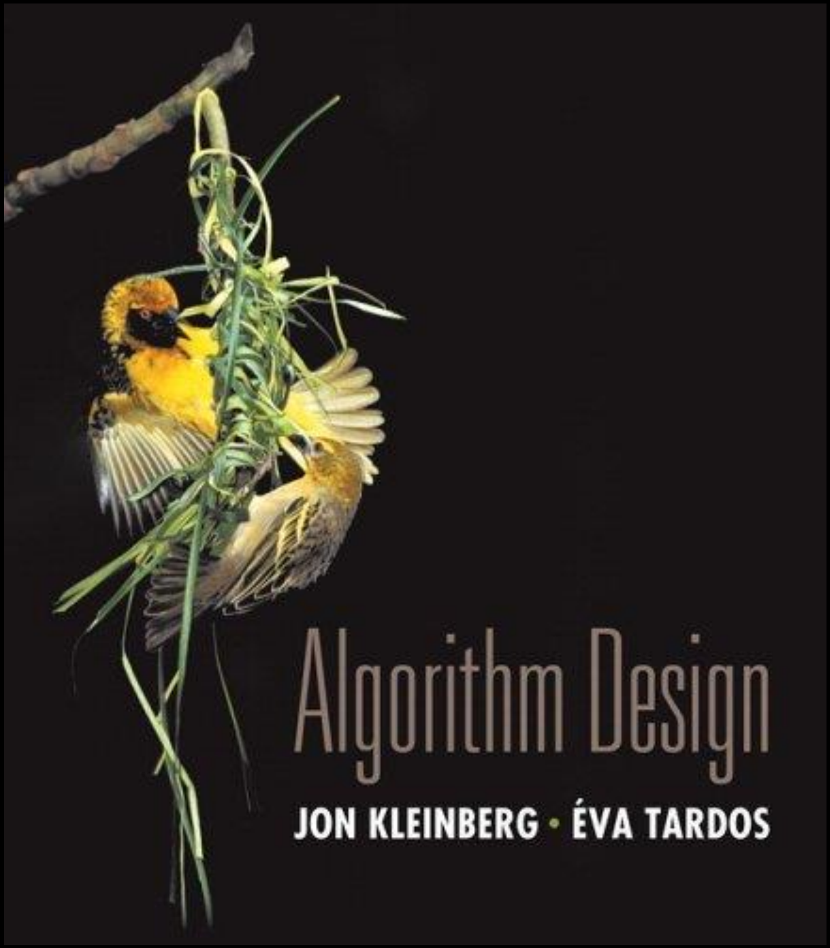


Chapter 8

NP and Computational Intractability

Slides by Kevin Wayne.

Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns.	Ex.
Greedy.	$O(n \log n)$ interval scheduling.
Divide-and-conquer.	$O(n \log n)$ FFT.
Dynamic programming.	$O(n^2)$ edit distance.
Duality.	$O(n^3)$ bipartite matching.
Reductions.	
Local search.	
Randomization.	

Algorithm design anti-patterns.	
NP-completeness.	$O(n^k)$ algorithm unlikely.
PSPACE-completeness.	$O(n^k)$ certification algorithm unlikely.
Undecidability.	No algorithm possible.

8.1 Polynomial-Time Reductions

Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

A working definition. [von Neumann 1953, Godel 1956, Cobham 1964, Edmonds 1965, Rabin 1966]

Those with polynomial-time algorithms.

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover

Primality testing	Factoring

Classify Problems

Desiderata. Classify problems according to those that can be solved in polynomial-time and those that cannot.

Provably requires exponential-time.

Given a Turing machine, does it halt in at most k steps?

Given a board position in an n-by-n generalization of chess, can black guarantee a win?

Frustrating news. Huge number of fundamental problems have defied classification for decades.

This chapter. Show that these fundamental problems are "computationally equivalent" and appear to be different manifestations of one really hard problem.

Polynomial-Time Reduction

Desiderata'. Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?

don't confuse with reduces from

Reduction. Problem X polynomial reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

computational model supplemented by special piece of hardware that solves instances of Y in a single step

Notation. $X \leq_P Y$.

Remarks.

- We pay for time to write down instances sent to black box $\Rightarrow$ instances of Y must be of polynomial size.
- Note: Cook reducibility.

in contrast to Karp reductions

Polynomial-Time Reduction

Purpose. Classify problems according to relative difficulty.

Design algorithms. If $X \leq_P Y$ and Y can be solved in polynomial-time, then X can also be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial time.

Establish equivalence. If $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$.

up to cost of reduction

Reduction By Simple Equivalence

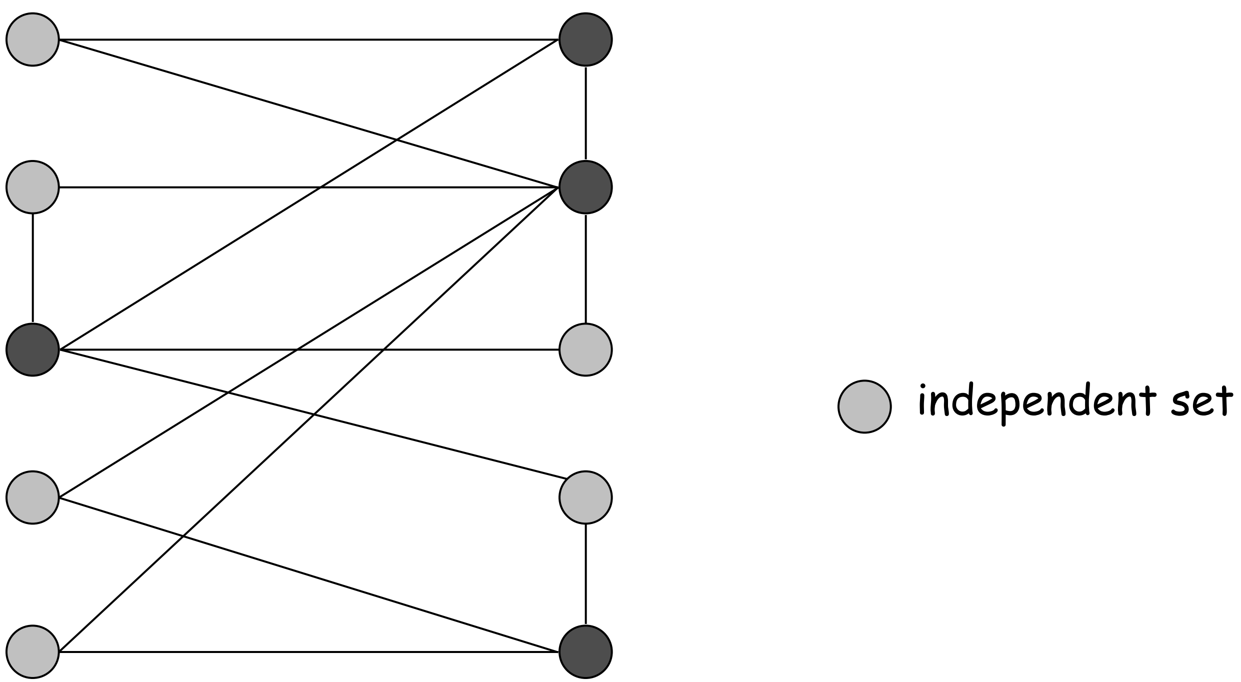
Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

Ex. Is there an independent set of size ≥ 6? Yes.
Ex. Is there an independent set of size ≥ 7? No.

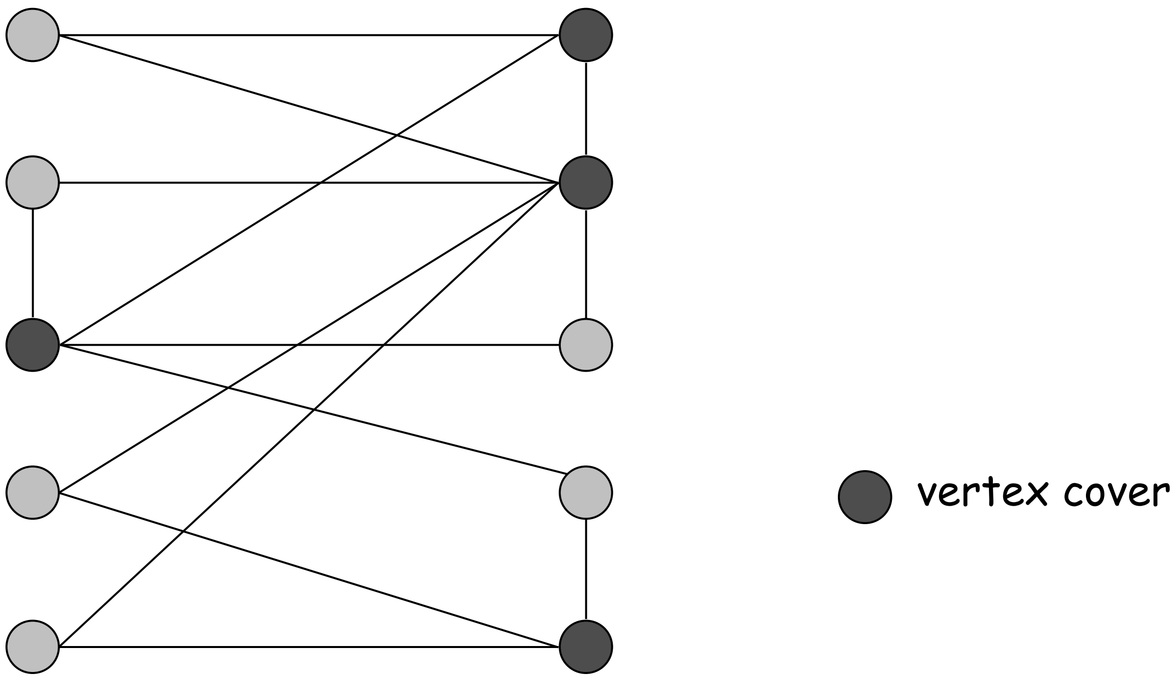


Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

Ex. Is there a vertex cover of size ≤ 4? Yes.

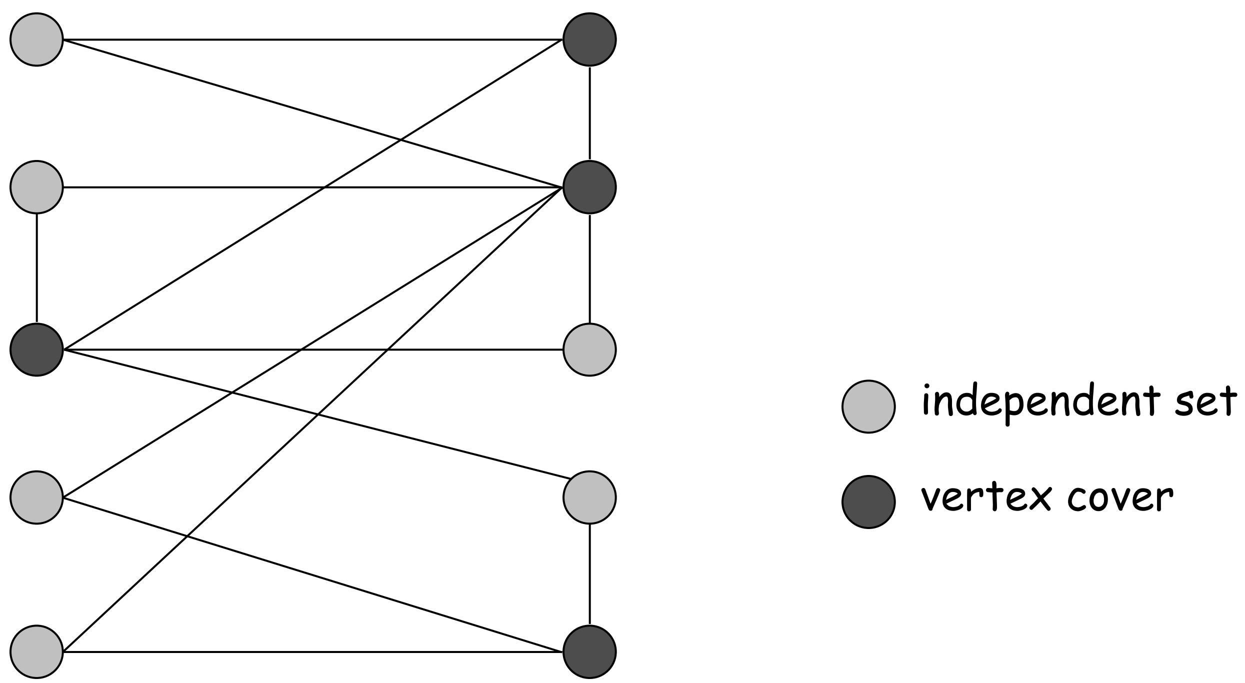
Ex. Is there a vertex cover of size ≤ 3? No.



Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.



Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.

$\Rightarrow$

Let S be any independent set.

Consider an arbitrary edge (u, v).

S independent $\Rightarrow$ $u \notin S$ or $v \notin S$ $\Rightarrow$ $u \in V - S$ or $v \in V - S$.

Thus, V – S covers (u, v).

$\Leftarrow$

Let V – S be any vertex cover.

Consider two nodes $u \in S$ and $v \in S$.

Observe that $(u, v) \notin E$ since V – S is a vertex cover.

Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set. ▪

Reduction from Special Case to General Case

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq k$ of these sets whose union is equal to U?

Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

Ex:

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$k = 2$

$S_1 = \{3, 7\}$	$S_4 = \{2, 4\}$
$S_2 = \{3, 4, 5, 6\}$	$S_5 = \{5\}$
$S_3 = \{1\}$	$S_6 = \{1, 2, 6, 7\}$

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\le_P$ SET-COVER.

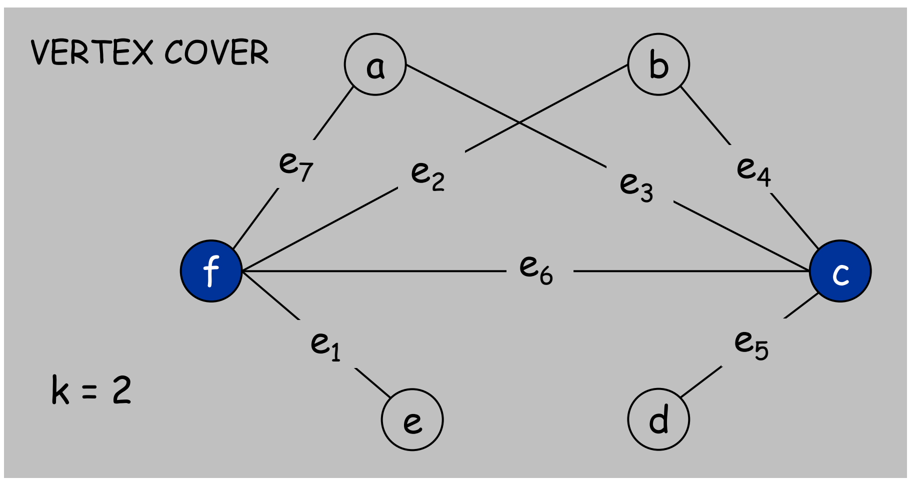
Pf. Given a VERTEX-COVER instance $G = (V, E)$, k, we construct a set cover instance whose size equals the size of the vertex cover instance.

Construction.

Create SET-COVER instance:

- $k = k$, $U = E$, $S_v = \{e \in E : e \text{ incident to } v\}$

Set-cover of size $\le k$ iff vertex cover of size $\le k$. ▪



SET COVER

$U = \{1, 2, 3, 4, 5, 6, 7\}$
$k = 2$

$S_a = \{3, 7\}$ $\quad$ $S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$ $\quad$ $S_d = \{5\}$
$S_e = \{1\}$ $\quad$ $S_f = \{1, 2, 6, 7\}$

Polynomial-Time Reduction

Basic strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

8.2 Reductions via "Gadgets"

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

each corresponds to a different variable

Ex: $\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_2 \vee x_3\right) \wedge \left(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}\right)$

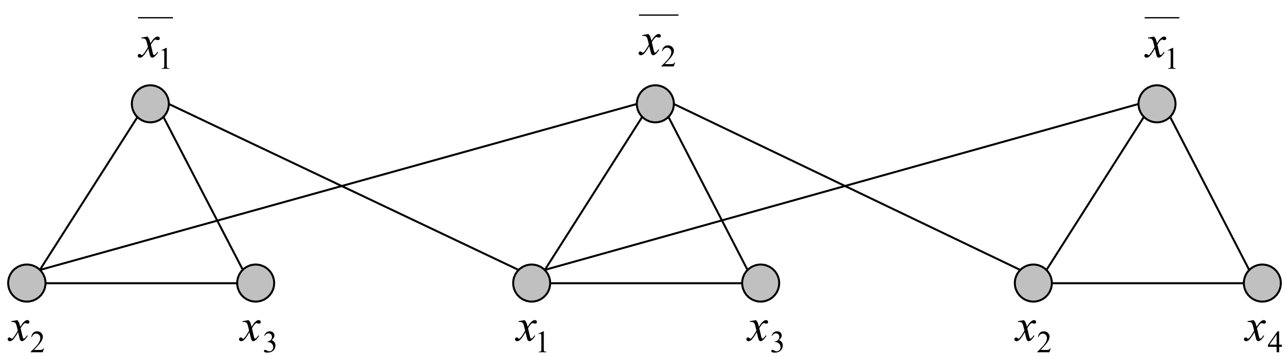
Yes: x_1 = true, x_2 = true x_3 = false.

