On Confidence Sequence from Universal Gambling

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Outline

Universal Gambling

Coin Betting

- Coin tosses $y_1, y_2, \ldots \in \{0,1\}$
- At each round t, a gambler distributes its wealth $\$\mathsf{W}_{t-1}$ according to $(q_t, 1-q_t)$
- For each \$1, earn \$1 if you hit, lose \$1 otherwise
- Causal strategy: $q_t := q(1|y^{t-1}) \in [0,1]$
- The recursive equation:

$$\mathsf{W}_t = \mathsf{W}_{t-1} 2 q_t^{\mathbb{1}\{y_t=1\}} (1-q_t)^{\mathbb{1}\{y_t=0\}} = \mathsf{W}_{t-1} 2 q(y_t|y^{t-1})$$

- Cumulative wealth: starting with $\$\mathsf{W}_0$,

$$\mathsf{W}_T = \mathsf{W}_0 \prod_{t=1}^{T} 2q(y_t|y^{t-1}) = \mathsf{W}_0 2^T q(y^T),$$

where $q(y^T) := \prod_{t=1}^T q(y_t|y^{t-1})$

Universality and Minimax Optimality

- Let $\mathsf{W}_t := \mathsf{W}^q(y^t)$ for a betting strategy $(q(\cdot|y^{t-1}))_{t=1}^{\infty}$
- For some $\mathcal{P} = \{\text{reference strategies } p\}$, track the best performance of $\mathcal{P}$ in hindsight
- Worst-case regret w.r.t. the best reference strategy

$$\max_{y^T} \max_{p \in \mathcal{P}} \log \frac{\mathsf{W}^p(y^T)}{\mathsf{W}^q(y^T)}$$

  If $o(T)$, the gambler q is said to be universal w.r.t. $\mathcal{P}$
- The best strategy is called minimax optimal

$$\min_{q} \max_{p \in \mathcal{P}} \max_{y^T} \log \frac{\mathsf{W}^p(y^T)}{\mathsf{W}^q(y^T)}$$

Coin Betting $\equiv$ Probability Assignment

- Note: $\dfrac{W^p(y^T)}{W^q(y^T)} = \dfrac{W_0 2^T p(y^T)}{W_0 2^T q(y^T)} = \dfrac{p(y^T)}{q(y^T)}$ by definition
- Binary prediction under log loss
  - At each round t, a learner assigns probability $q(\cdot|y^{t-1})$ over $\{0,1\}$
  - After observing $y_t \in \{0,1\}$, suffer loss $\log \frac{1}{q(y_t|y^{t-1})}$
  - The cumulative regret w.r.t. a reference probability $p(y^t)$ is

$$\sum_{t=1}^{T} \log \frac{1}{q(y_t|y^{t-1})} - \sum_{t=1}^{T} \log \frac{1}{p(y_t|y^{t-1})} = \log \frac{p(y^T)}{q(y^T)}$$

$\therefore$ coin betting $\equiv$ binary prediction under log loss ($\equiv$ lossless binary compression)

$\therefore$ universal compression $\rightarrow$ universal betting!

Example: Constant Bettors

- $\mathcal{P} = \{p_\theta(\cdot) \colon \theta \in [0,1]\}$, where $p_\theta(1|y^{t-1}) = \theta$
- Cumulative wealth:

$$\mathsf{W}^\theta(y^T) := \mathsf{W}_0 2^T p_\theta(y^T),$$

where $p_\theta(y^T)$ is the "probability" under $y^T \sim$ i.i.d. $\mathrm{Bern}(\theta)$

- **Fact**: p_{θ^*} is *optimal* if $y^T \sim$ i.i.d. $\mathrm{Bern}(\theta^*)$ (a.k.a. Kelly betting)
- Krichevsky–Trofimov (KT) probability assignment (Krichevsky and Trofimov, 1981)

$$q_{\mathsf{KT}}(1|y^{t-1}) := \frac{1}{t}\Bigl(\sum_{i=1}^{t-1} y_i + \frac{1}{2}\Bigr)$$

- Asymptotically minimax optimal (Xie and Barron, 2000)

$$\max_{\theta \in [0,1]} \max_{y^T} \log \frac{p_\theta(y^T)}{q_{\mathsf{KT}}(y^T)} = \frac{1}{2}\log T + \frac{1}{2}\log\frac{\pi}{2} + o(1)$$

Mixture Probability

- The KT probability $q_{\mathsf{KT}}(\cdot|y^{t-1})$ is induced by a mixture probability, i.e.,

$$q_{\mathsf{KT}}(y^T) \equiv \int_0^1 p_\theta(y^T)\,\mathrm{d}\pi(\theta)$$

for $\pi(\theta) = \mathsf{Beta}(\theta|\frac{1}{2}, \frac{1}{2})$

- In other words, KT strategy attains the mixture wealth,

$$\mathsf{W}^{\mathsf{KT}}(y^T) = \mathsf{W}_0 2^T q_{\mathsf{KT}}(y^T) = \int_0^1 \mathsf{W}^\theta(y^T)\,\mathrm{d}\pi(\theta)$$

- So, mixture is nice!

