Wealth, Returns and Taxation: A Tale of two Dependencies*

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Abstract

Recent research shows the important role of persistent heterogeneous rate of returns in explaining the high concentration of wealth. This paper studies the macroeconomic consequences of wealth taxation of wealth-rich households in an economy in which wealth portfolio and returns are heterogenous and correlate with wealth. We distinguish two forces behind this heterogeneity. First, agents with high investment skills tend to select over time at the top of the wealth distribution, i.e. *type* dependence. Second, portfolio and associated returns may be wealth-driven, i.e. *scale* dependence. We find that both mechanisms generate realistic wealth inequality, but the degree of *type* and *scale* dependence together with the extend to which wealth returns reflect the productivity of capital investments are key to characterize the optimal wealth redistribution. The macroeconomic responses are substantially amplified under *scale*-dependence, as investments are a function of wealth. Under *type* dependence, the selection of skill-types is endogenous to the wealth tax. We find that a top marginal wealth tax rates for top 1% wealthiest households of 4.2% is optimal as long as the model display *type* and *wealth* dependencies consistent with the US data.

Keywords: Wealth Inequality, Type and Scale Dependence, Wealth Taxation.

JEL codes: E22, E61, O41.

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"The rich are different from you and me." F. Scott Fitzgerald

"Yes, they have more money." Ernest Hemingway

1 Introduction

In the last 40 years, wealth concentration have increased substantially in the U.S. at the top end of distribution. For example, Saez and Zucman (2016) reports that the top 0.1% wealth share has risen from 7% in 1978 to 22% in 2012. Recent empirical evidence (Piketty et al., 2014; Smith et al., 2019a; Kuhn et al., 2020) and theoretical work (Benhabib et al., 2011, 2019) point to the important role of heterogeneous households rate of returns on wealth along the wealth distribution in explaining this high concentration, as well as its dynamics (Hubmer et al., 2020). At the same time, there have been popular and academic calls to tax wealth at the top of the wealth distribution, with the objective of mitigating wealth inequality (Piketty et al., 2014; Saez and Zucman, 2019).

To analyse the implications of top wealth inequality on macroeconomic aggregates, the literature have undergone substantial refinements over the past years to endogenously generate a realistic wealth inequality, using two common features in the spirit of Gabaix et al. (2016): *type-dependence* and *wealth-dependence*.[1] The former captures the fact that wealth rich individuals are rich because they differ in their abilities and skills relative to the wealth poor. This may condition, among other things, their decision to select into entrepreneurship, sort into jobs, save or allocate asset holdings. The latter captures the fact that wealth itself may drive the intensive and extensive margins of the aforementioned decisions. While recent empirical evidence show that both dependencies may lead to a strong correlation between wealth returns and wealth (Piketty et al., 2014; Smith et al., 2019a; Fagereng et al., 2020; Bach et al., 2020), their distinction has received little attention in the literature so far. However, understanding how do households becomes rich, and how do rich households becomes even richer, is essential for understanding the macroeconomic consequences of wealth redistribution policies on economic activity, growth and welfare.

In this paper, we ask what are the macroeconomic and welfare implications of taxing the top rich households in an economy featuring a realistic level of wealth inequality due to *type* and/or *scale* dependence in wealth portfolio and returns. Our first contribution is to isolate and clarify the key parameters that characterize these implications in an analytical two-period economy. Our second contribution is to quantitatively evaluate the role of *type* and *scale* dependence in determining the effects of a top marginal wealth tax on the top 1% wealthiest households and its optimal level. For this, we extend the standard incomplete-markets model with heterogeneous agents framework (Bewley, 1986; Huggett, 1993; Aiyagari, 1994) with ex-ante investment skills,

[1] We use *scale* and *wealth* dependence interchangeably throughout the paper.

wealth-dependent portfolio and returns, and endogenous labor supply and savings choices.

This paper makes two main contributions. explicitly incorporates *type* and *scale* dependence into household capital investments and introduces a theoretical decomposition of their joint effects. An important result that emerges is that various combinations of *type* and *scale* are consistent with key macroeconomic moments, but the relative importance of each component implies distinct output responses to a wealth inequality shock, conditioning optimal wealth redistribution. Quantitatively, we unravel and clarify new findings regarding wealth redistribution. Notably, the degree of *type*/*scale* dependence behind heterogeneous capital investments and their associated returns, together with the degree to which differential wealth returns reflect the difference in productivity of capital investments, are shown to determine the way wealth should be taxed. Finally, using US microdata, we assess the contributing role of both dependencies for wealth inequality, the output response to wealth redistribution, and derive the US welfare-maximizing wealth tax rate.

Our paper is motivated by a growing body of recent empirical evidence documenting the large heterogeneity in wealth returns (Piketty (2018), Xavier (2020)), and its key role in generating theoretically and quantitatively the thick right tail of the wealth distribution (Benhabib et al. (2011, 2019), Hubmer et al. (2020)). In particular, Bach et al. (2020) and Fagereng et al. (2020) use administrative panel data from Sweden and Norway and find that *type* and *scale* dependence account for the large heterogeneity in individual wealth returns. These pieces of evidence are arguably related to the fact that wealth-rich and wealth-poor individuals substantially differ in their portfolio allocation (Calvet et al. (2019), Smith et al. (2019b), Meeuwis (2019)).[2] In the US, the 1% wealthiest households owned around 36% of the total wealth over the last twenty years (Saez and Zucman, 2016) and, according to estimates from the Survey of Consumer Finances (SCF), an average share of 65% of their wealth was held in private and public equity investments.

In a tractable two-period model, we first derive a general decomposition of the link between the level of wealth inequality, aggregate investment and output based on *type* and *scale* dependence. Our setup introduces heterogeneity in portfolio choices due to both components. Additionally, returns may differ across asset classes because of different marginal products of capital (MPK) or a return wedge arising, for example, from some form of rent-extraction due to bargaining, market power, or political connections (Piketty et al. (2014); Rothschild and Scheuer (2016) among others). Our model first builds on the incomplete markets growth economy of Angele-

[2] Other related work are worth mentioning. Using Norwegian administrative data, Halvorsen et al (2021) show that private and public equity investments are responsible for the high return heterogeneity at the upper end of the wealth distribution. Fagereng et al. (2019) show the importance of capital gains and heterogeneity in portfolio composition for gross savings, and Kuhn et al. (2020) document the importance of business equity at the very top.

tos and Calvet (2005, 2006) in which households allocate their wealth into a risk-free technology and into a riskier technology subject to idiosyncratic shocks with high expected return premium. We incorporate wealth and type dependent risk-taking into their framework. To achieve this, we introduce *generalized* CARA preferences, that let us capture non-linear scale dependence on household investment decisions while keeping the model tractable. Those preferences introduce innate heterogeneity in risk tolerance, or abilities, that aim to reflect capitalist-entrepreneur skills or investment preferences, along the lines of Moll (2014). High types, in our setup, are willing to undergo higher capital income risk irrespectively of their wealth level. As a result, the whole economy is populated by heterogeneous households with different degrees of marginal propensity to take risks. Because the supply of risky and riskless projects is perfectly elastic, aggregate productivity (TFP) is determined in equilibrium by aggregating efficiency units of household capital investments. Therefore, effects of wealth redistribution on aggregates depend, in our setting, on the joint distribution of innate risk-taking types and wealth holdings.

In this environment, the inequality-efficiency relation depends on four statistics, (i) the shape of the wealth distribution, (ii) the risk premium augmented by the degree to which differential returns to capital reflect differences in the productivity of capital investments, (iii) the elasticity of risk-taking with respect to wealth, and (iv) a dependence parameter that captures the selection or sorting of individuals with different investment types across the wealth distribution. We find that while both dependencies can consistently match the distribution of capital allocation and wealth return in the static model, they generate substantially different output responses. More generally, the relative strength of *type* versus *scale* dependence, and their interaction, determines the inequality-efficiency relation of the economy: for a given initial level of wealth inequality, the effects of a marginal variation in wealth inequality on growth depends on the space governing *type* and *scale* dependence. Inequality variations are growth neutral only when both dependencies are inexistent or exactly offset each other. In such a case, the economy is said to be on the *Growth Irrelevance Frontier*, a special *iso-growth* curve that separates the growth-hampering and the growth-enhancing regions.

Using our analytical framework, we then characterize the welfare-maximizing wealth redistribution when household preferences are aggregated by a social welfare function. The key trade-offs are as follows. First, on the efficiency side, wealth redistribution affects the allocation of capital investment and thus aggregate productivity which trickles down to equilibrium wage. Second, whenever returns to capital do not entirely reflect their marginal product due to some forms of rent-extraction, equilibrium returns to wealth adjust to ensure that the total capital income distributed to households equalizes the total product of capital generated on the supply side. Third, there is a standard equity motive that arises as households differ in their wealth holdings and thus in their marginal utility of consumption. Importantly, the first two effects go in opposite directions and their magnitude depends on the degree of *type* and *scale* dependence.

Our analytical insights generalize to heterogeneous saving decisions, and not only to capital allocation, that interact with uninsurable idiosyncratic labor productivity shocks. Finally, while we focus on risk-taking, our insights can be reframed with other *type* and *scale* mechanisms.

For illustrative purposes, the above static analytical representation assumed an exogenous joint distribution of wealth and risk-taking types. In the second part of the paper, our objective is to explore the implications of *type* and *scale* dependence for aggregates in a richer quantitative dynamic environment in which the joint distribution arises endogenously as households differ in their capital investment decisions and their sequence of labor productivity. The former is captured in a reduced form portfolio choice that depends on risk-taking types, wealth holdings, and an element of *luck* capturing investment idiosyncratic risks. A government taxes the stock and income flows of capital as well as labor income to finance an exogenous amount of public expenditures.

Using evidence from the SCF and the PSID, we calibrate the stationary equilibrium to the US economy under three different alternatives to isolate the distinct quantitative effects of *type* and *scale* dependence. In the first, portfolio allocations are only driven by *scale* dependence, where the level of household wealth determines investments. In the second, they reflect only *type* dependence which we introduce using two *persistent* realization of risk-taking types: households who invest part of their wealth into risky equity and non-investors who do not invest. The third alternative describes a *hybrid* version that use additional novel moments to put more structure on *scale* and *type* dependence in equity investment. In this model, we calibrate the transition matrix between types allowing for some degree of *scale* dependence consistent with estimates in the PSID: conditional on household characteristics, the probability to select as a non-investor becoming an investor increases with wealth. Furthermore, consistent with detailed portfolio composition in the SCF, we allow the share of risky equity, conditional of being investor, to increase with wealth at the top of the wealth distribution. Specifically, we use the fact that an important fraction of the observed increase in risky equity of investors is due to diversification into new or recent private equity business investments and attribute this margin to *scale* dependence.

We show that while the three models can be consistent with the upper tail thickness of the wealth distribution, they have very different normative implications. In particular, the response of output to wealth redistribution substantially depends on the underlying factor that "explains" wealth inequality. First, the concentration of wealth at the top is primarily driven by the empirically consistent distribution of returns to wealth in the three alternatives. Under *type* dependence, households extracting higher capital returns tend to self-select at the top of the wealth distribution in the stationary equilibrium as long as "types" are persistent. A more dispersed distribution of returns across types induces a stronger selection and leads to higher wealth inequality. In contrast, *scale* dependence simply implies that wealth-rich households invest differently relative to wealth-poor households. With a positive effect of wealth on the propensity of risky investments and returns, the model is able to generate the top wealth shares. Second, in a model with only *scale* dependence, the response of output and of subsequent inequality to a 1% marginal wealth tax on the 1% wealthiest households is almost two times higher relative to a model with only *type* dependence. Consistently, the long-run response in the *hybrid* model falls in between. The reason is that *scale* dependence implies a dynamic self-enforcing behavioral multiplier as equity investment, and thus future expected wealth, is a function of wealth itself.

We finally show that the distinction between *type* and *scale* dependence is pivotal when considering the effects of wealth redistribution through a wealth tax on the wealthy. If wealth heterogeneity is the result of *scale* dependence where wealth-rich agents extract more rent without any higher marginal efficiency in their investments, then a planner wants to heavily tax the stock of capital at the top.[3] If this *scale* dependence comes across higher marginal efficiency of capital investments, then a planner wants to not tax the stock of capital as this translates into rather large efficiency losses. On the other hand, if wealth inequality comes from different investment skills, then taxing the wealth of top investors is welfare improving. The intuition is as follows: by reducing the stock of wealth at the top, the planner creates an environment where "only the fittest survives at the top" and hence allows for a better selection of agents with higher capital gains at the top of the wealth distribution. This better selection is contrasted by whether returns reflect the marginal efficiency of capital investments. When the size of the rents in returns increases, the tax on wealth decreases to limit the selection of top capital income earners at the top. As a result, the relative strength of *type* and *scale* dependence together with the degree to which wealth returns reflect the productivity of capital investments are thus two crucial statistics that condition the welfare gains of a wealth tax.

Moreover, if realized returns and wealth inequality depends on a mix of heterogeneous skills and scale effects calibrated to the US microdata, our model shows that a planner has an incentive to tax the stock of capital at the top. Neglecting rent-extraction motives, we find an optimal top marginal tax rate of 3.5%. For the sake of illustration, we then attribute returns from equity investments into finance and law sectors as rent-extraction following the insights of Lockwood

[3] We study optimal capital tax and sidestep the question of whether the taxation of labor income, consumption, or bequests are useful redistribution policy tools. We view the taxation of capital as a natural and transparent experiment to characterize the relationship between the wealth distribution, capital heterogeneity, and macroeconomic outcomes.

et al. (2017). Under this scenario, the welfare-maximizing top marginal wealth tax *rises*, at 4.2%, indicating that *scale-dependence* seems to dominate the direction of the tax in this economy: it is optimal to tax at a higher rate the stock of capital when the size of the rent increases.

Structure of the paper The remainder proceeds as follows. After discussing the relation to the literature, Section 2 analytical characterizes the main tradeoffs and the inequality-efficiency decomposition of an economy with heterogeneous capital investments on the household side. We present our quantitative dynamic model in Section 3. Section 4 provides the calibration procedure while section 5 discusses the properties of the alternative models. Finally, in Section 6, we use our model to study the welfare-maximizing top marginal wealth tax and section 7 concludes. The appendix contains all analytical proofs, empirical analysis, and computational details.

Related literature This paper provides a unified framework to understand and clarify the effects of a number of mechanisms with a *type* or *scale* dependence representation, either in isolation or in combination.[4] Especially relevant to our work are the theoretical papers by Gabaix et al. (2016) who show that both elements are capable of explaining the recent rise of top income inequality, and Moll (2014) who develops a tractable model of types in investment productivity. Market incompleteness and capital income risk link our paper to Angeletos and Calvet (2005, 2006). Their setups, however, feature linear investment policy functions which rule out *distributional relevance*, a feature that we precisely aim to study in this paper. By focusing on heterogeneity in efficient investment through risk-taking behaviors, this paper is related to the risk preference literature (see Peress (2004), Brunnermeier and Nagel (2008), Calvet et al. (2009) and Meeuwis (2019) among others) and to the quantitative application of Robinson (2012) and Herranz et al. (2015). Those papers rationalize a number of mechanisms that could indeed lead to *type* and *scale* dependence in household capital investments, while we more generally focus on the inequality-efficiency tradeoff and the effects of wealth redistribution.

The importance of heterogeneity in capital returns in shaping the observed high wealth concentration is studied theoretically in Benhabib, Bisin and Zhu (2011) and quantitatively in Benhabib, Bisin and Luo (2019), with both papers focusing on *type* dependence. Hubmer, Krusell and

[4] Among many others, Kaldor (1956), Stiglitz (1969) and Bourguignon (1981) study the role of wealth inequality in a neoclassical economy with convex *scale* dependence in saving behaviors. Other mechanisms include non-convex investment cost and DRS (Banerjee and Newman (1993), Galor and Zeira (1993), Blaum et al. (2013)), economies of scale in wealth management (Kacperczyk et al., 2019), social status derived from wealth holdings (Roussanov, 2010), investment in financial sophistication (Lusardi et al., 2017), wealth-dependent offshore investments (Alstadsæter et al., 2018) or non-convex investment costs in high return assets (Kaplan et al., 2018). In contrast, Kihlstrom and Laffont (1979), Moll (2014) or Moll and Itskhoki (2019) introduce *type* dependence in which the distribution of types, and their persistence, is crucial to derive aggregate efficiency. Combined dependencies arise in Quadrini (2000) and Cagetti and De Nardi (2006) through *type* dependence because entrepreneurs/business owners self-select at the top of the wealth distribution, and through *scale* dependence due to a wealth-driven occupational choice and a DRS technology.

Smith Jr (2020) use return estimates from Bach et al. (2020) and interpret this observed heterogeneity as only *scale* effects in portfolio allocation to study the recent rise of wealth inequality in the US. To the best of our knowledge, however, this paper is the first to clarify the distinct role of *type* and *scale* dependence in capital investment heterogeneity in shaping the wealth distribution, the response of aggregate variables to wealth redistribution, and optimal wealth taxation in a general equilibrium context.

Our paper is finally related to those quantifying optimal taxation in models with heterogeneity in household capital investments (see among others Kitao (2008); Shourideh et al. (2012); Guvenen et al. (2019); Brüggemann (2020); Boar and Midrigan (2020); Boar and Knowles (2020)). Despite the rapid growth of this literature, few papers discuss the role of *type* and *scale* dependence for capital taxation. Gerritsen et al. (2020) find that capital taxation is positive with heterogeneity in the rate of returns under both dependencies and Schulz (2021) shows that the degree of *scale* dependence in returns significantly affect the optimal capital income tax. Relative to them, we notably find that the underlying force behind return heterogeneity, whether coming from an endogenous selection of types and/or *scale* effects, and to what extend returns to capital reflect their marginal product are pivotal when considering the effects of a wealth tax.

2 Illustrative Analytical Two-Period Model

We begin with an analytical two-period framework with capital investment heterogeneity to illustrate the concepts of *type* and *scale* dependence and the main trade-offs. We will later extend the analysis within a rich quantitative model in which the distribution of wealth arises endogenously.

2.1 Environment

Households A unit mass of heterogeneous households indexed i lives for two periods, $t \in \{1, 2\}$, with initial wealth a_0^i and innate risk-taking type ϑ^i drawn from the joint distribution $\mathcal{G}_0(\vartheta, a_0)$, with marginal distributions $g_\vartheta(\vartheta)$ and $g_{a_0}(a_0)$ defined over the support $\Theta \subset \mathbb{R}_+$ and $\mathcal{A}_0 \subset \mathbb{R}_+$. Preferences over consumption c_1^i and c_2^i are recursive: $u_1 = U(c_1) + \beta U\Big(G^{-1}\Big(\mathbb{E}\Big[G\Big(U^{-1}(u_2)\Big)\Big)\Big)$, where $u_2 = U(c_2)$, $U = \frac{c^{1-1/\sigma}}{1-1/\sigma}$, $\beta \in (0, 1)$ and $\sigma > 0$. G introduces *generalized* constant absolute risk aversion (G-CARA) preferences that aggregate consumption across states according to

$$G(c^i; \vartheta^i, a_0^i) = \Big(1/\alpha^i\Big)\left[1 - e^{-\alpha^i c^i}\right], \quad \text{and} \quad \alpha^i := \overline{\vartheta} \cdot \left[\vartheta^i (a_0^i)^\gamma\right]^{-1}, \tag{1}$$

where the absolute risk aversion α^i depends on the average economy-wide risk tolerance $\overline{\vartheta} \equiv \mathbb{E}[\vartheta]$, agent's innate risk-taking type ϑ^i and initial wealth a_0^i. The parameter $\gamma \geq 0$ governs the shape of the wealth-dependence risk-taking. G-CARA preferences thus capture in a reduced form several mechanisms driving type and wealth dependence in portfolio choices and capital returns mentioned in the related literature.[5,6]

In period $t = 1$, households consume c_1^i and invest optimally a share ω_1^i of their savings a_1^i into a *risky* innovative asset with stochastic gross return R_r^i, and a share $(1-\omega_1^i)$ into a *risk-free* asset with certain gross return R_f. Agents inelastically supply one labor unit with productivity $h^i \sim \mathcal{N}(1, \sigma_h)$. In $t = 2$, production, returns $\{R_r^i, R_f\}$ and wage w realize and households consume c_2^i. The objective of household i is given by

$$v_0^i = \max_{c_1^i, w_1^i, a_1^i} \frac{1}{1-1/\sigma}\left((c_1^i)^{1-1/\sigma} + \beta\left\{-\frac{1}{\alpha^i}\log\left(\mathbb{E}\left[e^{-\alpha^i c_2^i}\right]\right)\right\}^{1-1/\sigma}\right), \tag{P1}$$

$$s.t. \qquad c_1^i + a_1^i \le a_0^i\,, \qquad c_2^i \le \underline{r}a_1^i + wh^i + R_f(1-\omega_1^i)a_1^i + R_r^i\omega_1^i a_1^i\,,$$

where $\underline{r}$ ensures that total capital income equals the total capital product in equilibrium.