3 Satisfiability Reduces to Independent Set

Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.
Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.
- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3 Satisfiability Reduces to Independent Set

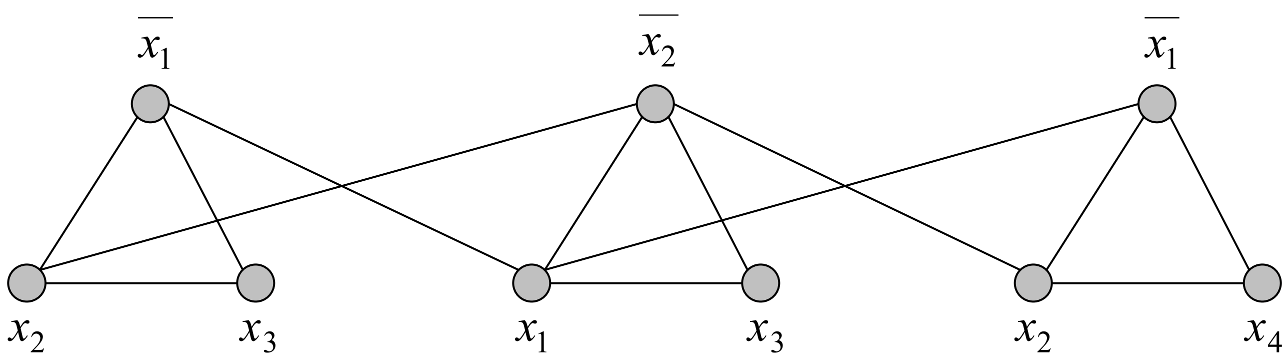
Claim. G contains independent set of size k = $|\Phi|$ iff Φ is satisfiable.

Pf. $\Rightarrow$ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true. $\leftarrow$ and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.

Pf idea. Compose the two algorithms.

Ex: 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Self-Reducibility

Decision problem. Does there exist a vertex cover of size $\leq$ k?
Search problem. Find vertex cover of minimum cardinality.

Self-reducibility. Search problem $\leq_P$ decision version.

- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

Ex: to find min cardinality vertex cover.

- (Binary) search for cardinality k* of min vertex cover.
- Find a vertex v such that $G - \{ v \}$ has a vertex cover of size $\leq$ k* - 1.
  - any vertex in any min vertex cover will have this property
- Include v in the vertex cover.
- Recursively find a min vertex cover in $G - \{ v \}$.

delete v and all incident edges

8.3 Definition of NP

Decision Problems

Decision problem.

- X is a set of strings.
- Instance: string s.
- Algorithm A solves problem X: $A(s) = \texttt{yes}$ iff $s \in X$.

Polynomial time. Algorithm A runs in poly-time if for every string s, A(s) terminates in at most $p(\underbrace{|s|}_{\text{length of s}})$ "steps", where $p(\cdot)$ is some polynomial.

PRIMES: X = { 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, …. }

Algorithm. [Agrawal-Kayal-Saxena, 2002] $p(|s|) = |s|^8$.

Definition of P

P. Decision problems for which there is a poly-time algorithm.

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is x a multiple of y?	Grade school division	51, 17	51, 16
RELPRIME	Are x and y relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is x prime?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5?	Dynamic programming	niether neither	acgggt ttttta
LSOLVE	Is there a vector x that satisfies Ax = b?	Gauss-Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

NP

Certification algorithm intuition.

Certifier views things from "managerial" viewpoint.

Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that $s \in X$.

Def. Algorithm $C(s, t)$ is a certifier for problem X if for every string s, $s \in X$ iff there exists a string t such that $C(s, t) =$ `yes`.

"certificate" or "witness"

NP. Decision problems for which there exists a poly-time certifier.

$C(s, t)$ is a poly-time algorithm and $|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.

Remark. NP stands for nondeterministic polynomial-time.

Certifiers and Certificates: Composite

COMPOSITES. Given an integer s, is s composite?

Certificate. A nontrivial factor t of s. Note that such a certificate exists iff s is composite. Moreover $|t| \le |s|$.

Certifier.

```
boolean C(s, t) {
   if (t ≤ 1 or t ≥ s)
      return false
   else if (s is a multiple of t)
      return true
   else
      return false
}
```

Instance. s = 437,669.

Certificate. t = 541 or 809. ← 437,669 = 541 × 809

Conclusion. COMPOSITES is in NP.

Certifiers and Certificates: 3-Satisfiability

SAT. Given a CNF formula Φ, is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

Ex.

$$\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_1 \vee x_2 \vee x_4\right) \wedge \left(\overline{x_1} \vee \overline{x_3} \vee \overline{x_4}\right)$$

instance s

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

certificate t

Conclusion. SAT is in NP.

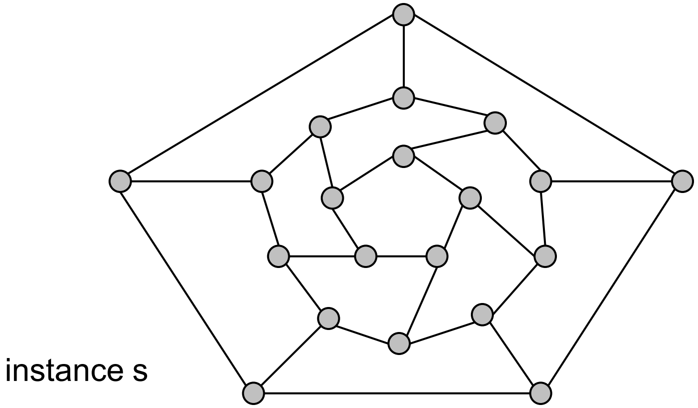
Certifiers and Certificates: Hamiltonian Cycle

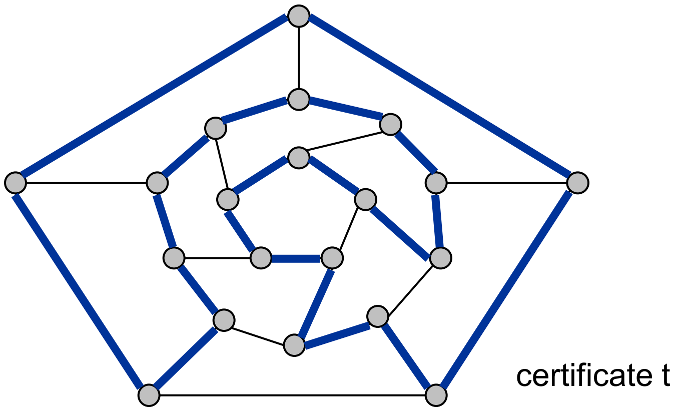
HAM-CYCLE. Given an undirected graph G = (V, E), does there exist a simple cycle C that visits every node?

Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

Conclusion. HAM-CYCLE is in NP.





P, NP, EXP

P. Decision problems for which there is a poly-time algorithm.
EXP. Decision problems for which there is an exponential-time algorithm.
NP. Decision problems for which there is a poly-time certifier.

Claim. P $\subseteq$ NP.
Pf. Consider any problem X in P.
By definition, there exists a poly-time algorithm $A(s)$ that solves X.
Certificate: $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

Claim. NP $\subseteq$ EXP.
Pf. Consider any problem X in NP.
By definition, there exists a poly-time certifier $C(s, t)$ for X.
To solve input s, run $C(s, t)$ on all strings t with $|t| \leq p(|s|)$.
Return `yes`, if $C(s, t)$ returns `yes` for any of these. ▪

The Main Question: P Versus NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

Is the decision problem as easy as the certification problem?

Clay $1 million prize.

would break RSA cryptography
(and potentially collapse economy)

If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, …

Consensus opinion on P = NP? Probably no.

8.4 NP-Completeness

Polynomial Transformation

Def. Problem X **polynomial reduces** (Cook) to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Def. Problem X **polynomial transforms** (Karp) to problem Y if *given* any input x to X, we can construct an input y such that x is a `yes` instance of X iff y is a `yes` instance of Y.

we require |y| to be of size polynomial in |x|

Note. Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X. Almost all previous reductions were of this form.

Open question. Are these two concepts the same with respect to NP?

we abuse notation $\leq_p$ and blur distinction

NP-Complete

NP-complete. A problem Y in NP with the property that for every problem X in NP, $X \leq_p Y$.

Theorem. Suppose Y is an NP-complete problem. Then Y is solvable in poly-time iff P = NP.

Pf. $\Leftarrow$ If P = NP then Y can be solved in poly-time since Y is in NP.

Pf. $\Rightarrow$ Suppose Y can be solved in poly-time.

Let X be any problem in NP. Since $X \leq_p Y$, we can solve X in poly-time. This implies NP $\subseteq$ P.

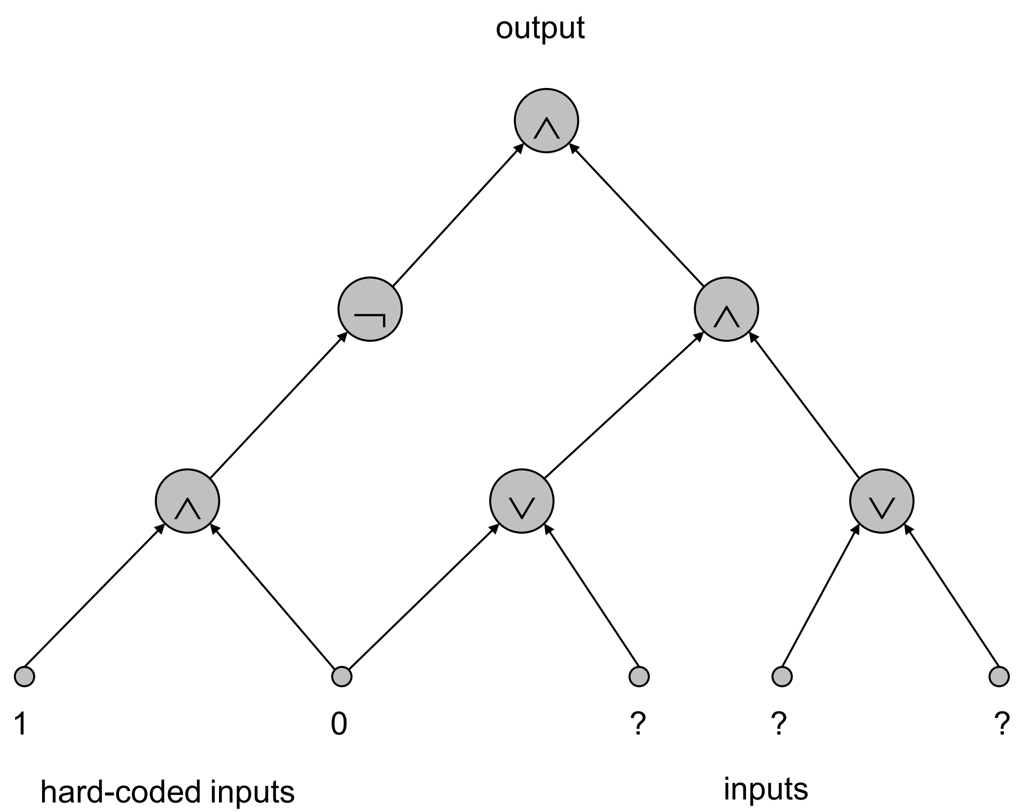
We already know P $\subseteq$ NP. Thus P = NP. ▪

Fundamental question. Do there exist "natural" NP-complete problems?

Circuit Satisfiability

CIRCUIT-SAT. Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?

yes: 1 0 1



The "First" NP-Complete Problem

Theorem. CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]

Pf. (sketch)

Any algorithm that takes a fixed number of bits n as input and produces a yes/no answer can be represented by such a circuit. Moreover, if algorithm takes poly-time, then circuit is of poly-size.

sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

Consider some problem X in NP. It has a poly-time certifier C(s, t). To determine whether s is in X, need to know if there exists a certificate t of length p(|s|) such that C(s, t) = `yes`.

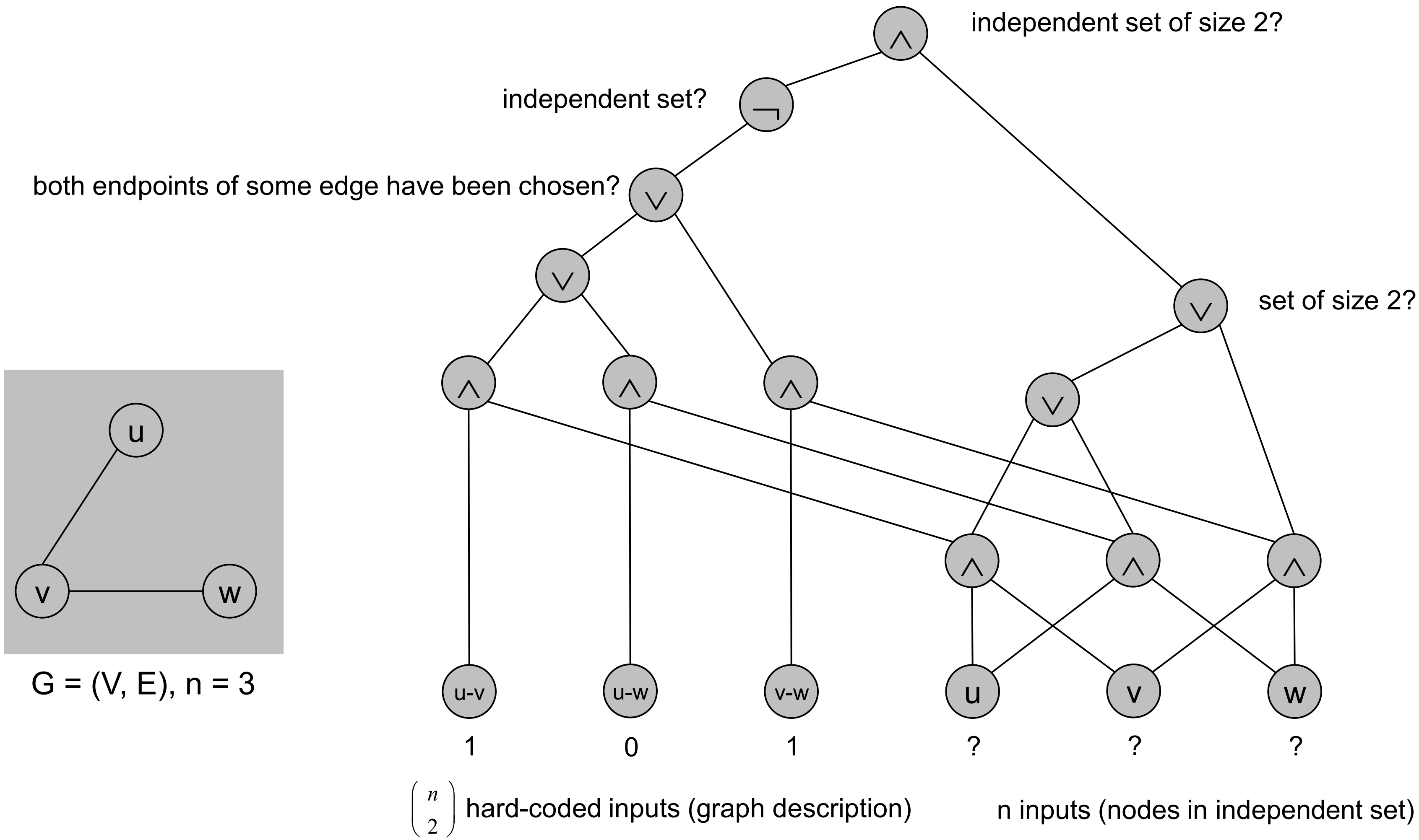
View C(s, t) as an algorithm on |s| + p(|s|) bits (input s, certificate t) and convert it into a poly-size circuit K.

- first |s| bits are hard-coded with s
- remaining p(|s|) bits represent bits of t

Circuit K is satisfiable iff C(s, t) = `yes`.

Example

Ex. Construction below creates a circuit K whose inputs can be set so that K outputs true iff graph G has an independent set of size 2.



Establishing NP-Completeness

Remark. Once we establish first "natural" NP-complete problem, others fall like dominoes.

Recipe to establish NP-completeness of problem Y.

- Step 1. Show that Y is in NP.
- Step 2. Choose an NP-complete problem X.
- Step 3. Prove that $X \leq_p Y$.

Justification. If X is an NP-complete problem, and Y is a problem in NP with the property that $X \leq_P Y$ then Y is NP-complete.

Pf. Let W be any problem in NP. Then $W \underbrace{\leq_P}_{\text{by definition of NP-complete}} X \underbrace{\leq_P}_{\text{by assumption}} Y$.

By transitivity, $W \leq_P Y$.

Hence Y is NP-complete. ▪

3-SAT is NP-Complete

Theorem. 3-SAT is NP-complete.

Pf. Suffices to show that CIRCUIT-SAT $\le_P$ 3-SAT since 3-SAT is in NP.

