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BERRY–ESSEEN BOUNDS FOR SELF-NORMALIZED SUMS OF LOCAL DEPENDENT RANDOM VARIABLES

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Abstract

In this paper, we prove Berry–Esseen bounds with optimal order for self-normalized sums of local dependent random variables under some mild dependence conditions. The proof is based on Stein's method and a randomized concentration inequality. As applications, we obtain optimal Berry–Esseen bounds for m-dependent random variables and graph dependency models.
Keywords: Berry—Esseen bounds; Self-normalized sums; local dependence; m-dependence; graph dependency

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1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent random variables, and let

$$S_n = \sum_{i=1}^n X_i, \quad V_n^2 = \sum_{i=1}^n X_i^2, \quad \text{and} \quad \widehat{\sigma}_n^2 = \frac{n-1}{n}\sum_{i=1}^n (X_i - \bar{X})^2,$$

where $\bar{X} = S_n/n$. We say S_n/V_n is a *self-normalized sum*. We note that $S_n/\widehat{\sigma}_n$ is the well-known Student's t-statistic, which is one of the most important tools in statistical testing when the standard deviation of the underlying distribution is unknown. Based on the fact that

$$\mathbb{P}(S_n/\widehat{\sigma}_n \geqslant x) = \mathbb{P}(S_n/V_n \geqslant x[n/(n+x^2-1)]^{1/2}), \quad x \geqslant 0,$$

it is common to study the self-normalized sum S_n/V_n. One of the most important advantages of self-normalized sums is that the range of Gaussian approximation can

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be much wider than their corresponding non-self-normalized sums under the same polynomial moment conditions. Berry–Esseen bounds of Gaussian approximation for self-normalized sums of independent random variables have been well-studied in the literature. For example, Berry–Esseen theorem for Student's t-statistics were proved in [4] and [3], and an exponential nonuniform Berry–Esseen bound was obtained in [13]. Differing from the method in [4] and [3], Stein's method can be used to prove Berry–Esseen bounds with explicit constants (see [15]).

If $(X_1, X_2, \dots)$ is a family of dependent random variables, asymptotic theories on self-normalized sums have also been studied in the literature. For example, [11] studied self-normalized processes, and [5], considered self-normalized sums for martingales. Recently, [**?**] proved a Berry–Esseen bound for self-normalized sums of martingales. Cramér-type moderate deviations for weak dependent random variables and martingales have also been studied. See [10] and [**?**]. For more details, we refer to [16] for a survey.

Local dependence is also a commonly used dependence structure in applications. A family of local dependent random variables means that certain subset of the random variables are independent of those outside their respective neighborhoods. There are several forms of local dependence assumptions in the literature. For example, *decomposable random variables* were considered in [2], where a L_1 bound was also obtained; *dependency neighborhoods* was introduced in [14] to study error bounds for multivariate normal approximation. Uniform and nonuniform Berry–Esseen bounds for a general local dependence structure were established in [8]. Recently, [12] proved an error bound of Wasserstein-2 distance for a generalized local dependence structure.

In this paper, our main purpose is to prove Berry–Esseen bounds for self-normalized sums of local dependent random variables. In Theorem 2.1 (see Section 2), under conditions (LD1) and (LD2), we provide a Berry–Esseen bound for self-normalized sums of local dependent random variables. Compared to the results for non-self-normalized sums in Theorem 2.2 in [8], under the same conditions (LD1) and (LD2), we do not require the fourth moment assumption to obtain the optimal convergence rate, which in turn shows robustness of self-normalized statistics. As applications, we obtain Berry–Esseen bounds for studentized statistics for m-dependent random variables and graph dependency models.

The proof of our main result is based on Stein's method and concentration inequality approach. The technique of concentration inequality approach has been applied to obtain sharp Berry–Esseen bounds for univariate and multivariate normal approximations in the literature, and we refer to [7, 6, 8, 9, 15, 18] and [17]. In this paper, we develop new randomized concentration inequalities for local dependent random variables under some mild dependence conditions. The ideas are based on [8] and [15].

The rest of this paper is organized as follows. We give our main results in Section 2. Applications are given in Section 3. Some useful preliminary lemmas and a new randomized concentration inequality are proved in Section 4. We give the proof of our main result in Section 5.

2. Main results

Let $\{X_i\}_{i\in\mathcal{J}}$ be a field of real-valued random variables satisfying that $\mathbb{E}\,X_i = 0$ for all $i \in \mathcal{J}$. We introduce the following local dependence conditions.

(LD1) For any $i \in \mathcal{J}$, there exists $A_i \subset \mathcal{J}$ such that X_i is independent of $\{X_j : j \notin A_i\}$;

(LD2) For any $i \in \mathcal{J}$, there exists $B_i \subset \mathcal{J}$ such that $B_i \supset A_i$ and $\{X_j : j \in A_i\}$ is independent of $\{X_j : j \notin B_i\}$.

These dependence assumptions have been commonly used in the literature. We remark that we do not assume any structures on the index set. Put

$$S = \sum_{i\in\mathcal{J}} X_i, \quad V = \sqrt{\Big(\sum_{i\in\mathcal{J}} (X_iY_i - \bar{X}\bar{Y})\Big)_+}, \quad W = S/V, \tag{2.1}$$

where $Y_i = \sum_{j\in A_i} X_i$, $\bar{X} = \sum_{i\in\mathcal{J}} X_i/|\mathcal{J}|$, $\bar{Y} = \sum_{j\in\mathcal{J}} Y_j/|\mathcal{J}|$ and $(x)_+ = \max(x, 0)$ is the positive part of x. Let $\sigma = (\mathrm{Var}(S))^{1/2}$. Here and in the sequel, we denote by $|A|$ the cardinality of A for any $A \subset \mathcal{J}$. We remark that if $\mathbb{E}\,X_i$ is not necessarily 0, then one can simply replace X_i by $X_i - \mathbb{E}\,X_i$ in (2.1).

Moreover, let κ be any number such that $\kappa \geqslant \max_{i\in\mathcal{J}}\{|\{j : B_i \cap A_j \neq \emptyset\}|, |\{j : i \in$

$B_j\}|$ and let

$$\begin{aligned}
\beta_0 &= \sum_{i\in\mathcal{J}} \mathbb{P}(|X_i| > \sigma/\kappa),\\
\beta_2 &= \frac{1}{\sigma^2}\sum_{i\in\mathcal{J}} \mathbb{E}\{|X_i|^2 \mathbb{I}(|X_i| > \sigma/\kappa)\},\\
\beta_3 &= \frac{1}{\sigma^3}\sum_{i\in\mathcal{J}} \mathbb{E}\{|X_i|^3 \mathbb{I}(|X_i| \leqslant \sigma/\kappa)\},\\
\theta &= \frac{1}{\sigma^2}\sum_{i\in\mathcal{J}}\sum_{j\in A_i} \mathbb{E}|X_i X_j \mathbb{I}(|X_i| \leqslant \sigma/\kappa, |X_j| \leqslant \sigma/\kappa)|.
\end{aligned}$$

We have the following theorem.

Theorem 1. *Under (LD1) and (LD2). We have*

$$\begin{aligned}
&\sup_{z\in\mathbb{R}}\big|\mathbb{P}(W \leqslant z) - \Phi(z)\big|\\
&\qquad \leqslant C\{(1+\theta)\kappa^2\beta_3 + \kappa\beta_2 + \beta_0\} + C\kappa^{1/2}(\theta+1)|\mathcal{J}|^{-1/2},
\end{aligned} \tag{2.2}$$

where $C > 0$ is an absolute constant and Φ is the standard normal distribution function.

Remark 1. We first make some remarks on β_j's. If $\max_i |X_i| \leqslant \sigma/\sigma$ almost surely, then $\beta_0 = 0$. Otherwise, by Chebyshev's inequality, we have

$$\beta_0 \leqslant \kappa^2\beta_2.$$

If $\mathbb{E}|X_i|^3 < \infty$, then

$$\kappa\beta_2 + \kappa^2\beta_3 \leqslant \frac{\kappa^2}{\sigma^3}\sum_{i\in\mathcal{J}} \mathbb{E}|X_i|^3.$$

Moreover, by Hölder's inequality, we have

$$\begin{aligned}
\theta &\leqslant \left(\mathbb{E}\left\{\frac{1}{\sigma^2}\sum_{i\in\mathcal{J}}\sum_{j\in A_i} |X_i X_j| \mathbb{I}(|X_i| \leqslant \sigma/\kappa, |X_j| \leqslant \sigma/\kappa)\right\}^{3/2}\right)^{2/3}\\
&\leqslant \kappa_1 |\mathcal{J}|^{1/3}\beta_3^{2/3} \leqslant \kappa|\mathcal{J}|^{1/3}\beta_3^{2/3},
\end{aligned} \tag{2.3}$$

where $\kappa_1 = \max_{i\in\mathcal{J}}|A_i|$. If β_3 is of order $O(|\mathcal{J}|^{-1/2})$, then θ is of order $O(1)$.

Remark 2. [8] proved a Berry–Esseen bound of the same order as in Theorem 1 for non-self-normalized sums. Specifically, assume further that for any $i \in \mathcal{J}$, there exists

$C_i \subset \mathcal{J}$ such that $C_i \supset B_i$ and $\{X_j : j \in B_i\}$ is independent of $\{X_j : j \notin C_i\}$, and it follows that (see Theorem 2.4 of [8])

$$\sup_{z\in\mathbb{R}}|\mathbb{P}(S/\sigma \leqslant z) - \Phi(z)| \leqslant 75(\kappa')^{p-1}\sum_{i\in\mathcal{J}}\mathbb{E}|X_i/\sigma|^p,$$

where $\kappa' \geqslant \max(|\{j : B_j \cap C_i \neq \emptyset\}|, |\{j : i \in C_j\}|)$. Under the same dependence condition as in Theorem 1.1, [8] proved that for $2 < p \leqslant 4$,

$$\begin{aligned}\sup_{z\in\mathbb{R}}\big|\mathbb{P}(S/\sigma \leqslant z) - \Phi(z)\big| \leqslant{}& C(1+\kappa)\sum_{i\in\mathcal{J}}\big(\mathbb{E}\,|X_i/\sigma|^{3\wedge p} + \mathbb{E}\,|Y_i/\sigma|^{3\wedge p}\big)\\ &+ C\kappa^{1/2}\left(\sum_{i\in\mathcal{J}}\Big(\mathbb{E}|X_i/\sigma|^p + \mathbb{E}|Y_i/\sigma|^p\Big)\right)^{1/2},\end{aligned}$$

which is of the best possible order $O(n^{-1/2})$ when $p = 4$ and $\kappa = O(1)$. For the self-normalized sum S/V, the best possible order can be obtained under a third moment condition.

Remark 3. Specially, if $\{X_i, i \in \mathcal{J}\}$ is a field of independent random variables, then it is not hard to see that $\kappa = \theta = 1$. Then, by (2.3), $|\mathcal{J}|^{-1/2} \leqslant \beta_3$. Hence, the right hand side of (2.2) reduces to $C(\beta_0 + \beta_2 + \beta_3)$, which is as same as the results in [3].