Production side In period $t = 2$, a competitive final good producer uses labor n and intermediate goods x_s^j from two technologies $s \in \{N, I\}$, an innovative technology I with risky returns and a safe non-innovative technology N. The aggregate production function is given by: $Y = Xn^\varphi$, where $X = \left(\sum_s \int_j x_s^j \, dj\right)$ and $\varphi \in [0, 1)$.[7] Profit maximization is given by

$$\max_{n, \{x_s^j\}_{j,s}} Xn^\varphi - wn - \sum_s \int_j p_s^j x_s^j \, dj \tag{2}$$

where p_s^j denotes the price of an intermediate good j in sector s. The first order condition with respect to labor yields $w = \varphi X n^{\varphi-1}$. We normalize $\int_i h^i di = n = 1$. Substituting for labor demand, the profit maximization is rewritten as $\max_{\{x_s^j\}_{j,s}} (1-\varphi)Xn^\varphi - \sum_s \int_j p_s^j x_s^j \, dj$.

An intermediate good producer uses household i investments into risk-free and risky capital to run projects with linear technologies. Risky *innovative* projects produce $x_I^j = (\phi\mu + A(1-\mu))\omega_1^i a_1^i$, with $\mu \in [0, 1]$. *Traditional* projects operate with technology $x_N^j = A(1-\omega_1^i)a_1^i$. Using the profit maximization equation of the final good producer, the revenue generated by a traditional safe

[5] G-CARA preferences extend the utility functions used in Alpanda and Woglom (2007) or Makarov and Schornick (2010) by specifying the wealth normalization with a power function. It is also ultimately linked to Guiso and Paiella (2000) and Gollier (2001), who specify the shape of risk tolerance in terms of consumption rather than wealth.

[6] One can always rewrite the demand for risky assets for an arbitrary von Neumann-Morgenstern utility function as approximately proportional to risk tolerance, i.e. $a_1^i \approx \frac{\mu_a^p}{var_a}\mathcal{T}(\psi_0^i R)$. Contrary, under G-CARA preferences risky investment is *exactly* proportional to our generalized risk tolerance $\mathcal{T}^i(\vartheta^i, \psi_0^i)$.

[7] In Appendix F.4 we derive the case with aggregate decreasing returns to scale on X. This is however at the cost of a fixed point problem. Moreover, in such case the portfolio choice is increasing in X, as higher X tends to depress the dispersion of returns. Therefore, the risky capital supply in this alternative model is upward-sloping.

project is $p^j_N x^j_N$, respectively $p^j_I x^j_I$ from a risky innovative project. As intermediate goods are homogenous, their prices are identical and $p^j_s = (1-\varphi)$. The intermediate producer redistributes revenues to household i as follows. Revenues from riskless assets are redistributed such that their returns equal the net of wage payments marginal product of capital, i.e. $R_f = (1-\varphi)A$. In contrast, we assume that the returns of innovative investments may deviate from the net marginal product, such that $R^i_r = (1-\varphi)\kappa^i$, where $\kappa^i \sim \mathcal{N}(\phi, \sigma^2_\kappa)$. That is, there exists a return wedge between the expected returns to wealth on the household side R^i_r and the net marginal product of innovative capital $(1-\varphi)(\phi\mu + A(1-\mu))$, henceforth MPK_r, on the production side.[8] When $\mu = 1$, expected returns to innovative capital investments equal the MPK_r. Whenever $\mu < 1$, households' expected risky returns are higher than the MPK_r. In the extreme case where $\mu = 0$, the risk premium $(1-\varphi)(\phi - A)$ observed on the household side is not related to differential productivities across asset classes. A rationale for $\mu < 1$ might come from the presence of rent-extraction motives due to some forms of bargaining, market power or political connections of investors, in the words of Rothschild and Scheuer (2016), "*the pursuit of personal enrichment by extracting a slice of the existing economic pie rather than by increasing the size of that pie*".[9] We view this return wedge as a stylized way to reconcile two approaches while acknowledging that empirically measured returns to wealth can not easily be partitioned into a rent component and the marginal product of capital. On one side, some work disentangles returns to wealth from MPK, either because of their partial equilibrium structure (Benhabib et al., 2019) or because of implicit full rent extraction (Hubmer et al., 2020). On the other side, models with capitalists often assume a perfect pass-through between MPK and returns (see among others Cagetti and De Nardi (2006, 2009) or Guvenen et al. (2019)). Instead, we derive results for a range of values for the return wedge μ.

With this structure, whenever $\mu < 1$, the rate $\underline{r}$ has to adjust in equilibrium in order to ensure that the total product of capital generated on the production side coincides with the total capital income redistributed to the households by the intermediate good producer, such that:

$$\int_i \Big[R_f a^i_1 (1-\omega^i_1) + R^i_r a^i_1 \omega^i_1 + \underline{r} a^i_1 \Big] di = \int_i \Big[A(1-\varphi) a^i_1 (1-\omega^i_1) + (\phi\mu + A(1-\mu))(1-\varphi) a^i_1 \omega^i_1 \Big] di \,.$$

[8] In our specification, idiosyncratic risk materializes as return risk on the investor side rather than idiosyncratic production risk. In this respect, our framework deviates from the seminal incomplete markets economy of Angeletos and Calvet (2006); Angeletos (2007). We impose this assumption out of tractability. If the idiosyncratic capital income risk is modeled as an idiosyncratic productivity shock, one needs to integrate over the joint distribution of wealth and types to obtain aggregate output and productivity, similar to Gabaix (2011). In Appendix F.1, we show that the policy functions are isomorphic in both cases. Aggregation follows under the additional assumption that there is a sub-continuum of agents in each state (ϑ, a_0).

[9] See notably the discussion in Scheuer and Slemrod (2020) and the work of Piketty et al. (2014) and Rothschild and Scheuer (2016). In a related work, Boar and Midrigan (2019) study a setup with entrepreneurs and workers in which entrepreneurial returns to capital investment reflect partially market power.

Distributional assumptions (κ, h) follows a multivariate Gaussian distribution with mean $[\phi\ 1]$, standard deviation $[\sigma_\kappa\ \sigma_h]$, and a correlation between the investment shock and the labor productivity, $\rho_{\kappa,h}$.[10] From this, Lemma 1 characterizes the distribution of *terminal wealth* c_2^i.

Lemma 1 (TERMINAL WEALTH DISTRIBUTION)**.** *Individual terminal wealth is affine in* (κ, h)*, i.e. its distribution is Gaussian:* $c_2^i \sim \mathcal{N}\left(\mu_{c_2}^i, \sigma_{c_2}^i\right)$ *with mean* $\mu_{c_2}^i = \varphi X + (A(1-\varphi) + \underline{r})a_1^i + \omega_1^i a_1^i (1-\varphi)(\phi - A)$ *and variance* $\sigma_{c_2}^i = (\varphi X)^2 \sigma_h^2 + (\omega_1^i a_1^i (1-\varphi))^2 \sigma_\kappa^2 + 2\rho_{\kappa,h} \varphi X \omega_1^i a_1^i (1-\varphi) \sigma_h \sigma_\kappa$*.*

Together with *generalized* CARA preferences, this property ensures tractability of the equilibrium allocation. Finally, the initial wealth is assumed to be Pareto distributed.

Assumption 1 (INITIAL WEALTH DISTRIBUTION)**.** *Initial wealth is drawn from a Pareto law with scale* $\underline{a}$ *and shape* $\eta > \max\{\gamma, 1\}$*, such that* $a_0 \sim \mathcal{P}a\left(\underline{a}, \eta\right)$ *with* $\mathbb{P}(A_0 \geq a_0) = \left(\underline{a}/a_0\right)^\eta$ *,* $\forall a_0 \geq \underline{a}$ *.*

While theoretically convenient, this assumption is consistent with well-known empirical evidence documenting that the wealth distribution is right-skewed and displays an heavy upper tail (Vermeulen, 2016; Klass et al., 2006). The shape parameter η is inversely related to wealth inequality.[11] Because the shape η leads *ceteris paribus* to a change in aggregate wealth, we sometimes study the effect of varying η (*redistribution* effect) while preserving the same aggregate wealth level by changing the scale $\underline{a}$ (*level* effect).

2.2 Efficiency Gains and Redistribution: A Special Case

We first consider a stylized case in which we rule out precautionary motives and focus on the aggregate allocation when investment decisions are driven by *type* and *scale* dependence.

Assumption 2 (SPECIAL CASE)**.** *There is no period one consumption and no labor income risk, i.e* $\sigma_h^2 = 0$*.*

Lemma 2 (POLICY FUNCTIONS)**.** *Let* $\widetilde{\omega} = \frac{\phi - A}{(1-\varphi)\sigma_\kappa^2}$ *denote the baseline CARA solution, under Assumption 2,* $a_1^i = a_0^i$ *and household i's risky asset and riskless asset holdings are respectively* $\omega_1^i a_0^i$ *and* $(1 - \omega_1^i) a_0^i$ *where the share of risky assets is given by*

$$\omega_1^i = \widetilde{\omega} \cdot \frac{\vartheta^i}{\overline{\vartheta}} \cdot (a_0^i)^{\gamma - 1} . \tag{3}$$

If $\gamma = 1$ and $\vartheta^i = 1\, \forall i$, Lemma 2 extends the Merton (1969) and Samuelson (1969) result without labor income risk: the share of risky asset holdings equals the baseline CARA solution captured by the risk premium over the variance of the risky return. Conditional on type ϑ^i, G-CARA preferences mimic IRRA (respectively DRRA) behavior if $\gamma < 1$ (respectively $\gamma > 1$).

[10] We thus impose that the mean ϕ is sufficiently large to guarantee $c_2^i > 0$ in most states of the world.
[11] The Pareto tail is also inversely related to the wealth share $q(p)$ of the p wealthiest households with: $q(p) = p^{1-1/\eta}$.

When $\gamma = 1$, G-CARA preferences nest CRRA behavior with constant risky asset shares, while $\gamma = 0$ implies CARA behavior with constant risky asset holdings. Therefore, the parameter γ pins down the elasticity of risky investments to initial wealth.

Using Lemma 2, we derive equilibrium quantities and prices.

Lemma 3 (AGGREGATE QUANTITIES). *Given the joint distribution of types and wealth $\mathcal{G}_0(\vartheta, a_0)$, aggregate risky capital K_I, output Y, productivity Z and the wage rate w satisfy*[12]

$$K_I = \int_{\Theta\times\mathcal{A}_0} \omega_1(\vartheta, a_0) a_0 \, d\mathcal{G}_0(\vartheta, a_0) = (\widetilde{\omega}/\overline{\vartheta})\Big(Cov(\vartheta, a_0) - \mathbb{E}\left[\vartheta\right]\mathbb{E}\left[a_0^\gamma\right]\Big) \tag{4}$$

$$Y = Z\mathbb{E}\left[a_0\right] , \quad \textit{with} \quad Z = \mu(\phi - A)\frac{K_I}{\mathbb{E}\left[a_0\right]} + A , \tag{5}$$

$$\textit{and} \quad w = \varphi Y , \quad \underline{r} = (\mu - 1)(\phi - A)(1-\varphi)\frac{K_I}{\mathbb{E}[a_0]} . \tag{6}$$

Due to the CRS structure of final good production, demand for intermediate goods is perfectly elastic. Yet, its supply is bounded as households are risk-averse. Consequently, the risky portfolio shares of households, together with the joint distribution of wealth and types $\mathcal{G}_0(\vartheta, a_0)$, determine aggregate productivity and output. Therefore, wealth redistribution impacts productivity to the extent that it alters household i's investment in risky assets. Condition (6) states that $\mu < 1$ implies $\underline{r} < 0$, i.e. the presence of rent-extraction from risky investments induces a general equilibrium effect that decreases the common component of wealth returns, $\underline{r} a_1^i$, for all households.

We now formalize the effect of wealth redistribution on aggregate risky capital investment.

Proposition 1 (DISTRIBUTIONAL RELEVANCE). *The effect of a mean preserving change in the Pareto tail, without loss of generality, to $\eta' > \eta$ on aggregate risky asset holdings K_I can be decomposed into*

$$\Delta K_I(\eta', \eta) = \underbrace{\Delta^w K_I(\eta', \eta)}_{\substack{\textit{scale dependence in} \\ \textit{portfolio holdings}}} + \underbrace{\Delta^s K_I(\eta', \eta)}_{\substack{\textit{type heterogeneity and} \\ \textit{selection}}} ,$$

where $\Delta^w K_I(\eta', \eta)$ is zero if $\gamma \in \{0, 1\}$, increasing in η' if $\gamma \in (0, 1)$ and decreasing in η' if $\gamma > 1$. A sufficient condition for $\Delta^s K_I(\eta', \eta)$ to decrease in η' is $\left(\frac{\partial corr(\vartheta, a_0^\gamma)}{\partial \eta} + \frac{\partial corr(\vartheta, a_0^\gamma)}{\partial \underline{a}} \frac{\underline{a}}{\eta(\eta-1)}\right) \frac{1}{corr(\vartheta, a_0^\gamma)} \le 0$.

Proposition 1 provides conditions under which the wealth distribution is a *distributional relevant* equilibrium object by decomposing the effect of a change in the shape of the wealth distribution η on aggregate risky capital K_I into two terms: (i) a *scale* dependence term $\Delta^w K_I(\eta', \eta)$ that hinges on the risk taking elasticity γ, and a (ii) *type* dependence term $\Delta^s K_I(\eta', \eta)$ which encapsulates the selection of ϑ-types across the wealth distribution. A change in the Pareto tail is *distributional relevant* if both effects do not offset each other.

[12] Condition (6) follows by assuming that there is a continuum of households in each state (ϑ, a_0^i).

In the absence of *type* dependence, e.g. $\vartheta^i = \overline{\vartheta}\,\forall i$, redistribution from the bottom to the top decreases (respectively increases) K_I if $0 < \gamma < 1$ (respectively $\gamma > 1$). When $\gamma = \{0, 1\}$, distributional *irrelevance* arises because aggregate variables can be determined without information on the wealth distribution, either because risky investments are constant ($\gamma = 0$), or because the share invested is constant ($\gamma = 1$). For the sake of clarity, we refer to *scale* dependence as a situation in which $\Delta^w K_I(\eta', \eta) \neq 0$, while a negative (respectively positive) *scale* dependence corresponds to the case where $\Delta^w K_I(\eta', \eta) > 0$ (respectively $\Delta^w K_I(\eta', \eta) < 0$).[13]

In the presence of type selection, the sufficient condition in Proposition 1 provides the bound on the change in the correlation between innate types and initial wealth such that $\Delta^s K_I(\eta', \eta) < 0$. Therefore, even if $\gamma = 1$, the distribution of wealth may be *relevant* through *type* dependence.[14]

2.2.1 The Efficiency-Inequality Decomposition: A Closed-form Representation

Although Proposition 1 is general, we subsequently study a tractable representation of the equilibrium and the effects of wealth redistribution by putting a structural assumption on $corr(\vartheta, a_0)$.

Assumption 3 (Joint Distribution). *Let $\vartheta \sim \mathcal{P}a(\underline{\vartheta}, \epsilon)$ and $\overline{\vartheta} \equiv E[\vartheta]$, the joint cdf $\mathcal{G}_0(\vartheta, a_0)$ is constructed based on the Farlie-Gumbel Morgenstern (FGM) copula with dependence parameter $\varrho \in [-1, 1]$.*

Under Assumption 3, when $\varrho > 0$ (respectively $\varrho < 0$), there is a positive (respectively negative) correlation of types and wealth, while $\varrho = 0$ induces no correlation.[15] The level of ϱ translates the degree of selection, which is, for simplicity, exogenous in this section. In the dynamic quantitative framework of Section 3, we will show that a positive selection arises endogenously if households with high types earn higher returns from their capital investments for multiple consecutive periods. As such, in equilibrium, they are more represented at the top of the wealth distribution. Under capital investment heterogeneity, the persistence and the dispersion of returns across types are thus two key components that generate a positive selection in equilibrium.[16]

The following result decomposes the trade-off between inequality and efficiency into three terms which capture *type* dependence, *scale* dependence and an interaction term.

Proposition 2 (Efficiency-Inequality Relation). *Given assumptions 1-3 and a positive riskless capital supply in equilibrium, such that $\frac{K_I}{\mathbb{E}[a_0]} < 1$, wealth-normalized output is given by $\widetilde{Y}(\eta) = A +$*

[13] Our notion of *scale*-dependence therefore corresponds to cases in which scale effects, through γ, generate a distributional relevant link between wealth redistribution and aggregate quantities. This arises when the individual policy function $\omega_1^i(a_0^1)$ depends non-linearly on initial wealth. Furthermore, recall that η is inversely related to inequality.

[14] Interestingly, the economy can however be represented by a representative household, as shown in Appendix F.6.

[15] The dependence parameter ϱ and the Spearman's correlation, ϱ^s, are under the FGM copula related by $\varrho^s = \varrho/3$.

[16] See Moll (2014) for highlighting the key role of persistence in types for aggregate TFP in a capitalist/entrepreneur framework. Using his model, he studies the case of an economy starting with negative dependence between entrepreneurial types and wealth and shows that it converges to an economy with a positive dependence, provided that types are persistent.

$\mu(\phi - A)\widetilde{\omega}\left(1 + \frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\frac{\eta-1}{\eta-\gamma}\underline{a}^{\gamma-1}$.[17] *The marginal effect of wealth redistribution on* $\widetilde{Y}(\eta)$ *is*

$$\frac{\partial \widetilde{Y}(\eta;\gamma,\varrho)}{\partial \eta} \propto -\mu(\phi - A)\Big(\underbrace{\Omega^{w}(\eta,\gamma)\cdot(\gamma-1)}_{\text{scale dependence}} + \underbrace{\Omega^{s}(\eta,\gamma)\cdot\varrho}_{\text{type dependence}} + \underbrace{\Omega^{ws}(\eta,\gamma)\cdot\rho(\gamma-1)}_{\text{interaction term}}\Big), \tag{7}$$

where $\Omega^{w}(\eta,\gamma)$, $\Omega^{s}(\eta,\gamma)$ *and* $\Omega^{ws}(\eta,\gamma)$ *are strictly positive inequality multipliers.*

A key property of Proposition 2 is the ambiguous effect of rising wealth inequality on (normalized) output that depends on the relative strength of *type* dependence, *scale* dependence and their interaction, respectively captured by the terms $\gamma - 1$, ϱ and $\varrho(\gamma-1)$. If G-CARA preferences mimic DRRA behavior ($\gamma > 1$) and there is a positive selection of types ($\varrho > 0$), then output unambiguously rises with higher wealth inequality, since it reallocates wealth to agents investing in riskier and possibly more productive assets. This effect is scaled to the degree to which returns to investment reflect differential capital productivity, captured by the term $\mu(\phi - A)$.

Importantly, variations of the Pareto tail exhibit highly nonlinear effects on output captured by the inequality multipliers $\Omega^{w}(\eta,\gamma)$, $\Omega^{s}(\eta,\gamma)$ and $\Omega^{ws}(\eta,\gamma)$. The precise decomposition and the associated inequality multipliers are obviously model-specific, but, as discussed below, the general idea and mechanisms unify a number of frameworks. In complete unequal economies, i.e. $\eta \to \max\{\gamma, 1\}$, small variations of the Pareto tail result in rather large output variations. In contrast, in a complete egalitarian societies, i.e. $\eta \to \infty$, small variations in η result in small output variations. Intuitively, in more unequal economies, *scale* dependence and selection effects are stronger in magnitude such that even small variations of η lead to strong investment reallocations.

Equation (7) also implies that, for a given level of inequality η, there exists an infinite number of possible combination of *type* and *scale* dependence on a bounded two-dimensional set consistent with a given marginal effect of wealth redistribution on output. In Definition 1, we specify the notion of *iso-growth* which describes all parameter pairs (γ, ϱ) for which a marginal variation of the Pareto shape η generates a given output response $\overline{g}$.

Definition 1 (ISO-GROWTH OF INEQUALITY)**.** *For a given wealth Pareto tail* η*, the iso-growth with level* $\overline{g}$ *is described by the pair* (γ, ϱ) *which satisfies* $isoG(\eta, \overline{g}) \equiv \left\{(\gamma,\varrho) \in \Gamma \times [-1,1] : -\frac{\partial \widetilde{Y}(\eta;\gamma,\varrho)}{\partial \eta} = \overline{g}\right\}$.

A special case ensues for $\overline{g} = 0$ for which the *iso-growth* curve separates the *growth enhancing region*, i.e. $-\frac{\partial \widetilde{Y}(\eta;\gamma,\varrho)}{\partial \eta} > 0$, from the *growth dampening region*, i.e. $-\frac{\partial \widetilde{Y}(\eta;\gamma,\varrho)}{\partial \eta} < 0$. For this reason, we label this special *iso-growth* curve the *Growth Irrelevance Frontier* of wealth inequality. Lemma 7 in Appendix A.1.7 provides conditions for its existence based on the strength of *type* and *scale* dependence and the wealth Pareto tail η. If these parameter restrictions de not apply, an infeasible

[17] The condition $\frac{K_I}{\mathbb{E}[a_0]} < 1$ is satisfied for $\underline{a}$ such that $\frac{\eta-\gamma}{\eta-1} \geq \widetilde{\omega}\left(1 + \frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\underline{a}^{\gamma-1}$.

