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Inverse Problems - Exercise Sheet 1

Publication date: 28 September, 2022

Due date: October 11, 2022

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $K \colon X \to Y$ be a linear continuous operator. For a given datum $f \in Y$ consider the inverse problem of finding $u \in X$ such that

$$Ku = f\,. \tag{1}$$

We say that the inverse problem (1) is *well-posed* in the sense of Hadamard if it admits unique solution for all $y \in Y$ and if solutions are continuous with respect to perturbations, i.e., if it holds

$$\|Ku_j - f\|_Y \to 0 \quad \implies \quad \|u_j - u\|_X \to 0\,.$$

Problem (1) is *ill-posed* if it is not well-posed.

Exercise 1.1 - Matrix inversion (30 pts)

Let $n \geq 1$ and consider $X = Y = \mathbb{R}^n$. Suppose that $K \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix. Then by the spectral theorem

$$K = \sum_{j=1}^n \lambda_j \, k_j \otimes k_j$$

with $\{k_j\}_{j=1}^n$ orthonormal basis of $\mathbb{R}^n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ eigenvectors, where the tensor product between two vectors $a, b \in \mathbb{R}^n$ is the matrix $a \otimes b := ab^T$, that is, $(a \otimes b)_{ij} = a_i b_j$.

(a) Suppose that $Ku = f$ and $Ku^\delta = f^\delta$. Show that

$$\left\|u - u^\delta\right\| \leq \frac{1}{\lambda_n} \left\|f - f^\delta\right\|\,,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

Hint: Note that $(a \otimes b)u = (b \cdot u)a$, where $\cdot$ is the scalar product in $\mathbb{R}^n$.

(b) In the setting of (a), prove the relative error bound

$$\frac{\left\|u - u^\delta\right\|}{\|u\|} \leq \frac{\lambda_1}{\lambda_n} \frac{\left\|f - f^\delta\right\|}{\|f\|}\,.$$

(c) Is the inverse problem $Ku = f$ well-posed?

(d) $\kappa := \lambda_1 \lambda_n^{-1}$ is called *condition number* of the matrix K. In terms of real-world reconstructions, i.e., in the presence of noise, is it better for κ to be small or large?

Exercise 1.2 - Differentiation (30 pts)

Let $X = C([0,1])$ be the space of continuous functions equipped with the supremum norm

$$\|u\|_\infty = \sup_{x\in[0,1]} |u(x)| \,, \qquad \text{for all} \;\; u \in X \,.$$

Let $Y = \{f \in C^1([0,1]) : \; f(0) = 0\}$, with $C^1([0,1])$ the space of continuously differentiable functions. Define the linear operator $K \colon X \to Y$ by

$$(Ku)(x) = \int_0^x u(y)\, dy \,,$$

for all $u \in X$ and $x \in [0,1]$.

(a) Show that the inverse problem $Ku = f$ admits a unique solution for all $f \in Y$.

(b) Prove that the inverse problem $Ku = f$ is ill-posed when Y is equipped with the supremum norm.

Hint: Consider noisy data of the form $f^\delta = f + n^\delta$ for some suitable noise $n^\delta \in Y$.

(c) Show that the inverse problem $Ku = f$ is well-posed when Y is equipped with the norm $\|\cdot\|_{C^1}$, where $\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty$ for all $u \in C^1([0,1])$.

Exercise 1.3 - Closed range (20 pts)

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $K : X \to Y$ be a bounded linear operator. On X we define the equivalence relation

$$x_1 \sim x_2 \iff x_1 - x_2 \in \ker(K) \,.$$

Define $\hat{X} = X/_\sim$ to be the quotient space w.r.t. $\sim$, with equivalence classes denoted by $[x]$. Introduce

$$\|[x]\|_{\hat{X}} := \inf_{y\in[x]} \|y\|_X \;,$$

and the operator $\hat{K} \colon \hat{X} \to \mathrm{rg}(K)$ defined by $\hat{K}[x] := Kx$, where $\mathrm{rg}(K)$ is the range of K.

(a) Show that $(\hat{X}, \|\cdot\|_{\hat{X}})$ is a Banach space.

Hint: Use that a normed space is complete if and only if every absolutely convergent series is convergent

(b) Show that $\hat{K}$ is well-defined, linear and bounded.

(c) Show that if $\mathrm{rg}(K)$ is closed then $\hat{K}^{-1}$ is continuous.

Hint: Note that $\hat{K}$ is bijective and use the open mapping theorem

(d) Prove that if $\hat{K}^{-1}$ is continuous then $\mathrm{rg}(K)$ is closed.