3. Applications

3.1. Self-normalized sums of m-dependent random variables

Let $d \geqslant 1$ and let $\mathbb{Z}^d$ denote the d-dimensional space of positive integers. For any $i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, we define the distance by $|i-j| := \max_{1\leqslant k\leqslant d}|i_k - j_k|$, and for $A, B \subset \mathbb{Z}^d$, we define the distance between A and B by $\rho(A, B) = \inf\{|i-j| : i \in A, j \in B\}$. Let $\mathcal{J}$ be a subset of $\mathbb{Z}^d$, and we say a field of random variables $\{X_i : i \in \mathcal{J}\}$ is an m-dependent random field if $\{X_i, i \in A\}$ and $\{X_j, j \in B\}$ are independent whenever $\rho(A, B) > m$ for any $A, B \subset \mathcal{J}$. Choose $A_i = \{j : |i-j| \leqslant m\}$, $B_i = \{j : |i-j| \leqslant 2m\}$. We have $|\{j : A_j \cap B_i\}| \leqslant (6m+1)^d$ for all i. Applying Theorem 1, we have the following theorem.

Theorem 2. *Let $\{X_i, i \in \mathcal{J}\}$ be an m-dependent field with $\mathbb{E}\,X_i = 0$ and assume that $\mathbb{E}\,|X_i|^3 < \infty$. Let $Y_i = \sum_{j\in A_i} X_i$. Assume that $\sigma^2 := \sum_{i\in\mathcal{J}}\mathbb{E}\{X_iY_i\} > 0$. Let W be* *as in (2.1). Then,*

$$\sup_{z\in\mathbb{R}}\big|\mathbb{P}(W\leqslant z)-\Phi(z)\big|\leqslant C(m+1)^{3d}(1+(m+1)^d|\mathcal{J}|^{1/3}\gamma^{2/3})(\gamma+|\mathcal{J}|^{-1/2}),$$

where $\gamma=\sum_{i\in\mathcal{J}}\mathbb{E}\,|X_i|^3/\sigma^3$.

3.2. Graph dependency

We now consider a field of random variables $\{X_i : i\in\mathcal{V}\}$ indexed by the vertices of a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$. We say $\mathcal{G}$ is a dependency graph if for any pair of disjoint sets Γ_1 and Γ_2 in $\mathcal{V}$ such that no edge has one endpoint in Γ_1 and the other in Γ_2, then the sets of random variables $\{X_i : i\in\Gamma_1\}$ and $\{X_j : j\in\Gamma_2\}$ are independent. Let $A_i=\{j\in\mathcal{V} : \text{there is an edge connecting } i \text{ and } j\}$, $A_{ij}=A_i\cup A_j$, $B_i=\cup_{j\in A_i}A_j$. Noting that $|\{j : A_j\cap B_i\}|\leqslant Cd^3$, and applying Theorem 1, we have the following theorem.

Theorem 3. *Let $\{X_i, i\in\mathcal{V}\}$ be a field of random variables indexed by the vertices of a dependency graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$. Assume that $\mathbb{E}\,X_i=0$. Let $S=\sum_{i\in\mathcal{V}}X_i$, $Y_i=\sum_{j\in A_i}X_j$ and let $V=\sqrt{(\sum_{i\in\mathcal{V}}(X_iY_i-\bar{X}\bar{Y}))_+}$. Put $W=S/V$ and $n=|\mathcal{V}|$. Let d be the maximal degree of $\mathcal{G}$. Then, we have*

$$\sup_{z\in\mathbb{R}}|\mathbb{P}(W\leqslant z)-\Phi(z)|\leqslant Cd^9(1+d^6n^{1/3}\gamma^{2/3})\big(n^{-1/2}+\gamma\big),$$

where $\gamma=\sum_{i\in\mathcal{V}}\mathbb{E}|X_i/\sigma|^3$, *and* $\sigma^2=\sum_{i\in\mathcal{V}}\mathbb{E}\{X_iY_i\}$.

Graph dependency was firstly discussed in [1], where a Berry–Esseen bound for non-self-normalized version of W is also proved. However, the self-normalized Berry–Esseen bound is new.

4. Some preliminary lemmas and propositions

In this section, we first prove some preliminary lemmas, and then prove an important concentration inequality, which is of independent interest.

4.1. Some Preliminary lemmas

Let

$$\begin{aligned}
&\bar{X}_i := X_i \mathbb{I}(|X_i| \leqslant \sigma/\kappa), \quad \bar{Y}_i := \sum_{j\in A_i} \bar{X}_i, \qquad \bar{\sigma} = \Big(\sum_{j\in\mathcal{J}} \mathbb{E}\{\bar{X}_j \bar{Y}_j\}\Big)^{1/2}, \\
&\bar{S} := \sum_{i\in\mathcal{J}} \bar{X}_i, \qquad \bar{V} := \psi\Big(\sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i\Big), \quad \bar{W} := \bar{S}/\bar{V},
\end{aligned} \tag{4.1}$$

where

$$\psi(x) = ((x \vee 0.25\sigma^2) \wedge 2\sigma^2)^{1/2}. \tag{4.2}$$

The following lemma provides bounds for $\bar{\sigma}^2$ and some tail and moment inequalities for $\{\bar{X}_i, i \in \mathcal{J}\}$.

Lemma 1. *Assume that (LD1) holds. We have*

$$\left|\bar{\sigma}^2 - \sigma^2\right| \leqslant 3\kappa\sigma^2\beta_2, \tag{4.3}$$

$$\mathbb{E}\bigg(\sum_{i\in\mathcal{J}} (\bar{X}_i\bar{Y}_i - \mathbb{E}\{\bar{X}_i\bar{Y}_i\})\bigg)^2 \leqslant \kappa^2\sigma^4\beta_3, \tag{4.4}$$

$$\mathbb{P}\bigg\{\bigg|\sum_{i\in\mathcal{J}} (\bar{X}_i\bar{Y}_i - \mathbb{E}\{\bar{X}_i\bar{Y}_i\})\bigg| \geqslant \frac{\sigma^2}{2}\bigg\} \leqslant 4\kappa^2\beta_3. \tag{4.5}$$

Specially, if $\beta_2 \leqslant 1/(150\kappa)$*, we have*

$$0.98\sigma^2 \leqslant \bar{\sigma}^2 \leqslant 1.02\sigma^2. \tag{4.6}$$

Proof of Lemma 1. We first prove (4.3). Note that for any $i, j \in \mathcal{J}$,

$$\begin{aligned}
&|\mathbb{E}\{X_iX_j\} - \mathbb{E}\{\bar{X}_i\bar{X}_j\}| \\
&\quad \leqslant \mathbb{E}|(X_i - \bar{X}_i)\bar{X}_j| + \mathbb{E}|\bar{X}_i(X_j - \bar{X}_j)| + \mathbb{E}|(X_i - \bar{X}_i)(X_j - \bar{X}_j)| \\
&\quad \leqslant \frac{\sigma}{\kappa}(\mathbb{E}|X_i\mathbb{I}(|X_i| > \sigma/\kappa)| + \mathbb{E}|X_j\mathbb{I}(|X_j| > \sigma/\kappa)|) \\
&\qquad + \mathbb{E}\{|X_iX_j|\mathbb{I}(|X_i| > \sigma/\kappa, |X_j| > \sigma/\kappa)\} \\
&\quad \leqslant 1.5\big(\mathbb{E}\{|X_i|^2\mathbb{I}(|X_i| > \sigma/\kappa)\} + \mathbb{E}\{|X_j|^2\mathbb{I}(|X_j| > \sigma/\kappa)\}\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\bar\sigma^2-\sigma^2\right| &= \left|\sum_{i\in\mathcal{J}}\sum_{j\in A_i}\mathbb{E}\{X_iX_j\} - \sum_{i\in\mathcal{J}}\sum_{j\in A_i}\mathbb{E}\{\bar X_i\bar X_j\}\right|\\
&\leqslant 1.5\sum_{i\in\mathcal{J}}\sum_{j\in A_i}\left(\mathbb{E}\{|X_i|^2\mathbb{I}(|X_i|>\sigma/\kappa)+|X_j|^2\mathbb{I}(|X_j|>\sigma/\kappa)\}\right)\\
&\leqslant 1.5\sum_{i\in\mathcal{J}}\sum_{j\in A_i}\mathbb{E}\{|X_i|^2\mathbb{I}(|X_i|>\sigma/\kappa)\}\\
&\quad + 1.5\sum_{j\in\mathcal{J}}\sum_{i:j\in A_i}\mathbb{E}\{|X_j|^2\mathbb{I}(|X_j|>\sigma/\kappa)\})\\
&\leqslant 3\sigma^2\kappa\beta_2.
\end{aligned}$$

This proves (4.3). Specially, (4.6) follows directly from (4.3).

Now, by Cauchy's inequality, we have with $c=\kappa^2$,

$$\begin{aligned}
\mathbb{E}\{\bar X_i^2\bar Y_i^2\} &\leqslant \frac{c}{2}\,\mathbb{E}\,\bar X_i^4 + \frac{\kappa^3}{2c}\sum_{j\in A_i}\mathbb{E}\,\bar X_j^4 = \frac{\kappa^2}{2}\,\mathbb{E}\,\bar X_i^4 + \frac{\kappa}{2}\sum_{j\in A_i}\mathbb{E}\,\bar X_j^4\\
&\leqslant \frac{\kappa\sigma}{2}\,\mathbb{E}\,\bar X_i^3 + \frac{\sigma}{2}\sum_{j\in A_i}\mathbb{E}\,\bar X_j^3,
\end{aligned}\tag{4.7}$$

where we used the fact that $|\bar X_i|\leqslant\sigma/\kappa$ in the last line. Taking summation over $i\in\mathcal{J}$ on both sides of (4.7) yields

$$\sum_{i\in\mathcal{J}}\mathbb{E}\{\bar X_i^2\bar Y_i^2\}\leqslant\kappa\sigma^4\beta_3.\tag{4.8}$$

By (4.8), we have the left hand side of (4.4) is

$$\begin{aligned}
&\mathbb{E}\left(\sum_{i\in\mathcal{J}}\sum_{j\in B_i}(\bar X_i\bar Y_i-\mathbb{E}\{\bar X_i\bar Y_i\})(\bar X_j\bar Y_j-\mathbb{E}\{\bar X_j\bar Y_j\})\right)\\
&\quad=\sum_{i\in\mathcal{J}}\sum_{j\in B_i}\left\{\frac12\,\mathbb{E}\{\bar X_i^2\bar Y_i^2\}+\frac12\,\mathbb{E}\{\bar X_j^2\bar Y_j^2\}\right\}\\
&\quad=\frac12\left(\sum_{i\in\mathcal{J}}\sum_{j\in B_i}\mathbb{E}\{\bar X_i^2\bar Y_i^2\}+\sum_{j\in\mathcal{J}}\sum_{i:j\in B_i}\mathbb{E}\{\bar X_j^2\bar Y_j^2\}\right)\\
&\quad\leqslant\kappa^2\sigma^4\beta_3.
\end{aligned}$$

This proves (4.4). Moreover, (4.5) follow immediately from (4.4) and the Markov inequality. □

The following lemma will be useful in the proof of our main result. Let $\xi_i = \bar X_i/\bar V, \eta_i=\bar Y_i/\bar V$ and $W^{(i)}=W-\eta_i$.