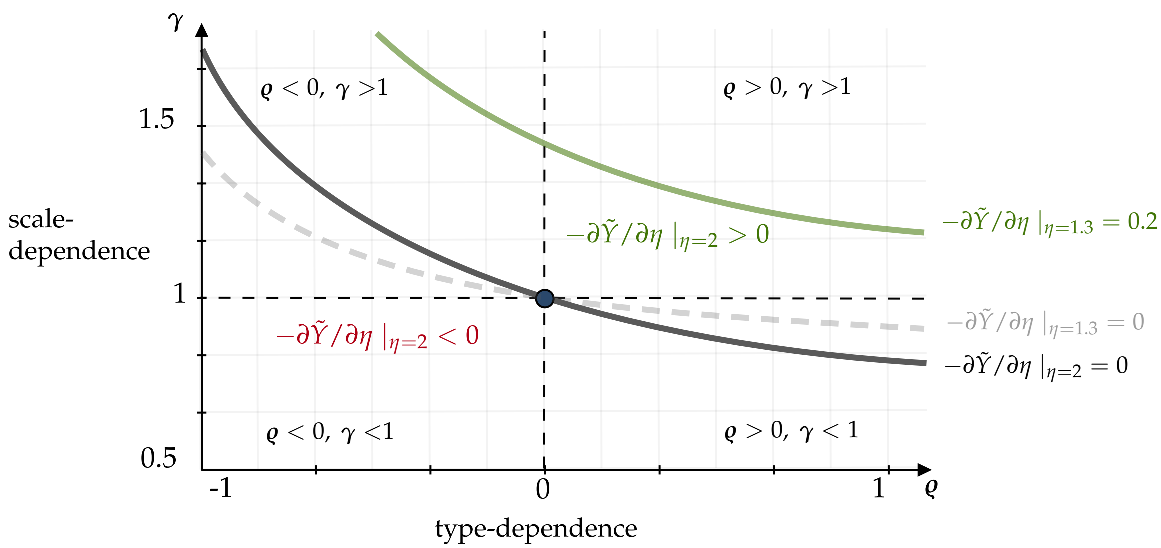
pair $(\gamma, \varrho) \notin \Gamma \times [-1, 1]$ would be required to obtain *growth neutrality*.[18] On the *Growth Irrelevance Frontier (GIF)* of wealth inequality, *type* and *scale* dependence exactly offset each other, or are absent. In Lemma 4, we provide key properties of the *iso-growth*: it is decreasing on the space (γ, ϱ) and rotates with the wealth Pareto shape η.

Lemma 4 (PROPERTIES OF THE GIF AND ISO-GROWTH)**.** *The GIF is strictly decreasing on the defined set of Lemma 7, i.e.* $\frac{\partial \gamma}{\partial \varrho}|_{d\eta=0} < 0$. *A higher tail* η *rotates the GIF such that* $\frac{d\gamma}{d\eta}|_{d\varrho=0} > 0$ *for* $\gamma > 1$ *and* $\frac{d\gamma}{d\eta}|_{d\varrho=0} \leq 0$ *for* $\gamma \leq 1$*. Also, for a higher level* $\overline{g}$*, isoG*$(\eta, \overline{g})$ *is the translation of the GIF and* $\frac{d\gamma}{d\overline{g}}|_{d\varrho=0} > 0$.

Figure 1 illustrates the *iso-growth* for two different Pareto tails η and output levels $\overline{g}$. There are four regions which delimit the sign of the *type* and *scale* dependence. In the top-right and bottom-left regions, wealth-dependent risk-taking and selection effects move in the same direction. An increase in inequality therefore unambiguously induces more (respectively less) economy-wide risk-taking and higher (respectively lower) productivity. In the top-left and bottom-right regions, wealth-dependent risk-taking and selection effects move in opposite directions. In these regions, there is a combination of *type* and *scale* dependence such that both effects offset each other such that the GIF exists. In the top-left region characterized by $\{\varrho < 0, \gamma > 1\}$, an increase in inequality leads to higher output only if the positive *scale* dependence is sufficiently strong. Contrary, in the bottom-right region characterized by $\{\varrho > 0, \gamma < 1\}$ an increase in inequality decreases output if the wealth-dependent risk-taking elasticity γ is sufficiently low for a given selection $\varrho > 0$.

Figure 1. The inequality–efficiency diagram.

Note: numerical parameter values are: $\varepsilon = 2.0, \underline{a} = \underline{\vartheta} = 1.0, A_r = 1.1, \sigma_\kappa = 0.2, A = 1.0$.



[18]Notice that the *isoG*$(\eta, 0)$ does not exist in a complete unegalitarian economy ($\eta = \max\{1, \gamma\}$), since in this case the absolute strength of scale dependence dominates the selection effect. Moreover, under relatively egalitarian economy, both effects are small in magnitude such that the *isoG*$(\eta, 0)$ exists on a bounded set.

For a higher effect of wealth inequality on the level of output (an increase in $\overline{g}$), the iso-growth moves upward in the (γ, ϱ) diagram: higher effect of a positive wealth inequality variation on output can only be rationalized with higher degrees of positive *type* and/or *scale* dependence. Finally, the level of wealth inequality changes the relative strength of *type* and *scale* dependence effects: higher inequality (lower η) reinforces the *scale* dependence effect relative to the selection effect and, as a result, the GIF (and the translated *iso-growth* curve) rotates counterclockwise through its anchor point ($\gamma = 1, \varrho = 0$). Finally, for $\overline{g} > 0$ (respectively $\overline{g} < 0$), a given $isoG(\overline{g}, \eta)$ shifts downward (respectively upward) with an increase in the productivity gap $\mu(\phi - A)$ as less reallocation between the two productive sectors is needed to achieve a level $\overline{g}$.

The decomposition derived here carries over many quantitative models, such as the one studied in Section 3. It shows that, in order to understand effects of wealth redistribution on the aggregates in a framework that accounts for heterogeneous capital investments, there are four key statistics required: (i) the shape of the wealth distribution, η, (ii) the risk premium augmented by the degree to which differential returns to capital reflect differences in the productivity of capital investments, $\mu(\phi - A)$, (iii) the elasticity of risk-taking with respect to wealth, γ, and (iv) the selection of types across the wealth distribution, captured by the dependence parameter ϱ.

Discussion and practical implications A number of frameworks with *type* and *scale* dependent mechanisms can be unified within the representation of the inequality-efficiency diagram of Figure 1 and equation (7). In the basic Aiyagari (1994) economy, the distribution of wealth is almost growth irrelevant due to the quasi linearity of the decision to save of the wealth-rich households. Angeletos (2007) study an economy with linear portfolio policy functions and no type heterogeneity, implying distribution *irrelevance*. Those models are located respectively at, and close to, the anchor point of the inequality-efficiency diagram ($\gamma = 1$ and $\varrho = 0$). Models with capitalists/entrepreneurs (Cagetti and De Nardi (2006, 2009), Guvenen et al. (2019), Brüggemann (2021)) often display positive *type* dependence at the stationary equilibrium, as entrepreneurs self-select at the top of the wealth distribution ($\varrho > 0$), but feature decreasing marginal product of capital investments which can be reinterpreted as negative *scale* dependence effects ($\gamma < 1$). In those models, wealth redistribution from the top to the bottom has thus an ambiguous effect on output. Therefore, they are positioned in the bottom-right region. Similarly, the seminal paper by Galor and Zeira (1993) with non-convex human capital investment costs can be reinterpreted as a form of DRS ($\gamma < 1$), but without type heterogeneity ($\varrho = 0$). Finally, the model of Moll (2014) displays only *type*-dependence in households capital productivity with linear investment policy functions.

Such models therefore locate on the locus with $\gamma = 1$.[19]

From the diagram, it is interesting to see that *type* and *scale* dependence cannot be simply identified by using information on the effect of inequality on growth, as an infinite combination of pairs (γ, ϱ) may rationalize the relationship.[20] It is also difficult to identify both dependencies based on cross-sectional data as both mechanisms may in principle generate consistent patterns regarding the portfolio allocation and associated returns to capital across the wealth distribution. This is striking given that both effects may result in substantially different elasticities of macroeconomic aggregates to wealth redistribution and, as shown in section 3, to differing implications for optimal wealth taxation.

A numerical example Consider two distinct models: the first with only *scale* dependence, and the second with only *type* dependence. We calibrate *type* and *scale* dependence to generate a consistent cross-sectional pattern of risky equity shares, targeting the share of the top 1% wealthiest households of 65% as in the 2010 SCF. Note that because the portfolio shape is increasing in wealth in the cross-sectional data, the two models are located respectively on the $\{\varrho = 0, \gamma > 1\}$ and $\{\varrho > 0, \gamma = 1\}$ loci. We normalize the productivity $A = 1$ and assume $\mu = 1$. We set the labor share $\varphi = 0.67$, the shape $\eta = 1.40$ consistent with estimates for the US (Vermeulen, 2016), and the risk premium such that $(1 - \varphi)(\phi - A) = 15\%$. The variance σ_κ^2 is set to 0.16, consistent with estimates in the PSID, that we discuss in section 3. In the *type* dependence model, we set the Pareto shape of types $\varepsilon = 2$ and vary the correlation between types and wealth to match the top 1% risky share, such that $corr(\vartheta, a_0) = 0.65$.[21] In the *scale* dependence model, we vary the wealth-dependent risk-taking elasticity γ and obtain $\gamma = 1.39$. In Appendix B.1, we show that both models reproduce well the overall cross-sectional pattern of portfolio shares, at the bottom and at the upper end of the wealth distribution. However, the responses of output to a proportional top marginal wealth tax of 1% on the top 1% wealthiest households, redistributed through lump-sum transfers, differ substantially. Output drops by 0.43% under *type* dependence, substan-

[19]In Appendix E.1, we show how a number of additional frameworks can be understood within this representation. Moreover, the representation above provides a rationale for the absence of clear evidence on the role of wealth inequality on growth (see for instance Perotti (1996); Forbes (2000); Barro (2008)). Recent panel data estimations tend to find a weak *positive* relationship in developed countries (Forbes, 2000; Voitchovsky, 2005; Barro, 2008; Frank, 2009) and a *negative* one in developing economies. The disparity of the estimates can be reconciled within our framework as the relation crucially depends on the underlying selection of agents as well as the direction of the *scale*-dependence, which may of course be country-specifc.

[20]There are, however, identified under particular conditions. Figure 1 shows that γ and ϱ are identified with *two* different couples of observation $(\eta_1, \bar{g}_1)$ and $(\eta_2, \bar{g}_2)$, but this requires that type and wealth dependence are constant over time. Moreover, estimating this relationship is somewhat complex as shown by the variety of empirical results in this related literature. See among others Forbes (2000), Voitchovsky (2005) or Barro (2008).

[21]Instead, it is possible to fix the correlation of type but vary the extend that individuals are different by varying the shape ϵ. There is marginal difference in considering one or the other, as long as the cross-sectional distribution of portfolio shares and wealth are well matched.

tially lower than the 0.70% reduction found under *scale* dependence. This difference originates from the behavioral response triggered by the *scale* dependence mechanisms as wealth varies (see Lemma 2).[22] This simple numerical example illustrates the importance of unraveling the economic forces behind heterogeneous capital investments. Fortunately, their relative importance has been recently investigated by Bach et al. (2020) and Fagereng et al. (2020) using Norwegian and Swedish administrative panel data. In section 3, we provide evidence for the US economy.

Evidence from cross-country data In the Online Appendix E.1, we investigate the relationship between inequality and GDP growth (henceforth *IG-slope*) using new estimates regarding wealth concentration. We extend previous analysis, relying mostly on income shares, to a larger sample by constructing and using wealth concentration measures for 29 countries (mostly developed). Some of those estimates, derived from survey data, are corrected for under-representation and under-reporting following Vermeulen (2016). We focus on the heterogeneous impact of inequality along wealth and income distributions on growth, using quantile shares at the top of the distribution. In contrast, previous studies mainly analyze the effect of the income Gini coefficient, a synthetic inequality measure revealing an average effect of inequality.[23] We document that an increase in top wealth and income shares has a significant positive impact on subsequent growth, averaged over the subsequent five years. A 1 percentage point increase in the top 1% wealth share is associated with a 0.27% increase in the average GDP growth. We then decompose the IG-slope and find that top inequality is mainly driving Solow residual and physical capital accumulation, while the relation is negative and insignificant for human capital. This is consistent with the reallocation channel between low productive and high productive risky capital outlined analytically. Finally, we point out that several other studies find a negative IG-slope, especially when considering low-income countries. In light of our theoretical framework, we rationalize these findings. The stage of development of a country may go along with different relative strength of type and scale dependence. Developed countries may sort themselves more likely into the growth-enhancing region with a positive correlation between types and wealth. On the contrary, less developed countries may display negative sorting, and fall into the growth dampening region.

[22]Notice that in a dynamic model, scale effects are amplified, as future wealth is a function of current investment.

[23]A wide range of empirical papers studies the link between inequality and growth. For instance, Forbes (2000), Barro (2000) and Halter et al. (2014) use the income Gini coefficient as measure of inequality, while Barro (2008) and Voitchovsky (2005) use quintile and decile shares. None of the previous papers explore the *IG-slope* using wealth concentration measures. To the best of our knowledge, only Voitchovsky (2005) looks at the impact of quantile income concentration at the relatively more aggregated decile level. An exception is Frank (2009), however, his study considers a panel of US states.

2.3 From Efficiency to Welfare

We now shift our focus to a welfare analysis of redistributing wealth. For tractability, we consider the case of a constant rate of progressivity (CRP) tax similar to Feldstein (1969) and Heathcote et al. (2017) on initial wealth such that $t_a(a_0^i) = a_i^0 - (a_i^0)^{1-p_a}$, where $p_a \in (-\infty, 1)$ captures the progressivity of the wealth tax. We denote with "$\sim$" the *post-tax* variables. Under this specification, aggregate conditions are isomorphic, replacing initial wealth a_i^0 with $\widetilde{a}_i^0 = a_i^0 - t_a(a_i^0)$, which implies an updated wealth Pareto tail $\widetilde{\eta} = \frac{\eta}{1-p_a}$ and scale $\underline{\widetilde{a}} = \underline{a}\left(1 + \frac{p_a}{\eta - 1}\right)$, where $p_a \to 1$ implies a complete egalitarian economy. We measure welfare in terms of consumption equivalents, defined as the amount $\Delta^{CE,i}$ that makes a household i in the reformed economy as well off as in the initial status quo economy, such that $E[u(\widetilde{c}_2^i - \Delta^{CE,i})] = E[u(c_2^i)]$. Under G-CARA preferences, this gives $\Delta^{CE,i} = \widetilde{x}^i_{c_2} - x^i_{c_2} + \frac{\Delta^i_c}{\widetilde{\alpha}_i}$, where $x^i_{c_2}$ and $\widetilde{x}^i_{c_2}$ denote certainty equivalents of the second period pre-tax and post-tax consumption, and the term Δ^i_c arises as the utility is a positive function of initial wealth through α^i. Given the equivalent variation-based welfare measure, the planner solves

$$\mathcal{W} = \arg\max_{p_a} \int s(\vartheta, a_0)\, \Delta^{CE}(\vartheta, a_0)\, d\mathcal{G}_0(\vartheta, a_0) \qquad \text{s.t.} \qquad T = \frac{\eta \underline{a}}{\eta - 1} - \frac{\eta \underline{a}^{1-p_a}}{\eta - 1 + p_a}\,. \tag{8}$$

where $s(\vartheta, a_0)$ defines the social welfare weight, such that $\int s(\vartheta, a_0)\, d\mathcal{G}_0(\vartheta, a_0) = 1$, and the last equality balances the government budget constraint, such that $\int t_a(a_i^0) di = T$.

Lemma 5 (Optimal wealth redistribution)**.** *Assume $x_r \equiv \phi - A - \frac{\sigma_\kappa^2}{2}(\widetilde{\omega}/\overline{\vartheta}) > 0$ and let $x^i_{c_2} \equiv \mu_{c_2^i} - \frac{\alpha^i}{2}\sigma^2_{c_2^i}$ be the certainty equivalent of c_2^i. The optimal progressivity p_a^* is implicitly given by*

$$\left(\underbrace{\varphi \frac{\partial Y(\widetilde{\eta}; \varrho, \gamma)}{\partial \widetilde{\eta}}}_{\substack{\textit{GE wage} \\ \textit{efficiency if } \mu > 0}} + \underbrace{\frac{\partial \underline{r}(\widetilde{\eta}; \varrho, \gamma)}{\partial \widetilde{\eta}} \int s(a_0, \vartheta) \widetilde{a}_0 \mathcal{G}_0(a_0, \vartheta)}_{\textit{GE rent extraction if } \mu < 1} \right) \frac{\partial \widetilde{\eta}}{\partial p_a} + \underbrace{\frac{\partial T}{\partial p_a}}_{\substack{\textit{lump-sum} \\ \textit{transfers}}} + \underbrace{\int s(a_0, \vartheta) \mathcal{R}(a_0, \vartheta) \mathcal{G}_0(a_0, \vartheta)}_{\textit{direct effects of the wealth tax}} = 0$$

where $\mathcal{R}(\vartheta, a_0)$ captures the direct effects of the wealth tax on second period consumptions and on risk aversion $\widetilde{\alpha}_i$. The sign of $\frac{\partial Y(\widetilde{\eta};\varrho,\gamma)}{\partial \widetilde{\eta}}$ is characterized in Proposition 2 and $sgn\left(\frac{\partial \underline{r}(\widetilde{\eta};\varrho,\gamma)}{\partial \widetilde{\eta}}\right) = -sgn\left(\frac{\partial Y(\widetilde{\eta};\varrho,\gamma)}{\partial \widetilde{\eta}}\right)$.

Lemma 5 characterizes the main tradeoffs of wealth redistribution on welfare. The redistribution channels are composed of welfare gains from lump-sum transfers T and welfare losses through the effects of taxes on terminal wealth. General equilibrium feedback effects affect welfare in two ways. On the one hand, a wealth tax affects efficiency and thus the wage rate w which is scaled to aggregate productivity. On the other hand, higher rent-extraction (a decrease of μ) lowers welfare through the return component $\underline{r}a_1^i$, as investors obtain private returns larger than their social value. The optimal proportional wealth tax thus balances efficiency *versus* rent-extraction and equity. Importantly to notice, in this static model with exogenous *type* dependence, a lower μ implies a higher progressivity of the wealth tax. Section 3 extends the results to a quantitative dynamic model in which the joint distribution of ϑ-types and wealth arises endogenously.

2.4 Generalization

The assumptions made throughout the special case allowed to isolate risk-taking decisions arising from *type* and *scale* dependence. We now extend the analysis to saving decisions under uninsurable labor income risk, alternative sources of *scale* dependence, and aggregate productivity shocks.

2.4.1 Saving decisions and uninsurable labor income risk

In this case, the joint heterogeneity in marginal propensities to save and marginal propensities to take risk (Kekre and Lenel, 2020) are key to study aggregate allocations and gives rise to a generalized *iso-growth* curve.

Lemma 6 (Individual Portfolio Choice). *Denote $\widetilde{\beta} = (\widetilde{R}\beta)^{\sigma} + \widetilde{R}$ where $\widetilde{R} = (\underline{r} + A(1-\varphi))^{\sigma}$ and let us assume an interior solution to (P1) such that $c_1^i > 0$ and $\sigma_h > 0$, individual portfolio choices of risky and riskless assets denoted respectively, $k_1^i = \omega_1^i a_1^i$ and $b_1^i = (1-\omega_1^i)a_1^i$ are given by*

$$k_1^i = \underbrace{\widetilde{\omega}\frac{\vartheta^i}{\overline{\vartheta}}(a_0^i)^{\gamma}}_{\text{G-CARA portfolio}} \underbrace{-\rho_{\kappa,h}\frac{\varphi Y}{1-\varphi}\frac{\sigma_h}{\sigma_\kappa}}_{\text{labor income risk}}, \quad b_1^i = \frac{1}{\widetilde{\beta}}\left(\underbrace{(\widetilde{R}\beta)^{\sigma}a_0^i - ((\widetilde{R}\beta)^{\sigma} + \underline{r} + \phi)k_1^i - \varphi Y}_{\text{Intertemporal substitution}} + \underbrace{\frac{1}{2}\alpha^i\sigma_{c_2^i}^2}_{\text{Prec. savings}}\right).$$

Lemma 6 is a generalized counterpart to Proposition 1 in Angeletos and Calvet (2006) derived under a baseline CARA specification. The optimal decision of k_1^i has now two components: the G-CARA portfolio choice of Lemma 2, and a term that arises due to the correlation between labor income and investment risks. If $\rho_{\kappa,h} > 0$, it discourages wealth-poor households from risky asset investments and encourages them simultaneously to risk-free asset investments. Intuitively, in absence of correlated shocks, labor income plays the role of a risk-free asset and encourages risk-taking.[24] The two-period structure also leads to an intertemporal substitution effect due to the risky asset holdings and a precautionary savings effect that arises from uncertainty about the realization of c_2^i. The effect of wealth increases on precautionary savings is *a priori* ambiguous and linked to the risk tolerance shape. On the one hand, increasing wealth decreases risk prudence. On the other hand, lower risk aversion enhances risk-taking such that $\sigma_{c_2}^i$ rises.