Horse Race

- Horses: $1, 2, \ldots, m$
- Odds: $o_1, o_2, \ldots, o_m$
- Outcome: $y_t \in [m]$
- Bets: $(q(1|y^{t-1}), \ldots, q(m|y^{t-1})) \in \Delta_{m-1}$
- Instantaneous gain: $o_{y_t} q(y_t|y^{t-1})$
- Cumulative wealth:

$$\mathsf{W}^q(y^T) = \mathsf{W}_0 \prod_{t=1}^{T} o_{y_t} q(y_t|y^{t-1}) = \mathsf{W}_0 \prod_{z \in [m]} o_z^{\sum_{t=1}^{T} \mathbb{1}\{y_t = z\}} q(y^T)$$

- Regret: $\log \frac{\mathsf{W}^p(y^T)}{\mathsf{W}^q(y^T)} = \log \frac{p(y^T)}{q(y^T)} \Rightarrow$ equivalent to m-ary prediction under log loss!
- KT strategy: $q_{\mathsf{KT}}(y^T) := \int_{\Delta_{m-1}} p_{\boldsymbol{\theta}}(y^T)\, \mathrm{d}\pi(\boldsymbol{\theta})$, where $\pi(\boldsymbol{\theta}) = \mathsf{Dir}(\boldsymbol{\theta}|\frac{1}{2}, \ldots, \frac{1}{2})$

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Stock Investment

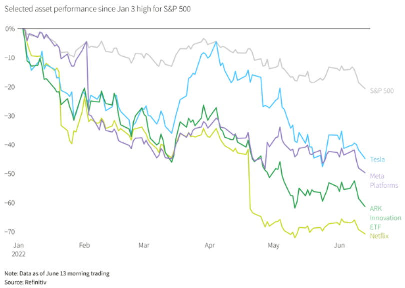
- Stocks: $1, 2, \ldots, m$
- Price relatives (market vector):

$$\mathbf{x}_t = (x_{t1}, \ldots, x_{tm}) \in \mathcal{M} \subseteq \mathbb{R}^m_{\geq 0},$$

$$x_{ti} := \frac{(\text{end price of stock } i \text{ on day } t)}{(\text{start price of stock } i \text{ on day } t)}$$

- Portfolio: $\mathbf{b}(\mathbf{x}^{t-1}) \in \Delta_{m-1}$
- Cumulative wealth: starting with $\$\mathsf{W}_0$,

$$\mathsf{W}(\mathbf{x}^T) = \mathsf{W}_0 \prod_{t=1}^{T} \langle \mathbf{b}(\mathbf{x}^{t-1}), \mathbf{x}_t \rangle$$



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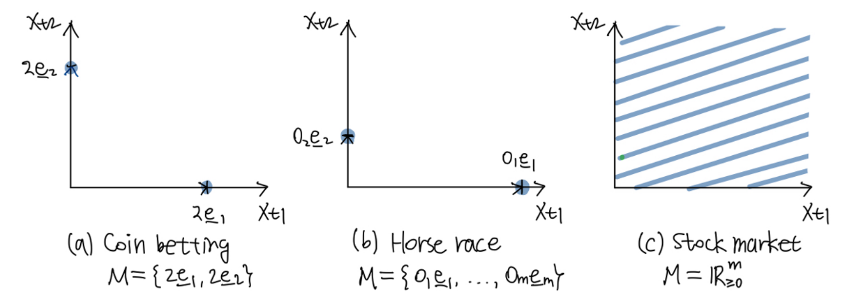
Special Cases



From Probability Assignment to Portfolio Selection

- By distributive law,

$$\mathsf{W}(\mathbf{x}^T) = \mathsf{W}_0 \prod_{t=1}^{T} \langle \mathbf{b}(\mathbf{x}^{t-1}), \mathbf{x}_t \rangle = \mathsf{W}_0 \sum_{y^T \in [m]^T} \Big(\prod_{t=1}^{T} b(y_t \,|\, \mathbf{x}^{t-1}) \Big) \mathbf{x}^T(y^T),$$

where $\mathbf{x}^T(y^T) := x_{1y_1} \ldots x_{Ty_T} =$ (multiplicative gain of the extremal portfolio y^T)

