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Source: *Mathematics of Operations Research*, Vol. 19, No. 1 (Feb., 1994), pp. 53-67
Published by: INFORMS
Stable URL: http://www.jstor.org/stable/3690376
Accessed: 06-12-2015 02:36 UTC

MATHEMATICS OF OPERATIONS RESEARCH
Vol. 19, No. 1, February 1994
Printed in U.S.A.

AN $O(\sqrt{n}L)$-ITERATION HOMOGENEOUS AND SELF-DUAL LINEAR PROGRAMMING ALGORITHM

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We present an $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming (LP) algorithm. The algorithm possesses the following features:

• It solves the linear programming problem without any regularity assumption concerning the existence of optimal, feasible, or interior feasible solutions.

• It can start at any positive primal-dual pair, feasible or infeasible, near the central ray of the positive orthant (cone), and it does not use any big M penalty parameter or lower bound.

• Each iteration solves a system of linear equations whose dimension is almost the same as that solved in the standard (primal-dual) interior-point algorithms.

• If the LP problem has a solution, the algorithm generates a sequence that approaches feasibility and optimality simultaneously; if the problem is infeasible or unbounded, the algorithm will correctly detect infeasibility for at least one of the primal and dual problems.

1. Introduction. We consider linear programs in the following standard form:

$$\text{(LP)} \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b,\ x \geqslant 0, \end{array}$$

where $c \in R^n$, $A \in R^{m \times n}$ and $b \in R^m$ are given, $x \in R^n$, and ^{T} denotes transpose. (LP) is said to be feasible if and only if its constraints are consistent; it is called unbounded if there is a sequence $\{x^k\}$ such that x^k is feasible for all k but $c^T x^k \to -\infty$.

The dual to (LP) can be written as

$$\text{(LD)} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leqslant c, \end{array}$$

where $y \in R^m$. The components of $s = c - A^T y \in R^n$ are called dual slacks. Denote by $\mathscr{F}$ the set of all (x, y, s) such that x and (y, s) are feasible for the primal and dual, respectively. Denote by $\mathscr{F}^0$ the set of points in $\mathscr{F}$ with $(x, s) > 0$.

There are several remaining issues concerning interior-point algorithms for LP. First, almost all interior-point algorithms solve the LP problem under the regularity assumption that $\mathscr{F}^0 \neq \varnothing$ or the optimal solution set for both (LP) and (LD) is bounded. Since no prior knowledge is usually available on the status of $\mathscr{F}^0$, one has to explicitly bound the feasible region of the LP problem. This bound has to be set to 2^L in the worst case, where L is the data length of the LP problem, assuming A, b,

Received July 24, 1992; revised April 28, 1993.
AMS 1980 subject classification. Primary: 90C05.
OR/MS Index 1978 subject classification. Primary: 643 Programming/Linear/Algorithms.
Key words. Linear programming, interior-point algorithms, homogeneity, self-dual.

0364-765X/94/1901/0053/$01.25

and c are integer. Obviously, this bounding is impossible to implement in real computation. Anstreicher (1992) proposed another technique to combine the primal and dual LP problems into an artificial linear feasibility problem. The disadvantage of this combination is that the dimension of the artificial problem is doubled.

Second, most interior-point algorithms have to start at an initial strictly feasible point. The complexity of obtaining such an initial point is the same as that of solving the LP problem itself. Several approaches have been proposed to resolve this difficulty:

· Combining the primal and dual into a single linear feasibility (LF) problem (e.g., Karmarkar (1984)). Theoretically, this approach can maintain the currently "best" $O(\sqrt{n}\,L)$-iteration complexity. Practically, the disadvantage of this approach is again the doubled dimension of the system of equations which must be solved at each iteration.

· The big M method, i.e., add one or more artificial column(s) and/or row(s) and a 2^L penalty parameter to force solutions to become feasible during the algorithm (e.g., Kojima et al. (1989) and Monteiro and Adler (1989)). Theoretically, this approach maintains $O(\sqrt{n}\,L)$ complexity. The major disadvantage of this approach is the numerical problems caused by the addition of coefficients of magnitude 2^L. It also makes the algorithms slow to converge. Recently, some revised big M methods have been proposed. One is the dynamic big M method (Kojima, Mizuno, and Yoshise (1993)), i.e., M is enlarged dynamically during the course of interior-point algorithms. The complexity of the method is $O(nL)$. Another is the primal-dual "exterior" or "infeasible" algorithm (Kojima, Megiddo, and Mizuno (1993) and Lustig (1990/91), Lustig et al. (1991)). An $O(n^2L)$ complexity can be established for this approach if the LP problem possesses an optimal solution and if the initial point is set to 2^Le, where e is the vector of all ones (Zhang (1992) and Mizuno (1992)). Thus, the big M difficulty remains in these polynomial infeasible-interior-point algorithms.

· Phase I-then-Phase II method, i.e., first try to find a feasible point (and possibly one for the dual problem), and then start to look for an optimal solution if the problem is feasible and bounded. Theoretically, this approach can maintain $O(\sqrt{n}\,L)$ complexity. The major disadvantage of this approach is that the two (or three) related LP problems are solved sequentially.

· Combined Phase I-Phase II method, i.e., approach feasibility and optimality simultaneously (e.g., Anstreicher (1989), de Ghellinck and Vial (1986), and Todd (1992)). To our knowledge, the currently "best" complexity of this approach is $O(nL)$. Other disadvantages of the method include the assumption of nonempty interior and/or the use of the big M lower bound. The method also works exclusively in either the primal or the dual form.

Finally, we feel that a complete LP algorithm should accomplish two tasks: (1) affirmatively detect the infeasibility or unboundedness status of the LP problem, then (2) generate an optimal solution if the problem is neither infeasible nor unbounded.

In this paper, we present an $O(\sqrt{n}\,L)$-iteration homogeneous and self-dual LP algorithm to resolve the issues mentioned above. The algorithm possesses the following features:

· It solves the linear programming problem without any regularity assumption concerning the existence of optimal, feasible, or interior feasible solutions.