In this economy, the effects of a change in the shape of the wealth distribution on output can be characterized by means of sufficient statistics.

[24] This argument originates from Merton and Samuelson (1992) or Viceira (2001) and is numerically illustrated in Cocco, Gomes and Maenhout (2005).

Corollary 1 (EFFICIENCY AND INEQUALITY). *The output in this economy is giben by* $Y = \mathbb{E}[b_1(\vartheta, a_0)]A + (\mu\phi + (1-\mu)A)\mathbb{E}[k_1(\vartheta, a_0)]$. *The effect of a change in inequality on wealth-normalized output is:*

$$\frac{\partial Y/\mathbb{E}[a_0]}{\partial \eta} = A \cdot Cov\left(mps^i, \frac{da_0^i}{\mathbb{E}[a_0]}; \eta, \varrho, \gamma\right) + \mu(\phi - A) \cdot Cov\left(mpr^i \times mps^i, \frac{da_0^i}{\mathbb{E}[a_0]}; \eta, \varrho, \gamma\right),$$

with $mps^i = \frac{\partial a_1^i}{\partial a_0^i}$ *and* $mpr^i = \left(\frac{\partial k_1^i}{\partial a_1^i}\right) / \left(\frac{\partial a_1^i}{\partial a_0^i}\right)$ *are the marginal propensity to save and to take risk.*

Corollary 1 characterizes the overall effect of a wealth inequality change on aggregate efficiency into sufficient statistics in an economy with capital accumulation. In such an economy, *type* and *scale* dependence determine the extent to which agents invest in risky assets (captured by the distribution of mpr^i), and how they accumulate wealth (captured by the distribution of mps^i). The quantitative setting in Section 3 incorporates both elements into a dynamic framework.

2.4.2 Extensions

Other sources of wealth dependence Appendix F.2.1 introduces an entrepreneurship type of model along the line of Cagetti and De Nardi (2006); Guvenen et al. (2019); Brüggemann (2021). The model is shown to map the representation in equation (7). In this setting, we show that wealth-normalized output depends negatively on wealth inequality due to decreasing returns to scale on private equity investments (negative wealth-dependence $\gamma < 1$), but positively with the selection of entrepreneurs at the top of the distribution (positive type-dependence $\varrho > 0$). Such models are located in the bottom-right area of the inequality-efficiency diagram of Figure 1. Second, we consider the case of wealth-dependent borrowing constraint and show that it generates similar results as the one derived above under wealth-dependent risk-aversion.

Aggregate shocks Throughout the paper, we assumed that investment return risks are idiosyncratic following the findings of Bach et al. (2020) on private equity, which represents the largest share of wealth of the wealth-rich. Under aggregate production risk, that we introduce by assuming that the productivity of risky projects is now given by $z(\mu\phi + (1-\mu)A)$ with $z \sim \mathcal{N}(1, \sigma_z^2)$, the main insights with respect to investment decisions and output are unchanged, but two additional features emerge. First, *a growth – variance trade-off* arises because increasing inequality affects expected growth, but also its volatility. A social planner seeking to redistribute wealth has an additional incentive to stabilize the wage rate, pushing towards less inequality when higher inequality is linked to more aggregate risky investments. Under the special case of Section 2.2, this creates a link between inequality and wealth-normalized output volatility $\sigma^2_{\tilde{Y}}(\eta)$, such that: $\frac{\partial \sigma^2_{\tilde{Y}}(\eta)}{\partial \eta} = -2\sigma_z^2 \widetilde{\omega}^2 (\phi\mu + A(1-\mu))^2 \left(\frac{\partial \Psi(\eta; \gamma, \varrho)}{\partial \eta}\right) \Psi(\eta; \gamma, \varrho)$ where $\Psi(\eta; \gamma, \varrho)$ is defined in Appendix

F.3 and captures type and wealth dependence. Second, *a GE precautionary saving effect* arises on household's portfolio allocation in the case of endogenous saving decisions as aggregate risks induces a correlation between the wage rate and risky returns. We view the latter extension particularly interesting, but given the focus of this paper, we leave their implications for future research.

3 A Dynamic Quantitative Model with Investment Heterogeneity

The previous section derived a representation of the link between wealth redistribution and aggregate output based on *type* and *scale* dependence. In a two-period framework, we show that the degree of *type* and *wealth* dependence, together with the extend that returns reflect investment productivity, determine the link between wealth inequality, output and welfare. We now evaluate a dynamic quantitative model in which the wealth distribution is endogenous. We analyze the role of the previous key statistics in shaping the wealth inequality, the responsiveness of output to a wealth taxation, and ultimately the optimal taxation of wealth.

3.1 Environment

The distribution of wealth arises endogenously from two empirically relevant features: heterogeneity in labor productivity as in standard incomplete markets model (Aiyagari, 1994), and heterogeneity in capital investments and associated returns. While Benhabib et al. (2011, 2019) and Hubmer et al. (2020) show that the latter is key to generate the right tail of the wealth distribution, we more generally study the role of *type* and *scale* dependence. Apart from those elements, the rest of the model is kept purposely parsimonious.

3.1.1 Preference and endowments

Households are infinitely lived and derive a flow of utility $\mathcal{V}_t^i$ from consumption c_t^i and labor supply ℓ_t^i. Agents differ in their wealth a_t^i, their permanent component of labor productivity h_t^i, and their innate risk taking or investment skill type ϑ_t^i. The state space is $\mathbf{s} = (a, \vartheta, h)$. They discount future periods at rate $\beta \in (0,1)$ and die with probability $d(h_t^i)$, such that

$$\mathcal{V}_t^i = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(1 - d(h_t^i)\right)^t u(c_t^i, \ell_t^i) \right] . \tag{9}$$

Unless necessary, we drop time and households indexes. Pre-tax labor income is $w\mathcal{H}(h)y\ell$, where w is the equilibrium wage rate. y and $\mathcal{H}(h)$ denote respectively the transitory and the persistent labor productivity components. Similar to Hubmer et al. (2020), we improve the fit of the earnings distribution by assuming that the persistent component $\mathcal{H}(h)$ follows a lognormal

AR(1) process with persistence ρ_h and variance σ_h^2. However, at the top of the income distribution, h is drawn from a Pareto Law with shape $\eta_h > 1$,

$$\mathcal{H}(h) = \begin{cases} e^h & \text{if} \quad F_h(h) \leq q_h, \\ F^{-1}_{Pareto(\eta^h)}\left(\frac{F_h(h)-q_h}{1-q_h}\right) & \text{otherwise.} \end{cases} \tag{10}$$

where $F_h(\cdot)$ is the cdf of h and $F^{-1}_{Pareto(\eta^h)}(\cdot)$ the inverse CDF for a Pareto distribution with lower bound $F_h^{-1}(q_h)$ with $q_h \in (0,1)$. The persistent component is discretized into bins $h \in \{h_1, ..., h_H\} \in \mathbb{H}$ and – while the agent is alive – its evolution follows a first order Markov chain with transition matrix $\pi_h(h'|h)$. Upon death, a newborn imperfectly inherits the persistent component of her parents. With probability p_h she draws her parent's persistent labor productivity, and with probability $(1-p_h)$ she draws her productivity from the invariant distribution $F_h(h)$ generated by $\pi_h(h'|h)$.

Following Sommer and Sullivan (2018), our model features stylized stochastic aging to capture the dynamics of income and wealth accumulation over the life-cycle without the need to explicitly incorporate age in the state space. We assume that there is a probability $\chi_h = \frac{1}{\Pi_h M}$ of transiting from h to h' due to an aging shock, where Π_h is the fraction of the population with productivity h, and M is a constant equal to the expected lifetime. For this reason, the probability to die is assumed to increase with income, i.e. $d(h)$ increases in h. Appendix D.1.1 provides details of the full transition matrix associated with our assumptions.

Risk-taking type follows a S-states Markov chain, $\vartheta \in \{\vartheta_1, ..., \vartheta_S\} \in \Theta$, with transition matrix $\boldsymbol{\pi}_\vartheta$. A newborn draws her parent's risk-taking type with probability p_ϑ and from the invariant distribution $F_\vartheta(\vartheta)$ otherwise.

Households are heterogeneous in their capital investments. They split their savings into risk-free assets (cash, savings, deposits, bonds, housing, and other safe assets) and risky productive assets (equity). An agent with risk-taking type ϑ and wealth a invests a share $\omega(a,\vartheta)$ in the risky asset. For the sake of clarity, our portfolio specification should be understood as a reduced form of a more elaborated portfolio choice, which is for example present in as a model with entrepreneurs/capitalists in which some households invest in private equity business investments. Let r_F and r_R, with $r_R > r_F$, respectively be the risk-free and risky rates determined in equilibrium. The *ex post* pre-tax return on investment is given by

$$r(a,\vartheta,\kappa) = \underline{r} + r_F \cdot (1 - \omega(a,\vartheta)) + (r_R \kappa) \cdot \omega(a,\vartheta)\,, \tag{11}$$

where $\underline{r}$ is an aggregate return component and $\kappa \sim \mathcal{N}(1,\sigma_\kappa)$ an element of *luck* approximated into m discrete bins $\kappa \in \{\kappa_1, ..., \kappa_m\}$ with corresponding probabilities $\pi_\kappa(\kappa)$. The variance of returns is therefore given by $(\sigma_r(a,\vartheta))^2 = (r_R\omega(a,\vartheta)\sigma_\kappa)^2$ and implies that households with higher equity shares experience higher portfolio risk. Such a feature is supported in the PSID, and documented by Bach, Calvet and Sodini (2020) and Fagereng, Guiso, Malacrino and Pistaferri (2020).[25] From equation (11), it is clear that returns are correlated over time through wealth itself, and through the process governing the evolution of risk-taking type ϑ.

Agents optimally choose their saving a' and cannot borrow. Their recursive program is

$$v(a,\vartheta,h) = \mathbb{E}_{\kappa,y}\left\{\max_{c>0,\,a'\geq 0,\,\ell\geq 0}\left\{u(c,\ell)+\beta(1-d(h))\mathbb{E}_{h',\vartheta'|h,\vartheta}\left[v(a',\vartheta',h')\right]\right\}\right\} \tag{12}$$

$$\text{s.t.}\quad c+a' = w\mathcal{H}(h)y\ell - t_w(w\mathcal{H}(h)y) + (1+r(a,\vartheta,\kappa))a - t_r(r(a,\vartheta,\kappa)a) - t_a(a), \tag{13}$$

where $t_r(.)$, $t_w(.)$ and $t_a(.)$ are respectively the tax schedule on capital income, labor income and wealth. Upon death, bequests are taxed such that $a^{child} = a^{parents} - t_b(a^{parents})$.[26]

3.1.2 Production, government, and equilibrium

Production An intermediate producer operates at no cost a continuum of projects j in sectors $s \in \{N, I\}$. Similar to the analytical model, each of them uses assets supplied by a household (with wealth a and skill type ϑ) to produce x intermediate goods with technology

$$x_j^N(a,\vartheta) = A[(1-\omega(a,\vartheta))a]^{\nu_f}, \qquad x_j^I(a,\vartheta) = (\phi\mu + A(1-\mu))[\omega(a,\vartheta)a]^{\nu_r}, \tag{15}$$

where $\phi \geq A$ holds and $\nu_r, \nu_f > 0$ are returns to scale on the technologies. Similar to section 2, μ is a wedge capturing the extend to which wealth returns on risky investments reflect the associated capital productivity. The intermediate producer sells units of intermediate goods to a final good producer at price $p_j^s(a,\vartheta)$ and, for a given project, obtain revenues $\Pi_j(a,\vartheta) = \sum_s p_j^s \cdot x_j^s(a,\vartheta)$. For each project, revenues are redistributed to households. Moreover, recall that intermediate producers do not face any risk, but investors are subject to the investment shock κ.

A competitive final good producer uses labor L and intermediate goods $X = \sum_s \left(\int_j x_j^s dj\right)$ to produce final goods with technology $Y = F(X, L)$, where $F(\cdot)$ satisfies the Inada conditions. Profit maximization, $\max_{x_j^s, L} Y - \sum_s \int_j p_j^s x_j^s\, dj - wL - \delta X$, yields the following prices: $p_j^s = \frac{\partial F(X,L)}{\partial X}\frac{\partial X}{\partial x_j^s} - \delta$, and $w = \frac{\partial F(X,L)}{\partial L}$, where $\delta \in (0,1)$ is the depreciation rate. As intermediate goods are homogeneous, it follows that $p_j^s = p\ \forall j, s$.

[25] Another version assumes that κ enters additively the risky component. Results are unchanged in this version.

[26] The outer expectation comes from the fact that y and κ are *iid*. An alternative way to write this value function is

$$v(a,\vartheta,h,\kappa,y) = \max_{c>0,\,a'\geq 0,\ell\geq 0}\left\{u(c,\ell)+\beta(1-d(h))\mathbb{E}_{\kappa',y',h',\vartheta'|h,\vartheta}\left[v(a',\vartheta',h',\kappa',y')\right]\right\}. \tag{14}$$

Given equations (15), the return wedge and the profit maximization, the returns to riskless and risky assets are given by

$$r_F := \frac{px^N(a,\vartheta)}{(1-\omega(a,\vartheta))a} = MPK_F = pA[(1-\omega(a,\vartheta))a]^{\nu_f-1}\,, \tag{16}$$

$$r_R := \frac{px^I(a,\vartheta)}{\omega(a,\vartheta)a} = p\phi[\omega(a,\vartheta)a]^{\nu_r-1} \geq MPK_R = p(\phi\mu + A(1-\mu))[\omega(a,\vartheta)a]^{\nu_r-1}\,, \tag{17}$$

where $\mu < 1$ implies $r_R > MPK_R$. It thus describes the case in which risky wealth returns do not only reflect investment productivity, but for example some forms of rent-extraction.

Government The government finances an external government expenditure level G using four tax instruments: capital income taxes T_r, bequest taxes T_b, wealth taxes T_a, and labor income taxes T_w. Capital income and labor income are respectively subject to a proportional tax, i.e. $t_r(\mathrm{x}) = \mathrm{x}\tau_r$, respectively $t_w(\mathrm{x}) = \mathrm{x}\tau_w$. There is no wealth tax in the baseline economy. We will however introduce a progressive wealth tax in the subsequent experiments. Specifically, we will assume that wealth is taxed only above a threshold of wealth $\underline{a}_{tax}$ at a top marginal wealth tax rate τ_a.[27] This implies $t_a(\mathrm{x}) = \mathbb{1}_{\mathrm{x}\geq\underline{a}_{tax}}\tau_a(\mathrm{x}-\underline{a}_{tax})$. The bequest tax is proportional, such that $t_b(\mathrm{x}) = \tau_b(\mathrm{x} - \tau_a(\mathrm{x}))$,where we assume that the wealth tax is paid first.

3.2 Equilibrium

Definition 2. *Denote the state space by $\boldsymbol{s} = (a,\vartheta,h) \in \mathbb{S} \equiv \mathbb{R}^+ \times \Theta \times \mathbb{H}$. A steady-state equilibrium of this economy is a vector of quantities $\{Y, X, L\}$, a set of policy functions $\{c(\boldsymbol{s}), a'(\boldsymbol{s}), \ell(\boldsymbol{s})\}$ a set of prices $\{\underline{r}, p, w\}$, a set of tax rates $\{\tau_w, \tau_r, \tau_b, \tau_a, \underline{a}_{tax}\}$, a probability distribution of households $\mathcal{G}$ defined over $\mathbb{S}$, such that: (1) the representative final producer maximizes profits,* $\max_{\{X,L\}} F(X,L) - pX - wL - \delta X$, *where p and w are given by their respective marginal products; (2) given prices, households solve the stationary version of their decision problem (14), giving rise to an invariant distribution $\mathcal{G}(\boldsymbol{s})$;[28] (3) the government budget constraint*

$$\begin{aligned} G =& w\tau_w \int_y \int_{(a,\vartheta,h)} \ell(h,y)n\, d\mathcal{G}(a,\vartheta,h)dy + \tau_r \int_{(a,\vartheta,h)} \sum_\kappa r(a,\vartheta,\kappa)a\, \pi_\kappa(\kappa)\, d\mathcal{G}(a,\vartheta,h) \\ &+ \int_{(a,\vartheta,h)} \tau_a(a) d\mathcal{G}(a,\vartheta,h) + \tau_b \int_{(a,\vartheta,h)} d(h) t_b(a) d\mathcal{G}(a,\vartheta,h)\,, \end{aligned} \tag{18}$$

[27] This type of wealth taxation is used in almost all countries that implements or implemented a wealth tax (e.g. Switzerland, France, Denmark, Norway).

[28] At each state (a,ϑ,h), there is a continuum of individuals experiencing the *iid* shock y and κ. The stationary distribution is obtained using non-stochastic simulations.

is satisfied. (4). labor market clear, i.e. $L = \int_y \int_s \mathcal{H}(h) y \ell(s) d\mathcal{G}(\boldsymbol{s}) dy$, *and intermediate goods market clear,*

$$X = \int_{(a,\vartheta,h)} \Big(A[(1-\omega(a,\vartheta))a]^{\nu_f} + (\phi\mu + A(1-\mu))[\omega(a,\vartheta)a]^{\nu_r} \Big) \, d\mathcal{G}(a,\vartheta,h), \tag{19}$$

and (5) the total capital product distributed by intermediate producer for each project $\Pi_j(a,\vartheta)$ *is consistent with the total capital income received by households,*

$$pX = \int_{(a,\vartheta,h)} \sum_{\kappa} \Big(\underline{r} + r_F(a,\vartheta)(1-\omega(a,\vartheta)) + r_R(a,\vartheta)\kappa\omega(a,\vartheta) \Big) a\pi_\kappa(\kappa) d\mathcal{G}(a,\vartheta,h). \tag{20}$$

Moreover, our measure of total wealth in this economy is given by $K = \int a(\boldsymbol{s}) \, d\mathcal{G}(\boldsymbol{s})$.

Condition (19) states that total capital efficiency units used in the production sector must correspond to the total capital supplied by households given their investment choice between risky and riskless assets. When $\mu < 1$, each unit of household risky investment yields wealth returns higher than its corresponding marginal product. As such, the equilibrium base return $\underline{r}$ must be negative to satisfy condition (20). Therefore, besides X/L that pins down p and w, the return component $\underline{r}$ in equation (11) is the second aggregate object that adjusts in equilibrium.

Numerical solution The model admits no analytical solution. We solve it numerically using a version of the endogenous grid method (Carroll, 2006). We discretize the κ, y shocks into bins using Gauss-Hermite quadratures. Under certain calibrations (for example, the one of the benchmark economy), the model induces a Pareto distribution of wealth. As a consequence, the support of the stationary distribution of wealth is unbounded from above. To circumvent this issue in practice, we use a large value for the upper bound on wealth in our numerical implementation. To check the size of the error implied by this truncation, we estimate the Pareto tail of the model-generated wealth distribution. Suppose that the wealth distribution is Pareto above the a_ξ threshold (with a mass $\mathcal{G}_\xi$ of households accounting for K_ξ/K of the total wealth). At the top, for $a \geq a_\xi$, the Pareto tail is inversely related to the wealth share $q(\xi; \eta_\xi) = \xi^{1-1/\eta_\xi}$ of the ξ% wealthiest households. We first estimate $\hat{\eta}_\xi$ using observations from the top of the model implied wealth distribution, and then use it to compute the economy-wide wealth share given by $q(\xi; \hat{\eta}_\xi)(K_\xi/K)$. We finally compare the top 0.1% wealth share generated by the truncated model with the one implied by the estimated Pareto tail and find that the difference is negligible.[29]

[29] The details of the estimation procedure are similar to the one implemented in Appendix C.2.

4 Taking the Model to the Data

We map the stationary equilibrium of the quantitative model into the US data in two steps. We first fix some parameters based on model-exogenous information. We then calibrate the remaining parameters endogenously by numerically simulating the model to match moments in the data.

4.1 Capital Heterogeneity: A First Glance into the Data

We begin with a discussion of the data used to calibrate and validate the model.

In all data sources, our concept of wealth is *net worth*, defined as the marketable value of all assets minus total liabilities. Assets comprise riskless assets (deposits, savings, cash), direct and indirectly held public equities (such as in mutual funds and individual retirement account), net private equity business investments, primary and secondary residences, and other non-financial assets. Liabilities comprise student loans, mortgages, consumer credits and other loans.