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On the space $L^2([-\pi,\pi]^2)$ we define the scalar product

$$\langle u,\, v\rangle := \int_{[-\pi,\pi]^2} u(x)\overline{v}(x)\, dx\,.$$

With respect to such scalar product, an orthonormal basis of $L^2([-\pi,\pi]^2)$ is given by the functions

$$e_l(x_1,x_2) := c_l\, e^{i(l_1x_1+l_2x_2)}\,,$$

for all $l=(l_1,l_2)\in\mathbb{Z}^2$, with $c_l\in\mathbb{R}$ suitable normalization constant. Any function $u\in L^2([-\pi,\pi]^2)$ admits a representation in terms of its Fourier series

$$u=\sum_{l\in\mathbb{Z}^2}\hat{u}_l\, e_l\,, \qquad \hat{u}_l := \langle u,\, e_l\rangle\,.$$

Recall that $\hat{u} := (\hat{u}_l)_l \in \ell^2(\mathbb{Z}^2)$. Also, an operator is compact if it is the limit of finite-range operators in operator norm. Recall that a space is finite dimensional if and only if its closed unit ball is compact.

Exercise 1.4 - Convolution (20 pts)

For $k\in L^2([-\pi,\pi]^2)$ define the convolution operator $K\colon L^2([-\pi,\pi]^2)\to L^2([-\pi,\pi]^2)$ by setting

$$Ku := k*u\,, \quad (k*u)(x) := \int_{[-\pi,\pi]^2} k(x-y)u(y)\, dy\,,$$

where we implicitly assume that k and u are extended periodically to the whole $\mathbb{R}^2$.

(a) Show that

$$\widehat{(Ku)}_l = \frac{1}{c_l}\,\hat{k}_l\hat{u}_l \quad \text{for all} \quad l\in\mathbb{Z}^2\,.$$

Moreover, provide the inverse of K in case $\hat{k}_l\neq 0$ for all $l\in\mathbb{Z}^2$.

(b) Prove that K is compact. Deduce that, in case $\hat{k}_l\neq 0$ for infinitely many $l\in\mathbb{Z}^2$, $\mathrm{rg}(K)$ is not closed.

Note: Thanks to Exercise 1.3, this shows that de-convolution is an ill-posed inverse problem

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Inverse Problems - Exercise Sheet 2

Publication date: 15 October, 2022 Due date: 25 October, 2022

Motivation: compact operators often arise in inverse problems applications. In this worksheet we highlight a few of their properties.

Let X be a Banach space with norm $\|\cdot\|_X$ and denote by B_X its unit ball, that is,

$$B_X := \{x \in X : \|x\|_X \leq 1\} .$$

Let Y be a Banach space with norm $\|\cdot\|_Y$. We denote by $\mathcal{L}(X,Y)$ the space of linear continuous operators $K : X \to Y$. Recall that $\mathcal{L}(X,Y)$ is a Banach space with the operator norm. We set $\mathcal{L}(X) := \mathcal{L}(X,X)$. For a subset $M \subset X$ we denote the distance of $x \in X$ to M by

$$\operatorname{dist}(x, M) := \inf_{m \in M} \|x - m\|_X .$$

Compact operators: We say that $K \in \mathcal{L}(X,Y)$ is a *compact operator* if the closure of $K(B_X)$ is compact in Y. We denote the space of compact operators from X to Y by $\mathcal{K}(X,Y)$, and $\mathcal{K}(X) := \mathcal{K}(X,X)$.

Finite rank operators: We say that $K \in \mathcal{L}(X,Y)$ has *finite rank* if $K(X)$ is finite dimensional. Note that finite rank operators are clearly compact.

Adjoint: Let $K \in \mathcal{L}(X,Y)$. The *adjoint* of K is the linear operator $K^* : Y^* \to X^*$ defined by

$$\langle K^* y^*, x \rangle_{X^*,X} = \langle y^*, Kx \rangle_{Y^*,Y} \quad \text{for all} \quad x \in X,\ y^* \in Y^* .$$

It is well-known that $K^* \in \mathcal{L}(Y^*, X^*)$, with $\|K\| = \|K^*\|$.

Summable sequences: For $p \geq 1$ denote by ℓ_p the space of square summable sequences with norm $\|\cdot\|_{\ell_p}$, that is,

$$\ell_p := \left\{ x = (x_j)_{j \in \mathbb{N}} : \sum_{j=1}^{\infty} |x_j|^p < \infty \right\} , \quad \|x\|_{\ell_p} := \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} .$$

Recall that ℓ_2 is a Hilbert space with scalar product

$$\langle x, y \rangle_{\ell_2} := \sum_{j=1}^{\infty} x_j y_j .$$

Exercise 2.1 (30 pts) - The identity is not compact in infinite dimensions. Let X be a normed space. Suppose $M \subset X$ is a closed linear subspace with $M \neq X$.