· It can start at any positive primal-dual pair, feasible or infeasible, near the central ray of the positive orthant (cone), and it does not use any big M penalty parameter or lower bound.

· Each iteration solves a system of linear equations whose dimension is almost the same as that solved in the standard (primal-dual) interior-point algorithms, but with three right-hand sides.

· If the LP problem has a solution, the algorithm generates a sequence that approaches feasibility and optimality simultaneously; if the problem is infeasible or unbounded, the algorithm will correctly detect infeasibility for at least one of the primal and dual problems.

2. A homogeneous and self-dual artificial linear program. Our algorithm is based on the construction of a homogeneous and self-dual artificial linear program related to (LP) and (LD). We now briefly explain the two major concepts, homogeneity and self-duality, used in our algorithm.

In the context of interior-point algorithms, the idea of attacking a standard-form LP by solving a related homogeneous artificial linear program can be traced to many earlier works. (By a homogeneous linear program, we do not mean that all constraints must be homogeneous, or equivalently all right-hand sides zero. We allow a single inhomogeneous constraint, often called a normalizing constraint.) Karmarkar's original canonical form (Karmarkar, 1984) is a homogeneous linear program. The transformation of the standard-form LP into this homogeneous form, via a projective transformation, has been carefully studied by many researchers (e.g., de Ghellinck and Vial (1986), Gonzaga (1989)). Recently, Anstreicher (1992) constructed a homogeneous LP and used the combined Phase I-Phase II method to solve it. His method does not require the assumption of a nonempty interior for the original LP problem. Ye (1991) analyzed some probabilistic complexity issues for interior-point algorithms via the construction of a homogeneous artificial LP problem. We must also mention the work of Nesterov and Nemirovskiy (1991), who have developed a complete theory for the "conic" or homogeneous formulation of convex optimization problems. One advantage of working in the homogeneous form is that we do not need to be concerned about the magnitude of solutions, since a solution is represented by a ray whose quality is scale-invariant. A disadvantage is that these related homogeneous problems, especially if they do not use any big M parameters, usually involve combining the primal and dual constraints and thus usually lead to algorithms requiring the solution of linear systems roughly twice as large as other methods.

Self-dual linear programs, meaning that the dual of the problem is equivalent to the primal, were introduced many years ago. We state the form of such problems, with inequality constraints, and their properties in the following result, whose proof is omitted.

PROPOSITION 1. *Let $\tilde{A} \in R^{p \times p}$ be skew-symmetric, and let $\tilde{b} = -\tilde{c} \in R^p$. Then the problem*

$$\text{(SDP)} \qquad \textit{minimize} \quad \tilde{c}^T\tilde{u}$$

$$\textit{subject to} \quad \tilde{A}\tilde{u} \geqslant \tilde{b},\ \tilde{u} \geqslant 0,$$

is equivalent to its dual. Suppose that (SDP) *has a feasible solution $\tilde{u}$. Then $\tilde{u}$ is also feasible in the dual problem, and the two objective values sum to zero. Moreover, in this case* (SDP) *has an optimal solution, and its optimal value is zero.* □

In the context of interior-point algorithms, Ye (1991) constructed an artificial homogeneous LP with an optimal solution set that is self-dual. Ye also discussed the relation between the solution of the original LP problem and that of the artificial problem, which we will use later in our analysis. His construction is based on the assumption that the LP problem has a solution. Recently, Kojima, Mizuno, and Yoshise (1993) have constructed a self-dual big M artificial problem to address the initialization issue of interior-point algorithms. The advantage of self-duality is that we can apply a primal-dual interior-point algorithm to solve the self-dual artificial problem *without* doubling the dimension of the linear system solved at each iteration.

We now present a homogeneous and self-dual (artificial) linear program (HLP) relating (LP) and (LD). Given any $x^0 > 0$, $s^0 > 0$, and y^0, we formulate

$$
\begin{array}{llllllll}
 & \text{(HLP)} & \min & & & & ((x^0)^T s^0 + 1)\theta & \\
(1) & & \text{s.t.} & & Ax & -b\tau & +\bar{b}\theta & = 0, \\
(2) & & & -A^T y & & +c\tau & -\bar{c}\theta & \geqslant 0, \\
(3) & & & b^T y & -c^T x & & +\bar{z}\theta & \geqslant 0, \\
(4) & & & -\bar{b}^T y & +\bar{c}^T x & -\bar{z}\tau & & = -(x^0)^T s^0 - 1, \\
 & & & y \text{ free}, & x \geqslant 0, & \tau \geqslant 0, & \theta \text{ free}, &
\end{array}
$$

where

$$(5) \qquad \bar{b} = b - Ax^0, \qquad \bar{c} = c - A^T y^0 - s^0, \qquad \bar{z} = c^T x^0 + 1 - b^T y^0.$$

Here $\bar{b}$, $\bar{c}$, and $\bar{z}$ represent the "infeasibility" of the initial primal point, dual point, and primal-dual "gap," respectively.

Note that (1)–(3), with $\tau = 1$ and $\theta = 0$, represent primal and dual feasibility (with $x \geqslant 0$) and reversed weak duality, so that they define primal and dual optimal solutions. Making τ a variable (homogenizing) adds the required variable dual to the constraint (3). Then, to achieve feasibility for $x = x^0$, $(y, s) = (y^0, s^0)$, we add the artificial variable θ with appropriate coefficients, and then the final constraint (4) is added to achieve self-duality.

Denote by s the slack vector for the second (inequality) constraint (2) and by κ the slack scalar for the third (inequality) constraint (3). Denote by $\mathscr{F}_h$ the set of all points $(y, x, \tau, \theta, s, \kappa)$ that are feasible for (HLP). Denote by $\mathscr{F}_h^0$ the set of strictly feasible points with $(x, \tau, s, \kappa) > 0$ in $\mathscr{F}_h$. Note that by combining the constraints, we can write the last (equality) constraint (4) as

$$(6) \qquad (s^0)^T x + (x^0)^T s + \tau + \kappa - \left((x^0)^T s^0 + 1\right)\theta = (x^0)^T s^0 + 1,$$

which serves as a normalizing constraint for (HLP). Also note that the constraints of (HLP) form a skew-symmetric system, which is basically why it is a self-dual linear program.