We use the SCF waves from 1989-2019 with detailed information on households' portfolio composition, comprising a number of very wealth-rich households. When computing moments related to wealth inequality, we will sometimes refer to the *adjusted SCF*. The adjustments are made following the method of Vermeulen (2016) for our entire sample period. First, we correct for underreporting of assets by adjusting survey estimates of real assets, financial assets, and liabilities such that they align with national balance sheets. Second, we adjust for under-representation at the top by merging the SCF with households in the Forbes rich lists and use it to compute wealth shares based on estimates of a Pareto Law for wealth. The procedure is detailed in Appendix C.2.

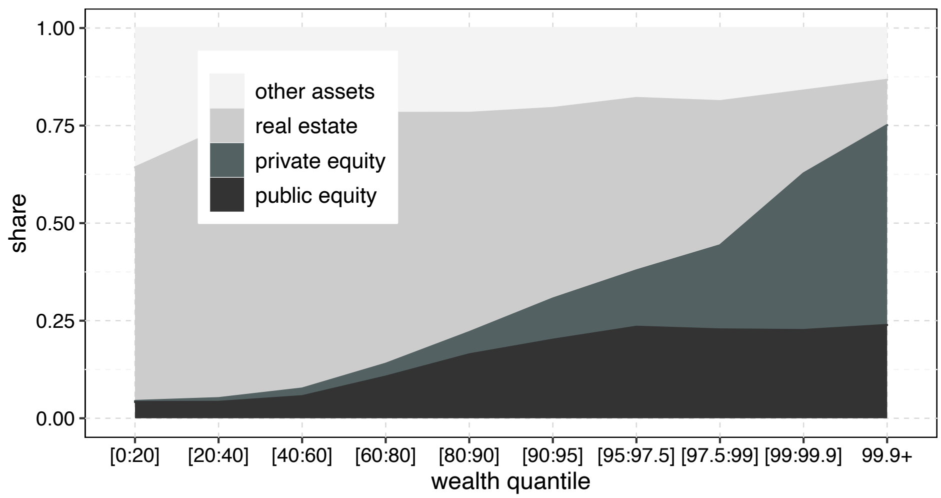
We also use the PSID (1998-2018) as it is a large and nationally representative biennial survey with a long panel dimension, containing details on capital income and costs (by broad classes of assets), asset prices, inflows and outflows for each asset.[30] We restrict to the population aged 20 to 70. Our definition of wealth is net worth and we consider risky equity (private and public assets) as "productive risky assets" while other assets (combining riskless assets, residential properties and other non-financial assets) are considered as safe productive assets.[31] This classification is not specific to this paper, Cagetti and De Nardi (2006) separate private equity entrepreneurial assets from other capital invested in a corporate firm, and Kaplan et al. (2018) consider only equity and commercial or business real estate as productive assets.

[30] See Pfeffer et al. (2016) for an excellent comparison between both surveys and Flavin and Yamashita (2002) for a discussion about returns estimated from the PSID. Appendix C.1 provides further details. Of interest, with its oversampling, the SCF's total capital estimate is close to the one in national accounts. A drawback of the PSID is the presence of only a small number of households at the very top (within the top 1%) of the wealth distribution.

[31] Our motivation for classifying housing in safe assets comes from the fact that returns from housing assets are less volatile than other assets. Moreover, their volatility relates mostly to differential mortgage interest rates.

Portfolio composition. In Figure 2 we first use the SCF data to get a full *cross-sectional* picture of the average household's portfolio composition across the US wealth distribution for our sample period. As already intensively documented in the literature, private and public equity investments strongly positively correlate with wealth in the cross-section. The top 1% in the US hold on average 65% of wealth into risky equity, while the corresponding share for the median household is 7%.[32] The noticeable increase in risky equity share at the very top of the distribution (above the 95th percentile) is mostly driven by assets held in private equity business investments.

Figure 2. Average portfolio share of gross assets by wealth percentile.



Source: adjusted SCF from 1989 to 2019, averaged from the different SCF imputations.

Returns to wealth. We then measure wealth returns in the US using the PSID. The full details regarding their construction (and robustness) are provided in Appendix C.1. We provide a measure for four asset classes: total net worth, safe assets (savings, bonds, checking accounts), financial risky assets (public equities), and non-financial risky assets (private equity businesses), with $l \in \{total, safe, fin, bus\}$. Return on asset l for household i in year t is given by

$$r_{i,l,t} = \frac{R^K_{i,l,t} + R^I_{i,l,t} - R^D_{i,l,t}}{(a^g_{i,l,t-2} + a^g_{i,l,t} + F_{i,l,t})/2}, \tag{21}$$

where $a^g_{i,l,t-2}$ and $a^g_{i,l,t}$ are the (positive) amount of assets l held in the previous $(t-2)$ and current wave (t) and $F_{i,l,t}$ are inflow minus outflow (net investment), that we divide by two due to the biennial nature of the sample. The values $R^K_{i,l,t}$ and $R^I_{i,l,t}$ and $R^D_{i,l,t}$ correspond respectively to capital gains, income (dividends and payments) and to the cost of debts (if any). Table 1 provides de-

[32]This is comparable to estimates using detailed Swedish administrative data according to Bach et al. (2020), who use a similar definition of risky assets.

scriptive statistics regarding returns to wealth in the PSID. Notably, private equity returns display the highest expected returns (15.6%) with substantial heterogeneity and skewness to the right. To a lower extend, public equity generates also substantial returns (5.8%). Despite the absence of very wealthy households in the PSID, our results are comparable to estimates in Fagereng et al. (2020) in Norway and Bach et al. (2020) in Sweden using administrative data. As a direct comparison with US estimates, Xavier (2020) evaluates, using cross-sectional information in the SCF, that aggregate returns are 13.6% for private equity, 6.4% for public equity, and between 0.4%-2.1% on the different safe assets.[33] A noticeable difference, however, is that our aggregate estimate of returns on net worth (before-tax) is 3.3%, substantially lower than the one evaluated by Xavier (2020) (6.8%) but closer to the ones estimated using a quantitative structural model of inequality by Benhabib et al. (2019) (3.1%) and to the empirical estimates in Fagereng et al. (2020) for Norway (3.8%). In comparison to Sweden, Bach et al. (2020) find a median return to net worth of 4.5% with a standard deviation of 13% per year. We attribute this discrepancy to the fact that the PSID does not account for the upper end of the wealth distribution and to using different methodologies.

Table 1. Wealth returns in the PSID (2000-2018).

WEALTH COMPONENT	DESCRIPTIVE STATISTICS					
	Mean	St.Dev.	Skewness	Kurtosis	P20	P80
Net worth (before-tax)	0.033	0.158	0.897	6.243	-0.035	0.089
Private equity	0.156	0.614	2.071	10.967	-0.225	0.500
Public equity	0.058	0.417	-0.122	0.085	-0.248	0.385
Safe assets	0.004	0.009	3.234	10.267	0.000	0.003

Note: we apply a trimming of 0.5% at the top and the bottom for each asset class.

Figure 3 shows the before-tax wealth returns by wealth quantile in the PSID.[34] Wealth returns strongly correlate with wealth in the cross-section. In Appendix C.1, we decompose the relation and find that this apparent correlation is not observed within asset class. Therefore, the increase in total returns is likely to be driven by heterogeneity in the household's portfolio composition, as documented above.[35] This feature is also consistent with existing work establishing the positive correlation between private equity ownership and wealth (Quadrini, 2000; Cagetti and De Nardi,

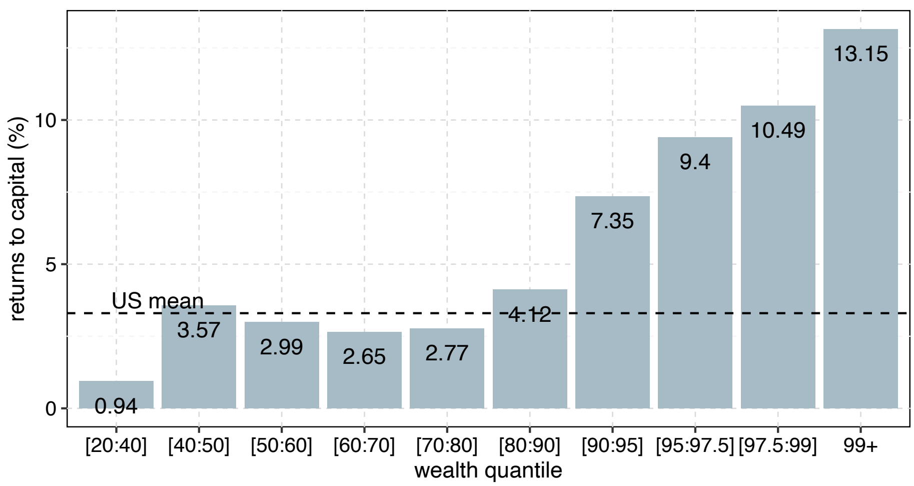
[33]Her methodology is very different from ours as we use the panel dimension of the PSID to compute returns of a given household over time, while she computes returns from the SCF waves in the cross-section using information on capital income and stocks, averaged by wealth percentile. A drawback of her analysis is sample selection, as individuals may in principle move in and out of a given wealth percentile over time, as shown in Gomez et al. (2018). A drawback of our analysis is the under-representation of the very wealthy in the PSID.

[34]In Appendix C.1, we perform a robustness using after-tax wealth returns. The pattern turns out to be similar.

[35]Such an observation is shared by Bach et al. (2020) while Fagereng et al. (2020) find that there is still a substantial heterogeneity within a broad asset class, which may reflect a key role for heterogeneity in skills. In fact, by focusing on a broad asset class, it is difficult to disentangle whether increased returns are due to specific skills or to higher risk-taking within a broad asset class. For private equity investments, this may arise due to diversification Penciakova (2018) or due to the interaction between business risk-taking and borrowing limits (Robinson, 2012).

2006). Of course, those numbers are not informative on whether it is *type* or *scale* dependence that drives the correlation. In practice, there is no obvious way to disentangle the two as both mechanisms are likely to drive the observed cross-sectional pattern, as we demonstrate below.

Figure 3. Average return on net worth by gross wealth quantile.



To sum up, the increasing share of risky assets at the top of the wealth distribution is substantially driven by private equity holdings, which also exhibit the highest wealth returns. The positive correlation between wealth returns and wealth appears to be mostly driven by the differing portfolio composition between rich and poor individuals.

4.1.1 Scale dependence in capital investments in the US economy

Exploiting the panel dimension of the PSID and controlling for individual characteristics, Hurst and Lusardi (2004) find evidence for *scale* dependence in the propensity to select into private equity business ownership among the top 5% wealthiest households. According to their estimates, above the top 5%, the probability to enter business ownership increases by 0.5 percentage points in normalized wealth: reaching around 4% for the 95th percentile and 7% for the 98th percentile. In comparison, for the bottom 80%, the average probability is flat, at 3%.

Using the information on returns on the capital endowments of US universities, Piketty (2018) (Chapter 12) finds that returns substantially increase with wealth and argue that it may come from economies of scale in portfolio management.[36] However, the wealth holdings of US universities are substantially larger than most households in the US (above 10 million dollars), and may be only representative of the behaviors at the upper end of the wealth distribution.

As an additional piece of evidence, Bach et al. (2020) use Swedish administrative panel data and test for *scale* dependence in wealth returns. They argue that, even within a sample of twins

[36]In Piketty's argument: "*The most obvious one is that a person with 10 million euros rather than 100,000, or 1 billion euros rather than 10 million, has greater means to employ wealth management consultants and financial advisors.*".

and controlling for twin-pair fixed effects, there is evidence of strong *scale* dependence, especially increasing at the top of the wealth distribution (Table 9, p 2738). They argue that *scale* dependence is likely driven by changes in the individual portfolio composition due to, for example, returns to scale on management costs or decreasing relative risk aversion behavior. Fagereng et al. (2020) use a Norwegian administrative panel on wealth tax records and regress the average wealth return on the individual's wealth percentile at the beginning of the period, with individual and year fixed effects. Both wealth (scale) and individual fixed effects are found statistically significant. Their estimates imply that *scale* dependence alone explains 48% of the 18 percentage point return difference between the 10% and 90% net worth percentiles.

Finally, Robinson (2012) shows that wealthier business owners tend to start relatively riskier private equity business investments, with higher expected profitability. Penciakova (2018) confirms a similar pattern with data on US firms using the Census Bureau's Longitudinal Business Database, patenting data, and Compustat. She documents that business owners who diversify tend to start riskier additional businesses. This diversification among business owners occurs in the data, but mostly among the wealthiest households in the US, as shown below.

All in all, this leads us to conclude that both *type* and *scale* dependencies are likely to drive the observed correlation between wealth returns and wealth.

4.2 Calibration

We now detail the calibration of the benchmark model and two alternatives that allow us to study the properties of *type* and *wealth* dependence. The benchmark model aims to capture the key features of the SCF and PSID described above. Specifically, we will parsimoniously account for *wealth* effects on the extensive margin (participation in risky investment) and on the intensive margin of risky portfolio choice. From this benchmark, two alternatives will be designed as follows. The first refers to the one with only *wealth* dependence (*Scale*-model) in line with the information acquisition model of Peress (2004) and the incomplete markets models of Meeuwis (2019) and Hubmer et al. (2020). In this alternative, we neutralize all possible *type*-dependence, and scale effects are recalibrated to match key moments. The second alternative refers to the one with only skill-*type* dependence (*Type*-model), in which we nullify *wealth* effects, such that we capture a stylized version of the capitalist/entrepreneur framework along the lines of Moll (2014) or Gomez et al. (2016).

4.2.1 Exogenously set common parameters

Preferences and technology The model is calibrated to the US economy and the period is one year. Preferences are given by a CRRA utility $u(c) = \frac{c^{1-\sigma_1}}{1-\sigma_1} - \chi \frac{\ell^{1+\sigma_2}}{1+\sigma_2}$. We set $\sigma_1 = 2.5$, and $\sigma_2 = 1.7$

following Brüggemann (2021). The death probability is set to $d(h) = 1.5\% + 0.0045\,h$, which implies a probability of death of 6% in the last income bracket and aims to capture the increasing death rate over the life cycle.

We specify $F(X, L) = X^{\alpha} L^{1-\alpha}$ and set $\alpha = 0.33$ and the depreciation rate to $\delta = 5\%$. We normalize $A = 1$. In the baseline, we set $\mu = 1.0$ (perfect pass-through between returns and MPK) and study later cases with $\mu < 1$ when studying the aggregate effects of wealth redistribution.

As already stressed above, micro returns to scale on investment (ν_r, ν_f) is another important wealth dependence channel on returns. To estimate this parameter, we use our PSID sample and estimate the effect of asset holdings $a_{i,l,t}$ on asset returns $r_{i,l,t}$, fixing the broad class of asset l, such that: $\log(r_{i,l,t}) = \beta_l \log(a_{i,l,t}) + FE_i + FE_t + \epsilon_{i,l,t}$, where β_l is an estimate of the returns to scale and FE_i and FE_t stand for household and year fixed effects. If β_l does not statistically differ from zero, returns exhibits CRS. Our results suggest that there is no significance for either IRS nor DRS, even for private equity business holdings. This result is supported by Bach et al. (2020). However, it contrasts with the variety of entrepreneurs/capitalists models assuming DRS technology on private equity businesses investment (Cagetti and De Nardi, 2006; Brüggemann, 2021; Guvenen et al., 2019). Therefore, as CRS is not a standard assumption, we will also investigate the case where $\nu_r < 1$ out of robustness reasons.

Tax system The labor income and capital income tax rates are set to $\tau_w = 22.5\%$ and $\tau_r = 25\%$, consistent with Guvenen et al. (2019). The bequest tax rate is fixed to $\tau_b = 40\%$. The baseline economy does not feature any wealth tax, therefore $t_a(x) = 0$.

Labor income process The processes for labor productivity aim to generate a realistic earning distribution and contribute to the overall wealth inequality. Starting with the persistent process h, we set the threshold of the Pareto Law to $q_h = 0.9$ and the shape to $\eta_h = 1.9$, consistent with 1990-2010 estimates for the US (Piketty and Saez, 2003). In line with Storesletten, Telmer and Yaron (2004), parameters of the persistent component of labor productivity are $\rho_h = 0.95$ and $\sigma_h = 0.2$.[37] The correlation between parents' labor productivity with the one of their heir is $p_h = 0.35$, consistent with Chetty et al. (2014). Finally, the expected lifetime is set to $M = 70$.

The transitory process follows $y \sim Log\text{-}\mathcal{N}(0, \sigma_y)$ where $\sigma_y = 0.15$, consistent with Heathcote et al. (2010). As Hubmer et al. (2020), we assume an unemployment state $y_0 = \overline{b}$ occurring with probability $\pi_y(y_0) = 7.5\%$, independently of (y, h) and over time, with $\overline{b}$ set to 0.4, in the range of

[37] For the sake of transparency, we reduce the computational burden by using a reduced transition matrix $\hat{\Pi}_h(h'|h)$ such that: $\hat{\Pi}_h(h'|h) = \begin{cases} \Pi_h(h'|h) & \text{if} \quad \Pi_h(h'|h) \geq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$ with $\epsilon = 10e^{-6}$ and normalizing the matrix $\sum_{h'} \hat{\Pi}_h(h'|h) = 1$. This allows us to exploit the sparsity of the transition matrix.

the unemployment replacement rate in the US.

4.2.2 Calibrating Capital Heterogeneity and Wealth Returns

We now calibrate the portfolio allocation and variables associated with wealth returns. The variance of risky returns is important as it adds to the dispersion of returns, and may generate substantial wealth mobility and wealth inequality (Gabaix et al., 2016; Hubmer et al., 2020). We set $\sigma_\kappa = 0.50$, which is in between our estimate for public and private equity from the PSID displayed in Table 1. As stressed earlier, the key adjustment margin at the top of the wealth distribution in terms of portfolio allocation arises from private equity business investments. We will therefore focus mostly on this margin for our calibration of the portfolio share $\omega(a, \vartheta)$.

***Benchmark* model** A difficulty that arises when distinguishing *type* and *wealth* dependence is that in order to identify carefully the distinction, one would need a large and long panel data on individual investment behavior. This requires to test for the dependence of portfolio allocation or wealth returns by controlling for individual characteristics and fixed effects. Following this, Fagereng et al. (2020) and Bach et al. (2020) use administrative panel data on wealth tax records. They provide strong support that returns feature both *type* and *wealth* dependence, with a crucial role of portfolio composition. However, the results are conditioned by the statistical model used to estimate *type* and *scale* dependence, e.g. a linear model with fixed effects (type) and wealth percentiles (scale).

We pursue another strategy that consists in recognizing that there are two common ways to generate *scale* dependence in models featuring private or public equity investments. On one hand, the extensive margin decision to invest, or to participate, might be wealth-dependent. This is the case in many occupational choice models in the presence of borrowing constraint (Cagetti and De Nardi, 2006; Brüggemann, 2021) or models with fixed participation cost (Fagereng et al. (2017)). Potential investors switch to an investor state only when their wealth level is high enough. On the other hand, once investing in equity, there is potentially an intensive margin effect such that the portfolio share of investors is itself a function of wealth.

To capture both margins, we first assume that there are two skill-ϑ types, $\vartheta \in \{\vartheta_1 = 0, \vartheta_2 = 1\}$, i.e. those with investment skills, and those without investment skills.[38] We let the probability to switch from unskilled to skilled investor type be a function of wealth. Specifically, the transition

[38]The results are qualitatively similar if we calibrate a version of the model with more than two types who differ in their propensity to invest in risky assets.

matrix associated with the skill process ϑ is given by:

$$\pi_\vartheta(\vartheta'|\vartheta, a) = \begin{bmatrix} 1 - \overline{\pi}_\vartheta - \lambda(a) & \overline{\pi}_\vartheta + \lambda(a) \\ \underline{\pi}_\vartheta & 1 - \underline{\pi}_\vartheta \end{bmatrix} . \tag{22}$$

The components $\underline{\pi}_\vartheta$ and $\overline{\pi}_\vartheta$ capture the switching probabilities that are unrelated to wealth. This may account for time-variations in skills (notably life-cycle aspects) or preferences, that are independent of household wealth. In the data, it should be noted that the fraction of public equity investors is substantially larger than the fraction of private equity holders. Moreover, the margin of adjustment at the top of the distribution is mostly due to private equity investments (Figure 2). As such, households investing in public equity and who do not invest in private equity are counted as investor only when they hold more than 50% of their wealth into public equity. Therefore, a household with 10% of her wealth in public equity and 90% in safe assets is not considered as a skilled investor in our economy. The exit probability $\underline{\pi}_\vartheta$ is estimated to be 15% in the PSID. The probability $\overline{\pi}_\vartheta$, instead, is adjusted such that the fraction of investors in the economy is 13% given the calibration of $\underline{\pi}_\vartheta$ and $\lambda(a)$.