Lemma 2. *Assume that* (LD1) *holds and assume that* $\beta_2 \leqslant 1/(150\kappa)$. *Let* f *be any absolutely continuous function such that* $\|f\| \leqslant 1$ *and* $\|f'\| \leqslant 1$. *We have*

$$\sum_{i\in\mathcal{J}} \big|\mathbb{E}\{\xi_i f(\bar{W} - \eta_i)\}\big| \leqslant 62\kappa^2\beta_3 + 2\kappa\beta_2. \tag{4.9}$$

Proof. Let $N(A_i) = \{j : A_i \cap A_j \neq \emptyset\}$, $N(B_i) = \{j : B_i \cap A_j \neq \emptyset\}$, and

$$\bar{V}^{(i)} = \psi\biggl(\sum_{k\in A_i^c} \sum_{l \in A_i^c \cap A_k} \bar{X}_k \bar{X}_l\biggr).$$

The left hand side of (4.9) can be bounded by

$$\begin{aligned}
\sum_{i\in\mathcal{J}} \big|\mathbb{E}\{\xi_i f(\bar{W} - \eta_i)\}\big| &\leqslant \sum_{i\in\mathcal{J}} \biggl|\mathbb{E}\biggl\{\frac{\bar{X}_i}{\bar{V}} f\Bigl(\frac{\bar{S} - \bar{Y}_i}{\bar{V}}\Bigr)\biggr\} - \mathbb{E}\biggl\{\frac{\bar{X}_i}{\bar{V}^{(i)}} f\Bigl(\frac{\bar{S} - \bar{Y}_i}{\bar{V}^{(i)}}\Bigr)\biggr\}\biggr| \\
&\quad + \sum_{i\in\mathcal{J}} \biggl|\mathbb{E}\biggl\{\frac{\bar{X}_i}{\bar{V}^{(i)}} f\Bigl(\frac{\bar{S} - \bar{Y}_i}{\bar{V}^{(i)}}\Bigr)\biggr\}\biggr| \\
&\coloneqq T_1 + T_2.
\end{aligned}$$

Recall that $\bar{V} \geqslant \sigma/2$ and $\bar{V}^{(i)} \geqslant \sigma/2$. Then, it follows that

$$\begin{aligned}
\biggl|\frac{1}{\bar{V}} - \frac{1}{\bar{V}^{(i)}}\biggr| &\leqslant \frac{1}{\bar{V}\bar{V}^{(i)}(\bar{V} + \bar{V}^{(i)})} \biggl|\sum_{j\in A_i} \sum_{k\in A_j} \bar{X}_j \bar{X}_k + \sum_{j\in A_i^c} \sum_{k \in A_i \cap A_j} \bar{X}_j \bar{X}_k\biggr| \\
&\leqslant \frac{4}{\sigma^3} \biggl(\sum_{j\in A_i} \sum_{k\in A_j} |\bar{X}_j \bar{X}_k| + \sum_{j\in A_i^c} \sum_{k\in A_j\cap A_i} |\bar{X}_j \bar{X}_k|\biggr).
\end{aligned} \tag{4.10}$$

By (4.10),

$$\begin{aligned}
T_1 &\leqslant \sum_{i\in\mathcal{J}} \mathbb{E}\biggl\{\biggl(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\biggr)|\bar{X}_i| \biggl|\frac{1}{\bar{V}} - \frac{1}{\bar{V}^{(i)}}\biggr|\biggr\} \\
&\leqslant \frac{4}{\sigma^3} \sum_{i\in\mathcal{J}} \sum_{j\in A_i} \sum_{k\in A_j} \mathbb{E}\biggl\{\biggl(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\biggr)|\bar{X}_i \bar{X}_j \bar{X}_k|\biggr\} \\
&\quad + \frac{4}{\sigma^3} \sum_{i\in\mathcal{J}} \sum_{k\in A_i} \sum_{j : k\in A_j} \mathbb{E}\biggl\{\biggl(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\biggr)|\bar{X}_i \bar{X}_j \bar{X}_k|\biggr\} \\
&\leqslant \frac{8\kappa^2}{3\sigma^3} \sum_{i\in\mathcal{J}} \mathbb{E}\biggl\{\biggl(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\biggr)|\bar{X}_i^3|\biggr\} \\
&\quad + \frac{8\kappa}{3\sigma^2} \sum_{j\in\mathcal{J}} \sum_{i : j\in A_i} \mathbb{E}\biggl\{\biggl(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\biggr)|\bar{X}_j^3|\biggr\} \\
&\quad + \frac{8}{3\sigma^2} \sum_{k\in\mathcal{J}} \sum_{j : k\in A_j} \sum_{i : j\in A_i} \mathbb{E}\biggl\{\biggl(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\biggr)|\bar{X}_k^3|\biggr\}.
\end{aligned} \tag{4.11}$$

Let $\bar{Y}_{ij} = \sum_{k\in A_i\cup A_j} \bar{X}_k$. Then, by $|\bar{Y}_{ij} - \bar{Y}_i| \leqslant \sum_{k\in A_j} |\bar{X}_k| \leqslant \sigma$, we have

$$\begin{aligned}
&\mathbb{E}\bigg\{\bigg(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\bigg)|\bar{X}_i|^3\bigg\} = \mathbb{E}\bigg\{\bigg(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\bigg)\bigg\}\,\mathbb{E}\{|\bar{X}_i|^3\},\\
&\mathbb{E}\bigg\{\bigg(1 + \frac{2|\bar{S} - \bar{Y}_i|}{\sigma}\bigg)|\bar{X}_j|^3\bigg\} \leqslant \mathbb{E}\bigg\{\bigg(1 + \frac{2|\bar{S} - \bar{Y}_{ij}|}{\sigma}\bigg)\bigg\}\,\mathbb{E}\{|\bar{X}_j|^3\} + 2\,\mathbb{E}|\bar{X}_j|^3.
\end{aligned} \tag{4.12}$$

As $|\bar{Y}_i| \leqslant \sigma, |\bar{Y}_{ij}| \leqslant 2\sigma$ and $\mathbb{E}\,|\bar{S}| \leqslant \bar{\sigma} \leqslant 1.02\sigma$, we have

$$\mathbb{E}|\bar{S} - \bar{Y}_i| \leqslant 2.02\sigma, \quad \mathbb{E}|\bar{S} - \bar{Y}_{ij}| \leqslant 3.02\sigma. \tag{4.13}$$

Substituting (4.12) and (4.13) to (4.11) yields

$$T_1 \leqslant 62\kappa^2\beta_3. \tag{4.14}$$

Moreover, observe that

$$\begin{aligned}
\sum_{i\in\mathcal{J}} |\mathbb{E}\,\bar{X}_i| &\leqslant \sum_{i\in\mathcal{J}} \mathbb{E}\{|X_i|\mathbb{I}(|X_i| > \sigma/\kappa)\}\\
&\leqslant \frac{\kappa}{\sigma}\sum_{i\in\mathcal{J}} \mathbb{E}\{|X_i|^2\mathbb{I}(|X_i| \geqslant \sigma/\kappa)\} = \kappa\sigma\beta_2.
\end{aligned} \tag{4.15}$$

For T_2, as $\bar{V}^{(i)} \geqslant \sigma/2$ and $\|f\| \leqslant 1$, by (4.15), we have

$$|T_2| \leqslant \frac{2}{\sigma}\sum_{i\in\mathcal{J}} |\mathbb{E}\{\bar{X}_i\}| \leqslant 2\kappa\beta_2. \tag{4.16}$$

Combining (4.14) and (4.16), we complete the proof. □

The next lemma provides upper bounds for the fourth moments of $\sum_{i\in\mathcal{J}} \bar{X}_i$ and $\sum_{i\in\mathcal{J}} \bar{Y}_i$.

Lemma 3. *Under (LD1) and (LD2), and assume that $\beta_2 \leqslant 1/(150\kappa)$. We have*

$$\mathbb{E}\bigg(\sum_{i\in\mathcal{J}} \bar{X}_i\bigg)^4 \leqslant 1161(\theta + 1)\sigma^4, \tag{4.17}$$

$$\mathbb{E}\bigg(\sum_{i\in\mathcal{J}} \bar{Y}_i\bigg)^4 \leqslant 1161\kappa^2(\theta + 1)\sigma^4. \tag{4.18}$$

As a consequence,

$$\mathbb{E}\bigg\{\bigg(\sum_{i\in\mathcal{J}} \bar{X}_i\bigg)^2\bigg(\sum_{j\in\mathcal{J}} \bar{Y}_j\bigg)^2\bigg\} \leqslant 1161\kappa(\theta + 1)\sigma^4. \tag{4.19}$$

Proof. We first prove the first inequality. Recall that $\mathbb{E}\, X_i = 0$ and thus

$$|\,\mathbb{E}\,\bar{S}| = \Big|\sum_{i\in\mathcal{J}} \mathbb{E}\,\bar{X}_i\Big| = \Big|\sum_{i\in\mathcal{J}} \mathbb{E}\{X_i \mathbb{I}(|X_i| > \sigma/\kappa)\}\Big| \leqslant \beta_2\kappa\sigma. \tag{4.20}$$

Let $\bar{X}_{i,0} = \bar{X}_i - \mathbb{E}\{\bar{X}_i\}$ and let $\bar{Y}_{i,0} = \sum_{j\in A_i} \bar{X}_{j,0}$. Under (LD1), we have

$$\begin{aligned}
\mathbb{E}|\bar{S} - \mathbb{E}\,\bar{S}|^4 &= \sum_{i\in\mathcal{J}} \mathbb{E}\bigg\{\bar{X}_{j,0}\bigg(\Big(\sum_{i\in\mathcal{J}} \bar{X}_{j,0}\Big)^3 - \Big(\sum_{i\in A_i^c} \bar{X}_{j,0}\Big)^3\bigg)\bigg\} \\
&= 3\sum_{j\in\mathcal{J}}\sum_{j\in A_i} \mathbb{E}\bigg\{\bar{X}_{i,0}\bar{X}_{j,0}\Big(\sum_{k\in\mathcal{J}} \bar{X}_{k,0}\Big)^2\bigg\} \\
&\quad - 3\sum_{i\in\mathcal{J}}\sum_{j\in A_i}\sum_{k\in A_i}\sum_{l\in\mathcal{J}} \mathbb{E}\{\bar{X}_{i,0}\bar{X}_{j,0}\bar{X}_{k,0}\bar{X}_{l,0}\} \\
&\quad + \sum_{i\in\mathcal{J}}\sum_{j\in A_i}\sum_{k\in A_i}\sum_{l\in A_i} \mathbb{E}\{\bar{X}_{i,0}\bar{X}_{j,0}\bar{X}_{k,0}\bar{X}_{l,0}\} \\
&\leqslant T_1 + T_2,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \frac{9}{2}\sum_{j\in\mathcal{J}} \mathbb{E}\bigg\{|\bar{X}_{i,0}\bar{Y}_{i,0}|\Big(\sum_{k\in\mathcal{J}} \bar{X}_{k,0}\Big)^2\bigg\}, \\
T_2 &= \frac{5}{2}\sum_{i\in\mathcal{J}}\sum_{j\in A_i}\sum_{k\in A_i}\sum_{l\in A_i} \mathbb{E}\{|\bar{X}_{i,0}\bar{X}_{j,0}\bar{X}_{k,0}\bar{X}_{l,0}|\}.
\end{aligned}$$

By (LD2), we have

$$\begin{aligned}
|T_1| &\leqslant 9\sum_{i\in\mathcal{J}} \mathbb{E}\bigg\{|\bar{X}_{i,0}\bar{Y}_{i,0}|\Big(\sum_{k\in B_i^c} \bar{X}_{k,0}\Big)^2\bigg\} + 9\sum_{i\in\mathcal{J}} \mathbb{E}\bigg\{|\bar{X}_{i,0}\bar{Y}_{i,0}|\Big(\sum_{k\in B_i} \bar{X}_{k,0}\Big)^2\bigg\} \\
&\coloneqq T_{11} + T_{12}.
\end{aligned}$$