With regard to the selection of (x^0, y^0, s^0), note that if x^0 (respectively, (y^0, s^0)) is feasible in (LP) ((LD)), then $\bar{b}$ ($\bar{c}$) is zero, and then every feasible solution to (HLP) with $\tau > 0$ has x/τ feasible in (LP) ($(y, s)/\tau$ feasible in (LD)). Conversely, if $\bar{z} < 0$, then every feasible solution to (HLP) with $\theta > 0$ and $\tau > 0$ has $c^T x - b^T y \leqslant \bar{z}\theta < 0$, so either x/τ or $(y, s)/\tau$ must be infeasible.

Now let us denote by (HLD) the dual of (HLP). Denote by y' the dual multiplier vector for constraint (1), by x' the dual multiplier vector for constraint (2), by τ' the dual multiplier scalar for constraint (3), and by θ' the dual multiplier scalar for constraint (4). Then, we have the following result.

THEOREM 2. (*i*). (HLD) *has the same form as* (HLP), *i.e.*, (HLD) *is simply* (HLP) *with* (y, x, τ, θ) *being replaced by* (y', x', τ', θ').

(ii). (HLP) *has a strictly feasible point*

$$y = y^0, \quad x = x^0 > 0, \quad \tau = 1, \quad \theta = 1, \quad s = s^0 > 0, \quad \kappa = 1.$$

(iii). (HLP) *has an optimal solution and its optimal solution set is bounded.*

(iv). *The optimal value of* (HLP) *is zero, and for any feasible point* $(y, x, \tau, \theta, s, \kappa) \in \mathscr{F}_h$,

$$\left((x^0)^T s^0 + 1\right)\theta = x^T s + \tau\kappa.$$

(v). *There is an optimal solution* $(y^*, x^*, \tau^*, \theta^* = 0, s^*, \kappa^*) \in \mathscr{F}_h$ *such that*

$$\begin{pmatrix} x^* + s^* \\ \tau^* + \kappa^* \end{pmatrix} > 0,$$

which we call a strictly self-complementary solution. (*Similarly, we sometimes call an optimal solution to* (HLP) *a self-complementary solution*; *the strict inequalities above need not hold.*)

PROOF. In what follows, denote the slack vector and scalar in (HLD) by s' and κ', respectively. The proof of (i) is based on the skew-symmetry of the linear constraint system of (HLP). We omit the details. Result (ii) can be easily verified. Then (iii) is due to the self-dual property: (HLD) is also feasible and it has nonempty interior. The proof of (iv) can be constructed as follows. Let $(y, x, \tau, \theta, s, \kappa)$ and $(y', x', \tau', \theta', s', \kappa')$ be feasible points for (HLP) and (HLD), respectively. Then the primal-dual gap is

$$\left((x^0)^T s^0 + 1\right)(\theta + \theta') = x^T s' + s^T x' + \tau\kappa' + \kappa\tau'.$$

Let $(y', x', \tau', \theta', s', \kappa') = (y, x, \tau, \theta, s, \kappa)$, which is possible since any feasible point $(y', x', \tau', \theta', s', \kappa')$ of (HLD) is a feasible point of (HLP) and vice versa. Thus, we have (iv). Note that (HLP) and (HLD) possess a strictly complementary solution pair: the primal solution is the solution for (HLP) in which the number of positive components is maximized, and the dual solution is the solution for (HLD) in which the number of positive components is maximized. Since the supporting set of positive components of a strictly complementary solution is invariant and since (HLP) and (HLD) are identical, the strictly complementary solution $(y^*, x^*, \tau^*, \theta^* = 0, s^*, \kappa^*)$ for (HLP) is also a strictly complementary solution for (HLD) and vice versa. Thus, we establish (v). □

Henceforth, we simply choose

(7) $$y^0 = 0, \qquad x^0 = e, \quad \text{and} \quad s^0 = e.$$

(It may seem that this choice is very scale-dependent. However, note that Xs is a dimensionless vector, so that assuming $X^0 s^0 = e$ is natural. We may then choose any positive x^0, s^0 satisfying this equation, and then scale the problem to make these initial solutions both equal to e. The result is a scale-invariant problem with x and s

dimensionless.) Then, (HLP) becomes

$$
\begin{array}{llllll}
\text{(HLP)} & \min & & & & (n+1)\theta \\
& \text{s.t.} & & Ax & -b\tau & +\bar{b}\theta & = 0, \\
& & -A^T y & & +c\tau & -\bar{c}\theta & \geqslant 0, \\
& & b^T y & -c^T x & & +\bar{z}\theta & \geqslant 0, \\
& & -\bar{b}^T y & +\bar{c}^T x & -\bar{z}\tau & & = -(n+1), \\
& & y \text{ free}, & x \geqslant 0, & \tau \geqslant 0, & \theta \text{ free}.
\end{array}
$$

where

$$\bar{b} = b - Ae, \qquad \bar{c} = c - e, \quad \text{and} \quad \bar{z} = c^T e + 1. \tag{8}$$

Again, combining the constraints we can write the last (equality) constraint as

$$e^T x + e^T s + \tau + \kappa - (n+1)\theta = n + 1. \tag{9}$$

Since $\theta^* = 0$ at every optimal solution for (HLP), we can see the normalizing effect of equation (9) for (HLP).

We now relate optimal solutions for (HLP) to those for (LP) and (LD).

THEOREM 3. *Let* $(y^*, x^*, \tau^*, \theta^* = 0, s^*, \kappa^*)$ *be a strictly self-complementary solution for* (HLP).

(i). (LP) *has a solution* (*feasible and bounded*) *if and only if* $\tau^* > 0$. *In this case,* x^*/τ^* *is an optimal solution for* (LP) *and* $(y^*/\tau^*, s^*/\tau^*)$ *is an optimal solution for* (LD).