The function $\lambda(a)$ is used to match the increasing probability of entering into the investor state, that we proxy in the data by the increasing probability of participating into private equity business investments. In practice, we approximate $\lambda(a)$ using a piecewise function. For wealth threshold $\mathbf{T} = [t_1 \cdots t_T]$ and nodes $\mathbf{N} = [n_1 \cdots n_T]$ we define the piecewise function $\mathcal{S}(a; \mathbf{T}, \mathbf{N})$ as:

$$\mathcal{S}(a; \mathbf{T}, \mathbf{N}) = \begin{cases} n_1 & \text{if } a < t_1 \\ n_1 + (n_2 - n_1)\left(\frac{a - t_1}{t_2 - t_1}\right) & \text{if } a \in [t_1, t_2[\\ \vdots & \\ n_{N-1} + (n_N - n_{N-1})\left(\frac{a - t_{N-1}}{t_N - t_{N-1}}\right) & \text{if } a \in [t_{N-1}, t_N[\\ n_N & \text{if } a \geq t_N \end{cases} . \tag{23}$$

The probability to participate in equity investment is an increasing function of wealth, such that $\lambda(a) = \mathcal{S}(a; \mathbf{T}_\lambda, \mathbf{N}_\lambda)$ where $\mathbf{T}_\lambda$ are set to particular wealth quantiles in the model and the $\mathbf{N}_\lambda$ are chosen such that the model entry rates across the wealth distribution match with their estimates in the PSID, based on the empirical study of Hurst and Lusardi (2004).[39] This piecewise portfolio function is intended to capture the possible highly non-linear dependence between the probability of selecting into risky equity investment and wealth along the wealth distribution. We detail the

[39] Notice that we provide similar results using recent PSID waves. We relegate the full analysis to Appendix C.3. Our approach is of course a short-cut. It can be however micro-founded with the presence of fixed costs to start a private equity business investment, or to participate in public equity investment.

resulting piecewise parameters in Appendix D.1.3.

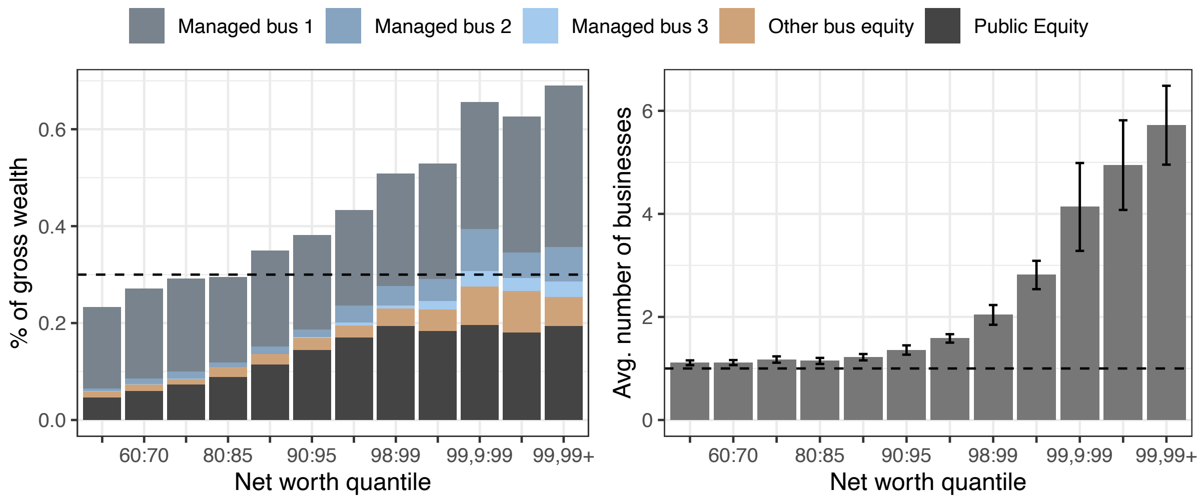
We calibrate the intensive margin of equity investment, conditional on being equity investor, as follows. In the SCF, conditioning on business ownership, the share of private equity to net worth increases along the wealth distribution (left panel of Figure 4). One practical issue, however, is to distinguish whether the share of private equity held by those owners increases because of systematic unrealized capital gains, which induces no investment decision, or whether it is the result of net inflows. Moreover, even if it results from additional investments, it is difficult to disentangle whether business owners who invest more are more likely to select at the top of the wealth distribution, or whether wealth itself induces higher investment. To circumvent those issues, we exploit detailed information on the number and the timing of private equity business investments in the SCF, including the share of wealth in each business and their acquisition dates.

Figure 4 shows the average share of private equity investments per net worth quantile, decomposed in different business investments (limited to the biggest three and others). In the left panel, the average share of private equity investment over total gross wealth appears to be strongly correlated with wealth (above the top 5%). This relation is driven by diversification at the top. From our standpoint, this pattern cannot be solely driven by *type* dependence. First, borrowing constraints are likely to prevent relatively wealth-poor households to invest in multiple businesses, limiting the concern regarding the possible reverse causality that an owner of multiple businesses is more likely to select over time at the top of the wealth distribution. Second, those additional businesses are generally newly founded businesses: 85% of the second businesses were created in the past ten years relative to the survey date and 67% within the past five years. As a comparison, 47% of the first main business of those private equity owners were created within the past ten years. Therefore, given that wealth accumulation is a slow process, it seems unlikely that multiple business owners at the top became rich due to their multiple private equity investments. Instead, the decision to open additional private equity investments is likely, among other determinants, to be wealth-driven.[40]

We further elaborate on this point using two additional pieces of evidence. First, as shown in the right panel of Figure 4, the average *number* of private equity business investments substantially rise at the upper end of the wealth distribution, consistent with our previous findings. Again,

[40] The result is not due to household composition. Even when focusing on single households, private equity investments increase with wealth through diversification. Moreover, additional businesses are different than the first established business: 70% of additional private equity investments are made in a different sector. Moreover, they appear only slightly more represented in finance-related industry, limiting the concern that it constitutes a financial affiliate company of the main business. Those facts challenge the widely held view that business owners are poorly diversified (Moskowitz and Vissing-Jørgensen (2002)) and are consistent with recent findings in Penciakova (2018). Diversification occurs, but only at the very top. In numbers, 12% of business owners own multiple managed businesses in the US. In the top 1%, they are 40%.

Figure 4. Average share held in equity (left panel) and average number of private equity business investments (right panel), conditioning on investors.



Source: SCF (1989-2019). The dashed line in the left panel indicates the average private equity share. The dashed line in the right panel indicates "one private equity business investment".

zooming at the timing of those additional businesses, we find that the last acquired business is particularly recent relative to the main business. Finally, despite the lack of observations at the very top in the PSID, we use its panel dimension and confirm that investment in additional private equity business investments, conditional on already being a business owner and on individual characteristics, is statistically positively correlated with net worth. To save on space, we defer this additional empirical evidence to Appendix C.4.

We attribute the part of the observed increase in equity investments due to diversification in private equity investments in Figure 4 (left panel) to *wealth* dependence in the model. We view this choice as conservative as it constitutes a lower bound on the effect of wealth on the equity share of investors. We denote respectively $\mathbf{T}_\omega$ and $\mathbf{N}_\omega$ the thresholds and share parameters of the piecewise portfolio function chosen to match the increasing equity portfolio share in the data, such that $\varpi(a) = \mathcal{S}(a; \mathbf{T}_\omega, \mathbf{N}_\omega)$. Given this, we specify:

$$\omega(a, \vartheta) = \vartheta(\underline{\omega} + \varpi(a)), \tag{24}$$

where $\underline{\omega} = 0.35$ is the fairly stable average share invested in equity outside the top 10% wealthiest households, conditional on being investor, as shown in the left panel of Figure 4.[41]

[41] In an alternative calibration strategy, we used the longitudinal information of returns to wealth in the PSID to calibrate portfolio shares $\omega(a, \vartheta)$ such that they are consistent with the shape of returns to net worth. However, there are important limitations to this approach. It is difficult, given the small number of observations in the PSID, to test for the non-linearity of *wealth*-dependence at the very top of the distribution. Moreover, evidence suggests that *wealth* dependence occurs in both the extensive and the intensive margins of equity investments, thus introducing important

As discussed above, we now describe two alternatives in which the correlation between the average portfolio share and wealth is driven in isolation by *type* or *wealth* dependence.

***Scale* model** In this alternative, skill-type heterogeneity is irrelevant. We rather assume that heterogeneity in risky capital investment is solely driven by *wealth* dependence. Specifically, all households in the economy now invest in risky capital (i.e. $\vartheta = 1$ for all households), but the amount depends solely on the wealth level. Therefore, $\underline{\omega}$ is set to 0 while $\mathbf{T}_\omega$ and $\mathbf{N}_\omega$ are recalibrated to match the average portfolio of risky equity observed in the SCF data of Figure 2.

***Type* model** In this alternative, there are no *wealth* effects: neither on the propensity to be selected as an investor, nor on the portfolio shares. As such, the wealth-dependent propensity to select as an investor $\lambda(a)$ is set to 0, and the fixed transition probability $\overline{\pi}_\vartheta$ is recalibrated to match the fraction of investors in the benchmark model. Furthermore, the wealth-dependent portfolio component $\varpi(a)$ is set to 0, and $\underline{\omega}$ is recalibrated to match the average economy-wide portfolio share in the data. As investment skill types are persistent, i.e. $\pi_\vartheta(\vartheta' = 0|\vartheta = 1) < 0.5$, the model with only *type* dependence will generate an endogenous positive correlation between wealth returns and wealth along the wealth distribution. This is the case as long as skilled-investor obtain, on average, higher returns on their wealth.

4.2.3 Other endogeneously picked parameters

The discount factor β helps replicate a capital-output ratio $\frac{K}{Y}$ of 2.6. The excess return parameter ϕ helps to generate the top 1% wealth share of 35% by controlling the dispersion in terms of wealth returns between households. Together with the piecewise threshold parameters $\mathbf{T}_\lambda$ and $\mathbf{T}_\omega$ which are endogenously chosen such that the model is able to match portfolio decisions and investment participation along the wealth distribution, we have to calibrate X parameters to match Y targets. Appendix **??** shows the model parameters and resulting moments.

Sensitive analysis Finally, it is worth noting that we explored the qualitative and quantitative implications of introducing heterogeneity in discount factors similar to Krusell and Smith (1998) and a calibration allowing for a positive amount of household debt. Those alternatives leave the conclusion of this paper materially unchanged.

interactions which we capture in our specification.

5 Properties of the Models

We now discuss key properties of the calibrated models regarding: (i) overall wealth inequality, portfolio choice, and wealth returns (section 5.1), (ii) mobility and persistence of wealth-rank (section 5.2), and (iii) the response of output to a wealth inequality shock (section 5.3).

5.1 Wealth Inequality and Wealth Returns

As demonstrated in the analytical framework of Section 2, it is important that the model captures well the shape of the wealth distribution, especially at the very top, as it conditions the relative strength of *wealth* and *type* dependence effects. In Table 2, we first assess the models' ability to generate an overall wealth distribution relative to its SCF counterpart. A striking result is that, outside the top 1% wealth share which is targeted, the benchmark model and the *Type* and *Scale* alternatives (respectively in the 1^{st}, 2^{nd} and 3^{rd} row) account remarkably well for the observed top wealth shares. This result may explain why Benhabib, Bisin and Luo (2019) (Table 9) find that including *wealth* dependence in wealth returns in a model featuring already *type* dependence in wealth returns does not provide further explanatory power (and are thus badly identified). Put differently, *wealth* and *type* dependence may be hardly distinguishable from their ability to generate high wealth concentration.

Table 2. Wealth distribution: data (1998:2019) and models.[a]

	Gini	Share of wealth (in %) held by the top x%						
		40	20	10	5	1	0.1	0.01
US data (World Inequality Database)	–	97.6	85.1	70.8	57.8	35.7	18.2	8.1
US data (adjusted SCF with Forbes World's Billionaires)[b]	0.80	97.2	86.4	72.7	59.7	37.2	17.8	7.3
1. *Benchmark* model	0.74	89.5	78.6	66.5	54.6	35.0	21.1	12.4
2. *Scale* model	0.81	93.1	85.1	75.5	62.4	35.2	16.1	7.3
3. *Type* model	0.75	90.4	78.5	66.5	55.1	35.1	19.2	10.2
Decomposition of the benchmark model								
4. – no pure *type* dependence, $\underline{w} = 0$	0.61	83.8	65.8	50.1	35.0	13.1	2.7	0.7
5. – no scale in portfolio, $\varpi(a) = 0$	0.69	87.0	73.2	59.3	46.4	24.3	10.8	5.0
6. – no scale in entry, $\lambda(a) = 0$	0.72	88.9	76.2	62.6	50.8	30.7	16.7	9.2
6. – no scale, $\lambda(a) = \varpi(a) = 0$ [TO DO]								
7. – no idiosyncratic risk, $\sigma_\kappa = 0$	0.71	86.1	75.4	61.0	50.2	29.6	15.9	8.9
8. – no portfolio heterogeneity	0.56	79.9	60.5	42.7	29.3	9.5	1.5	0.2

[a] The top one percent wealth share is targeted.
[b] Adjusted for under-representation and underreporting following the procedure in Vermeulen (2016). See Appendix C.2 for details.

This high concentration of wealth can be traced back to the large heterogeneity in wealth returns implied by the different equity portfolio allocations between households. Table 3 shows that consistent with estimates from different data sources (columns (1) to (4)), our three models (benchmark, *Type*-model and *Scale*-model) produce an average return to wealth which is increasing along the wealth distribution. In the *Type*-model (column (6)), this is driven by selection only. As risk-taking ϑ-type are persistent, households with a high propensity to invest in equities have higher expected returns for several periods and are thus more likely to be selected at the top of the distribution, hence driving the observed cross-sectional relationship.[42] In the *Scale*-model (column (5)), the relationship is generated by construction, such that a higher level of wealth is associated with higher risk-taking and higher expected returns. In the *benchmark* model (columns (9) to (11)), both *wealth* and *type* dependence drive the observed pattern. To see this, we compute in the last two columns the contribution of *wealth* and *type* dependence by reporting the average returns to wealth across the wealth distribution assuming that the type component $\underline{\omega} = 0$ and assuming no scale dependence in the intensive margin, i.e. $\varpi(a) = 0$. Note that we do not recompute the model in those cases but simply report the part of returns explained by difference in types ($\underline{\omega}$) and the part explained by the intensive margin in portfolio choice ($\varpi(a)$).[43] We find that both forces contribute significantly to the observed increase in average returns across the wealth distribution. This result is supported by aforementioned empirical evidence in Fagereng et al. (2020) and Bach et al. (2020) who find evidence that wealth returns are driven by both *wealth* dependence and *type* dependence. Finally, and again consistent with the data, the cross-sectional standard deviation of returns to wealth sharply increases with net worth due to the high idiosyncratic risks in equity investments.[44] This implies that in the *Hybrid*-model, the standard deviation of returns to wealth is 12.5%, against 15% in the PSID.

Decomposition In Table 2, we then ask how much *scale* dependence, *type* dependence and *luck* (the idiosyncratic risk κ) contribute to the observed steady-state wealth inequality in the calibrated *benchmark* model. To answer this question, we decompose their respective role. Specifically, we shut down one component at a time and then recompute the steady-state stationary distribution without this component, keeping everything else unchanged. A first counterfactual assume no pure *type* dependence (column (4)), such that only investment driven by wealth are taken into account, i.e. $\underline{\omega} = 0$. Second, we investigate the effects of *wealth* dependence by shutting down first the intensive margin component $\varpi(a) = 0$ (column (5)), and then the extensive margin component $\lambda(a) = 0$ (column (6)). Third, we assume no risk on equity investment, such that $\sigma_\kappa = 0$ (column (7)). Fourth, we remove all components linked to heterogeneous investments (column (8)), such

[42] See Benhabib et al. (2011) and Moll (2014) for a theoretical illumination on the role of persistence in capital returns.

[43] It is however difficult to isolate the role of scale in the probability to select as an investor, as it would require to recalibrate a model.

[44] Moreover, notice that the role of private equity is substantial in Fagereng et al. (2020): *"All in all, heterogeneity in our most comprehensive measure of returns to wealth can be traced in the first place to heterogeneity in returns to private equity and the cost of debt and only partially to heterogeneity in returns to financial wealth."*.

Table 3. Mean wealth returns across the wealth distribution: data and model.

<table>
<tr><th rowspan="3">Wealth group</th><th colspan="4">Data[a]</th><th>Scale-model</th><th colspan="3">Type-model</th><th colspan="3">Benchmark model[b]</th></tr>
<tr><th rowspan="2">PSID</th><th rowspan="2">SCF</th><th rowspan="2">Norway</th><th rowspan="2">Sweden</th><th rowspan="2">base.</th><th rowspan="2">base.</th><th colspan="2">with DRS</th><th rowspan="2">base.</th><th rowspan="2">scale $\underline{\omega}$=0</th><th rowspan="2">type $\varpi(a)$=0</th></tr>
<tr><th>base.</th><th>scale</th></tr>
<tr><th></th><th>(1)</th><th>(2)</th><th>(3)</th><th>(4)</th><th>(5)</th><th>(6)</th><th>(7)</th><th>(8)</th><th>(9)</th><th>(10)</th><th>(11)</th></tr>
<tr><td>P40-P50</td><td>REF.</td><td>REF.</td><td>REF.</td><td>REF.</td><td>REF.</td><td>REF.</td><td>REF.</td><td>REF.</td><td>REF.</td><td>REF.</td><td>REF.</td></tr>
<tr><td>P50-P60</td><td>−0.6</td><td rowspan="2">−0.4</td><td rowspan="2">1.5</td><td>0.2</td><td>0.0</td><td>0.4</td><td>0.9</td><td>−0.1</td><td>0.3</td><td>0.0</td><td>0.3</td></tr>
<tr><td>P60-P70</td><td>−0.9</td><td>0.3</td><td>0.0</td><td>1.0</td><td>2.4</td><td>−0.2</td><td>0.6</td><td>0.0</td><td>0.6</td></tr>
<tr><td>P70-P80</td><td>−0.8</td><td>0.0</td><td rowspan="2">2.6</td><td>0.3</td><td>0.0</td><td>2.0</td><td>4.4</td><td>−0.4</td><td>1.0</td><td>0.0</td><td>1.0</td></tr>
<tr><td>P80-P90</td><td>0.5</td><td>0.2</td><td>0.5</td><td>0.7</td><td>3.1</td><td>6.2</td><td>−0.6</td><td>1.6</td><td>0.0</td><td>1.6</td></tr>
<tr><td>P90-P95</td><td>3.8</td><td>1.4</td><td>3.6</td><td>0.8</td><td>2.6</td><td>3.2</td><td>5.6</td><td>−0.9</td><td>1.6</td><td>0.7</td><td>1.5</td></tr>
<tr><td>P95-P97.5</td><td>5.8</td><td>2.6</td><td rowspan="2">5.2</td><td>1.1</td><td>3.8</td><td>3.1</td><td>8.1</td><td>−1.0</td><td>1.7</td><td>2.1</td><td>1.3</td></tr>
<tr><td>P97.5-P99</td><td>6.9</td><td>3.8</td><td>1.5</td><td>5.4</td><td>5.2</td><td>9.2</td><td>−1.2</td><td>3.2</td><td>2.9</td><td>2.5</td></tr>
<tr><td>Top 1%</td><td>9.6</td><td>4.6</td><td>8.3</td><td>2.5</td><td>7.3</td><td>6.7</td><td>8.1</td><td>−1.5</td><td>5.7</td><td>4.2</td><td>4.3</td></tr>
<tr><td>Top 0.1%</td><td>–</td><td>–</td><td>–</td><td>3.7</td><td>8.6</td><td>6.1</td><td>6.4</td><td>−1.8</td><td>6.0</td><td>5.0</td><td>4.2</td></tr>
</table>

Note: REF. stands for reference wealth bracket. Average returns are computed as the difference to the REF.
[a] Estimates are our own for the PSID. They are taken from Xavier (2020) for the SCF, from Bach et al. (2020) for Sweden and from Fagereng et al. (2020) and Halvorsen et al. (2021) for Norway.
[b] The returns are computed assuming $\underline{\omega} = 0$ in the no type model and $\varpi(a) = 0$ in the no scale model.

that the wealth distribution is driven by heterogeneity in labor income only. We report the results in the last rows of Table 2. We find that *wealth* and *type* dependence are both important components of wealth inequality, as evidenced by the lower top wealth shares generated under those counterfactuals. By generating large dispersion in returns, the *luck* component κ explains part of the wealth concentration. As expected, a model without heterogeneity in capital investments fails to account for the high wealth concentration observed in the data. As previously shown by Benhabib et al. (2011), in such cases, the tail of the wealth distribution inherits the tail of the labor income distribution. In the model, the Pareto tail of the labor income distribution is $\eta_h = 1.9$, a value much higher than the 1.4 estimated for wealth using the adjusted SCF.