Let K be any circuit.

Create a 3-SAT variable x_i for each circuit element i.

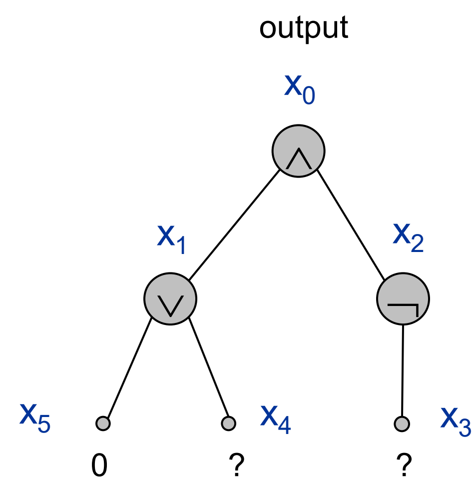
Make circuit compute correct values at each node:

- $x_2 = \neg\, x_3 \;\Rightarrow$ add 2 clauses:
- $x_1 = x_4 \vee x_5 \;\Rightarrow$ add 3 clauses: $x_1 \vee \overline{x_4}\,,\;\; x_1 \vee \overline{x_5}\;,\;\; \overline{x_1} \vee x_4 \vee\; x_5$
- $x_0 = x_1 \wedge x_2 \;\Rightarrow$ add 3 clauses: $\overline{x_0} \vee x_1\,,\;\; \overline{x_0} \vee x_2\,,\;\; x_0 \vee \overline{x_1} \;\vee\; \overline{x_2}$

Hard-coded input values and output value.

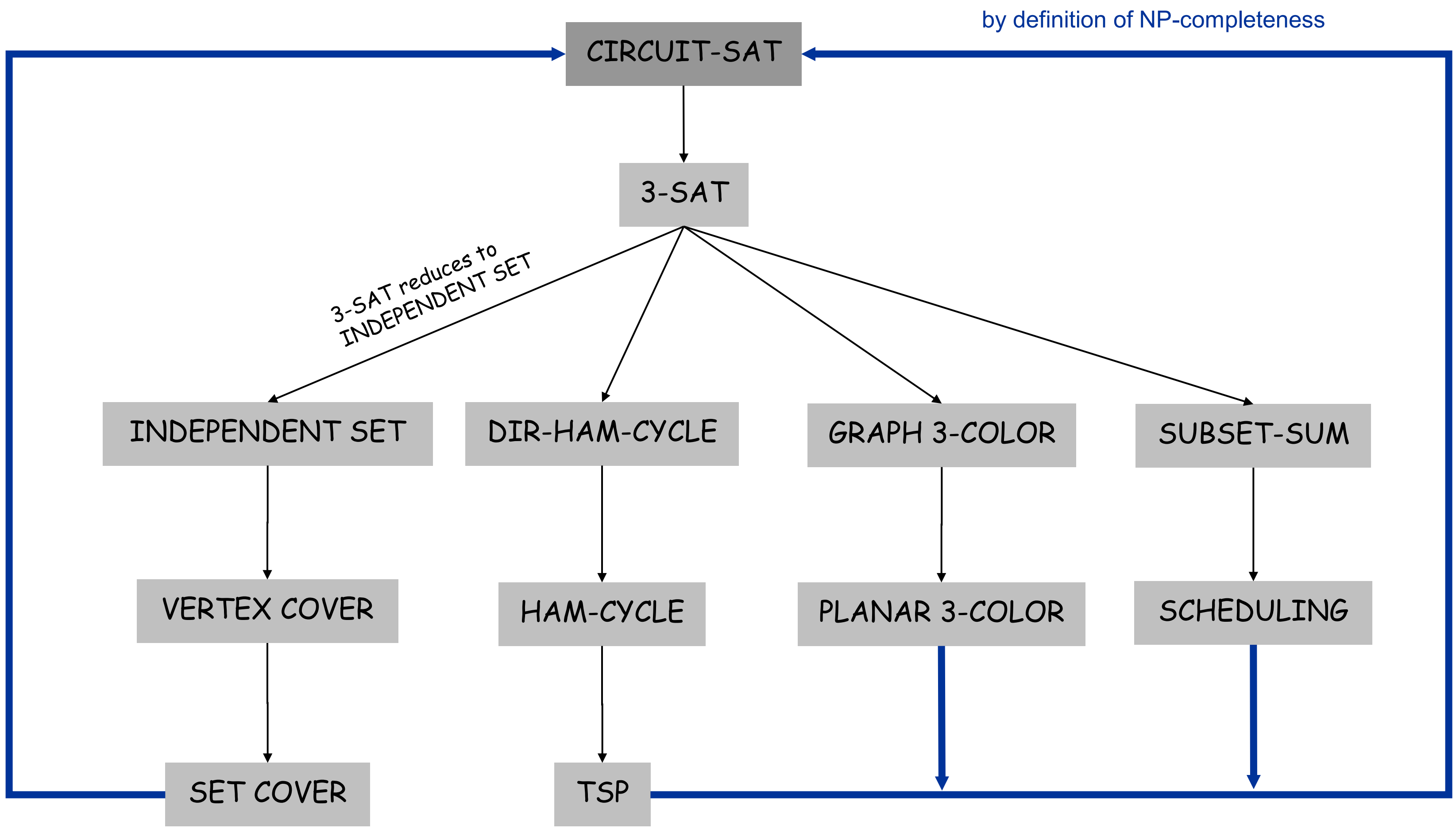
- $x_5 = 0 \;\Rightarrow$ add 1 clause:
- $x_0 = 1 \;\Rightarrow$ add 1 clause:

Final step: turn clauses of length < 3 into clauses of length exactly 3. ▪



NP-Completeness

Observation. All problems below are NP-complete and polynomial reduce to one another!



Some NP-Complete Problems

Six basic genres of NP-complete problems and paradigmatic examples.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Practice. Most NP problems are either known to be in P or NP-complete.

Notable exceptions. Factoring, graph isomorphism, Nash equilibrium.

Extent and Impact of NP-Completeness

Extent of NP-completeness. [Papadimitriou 1995]

- Prime intellectual export of CS to other disciplines.
- 6,000 citations per year (title, abstract, keywords).
  - more than "compiler", "operating system", "database"
- Broad applicability and classification power.
- "Captures vast domains of computational, scientific, mathematical endeavors, and seems to roughly delimit what mathematicians and scientists had been aspiring to compute feasibly."

NP-completeness can guide scientific inquiry.

- 1926: Ising introduces simple model for phase transitions.
- 1944: Onsager solves 2D case in tour de force.
- 19xx: Feynman and other top minds seek 3D solution.
- 2000: Istrail proves 3D problem NP-complete.

More Hard Computational Problems

Aerospace engineering: optimal mesh partitioning for finite elements.
Biology: protein folding.
Chemical engineering: heat exchanger network synthesis.
Civil engineering: equilibrium of urban traffic flow.
Economics: computation of arbitrage in financial markets with friction.
Electrical engineering: VLSI layout.
Environmental engineering: optimal placement of contaminant sensors.
Financial engineering: find minimum risk portfolio of given return.
Game theory: find Nash equilibrium that maximizes social welfare.
Genomics: phylogeny reconstruction.
Mechanical engineering: structure of turbulence in sheared flows.
Medicine: reconstructing 3-D shape from biplane angiocardiogram.
Operations research: optimal resource allocation.
Physics: partition function of 3-D Ising model in statistical mechanics.
Politics: Shapley-Shubik voting power.
Pop culture: Minesweeper consistency.
Statistics: optimal experimental design.

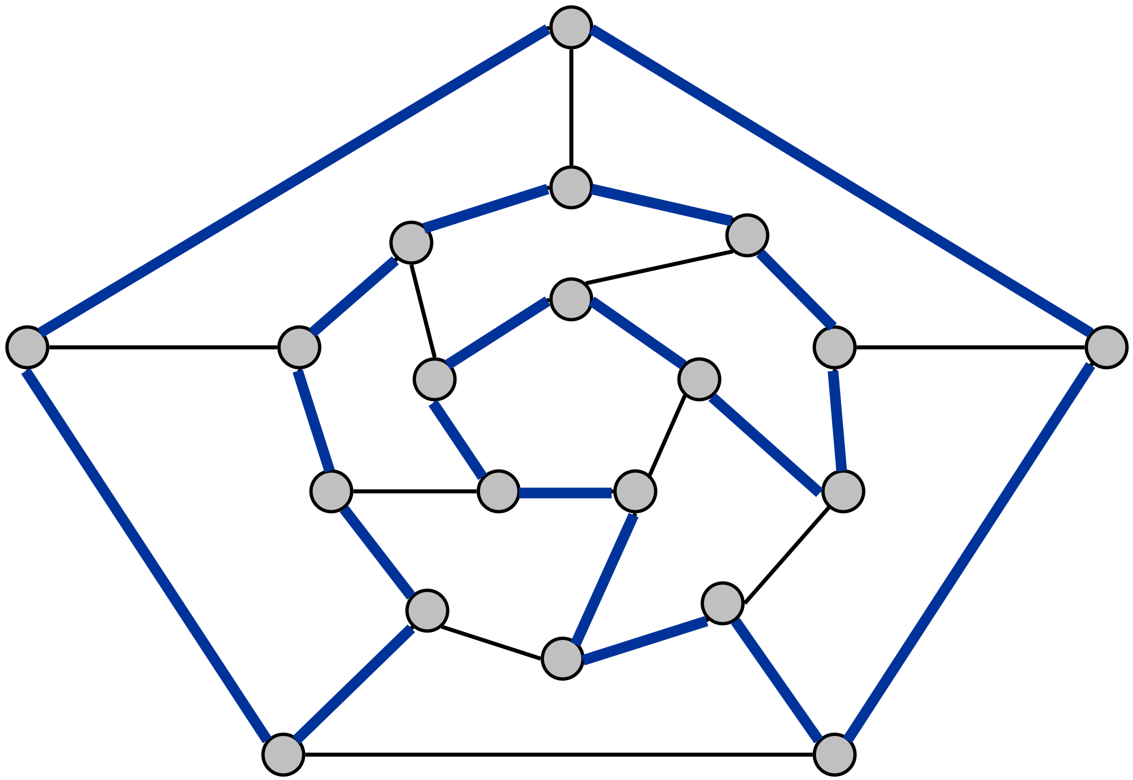
8.5 Sequencing Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Hamiltonian Cycle

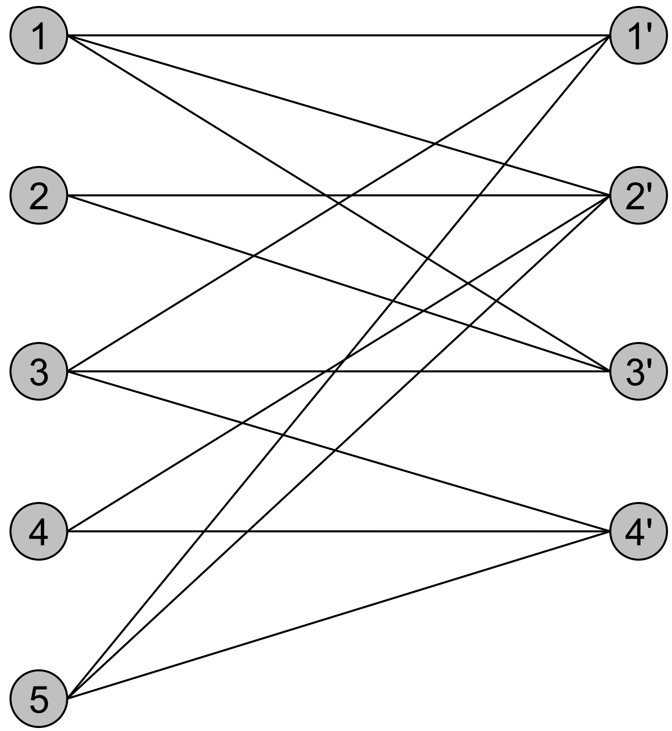
HAM-CYCLE: given an undirected graph $G = (V, E)$, does there exist a simple cycle Γ that contains every node in V.



YES: vertices and faces of a dodecahedron.

Hamiltonian Cycle

HAM-CYCLE: given an undirected graph $G = (V, E)$, does there exist a simple cycle Γ that contains every node in V.



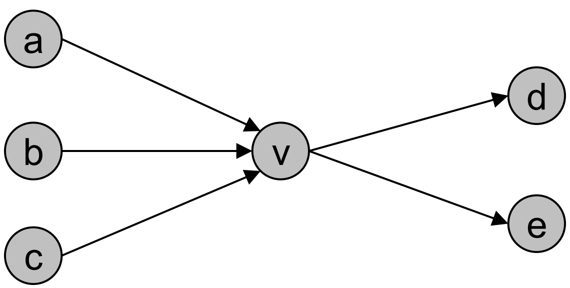
NO: bipartite graph with odd number of nodes.

Directed Hamiltonian Cycle

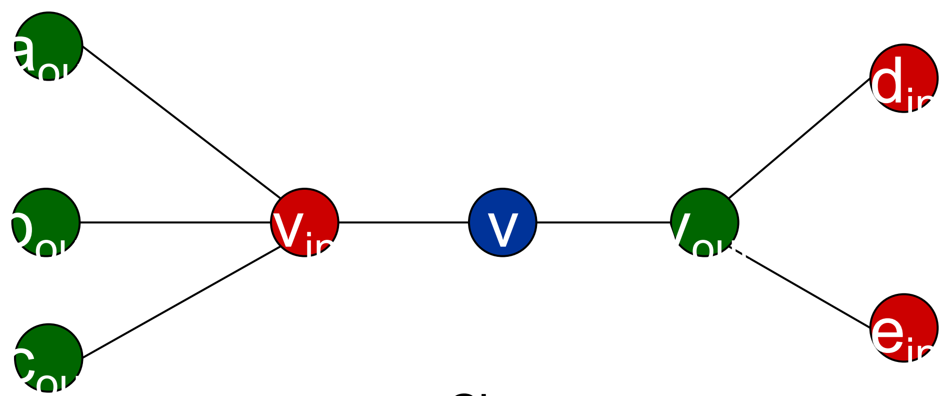
DIR-HAM-CYCLE: given a digraph $G = (V, E)$, does there exists a simple directed cycle Γ that contains every node in V?

Claim. DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE.

Pf. Given a directed graph $G = (V, E)$, construct an undirected graph G' with $3n$ nodes.



G



G'

Directed Hamiltonian Cycle

Claim. G has a Hamiltonian cycle iff G' does.

Pf. $\Rightarrow$

Suppose G has a directed Hamiltonian cycle Γ.
Then G' has an undirected Hamiltonian cycle (same order).

Pf. $\Leftarrow$

Suppose G' has an undirected Hamiltonian cycle Γ'.
Γ' must visit nodes in G' using one of following two orders:

…, B, G, R, B, G, R, B, G, R, B, …

…, B, R, G, B, R, G, B, R, G, B, …

Blue nodes in Γ' make up directed Hamiltonian cycle Γ in G, or reverse of one. ▪

3-SAT Reduces to Directed Hamiltonian Cycle

Claim. 3-SAT $\leq_P$ DIR-HAM-CYCLE.

Pf. Given an instance Φ of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff Φ is satisfiable.

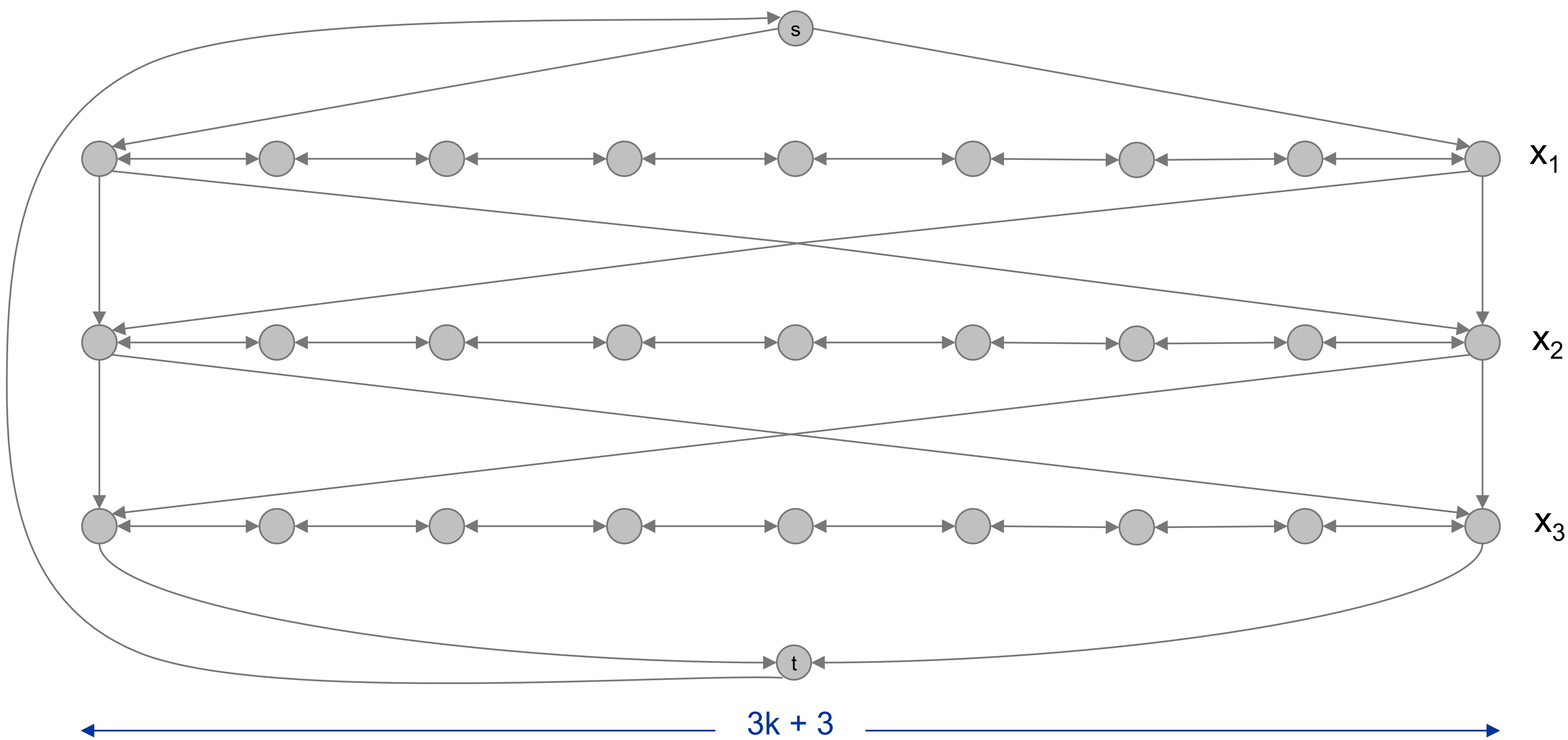
Construction. First, create graph that has 2^n Hamiltonian cycles which correspond in a natural way to 2^n possible truth assignments.