(a) Let $x \in X \setminus M$. Prove that $\operatorname{dist}(x, M) > 0$.

(b) (Riesz's Lemma) Prove that for all $\varepsilon > 0$ there exists $x \in X$ such that

$$\|x\|_X = 1 , \quad \operatorname{dist}(x, M) \geq 1 - \varepsilon .$$

(c) Suppose in addition that X is a Banach space. Prove that the identity map $I : X \to X$ is compact if and only if $\dim X < \infty$.

Hint: You can use point (b).

Exercise 2.2 (20 pts) - Image through compact operators. Let X, Y be Banach spaces and assume that X is reflexive. Let $K \in \mathcal{L}(X, Y)$ and $M \subset X$ be closed, convex and bounded.

(a) Show that $K(M)$ is closed in Y.

Hint: Recall that a linear operator $K: X \to Y$ between normed spaces is continuous if and only if it is weak-to-weak continuous. Also recall that if $M \subset X$ is convex then the closure of M coincides with its weak closure.

(b) In addition, assume that $K \in \mathcal{K}(X, Y)$. Show that $K(M)$ is compact.

Exercise 2.3 (20 pts) - Compact operators on Hilbert spaces. Let X, Y be real Hilbert spaces and $K \in \mathcal{L}(X, Y)$. Let $x_n, x \in X$ for $n \in \mathbb{N}$.

a) Show that $x_n \to x$ strongly in X if and only if

$$x_n \rightharpoonup x \quad \text{weakly in} \quad X \quad \text{and} \quad \|x_n\|_X \to \|x\|_X \, .$$

b) Show that K is compact if and only if the following condition holds:

$$\text{If} \quad x_n \rightharpoonup x \quad \text{weakly in} \quad X \, , \quad \text{then} \quad Kx_n \to Kx \quad \text{strongly in} \quad Y \, .$$

Hint: The statements in Exercise 2.2 could be useful.

Exercise 2.4 (10 pts) - Range of compact operators.

a) Let X be a Banach space with $\dim X = +\infty$ and let $K \in \mathcal{K}(X)$. Show that K cannot be surjective, that is, there exists $y \in Y$ such that the equation

$$Kx = y$$

has no solution in X.

b) Let X, Y be Banach spaces. Let $K \in \mathcal{K}(X, Y)$. Show that $\operatorname{rg}(K)$ is closed if and only if $\dim \operatorname{rg}(K) < \infty$.

Hint: You might find Exercise 2.1 point (c) and Exercise 1.3 useful. Also recall that the composition (in whichever order) of a compact operator with a bounded operator is compact.

Exercise 2.5 (20 pts) Define the operator $K: \ell_2 \to \ell_2$ by

$$(Kx)_j := \frac{x_j}{j} \, , \qquad \text{for all } \; j \in \mathbb{N} \, .$$

(a) Show that K is well-defined, linear and compact.

Hint: ℓ_2 is a Hilbert space, therefore you can use Exercise 2.3 to prove compactness. Also, you could use the dominated convergence Theorem in the ℓ_1 setting.

(b) Show that range of K is not closed by finding some element $y \in \overline{\operatorname{rg}(K)} \setminus \operatorname{rg}(K)$.

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Inverse Problems - Exercise Sheet 3

Publication date: 2 November, 2022 Due date: 15 November, 2022

Exercise 3.1 (25 pts) - Compact operators are not coercive in infinite dimensions. Let X, Y be Banach spaces, $K \in \mathcal{L}(X, Y)$. Assume there exists a constant $c > 0$ such that

$$\|Kx\|_Y \geq c\,\|x\|_X \quad \text{for all} \quad x \in X\,. \tag{1}$$

Show that K is compact if and only if $\dim X < +\infty$.

Remark: An operator K satisfying (1) is called coercive. Notice that coercivity implies injectivity.

Exercise 3.2 (25 pts) - Non-injective compact operators form a dense set. Let X, Y be Banach spaces, with $\dim X = \infty$. Suppose that $K \in \mathcal{K}(X, Y)$

(a) Show that there exists a sequence $\{x_n\}_n$ in X such that $\|x_n\|_X = 1$ for all $n \in \mathbb{N}$ and $Kx_n \to 0$ strongly.