For T_{11}, noting that $(\bar{X}_{i,0}, \bar{Y}_{i,0})$ is independent of $\{\bar{Y}_{k,0} : j \in B_i^c\}$, then

$$T_{11} = 9\sum_{i\in\mathcal{J}} \mathbb{E}|\bar{X}_{i0}\bar{Y}_{j,0}| \times \mathbb{E}\bigg\{\Big(\sum_{k\in B_i^c} \bar{X}_{k,0}\Big)^2\bigg\}. \tag{4.21}$$

Note that $|\bar{X}_{i,0}| \leqslant 2\sigma/\kappa$. By Lemma 1, the second expectation of (4.21) can be bounded by

$$\begin{aligned}
\mathbb{E}\bigg\{\Big(\sum_{k\in B_i^c} \bar{X}_{k,0}\Big)^2\bigg\} &\leqslant 2\,\mathbb{E}\bigg\{\Big(\sum_{k\in\mathcal{J}} \bar{X}_{k,0}\Big)^2\bigg\} + 2\,\mathbb{E}\bigg\{\Big(\sum_{k\in B_i} \bar{X}_{k,0}\Big)^2\bigg\} \\
&\leqslant 2\,\mathbb{E}\,|\bar{S}|^2 + 8\sigma^2 \leqslant 10.04\sigma^2.
\end{aligned} \tag{4.22}$$

Recalling that $\theta = \sum_{i\in\mathcal{J}}\sum_{j\in A_i}\mathbb{E}|\bar{X}_i\bar{X}_j/\sigma^2|$, and noting that $|\bar{Y}_i| \leqslant 1$, we have the first expectation of (4.21) is bounded by

$$\begin{aligned}\sum_{i\in\mathcal{J}}\mathbb{E}\,|\bar{X}_{i,0}\bar{Y}_{i,0}| &\leqslant \sum_{i\in\mathcal{J}}(\mathbb{E}\,|X_iY_i| + 3\kappa|\,\mathbb{E}\,\bar{X}_i|)\\ &\leqslant \theta\sigma^2 + 3\sigma^2\kappa\beta_2 \leqslant (\theta+0.02)\sigma^2,\end{aligned} \tag{4.23}$$

where we used (4.15) and the assumption that $\beta_2 \leqslant 1/(150\kappa)$ in the last line. Substituting (4.22) and (4.23) to (4.21) gives

$$T_{11} \leqslant 99(\theta+1)\sigma^4. \tag{4.24}$$

For T_{12}, as $\sum_{k\in B_i}|\bar{X}_{k,0}| \leqslant 2\sigma$, we have $(\sum_{k\in B_i}|\bar{X}_{k,0}|)^2 \leqslant 4\sigma^2$ almost surely, and thus

$$T_{12} \leqslant 36\sum_{i\in\mathcal{J}}\mathbb{E}\,|\bar{X}_{i,0}\bar{Y}_{i,0}| \leqslant 36(\theta+1)\sigma^4. \tag{4.25}$$

By (4.23) and noting that $(\sum_{k\in A_i}\bar{X}_{k,0})^2 \leqslant 4\sigma^2$ almost surely, we have

$$|T_2| \leqslant 10(\theta+1)\sigma^4. \tag{4.26}$$

By (4.20) and (4.24)–(4.26), and recalling that $\beta_2 \leqslant 1/(150\kappa^2)$ and $\kappa \geqslant 1$, we have

$$\begin{aligned}\mathbb{E}\,|\bar{S}|^4 &\leqslant 8\big(\mathbb{E}|\bar{S} - \mathbb{E}\{\bar{S}\}|^4 + |\mathbb{E}\,\bar{S}|^4\big)\\ &\leqslant 8\big(145(\theta+1)\sigma^4 + \beta_2^4\sigma^4\big)\\ &\leqslant 1161(\theta+1)\sigma^4.\end{aligned} \tag{4.27}$$

This proves (4.17).

For the second inequality, observe that

$$\sum_{i\in\mathcal{J}}\bar{Y}_i = \sum_{i\in\mathcal{J}}\sum_{j\in A_i}\bar{X}_j = \sum_{j\in\mathcal{J}}\lambda_j\bar{X}_j,$$

where $\lambda_j \coloneqq |\{i : j \in A_i\}|$. Then, we have $\lambda_j \leqslant \kappa$ for all $j \in \mathcal{J}$. Using a similar argument, we have (4.18) holds. The inequality (4.19) follows from (4.17) and (4.18) by applying Cauchy's inequality. □

4.2. Concentration inequalities

In this subsection, we prove a concentration inequality under (LD1) and (LD2). We denote by $C, C_1, C_2, \ldots$ absolute constants.

Proposition 1. *Assume that (LD1) and (LD2) holds and assume that* $\beta_2 \leqslant 1/(150\kappa)$ *and* $\beta_3 \leqslant 1/(150\kappa^2)$. *Let* $\mathcal{F}_i = \sigma(X_j : j \in A_i)$. *For any* $z \in \mathbb{R}$ *and* $\mathcal{F}_i$*-measurable random variables* a *and* b *such that* $a \leqslant b$, *we have*

$$\begin{aligned}\mathbb{P}^{\mathcal{F}_i}&\Big(z + \frac{a}{\bar{V}} \leqslant \bar{W} \leqslant z + \frac{b}{\bar{V}}\Big)\\ &\leqslant \frac{2(b-a)}{\sigma} + C_1\kappa^2\Big(\beta_3 + \sum_{j\in N(B_i)} \big(\mathbb{E}^{\mathcal{F}_i}\,|\bar{X}_j/\sigma|^3 + \mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_j^3/\sigma^4|\big)\Big).\end{aligned}$$

where $\mathbb{P}^{\mathcal{F}_i}$ *and* $\mathbb{E}^{\mathcal{F}_i}$ *denote the conditional probability and conditional expectation given* $\mathcal{F}_i$, *respectively.*

Proof of Proposition 1. We use the idea of Proposition 3.2 in [8] to prove this proposition. Let $\alpha = 20\kappa^2\sigma\beta_3$. Without loss of generality, we assume that $b-a \leqslant \sigma/2$, otherwise the result is trivial. As $\kappa^3 \geqslant 1$ and by assumption we have $\kappa^2\beta_3 \leqslant 1/150$. Therefore, we have

$$b - a + \alpha \leqslant 0.8\sigma. \tag{4.28}$$

Let

$$f_{a,b,v}(w) = \begin{cases} -\dfrac{b-a+\alpha}{2v} & \text{if } w \leqslant z + a/v - \alpha/v,\\[2mm] \dfrac{1}{2\alpha}\Big(w - \dfrac{\alpha+\alpha}{v}\Big)^2 - \dfrac{b-a+\alpha}{2v} & \text{if } z + \dfrac{a-\alpha}{v} < w \leqslant z + \dfrac{a}{v},\\[2mm] w - \dfrac{a+b}{2v} & \text{if } z + a/v < w \leqslant z + b/v,\\[2mm] -\dfrac{1}{2\alpha}\Big(w - \dfrac{b-\alpha}{v}\Big)^2 + \dfrac{b-a+\alpha}{2v} & \text{if } z + \dfrac{b}{v} < w \leqslant z + \dfrac{b+\alpha}{v},\\[2mm] \dfrac{b-a+\alpha}{2v} & \text{if } w > z + (b+\alpha)/v. \end{cases} \tag{4.29}$$

Then, we have

$$f'_{a,b,v}(w) = \begin{cases} 1 & \text{if } z + a/v \leqslant w \leqslant z + b/v,\\ 0 & \text{if } w < z + a/v - \alpha/v \text{ or } w > z + b/v + \alpha/v,\\ \text{linear} & \text{otherwise.} \end{cases} \tag{4.30}$$

Let $f := f_{a,b,\bar{V}}$. Then, it follows from $\bar{V} \geqslant \sigma/2$ that $\|f\| \leqslant (b-a+\alpha)/(2\bar{V}) \leqslant (b-a+\alpha)/\sigma$. Fix i. Recall that $N(A_i) = \{j : A_i \cap A_j \neq \emptyset\}$ and $N(B_i) = \{j : B_i \cap A_j \neq \emptyset\}$. Let $\sigma_i = (\mathbb{E}\{\sum_{j\in N(B_i)^c} \bar{X}_j\}^2)^{1/2}$. Then, recalling that $\beta_3 \leqslant 1/(150\kappa^2)$ and by Hölder's inequality, we have

$$\sigma_i \leqslant \Big(|N(B_i)|^2 \sum_{j\in N(B_i)} \mathbb{E}|\bar{X}_j|^3\Big)^{1/3} \leqslant \big(\kappa^2\sigma^3\beta_3\big)^{1/3} \leqslant 0.2\sigma.$$

Recall that ξ_i is as defined in Lemma 2. On one hand, as $\|f\| \leqslant (b-a+\alpha)/\sigma$, $\bar{V} \geqslant \sigma/2$, and $\{X_j : j \notin N(B_i)\}$ is independent of $\mathcal{F}_i$, we have

$$\begin{aligned}\bigg|\mathbb{E}^{\mathcal{F}_i}\Big\{\Big(\sum_{j\notin N(B_i)} \xi_j\Big) f(\bar{W})\Big\}\bigg| &\leqslant \frac{2(b-a+\alpha)}{\sigma^2}\,\mathbb{E}\Big|\sum_{j\notin N(B_i)} \bar{X}_i\Big| \\ &\leqslant \frac{2\sigma_i}{\sigma^2}(b-a+\alpha) \leqslant \frac{b-a+\alpha}{2\sigma}.\end{aligned} \tag{4.31}$$

Let

$$\begin{aligned}\widehat{M}_j(t) &= \bar{X}_j\big(\mathbb{I}(-\bar{Y}_j \leqslant t \leqslant 0) - \mathbb{I}(0 < t \leqslant -\bar{Y}_j)\big), & M_j(t) &= \mathbb{E}\{\widehat{M}_j(t)\},\\ \widehat{M}(t) &= \sum_{j\in N(B_i)^c} \widehat{M}_j(t), & M(t) &= \mathbb{E}\{\widehat{M}(t)\}.\end{aligned}$$

For the lower bound of the left hand side of (4.31), observe that

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_i}\Big\{\Big(\sum_{j\notin N(B_i)} \xi_j\Big) f(\bar{W})\Big\} &= \sum_{j\notin N(B_i)} \mathbb{E}^{\mathcal{F}_i}\big\{\xi_j(f(\bar{W}) - f(\bar{W}-\eta_j)\big\}\\ &\quad + \sum_{j\notin N(B_i)} \mathbb{E}^{\mathcal{F}_i}\{\xi_j f(\bar{W}-\eta_j)\}\\ &\coloneqq Q_1 + Q_2 + Q_3 + Q_4,\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}Q_1 &= \mathbb{E}^{\mathcal{F}_i}\Big\{\frac{f'(\bar{W})}{\bar{V}^2}\int_{-\infty}^{\infty} M(t)dt\Big\},\\ Q_2 &= \mathbb{E}^{\mathcal{F}_i}\Big\{\frac{1}{\bar{V}^2}\int_{-\infty}^{\infty}\Big(f'\Big(\frac{\bar{S}+t}{\bar{V}}\Big) - f'\Big(\frac{\bar{S}}{\bar{V}}\Big)\Big)M(t)dt\Big\},\\ Q_3 &= \mathbb{E}^{\mathcal{F}_i}\Big\{\frac{1}{\bar{V}^2}\int_{-\infty}^{\infty} f'\Big(\frac{\bar{S}+t}{\bar{V}}\Big)\big(\widehat{M}(t) - M(t)\big)dt\Big\},\\ Q_4 &= \sum_{j\in N(B_i)^c} \mathbb{E}^{\mathcal{F}_i}\big\{\xi_j f(\bar{W}-\eta_j)\big\}.\end{aligned}$$

We now bound Q_1 to Q_4 one by one.