Comparison with alternative models We showed that a model with positive *type* and *wealth* dependence can replicate the distribution of returns, while at the same time it also replicates well the distribution of wealth. Yet, one may still wonder how other frameworks can replicate their joint distribution. We provide the results for an capitalist/entrepreneur type of model along the lines of Cagetti and De Nardi (2006) and Guvenen et al. (2019), where the key difference stems from the assumption of decreasing returns to scale (DRS) on private equity investments. Apart from the latter, those economies allow for limited *wealth* dependence in private equity investment, often capped to a borrowing constraint which is proportional to wealth.[45] The comparison is worth noting as it constitutes one of the most used framework featuring heterogeneity in household

[45]To be precise, those frameworks may feature additional *wealth* dependence in the selection into private equity business investments through an occupational choice (worker versus entrepreneur). A given entrepreneur, however, can invest at most a fixed fraction of her wealth, $k \leq \lambda a$, where λ captures the tightness of the borrowing constraint. In Cagetti and De Nardi (2006), this borrowing limit is endogenous, but turns out to be quasi-linear in wealth.

capital investments in the literature. To do this, we first assume that investor-type invest all their wealth in private equity, $\underline{\omega} = 1$. Second, we assume that the equity returns are non-risky, $\sigma_\kappa = 0$, and that the innovative technology features DRS with $\nu_r = 0.9$.[46] We refer to this model as a *Type*-model with DRS and calibrate it to match the top 1% wealth share and the K/Y ratio using the excess return ϕ and the discount factor β. It turns out that the *wealth* dependence shifts sign due to the DRS specification, and the overall shape of returns across the wealth distribution is now hump-shaped. Such pattern is not observed in the data. Focusing on private-equity business returns, there is no evidence of DRS in Fagereng et al. (2020) nor in Bach et al. (2020).[47]

5.2 Wealth Mobility

Benhabib et al. (2019) stress that including *wealth* dependence in wealth returns slightly improve the fit of the social mobility matrix in their structural quantitative model relative to the US economy. We investigate how the intra-generational wealth mobility matrix in our model alternatives compares with its empirical counterpart taken from Klevmarken et al. (2003) who estimate a five-state (quintiles) five-year transition matrix from the 1994–1999 PSID waves. Table 4 reports the results for the *benchmark* economy, the *Type*-model and the *Scale*-model. We find that the three models over-state the persistence of wealth-rank in the top quintiles while being broadly consistent with the U-shaped diagonal transition rate. This discrepancy may be due to measurement issues in the data or to the absence of deterministic age states in the models. Interestingly, adding portfolio heterogeneity helps in generating an empirically consistent wealth mobility. Overall, we find that it is hard to distinguish the three models based on the resulting wealth mobility matrix.

Table 4. Wealth mobility: data and model

5-years transition	Diagonal element (quintile – quintile)				
	Q1 – Q1	Q2 – Q2	Q3 – Q3	Q4 – Q4	Q5 – Q5
PSID (1994-1999), Klevmarken et al. (2003)	0.58	0.44	0.42	0.48	0.71
Scale-model	0.59	0.32	0.32	0.55	0.89
Type-model	0.63	0.37	0.38	0.56	0.87
Hybrid-model	0.54	0.38	0.40	0.60	0.88
No portfolio heterogeneity	0.71	0.51	0.60	0.73	0.97

[46] In our view, this specification is close to the traditional entrepreneur/capitalist type of model where entrepreneurs invest the total amount of their assets in private equity business investments subject to a DRS technology. This induces a maximum amount of equity that a household would like to optimally invest. This maximum is never reached under our parameterization. Notice that the DRS is often imposed as the relevant assumption in the firm dynamics literature. Gaillard (2021) shows that DRS on the firm side and CRS on the households side can be reconciled once we take into account diversification in multiple private equity investments at the very top of the wealth distribution.

[47] In Bach et al. (2020), there is a slight decrease in private equity returns with wealth due to leverage effects.

5.3 Wealth Inequality Shock: the Role of *Type* and *Scale* dependence

We now analyze the response of the model to a wealth inequality shock. The exercise we perform is to compute the long-run steady-state adjustment of the economy following a 1% top marginal wealth tax on the wealth-rich. We set the marginal tax rate to $\tau_a = 1\%$, and the wealth threshold above which wealth is taxed, i.e. $\underline{a}_{tax}$ to the wealth level corresponding to the top 1% wealthiest household. The goal is to give a sense on how our different model alternatives locate relative to a situation in which inequality is *growth irrelevant* or neutral (c.f. the GIF represented in Figure 1) depending on the strength of *type* and *scale* dependence in household investments.

Table 5 shows the results. While each model produces consistent wealth inequality patterns, they differ greatly in terms of output response to a wealth inequality shock. Moving from the *Scale* model (column (1)) to the *Type* model (column (2)) reduces the output response from -1.92% to -0.89%, by more than one half. This is because under *scale*-dependence, risky asset holdings (and thus future wealth) are a function of the current wealth level. If wealth is taxed, it does not only change saving decisions but also how households allocate their wealth between risky and safe assets, generating a dynamic self-enforcing behavioral multiplier. The *benchmark* model (column (4)) falls in between the two alternatives, with a response of -1.12%. In the *Type* model, the response of wealth inequality (proxied by the top 1% wealth share) is substantially lower, mostly because individuals keep on investing a given share of their wealth into equity, and thus continue to experience high capital returns. [missing discussion selection] This shows that accounting for an appropriate degree of *type* and *scale* dependence is quantitatively important.

We also report several other models. The *Type* model with DRS that aims to represent the capitalist/entrepreneur models (column (3)) produces, consistent with previous results, a lower output-response of -0.58%. As such, those models may understate the effect of wealth redistribution on the aggregate, specifically because the marginal product of capital in those models is a decreasing function of agent's wealth and risky private equity business investment. Column (5) and (6) provide the results in a recalibrated version of the *benchmark* economy in which we remove respectively the intensive margin in portfolio choices, setting $\varpi(a) = 0$, and the type dependence, setting $\underline{\omega} = 0$. Again, a higher degree of scale dependence is related to stronger output-response. Finally, a model without portfolio heterogeneity (column (7)) produces a slight reduction in output of -0.18% and is much closer to growth neutrality. Without capital heterogeneity, the response is only related to change in overall wealth accumulation (the aggregate stock of capital K). This result is consistent with the standard Aiyagari (1994) type of model, in which effects of wealth redistribution are significantly lowered due to the quasi-linearity of the saving decisions for individuals sufficiently far away from the borrowing constraint.

Rent extraction If $\mu < 1$. TO DO. missing discussion

Table 5. Long-run response (in %) to a 1% top marginal wealth tax

		Scale-model	*Type*-model		*Hybrid*-model [a]			no portfolio heterogeneity, $\omega(a,\vartheta)=0$
			$\nu = 1.0$	$\nu = 0.9$	base.	no scale $\varpi(a) = 0$	no type $\underline{\omega} = 0$	
		(1)	(2)	(3)	(4)	(5)	(6)	(7)
$\mu = 1$	ΔY	-1.92	-0.89	-0.58	-1.12	-0.99	-1.53	-0.18
	Δtop1%	-21.9	-11.9	-7.2	-13.6	-12.4	-16.3	-8.99
	$\Delta(X/K)$	-4.44	0.8		-0.1			
	$\Delta \underline{r}$	–	–	–	–	–	–	–
$\mu = 0$	ΔY							
	Δtop1%							
	$\Delta(X/K)$	–	–	–	–	–	–	–
	$\Delta \underline{r}$							

[a] model without scale or type dependence are recalibrated to match the top 1% and the K/Y ratio.

Empirical counterpart It is interesting to connect our model estimates with empirical cross-country evidence. In Online Appendix E.1, we estimate the link between a change of top wealth shares and subsequent output growth. We find that an increase by 1 percentage point (pp) of the top 1% wealth share is associated with a 0.27 increase in subsequent GDP growth, averaged over the next 5 years.[48] Using estimates in Table 5, the association between the change of the top 1% wealth share and long-run GDP has a similar order of magnitude. Recall that in the benchmark and the two alternatives, the initial top 1% wealth share is 35%. Therefore, in the type and scale dependence models, a 1 pp decrease in the top 1% wealth share is respectively associated with a decrease in long-run GDP of 0.24% and 0.23%. In contrast, in absence of portfolio heterogeneity (7th column), the relation falls by an order of magnitude. In this model, the top wealth share is 22% in the initial steady-state, meaning that a 8.99% fall in the top 1% wealth share corresponds to a 2 pp decrease. Therefore, a decrease by a 1 pp of the top 1% wealth share is associated with a 0.09% decrease in long-run GDP. Consequently, heterogeneity in portfolio choice and associated returns brings the model responses of a wealth inequality change closer to its empirical counterpart.

Interestingly, this also indicates that the benchmark choice of $\mu = 1$ regarding the pass-through between capital productivity and wealth returns does not seem to generate extreme results regarding the association between wealth inequality and GDP growth. Of course, we acknowledge that those empirical results may be subject to measurement errors.

The results above show that the degree of *scale*/*type* dependence in capital investments together with the return wedge μ condition the aggregate response to top wealth redistribution. In

[48]Note that we do not claim any causal relationship. Even in the model, a negative wealth shock at the top of the distribution affects inequality and GDP, and their combined change in turn feeds back into inequality and GDP. Therefore, it is hard to identify causality based on our simulated results.

the subsequent section, we show the key interacting role that they play in determining how to optimally tax wealth.

6 Wealth Taxation, Endogenous Selection and Scale effects

The foregoine discussion highlighted the interplay between wealth inequality, the degree of *type* versus *scale* dependence, and aggregate responses to wealth redistribution. Recently, Guvenen et al. (2019) and Boar and Midrigan (2020) have studied the role of wealth taxation in general equilibrium with capitalists/entrepreneurs with almost no role for *scale* dependence (apart from DRS in private equity investments). Notably, return heterogeneity breaks the equivalence between a capital income tax and a wealth tax that holds under homogeneous returns. In this section, we assess the normative implications of taxing the stock of wealth with *type* and *scale* dependence. Notably, we show that a wealth tax has non-trivial effects on the aggregates depending on the degree of *type* and *scale* dependence.

We start by describing our thought experiment: a wealth tax on the wealthiest households in the economy. The main tradeoffs at play are those identified in Section 2. The wealth tax balances (i) equity by reallocating wealth from the top to the bottom, (ii) efficiency as wealth-rich households trigger potentially important general equilibrium effects on productivity and thus wages, and (iii) rent-extraction, as whenever $\mu < 1$ higher amount of risky investments lead to an overall decrease in the return component $\underline{r}$. We show that the degree of *type* versus *scale* dependence, together with the degree to which returns to wealth reflect productivities of capital investments, are pivotal to understand the effects of wealth redistribution, and therefore to derive the welfare-maximizing wealth tax.

Thought experiment The objective of the government is to impose a wealth tax on the top 10% wealthiest households in the economy. It chooses optimally the top marginal wealth tax (henceforth TMWT), denoted τ_a, by adjusting the labor income tax τ_w to balance the government budget defined in condition (18).[49]

We measure welfare using the consumption-equivalent variation. It is constructed as follows. After solving for the stationary equilibrium of a specific tax reform, we compute the variation Δ^{CEV} of consumption that makes every newborn households in state $\mathbf{s} \in \mathbb{S}$ in the post-reform economy as well-off as in the pre-reform economy. With this criterion, aggregate welfare in the post-reform steady state, $\mathcal{W}^{post}(\tau_a)$, has to be equal to aggregate welfare in the initial status quo steady state $\mathcal{W}^*$ without wealth tax, where optimal consumption has been changed by Δ^{CEV} per-

[49] In Appendix F.5, we provide the results for a wealth tax without exemption threshold and using alternative tax instrument to balance the government budget constraint and using alternative threshold $\underline{a}_{max}$.

cent, such that

$$\mathcal{W}^{post}(\tau_a) = \mathcal{W}^{pre}(\tau_a = 0, \Delta^{CEV})$$

$$\int_{\mathbb{S}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \widehat{\beta}^t u\Big(c_t^{post}(\mathbf{s}), \ell^{post}(\mathbf{s})\Big) \right] \Gamma^{post}(\mathbf{s}) = \int_{\mathbb{S}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \widehat{\beta}^t u\Big((1+\Delta^{CEV}) c_t^{pre}(\mathbf{s}), \ell^{pre}(\mathbf{s})\Big) \right] \Gamma^{pre}(\mathbf{s}),$$

where $\widehat{\beta} = \beta(1 - d(h))$. This utilitarian welfare measure is widely used in the quantitative macroeconomics literature, see among others Conesa et al. (2009), Brüggemann (2021), Guvenen et al. (2019). As an alternative measure, we also tried to aggregate the micro consumption-equivalent variation $\Delta^{CEV}(\mathbf{s})$ of each individuals between the status-quo and the policy reform, using the initial stationary distribution $\Gamma(\mathbf{s})$. The results are qualitatively similar. Finally, note that our experiment is run under the assumption that the portfolio allocations respond to the wealth tax only through its direct effects on saving decisions. Moreover, we do not claim that we capture all the welfare effects associated with the wealth tax. In reality, the welfare gains depends on the underlying factor behind *wealth* and *type* dependence, especially if differences in preferences (rather than skills) play a key role. Therefore, our welfare criterion should be reinterpreted as a representative utilitarian welfare criterion that allows us to compare the different studied economies.

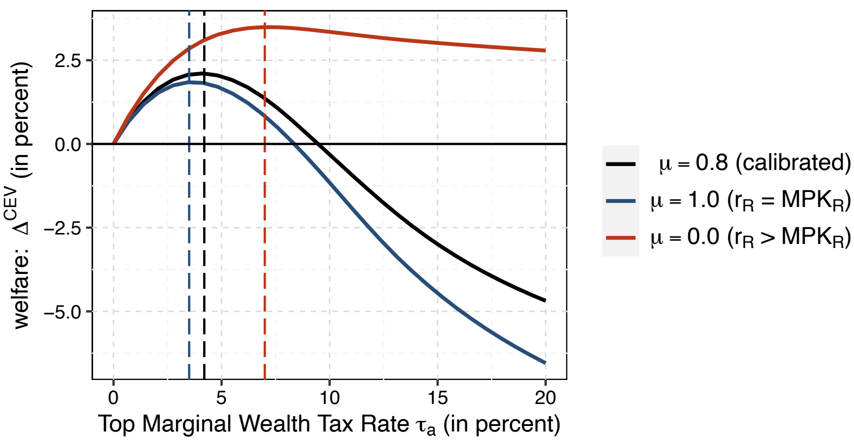
Subsequently, we perform the analysis under the two polar assumptions that differences in wealth returns reflect differences in the marginal product of capital ($\mu = 1$), and when they are disconnected from the marginal product due to full rent-extraction ($\mu = 0$).

6.1 Optimal Top Marginal Wealth Tax Rate (TMWTR)

We first assess the effects of a wealth tax in the *benchmark* model. Figure 5 displays the welfare gains from varying the top marginal wealth tax. We find that the welfare gain is maximal for a positive TMWTR of 3.5% when the return wedge $\mu = 1$ (perfect pass-through between MPK and returns) and of 7.5% when $\mu = 0$ (full rent-extraction). Therefore, the TMWTR is *decreasing* with the degree μ to which wealth returns reflect the productivity of capital investments. ADD the path of tax (i.e. the sensitivity).

To what extend do differential returns reflect the productivity of capital investments? The answer to this question is particularly challenging, as it would require data allowing to link household capital investments to their corresponding productivity. To illustrate how a realistic amount of rent-extraction may modify the resulting TMWTR, we report the case where the parameter μ is calibrated such that wealth returns obtained from capital investments made into law and finance sectors are attributed to rents. We motivate this choice following Lockwood et al. (2017) who report that those two sectors produce negative externalities on aggregate income when their respective aggregate income share increases. Using the SCF microdata, households equity investments in law and finance sectors account for 15% to 20% of the total equity investment. We therefore attribute 20% of equity investments to rents, and set $\mu = 0.8$.[50] Under this calibration, we find that the optimal TMWTR, identified using the black line in Figure 5, significantly increases relative to the case without rent-extraction, at $\tau_a = 4.2\%$, and leads to sizable welfare gains.

Figure 5. Welfare measure as function of τ_a, the case of the *benchmark* model.



Note: the results are derived by comparing the welfare measure among different long-run stationary equilibrium in which the wealth tax rate τ_a varies. The labor income tax rate τ_w balances the government budget constraint.

6.1.1 Understanding the distinct role of *scale* and *type* dependence.

To understand the key forces behind the result derived above, we now investigate what would be the optimal wealth tax rate in the only *Type* and *Scale* model.

***Scale* model** In Figure **??**, we show the welfare gains/losses associated with a given TMWTR (expressed in percent). When returns to wealth are driven by *scale*-dependence and reflect differences in the productivity of capital, $\mu = 1$, we find that the TMWTR is numerically zero (in fact, the optimal wealth tax is even negative do it). This result is mostly driven by the high long-run elasticity of aggregate output to the TMWTR under this specification, which mirrors the aforementioned dynamic self-enforcing mechanism. A wealth tax lowers the wealth accumulated by the wealth-rich households, translating into lower risky investments, less wealth accumulation in the future, and so on and so forth. This induces a large reduction in aggregate productivity and thus on the

[50]The share of equity in those two sectors display significant heterogeneity across the wealth distribution. Notably, it is particularly important at the top of the wealth distribution, reaching around 22% within the top 1% wealthiest households. In an alternative experiment, we capture this non-linearity by assuming that the return wedge is wealth dependent, such that $\mu(a) = \mu_1 a^{\mu_2}$, where μ_1 and μ_2 are calibrated to match the corresponding fraction of capital investments made in law and finance sectors across the wealth distribution.

equilibrium wage rate w. At the end, the loss implied by this large response counter-balance the welfare gains induced by the redistribution from lower labor income tax.

When returns reflect full rent-extraction, such that $\mu = 0$, then the welfare-maximizing TMWTR increases substantially, around 10%. In that case, taxing wealth-rich households decreases the overall rent extracted and rises the equilibrium return component $\underline{r}$, benefiting the whole population. This pushes toward higher wealth accumulation (except at the top) which further benefits the whole economy.

As a result, under *scale*-dependence, the TMWTR is *decreasing* in the degree to which returns to capital reflect their marginal product.

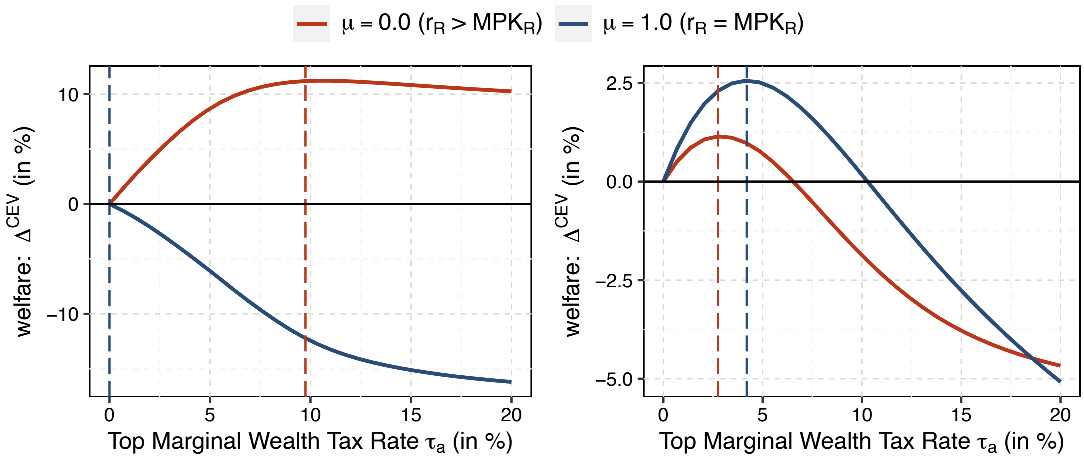
***Type* model** Figure 6 shows the welfare gains/losses as a function of the TMWTR in the *type*-dependence model. We find that under the two polar cases regarding the pass-through between the productivity of capital and returns, μ, the TMWTR is positive and significant. Why this is the case? To understand the mechanics, it is important to see that when types are persistent, households with high returns are more likely to be represented at the top of the wealth distribution as they experience during several periods high returns on their wealth. When the wealth-rich are taxed on their capital stock, those with high returns to wealth are relatively less affected by the TMWTR than those with low returns to wealth. In fact, individuals with low capital income returns desaccumulate faster than those with high capital returns. Therefore, the wealth tax selects individuals with high returns even further at the top of the distribution, and generates downward wealth mobility for those with low returns. In the end, the wealth tax has the effect of rebalancing aggregate capital in the hand of households with higher propensity to take risk.

When returns reflect differences in the productivity of capital, such that $\mu = 1$, the optimal TMWTR is $\tau_a = 4.2\%$. In that case, the selection effect limits efficiency losses from taxing the wealth-rich households by reallocating wealth to households investing in risky and more productive assets, while the redistribution channels generate welfare gains for wealth-poor households. This result is in line with the *use-it-or-lose-it* mechanism highlighted in Guvenen et al. (2019), who argue that replacing the capital income tax with a wealth tax is welfare improving in a framework in which differential returns are primarily driven by some individuals entitled with the ability to run private equity business investments (that is, under *type* dependence).

In contrast, the results flip side when differential returns do not reflect differences in the productivity of capital but some form of rent-extraction ($\mu = 0$). The positive selection of high capital income at the top of the wealth distribution has the tendency to increase overall rent-extraction, i.e. $\underline{r}$ decreases even further. As such, it is optimal to lower the TMWTR to avoid selecting rent-extractors at the top of the distribution.