- A probability induced portfolio: for a probability $q(y^T)$, define

$$\mathsf{W}^q(\mathbf{x}^T) := \mathsf{W}_0 \sum_{y^T \in [m]^T} q(y^T) \mathbf{x}^T(y^T),$$

which is achieved by a causal bettor $\mathbf{b}^q$ defined to satisfy

$$\mathsf{W}^q(\mathbf{x}^t) = \mathsf{W}^q(\mathbf{x}^{t-1}) \langle \mathbf{b}^q(\mathbf{x}^{t-1}), \mathbf{x}_t \rangle$$

Portfolio Selection $\equiv$ Probability Assignment

Theorem

$$\sup_{p\in\mathcal{P}}\sup_{\mathbf{x}^T}\frac{\mathsf{W}^p(\mathbf{x}^T)}{\mathsf{W}^q(\mathbf{x}^T)}=\sup_{p\in\mathcal{P}}\sup_{y^T}\frac{p(y^T)}{q(y^T)}$$

Proof

$$\sup_{\mathbf{x}^n}\sup_{p\in\mathcal{P}}\frac{\mathsf{W}^p(\mathbf{x}^n)}{\mathsf{W}^q(\mathbf{x}^n)}\ge\sup_{y^n\in[m]^n}\sup_{p\in\mathcal{P}}\frac{\mathsf{W}^p(\mathbf{e}_{y_1}\ldots\mathbf{e}_{y_n})}{\mathsf{W}^q(\mathbf{e}_{y_1}\ldots\mathbf{e}_{y_n})}=\sup_{y^n\in[m]^n}\sup_{p\in\mathcal{P}}\frac{p(y^n)}{q(y^n)}$$

$$\sup_{\mathbf{x}^n}\sup_{p\in\mathcal{P}}\frac{\mathsf{W}^p(\mathbf{x}^n)}{\mathsf{W}^q(\mathbf{x}^n)}=\sup_{\mathbf{x}^n}\sup_{p\in\mathcal{P}}\frac{\sum_{y^n}p(y^n)\mathbf{x}(y^n)}{\sum_{y^n}q(y^n)\mathbf{x}(y^n)}\overset{(\star)}{\le}\sup_{p\in\mathcal{P}}\sup_{y^n}\frac{p(y^n)}{q(y^n)}$$

Lemma $\star$ (Cover, 2006, Lemma 16.7.1)

For $a_i, b_i \ge 0$, we have $\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\le\max_{j\in[n]}\frac{a_j}{b_j}$, where $\frac{0}{0}:=0$

Example: Constant Rebalanced Portfolios

- $\mathcal{P}_{\text{i.i.d.}} = \{\text{i.i.d. categorical probabilities}\} = \{p_{\boldsymbol{\theta}}(\cdot) \colon \boldsymbol{\theta} \in \Delta_{m-1}\}$
- For each $\boldsymbol{\theta} \in \Delta_{m-1}$, $\mathbf{b}^{\boldsymbol{\theta}} := \mathbf{b}^{p_{\boldsymbol{\theta}}}$ is called a constant rebalanced portfolio (CRP)
- **Fact**: for an i.i.d. market $(\mathbf{x}_t)_{t=1}^{\infty}$, the log-optimal portfolio is a CRP for some $\boldsymbol{\theta}^*$
- **Example**: Consider a market vector sequence $(1, \frac{1}{2}), (1, 2), (1, \frac{1}{2}), \ldots$
- To track the best performance of CRPs, we can plug-in the KT probability!
- Cover's universal portfolio (Cover, 1991; Cover and Ordentlich, 1996): $\mathbf{b}^{\mathsf{UP}} := \mathbf{b}^{q_{\mathsf{KT}}}$

$$\sup_{p \in \mathcal{P}_{\text{i.i.d.}}} \sup_{\mathbf{x}^T} \log \frac{\mathsf{W}^p(\mathbf{x}^T)}{\mathsf{W}^{\mathsf{UP}}(\mathbf{x}^T)} = \sup_{p \in \mathcal{P}_{\text{i.i.d.}}} \sup_{y^T} \log \frac{p(y^T)}{q_{\mathsf{KT}}(y^T)}$$

- Time complexity: $O(t^{m-1})$ at round t
- Note: for horse race, UP is equivalent to the simple KT strategy