(b) Using (a), prove that the set

$$\{K \in \mathcal{K}(X, Y) : \ K \ \text{ is not injective }\}$$

is dense in $\mathcal{K}(X, Y)$ with respect to the operator norm.

Exercise 3.3 (25 pts) - Existence of minimal norm elements. Let X be a Banach space. Let $A \subset X$ be convex, closed and non-empty.

(a) Prove that the function

$$x \in X \mapsto \operatorname{dist}(x, A)$$

is lower semicontinuous with respect to the weak topology on X.

(b) Assume that X is reflexive. Prove that A has a minimal norm element, that is, show that there exists $\hat{x} \in A$ such that

$$\|\hat{x}\| = \inf_{x \in A} \|x\|$$

(c) Assume that X is reflexive. Let $M \subset X$ be a closed linear subspace. Prove that in Riesz's Lemma of Exercise 2.1 point (b) one can choose $\varepsilon = 0$, that is, show that there exists $x \in X$ such that

$$\|x\|_X = 1\,, \quad \operatorname{dist}(x, M) \geq 1\,.$$

Exercise 3.4 (25 pts) - Computation of Moore-Penrose Inverse For the following operators T in $\mathcal{L}(X, Y)$ compute the Moore-Penrose inverse of T, and check whether $\mathrm{Dom}(T^\dagger) = Y$.

(a) For Hilbert spaces X, Y and fixed $u \in X$ with $u \neq 0$ and $v \in Y$ with $v \neq 0$ let $T\colon X \to Y$ be defined according to $Tx = v\langle u, x\rangle$ (where $\langle \cdot, \cdot \rangle$ denote the scalar product on X).

(b) Let $T\colon L^2([0,1]) \to L^2([0,1])$ according to

$$[Tx](t) = \int_0^t x(s)\,ds \qquad \text{for almost every } t \in [0,1].$$

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Inverse Problems - Exercise Sheet 4

Publication date: 18 November, 2022

Due date: 22 November, 2022

Exercise 4.1 (50 pts) - Radon transform. For $f\colon B \to \mathbb{R}$ continuous, where $B := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, we define the Radon transform as

$$\mathcal{R}[f](\theta, s) = \int_{-\infty}^{\infty} f\left(s\ \mathbf{w}(\theta) + t\ \mathbf{w}^{\perp}(\theta)\right) dt, \quad (\theta, s) \in [0, 2\pi] \times \mathbb{R},$$

with f extended by 0 outside B and $\mathbf{w}(\theta) = (\cos(\theta), \sin(\theta))^t$, $\mathbf{w}^{\perp}(\theta) = (\text{-}\sin(\theta), \cos(\theta))^t$.

(a) Show that the Radon transform can be extended to a linear continuous operator from $L^2(B)$ into $L^2(Q)$ with $Q := [0, 2\pi] \times [-1, 1]$

(b) Prove that the adjoint of the Radon transform $\mathcal{R}^*$, also called **backprojection operator**, has the form

$$\mathcal{R}^*[g](x) = \int_0^{2\pi} g\left(\theta, x \cdot \mathbf{w}(\theta)\right) d\theta\,. \tag{1}$$

Note: Adjoint means that $\langle \mathcal{R}[f], g\rangle_{L^2(Q)} = \langle f, \mathcal{R}^*[g]\rangle_{L^2(B)}, \quad \forall f \in L^2(B), g \in L^2(Q)$.

Instructions for next exercise: Please bring your code on a pendrive on the day of the class

Exercise 4.2 (50 pts) - Numerical implementation of the Radon transform. We will now work in the discrete case where functions defined on $[-1, 1] \times [-1, 1]$ will be represented by square matrices.

(a) Using the provided template in Matlab `radontrans.m`, implement the Radon transform.

Hint: Consider using the `imrotate` function. If the code is taking more than 10 lines, reconsider your approach.

(b) The output of a Radon transform is called a sinogram, denoted by

$$g(\theta, s) = \mathcal{F}[f](\theta, s)\,.$$

From the sinogram at the left hand side of Figure 1, which could be the underlying source f that produces it?

(c) Using the provided template in Matlab `backprojection.m`, implement the adjoint of the Radon transform defined in (1).

Hint: Notice that for fixed θ, the function $G(x) = g(\theta, x \cdot \mathbf{w}(\theta)), x \in \mathbb{R}^2$, is constant along lines parallel to $\mathbf{w}^{\perp}(\theta)$.

Hint 2: Consider using the `imrotate` function. If the code is taking more than 10 lines, reconsider your approach.