Observe that

$$\int_{-\infty}^{\infty} M(t)dt = \sum_{j\in N(B_i)^c} \mathbb{E}\{\bar{X}_j\bar{Y}_j\} = \bar{\sigma}^2 - \sum_{j\in N(B_i)} \mathbb{E}\{\bar{X}_j\bar{Y}_j\}. \tag{4.33}$$

Now, by Hölder's inequality, the second term of the R.H.S. of (4.33) is bounded by

$$\begin{aligned}\Big|\sum_{j\in N(B_i)}\mathbb{E}\{\bar X_j\bar Y_j\}\Big| &\leqslant \Big(|N(B_i)|^{1/2}\sum_{j\in N(B_i)}\mathbb{E}|\bar X_j\bar Y_j|^{3/2}\Big)^{2/3}\\ &\leqslant \Big(\kappa^{1/2}\sum_{j\in N(B_i)}\Big[\frac{\kappa^{3/2}}{2}\,\mathbb{E}|\bar X_j|^3+\frac{1}{2\kappa^{3/2}}\,\mathbb{E}|\bar Y_j|^3\Big]\Big)^{2/3}\\ &\leqslant (\kappa^2\beta_3)^{2/3}\sigma^2\leqslant (1/150)^{2/3}\sigma^2\leqslant 0.05\sigma^2.\end{aligned}$$

Thus, by (4.6) we have

$$\Big|\int_{-\infty}^{\infty}M(t)dt-\sigma^2\Big|\leqslant 0.1\sigma^2.$$

Note that $\sigma^2/4\leqslant \bar V^2\leqslant 2\sigma^2$. For Q_1, we have

$$\begin{aligned}Q_1&\geqslant 0.9\,\mathbb{E}^{\mathcal F_i}\Big\{\frac{\sigma^2}{\bar V^2}\mathbb{I}(z+a/\bar V\leqslant \bar W\leqslant a+b/\bar V)\Big\}\\ &\geqslant 0.45\,\mathbb{P}^{\mathcal F_i}\big(z+a/\bar V\leqslant \bar W\leqslant a+b/\bar V\big).\end{aligned}\tag{4.34}$$

For Q_3,

$$|Q_3|\leqslant Q_{31}+Q_{32},$$

where

$$\begin{aligned}Q_{31}&=\mathbb{E}^{\mathcal F_i}\int_{|t|\leqslant\sigma}\frac{1}{\bar V^2}f'\Big(\frac{\bar S+t}{\bar V}\Big)|\widehat M(t)-M(t)|dt,\\ Q_{32}&=\mathbb{E}^{\mathcal F_i}\int_{|t|>\sigma}\frac{1}{\bar V^2}|\widehat M(t)-M(t)|dt.\end{aligned}$$

For Q_{31}, noting that $\|f'\|\leqslant 1$, we have

$$\frac{1}{v}\int_{|t|\leqslant\sigma}\Big|f'\Big(\frac{s+t}{v}\Big)\Big|^2dt\leqslant \int_{|t|\leqslant\sigma/v}f'\Big(\frac{s}{v}+t\Big)dt\leqslant b-a+\alpha,$$

and thus, by Cauchy's inequality,

$$\begin{aligned}Q_{31}&\leqslant \frac{1}{8}\,\mathbb{E}^{\mathcal F_i}\int_{|t|\leqslant\sigma}\Big|\frac{1}{\bar V^2}f'\Big(\frac{\bar S+t}{\bar V}\Big)\Big|^2dt+2\,\mathbb{E}^{\mathcal F_i}\int_{|t|\leqslant\sigma}\frac{1}{\bar V^2}|\widehat M(t)-M(t)|^2dt\\ &\leqslant \frac{b-a+\alpha}{4\sigma}+\frac{8}{\sigma^2}\int_{|t|\leqslant\sigma}\mathbb{E}|\widehat M(t)-M(t)|^2dt,\end{aligned}\tag{4.35}$$

where we used the fact that $\bar V\geqslant\sigma/2$ and $\widehat M(t)$ is independent of $\mathcal F_i$. For the second term of (4.35), letting $(\bar{X}_j^*, \bar{Y}_j^*)$ be an independent copy of $(\bar{X}_j, \bar{Y}_j)$, we have

$$\begin{aligned}
&\frac{1}{\sigma^2}\int_{|t|\leqslant\sigma}\mathbb{E}|\widehat{M}(t)-M(t)|^2dt \\
&\quad=\frac{1}{\sigma^2}\sum_{j\in N(B_i)^c}\sum_{l\in N(B_j)\cap N(B_i)^c}\int_{|t|\leqslant\sigma}\mathrm{Cov}\big\{\widehat{M}_j(t),\widehat{M}_l(t)\big\}dt \\
&\quad=\frac{1}{\sigma^3}\sum_{j\in N(B_i)^c}\sum_{l\in N(B_j)\cap N(B_i)^c}\mathbb{E}\big\{\bar{X}_j\bar{X}_l(|\bar{Y}_j|\wedge|\bar{Y}_l|)1(\bar{Y}_j\bar{Y}_l\geqslant 0) \qquad (4.36)\\
&\qquad\qquad -\bar{X}_j\bar{X}_l^*(|\bar{Y}_j|\wedge|\bar{Y}_l^*|)1(\bar{Y}_j\bar{Y}_l^*\geqslant 0)\big\} \\
&\quad\leqslant\frac{1}{\sigma^3}\sum_{j\in\mathcal{J}}\sum_{l\in N(B_i)}\mathbb{E}\big\{|\bar{X}_j\bar{X}_l|(|\bar{Y}_j|\wedge|\bar{Y}_l|)+|\bar{X}_j\bar{X}_l^*|(|\bar{Y}_j|\wedge|\bar{Y}_l^*|)\big\}\leqslant 2\kappa^2\beta_3.
\end{aligned}$$

Substituting (4.36) to (4.35) yields

$$Q_{31}\leqslant\frac{0.25(b-a+\alpha)}{\sigma}+16\kappa^2\beta_3. \qquad (4.37)$$

For Q_{32}, noting that

$$\mathbb{E}\int_{-\infty}^{\infty}|t\widehat{M}(t)|dt\leqslant\frac{1}{2}\sum_{i\in\mathcal{J}}\mathbb{E}\{|\bar{X}_i\bar{Y}_i^2|\}\leqslant\frac{\kappa^2}{2}\sum_{i\in\mathcal{J}}\mathbb{E}|\bar{X}_i|^3, \qquad (4.38)$$

we have

$$Q_{32}\leqslant\frac{4}{\sigma^3}\,\mathbb{E}\int_{|t|>\sigma}|t||\widehat{M}(t)-M(t)|dt\leqslant\frac{8}{\sigma^3}\,\mathbb{E}\int_{-\infty}^{\infty}|t\widehat{M}(t)|dt\leqslant 4\kappa^2\beta_3. \qquad (4.39)$$

Combining (4.37) and (4.39), we have

$$|Q_3|\leqslant\frac{0.125(b-a+\alpha)}{\sigma}+20\kappa^2\beta_3. \qquad (4.40)$$

Let $f_{a,b,\bar{V}^{(j)}}$ be defined as in (4.30) by taking $v=\bar{V}^{(j)}$. For Q_4, we have

$$\begin{aligned}
Q_4&=\sum_{j\in N(B_i)^c}\mathbb{E}^{\mathcal{F}_i}\left\{\bar{X}_j\left[\frac{1}{\bar{V}}f\left(\frac{\bar{S}-\bar{Y}_j}{\bar{V}}\right)-\frac{1}{\bar{V}^{(j)}}f\left(\frac{\bar{S}-\bar{Y}_j}{\bar{V}^{(j)}}\right)\right]\right\} \\
&\quad+\sum_{j\in N(B_i)^c}\mathbb{E}^{\mathcal{F}_i}\left\{\frac{\bar{X}_j}{\bar{V}^{(j)}}f\left(\frac{\bar{S}-\bar{Y}_j}{\bar{V}^{(j)}}\right)-\frac{\bar{X}_j}{\bar{V}^{(j)}}f_{a,b,\bar{V}^{(j)}}\left(\frac{\bar{S}-\bar{Y}_j}{\bar{V}^{(j)}}\right)\right\} \\
&\quad+\sum_{j\in N(B_i)^c}\mathbb{E}^{\mathcal{F}_i}\left\{\frac{\bar{X}_j}{\bar{V}^{(j)}}f_{a,b,\bar{V}^{(j)}}\left(\frac{\bar{S}-\bar{Y}_j}{\bar{V}^{(j)}}\right)\right\} \\
&:=Q_{41}+Q_{42}+Q_{43}.
\end{aligned}$$

For any $j \in N(B_i)^c$, by (4.10) and (4.11), we have

$$
\begin{aligned}
Q_{41} &\leqslant \sum_{j \in N(B_i)^c} \mathbb{E}^{\mathcal{F}_i} \left| \left\{ \frac{\bar{X}_j}{\bar{V}} f\Big(\frac{\bar{S}-\bar{Y}_j}{\bar{V}}\Big) \right\} - \left\{ \frac{\bar{X}_j}{\bar{V}^{(j)}} f\Big(\frac{\bar{S}-\bar{Y}_j}{\bar{V}^{(j)}}\Big) \right\} \right| \\
&\leqslant \sum_{j \in N(B_i)^c} \mathbb{E}^{\mathcal{F}_i} \left\{ \left(\|f\| + \frac{2|\bar{S}-\bar{Y}_j|}{\sigma} \right) |\bar{X}_j| \left| \frac{1}{\bar{V}} - \frac{1}{\bar{V}^{(j)}} \right| \right\} \\
&\leqslant \frac{4}{\sigma^4} \sum_{j \in N(B_i)^c} \mathbb{E}^{\mathcal{F}_i} \left\{ \big((b-a+\alpha) + 2|\bar{S}-\bar{Y}_j| \big) |\bar{X}_j| \sum_{l \in N(A_j)} |\bar{X}_l \bar{Y}_l| \right\} \\
&\leqslant \frac{4(b-a+\alpha)}{3\sigma^4} Q_{44} + \frac{8}{3\sigma^4} Q_{45},
\end{aligned} \tag{4.41}
$$

where

$$
\begin{aligned}
Q_{44} &= \sum_{j \in N(B_i)^c} \sum_{l \in N(A_j)} \sum_{k \in A_l} \mathbb{E}^{\mathcal{F}_i} \{ (|\bar{X}_j|^3 + |\bar{X}_l|^3 + |\bar{X}_k|^3) \}, \\
Q_{45} &= \sum_{j \in N(B_i)^c} \sum_{l \in N(A_j)} \sum_{k \in A_l} \mathbb{E}^{\mathcal{F}_i} \big\{ \big(|\bar{S}-\bar{Y}_j|\big) (|\bar{X}_j|^3 + |\bar{X}_l|^3 + |\bar{X}_k|^3) \big\}.
\end{aligned}
$$