Time-Uniform Confidence Intervals

Confidence Intervals

- Consider a $[0, 1]$-valued stochastic process $Y_1, Y_2, \ldots$ such that

$$\mathsf{E}[Y_t | Y^{t-1}] \equiv \mu \in (0, 1)$$

- At time t, $C_t = (\ell_t, u_t)$ is said to be a confidence interval for μ with level $1 - \delta$ if

$$\mathsf{P}(\mu \in C_t) \geq 1 - \delta$$

- **Example**: for each $t \geq 1$, Hoeffding inequality gives

$$C_t^{\mathsf{H}} := \Bigl(\frac{1}{t}\sum_{i=1}^{t} Y_i - \sqrt{\frac{1}{2t}\log\frac{2}{\delta}}, \frac{1}{t}\sum_{i=1}^{t} Y_i + \sqrt{\frac{1}{2t}\log\frac{2}{\delta}}\Bigr)$$

  as a confidence interval with level $1 - \delta$, i.e.,

$$\mathsf{P}(\mu \in C_t^{\mathsf{H}}) \geq 1 - \delta, \ \forall t \geq 1$$

- However, we must choose t ahead of time to make a probabilistic statement

Time-Uniform Confidence Intervals

- Wish to decide to keep or stop sampling Y_t to estimate μ given confidence level on the fly (sequentially)
- Time-uniform confidence intervals (a.k.a. confidence sequence)

$$\mathsf{P}(\mu \in C_t,\ \forall t \geq 1) \geq 1 - \delta$$

- Contrast with

$$\mathsf{P}(\mu \in C_t^{\mathsf{H}}) \geq 1 - \delta,\ \forall t \geq 1$$

- Originally studied by Darling and Robbins (1967); Lai (1976), and recently resurrected by some statisticians (Ramdas et al., 2020; Waudby-Smith and Ramdas, 2020a,b; Howard et al., 2021) and computer scientists (Jun and Orabona, 2019; Orabona and Jun, 2021)

A Tool from Martingale Theory

- Many standard concentration inequalities (such as Hoeffding) rely on

Markov's inequality

For a nonnegative random variable W,

$$\mathsf{P}\left(\frac{W}{\mathsf{E}[W]} \geq \frac{1}{\delta}\right) \leq \delta$$

- In martingale theory, there is a time-uniform counterpart:

Ville's inequality (Ville, 1939)

For a nonnegative supermartingale sequence $(W_t)_{t=0}^{\infty}$ with $W_0 > 0$,

$$\mathsf{P}\left\{\sup_{t\geq 1} \frac{W_t}{W_0} \geq \frac{1}{\delta}\right\} \leq \delta$$

Supermartingales from Gambling

- A (super)martingale naturally arises as a wealth process from a (sub)fair gambling
- We call a gambling subfair, if $\mathsf{E}[\mathbf{x}_t|\mathbf{x}^{t-1}] \le \mathbb{1}$ for every t (and fair if "$=$")

Proposition

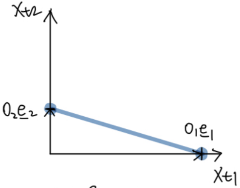
If $(\mathbf{x}_t)_{t=1}^{\infty}$ is (sub)fair, then $(\mathsf{W}_t)_{t=1}^{\infty}$ of any causal strategy is (super)martingale

Proof.

For every t, $\mathsf{E}[\mathsf{W}_t|\mathbf{x}^{t-1}] = \mathsf{W}_{t-1}\langle \mathbf{b}_t, \mathsf{E}[\mathbf{x}_t|\mathbf{x}^{t-1}]\rangle \le \mathsf{W}_{t-1}\langle \mathbf{b}_t, \mathbb{1}\rangle = \mathsf{W}_{t-1}$

Examples

- Coin betting: $\mathbf{x}_t = (2Y_t, 2(1-Y_t))$, $Y_t \in \{0,1\}$
  - fair if $\mathsf{E}[Y_t|Y^{t-1}] = \frac{1}{2}$ (e.g., $Y_t \sim$ i.i.d. $\mathrm{Bern}(\frac{1}{2})$)
- Two-horse race: $\mathbf{x}_t = (o_1 Y_t, o_2(1-Y_t))$, $Y_t \in \{0,1\}$
  - fair if $\frac{1}{o_1} + \frac{1}{o_2} = 1$ and $\mathsf{E}[Y_t|Y^{t-1}] = \frac{1}{o_1}$ (e.g., $Y_t \sim$ i.i.d. $\mathrm{Bern}(\frac{1}{o_1})$)
- Continuous two-horse race: $\mathbf{x}_t = (o_1 Y_t, o_2(1-Y_t))$, $Y_t \in [0,1]$
  - fair if $\frac{1}{o_1} + \frac{1}{o_2} = 1$ and $\mathsf{E}[Y_t|Y^{t-1}] = \frac{1}{o_1}$;
  - more like a structured stock market