3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

Construct G to have 2^n Hamiltonian cycles.

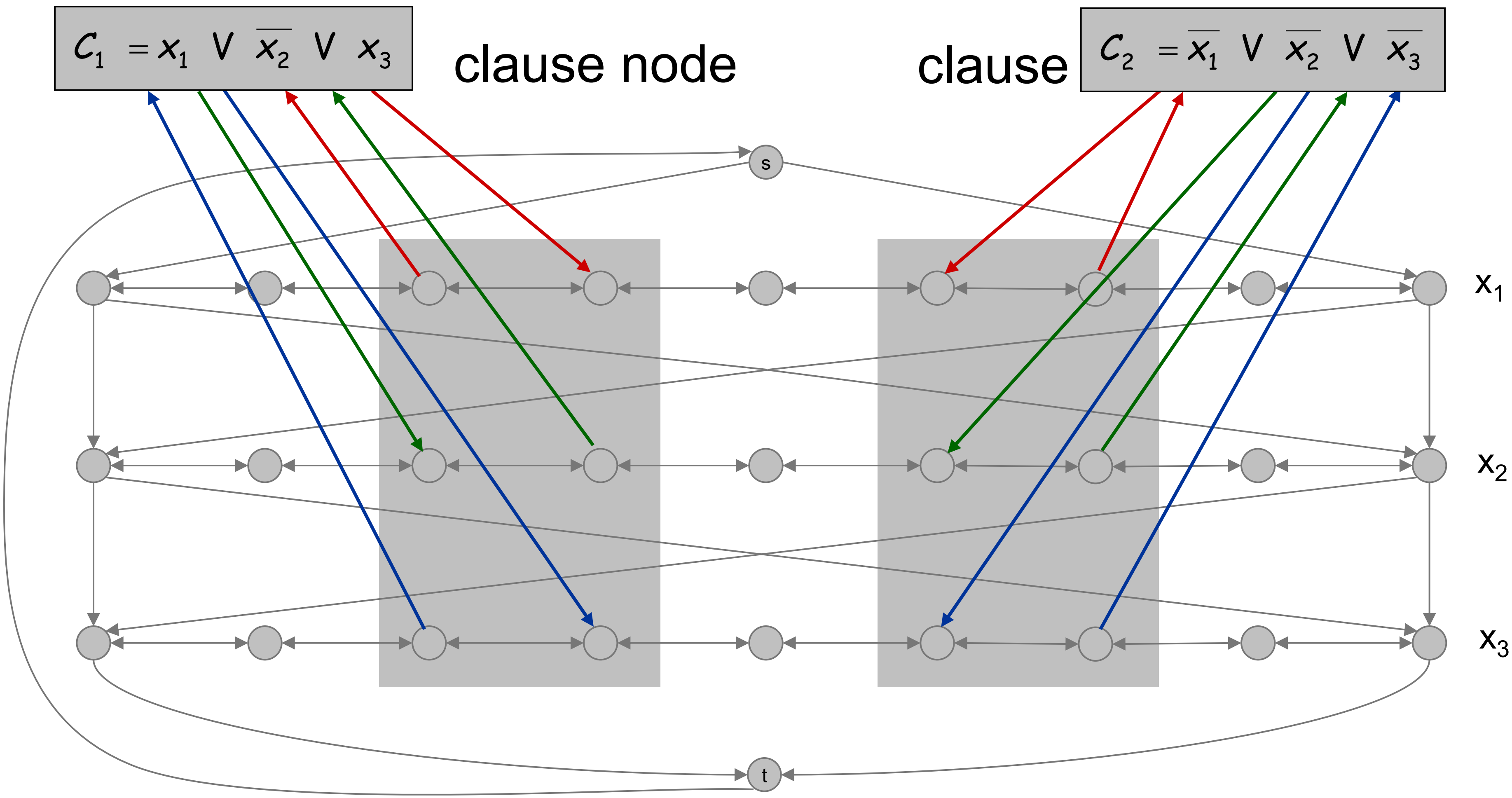
Intuition: traverse path i from left to right $\Leftrightarrow$ set variable $x_i = 1$.



3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

For each clause: add a node and 6 edges.



3-SAT Reduces to Directed Hamiltonian Cycle

Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. $\Rightarrow$

Suppose 3-SAT instance has satisfying assignment x^*.

Then, define Hamiltonian cycle in G as follows:

- if $x^*_i = 1$, traverse row i from left to right
- if $x^*_i = 0$, traverse row i from right to left
- for each clause C_j , there will be at least one row i in which we are going in "correct" direction to splice node C_j into tour

3-SAT Reduces to Directed Hamiltonian Cycle

Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. $\Leftarrow$

Suppose G has a Hamiltonian cycle Γ.

If Γ enters clause node C_j, it must depart on mate edge.

- thus, nodes immediately before and after C_j are connected by an edge e in G
- removing C_j from cycle, and replacing it with edge e yields Hamiltonian cycle on $G - \{ C_j \}$

Continuing in this way, we are left with Hamiltonian cycle Γ' in $G - \{ C_1 , C_2 , \ . \ . \ . \ , C_k \}$.

Set $x^*_i = 1$ iff Γ' traverses row i left to right.

Since Γ visits each clause node C_j, at least one of the paths is traversed in "correct" direction, and each clause is satisfied. ▪

Longest Path

SHORTEST-PATH. Given a digraph G = (V, E), does there exists a simple path of length at most k edges?

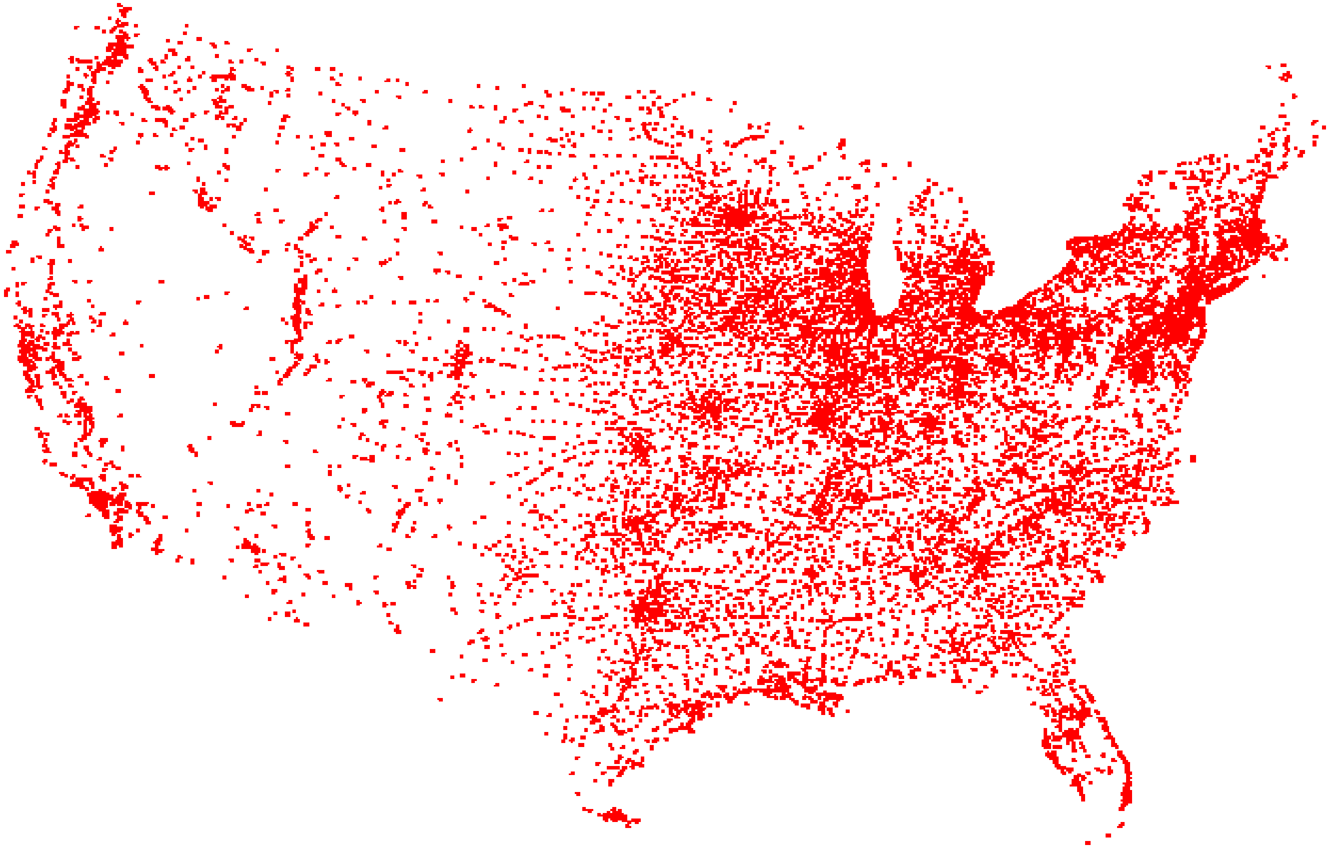
LONGEST-PATH. Given a digraph G = (V, E), does there exists a simple path of length at least k edges?

Claim. 3-SAT $\leq_P$ LONGEST-PATH.

Pf 1. Redo proof for DIR-HAM-CYCLE, ignoring back-edge from t to s.
Pf 2. Show HAM-CYCLE $\leq_P$ LONGEST-PATH.

Traveling Salesperson Problem

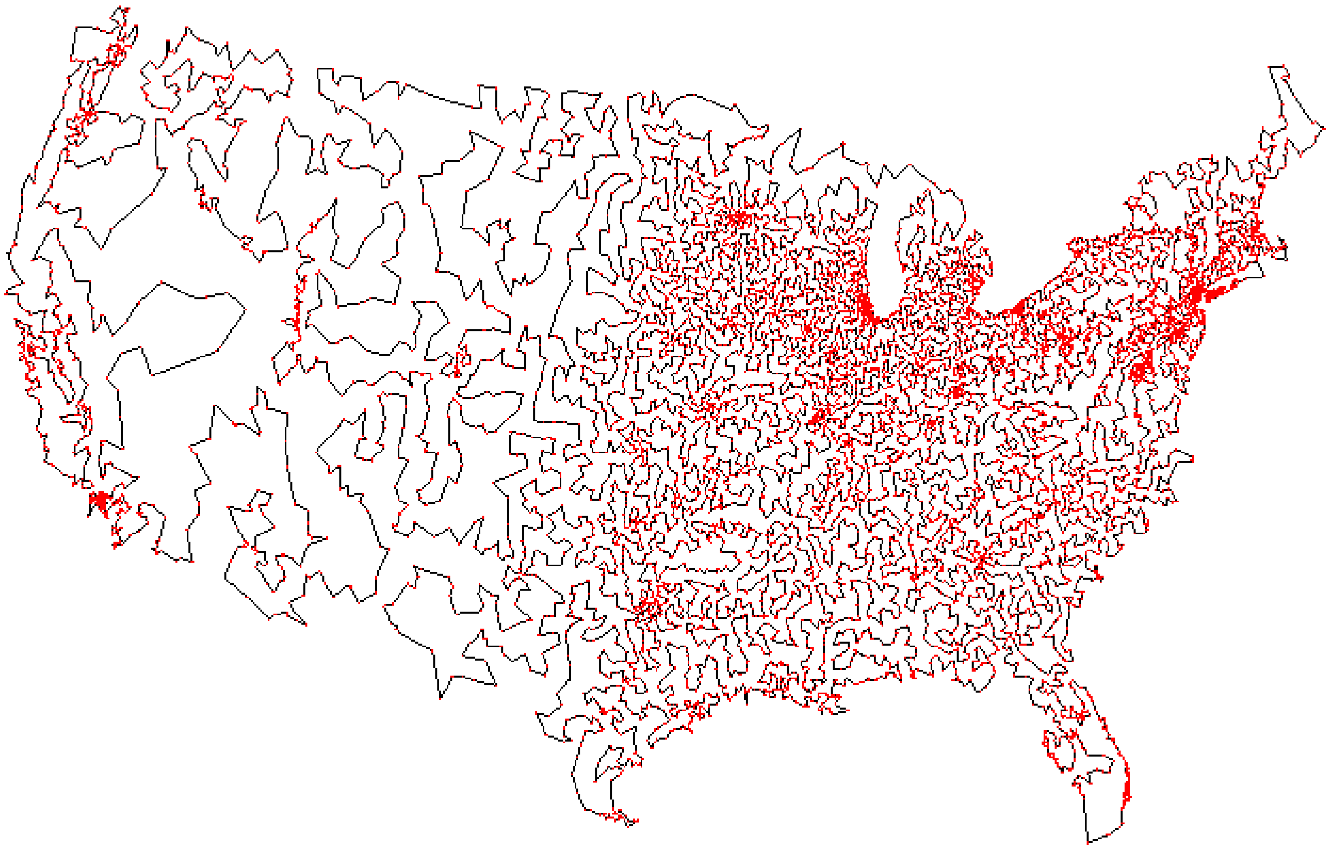
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



All 13,509 cities in US with a population of at least 500
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?

HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V?

Claim. HAM-CYCLE $\leq_P$ TSP.

Pf.

Given instance G = (V, E) of HAM-CYCLE, create n cities with distance function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

TSP instance has tour of length $\leq$ n iff G is Hamiltonian. ▪

Remark. TSP instance in reduction satisfies Δ-inequality.

8.6 Partitioning Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

3-Dimensional Matching

3D-MATCHING. Given n instructors, n courses, and n times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

Instructor	Course	Time
Wayne	COS 423	MW 11-12:20
Wayne	COS 423	TTh 11-12:20
Wayne	COS 226	TTh 11-12:20
Wayne	COS 126	TTh 11-12:20
Tardos	COS 523	TTh 3-4:20
Tardos	COS 423	TTh 11-12:20
Tardos	COS 423	TTh 3-4:20
Kleinberg	COS 226	TTh 3-4:20
Kleinberg	COS 226	MW 11-12:20
Kleinberg	COS 423	MW 11-12:20

3-Dimensional Matching

3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

Claim. 3-SAT $\leq_P$ 3D-MATCHING.

Pf. Given an instance Φ of 3-SAT, we construct an instance of 3D-matching that has a perfect matching iff Φ is satisfiable.