Noting that $|N(B_i)| \leqslant \kappa$, $|N(A_i)| \leqslant \kappa$, $|\{j : A_j \cap A_l \neq \emptyset\}| \leqslant \kappa$ and $|A_l| \leqslant \kappa$, we have

$$
\begin{aligned}
Q_{44} &\leqslant \sum_{j \in N(B_i)^c} \sum_{l \in N(A_j)} \sum_{k \in A_l} \mathbb{E}^{\mathcal{F}_i} |\bar{X}_j|^3 + \sum_{k \in N(B_i)^c} \sum_{l : k \in A_l} \sum_{j : A_j \cap A_l \neq \emptyset} \mathbb{E}^{\mathcal{F}_i} |\bar{X}_k|^3 && (4.42) \\
&\quad + \sum_{l \in N(B_i)^c} \sum_{j : A_j \cap A_l \neq \emptyset} \sum_{k \in A_l} \mathbb{E}^{\mathcal{F}_i} |\bar{X}_l|^3 + \sum_{k \in N(B_i)} \sum_{l : k \in A_l} \sum_{j : A_j \cap A_l \neq \emptyset} \mathbb{E}^{\mathcal{F}_i} |\bar{X}_k|^3 && (4.43) \\
&\quad + \sum_{l \in N(B_i)} \sum_{j : A_j \cap A_l \neq \emptyset} \sum_{k \in A_l} \mathbb{E}^{\mathcal{F}_i} |\bar{X}_l|^3 && (4.44) \\
&\leqslant 3\kappa^2 \sigma^3 \beta_3 + 2\kappa^2 \sum_{j \in N(B_i)} \mathbb{E}^{\mathcal{F}_i} |\bar{X}_j|^3, && (4.45)
\end{aligned}
$$

where we used the fact that $\mathbb{E}^{\mathcal{F}_i} |\bar{X}_j|^3 = \mathbb{E}|\bar{X}_j|^3$ for $j \in N(B_i)^c$.

Now, we bound Q_{45}. For any l, let $\bar{S}_l = \bar{S} - \sum_{k \in N(B_i) \cup A_j \cup A_l} \bar{X}_k$, then $\big| |\bar{S} - \bar{Y}_j| - |\bar{S}_l| \big| \leqslant 2\sigma$. Also, if $l \in N(B_i)^c$, we have $\bar{S}_l$ is independent of $\bar{X}_l$ and $(\bar{S}_l, \bar{X}_l)$ is independent of $\mathcal{F}_i$. Noting that $\bar{\sigma}^2 \leqslant 1.02\sigma^2$ by (4.6) and that $\kappa \geqslant 1$, we have

$$
\mathbb{E}|\bar{S}_l| \leqslant \mathbb{E}|\bar{S}| + 3\kappa\sigma \leqslant \bar{\sigma} + 3\sigma \leqslant 4.02\sigma.
$$

Therefore, for $l \in N(B_i)^c$,

$$
\begin{aligned}
\mathbb{E}^{\mathcal{F}_i} |(\bar{S}-\bar{Y}_j)\bar{X}_l^3| &\leqslant \mathbb{E}|\bar{S}_l \bar{X}_l^3| + 2\sigma\, \mathbb{E}|\bar{X}_l|^3 \\
&\leqslant \mathbb{E}|\bar{S}_l|\, \mathbb{E}|\bar{X}_l|^3 + 2\sigma\, \mathbb{E}|\bar{X}_l|^3 \leqslant 6.02\sigma\, \mathbb{E}|\bar{X}_l|^3.
\end{aligned} \tag{4.46}
$$

For $l \in N(B_i)$, we have

$$\mathbb{E}^{\mathcal{F}_i}|(\bar{S}-\bar{Y}_j)\bar{X}_l^3| \leqslant \sigma\,\mathbb{E}^{\mathcal{F}_i}|\bar{X}_l|^3 + \mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_l^3|. \tag{4.47}$$

Hence, by (4.46) and (4.47), we then obtain

$$\begin{aligned} Q_{45} \leqslant{} & \mathbb{E}^{\mathcal{F}_i}\sum_{j\in N(B_i)^c}\sum_{l\in N(A_j)}\sum_{k\in A_l}|\bar{S}-\bar{Y}_j||\bar{X}_j|^3 \\ & + \mathbb{E}^{\mathcal{F}_i}\sum_{k\in N(B_i)^c}\sum_{l:k\in A_l}\sum_{j:A_j\cap A_l\neq\emptyset}|\bar{S}-\bar{Y}_j||\bar{X}_k|^3 \\ & + \mathbb{E}^{\mathcal{F}_i}\sum_{l\in N(B_i)^c}\sum_{j:A_j\cap A_l\neq\emptyset}\sum_{k\in A_l}|\bar{S}-\bar{Y}_j||\bar{X}_l|^3 \\ & + \mathbb{E}^{\mathcal{F}_i}\sum_{k\in N(B_i)}\sum_{l:k\in A_l}\sum_{j:A_j\cap A_l\neq\emptyset}|\bar{S}-\bar{Y}_j||\bar{X}_k|^3 \\ & + \mathbb{E}^{\mathcal{F}_i}\sum_{l\in N(B_i)}\sum_{j:A_j\cap A_l\neq\emptyset}\sum_{k\in A_l}|\bar{S}-\bar{Y}_j||\bar{X}_l|^3 \\ \leqslant{} & C_2\sigma\kappa^2\Bigg(\sum_{j\in\mathcal{J}}\mathbb{E}|\bar{X}_j|^3 + \sum_{j\in N(B_i)}\mathbb{E}^{\mathcal{F}_i}|\bar{X}_j|^3\Bigg) + C_3\kappa^2\sum_{j\in N(B_i)}\mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_j^3|. \end{aligned} \tag{4.48}$$

By (4.45) and (4.48), we have

$$Q_{41} \leqslant C_4\kappa^2\Bigg(\beta_3 + \sum_{j\in N(B_i)}\mathbb{E}^{\mathcal{F}_i}|\bar{X}_j/\sigma|^3 + \sum_{j\in N(B_i)}\mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_j^3/\sigma^4|\Bigg). \tag{4.49}$$

For Q_{42}, observing that

$$\big|f(w) - f_{a,b,\bar{V}^{(j)}}\big| \leqslant \left|\frac{1}{\bar{V}} - \frac{1}{\bar{V}^{(j)}}\right|,$$

and by (4.10) and (4.45), we obtain

$$\begin{aligned} Q_{42} & \leqslant \frac{2}{\sigma}\sum_{j\in N(B_i)^c}\mathbb{E}\left\{|\bar{X}_j|\left|\frac{1}{\bar{V}} - \frac{1}{\bar{V}^{(j)}}\right|\right\} \\ & \leqslant \frac{8}{\sigma^3}\sum_{j\in N(B_i)^c}\sum_{k\in N(A_j)}\mathbb{E}^{\mathcal{F}_i}\big\{|\bar{X}_j\bar{X}_k\bar{Y}_k|\big\} \\ & \leqslant \frac{8}{3\sigma^3}\mathbb{E}^{\mathcal{F}_i}\sum_{j\in N(B_i)^c}\sum_{l\in N(A_j)}\sum_{k\in A_l}\Big(|\bar{X}_j|^3 + |\bar{X}_k|^3 + |\bar{X}_l|^3\Big) \\ & \leqslant 8\kappa^2\beta_3 + 8\kappa^2\sum_{j\in N(B_i)}\mathbb{E}^{\mathcal{F}_i}|\bar{X}_j/\sigma|^3. \end{aligned} \tag{4.50}$$

For Q_{43}, as $\beta_2 \leqslant 1/(150\kappa) \leqslant 1/150$, $j \in N(B_i)^c$, $V^{(j)} \geqslant 0.5\sigma$ and $\|f_{a,b,\bar{V}^{(j)}}\| \leqslant$

$(b-a+\alpha)/\sigma$, by (4.15), we have

$$\begin{aligned}|Q_{43}| &\leqslant \frac{2\|f_{a,b,\bar V^{(j)}}\|}{\sigma}\sum_{j\in N(B_i)^c}|\mathbb{E}^{\mathcal{F}_i}\,\bar X_j| = \frac{2}{\sigma^2}(b-a+\alpha)\sum_{j\in N(B_i)^c}|\mathbb{E}\,\bar X_j| \\ &\leqslant 2(b-a+\alpha)\beta_2/\sigma \leqslant 0.2(b-a+\alpha)/\sigma.\end{aligned} \tag{4.51}$$

By (4.49)–(4.51), we have

$$\begin{aligned}|Q_4| &\leqslant 0.2(b-a+\alpha)/\sigma \\ &\quad + C_5\kappa^2\biggl(\beta_3 + \sum_{j\in N(B_i)}\mathbb{E}^{\mathcal{F}_i}\,|\bar X_j/\sigma|^3 + \sum_{j\in N(B_i)}\mathbb{E}^{\mathcal{F}_i}|\bar S\bar X_j^3/\sigma^4|\biggr).\end{aligned} \tag{4.52}$$

To bound Q_2, we note that

$$\begin{aligned}&\biggl(f'\Bigl(\frac{\bar S+t}{\bar V}\Bigr) - f'\Bigl(\frac{\bar S}{\bar V}\Bigr)\biggr) \\ &= \begin{cases} \frac{1}{\alpha}\int_0^t(\mathbb{I}\{z+a/\bar V-\alpha/\bar V\leqslant(\bar S+s)/\bar V\leqslant z+a/\bar V\} \\ \qquad -\mathbb{I}\{z+b/\bar V\leqslant(\bar S+s)/\bar V\leqslant z+b/\bar V+\alpha/\bar V\})ds & \text{if } t\geqslant 0, \\ -\frac{1}{\alpha}\int_t^0(\mathbb{I}\{z+a/\bar V-\alpha/\bar V\leqslant(\bar S+s)/\bar V\leqslant z+a/\bar V\} \\ \qquad -\mathbb{I}\{z+b/\bar V\leqslant(\bar S+s)/\bar V\leqslant z+b/\bar V+\alpha/\bar V\})ds & \text{if } t<0.\end{cases}\end{aligned}$$

Then, recalling that $\alpha = 20\kappa^2\sigma\beta_3$, we have

$$\begin{aligned}|Q_2| &\leqslant \frac{4}{\sigma^2}\int_{|t|\geqslant\sigma}M(t)dt + \frac{8}{\alpha\sigma^2}\int_0^\sigma\int_0^t L(\alpha)ds|M(t)|dt \\ &\quad + \frac{8}{\alpha\sigma^2}\int_{-\sigma}^0\int_t^0 L(\alpha)ds|M(t)|dt \\ &\leqslant \frac{8}{\sigma^3}(\sigma\alpha^{-1}L(\alpha)+1)\int_{-\infty}^{\infty}|tM(t)|dt \\ &\leqslant 4\kappa^2(\sigma\alpha^{-1}L(\alpha)+1)\beta_3 \\ &\leqslant 0.2L(\alpha) + 4\kappa^2\beta_3,\end{aligned} \tag{4.53}$$

where

$$L(\alpha) = \lim_{n\to\infty}\sup_{y,x\in\mathbb{Q}}\mathbb{P}^{\mathcal{F}_i}(y+(x-1/n)/\bar V\leqslant\bar W\leqslant y+(x+1/n)/\bar V+\alpha/\bar V+1/n),$$

the notation $\mathbb{Q}$ denotes the set of rational numbers, and we applied (4.38) and Jansen's inequality in the last line.