(d) Continuous two-horse race

Martingales from Continuous Two-Horse Race

- Recall: Assume $\mathsf{E}[Y_t|Y^{t-1}] \equiv \mu$ for some $\mu \in (0,1)$
- Denote as CTHR(m) the **C**ontinuous **T**wo-**H**orse **R**ace defined by the market vector

$$\mathbf{x}_t = \left(\frac{Y_t}{m}, \frac{1-Y_t}{1-m}\right)$$

Proposition

- If $m = \mu$, any wealth process from CTHR(m) is martingale
- If $m \neq \mu$, there exists a causal betting strategy whose wealth process from CTHR(m) is strictly submartingale

Remark on the Alternative, Equivalent Convention

- CTHR(m) is equivalent to the gambling considered in (Waudby-Smith and Ramdas, 2020b; Orabona and Jun, 2021)
- For the two-horse race setting with odds $\frac{1}{m}$ and $\frac{1}{1-m}$ and a betting strategy $(b_t)_{t=1}^{\infty}$, the multiplicative gain can be written as

$$\frac{1}{m}y_t b_t + \frac{1}{1-m}(1-y_t)(1-b_t) = 1 + \lambda_t(m)(y_t - m),$$

by viewing the single number $y_t - m \in [-m, 1-m]$ as an outcome of the horse race and defining a scaled betting

$$\lambda_t(m) := \frac{b_t}{m(1-m)} - \frac{1}{1-m} \in \left[-\frac{1}{1-m}, \frac{1}{m}\right]$$

- Unlike $b_t \in [0,1]$, the scaled betting $\lambda_t(m)$ inherently depends on the underlying odds (and thus m) by the range it can take

High-Level Intuition (Waudby-Smith and Ramdas, 2020b)

- For CTHR(m), we play a strategy $(\mathbf{b}(Y^{t-1};m))_{t=1}^{\infty}$ and get $(\mathsf{W}(Y^t;m))_{t=1}^{\infty}$
- Since $(\mathsf{W}(Y^t;\mu))_{t=1}^{\infty}$ is martingale, by Ville's inequality, w.p. $\geq 1-\delta$,

$$\sup_{t\geq 1} \frac{\mathsf{W}(Y^t;\mu)}{\mathsf{W}_0} < \frac{1}{\delta}$$

- Assume this high-probability event happens (w.r.t. the randomness in $(Y_t)_{t=0}^{\infty}$)
- Suppose we "play" CTHR(m) for each $m \in (0,1)$ in parallel
- At round t, if the cumulative wealth from CTHR(m) exceeds the threshold W_0/δ, i.e.,

$$\frac{\mathsf{W}(Y^t;m)}{\mathsf{W}_0} \geq \frac{1}{\delta},$$

  then this means that m cannot be μ, and thus exclude m from the candidate list
- If we collect all m whose corresponding wealth never exceeds W_0/δ by then, it forms a time-uniform confidence set with level $1-\delta$

Confidence Sequence from CTHR(m)

- Formally, if we define

$$C_t(Y^t;\delta) := \left\{ m \in (0,1) \colon \sup_{1\le i\le t} \frac{\mathsf{W}(\mathbf{x}^i; m)}{\mathsf{W}_0} < \frac{1}{\delta} \right\},$$

  then

$$\mathsf{P}\{\mu \in C_t(Y^t;\delta),\ \forall t \ge 1\} \ge 1-\delta$$

- Intuitively, a better betting strategy gives a tighter confidence sequence, by growing wealth faster from CTHR(m) for $m \neq \mu$
- We can plug-in any (causal) strategies, so why shouldn't we try universal gambling strategies?
- Orabona and Jun (2021) empirically showed that applying Cover's UP gives tight confidence sequences

A Special Case: $\{0,1\}$-Valued Sequences

- CTHR(m) becomes the standard horse race THR(m) if $Y_t \in \{0,1\}$
- Recall: for the standard horse race, the KT strategy has asymptotic minimax optimality against constant bettors
- For THR(m), the KT strategy yields the cumulative wealth

$$\mathsf{W}^{\mathsf{KT}}(Y^t; m) = \mathsf{W}_0 \phi_t\Bigl(\sum_{i=1}^t Y_i; \frac{1}{m}, \frac{1}{1-m}\Bigr) q_{\mathsf{KT}}(Y^t),$$

  where $\phi_t(x; o_1, o_2) := o_1^x o_2^{t-x}$ for $x \in [0,t]$ and $q_{\mathsf{KT}}(y^t)$ is the KT probability
- Define

$$C_t^{\mathsf{KT}}(y^t; \delta) := \Bigl\{ m \in [0,1] \colon \sup_{1 \le i \le t} \frac{\mathsf{W}^{\mathsf{KT}}(y^i; m)}{\mathsf{W}_0} < \frac{1}{\delta} \Bigr\}$$

Confidence Sequence from KT Betting

Theorem

$(C_t^{\mathsf{KT}}(Y^t;\delta))_{t=1}^{\infty}$ is a time-uniform confidence interval with level $1-\delta$

Proof.