(ii). *If* $\tau^* = 0$, *then* $\kappa^* > 0$, *which implies that* $c^T x^* - b^T y^* < 0$, *i.e., at least one of* $c^T x^*$ *and* $-b^T y^*$ *is strictly less than zero. If* $c^T x^* < 0$ *then* (LD) *is infeasible; if* $-b^T y^* < 0$ *then* (LP) *is infeasible; and if both* $c^T x^* < 0$ *and* $-b^T y^* < 0$ *then both* (LP) *and* (LD) *are infeasible.*

PROOF. If (LP) and (LD) are both feasible, then they have a strictly complementary solution pair $\bar{x}$ and $(\bar{y}, \bar{s})$ for (LP) and (LD), respectively, such that

$$(\bar{x})^T \bar{s} = 0 \quad \text{and} \quad \bar{x} + \bar{s} > 0.$$

Let

$$\alpha = \frac{n+1}{e^T \bar{x} + e^T \bar{s} + 1} > 0$$

(see (9)). Then, one can verify that

$$\tilde{y}^* = \alpha \bar{y}, \quad \tilde{x}^* = \alpha \bar{x}, \quad \tilde{\tau}^* = \alpha, \quad \bar{\theta}^* = 0, \quad \tilde{s}^* = \alpha \bar{s}, \quad \tilde{\kappa}^* = 0$$

is a strictly self-complementary solution for (HLP) (Ye 1991). Since the supporting set of a strictly complementary solution for (HLP) is unique, $\tau^* > 0$ at any strictly complementary solution for (HLP).

Conversely, if $\tau^* > 0$, then $\kappa^* = 0$, which implies that

$$Ax^* = b\tau^*, \qquad A^T y^* + s^* = c\tau^*, \quad \text{and} \quad (x^*)^T s^* = 0.$$

Thus, x^*/τ^* is an optimal solution for (LP) and $(y^*/\tau^*, s^*/\tau^*)$ is an optimal solution for (LD). This concludes the proof of (i) in the theorem.

If $\tau^* = 0$, then $\kappa^* > 0$, which implies that $c^T x^* - b^T y^* < 0$, i.e., at least one of $c^T x^*$ and $-b^T y^*$ is strictly less than zero. Let us say $c^T x^* < 0$. In addition, we have

$$Ax^* = 0, \qquad A^T y^* + s^* = 0, \qquad (x^*)^T s^* = 0 \quad \text{and} \quad x^* + s^* > 0.$$

Suppose that (LD) is feasible, so that we have a solution $(\bar{y}, \bar{s} \geqslant 0)$ such that $A^T \bar{y} + \bar{s} = c$. Multiplying this equation by x^*, we have

$$0 \leqslant \bar{s}^T x^* = c^T x^* < 0$$

which is a contradiction. Thus, (LD) must be infeasible. The other results in (ii) hold similarly. □

From the proof of the theorem, we deduce the following

COROLLARY 4. *Let* $(\bar{y}, \bar{x}, \bar{\tau}, \bar{\theta} = 0, \bar{s}, \bar{\kappa})$ *be any optimal solution for* (HLP). *Then if* $\bar{\kappa} > 0$, *either* (LP) *or* (LD) *is infeasible.* □

3. Interior-point algorithms for solving (HLP). The following theorem resembles the central path analyzed for (LP) and (LD) (Bayer and Lagarias (1989), Megiddo (1988), Renegar (1988), and Sonnevend (1985)).

THEOREM 5. (i) *For any* $\mu > 0$, *there is a unique* $(y, x, \tau, \theta, s, \kappa)$ *in* $\mathcal{F}_h^0$ *such that*

$$\begin{pmatrix} Xs \\ \tau\kappa \end{pmatrix} = \mu e, \tag{10}$$

where $X = \operatorname{diag}(x)$ *denotes the diagonal matrix with diagonal entries equal to the components of* x.

(ii) *Let* $(d_y, d_x, d_\tau, d_\theta, d_s, d_\kappa)$ *be in the null space* Q *of the constraint matrix of* (HLP) *after adding surplus variables* s *and* κ, *i.e.*,

$$\begin{array}{ccccccc}
 & Ad_x & -bd_\tau & +\bar{b}d_\theta & & & = 0, \\
-A^T d_y & & +cd_\tau & -\bar{c}d_\theta & -d_s & & = 0, \\
b^T d_y & -c^T d_x & & +\bar{z}d_\theta & & -d_\kappa & = 0, \\
-\bar{b}^T d_y & +\bar{c}^T d_x & -\bar{z}d_\tau & & & & = 0.
\end{array}$$

Then

$$(d_x)^T d_s + d_\tau d_\kappa = 0.$$

PROOF. For any $\mu > 0$, there is (see Megiddo (1988), for instance) a unique feasible point $(y, x, \tau, \theta, s, \kappa)$ for (HLP) and a unique feasible point $(y', x', \tau', \theta', s', \kappa')$ for (HLD) such that

$$Xs' = \mu e, \qquad Sx' = \mu e, \qquad \tau\kappa' = \mu, \qquad \kappa\tau' = \mu.$$

However, if we switch the positions of $(y, x, \tau, \theta, s, \kappa)$ and $(y', x', \tau', \theta', s', \kappa')$ we satisfy the same equations. Thus, we must have

$$(y', x', \tau', \theta', s', \kappa') = (y, x, \tau, \theta, s, \kappa),$$

since (HLP) and (HLD) have the identical form. This concludes the proof of (i) in the theorem.

The proof of (ii) is simply due to the skew-symmetry of the constraint matrix. Multiply the first set of equations by d_y^T, the second set by d_x^T, the third equation by d_τ and the last by d_θ and add. This leads to the desired result. □

We see that Theorem 5 defines a path in (HLP):

$$\mathscr{C} = \left\{ (y, x, \tau, \theta, s, \kappa) \in \mathscr{F}_h^0 : \begin{pmatrix} Xs \\ \tau\kappa \end{pmatrix} = \frac{x^T s + \tau\kappa}{n+1} e, \right\}$$

which we may call the (self-)central path for (HLP). Obviously, if $X^0 s^0 = e$, then the initial interior feasible point proposed in Theorem 2 is on the path with $\mu = 1$. Our choice (7) for x^0 and s^0 satisfies this requirement. We can define a neighborhood of the path as

$$\mathscr{N}(\beta) = \left\{ (y, x, \tau, \theta, s, \kappa) \in \mathscr{F}_h^0 : \left\| \begin{pmatrix} Xs \\ \tau\kappa \end{pmatrix} - \mu e \right\| \leqslant \beta\mu \text{ where } \mu = \frac{x^T s + \tau\kappa}{n+1} \right\}$$

for some $\beta \in (0, 1)$. Here $\|\cdot\|$ without subscript designates the l_2-norm. Note that from statement (iv) of Theorem 2 we have $\theta = \mu$ for any feasible point in $\mathscr{F}_h$.