Combining (4.31), (4.32), (4.34), (4.40), (4.52) and (4.53), we have

$$\begin{aligned} &0.45\,\mathbb{P}^{\mathcal{F}_i}\big(z + a/\bar{V} \leqslant \bar{W} \leqslant z + b/\bar{V}\big) \\ &\quad \leqslant 0.9\frac{b-a+\alpha}{\sigma} + 0.2L(\alpha) \\ &\qquad + C_6\kappa^2\bigg(\beta_3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}\,|\bar{X}_j/\sigma|^3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_j^3/\sigma^4|\bigg). \end{aligned} \tag{4.54}$$

Rearranging (4.54) yields

$$\begin{aligned} &\mathbb{P}^{\mathcal{F}_i}\big(z + a/\bar{V} \leqslant \bar{W} \leqslant z + b/\bar{V}\big) \\ &\quad \leqslant \frac{2(b-a+\alpha)}{\sigma} + 0.5L(\alpha) \\ &\qquad + C_7\kappa^2\bigg(\beta_3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}\,|\bar{X}_j/\sigma|^3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_j^3/\sigma^4|\bigg). \end{aligned} \tag{4.55}$$

Letting $a = x - 1/n$ and $b = x + 1/n + \alpha$, and $z = y$ in (4.54) and taking supremum over $y, x \in \mathbb{Q}$ and letting $n \to \infty$, we have

$$L(\alpha) \leqslant 0.5L(\alpha) + C_8\kappa^2\bigg(\beta_3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}\,|\bar{X}_j/\sigma|^3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_j^3/\sigma^4|\bigg),$$

and hence,

$$L(\alpha) \leqslant C_9\kappa^2\bigg(\beta_3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}\,|\bar{X}_j/\sigma|^3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_j^3/\sigma^4|\bigg). \tag{4.56}$$

Substituting (4.56) to (4.55), and recalling that $\alpha = 20\kappa^2\beta_3$, we have

$$\begin{aligned} &\mathbb{P}^{\mathcal{F}_i}\big(z + a/\bar{V} \leqslant \bar{W} \leqslant z + b/\bar{V}\big) \\ &\quad \leqslant \frac{2(b-a)}{\sigma} + C_{10}\kappa^2\bigg(\beta_3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}\,|\bar{X}_j/\sigma|^3 + \sum_{j\in N(B_i)} \mathbb{E}^{\mathcal{F}_i}|\bar{S}\bar{X}_j^3/\sigma^4|\bigg). \end{aligned}$$

This completes the proof. □

5. Proof of Main results

In this subsection, let $\bar{X}_i, \bar{Y}_i, \bar{S}, \bar{V}$ and $\bar{W}$ be defined as in (4.1). We use a truncation argument to prove Theorem 1. Specifically, we first prove a Berry–Esseen bound for $\bar{W}$, and then prove an error bound for $\sup_{z\in\mathbb{R}}|\mathbb{P}(W \leqslant z) - \mathbb{P}(\bar{W} \leqslant z)|$. Again, we denote by $C, C_1, C_2, \ldots$ absolute positive constants that may take different values in different places.

Now, we give the following proposition, which provides a Berry–Esseen bound for $\bar{W}$, and the proof is based on Stein's method and the concentration inequality approach.

Proposition 2. *Under* (LD1) *and* (LD2). *We have*

$$\sup_{z\in\mathbb{R}}\bigl|\mathbb{P}(\bar{W}\leqslant z)-\Phi(z)\bigr|\leqslant C\bigl((\theta+1)\kappa^2\beta_3+\kappa\beta_2\bigr).$$

Proof. Without loss of generality, noting that $\kappa\geqslant 1$, we assume that $\beta_2\leqslant 1/(150\kappa)$ and $\beta_3\leqslant 1/(150\kappa^2)$, otherwise the inequality is trivial. Let f_z be the solution to the Stein equation

$$f'(w)-wf(w)=\mathbb{I}(w\leqslant z)-\mathbb{P}(Z\leqslant z).$$

In what follows, we simply write $f\coloneqq f_z$. It can be shown that for all $w,w'\in\mathbb{R}$,

$$0\leqslant f(w)\leqslant 1,\quad |f'(w)|\leqslant 1,\quad |f'(w)-f(w')|\leqslant 1, \tag{5.1}$$

and for all w and t,

$$|f'(w+t)-f'(w)|\leqslant(|w|+1)|t|+1(z-|t|\leqslant w\leqslant z+|t|). \tag{5.2}$$

Note that by the Stein equation, we have

$$\mathbb{P}(\bar{W}\leqslant z)-\mathbb{P}(Z\leqslant z)=\mathbb{E}\{f'(\bar{W})-\bar{W}f(\bar{W})\}.$$

Let $\bar{X}_i$ and $\bar{Y}_i$ be defined as in (4.1), and recall that

$$\xi_i=\frac{\bar{X}_i}{\bar{V}},\quad \eta_i=\frac{\bar{Y}_i}{\bar{V}},\quad \bar{W}^{(i)}=\bar{W}-\eta_i. \tag{5.3}$$

Then, it follows that

$$\begin{aligned}\mathbb{E}\{\bar{W}f(\bar{W})\}&=\sum_{i\in\mathcal{J}}\mathbb{E}\{\xi_i(f(\bar{W})-f(\bar{W}^{(i)}))\}+\sum_{i\in\mathcal{J}}\mathbb{E}\{\xi_i f(\bar{W}^{(i)})\}\\&=\mathbb{E}\biggl\{\frac{1}{\bar{V}^2}\int_{-\infty}^{\infty}f'\Bigl(\frac{\bar{W}+u}{\bar{V}}\Bigr)\widehat{K}(u)du\biggr\}+\sum_{i\in\mathcal{J}}\mathbb{E}\{\xi_i f(\bar{W}^{(i)})\},\end{aligned}$$

where

$$\widehat{K}(u)=\sum_{i\in\mathcal{J}}\bar{X}_i\bigl\{\mathbb{I}(-\bar{Y}_i\leqslant u\leqslant 0)-\mathbb{I}(0<u\leqslant-\bar{Y}_i)\bigr\}.$$

By (5.1) and (5.2), we then obtain

$$
\begin{aligned}
&\mathbb{P}(\bar{W} \leqslant z) - \mathbb{P}(Z \leqslant z) \\
&\quad = \mathbb{E}\bigg\{ f'(\bar{W}) \bigg(1 - \frac{\sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i}{\bar{V}^2} \bigg) \bigg\} \\
&\qquad - \mathbb{E}\bigg\{ \frac{1}{\bar{V}^2} \int_{-\infty}^{\infty} \bigg(f'\Big(\frac{\bar{S}+t}{\bar{V}} \Big) - f'\Big(\frac{\bar{S}}{\bar{V}} \Big) \bigg) \widehat{K}(t) dt \bigg\} - \sum_{i\in\mathcal{J}} \mathbb{E}\{\xi_i f(\bar{W}^{(i)})\} \\
&\quad \leqslant R_1 + R_2 + R_3 + R_4,
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
R_1 &= \mathbb{E}\bigg| 1 - \frac{1}{\bar{V}^2} \sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i \bigg|, \\
R_2 &= \frac{8}{\sigma^3} \mathbb{E}\bigg\{ (|\bar{W}|+1) \int_{-\infty}^{\infty} |t\widehat{K}(t)| dt \bigg\}, \\
R_3 &= \frac{4}{\sigma^2} \sum_{i\in\mathcal{J}} \mathbb{E}\Big\{ |\bar{X}_i \bar{Y}_i| \mathbb{I}\big(z - |\bar{Y}_i|/\bar{V} \leqslant \bar{W} \leqslant z + |\bar{Y}_i|/\bar{V} \big) \Big\}, \\
R_4 &= \bigg| \sum_{i\in\mathcal{J}} \mathbb{E}\{\xi_i f(\bar{W}^{(i)})\} \bigg|.
\end{aligned}
$$

For R_1, by (4.4)–(4.6), we have

$$
\begin{aligned}
R_1 &\leqslant \mathbb{E}\bigg\{ \bigg| 1 - \frac{1}{2\sigma^2} \sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i \bigg| \mathbb{I}\bigg(\sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i > 2\sigma^2 \bigg) \bigg\} \\
&\quad + \mathbb{E}\bigg\{ \bigg| 1 - \frac{4}{\sigma^2} \sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i \bigg| \mathbb{I}\bigg(\sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i < \sigma^2/4 \bigg) \bigg\} \\
&\leqslant \Big(1 + 4\bar{\sigma}^2/\sigma^2 \Big) \mathbb{P}\bigg(\bigg| \sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i - \bar{\sigma}^2 \bigg| > \sigma^2/2 \bigg) \\
&\quad + \frac{4}{\sigma^2} \mathbb{E}\bigg| \sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i - \bar{\sigma}^2 \bigg| \mathbb{I}\bigg(\bigg| \sum_{i\in\mathcal{J}} \bar{X}_i \bar{Y}_i - \bar{\sigma}^2 \bigg| > \sigma^2/2 \bigg) \\
&\leqslant 100\kappa^2 \beta_3.
\end{aligned}
\tag{5.5}
$$

For R_2, we have

$$
R_2 \leqslant \frac{4}{\sigma^3} \sum_{i\in\mathcal{J}} \mathbb{E}\big\{ (|\bar{W}|+1) |\bar{X}_i \bar{Y}_i^2| \big\} \leqslant \frac{4}{\sigma^3} \sum_{i\in\mathcal{J}} \mathbb{E}\bigg\{ \Big(\frac{2}{\sigma} |\bar{S}| + 1 \Big) |\bar{X}_i \bar{Y}_i^2| \bigg\}.
$$

Let

$$
\bar{S}^{(B_i)} = \sum_{j\notin B_i} \bar{X}_j.
$$

Then, $\bar{S}^{(B_i)}$ is independent of $(\bar{X}_i, \bar{Y}_i)$, and $|\bar{S} - \bar{S}^{(B_i)}| \leqslant \sigma$. Moreover, by (4.6),

$$\mathbb{E}|\bar{S}^{(B_i)}| \leqslant \mathbb{E}|\bar{S}| + \sigma \leqslant \bar{\sigma} + \sigma \leqslant 2.02\sigma.$$

Thus, we have

$$\mathbb{E}|\bar{S}\bar{X}_i\bar{Y}_i^2| \leqslant \mathbb{E}|\bar{S}^{(B_i)}|\,\mathbb{E}|\bar{X}_i\bar{Y}_i^2| + \sigma\,\mathbb{E}|\bar{X}_i\bar{Y}_i^2| \leqslant 3.02\sigma\,\mathbb{E}|\bar{X}_i\bar{Y}_i^2|.$$

Moreover, by Hölder's inequality, for any $c > 0$,

$$\begin{aligned}\sum_{i\in\mathcal{J}} \mathbb{E}|\bar{X}_i\bar{Y}_i^2| &\leqslant \sum_{i\in\mathcal{J}} \mathbb{E}\biggl(\frac{c^3|\bar{X}_i|^3}{3} + \frac{2|\bar{Y}_i^3|}{3c^{3/2}}\biggr)\\ &\leqslant \sum_{i\in\mathcal{J}} \mathbb{E}\biggl(\frac{c^3|\bar{X}_i|^3}{3} + \kappa^2 \sum_{j\in A_i} \frac{2|\bar{X}_j^3|}{3c^{3/2}}\biggr)\\ &\leqslant \frac{c^3\sigma^3\beta_3}{3} + \frac{2\kappa^3\sigma^3\beta_3}{3c^{3/2}}.\end{aligned}$$