- Apply Ville's inequality
- The set is an interval, since $m \mapsto \phi_t(x; \frac{1}{m}, \frac{1}{1-m})$ is log-convex

- **Note**: the size of the interval behaves as $\sqrt{\frac{2}{t}\log\frac{1}{\delta} + \frac{1}{t}\log t + o(1)}$ for $t \gg 1$, which is comparable to $\sqrt{\frac{2}{t}\log\frac{1}{\delta}}$ from the standard Hoeffding[1]

[1] The optimal order is $\frac{1}{t}\log\log t$, which is implied by the law of iterated logarithm (LIL)

A General Case: $[0,1]$-Valued Sequences

- One may still employ the KT strategy, but strictly suboptimal
- Cover's UP for CTHR(m) gives empirically very tight confidence sequence in general (Orabona and Jun, 2021); but $O(t)$ complexity at round t
- Orabona and Jun (2021) proposed an algorithm that approximates Cover's UP based on a regret analysis

Q. Can there be a conceptually simpler way to approximate Cover's UP with $O(1)$ complexity per round?

- An alternative approach (Ryu and Bhatt, 2022)
  - Recall that Cover's UP is defined as a mixture of wealths of CRPs
  - Consider a tight lower bound of the CRP wealth and take a mixture over the lower bounds

A Lower Bound on the Wealth of CRP

- Let $\bar{a} := 1 - a$ for any $a \in \mathbb{R}$
- For CTHR(m), we can lower-bound the multiplicative gain with CRP(b) as

Lemma (Generalization of (Waudby-Smith and Ramdas, 2020b, Lemma 1))

For any $n \in \mathbb{N}$ and $m \in (0,1)$, we have

$$\log\Bigl(b\frac{y}{m} + \bar{b}\frac{\bar{y}}{\bar{m}}\Bigr) \ge \log \phi_n\biggl(\frac{\bar{b}}{\bar{m}}; \biggl(\Bigl(1-\frac{y}{m}\Bigr)^{2n} - \Bigl(1-\frac{y}{m}\Bigr)^{k}\biggr)_{k=1}^{2n-1}, \Bigl(1-\frac{y}{m}\Bigr)^{2n}\biggr)$$

if $b \in [m,1)$ and $y \ge 0$, where

$$\phi_n(x;\boldsymbol{\rho},\eta) := \exp\biggl(\sum_{k=1}^{2n-1} \frac{(1-x)^k}{k}\rho_k + \eta \log x\biggr)$$

- Can view $\phi_n(x;\boldsymbol{\rho},\eta)$ as an unnormalized exponential-family distribution
- Lower-bound the logarithm by moments of y, i.e., $(1, y, \ldots, y^{2n})$

Key Lemma for the Proof

Lemma (Generalization of (Fan et al., 2015, Lemma 4.1))

For an integer $\ell \geq 1$, if we define

$$f_\ell(t) := \begin{cases} \left(\log(1+t) - \sum_{k=1}^{\ell-1} (-1)^{k+1} \frac{t^k}{k}\right) \Big/ \left((-1)^\ell \frac{t^\ell}{\ell}\right) & \text{if } t > -1 \text{ and } t \neq 0, \\ -1 & \text{if } t = 0, \end{cases}$$

then $t \mapsto f_\ell(t)$ is continuous and strictly increasing over $(-1, \infty)$

- Fan et al. (2015) considered $\ell = 2$, i.e.,

$$f_2(t) = \begin{cases} \frac{\log(1+t) - t}{t^2/2} & \text{if } t > -1 \text{ and } t \neq 0, \\ -1 & \text{if } t = 0 \end{cases}$$

A Lower Bound on the Cumulative Wealth of CRP

- Since it is easy to check $\phi_n(x;\boldsymbol{\rho},\eta)\phi_n(x;\boldsymbol{\rho}',\eta') = \phi_n(x;\boldsymbol{\rho}+\boldsymbol{\rho},\eta+\eta')$,