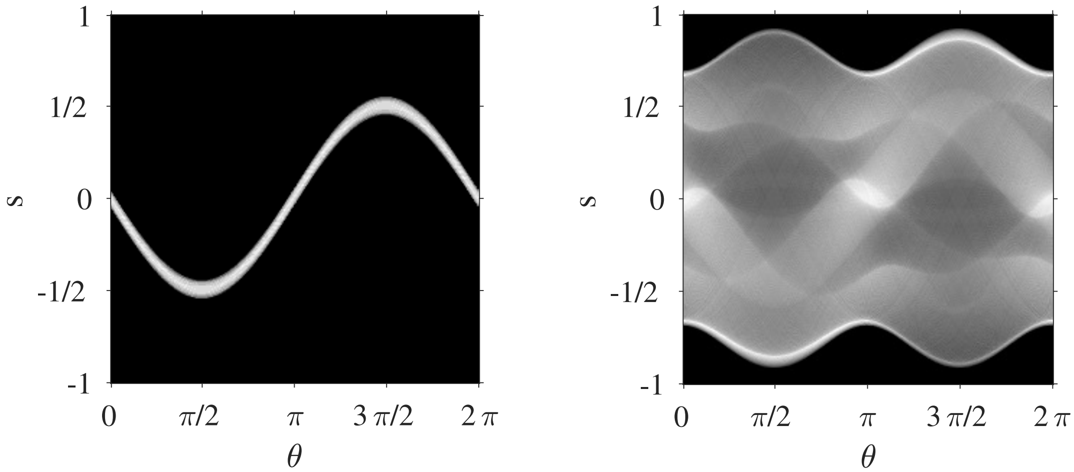

Figure 1: Example sinograms of unknown sources

(d) The **backprojection algorithm** corresponds to simply applying the adjoint operator $\mathcal{R}^*$ to the sinogram data. Write a script where you apply your implemented adjoint to the provided data `sinogram1.mat` and `sinogram2.mat`, that corresponds to the ones presented in Figure 1. What do you obtain?

Remark: It is required to use the same family of angles θ for the Radon transform and its adjoint. For the provided data $\theta =$ `linspace(0,360,256)`.

(e) Could you explain or interpret what is the backprojection algorithm $\mathcal{R}^*[\mathcal{R}[f]](x)$ doing?

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Inverse Problems - Exercise Sheet 5

Publication date: 1 December, 2022 Due date: 13 December, 2022

Exercise 5.1 (50 pts) - Singular values of the Radon transform For this exercise we fix the weight function $w(s) = \sqrt{1-s^2}$. Furthermore, we will require the Chebyshev polynomials

$$U_m(s) = \frac{\sin[(m+1)\arccos(s)]}{\sin(\arccos(s))}.$$

They satisfy the following properties:

- They are orthogonal in the $L^2([-1,1], w)$, the $w-$ weighted space. That is

$$\int_{-1}^{1} w(s) U_m(s) U_{m'}(s) ds = \frac{\pi}{2} \delta_{m,m'}.$$

- They form an orthogonal basis for $L^2([-1,1], w)$. Consequently, the functions $U_m(s)w^{-1}(s)$ form a basis of $Z = L^2([-1,1], w^{-1})$.
- They satisfy the following integral property

$$\int_{-w(s)}^{w(s)} U_m\left(\mathbf{w}(\theta') \cdot (s\mathbf{w}(\theta) + t\mathbf{w}^{\perp}(\theta))\right) dt = \frac{2}{m+1} \frac{\sin((m+1)(\theta-\theta'))\sin((m+1)\arccos(s))}{\sin(\theta-\theta')}$$

a) Prove that the Radon transform is continuous from $L^2(B)$ to $Z = L^2(\Omega, w(s)^{-1})$, the L^2 space weighted with w^{-1}, that is, show the following inequality

$$\|\mathcal{R}f\|_{L^2(\Omega,w^{-1})} = \int_0^{2\pi} \int_{-1}^{1} |\mathcal{R}[f](\theta,s)|^2 w(s)^{-1} ds d\theta \leq 4\pi \|f\|_{L^2(B)}$$

Hint 1: Notice that the interval $[-w(s), w(s)]$ corresponds to the effective interval in which the Radon transform integrates, for any direction θ and displacement s.
Hint 2: To integrate, reuse the same idea presented in the last exercise sheet.

b) Under this new metric on the image space of the Radon transform, find the adjoint operator $\mathcal{R}^*$.
Hint: you can use the results from the previous exercise class.

