and $u \in L^p(\mathbb{R})$. Prove that they are equivalent:

(I) $u \in W^{1,p}(\mathbb{R})$,

(II) There exists a constant $C > 0$ such that

$$\|\tau_h u - u\|_{L^p} \leq C|h| \qquad \text{for all} \quad h \in \mathbb{R}\,,$$

where $(\tau_h u)(x) := u(x+h)$.

Hint: For (I) $\implies$ (II) use that if $u \in W^{1,p}(\mathbb{R})$ then $u(y) - u(x) = \int_x^y u'(t) \, dt$ for a.e. $x, y \in \mathbb{R}$. For (II) $\implies$ (I), show that (II) implies (ii) in Exercise 5.2. The following identity may be useful: for all $\varphi \in C^1_c(\mathbb{R}), u \in L^p(\mathbb{R})$ one has $\int_{\mathbb{R}} u(x)[\varphi(x+h) - \varphi(x)] \, dx = \int_{\mathbb{R}} [u(x-h) - u(x)]\varphi(x) \, dx$.

Problem 5.4 (20 pts) - Generalized Poincaré Inequality. Let $I = (a, b)$ be bounded and $1 \leq p < +\infty$. Let $V \subset W^{1,p}(I)$ be a subspace such that

(i) V is closed in $W^{1,p}(I)$,

(ii) If $u \in V$ is constant, then $u \equiv 0$.

Prove that there exists a constant $C > 0$ such that

$$\|u\|_{W^{1,p}} \leq C \|u'\|_{L^p} \qquad \text{for all} \quad u \in V\,.$$

Hint: Follow the lines of the Proof by Contradiction of Theorem 7.35 in the Lecture Notes.

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Problem Sheet 6

Due date: 11.06.2021

Problem 6.1 (20 pts) - Assumptions of Theorem 9.9 are optimal.

Define the functional $F\colon W_0^{1,4}(0,1) \to \mathbb{R}$ by

$$F(u) := \int_0^1 (\dot{u}^2 - 1)^2 + u^4 \, dx \,,$$

and set

$$m := \inf\{F(u) : \ u \in W_0^{1,4}(0,1)\} \,.$$

(a) Prove that $m = 0$.

(b) Prove that F admits no minimizers in $W_0^{1,4}(0,1)$.

(c) Does the non existence of a minimizer contradict Theorem 9.9 in the Lecture Notes?

Hint: (a) Define a suitable bounded function $s\colon [0,1] \to \mathbb{R}$ such that $s(0) = s(1) = 0$ and $|s'| = 1$ a.e. in $(0,1)$. Then define a sequence $\{u_n\} \subset W_0^{1,4}(0,1)$ by suitably rescaling s, and show that $F(u_n) \to 0$.

Problem 6.2 (20 pts) - Assumptions of Theorem 9.9 are optimal.

Define the space

$$X := \{u \in W^{1,1}(0,1) : \ u(0) = 0, \ u(1) = 1\}$$

and the functional $F\colon W^{1,1}(0,1) \to \mathbb{R}$ by

$$F(u) := \int_0^1 \sqrt{u^2 + \dot{u}^2} \, dx \,.$$

Set

$$m := \inf\{F(u) : \ u \in X\} \,.$$

(a) Prove that $m = 1$.

(b) Prove that F admits no minimizers in X.

(c) Does the non existence of a minimizer contradict Theorem 9.9 in the Lecture Notes?

Hint: (a), (b) Can be done with similar ideas to the ones used for the minimization of the functionals G and H in the “Three Examples” of Lesson 3. Also recall that Sobolev functions are continuous.

Problem 6.3 (30 pts). Define the space

$$X := \{u \in W^{1,2}(0,2) : \; u(0) = 0, \; u(2) = 3\}$$

and the functional $F\colon W^{1,2}(0,2) \to \mathbb{R}$ by

$$F(u) := \int_0^2 \frac{1}{2}\,\dot{u}^2 + g(u)\,dx\,, \qquad g(s) := \int_0^s \arctan(t)\,dt\,.$$

(a) Verify the assumptions of Theorem 9.9 in the Lecture Notes to prove that F admits at least one minimizer over X.
(b) Is the minimizer unique?
(c) Prove that if $u_0 \in X$ minimizes F over X, then $u_0 \in C^\infty([0,2])$.

Hint: (b) Use Theorem 9.9 and Theorem 5.2 in the Lecture Notes. (c) Write down the weak ELE (using Theorem 8.4 in the Lecture Notes) and then use the bootstrap argument employed in Example 9.8 in the Lecture Notes.

Problem 6.4 (30 pts) - The Brachistochrone problem. The problem of the brachistochrone, formulated by Galileo in 1638, had a very strong influence on the development of the calculus of variations. It was resolved by John Bernoulli in 1696 and almost immediately after also by James, his brother, Leibniz and Newton. A decisive step was achieved with the work of Euler and Lagrange who found a systematic way of dealing with problems in this field by introducing what is now known as the Euler–Lagrange equation.
The aims to find the shortest path between two points that follows a point mass moving under the influence of gravity.

a) First, solve the physical problem by expressing the running time of the point mass in terms of its trajectory. Let the starting and ending points of the trajectory be the origin $(0,0)$ and (b,B) of a coordinate system whose y-axis points downwards (see Figure 1). This gives the functional to be minimized.

b) Compute the first variation on an interval $[\delta, b]$ where δ is small. Then the trajectory $\tilde{y}$ (parametrized by $x \in [0,b]$) solves the Euler–Lagrange equation piecewise on $[\delta, b]$. Show also that $\tilde{y}$ solves the Euler–Lagrange equation in (0,b] and $\tilde{y} \in C^2(0,b)$. Then you can find a differential equation, whose solution is the cycloid.

b) We skip the proof that the cycloid gives indeed the minimal value. However, for $B = 2\pi/b$, compare the running time of a point mass acted on by gravity on the line segment and on the cycloid from $(0,0)$ to (b,B). Compute the ratio of the running times.