We now apply the Mizuno-Todd-Ye predictor-corrector algorithm (Mizuno, Todd and Ye 1990) to solving (HLP).

Predictor-Corrector Algorithm. Given an interior feasible point $(y^k, x^k, \tau^k, \theta^k, s^k, \kappa^k) \in \mathscr{F}_h^0$, consider solving the following system of linear equations for $(d_y, d_x, d_\tau, d_\theta, d_s, d_\kappa)$:

$$(d_y, d_x, d_\tau, d_\theta, d_s, d_\kappa) \in Q, \tag{11}$$

$$\begin{pmatrix} X^k d_s + S^k d_x \\ \tau^k d_\kappa + \kappa^k d_\tau \end{pmatrix} = \gamma\mu^k e - \begin{pmatrix} X^k s^k \\ \tau^k \kappa^k \end{pmatrix}. \tag{12}$$

(Recall that Q denotes the null space of the coefficient matrix in (HLP)—see Theorem 5.)

Predictor step. For any even number k, we have $(y^k, x^k, \tau^k, \theta^k, s^k, \kappa^k) \in \mathscr{N}(\beta)$ with $\beta = 1/4$. We solve the linear system (11, 12) with $\gamma = 0$. Then let

$$y(\alpha) := y^k + \alpha d_y, \qquad x(\alpha) := x^k + \alpha d_x,$$

$$\tau(\alpha) := \tau^k + \alpha d_\tau, \qquad \theta(\alpha) := \theta^k + \alpha d_\theta,$$

$$s(\alpha) := s^k + \alpha d_s, \qquad \kappa(\alpha) := \kappa^k + \alpha d_\kappa.$$

We determine the step size using

$$\bar{\alpha} := \max\{\alpha : (y(\alpha), x(\alpha), \tau(\alpha), \theta(\alpha), s(\alpha), \kappa(\alpha)) \in \mathscr{N}(2\beta)\}. \tag{13}$$

Then compute the next points by $y^{k+1} = y(\bar{\alpha})$, $x^{k+1} = x(\bar{\alpha})$, $\tau^{k+1} = \tau(\bar{\alpha})$, $\theta^{k+1} = \theta(\bar{\alpha})$, $s^{k+1} = s(\bar{\alpha})$, and $\kappa^{k+1} = \kappa(\bar{\alpha})$.

Corrector step. For any odd number k, we solve the linear system (11, 12) with $\gamma = 1$. Then let $y^{k+1} = y^k + d_y$, $x^{k+1} = x^k + d_x$, $\tau^{k+1} = \tau^k + d_\tau$, $\theta^{k+1} = \theta^k + d_\theta$, $s^{k+1} = s^k + d_s$, and $\kappa^{k+1} = \kappa^k + d_\kappa$. We have

$$\left(y^{k+1}, x^{k+1}, \tau^{k+1}, \theta^{k+1}, s^{k+1}, \kappa^{k+1}\right) \in \mathcal{N}(\beta).$$

Termination. We use the termination technique described in Ye (1992) and Mehrotra and Ye (1991) to terminate the predictor-corrector algorithm. Define σ^k be the index set $\{j\colon x_j^k \geqslant s_j^k,\ j = 1, 2, \dots, n\}$, and denote by B those columns in A corresponding to σ^k and by N the rest of the columns in A. (Again, it may seem that this choice of σ^k is very scale-dependent; but recall the comments below (7).) Then, we use a least-squares projection to create an optimal solution that is strictly self-complementary for (HLP).

Case 1. If $\tau^k \geqslant \kappa^k$, we solve for y, x_B, and τ from

$$\begin{aligned} \min \quad & \|y^k - y\|^2 + \|x_B^k - x_B\|^2 + (\tau^k - \tau)^2 \\ \text{s.t} \quad & Bx_B - b\tau = 0, \\ & -B^T y + c_B \tau = 0, \\ & b^T y - c_B^T x_B = 0; \end{aligned}$$

otherwise,

Case 2. $\tau^k < \kappa^k$, and we solve for y, x_B, and κ from

$$\begin{aligned} \min \quad & \|y^k - y\|^2 + \|x_B^k - x_B\|^2 + (\kappa^k - \kappa)^2 \\ \text{s.t.} \quad & Bx_B = 0, \\ & -B^T y = 0, \\ & b^T y - c_B^T x_B - \kappa = 0. \end{aligned}$$

This projection guarantees that the resulting x_B^* and s_N^* ($s_N^* = c_N \tau^* - N^T y^*$ in Case 1 or $s_N^* = -N^T y^*$ in Case 2) are positive, and τ^* is positive in Case 1 and κ^* is positive in Case 2, as long as $(x^k)^T s^k + \tau^k \tau^k$ is small enough in the predictor-corrector algorithm.