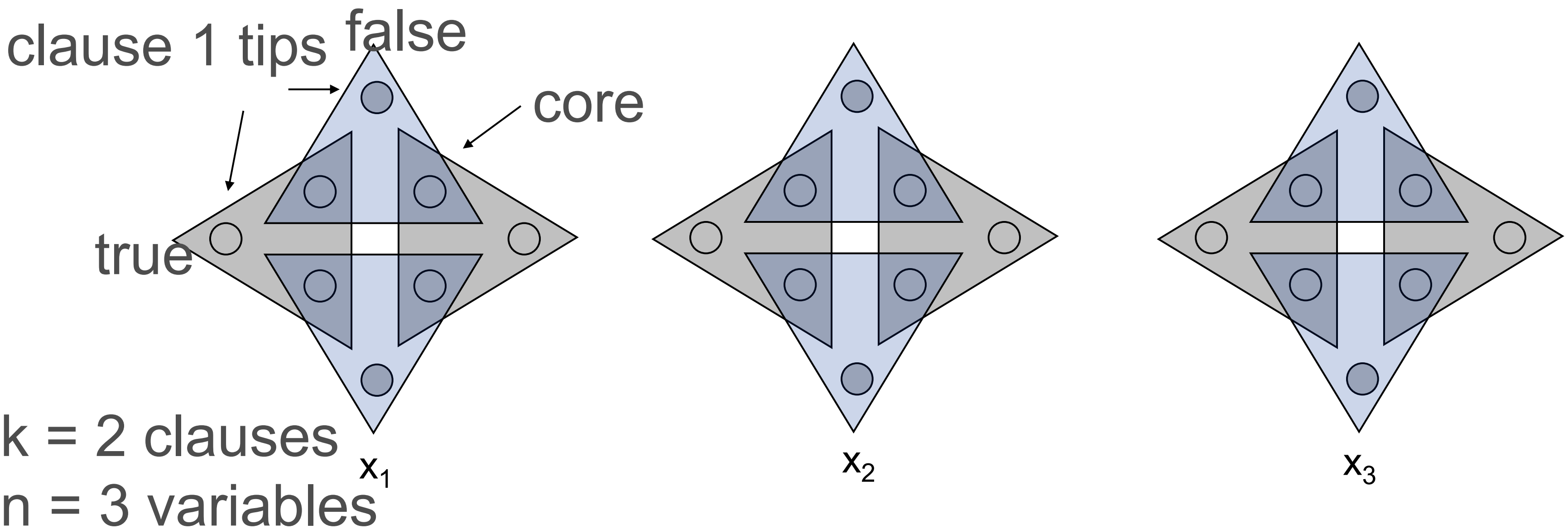
3-Dimensional Matching

number of clauses

Construction. (part 1)

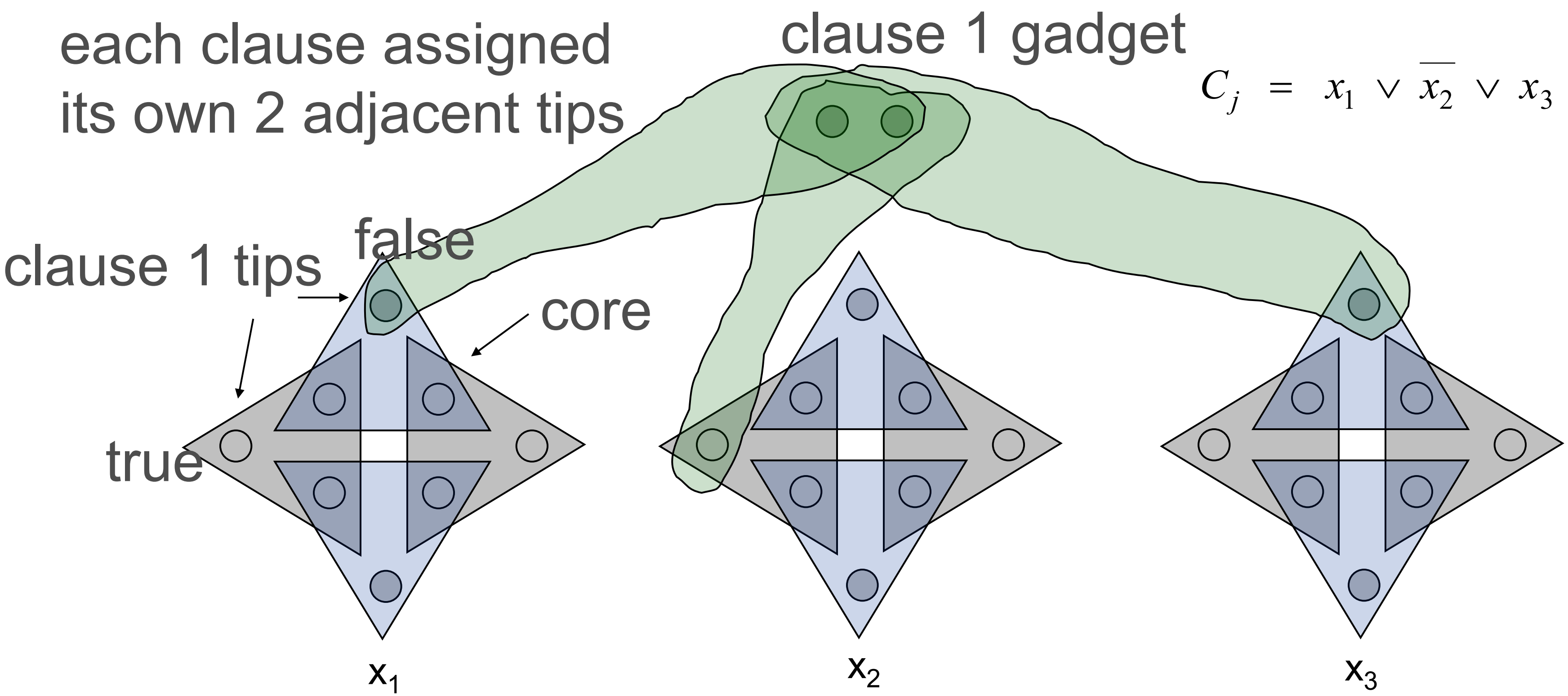
- Create gadget for each variable x_i with 2k core and tip elements.
- No other triples will use core elements.
- In gadget i, 3D-matching must use either both grey triples or both blue ones.

set x_i = true   set x_i = false



Construction. (part 2)

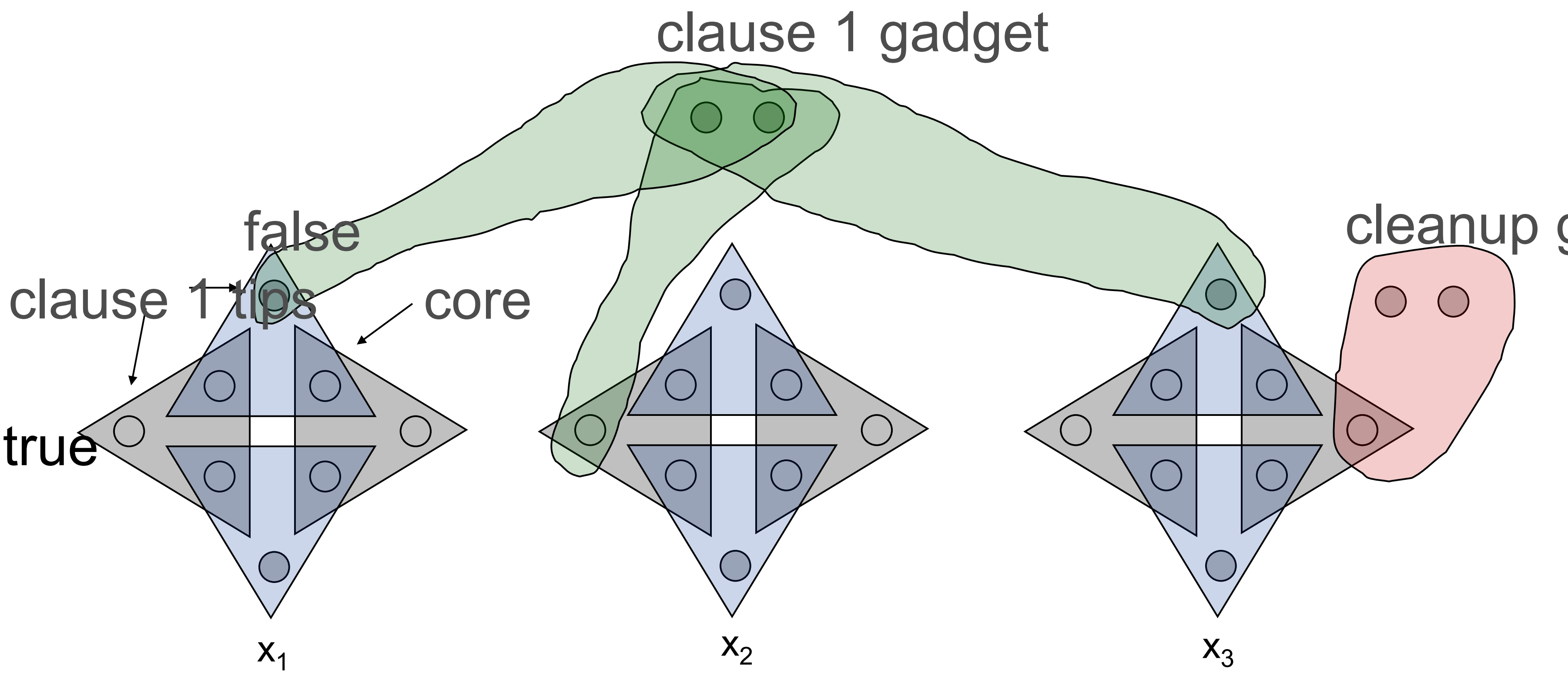
- For each clause C_j create two elements and three triples.
- Exactly one of these triples will be used in any 3D-matching.
- Ensures any 3D-matching uses either (i) grey core of x_1 or (ii) blue core of x_2 or (iii) grey core of x_3.



3-Dimensional Matching

Construction. (part 3)

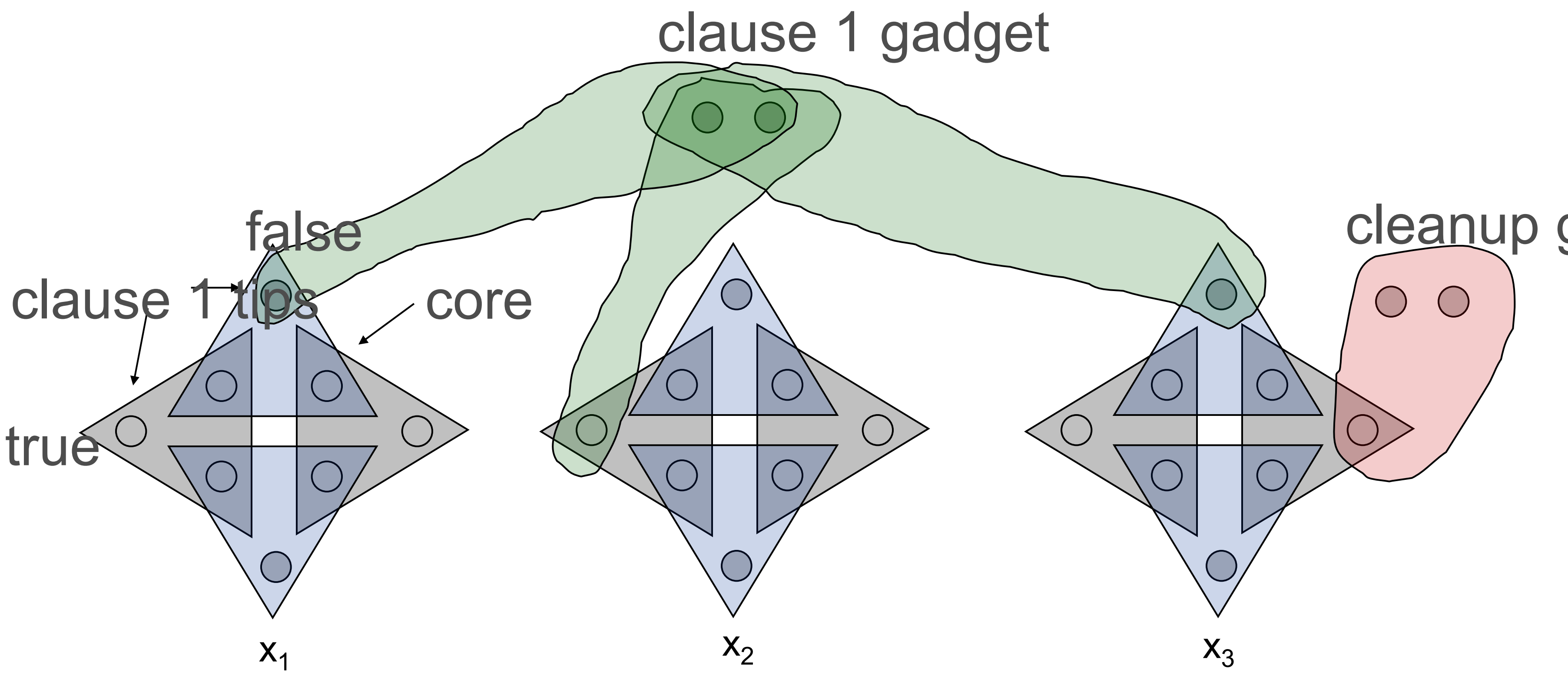
For each tip, add a cleanup gadget.



3-Dimensional Matching

Claim. Instance has a 3D-matching iff Φ is satisfiable.

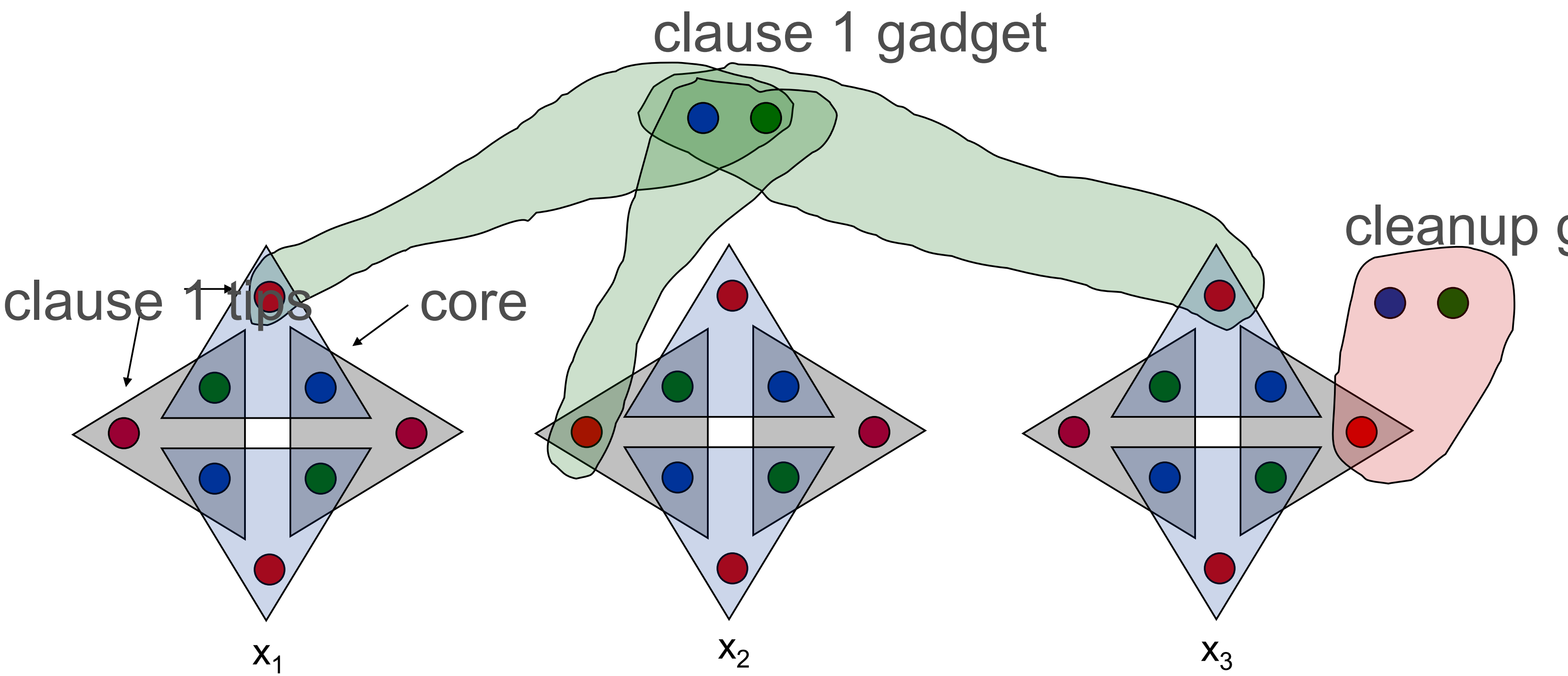
Detail. What are X, Y, and Z? Does each triple contain one element from each of X, Y, Z?



3-Dimensional Matching

Claim. Instance has a 3D-matching iff Φ is satisfiable.

Detail. What are X, Y, and Z? Does each triple contain one element from each of X, Y, Z?