Lemma

For any $n \in \mathbb{N}$, $m \in (0,1)$, $b \in [0,1]$, and $y^t \in [0,1]^t$, we have

$$\log \frac{\mathsf{W}_t^b(y^t;m)}{\mathsf{W}_0} \geq \log \phi_n\Big(\frac{\bar{b}}{\bar{m}};\boldsymbol{\rho}_n(y^t;m),\eta_n(y^t;m)\Big)$$

if $m < b < 1$, where $\eta_n(y^t;m) := \sum_{i=1}^t (1-\frac{y_i}{m})^{2n}$ and

$$(\boldsymbol{\rho}_n(y^t;m))_k := \sum_{i=1}^t \Big\{\Big(1-\frac{y_i}{m}\Big)^{2n} - \Big(1-\frac{y_i}{m}\Big)^k\Big\} \quad \text{for } k = 1,\ldots,2n-1$$

- Lower-bound the logarithm by moments of y^t, i.e., $(\sum_{i=1}^t y_i^j)_{j=1}^{2n}$
- Complexity from $O(t)$ to $O(n)$

A Mixture of Lower Bounds Approach

- Take a mixture of lower bounds with the conjugate prior of $\phi_n(x; \boldsymbol{\rho}, \eta)$
- In general, this prior is different from the Beta priors used for universal strategies
- For a special case, it subsumes the uniform distribution
- For example, with the uniform prior, the mixture of wealth lower bounds becomes

$$\bar{m} Z_n(\boldsymbol{\rho}_n(y^t; m), \eta_n(y^t; m)) + m Z_n(\boldsymbol{\rho}_n(\bar{y}^t; \bar{m}), \eta_n(\bar{y}^t; \bar{m})),$$

  where $Z_n(\boldsymbol{\rho}, \eta) := \int_0^1 \phi_n(x; \boldsymbol{\rho}, \eta)\, \mathrm{d}x$
- We can construct a time-uniform confidence interval using this "mixture of wealth lower bounds"!
- We call this LBUP(n), where n is the approximation order

Caveats

- Computational bottleneck: computing the normalization constant $Z_n(\boldsymbol{\rho}, \eta)$ of the form

$$\int_0^1 x^\eta \exp\Big(\sum_{k=0}^{2n-1} a_k x^k\Big)\, \mathrm{d}x$$

  - Hence, $O(1)$ per round in principle, but may take longer than running exact UP due to numerical integration steps
- Larger n leads to better approximation, but with increased numerical instability; $n = 2$ or $n = 3$ empirically work well
- Bad approximation in a small sample regime
  - Hybrid UP: run UP for the first few samples and switch to LBUP

Evolution of Wealth Processes

- The horizontal lines indicate an example threshold $\ln \frac{1}{\delta} \approx 2.996$ for $\delta = 0.05$

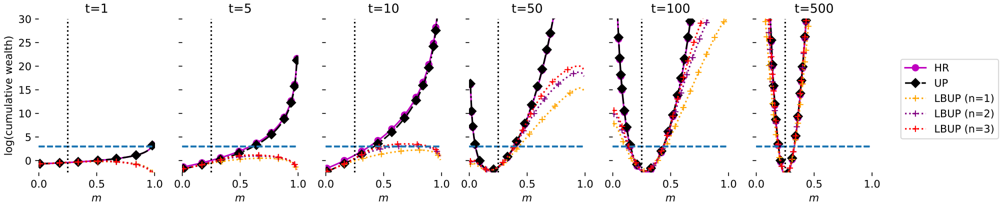


Figure: An i.i.d. Bern(0.25) process

- The horizontal lines indicate an example threshold $\ln \frac{1}{\delta} \approx 2.996$ for $\delta = 0.05$

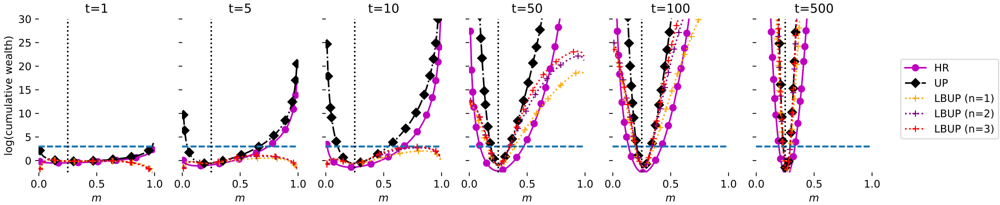


Figure: An i.i.d. Beta(1,3) process

Evolution of Wealth Processes

- The horizontal lines indicate an example threshold $\ln\frac{1}{\delta} \approx 2.996$ for $\delta = 0.05$