Choosing $c = \kappa^{2/3}$, we have

$$\sum_{i\in\mathcal{J}} \mathbb{E}|\bar{X}_i\bar{Y}_i^2| \leqslant \kappa^2\sigma^3\beta_3.$$

Therefore,

$$R_2 \leqslant 16 \sum_{i\in\mathcal{J}} \mathbb{E}|\bar{X}_i\bar{Y}_i^2/\sigma^3| \leqslant 16\kappa^2\beta_3. \tag{5.6}$$

For R_3, by Proposition 1 and noting that $|\bar{X}_i| \leqslant \sigma/\kappa$, we have

$$\begin{aligned}R_3 &\leqslant C_1\biggl(\sum_{i\in\mathcal{J}} \mathbb{E}|\bar{X}_i\bar{Y}_i^2/\sigma^3| + \kappa^2\beta_3 \sum_{i\in\mathcal{J}} \mathbb{E}|\bar{X}_i\bar{Y}_i/\sigma^2|\\ &\qquad + \kappa^2 \sum_{i\in\mathcal{J}} \sum_{j\in N(B_i)} \mathbb{E}|\bar{X}_i\bar{Y}_i\bar{X}_j/\sigma^3| + \kappa^2 \sum_{i\in\mathcal{J}} \sum_{j\in N(B_i)} \mathbb{E}|\bar{S}\bar{X}_i\bar{Y}_i\bar{X}_j/\sigma^4|\biggr)\\ &\leqslant C_2\kappa^2\bigl(\beta_3 + \kappa^2\theta\beta_3 + \beta_3 + \beta_3\bigr) \leqslant C_3(\theta+1)\kappa^2\beta_3.\end{aligned} \tag{5.7}$$

For R_4, by Lemma 2, we have

$$R_4 \leqslant C_4(\kappa^2\beta_3 + \kappa\beta_2). \tag{5.8}$$

By (5.4)–(5.8), we have

$$\mathbb{P}(\bar{W} \leqslant z) - \mathbb{P}(Z \leqslant z) \leqslant C_5(\theta+1)\kappa^2\beta_3 + C_6\kappa\beta_2.$$

The lower bound can be obtained similarly. This completes the proof. □

Now we are ready to give the proof of Theorem 1.

Proof of Theorem 1. Assume without loss of generality that

$$\beta_2 \leqslant \frac{1}{150\kappa}, \quad \beta_3 \leqslant \frac{1}{150\kappa^2}.$$

Recall the function ψ in (4.2) and $\bar{S}$ in (4.1), and define

$$\tilde{V} = \psi\Big(\sum_{i\in\mathcal{J}}(\bar{X}_i\bar{Y}_i - \bar{X}\bar{Y})\Big), \quad \tilde{W} = \bar{S}/\tilde{V}. \tag{5.9}$$

Then, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{R}}\big|\mathbb{P}(\tilde{W}\leqslant z) - \mathbb{P}(W\leqslant z)\big| \\
&\quad\leqslant \mathbb{P}(\max_{i\in\mathcal{J}}|X_i| > \sigma/\kappa) + \mathbb{P}\bigg(\sum_{i\in\mathcal{J}}(\bar{X}_i\bar{Y}_i - \bar{X}\bar{Y}) \leqslant \sigma^2/4\bigg) \\
&\qquad + \mathbb{P}\bigg(\sum_{i\in\mathcal{J}}(\bar{X}_i\bar{Y}_i - \bar{X}\bar{Y}) > 2\sigma^2\bigg).
\end{aligned}$$

Now,

$$\mathbb{P}(\max_{i\in\mathcal{J}}|X_i| > \sigma/\kappa) \leqslant \sum_{i\in\mathcal{J}}\mathbb{P}(|X_i| > \sigma/\kappa) = \beta_0.$$

Also, by (4.5) and Lemma 3, we have

$$\begin{aligned}
&\mathbb{P}\bigg\{\sum_{i\in\mathcal{J}}(\bar{X}_i\bar{Y}_i - \bar{X}\bar{Y}) \leqslant \sigma^2/4\bigg\} \\
&\quad\leqslant \mathbb{P}\bigg\{\sum_{i\in\mathcal{J}}(\bar{X}_i\bar{Y}_i - \bar{X}\bar{Y}) \leqslant \sigma^2/4, |\mathcal{J}||\bar{X}\bar{Y}| \leqslant \sigma^2/8\bigg\} + \mathbb{P}\big\{|\mathcal{J}||\bar{X}\bar{Y}| \geqslant \sigma^2/8\big\} \\
&\quad\leqslant \mathbb{P}\left\{\bigg|\sum_{i\in\mathcal{J}}(\bar{X}_i\bar{Y}_i - \mathbb{E}\{\bar{X}_i\bar{Y}_i\})\bigg| \geqslant \frac{\sigma^2}{2}\right\} + \frac{8|\mathcal{J}|}{\sigma^2}\,\mathbb{E}|\bar{X}\bar{Y}| \\
&\quad\leqslant 4\kappa^2\beta_3 + C\kappa^{1/2}(1+\theta)^{1/2}|\mathcal{J}|^{-1}.
\end{aligned}$$

Similarly,

$$\mathbb{P}\bigg\{\sum_{i\in\mathcal{J}}(\bar{X}_i\bar{Y}_i - \bar{X}\bar{Y}) \leqslant \sigma^2/4\bigg\} \leqslant 4\kappa^2\beta_3 + C\kappa^{1/2}(1+\theta)^{1/2}|\mathcal{J}|^{-1}.$$

Then,

$$\sup_{z\in\mathbb{R}}\big|\mathbb{P}(\tilde{W}\leqslant z) - \mathbb{P}(W\leqslant z)\big| \leqslant 8\kappa^2\beta_3 + \beta_0 + C\kappa^{1/2}(1+\theta)^{1/2}|\mathcal{J}|^{-1}. \tag{5.10}$$

For $\epsilon > 0$, let

$$h_{z,\epsilon}(w) = \begin{cases} 1 & \text{if } w \leqslant z, \\ 0 & \text{if } w > z + \epsilon, \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then, it follows that $h'_{z,\epsilon}(w) = \mathbb{I}(z \leqslant w \leqslant z + \epsilon)/\epsilon$. Recall that $\bar{W}$ and $\bar{V}$ as in (4.1), and observe that

$$\begin{aligned} \mathbb{P}(\tilde{W} \leqslant z) - \mathbb{P}(\bar{W} \leqslant z) &\leqslant \mathbb{E}\{h_{z,\epsilon}(\tilde{W}) - h_{z,\epsilon}(\bar{W})\} + \mathbb{P}(z \leqslant \bar{W} \leqslant z + \epsilon) \\ &\leqslant \mathbb{E}\{h_{z,\epsilon}(\tilde{W}) - h_{z,\epsilon}(\bar{W})\} + \big(\Phi(z+\epsilon) - \Phi(z)\big) \\ &\quad + 2\sup_{z\in\mathbb{R}} |\mathbb{P}(\bar{W} \leqslant z) - \Phi(z)|. \end{aligned} \tag{5.11}$$

Note that

$$\left| \frac{1}{\bar{V}} - \frac{1}{\tilde{V}} \right| \leqslant \frac{|\mathcal{J}||\bar{X}\bar{Y}|}{\bar{V}\tilde{V}(\bar{V} + \tilde{V})},$$

and thus

$$\begin{aligned} |\mathbb{E}\{h_{z,\epsilon}(\bar{W}) - h_{z,\epsilon}(\tilde{W})\}| &\leqslant \|h'_{z,\epsilon}\| \, \mathbb{E}\,|\bar{W} - \tilde{W}| \\ &\leqslant \frac{1}{\epsilon}\,\mathbb{E}\left\{ \frac{|\bar{S}^2\bar{Y}|}{\bar{V}\tilde{V}(\bar{V}+\tilde{V})} \right\} \\ &\leqslant \frac{4}{\epsilon\sigma^3}\,\mathbb{E}|\bar{S}^2\bar{Y}| \\ &= \frac{4}{\epsilon\sigma^3|\mathcal{J}|^{-1}}\,\mathbb{E}\left| \left(\sum_{i\in\mathcal{J}} \bar{X}_i\right)^2 \left(\sum_{j\in\mathcal{J}} \bar{Y}_j\right) \right|. \end{aligned} \tag{5.12}$$

By the Hölder inequality and Lemma 3, we have for any $c > 0$,

$$\begin{aligned} \mathbb{E}\left| \left(\sum_{i\in\mathcal{J}} \bar{X}_i\right)^2 \left(\sum_{j\in\mathcal{J}} \bar{Y}_j\right) \right| &\leqslant \frac{c}{2}\,\mathbb{E}|\bar{S}^2| + \frac{1}{2c}\,\mathbb{E}\left| \left(\sum_{i\in\mathcal{J}} \bar{X}_i\right)^2 \left(\sum_{j\in\mathcal{J}} \bar{Y}_j\right)^2 \right| \\ &\leqslant \frac{c}{2}(35(\theta+1)\sigma^2) + \frac{1}{2c}\{1161\kappa^2(\theta+1)\sigma^4\}. \end{aligned}$$

Choosing $c = 6\kappa\sigma$, we have the expectation term of the right hand side of (5.12) is bounded by

$$\mathbb{E}\left| \left(\sum_{i\in\mathcal{J}} \bar{X}_i\right)^2 \left(\sum_{j\in\mathcal{J}} \bar{Y}_j\right) \right| \leqslant 210\kappa(\theta+1)\sigma^3. \tag{5.13}$$

Substituting (5.13) to (5.12) yields

$$|\mathbb{E}\{h_{z,\epsilon}(\bar{W}) - h_{z,\epsilon}(\tilde{W})\}| \leqslant 840\kappa(\theta+1)/(\epsilon|\mathcal{J}|).$$

Moreover, for the second term of the right hand side of (5.11),

$$\Phi(z+\epsilon)-\Phi(z) \leqslant 0.4\epsilon.$$

Choosing $\epsilon = (\kappa^{1/2}(\theta+1))/|\mathcal{J}|^{1/2}$, and by (5.11), we have

$$\begin{aligned} &\mathbb{P}(\tilde{W} \leqslant z) - \mathbb{P}(\bar{W} \leqslant z) \\ &\quad \leqslant C\kappa^{1/2}(\theta+1)|\mathcal{J}|^{-1/2} + 2\sup_{z\in\mathbb{R}}|\mathbb{P}(\bar{W} \leqslant z) - \Phi(z)|. \end{aligned} \tag{5.14}$$

The same lower bound also holds by the same argument. Then, we have

$$\begin{aligned} &\sup_{z\in\mathbb{R}} | \, \mathbb{P}(\tilde{W} \leqslant z) - \mathbb{P}(\bar{W} \leqslant z)| \\ &\quad \leqslant C\kappa^{1/2}(\theta+1)|\mathcal{J}|^{-1/2} + 2\sup_{z\in\mathbb{R}}|\mathbb{P}(\bar{W} \leqslant z) - \Phi(z)|. \end{aligned} \tag{5.15}$$

By (5.10) and (5.15), and applying Proposition 2, we completes the proof. □

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