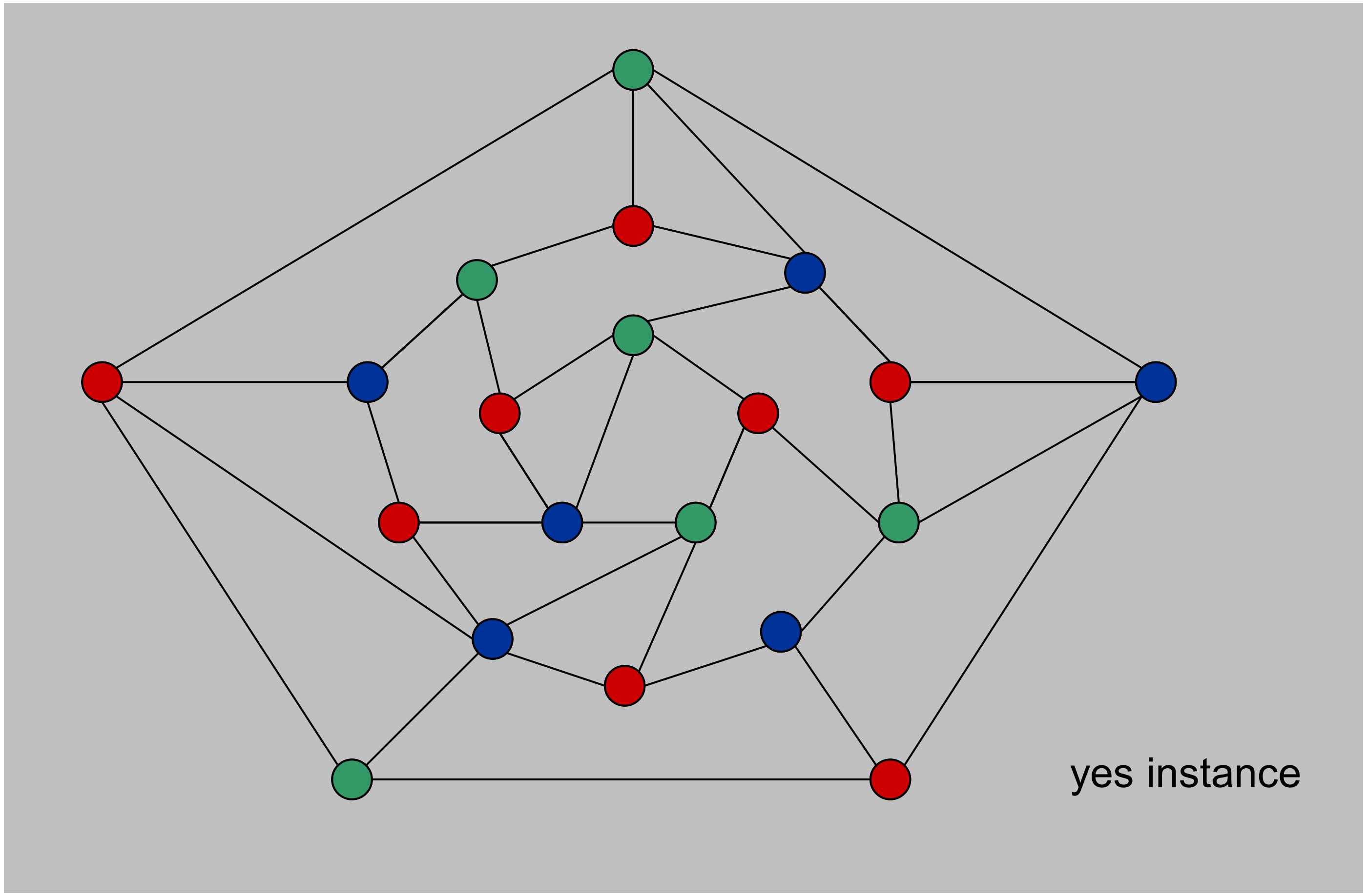
8.7 Graph Coloring

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

3-Colorability

3-COLOR: Given an undirected graph G does there exists a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?



Register Allocation

Register allocation. Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR $\leq_P$ k-REGISTER-ALLOCATION for any constant $k \geq 3$.

3-Colorability

Claim. 3-SAT $\leq_P$ 3-COLOR.

Pf. Given 3-SAT instance Φ, we construct an instance of 3-COLOR that is 3-colorable iff Φ is satisfiable.

Construction.

i. For each literal, create a node.

ii. Create 3 new nodes T, F, B; connect them in a triangle, and connect each literal to B.

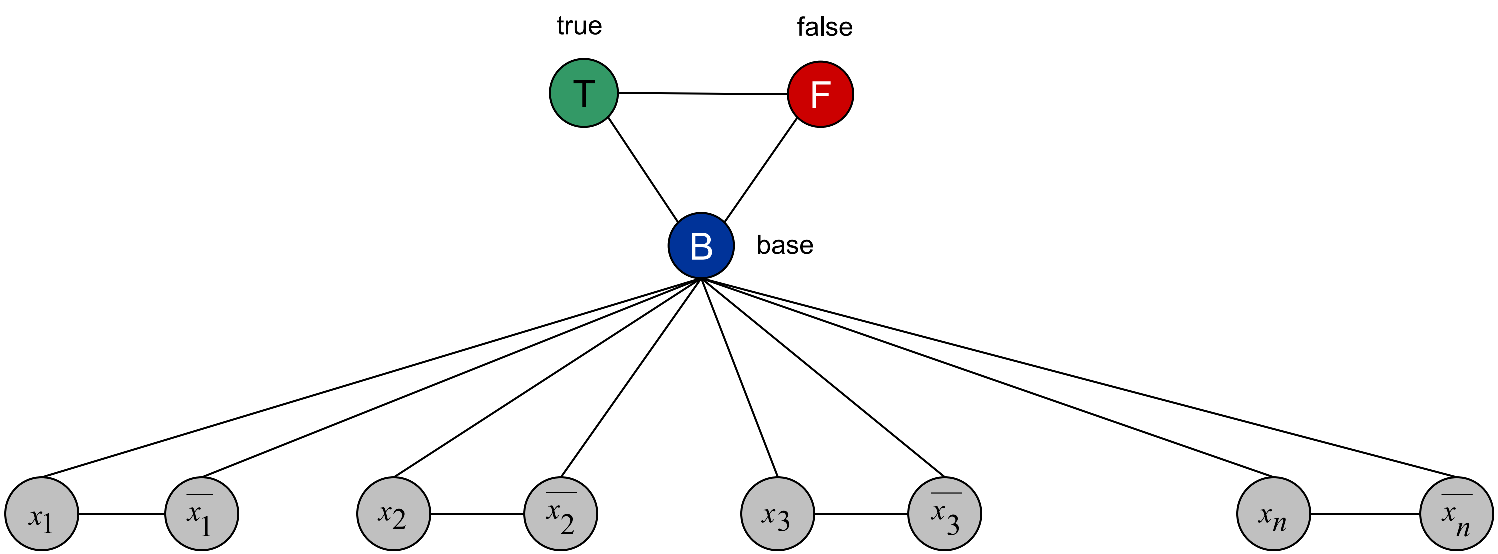
iii. Connect each literal to its negation.

iv. For each clause, add gadget of 6 nodes and 13 edges.

↑ to be described next

Claim. Graph is 3-colorable iff Φ is satisfiable.

Pf. $\Rightarrow$ Suppose graph is 3-colorable.
Consider assignment that sets all T literals to true.
(ii) ensures each literal is T or F.
(iii) ensures a literal and its negation are opposites.



3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

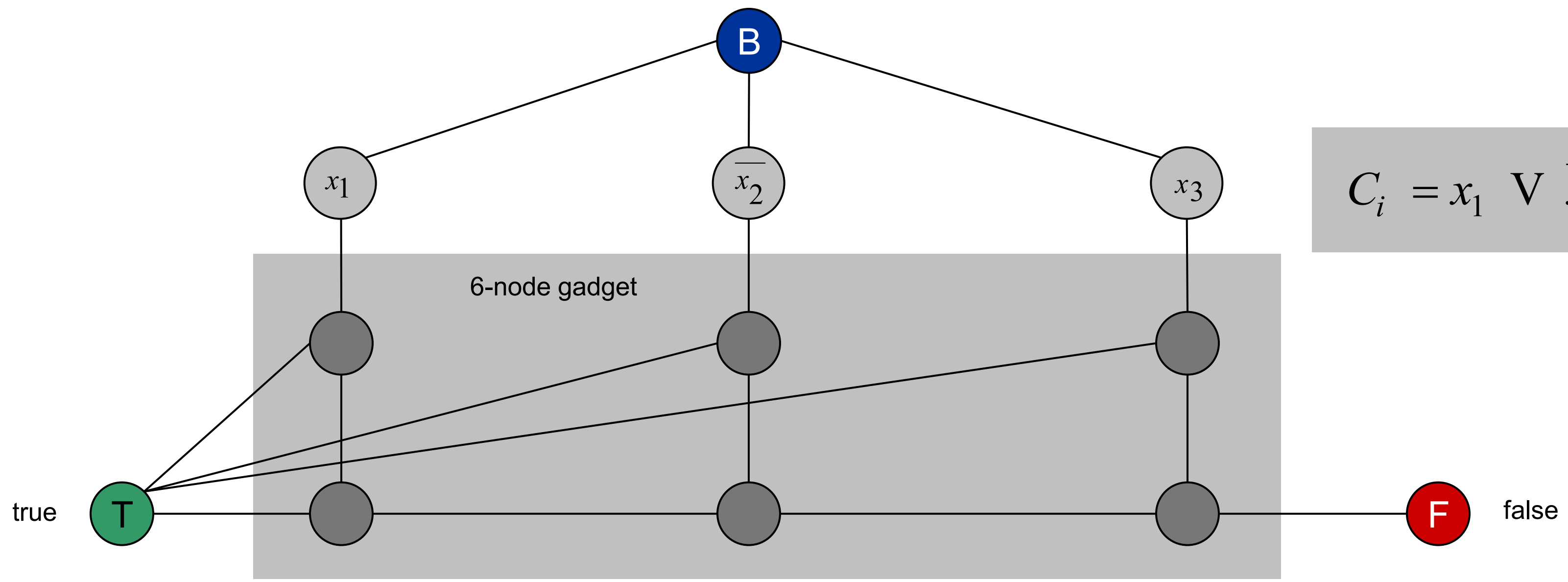
Pf. $\Rightarrow$ Suppose graph is 3-colorable.

Consider assignment that sets all T literals to true.

(ii) ensures each literal is T or F.

(iii) ensures a literal and its negation are opposites.

(iv) ensures at least one literal in each clause is T.



$$C_i = x_1 \vee \overline{x_2} \vee x_3$$

3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

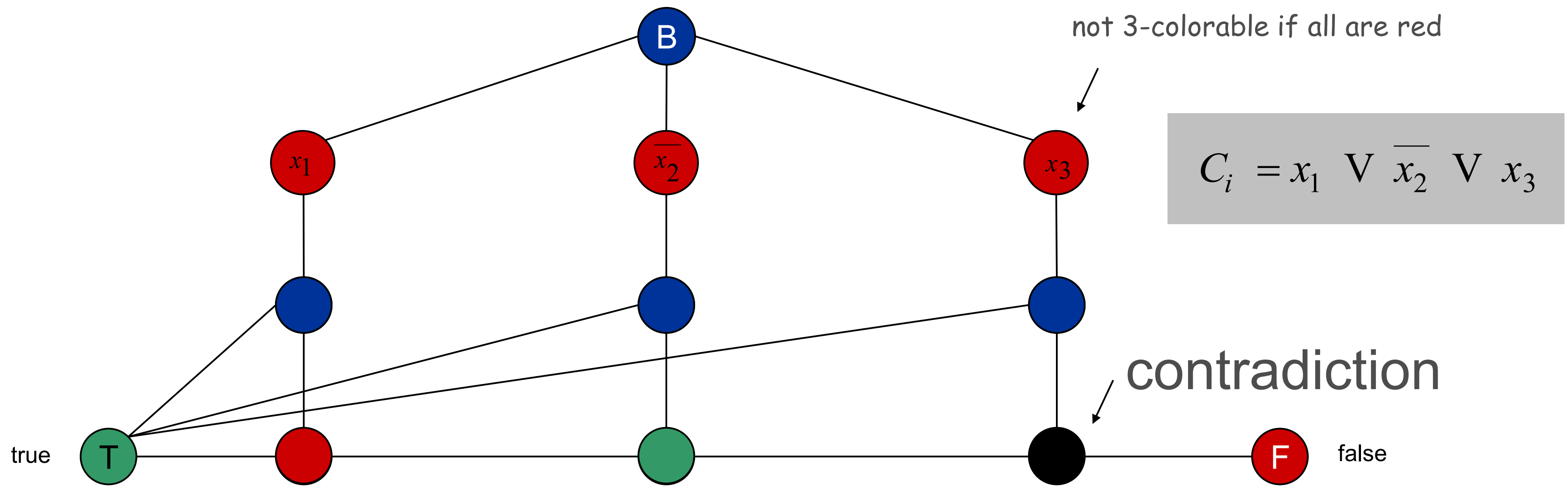
Pf. $\Rightarrow$ Suppose graph is 3-colorable.

Consider assignment that sets all T literals to true.

(ii) ensures each literal is T or F.

(iii) ensures a literal and its negation are opposites.

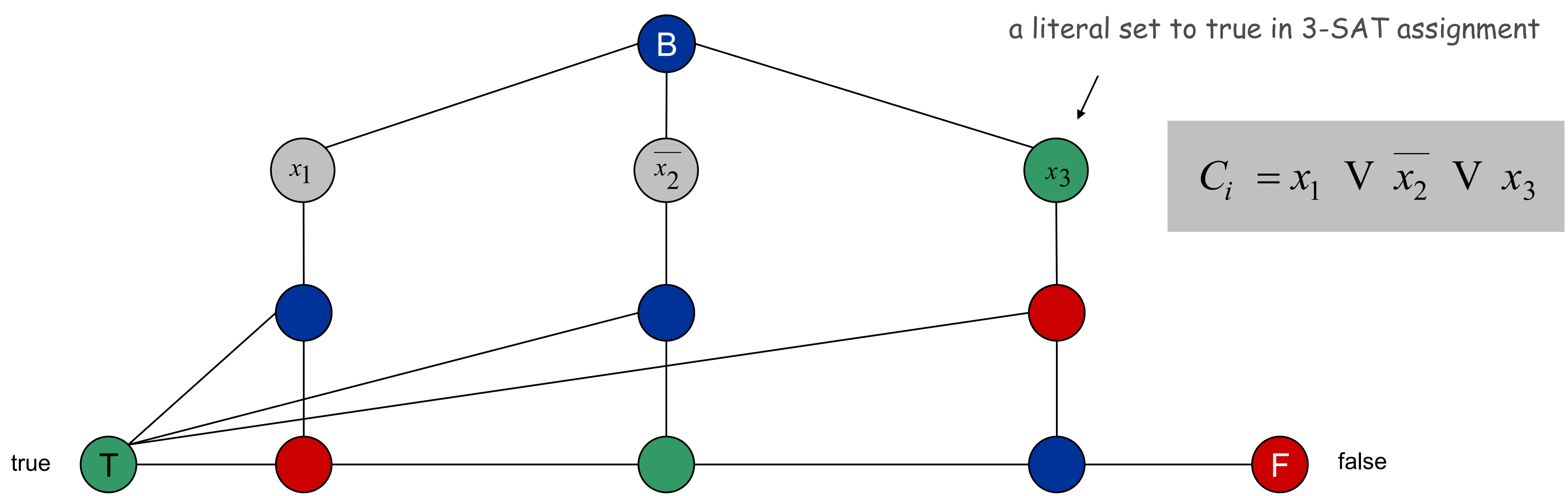
(iv) ensures at least one literal in each clause is T.



Claim. Graph is 3-colorable iff Φ is satisfiable.

Pf. $\Leftarrow$ Suppose 3-SAT formula Φ is satisfiable.

- Color all true literals T.
- Color node below green node F, and node below that B.
- Color remaining middle row nodes B.
- Color remaining bottom nodes T or F as forced. ▪



8.8 Numerical Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, 3D-MATCHING.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Subset Sum

SUBSET-SUM. Given natural numbers w_1, …, w_n and an integer W, is there a subset that adds up to exactly W?

Ex: { 1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344 }, W = 3754.
Yes. 1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754.

Remark. With arithmetic problems, input integers are encoded in binary. Polynomial reduction must be polynomial in binary encoding.

Claim. 3-SAT $\leq_P$ SUBSET-SUM.
Pf. Given an instance Φ of 3-SAT, we construct an instance of SUBSET-SUM that has solution iff Φ is satisfiable.

Subset Sum

Construction. Given 3-SAT instance Φ with n variables and k clauses, form 2n + 2k decimal integers, each of n+k digits, as illustrated below.

Claim. Φ is satisfiable iff there exists a subset that sums to W.
Pf. No carries possible.

$$C_1 = \bar{x} \vee y \vee z$$
$$C_2 = x \vee \bar{y} \vee z$$
$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

	x	y	z	C_1	C_2	C_3	
x	1	0	0	0	1	0	100,010
¬ x	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
¬ y	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
¬ z	0	0	1	0	0	1	1,001
	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

dummies to get clause columns to sum to 4

Scheduling With Release Times

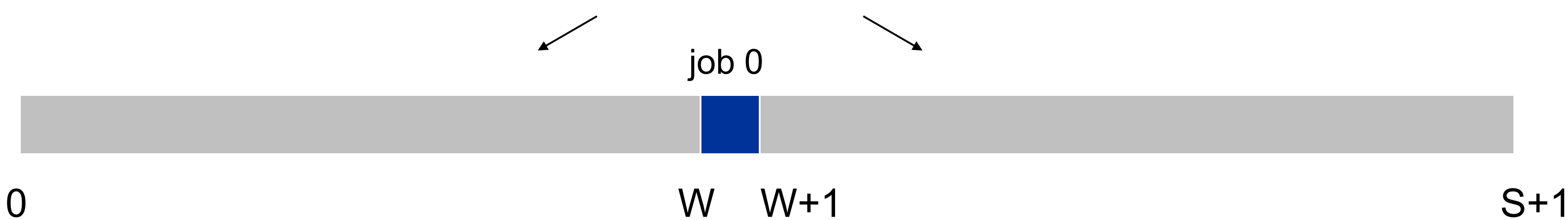
SCHEDULE-RELEASE-TIMES. Given a set of n jobs with processing time t_i, release time r_i, and deadline d_i, is it possible to schedule all jobs on a single machine such that job i is processed with a contiguous slot of t_i time units in the interval $[r_i, d_i]$?