Hint: See H. Kielhoefer, Calculus of Variations: An Introduction to the One-dimensional Theory with Examples and Exercises, 2018. Texts in Applied Mathematics, vol. 67, Springer.

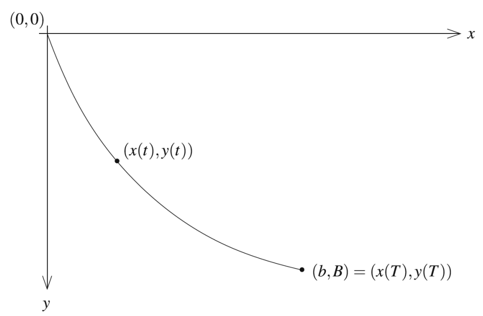

Figure 1: Coordinate system whose y-axis points downwards. The starting and ending points of the trajectory are the origin $(0,0)$ and (b,B), respectively.

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Problem Sheet 7

Due date: 25.06.2021

Problem 7.1 (20 pts). Let (X, d) be a metric space, $f\colon X \to \overline{\mathbb{R}}$. Recall that $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and that the LSC Envelope of f is defined by

$$\bar{f}(x) := \inf \left\{ \liminf_{n \to +\infty} f(x_n) : \{x_n\} \subset X,\ x_n \to x \right\}.$$

(a) Prove that f is LSC at x_0 if and only if

$$f(x_0) = \bar{f}(x_0).$$

(b) Let $g\colon X \to \overline{\mathbb{R}}$. Prove that

$$\overline{f+g} \geq \bar{f} + \bar{g}.$$

(c) Let $g\colon X \to \mathbb{R}$ be continuous. Prove that

$$\overline{f+g} = \bar{f} + g.$$

(d) Find one example of metric space (X, d) and functions $f, g\colon X \to \overline{\mathbb{R}}$ such that

$$\overline{f+g} > \bar{f} + \bar{g}$$

for at least one $x \in X$.

Hint: (c) Use Propositions 10.15 and 10.8 in the Lecture Notes.

Problem 7.2 (20 pts). Let $X := L^2(a, b)$ equipped with the metric d induced by the L^2 norm. Define $F, G\colon X \to \overline{\mathbb{R}}$ by

$$F(u) := \begin{cases} \int_a^b \dot{u}^2\, dx & \text{if } u \in C^1[a,b] \\ +\infty & \text{if } u \in X \smallsetminus C^1[a,b] \end{cases} \qquad G(u) := \begin{cases} \int_a^b \dot{u}^2\, dx & \text{if } u \in H^1(a,b) \\ +\infty & \text{if } u \in X \smallsetminus H^1(a,b) \end{cases}$$

Prove that $\bar{F} = G$ by using Proposition 10.18 in the Lecture Notes applied with X, d, $f = F$, $g = G$ and $D = C^1[a, b]$.

Hint: You may find the following results in the Lecture Notes useful: Theorem 7.27, Theorem 7.24, Corollary 7.10.

Problem 7.3 (20 pts). Let (X,d) be a metric space, $f_n, f\colon X \to \overline{\mathbb{R}}$. Suppose that

(i) $f_n \to f$ uniformly on compact sets of X,

(ii) f is LSC.

Prove that $f_n \to f$ in the sense of Γ-convergence.

Hint: If $x_n \to x_0$ then the set $K := \{x_n,\, n \in \mathbb{N}\} \cup \{x_0\}$ is compact in (X,d).

Problem 7.4 (20 pts).

(a) (X,d) metric space. Suppose that $f_n, f\colon X \to \overline{\mathbb{R}}$ are such that $f_n \to f$ in the sense of Γ-convergence. Moreover assume that $g_n, g\colon X \to \mathbb{R}$ are such that

   (i) $g_n \to g$ uniformly on compact sets of X,

   (ii) g is continuous.

   Prove that
$$f_n + g_n \to f + g$$
   in the sense of Γ-convergence.

(b) Let $X = \mathbb{R}$. Define
$$f_n(x) := \arctan(nx) + \frac{x^2}{n} \qquad \text{for} \quad x \in \mathbb{R}\,.$$
   Compute the Γ-limit of f_n in $\mathbb{R}$ as $n \to +\infty$.

Problem 7.5 (20 pts). Let $a < b$ and $A_n\colon [a,b] \to \mathbb{R}$ be a sequence of functions such that

(i) $A_n \geq 0$ a.e. in (a,b), for all $n \in \mathbb{N}$,

(ii) $A_n \leq M$ a.e. in (a,b), for all $n \in \mathbb{N}$, for some $M > 0$,

(iii) $A_n \rightharpoonup A$ weakly in $L^2(a,b)$.

Define the functionals $F_n, F\colon L^2(a,b) \to \overline{\mathbb{R}}$ by

$$F_n(u) := \begin{cases} \int_a^b \dot{u}^2 + A_n u^2 \, dx & \text{if } u \in C^1[a,b] \\ +\infty & \text{otherwise} \end{cases} \qquad F(u) := \begin{cases} \int_a^b \dot{u}^2 + A u^2 \, dx & \text{if } u \in H^1(a,b) \\ +\infty & \text{otherwise} \end{cases}$$

Prove that $F_n \to F$ in the sense of Γ-convergence with respect to the $L^2(a,b)$ metric.

Hint: You may find the following results in the Lecture Notes useful: Theorem 7.27, Theorem 7.24.