c) Consider functions of the form $g(s,\theta) = U(s)w(s)v(\theta)$, for $U \in L^2([-1,1]),\ v \in L^2([0,2\pi))$. Check that

$$(\mathcal{R}\mathcal{R}^* g)(\theta, s) = \int_{-w(s)}^{w(s)} \int_0^{2\pi} U\left(\mathbf{w}(\theta') \cdot (s\mathbf{w}(\theta) + t\mathbf{w}^{\perp}(\theta))\right) v(\theta') d\theta' dt$$

d) Check that

$$(\mathcal{R}\mathcal{R}^* g_m)(\theta, s) = \frac{4\pi}{m+1} w(s) U_m(s) \overline{v}(\theta),$$

where U_m are the Chebyshev polynomials, and $\overline{v}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin((m+1)(\theta-\theta'))}{\sin(\theta-\theta')} v(\theta') d\theta'$.

e) Consider the family $Y_l(\theta) = \frac{1}{\sqrt{2\pi}} e^{-il\theta}$ of functions that form a basis for $L^2([0, 2\pi))$ and satisfy

$$\overline{Y}_l = \begin{cases} Y_l & \text{if } -m \leq l \leq m \text{ and } l - m \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Write a basis for Z. From it, find the eigenfunctions and nonzero eigenvectors of $\mathcal{R}\mathcal{R}^*$.

f) Obtain the singular values for the Radon transform $\mathcal{R}$. How ill-posed is the respective inverse problem? Why there seem to be many zero singular values?

Exercise 5.2 (50 pts) - Singular values of the Radon transform and the limited angle problem.

a) Write a code that returns the matrix associated to the Radon transform of a function with support in the unitary ball. For it use the provided template `RadonMatrix.py` (in Python, feel free to do an equivalent Matlab version). Take into consideration the following points

- The Radon transform goes from images in two dimensions to a function with two variables. Since it is written as a matrix, it will take vectors into vectors. Take care to be consistent in these transformations.
- The obtained matrix will be a $|S||\Theta| \times N^2$ sized matrix. It is recommended that you use a sparse format for it.
- The Radon transform is such that it acts only on objects that are supported in the unitary ball, so no weight should be given to pixels outside that area.
- Feel free to consider reasonable approximations. One suggested example could be to approximate the line integral by an uniform Riemann sum.
- Save the output results in the following sections, as computations could be a bit length (order of minutes).

b) In the template `RadonMatrix.py` there is a script to test the Matrix with an image composed of a non-centered disk, use it to check your implementation.

c) In the following points, fix $N = 40$, $S = 40$, $T = 60$. Use template `InvertData.py` to invert the operator for the cases $\theta \in [0, \pi], \theta \in [0, \pi/4]$, and noiseless and 8% of noise level. What do you observe when plotting the reconstructed images?

d) Write a script in which you plot the singular values (in a non-increasing fashion) of the matrix for the cases $\theta \in [0, 2\pi], [0, \pi], [0, \pi/2], [0, \pi/4]$. What do you see and what would it explain?

e) Write a script, in which for the case of $\theta \in [0, \pi/4]$ and 10% of noise, an approximate inversion is obtained by using all the singular values above: 0.001, 0.01, and 0.05. Comment on the results.

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Inverse Problems - Exercise Sheet 6

Publication date: 12 January, 2023 Due date: 24 January, 2023

Exercise 6.1 (20 pts) - Tikhonov regularization with semi-norms. Let X, Y, and Z be Banach spaces, where X is additionally reflexive and Z is separable. Further, let $T : X \to Y$ be a linear and continuous operator, and $A : \operatorname{dom}(A) \to Z^*$ be a densely defined, linear operator on the Banach space X. Moreover, assume that the range of A is closed in Z^*, the kernel of A is finite-dimensional, and that A is weakly-weakly$*$ closed, which means that $x_n \rightharpoonup x$ and $Ax_n \overset{*}{\rightharpoonup} z$ implies $x \in \operatorname{dom}(A)$ and $Ax = z$.
For each $y^\delta \in Y, p, q \geq 1$, and $\alpha > 0$, show that there exists a solution of the minimization problem

$$\min_{x \in X} \frac{1}{q} \left\| Tx - y^\delta \right\|_Y^q + \frac{\alpha}{p} \left\| Ax \right\|_{Z^*}^p .$$

Hint: You may exploit: if the linear operator $S : \operatorname{dom}(S) \to Y$ between the dense subspace $\operatorname{dom}(S) \subset X$ and Y is closed and bijective, then S^{-1} is continuous. Further, note that the finite-dimensional subspace $\ker(A)$ is complemented in X.