THEOREM 6. *Let* (LP) *and* (LD) *have integer data with bit length L. Then by the construction of* (HLP), *the data of* (HLP) *remains integral and its length is $O(L)$. Therefore, the predictor-corrector algorithm, coupled with the termination technique described above, generates a strictly self-complementary solution for* (HLP) *in $O(\sqrt{n}\, L)$ iterations.*

PROOF. Let $\hat{x}, \hat{s}, \hat{d}_x$, and $\hat{d}_s$ be $\hat{n}$-dimensional vectors satisfying

$$(14) \qquad \hat{x} > 0, \qquad \hat{s} > 0, \qquad \hat{X}\hat{d}_s + \hat{S}\hat{d}_x = -\left(\hat{X}\hat{s} - \gamma\hat{\mu}e\right), \qquad \hat{d}_x^T \hat{d}_s = 0,$$

where $\hat{X} = \operatorname{diag}(\hat{x})$, $\hat{S} = \operatorname{diag}(\hat{s})$, $\gamma \in [0, 1]$, and $\hat{\mu} = \hat{x}^T \hat{s}/\hat{n}$. We define $\hat{x}(\alpha) = \hat{x} + \alpha\hat{d}_x$, $\hat{s}(\alpha) = \hat{s} + \alpha\hat{d}_s$, and $\hat{\mu}(\alpha) = \hat{x}(\alpha)^T \hat{s}(\alpha)/\hat{n}$. By using the analysis employed in

Mizuno, Todd and Ye (1990), we can prove that:

• if $\|\hat{X}\hat{s} - \hat{\mu}e\| \leqslant (1/4)\hat{\mu}$ and $\gamma = 0$ then $\hat{x}(\alpha) > 0$, $\hat{s}(\alpha) > 0$, $\|\hat{X}(\alpha)\hat{s}(\alpha) - \hat{\mu}(\alpha)e\| \leqslant (1/2)\hat{\mu}(\alpha)$, and $\hat{\mu}(\alpha) = (1-\alpha)\hat{\mu}$ for $\alpha \in [0, 8^{-.25}/\sqrt{\hat{n}}]$;

• if $\|\hat{X}\hat{s} - \hat{\mu}e\| \leqslant (1/2)\hat{\mu}$ and $\gamma = 1$ then $\hat{x}(1) > 0$, $\hat{s}(1) > 0$, $\|\hat{X}(1)\hat{s}(1) - \hat{\mu}(1)e\| \leqslant (1/4)\hat{\mu}(1)$, and $\hat{\mu}(1) = \hat{\mu}$.

We have (14) for $\hat{x} = (x^k, \tau^k)$, $\hat{s} = (s^k, \kappa^k)$, $\hat{d}_x = (d_x, d_\tau)$, and $\hat{d}_s = (d_s, d_\kappa)$ with $\hat{n} = n + 1$. By using the results above and (11), we have that in the predictor step $(y^k, x^k, \tau^k, \theta^k, s^k, \kappa^k) \in \mathcal{N}(1/4)$, $(y^{k+1}, x^{k+1}, \tau^{k+1}, \theta^{k+1}, s^{k+1}, \kappa^{k+1}) \in \mathcal{N}(1/2)$, and

$$\frac{\theta^{k+1}}{\theta^k} = \frac{(x^{k+1})^T s^{k+1} + \tau^{k+1}\kappa^{k+1}}{(x^k)^T s^k + \tau^k \kappa^k} \leqslant \left(1 - 8^{-.25}/\sqrt{n+1}\right),$$

and in the corrector step $(y^k, x^k, \tau^k, \theta^k, s^k, \kappa^k) \in \mathcal{N}(1/2)$, $(y^{k+1}, x^{k+1}, \tau^{k+1}, \theta^{k+1}, s^{k+1}, \kappa^{k+1}) \in \mathcal{N}(1/4)$, and

$$\frac{\theta^{k+1}}{\theta^k} = \frac{(x^{k+1})^T s^{k+1} + \tau^{k+1}\kappa^{k+1}}{(x^k)^T s^k + \tau^k \kappa^k} = 1.$$

Also note that the initial gap is $(x^0)^T s^0 + \tau^0 \kappa^0 = n + 1$. Using the results of Mehrotra and Ye (1991), and Ye (1993), we will generate an (exact) strictly self-complementary solution for (HLP) before $(x^k)^T s^k + \tau^k \kappa^k \leqslant 2^{-O(L)}$ while $(y^k, x^k, \tau^k, \theta^k, s^k, \kappa^k) \in \mathcal{N}(2\beta)$. Thus, the total number of iterations is at most $O(\sqrt{n}L)$. □

Theorem 6 shows that our algorithm generates a strictly self-complementary solution to (HLP) in $O(\sqrt{n}L)$ iterations. Then using Theorem 3 we obtain

COROLLARY 7. *Within $O(\sqrt{n}L)$ iterations, the predictor-corrector algorithm, coupled with the termination technique described above, generates either optimal solutions to* (LP) *and* (LD) *or an indication that* (LP) *or* (LD) *is infeasible.* □

Note that the algorithm may not detect the infeasibility status of both (LP) and (LD). Consider the example where

$$A = (-1 \quad 0 \quad 0), \qquad b = 1, \quad \text{and} \quad c = (0 \quad 1 \quad -1).$$

Then,

$$y^* = 2, \quad x^* = (0,2,1)^T, \quad \tau^* = 0, \quad \theta^* = 0, \quad s^* = (2,0,0)^T, \quad \kappa^* = 1$$

could be a strictly self-complementary solution generated for (HLP) with

$$c^T x^* = 1 > 0, \qquad b^T y^* = 2 > 0.$$

Thus (y^*, s^*) demonstrates infeasibility of (LP), but x^* doesn't show infeasibility of (LD). Of course, if the algorithm generates instead $x^* = (0,1,2)^T$, then we get demonstrated infeasibility of both.

4. Analysis of the output. In practice, we may wish to stop the algorithm at an approximate solution. Thus, we wish to analyze the asymptotic behavior of τ^k vs. θ^k.

THEOREM 8. *If* (LP) *possesses an optimal solution, then*

$$\tau^k \geqslant \frac{1-2\beta}{(e^T\bar{x}+e^T\bar{s}+1)} \quad \text{for all } k,$$

where $\bar{x}$ *and* $(\bar{y}, \bar{s})$ *are any optimal solution pair for* (LP) *and* (LD); *otherwise,*

$$\kappa^k \geqslant \frac{(1-2\beta)\epsilon}{n+1}$$

and

$$\frac{1-2\beta}{2(n+1)} \leqslant \frac{1-2\beta}{\kappa^k} \leqslant \frac{\tau^k}{\theta^k} \leqslant \frac{1+2\beta}{\kappa^k} \leqslant \frac{(n+1)(1+2\beta)}{(1-2\beta)\epsilon} \quad \text{for all } k,$$

where ϵ *is a fixed positive number independent of* k.