Claim. SUBSET-SUM $\leq_P$ SCHEDULE-RELEASE-TIMES.

Pf. Given an instance of SUBSET-SUM $w_1, \ldots, w_n$, and target W,

- Create n jobs with processing time $t_i = w_i$, release time $r_i = 0$, and no deadline ($d_i = 1 + \Sigma_j w_j$).
- Create job 0 with $t_0 = 1$, release time $r_0 = W$, and deadline $d_0 = W+1$.

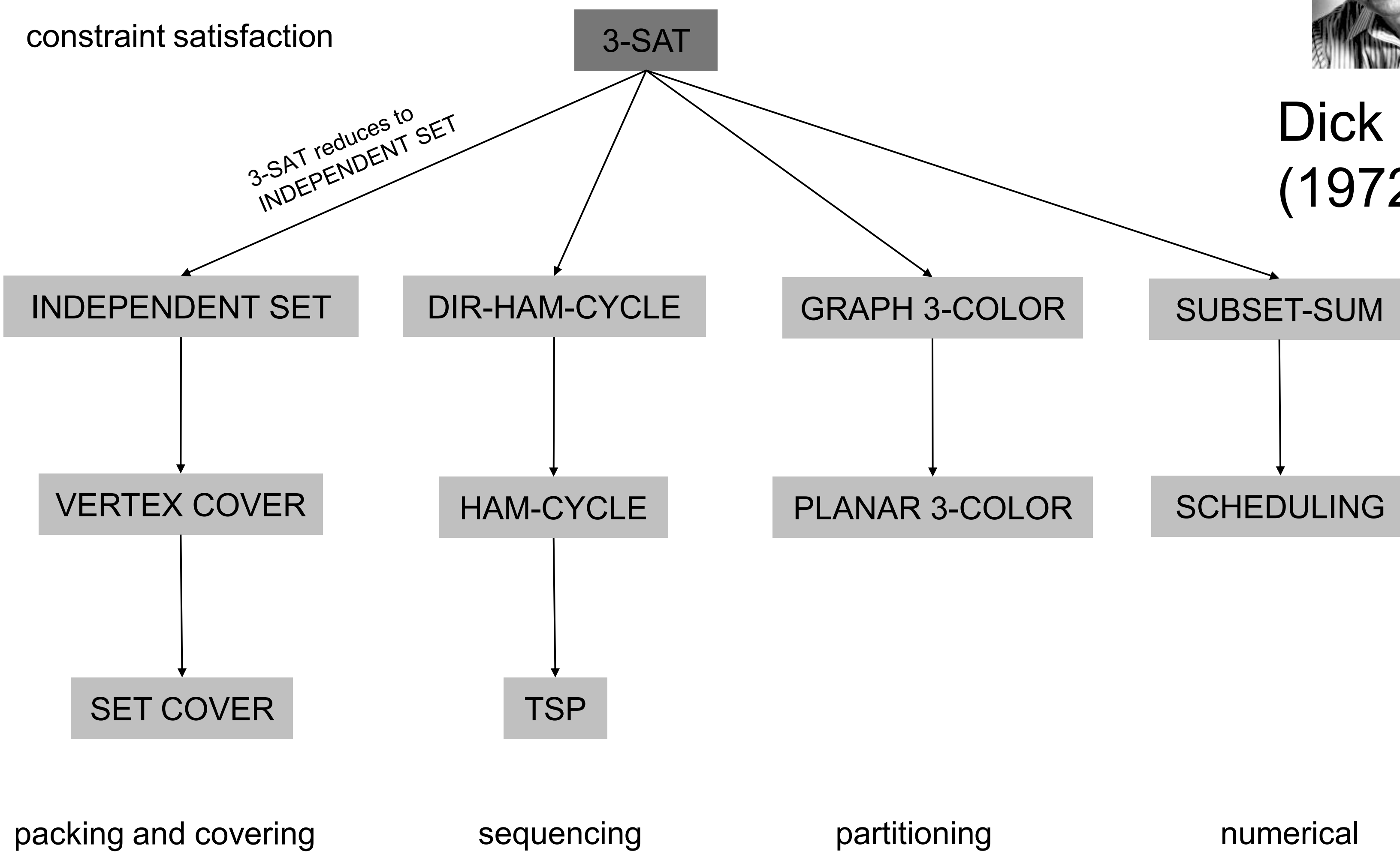
Can schedule jobs 1 to n anywhere but [W, W+1]



8.10 A Partial Taxonomy of Hard Problems

Polynomial-Time Reductions

Dick Karp
(1972)



Partition

SUBSET-SUM. Given natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

PARTITION. Given natural numbers $v_1, \ldots, v_m$, can they be partitioned into two subsets that add up to the same value?

$\frac{1}{2} \Sigma_i \, v_i$

Claim. SUBSET-SUM $\leq_P$ PARTITION.

Pf. Let $W, w_1, \ldots, w_n$ be an instance of SUBSET-SUM.

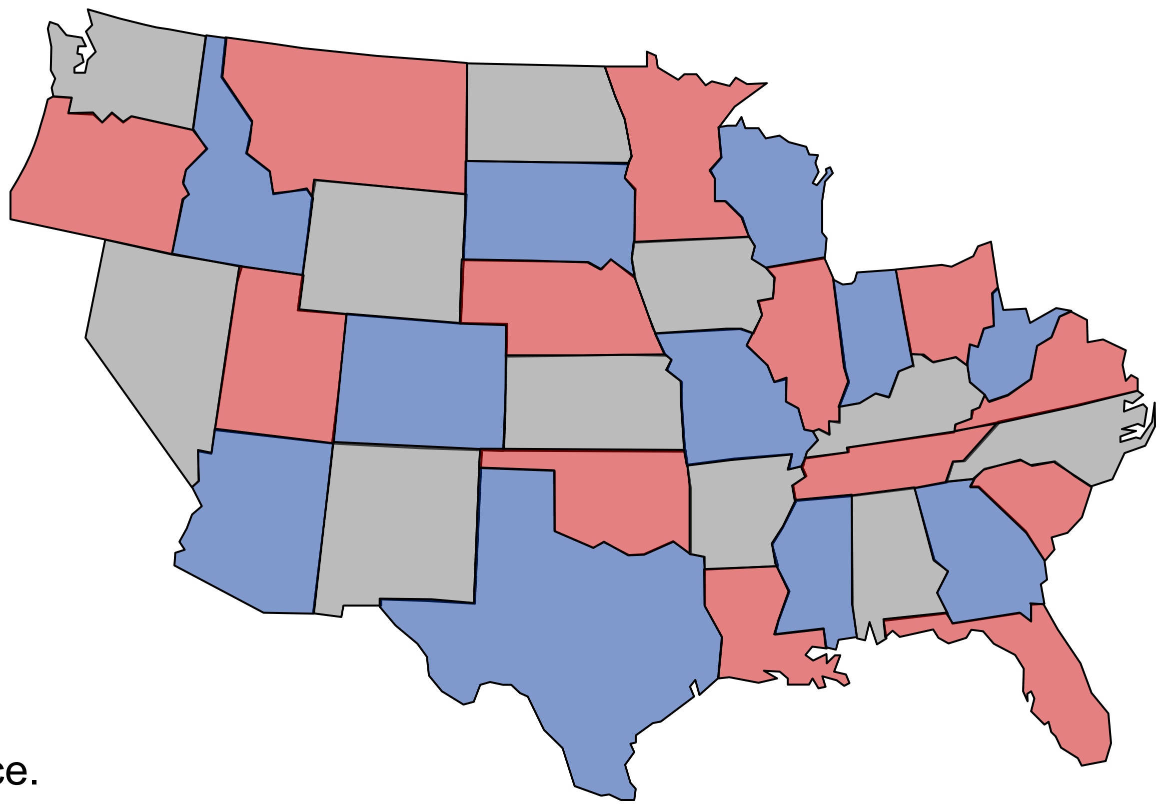
Create instance of PARTITION with $m = n+2$ elements.

- $v_1 = w_1, v_2 = w_2, \ldots, v_n = w_n, \quad v_{n+1} = 2 \Sigma_i \, w_i - W, \quad v_{n+2} = \Sigma_i \, w_i + W$

There exists a subset that sums to W iff there exists a partition since two new elements cannot be in the same partition. ▪

Planar 3-Colorability

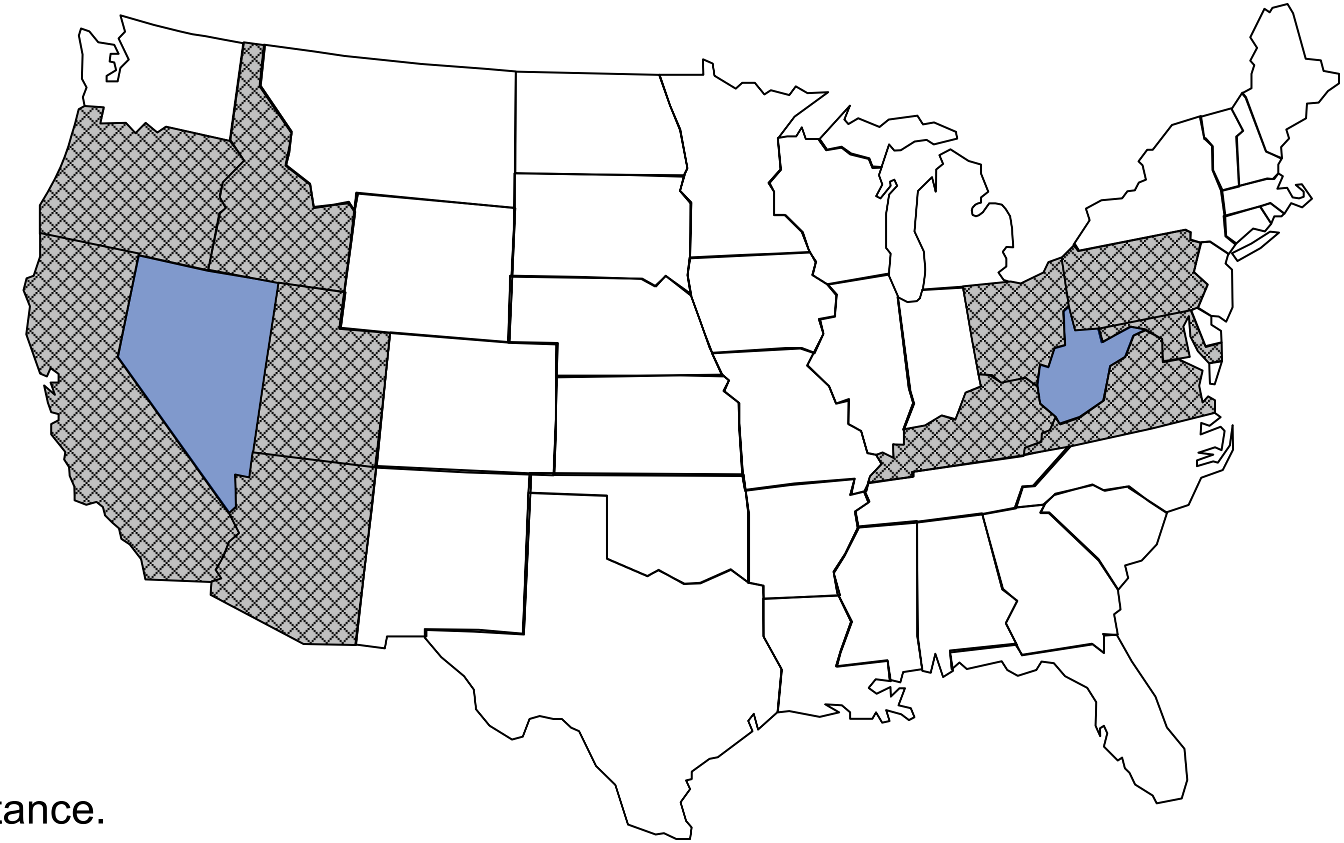
PLANAR-3-COLOR. Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?



YES instance.

Planar 3-Colorability

PLANAR-3-COLOR. Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

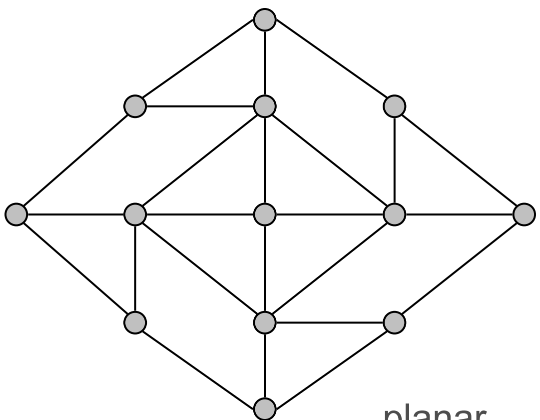


NO instance.

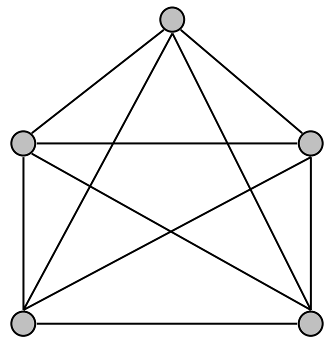
Planarity

Def. A graph is planar if it can be embedded in the plane in such a way that no two edges cross.

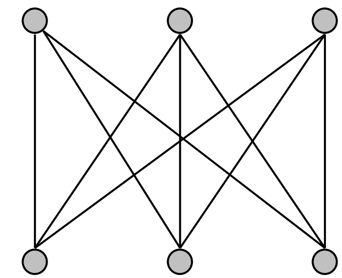
Applications: VLSI circuit design, computer graphics.



planar



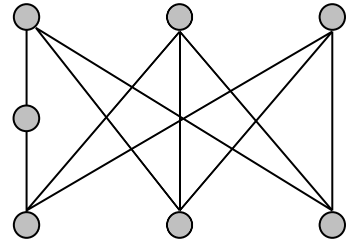
K_5: non-planar



$K_{3,3}$: non-planar

Kuratowski's Theorem. An undirected graph G is non-planar iff it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.

homeomorphic to $K_{3,3}$



Planarity Testing

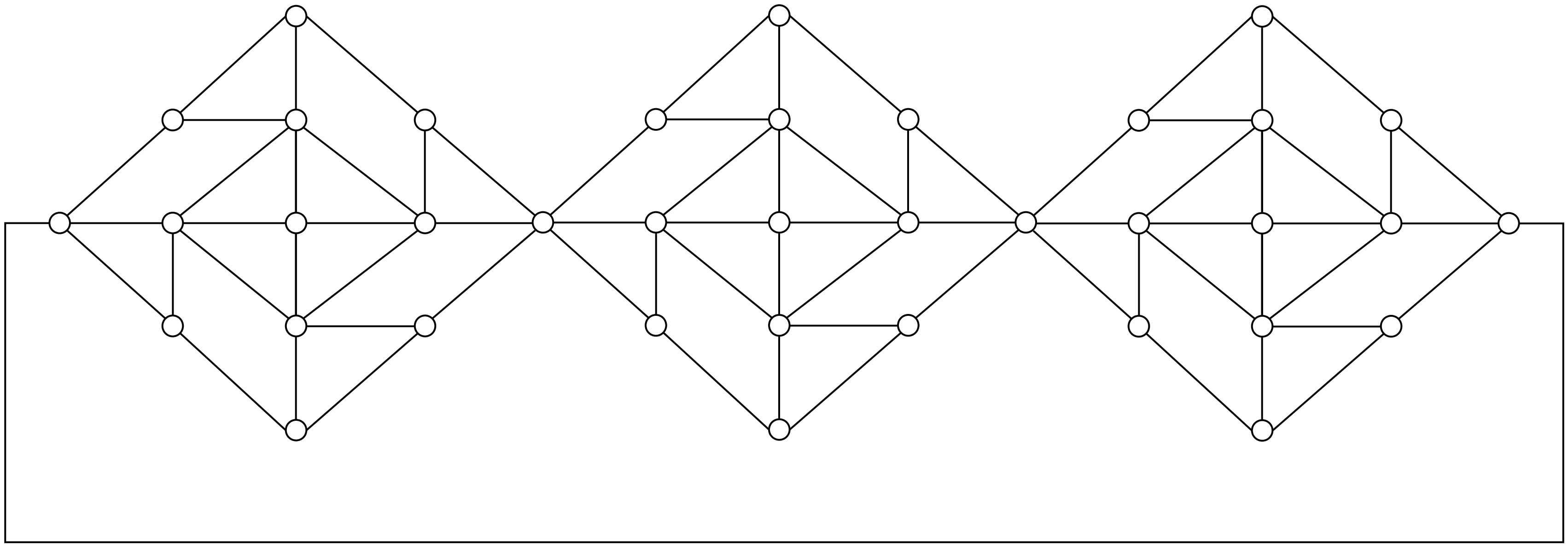
Planarity testing. [Hopcroft-Tarjan 1974] $O(n)$.

simple planar graph can have at most 3n edges

Remark. Many intractable graph problems can be solved in poly-time if the graph is planar; many tractable graph problems can be solved faster if the graph is planar.