Convex analysis. Let X be a Banach space, X^* its dual, and $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ a function. We define the convex conjugate of f as the function $f^* : X^* \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in X} \langle x^*, x \rangle - f(x) , \tag{1}$$

where $\langle x^*, x \rangle$ denotes the dual pairing between X and X^*. The convex conjugate satisfies the following properties:

- f^* is always convex and lower-semicontinuous.
- The convex biconjugate f^{**} (that is to apply the convex conjugate twice), corresponds to the closed convex hull of f, i.e. the largest lower semi-continuous convex function such that $f^{**} \leq f$. If f is proper, convex and lower semicontinuous, then $f^{**} = f$.
- **Fenchel duality:** For $f \colon X \to \mathbb{R} \cup \{\pm\infty\}$, $g \colon Y \to \mathbb{R} \cup \{\pm\infty\}$ convex functions and $A \colon X \to Y$ bounded linear operator, the following inequality holds

  $$\inf_{x \in X} f(Ax) + g(x) \geq \sup_{y^* \in Y^*} -f^*(-y^*) - g^*(A^* y^*) .$$

  Equality holds under certain regularity conditions. The above inequality is also named the weak duality of the *primal* and *dual* optimization problems. In the case of equality of the two optimization problems, it is said that there is strong duality.

Exercise 6.2 (30 pts) - The convex conjugate.

a) Consider a Banach space X with norm $\|\cdot\|_X$. Prove that the convex conjugate (1) of the norm is the convex indicator function over the unitary ball in the dual space, that is,

$$\left(\|\cdot\|_X\right)^* (x^*) = \mathbb{1}_{B^*}(x^*) = \begin{cases} 0 & \text{if } \|x^*\|_{X^*} \leq 1, \\ \infty & \text{if } \|x^*\|_{X^*} > 1 . \end{cases}$$

b) Consider the space $X = \mathbb{R}^n \times \mathbb{R}^n$ endowed with the norm $||x|| = \sum_{i=1}^n \sqrt{y_i^2 + z_i^2}$, for $x = (y, z)$. Prove that for $x^* = (y^*, z^*) \in X$ one has

$$(|| \cdot ||)^* \,(x^*) = \begin{cases} 0 & \text{if} \quad \max\limits_i \sqrt{(y_i^*)^2 + (z_i^*)^2} \leq 1, \\ \infty & \text{if} \quad \max\limits_i \sqrt{(y_i^*)^2 + (z_i^*)^2} > 1, \end{cases} \tag{2}$$

The primal-dual algorithm. We want to compute a solution to the problem

$$\min_{x \in X} \; f(Ax) + g(x)\,, \tag{3}$$

where:

- X, Y are two finite-dimensional real vector spaces equipped with an inner product $\langle \cdot, \cdot \rangle$ and norm.
- $A : X \to Y$ is a continuous linear operator
- $g : X \to [0, \infty)$ and $f : Y \to [0, \infty)$ are proper, convex, lower-semicontinuous functions.

For a function h the *resolvent operator* is defined as

$$(I + \tau \partial h)^{-1}\,(y) = \arg\min_x \left\{ \frac{\|x - y\|^2}{2\tau} + h(x) \right\}.$$

Then, for selected constants $\tau, \sigma > 0, \theta \in [0, 1]$, initial guess $(x^0, y^0) \in X \times Y$, f^* the convex conjugate of f, and A^* the adjoint operator of A, the *primal-dual algorithm* stopped at N iterations is

Algorithm 1 Primal-dual Algorithm

procedure PRIMAL-DUAL$(\tau, \sigma, \theta, N, x^0, y^0)$
  $\overline{x}^0 = x^0$
  for $n = 0, 1, 2, \ldots, N$ **do**
    $y^{n+1} = (I + \sigma \partial f^*)^{-1}\,(y^n + \sigma A \overline{x}^n)$
    $x^{n+1} = (I + \tau \partial g)^{-1}\,(x^n - \tau A^* y^{n+1})$
    $\overline{x}^{n+1} = x^{n+1} + \theta(x^{n+1} - x^n)$
  end for
  return x^{N+1}.
end procedure