PROOF. Note that the sequence generated by the predictor-corrector algorithm is in $\mathcal{N}(2\beta)$. We follow the proof technique of Güler and Ye (1993). Note that

$$y^* = \alpha\bar{y}, \quad x^* = \alpha\bar{x}, \quad \tau^* = \alpha, \quad \theta^* = 0, \quad s^* = \alpha\bar{s}, \quad \kappa^* = 0,$$

where

$$\alpha = \frac{n+1}{e^T\bar{x}+e^T\bar{s}+1} > 0,$$

is a self-complementary solution for (HLP). Now we use

$$(x^k - x^*)^T(s^k - s^*) + (\tau^k - \tau^*)(\kappa^k - \kappa^*) = 0,$$

which follows by subtracting the constraints of (HLP) for $(y^*, \dots, \kappa^*)$ from those for $(y^k, \dots, \kappa^k)$ and then multiplying by $((y^k - y^*)^T, \dots, \kappa^k - \kappa^*)$. This can be rewritten as

$$(x^k)^T s^* + (s^k)^T x^* + \kappa^k \tau^* = (n+1)\mu^k = (n+1)\theta^k.$$

Thus,

$$\tau^k \geqslant \frac{\tau^k\kappa^k}{(n+1)\mu^k}\tau^* \geqslant \frac{1-2\beta}{n+1}\tau^* = \frac{1-2\beta}{(e^T\bar{x}+e^T\bar{s}+1)}.$$

The second statement follows from a similar argument. We know that there is an optimal solution for (HLP) with $\kappa^* \geqslant \epsilon > 0$. Thus $\kappa^k \geqslant (1-2\beta)\epsilon/(n+1)$ for all k. In addition, from relation (9) we have $\kappa^k \leqslant (n+1) + (n+1)\theta^k \leqslant 2(n+1)$ for all k. □

Theorem 8 indicates that either τ^k stays bounded away from zero for all k, which implies that (LP) has an optimal solution, or τ^k and θ^k converge to zero at the same rate, which implies that (LP) does not have an optimal solution. We have the following bound for ϵ in Theorem 8 if the original (LP) is either infeasible or unbounded: let us say there is $x^* \geqslant 0$ such that $Ax^* = 0$ and $c^Tx^* < 0$. Denote by $z^*(> 0)$ the normalized value $-c^Tx^*/e^Tx^*$. Then we have an optimal solution for

(HLP) as follows:

$$y = 0, \quad x = \frac{(n+1)x^*}{(1+z^*)e^Tx^*}, \quad \tau = 0, \quad \theta = 0, \quad s = 0, \quad \kappa = \frac{(n+1)z^*}{1+z^*}.$$

Thus, ϵ in Theorem 8 is bounded below by $(n+1)z^*/(1+z^*)$.

In practice, for example, we can adopt the following two convergence criteria:

- $(x^k/\tau^k)^T(s^k/\tau^k) \leqslant \epsilon_1$, and $(\theta^k/\tau^k)\|(\bar{b}, \bar{c})\| \leqslant \epsilon_2$;
- $\tau^k \leqslant \epsilon_3$.

Here ϵ_1, ϵ_2, and ϵ_3 are small positive constants. Since both $(x^k)^Ts^k + \tau^k\kappa^k$ and θ^k decrease at least by the ratio $(1 - 8^{-.25}/\sqrt{n+1})$ in every two iterations, one of the above convergence criteria holds in $O(\sqrt{n}\,t)$ iterations for $t = \max\{\ln((x^0)^Ts^0/(\epsilon_1\epsilon_3^2)), \ln(\|\bar{b}, \bar{c}\|/(\epsilon_2\epsilon_3))\}$. If the algorithm terminates by the first criterion then we get approximate optimal solutions of (LP) and (LD); otherwise we detect that (LP) and (LD) have no optimal solutions such that $\|(\bar{x}, \bar{s})\|_1 \leqslant (1-2\beta)/\epsilon_3 - 1$ from Theorem 8.

The above theorems use the fact that the predictor-corrector algorithm generates a strictly self-complementary solution. If somehow we terminate the algorithm and obtain an optimal solution for (HLP) that is not necessarily strictly self-complementary, what can we say about the solution to the original LP problem?

THEOREM 9. *Suppose the following regularity assumption holds for the original* LP *problem*: *both* (LP) *and* (LD) *are feasible and have nonempty interior*. *Then* $\tau^* > 0$ *at any self-complementary solution* $(y^*, x^*, \tau^*, \theta^* = 0, s^*, \kappa^*)$ *for* (HLP).

PROOF. Note that by Corollary 4 and the assumption $\kappa^* = 0$ at any optimal solution for (HLP). Suppose to the contrary that $\tau^* = 0$. Then we have

$$Ax^* = 0, \quad A^Ty^* + s^* = 0, \quad c^Tx^* - b^Ty^* = 0 \quad \text{and} \quad e^Tx^* + e^Ts^* = n + 1.$$

Let $\bar{x}$ and $(\bar{y}, \bar{s})$ be an optimal solution pair for (LP) and (LD), respectively, so that

$$c^Tx^* = \bar{s}^Tx^* \geqslant 0 \quad \text{and} \quad -b^Ty^* = \bar{x}^Ts^* \geqslant 0.$$

Thus, we have

$$c^Tx^* = b^Ty^* = 0.$$

Since $e^Tx^* + e^Ts^* = n + 1$, x^* and s^* cannot both be zero. Let us say $x^* \neq 0$. Then, $\bar{x} + \alpha x^*$ is an optimal solution of (LP) for any $\alpha \geqslant 0$. This indicates that the optimal solution set of (LP) is unbounded or (LD) has empty interior, which is a contradiction. □

5. Further discussion. Since the (HLP) model constructed and analyzed in §2 does not rely on any particular algorithm for solving it, we may use many other interior-point algorithms, as long as they generate a strictly complementary solution (Güler and Ye (1993) show that most do). For example, as described in Mizuno et al. (1993), we may use wider neighborhoods to control the movement of the iteration sequence, which allow larger step sizes to be taken.