Planar Graph 3-Colorability

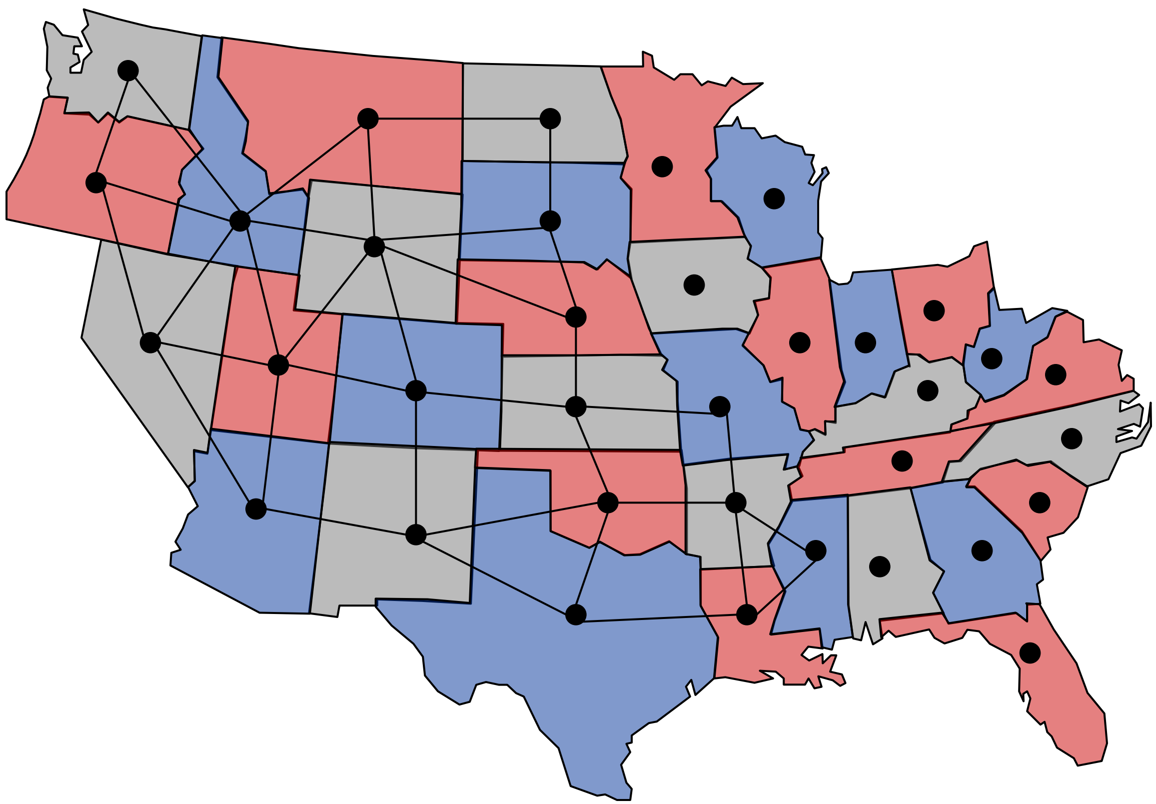
Q. Is this planar graph 3-colorable?



Planar 3-Colorability and Graph 3-Colorability

Claim. PLANAR-3-COLOR $\leq_P$ PLANAR-GRAPH-3-COLOR.

Pf sketch. Create a vertex for each region, and an edge between regions that share a nontrivial border.



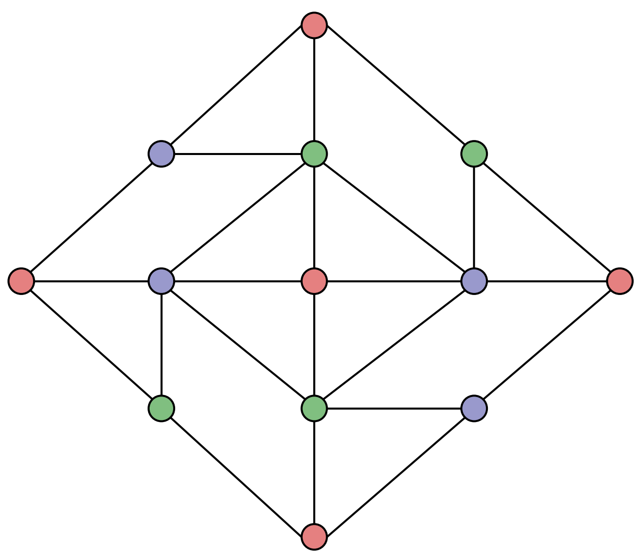
Planar Graph 3-Colorability

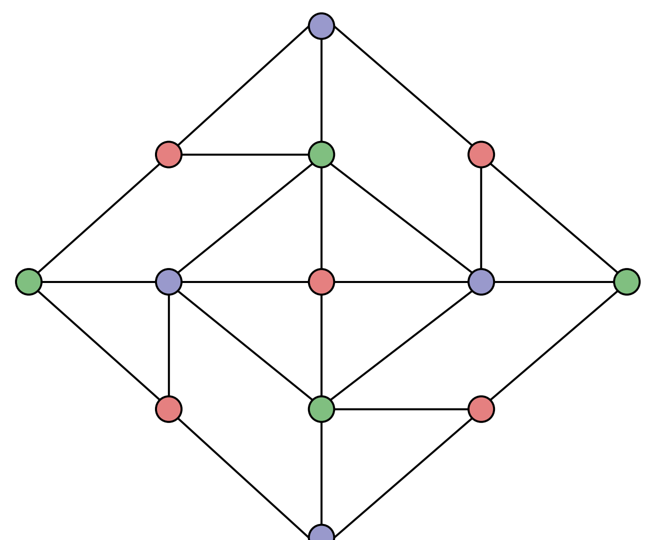
Claim. W is a planar graph such that:

In any 3-coloring of W, opposite corners have the same color.

Any assignment of colors to the corners in which opposite corners have the same color extends to a 3-coloring of W.

Pf. Only 3-colorings of W are shown below (or by permuting colors).



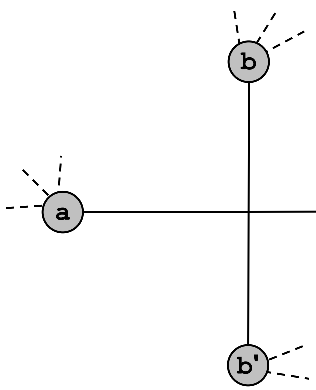


Planar Graph 3-Colorability

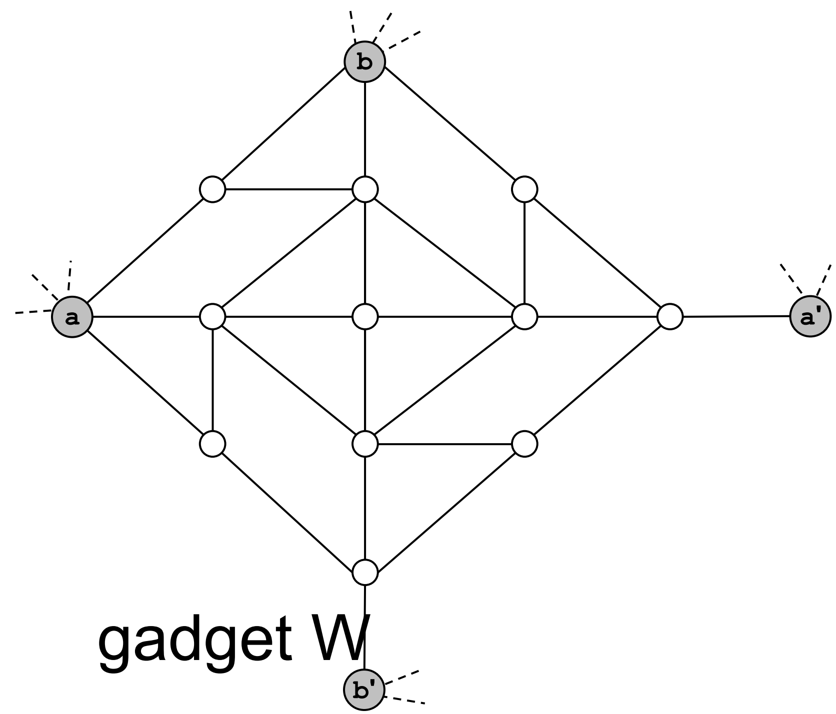
Claim. 3-COLOR $\leq_P$ PLANAR-GRAPH-3-COLOR.

Pf. Given instance of 3-COLOR, draw graph in plane, letting edges cross.

- Replace each edge crossing with planar gadget W.
- In any 3-coloring of W, $a \neq a'$ and $b \neq b'$.
- If $a \neq a'$ and $b \neq b'$ then can extend to a 3-coloring of W.



a crossing

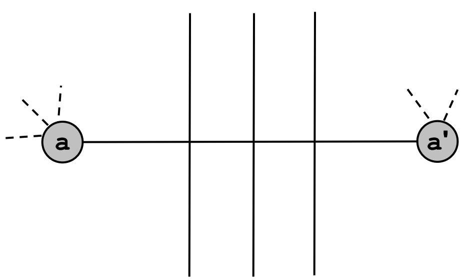


gadget W

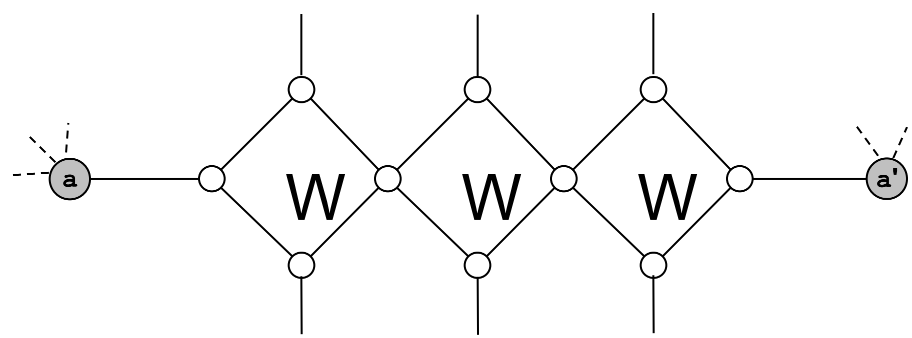
Claim. 3-COLOR $\le_P$ PLANAR-GRAPH-3-COLOR.

Pf. Given instance of 3-COLOR, draw graph in plane, letting edges cross.

- Replace each edge crossing with planar gadget W.
- In any 3-coloring of W, $a \ne a'$ and $b \ne b'$.
- If $a \ne a'$ and $b \ne b'$ then can extend to a 3-coloring of W.



multiple crossings



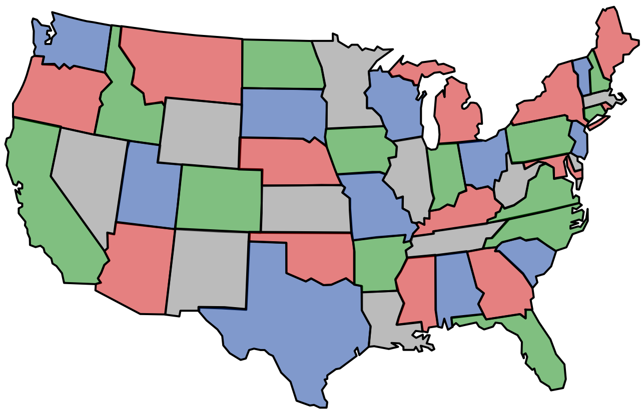
gadget W

Planar k-Colorability

PLANAR-2-COLOR. Solvable in linear time.

PLANAR-3-COLOR. NP-complete.

PLANAR-4-COLOR. Solvable in O(1) time.



Theorem. [Appel-Haken, 1976] Every planar map is 4-colorable.

- Resolved century-old open problem.
- Used 50 days of computer time to deal with many special cases.
- First major theorem to be proved using computer.

False intuition. If PLANAR-3-COLOR is hard, then so is PLANAR-4-COLOR and PLANAR-5-COLOR.

8.9 co-NP and the Asymmetry of NP

Asymmetry of NP

Asymmetry of NP. We only need to have short proofs of `yes` instances.

Ex 1. SAT vs. TAUTOLOGY.

Can prove a CNF formula is satisfiable by giving such an assignment.

How could we prove that a formula is not satisfiable?

Ex 2. HAM-CYCLE vs. NO-HAM-CYCLE.

Can prove a graph is Hamiltonian by giving such a Hamiltonian cycle.

How could we prove that a graph is not Hamiltonian?

Remark. SAT is NP-complete and SAT $\equiv_P$ TAUTOLOGY, but how do we classify TAUTOLOGY?

not even known to be in NP

NP and co-NP

NP. Decision problems for which there is a poly-time certifier.
Ex. SAT, HAM-CYCLE, COMPOSITES.

Def. Given a decision problem X, its complement $\overline{X}$ is the same problem with the `yes` and `no` answers reverse.

Ex. $\overline{X}$ = { 0, 1, 4, 6, 8, 9, 10, 12, 14, 15, … }
X = { 2, 3, 5, 7, 11, 13, 17, 23, 29, … }

co-NP. Complements of decision problems in NP.
Ex. TAUTOLOGY, NO-HAM-CYCLE, PRIMES.

NP = co-NP ?

Fundamental question. Does NP = co-NP?

- Do `yes` instances have succinct certificates iff `no` instances do?
- Consensus opinion: no.

Theorem. If NP $\neq$ co-NP, then P $\neq$ NP.

Pf idea.

- P is closed under complementation.
- If P = NP, then NP is closed under complementation.
- In other words, NP = co-NP.
- This is the contrapositive of the theorem.

Good Characterizations

Good characterization. [Edmonds 1965] NP $\cap$ co-NP.

If problem X is in both NP and co-NP, then:

- for `yes` instance, there is a succinct certificate
- for `no` instance, there is a succinct disqualifier

Provides conceptual leverage for reasoning about a problem.

Ex. Given a bipartite graph, is there a perfect matching.

If yes, can exhibit a perfect matching.

If no, can exhibit a set of nodes S such that $|N(S)| < |S|$.

Good Characterizations

Observation. $P \subseteq NP \cap$ co-NP.

Proof of max-flow min-cut theorem led to stronger result that max-flow and min-cut are in P.

Sometimes finding a good characterization seems easier than finding an efficient algorithm.

Fundamental open question. Does P = NP $\cap$ co-NP?

Mixed opinions.

Many examples where problem found to have a non-trivial good characterization, but only years later discovered to be in P.

- linear programming [Khachiyan, 1979]
- primality testing [Agrawal-Kayal-Saxena, 2002]

Fact. Factoring is in NP $\cap$ co-NP, but not known to be in P.

if poly-time algorithm for factoring, can break RSA cryptosystem

Theorem. PRIMES is in NP $\cap$ co-NP.
Pf. We already know that PRIMES is in co-NP, so it suffices to prove that PRIMES is in NP.

Pratt's Theorem. An odd integer s is prime iff there exists an integer $1 < t < s$ s.t.

$$t^{s-1} \equiv 1 \pmod{s}$$
$$t^{(s-1)/p} \not\equiv 1 \pmod{s}$$

for all prime divisors p of s-1

Input. s = 437,677
Certificate. t = 17, $2^2 \times 3 \times 36{,}473$

prime factorization of s-1
also need a recursive certificate
to assert that 3 and 36,473 are prime

Certifier.
- Check s-1 = $2 \times 2 \times 3 \times 36{,}473$.
- Check 17^{s-1} = 1 (mod s).
- Check $17^{(s-1)/2} \equiv 437{,}676$ (mod s).
- Check $17^{(s-1)/3} \equiv 329{,}415$ (mod s).
- Check $17^{(s-1)/36{,}473} \equiv 305{,}452$ (mod s).

use repeated squaring

FACTOR is in NP $\cap$ co-NP

FACTORIZE. Given an integer x, find its prime factorization.
FACTOR. Given two integers x and y, does x have a nontrivial factor less than y?

Theorem. FACTOR $\equiv_P$ FACTORIZE.

Theorem. FACTOR is in NP $\cap$ co-NP.
Pf.

- Certificate: a factor p of x that is less than y.
- Disqualifier: the prime factorization of x (where each prime factor is less than y), along with a certificate that each factor is prime.

Primality Testing and Factoring

We established: PRIMES $\leq_P$ COMPOSITES $\leq_P$ FACTOR.

Natural question: Does FACTOR $\leq_P$ PRIMES ?

Consensus opinion. No.

State-of-the-art.

- PRIMES is in P. $\leftarrow$ proved in 2001
- FACTOR not believed to be in P.

RSA cryptosystem.

- Based on dichotomy between complexity of two problems.
- To use RSA, must generate large primes efficiently.
- To break RSA, suffixes to find efficient factoring algorithm.