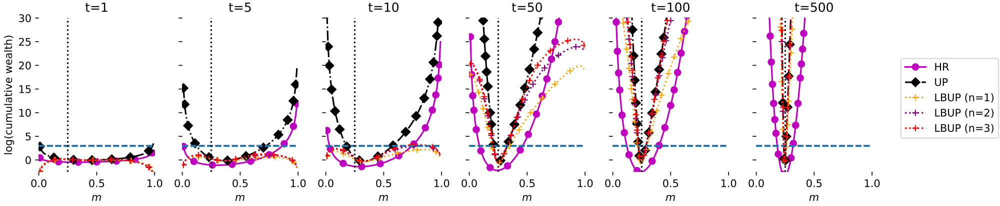


Figure: An i.i.d. Beta(10,30) process

Experiments

- Confidence sequences with level 0.95 (i.e., $\delta = 0.05$)
- CB: betting strategy from another gambling construction
- HR: KT strategy
- UP: exact Cover's UP strategy
- LBUP: proposed lower-bound approach
- HybridUP: run exact UP for the first few steps and switch to LBUP
- PRECiSE (Orabona and Jun, 2021)

Experiments

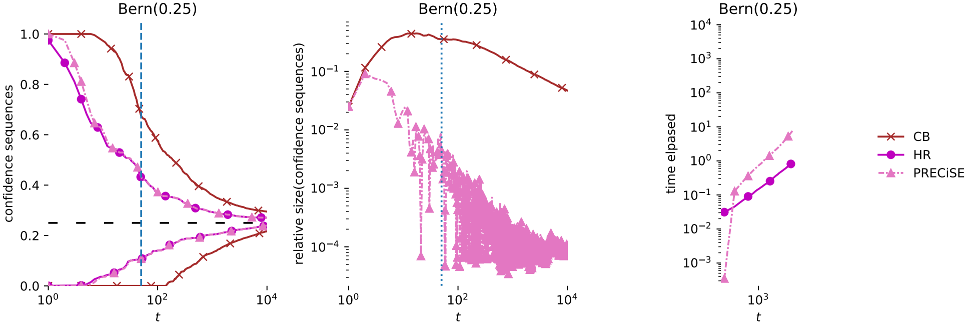


Figure: With i.i.d. Bern(0.25) processes

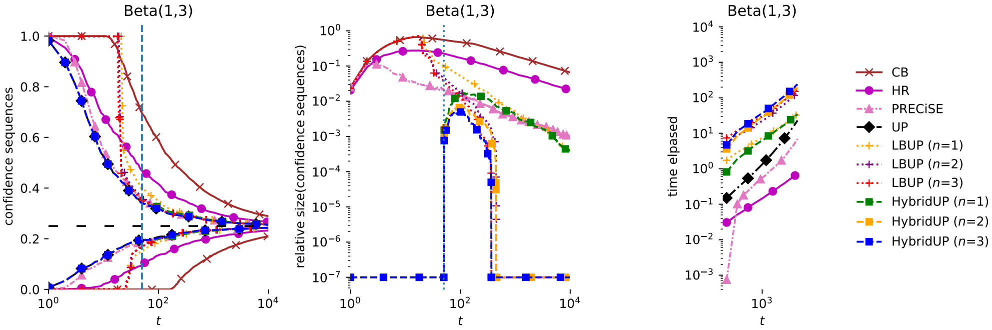


Figure: With i.i.d. Beta(1,3) processes

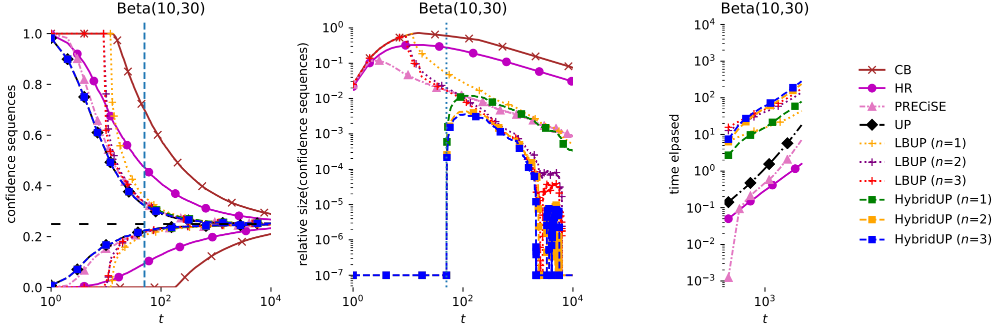


Figure: With i.i.d. Beta(10,30) processes

Concluding Remarks

- Gambling with respect to probability induced strategies $\equiv$ probability assignment
- Confidence sequence induced by universal portfolios can be "efficiently" approximated by a mixture of lower bounds approach
- Orabona and Jun (2021) provides an explicit analysis of the confidence sequence of UP based on the regret analysis

Q. Can we construct a time-uniform confidence set for bounded vectors?

Q. Can there be a gambling other than CTHR(m) that corresponds to some other statistics applications?

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