We can also adopt Karmarkar's (primal) potential function for (HLP) as

$$(15) \qquad \phi(y, x, \tau, \theta, s, \kappa) = \rho \log(x^Ts + \tau\kappa) - \sum_{j=1}^{n} \log(x_js_j) - \log(\tau\kappa).$$

Although this potential function is defined for (HLP) only, it resembles the Tanabe-Todd-Ye primal-dual potential function. We may use a potential-reduction algorithm with $\rho = (n+1) + \nu\sqrt{n+1}$, $1 \leqslant \nu = O(1)$, and choose $\gamma = (n+1)/\rho$ in system (11, 12) (e.g., Kojima, Mizuno and Yoshise (1991)). Then, the potential function ϕ will be reduced by a constant at any iteration, by using the analysis employed in Kojima, Mizuno and Yoshise (1991). Note that the Tanabe-Todd-Ye primal-dual potential function for (HLP) and (HLD) is

$$\psi(y, x, \tau, \theta, s, \kappa) = \rho' \log 2(x^T s + \tau\kappa) - 2\sum_{j=1}^{n} \log(x_j s_j) - 2\log(\tau\kappa)$$

when $(y', x', \tau', \theta', s', \kappa') = (y, x, \tau, \theta, s, \kappa)$. If $\rho = n + 1 + \sqrt{n+1}$ and $\rho' = (2n+2) + \sqrt{2(2n+2)}$, we see $\psi = 2\phi + \rho' \log 2$.

Observe that the direction generated by our algorithm satisfies

$$Ad_x = bd_\tau - (b - Ax^0)d_\theta$$

and

$$A^T d_y + d_s = cd_\tau - (c - A^T y^0 - s^0)d_\theta.$$

This is in contrast to the "exterior" algorithm of Kojima, Megiddo and Mizuno (1993), where d_τ is set to zero in all iterations and Ax^k and $A^T y^k + s^k$ each follow a line towards feasibility.

From the implementation point of view, each iteration of our algorithm solves the linear system (11, 12). By eliminating d_s and d_κ, we face the KKT system of linear equations:

$$(16) \qquad \begin{pmatrix} X^k S^k & -X^k A^T & X^k c \tau^k & -X^k \bar{c} \\ AX^k & 0 & -\tau^k b & \bar{b} \\ -\tau^k c^T X^k & \tau^k b^T & \tau^k \kappa^k & \tau^k \bar{z} \\ \bar{c}^T X^k & -\bar{b}^T & -\tau^k \bar{z} & 0 \end{pmatrix} \begin{pmatrix} (X^k)^{-1} d_x \\ d_y \\ (\tau^k)^{-1} d_\tau \\ d_\theta \end{pmatrix}$$

$$= \begin{pmatrix} \gamma\mu^k e - X^k s^k \\ 0 \\ \gamma\mu^k - \tau^k \kappa^k \\ 0 \end{pmatrix}.$$

One can also decompose the system by first expressing it as

(17)

$$\begin{pmatrix} X^k S^k & -X^k A^T \\ AX^k & 0 \end{pmatrix} \begin{pmatrix} (X^k)^{-1} d_x \\ d_y \end{pmatrix} = \begin{pmatrix} \gamma\mu^k e - X^k s^k \\ 0 \end{pmatrix} - \begin{pmatrix} X^k c & -X^k \bar{c} \\ -b & \bar{b} \end{pmatrix} \begin{pmatrix} d_\tau \\ d_\theta \end{pmatrix}.$$

We solve this system against the three right-hand-side vectors in (17), to obtain d_x and d_y as functions of d_τ and d_θ. Then, we use the other two equations in (16) to determine d_τ and d_θ. Finally, we determine d_x and d_y by substituting in these values. Thus, we need to solve a system of linear equations whose size is the same as that solved by standard interior-point algorithms, but with three different right-hand sides, calculate a few inner products, and then solve a 2×2 system.

Finally, we construct (HLP) for (LP) and (LD) in symmetric form. Consider the linear program pair:

$$\text{(LP)} \qquad \text{minimize} \quad c^T x$$
$$\text{subject to} \quad Ax \geqslant b,\ x \geqslant 0,$$

and

$$\text{(LD)} \qquad \text{maximize} \quad b^T y$$
$$\text{subject to} \quad A^T y \leqslant c,\ y \geqslant 0.$$

Let $t = Ax - b$ and $s = c - A^T y$. Given any $x^0 > 0$, $t^0 > 0$, $s^0 > 0$, and $y^0 > 0$, we can write the homogeneous and self-dual (HLP) as

$$\begin{array}{llllll}
\text{(HLP) min} & & & & ((x^0)^T s^0 + (t^0)^T y^0 + 1)\theta & \\
\text{s.t.} & & Ax & -b\tau & +\bar{b}\theta & \geqslant 0, \\
& -A^T y & & +c\tau & -\bar{c}\theta & \geqslant 0, \\
& b^T y & -c^T x & & +\bar{z}\theta & \geqslant 0, \\
& -\bar{b}^T y & +\bar{c}^T x & -\bar{z}\tau & & = -(x^0)^T s^0 - (t^0)^T y^0 - 1, \\
& y \geqslant 0, & x \geqslant 0, & \tau \geqslant 0, & \theta \text{ free}, &
\end{array}$$

where

$$\bar{b} = b - Ax^0 + t^0, \quad \bar{c} = c - A^T y^0 - s^0, \quad \text{and} \quad \bar{z} = c^T x^0 + 1 - b^T y^0.$$

This is another skew-symmetric linear programming problem.

Acknowledgement. We thank the referees for making many helpful suggestions to improve the presentation. Research of the first author was supported in part by NSF Grant DDM-9207347, an Iowa Business School Summer Grant, and K. C. Wong Educational Foundation, Hong Kong, through Academia Sinica, Beijing, China. Research of the second author was supported in part by NSF, AFOSR and ONR through NSF Grant DMS-8920550.

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