The ROF model: We now introduce the ROF model for image denoising. We model a 2D image u by a $N \times N$ matrix, $N > 0$ integer. Thus $u \in \mathbb{R}^{N \times N}$. The discrete gradient of u with Neumann boundary conditions is defined by $\nabla u = (\nabla_x u, \nabla_y u)$, where $\nabla_x u, \nabla_y u \in \mathbb{R}^{N \times N}$ are matrices defined by

$$(\nabla_x u)_{i,j} := \begin{cases} u_{i+1,j} - u_{i,j} & \text{if} \quad i < N-1\,, \\ 0 & \text{if} \quad i = N-1\,, \end{cases} \qquad (\nabla_y u)_{i,j} := \begin{cases} u_{i,j+1} - u_{i,j} & \text{if} \quad j < N-1\,, \\ 0 & \text{if} \quad j = N-1\,. \end{cases} \tag{4}$$

For $u \in \mathbb{R}^{N \times N}$, $p = (p^x, p^y)$, $p^x, p^y \in \mathbb{R}^{N \times N}$ we define

$$\|u\|_2 := \sqrt{\sum_{i,j=1}^{N} u_{i,j}^2}\,, \qquad \|p\|_{1,2} := \|p^x\|_2 + \|p^y\|_2\,. \tag{5}$$

Given a noisy image $D \in \mathbb{R}^{N \times N}$, the denoised image $u \in \mathbb{R}^{N \times N}$ according to the ROF model is a solution to the minimization problem

$$\min_{u \in \mathbb{R}^{N \times N}} \|\nabla u\|_{1,2} + \frac{\lambda}{2} \|u - D\|_2^2\,, \tag{6}$$

where $\lambda > 0$ is a fixed regularization parameter.

Exercise 6.3 (50 pts) - Implementing the primal-dual algorithm for ROF. In this exercise we will implement the primal-dual algorithm to compute solutions of the ROF model for image denoising at (6). We define the following:

- $X := \mathbb{R}^{N \times N}$ normed by $\|\cdot\|_X := \|\cdot\|_2$ as defined in (5). Notice that such norm is induced by the Hilbert product
$$\langle u, v \rangle_X := \langle u, v \rangle_2 := \sum_{i,j=1}^{N} u_{i,j} v_{i,j}\,,$$
for $u, v \in X$.
- $Y := \mathbb{R}^{N \times N} \times \mathbb{R}^{N \times N}$ normed by $\|\cdot\|_Y := \|\cdot\|_{1,2}$ as defined in (5). This norm is induced by the Hilbert product
$$\langle p, q \rangle_Y := \langle p, q \rangle_{1,2} := \langle p^x, q^x \rangle_2 + \langle p^y, q^y \rangle_2$$
for $p = (p^x, p^y), q = (q^x, q^y) \in Y$.
- $A \colon X \to Y$ linear continuous operator defined by $Au := \nabla u$ where $\nabla = (\nabla_x, \nabla_y)$ is as in (4).
- $f : Y \to [0, \infty)$ defined by $f(p) := \|p\|_{1,2}$.
- Given $\lambda > 0$ and $D \in X$, define $g_\lambda : X \to [0, \infty)$ as $g_\lambda(u) := \frac{\lambda}{2} \|u - D\|_2^2$.

With the above definitions, please address the following questions.

a) Compute A^*, the adjoint of the operator A.

b) Following the provided template `primal_dual.py`, implement the operators A and A^*.

c) By Exercise 6.2 we have that

$$f^*(p) = \begin{cases} 0 & \text{if} \quad \max_{i,j} \sqrt{(p_{i,j}^x)^2 + (p_{i,j}^y)^2} \le 1, \\ \infty & \text{if} \quad \max_{i,j} \sqrt{(p_{i,j}^x)^2 + (p_{i,j}^y)^2} > 1. \end{cases}$$

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Prove that the resolvent of f^* has the following form:

$$p = \left(I + \sigma \partial f^*\right)^{-1}(\tilde{p}) \qquad \Longleftrightarrow \qquad p_{i,j} = \frac{\tilde{p}_{i,j}}{\max\left(1, \sqrt{(\tilde{p}^x_{i,j})^2 + (\tilde{p}^y_{i,j})^2}\right)}\,.$$

d) Following the provided template `primal_dual.py`, implement the resolvent of f^*.

e) Prove that the resolvent of g_λ has the following form:

$$\left(I + \tau \partial g_\lambda\right)^{-1}(\tilde{u}) = \frac{\tilde{u} + \lambda \tau D}{1 + \lambda \tau}\,.$$

f) Following the provided template `primal_dual.py`, implement the resolvent of g_λ.

g) Following the provided template `primal_dual.py`, implement the primal-dual algorithm. Then run the code to obtain reconstructions for two noise cases. Feel free to change the parameters to see different results (λ, the number of iterations, noise level). Changing the values of σ and τ is discouraged, as it is related to the convergence properties of this algorithm, topic not discussed in this exercise.