As a result, under *type*-dependence, the TMWTR is *increasing* in the degree to which returns to capital reflect their marginal product.

Figure 6. Welfare measure as function of τ_a, the case of *type* and *wealth* dependence



The results are derived by comparing the welfare measure within different long-run stationary economy in which the wealth tax rate τ_a varies. The labor income tax rate τ_w balances the government budget constraint.

Overall, our results show that distinguishing *type* and *scale* dependence in wealth returns has important normative implications on the optimal wealth redistribution, especially when returns do not necessarily equalize the productivity of capital. In the benchmark model, the scale effects are strong enough to make the wealth tax decreasing in the return wedge μ.

We hope that the discussion derived here clarifies an important distinction regarding the optimal wealth redistribution between *scale* dependence, which induces a dynamic self-enforcing multiplier, and *type* dependence, which induces selection effects that are endogenous to the tax change. Finally, to save on space, we do not report a number of additional results. Specifically, Appendix F.5 contains results where we used the capital income tax τ_r or a lump-sum transfer to balance the government budget constraint. Additionally, we provide various experiments in which we optimally choose the labor income tax τ_w and the wealth tax τ_a (assuming $A_\tau = 0$), or the labor income tax τ_w and the capital income tax τ_r. In most cases, the resulting optimal redistribution highly depends on the degree of *scale* or *type* dependence in risky investments, but the results are substantially distinct in the case of the wealth tax.

7 Conclusion

Whether these conditions are satisfied or not remains an empirical question.

In this paper, we derived a general and unified representation of the response of aggregate output to wealth redistribution, unraveling and clarifying the economic forces behind a large class of heterogeneous agents incomplete markets models that account for the empirically relevant heterogeneity in capital investments and returns to wealth. We find that four statistics are key: (i) the shape of the wealth distribution, (ii) the risk premium augmented by the degree to which differential returns to capital reflect differences in the productivity of capital investments, (iii) *scale* dependence effects, and (iv) the selection or sorting of individuals with different investment types across the wealth distribution.

Quantitatively, once endogeneizing the joint distribution of wealth and types, we first showed that both dependencies can account for the observed large concentration of wealth at the upper end of the wealth distribution. This concentration is generated by large returns associated to risky investments and is supported with microdata. Second, the aggregate response of wealth redistribution is sensible to the degree of *scale* dependence relative to *type* dependence. We find that the response is more than two times larger in an economy with only *scale* dependence relative to an economy with only *type* dependence. Under *scale* dependence, a dynamic self-enforcing multiplier is the key driving force behind the large response. Our hybrid model, calibrated with additional moments of the data, falls in between. Third, although the paper is positive in essence, we derive few normative points. Considering the case of a wealth tax, we show that the importance of *scale* and *type* dependence together with the degree to which wealth returns reflect the productivity of capital are pivotal to evaluate the effects of a wealth tax, conditioning the welfare-improving top marginal tax rate.

Future research, especially on the empirical front, is needed to understand the relationship between the joint distribution of wealth, returns, and portfolio allocation. Specifically, much more investigation is needed to disentangle *scale* and *type* dependence in agent's decision. Moreover, future research should attempt to empirically evaluate the pass-through between the productivity of capital and wealth returns.

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A Appendix

A.1 Appendix for Section 2

A.1.1 Final producer maximisation

For clarity, we detail the steps of the final good producer who maximizes the use of labor n and intermediate goods x_s^j, with

$$\max_{\{n, x_s^j\}} \left(\sum_s \int_j x_s^j \, dj \right) n^{\varphi} - wn - \sum_s \int_j p_s^j \, x_s^j \, dj \, .$$

Taking the first order condition with respect to labor gives $w = \varphi \left(\sum_s \int_j x_s^j \, dj \right) n^{\varphi - 1}$. Plugging this condition together with the assumption that $n = \int_i h^i di = 1$ into the profit function yields

$$\Pi^f = (1 - \varphi) \left(\sum_s \int_j x_s^j \, dj \right) n^{\varphi} - \sum_s \int_j p_s^j \, x_s^j dj = (1 - \varphi) \left(\sum_s \int_j x_s^j \, dj \right) - \sum_s \int_j p_s^j \, x_s^j dj \, .$$

A.1.2 Proof of Lemma 1

Proof. Because of $u'(c_2^i) > 0$, we know that the second period budget constraint holds in equilibrium with equality. Thus, second period consumption is given by $c_2^i = \underline{r} a_1^i + wh^i + R_f a_1^i + \omega_1^i a_1^i (R_f - R_r^i)$. Substituting from the wage rate $w = \varphi Y$ and returns R_f and R_r^i, we get

$$c_2^i = \underline{r} a_1^i + \varphi Y h^i + A(1 - \varphi)(1 - \omega_1^i) a_1^i + \kappa^i (1 - \varphi) \omega_1^i a_1^i \, .$$

We obtain that $c_2^i \sim \mathcal{N} \left(\mu_{c_2}^i, \sigma_{c_2}^i \right)$, with

$$\begin{aligned} \mu_{c_2}^i &= \underline{r} a_1^i + \varphi Y + A(1 - \varphi)(1 - \omega_1^i) a_1^i + \phi (1 - \varphi) \omega_1^i a_1^i, \\ \sigma_{c_2}^i &= \varphi Y \sigma_h^2 + \sigma_\kappa^2 (1 - \varphi)^2 (\omega_1^i a_0^i)^2 + 2 \rho_{\kappa,h} \sigma_\kappa \sigma_h \varphi Y (1 - \varphi) \omega_1^i a_0^i, \end{aligned}$$

□

A.1.3 Proof of Lemma 2

Proof. We first need to derive $\mathbb{E} \left[u(c_2^i) | \mathcal{I}_1 \right]$ analytically. To do so, we use an arbitrary Gaussian distribution with mean $\mu_{c_2}^i$ and variance $(\sigma_{c_2}^i)^2$. Using Lemma 1, we obtain

$$
\begin{aligned}
\mathbb{E}\left[u(c_2^i)|\mathcal{I}_1\right] &= \frac{1}{\alpha_i}\int\left(1-e^{-\alpha_i c_2^i}\right)\times\frac{1}{\sqrt{2\pi(\sigma_{c_2}^i)^2}}e^{-\frac{1}{2(\sigma_{c_2}^i)^2}\left(c_2^i-\mu_{c_2}^i\right)^2}dc_2^i \\
&= \frac{1}{\alpha_i}-\frac{1}{\alpha_i}\int\frac{1}{\sqrt{2\pi(\sigma_{c_2}^i)^2}}e^{-\frac{1}{2(\sigma_{c_2}^i)^2}\left[\left(c_2^i-\mu_{c_2}^i\right)^2+2\alpha_i(\sigma_{c_2}^i)^2c_2^i\right]}dc_2^i \\
&= \frac{1}{\alpha_i}-\frac{1}{\alpha_i}e^{-\alpha_i\mu_{c_2}^i+\frac{1}{2}\alpha_i^2(\sigma_{c_2}^i)^2}\int\frac{1}{\sqrt{2\pi(\sigma_{c_2}^i)^2}}e^{-\frac{1}{2(\sigma_{c_2}^i)^2}\left(c_2^i-(\mu_{c_2}^i-\alpha_i(\sigma_{c_2}^i)^2)\right)^2}dc_2^i
\end{aligned}
$$

Recognizing that the term in the integral is the pdf of a normally distributed random variable with mean $\mu_{c_2}^i-\alpha_i(\sigma_{c_2}^i)^2$ and variance $(\sigma_{c_2}^i)^2$, we finally obtain

$$
\mathbb{E}\left[u(c_2^i)|\mathcal{I}_1\right] = \frac{1-e^{-\alpha_i\mu_{c_2}^i+\frac{1}{2}\alpha_i^2(\sigma_{c_2}^i)^2}}{\alpha_i} \; .
$$

Under the additional set of assumptions within the special case section, we have $a_1^i \equiv a_0^i$ and $c_2^i = \varphi Y + R_f(1-\omega_1^i)a_0^i + R_r^i\omega_1^i a_0^i$ such that $\mu_{c_2}^i = \varphi Y + A(1-\varphi)(1-\omega_1^i)a_0^i + \phi(1-\varphi)\omega_1^i a_0^i$ and $\sigma_{c_2}^i = \omega_1^i a_0^i\sigma_\kappa(1-\varphi)$. We solve for the maximization problem given by

$$
\max_{\{\omega_1^i\}}\left[1-exp\Big\{-\alpha_i\Big(\mu_{c_2}^i-\frac{\alpha_i}{2}\sigma_{c_2}^i\Big)\Big\}\right]\alpha_i^{-1} \; ,
$$

Denoting $\mathbb{V} = exp\Big\{-\alpha_i\Big(\mu_{c_2}^i-\frac{\alpha_i}{2}\sigma_{c_2}^i\Big)\Big\}$, the corresponding first order condition is given by

$$
-\frac{1}{\alpha_i}\mathbb{V}\left[-\alpha_i(\phi-A)(1-\varphi)a_0^i+\alpha_i^2(a_0^i)^2\omega_1^i\sigma_\kappa^2(1-\varphi)^2\right]=0 \; ,
$$

which results after rearranging

$$
\omega_1^i = \frac{\phi-A}{(1-\varphi)\alpha_i\sigma_\kappa^2}(a_0^i)^{-1} = \frac{\phi-A}{(1-\varphi)\sigma_\kappa^2}\frac{\vartheta^i}{\overline{\vartheta}}(a_0^i)^{\gamma-1} \; .
$$

To ensure that the solution is indeed a maximum, we derive the second order condition as

$$
-\frac{1}{\alpha_i}\mathbb{V}\left[-\alpha_i(\phi-A)(1-\varphi)a_0^i+\alpha_i^2(a_0^i)^2\omega_1^i\sigma_\kappa^2(1-\varphi)^2\right]^2-\frac{1}{\alpha_i}\mathbb{V}\alpha_i^2\sigma_\kappa^2(1-\varphi)^2(a_0^i)^2<0 \; .
$$

which completes the proof. □

A.1.4 Proof of Proposition 3

Proof. The expression for aggregate innovative asset holdings follows straightforward from integrating over household dynamics while applying the covariance formula

$$\begin{aligned}
K_I = (\widetilde{\omega}/\overline{\vartheta})\mathbb{E}\left[\vartheta a_0^{\gamma}\right] &= \frac{\phi - A}{(1-\varphi)\overline{\vartheta}\sigma_\kappa^2}\left(cov(\vartheta, a_0^{\gamma}) + \mathbb{E}\left[\vartheta\right]\mathbb{E}\left[a_0^{\gamma}\right]\right) \\
&= \frac{\phi - A}{(1-\varphi)\overline{\vartheta}\sigma_\kappa^2}\left(\rho_{\vartheta, a_0^{\gamma}}\sigma_\vartheta\sigma_{a_0^{\gamma}} + \mu_\vartheta\mu_{a_0^{\gamma}}\right) .
\end{aligned}$$

Aggregate output is given by $Y = \left(\sum_s \int_j x_s^j \, dj\right) n^{\varphi}$, where $n = \int_i e^i \, di = 1$. Given that $K_N + K_I = \mathbb{E}[a_0]$, this can be rewritten after integrating over intermediate goods x_s^j as

$$\begin{aligned}
Y &= \int_j x_I^j \, dj + \int_j x_N^j \, dj \\
&= \left[\phi\mu + A(1-\mu)\right]\int_i \omega_1^i a_0^i \, di + A\int_i (1-\omega_1^i) a_0^i \, di \\
&= \mu(\phi - A)K_I + A\mathbb{E}[a_0] = \underbrace{\Big[\mu(\phi - A)(K_I/\mathbb{E}[a_0]) + A\Big]}_{:=Z}\mathbb{E}[a_0]
\end{aligned}$$

The price $\underline{r}$ ensures that total capital distributed to households coincide with the total revenue distributed by the intermediate producer, such that

$$\begin{aligned}
R_f \int_i a_1^i(1-\omega_1^i)di + \int_i R_r^i a_1^i \omega_1^i di + \underline{r}\int_i a_1^i di &= A(1-\varphi)\int_i a_1^i(1-\omega_1^i)di + (\phi\mu + A(1-\mu))(1-\varphi)\int_i a_1^i\omega_1^i di \, , \\
\phi(1-\varphi)K_I + \underline{r}\mathbb{E}[a_0] &= (\phi\mu + A(1-\mu))(1-\varphi)K_I \, , \\
\underline{r} &= (\mu - 1)(\phi - A)(1-\varphi)(K_I/\mathbb{E}[a_0]) \, .
\end{aligned}$$

where the last equality, regarding the integration of $\int_i R_r^i a_1^i \omega_1^i di$, follows from the simplifying assumption that there is a sub-continuum of households in each state (ϑ^i, a_0^i). □

A.1.5 Proof of Proposition 1

Proof. To prove the result regarding the effect of a mean preserving change in wealth inequality on K_I, we compare the aggregate innovate asset holdings for two economies with different Pareto tails $\eta' \neq \eta$ while keeping aggregate wealth $\mu_{a_0} = \mathbb{E}[a_0]$ constant. We then proceed by case distinction.

Case 1: $\rho_{\vartheta, a_0^{\gamma}} = 0$

The difference in aggregate innovative asset holdings between the two economies is written as

$$\begin{aligned}\Delta^w K_I(\eta',\eta) &= (\widetilde{\omega}/\overline{\vartheta})\mu_\vartheta\left(\frac{\eta'}{\eta'-\gamma}(\underline{a}')^\gamma - \frac{\eta}{\eta-\gamma}\underline{a}^\gamma\right)\\ &= \widetilde{\omega}\frac{\eta}{\eta-\gamma}\underline{a}^\gamma\left(\frac{\eta'}{\eta'-\gamma}\frac{\eta-\gamma}{\eta}\left(\frac{\underline{a}'}{\underline{a}}\right)^\gamma - 1\right) .\end{aligned}$$

Making use of the mean preserving assumption, i.e. $\underline{a}\frac{\eta}{\eta-1} = \underline{a}'\frac{\eta'}{\eta'-1}$, we obtain

$$\Delta^w K_I(\eta',\eta) = \widetilde{\omega}\frac{\eta}{\eta-\gamma}\underline{a}^\gamma\left(\frac{\eta'}{\eta'-\gamma}\frac{\eta-\gamma}{\eta}\left(\frac{\eta'-1}{\eta'}\frac{\eta}{\eta-1}\right)^\gamma - 1\right) .$$

Defining $\chi(\eta,\gamma) \equiv \frac{\eta-\gamma}{\eta}\left(\frac{\eta}{\eta-1}\right)^\gamma$ and taking the derivative of the inner expression w.r.t. η', we get

$$\chi(\eta,\gamma)\left(\frac{\eta'-1}{\eta'}\right)^\gamma\left[-\frac{\gamma}{(\eta'-\gamma)^2} + \frac{\gamma}{(\eta'-\gamma)(\eta'-1)}\right] = \chi(\eta,\gamma)\left(\frac{\eta'-1}{\eta'}\right)^\gamma\frac{\gamma(1-\gamma)}{(\eta'-\gamma)^2(\eta'-1)} .$$

As a result, we obtain finally

$$\frac{\partial\Delta^w K_I(\eta',\eta)}{\partial\eta'} = \widetilde{\omega}\mu_{a_0^\gamma}\chi(\eta,\gamma)\left(\frac{\eta'-1}{\eta'}\right)^\gamma\frac{\gamma(1-\gamma)}{(\eta'-\gamma)^2(\eta'-1)} .$$

The Lemma then follows by recognizing that $\frac{\partial\Delta^w K_I(\eta',\eta)}{\partial\eta'} = 0$ if $\gamma \in \{0,1\}$. Similarly, we obtain $\frac{\partial\Delta^w K_I(\eta',\eta)}{\partial\eta'} > 0$ if $\gamma \in (0,1)$ and $\frac{\partial\Delta^w K_I(\eta',\eta)}{\partial\eta'} < 0$ if $\gamma > 1$.

CASE 2: $\rho_{\vartheta,a_0^\gamma} \neq 0$

In the case of an arbitrary correlation between innate risk aversion types and wealth, we obtain

$$\Delta K_I(\eta',\eta) = \Delta^s K_I(\eta',\eta) + \Delta^w K_I(\eta',\eta) ,$$

where the first term denotes distributional relevance arising from the selection effect, whereas the second term resembles distributional relevance arising from wealth dependent risk taking. Notice that the latter is equivalent to Case 1. Contrary, the first effect can be written as

$$\Delta^s K_I(\eta',\eta) = (\widetilde{\omega}/\overline{\vartheta})\rho_{\vartheta,a_0^\gamma}\sigma_\vartheta\sigma_{a_0^\gamma}\left(\frac{\rho_{\vartheta,a_0^\gamma}(\eta',\underline{a}',\cdot)}{\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}\frac{\sigma_{a_0^\gamma}(\eta',\underline{a}')}{\sigma_{a_0^\gamma}(\eta,\underline{a})} - 1\right) .$$

Using the relation $\underline{a}\frac{\eta}{\eta-1} = \underline{a}'\frac{\eta'}{\eta'-1}$, a change in the Pareto tail η' preserves the mean wealth if $\underline{a}'(\eta') = \underline{a}(\frac{\eta}{\eta-1})\left(\frac{\eta'-1}{\eta'}\right)$. Using a first order Taylor approximation of $\rho_{\vartheta,a_0^\gamma}(\eta',\underline{a}'(\eta'),\cdot)$ around η, we obtain:

$$\rho_{\vartheta,a_0^\gamma}(\eta',\underline{a}'(\eta'),\cdot) \approx \rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot) + \left| \frac{\partial\rho^1_{\vartheta,a_0^\gamma}(\eta',\underline{a}'(\eta'),\cdot)}{\partial\eta} + \frac{\partial\rho^2_{\vartheta,a_0^\gamma}(\eta',\underline{a}'(\eta'),\cdot)}{\partial\underline{a}'(\eta')}\frac{\partial\underline{a}'(\eta')}{\partial\eta'} \right|_{\eta'=\eta} (\eta'-\eta)$$
$$\approx \rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot) + \left(\frac{\partial\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}{\partial\eta} + \frac{\partial\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}{\partial\underline{a}}\frac{\underline{a}}{\eta(\eta-1)} \right)(\eta'-\eta) \ .$$

Substituting the previous expression into the one for $\Delta^s K_I(\eta',\eta)$ we arrive at

$$\Delta^s K_I(\eta',\eta) \approx (\widetilde{\omega}/\overline{\vartheta})\rho_{\vartheta,a_0^\gamma}\sigma_\vartheta\sigma_{a_0^\gamma}\left(\left[1+\frac{\left(\frac{\partial\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}{\partial\eta}+\frac{\partial\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}{\partial\underline{a}}\frac{\underline{a}}{\eta(\eta-1)}\right)}{\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}(\eta'-\eta)\right]\frac{\sigma_{a_0^\gamma}(\eta',\underline{a}')}{\sigma_{a_0^\gamma}(\eta,\underline{a})}-1\right) \ .$$

With a slight abuse of notation, we can take the derivative w.r.t. η' to obtain

$$\frac{\partial\Delta^s K_I(\eta',\eta)}{\partial\eta'} \approx (\widetilde{\omega}/\overline{\vartheta})\rho_{\vartheta,a_0^\gamma}\sigma_\vartheta\sigma_{a_0^\gamma}\left(\frac{1}{\sigma_{a_0^\gamma}}\frac{\partial\sigma'_{a_0^\gamma}}{\partial\eta'}+\frac{\frac{\partial\rho_{\vartheta,a_0^\gamma}}{\partial\eta}+\frac{\partial\rho_{\vartheta,a_0^\gamma}}{\partial\underline{a}}\frac{\underline{a}}{\eta(\eta-1)}}{\rho_{\vartheta,a_0^\gamma}}\left(\frac{\sigma'_{a_0^\gamma}}{\sigma_{a_0^\gamma}}+(\eta'-\eta)\frac{1}{\sigma_{a_0^\gamma}}\frac{\partial\sigma'_{a_0^\gamma}}{\partial\eta'}\right)\right) \ .$$

To get an impression about the sign of the previous derivative, let us first analyze the sign of $\frac{\partial\sigma'_{a_0^\gamma}}{\partial\eta'}$. Notice that it is straightforward to show that a_0^γ follows a $\mathcal{P}a(\underline{a}^\gamma,\frac{\eta}{\gamma})$ distribution. As a result, the variance is given by

$$\sigma'_{a_0^\gamma} = (\underline{a}')^{2\gamma}\frac{\frac{\eta'}{\gamma}}{\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)} = \left(\underline{a}\frac{\eta}{\eta-1}\right)^{2\gamma}\left(\frac{\eta'-1}{\eta'}\right)^{2\gamma}\frac{\frac{\eta'}{\gamma}}{\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)} \ ,$$

where again the last equality follows from using $\underline{a}\frac{\eta}{\eta-1} = \underline{a}'\frac{\eta'}{\eta'-1}$. Defining the auxiliary variable $\tilde{\chi}(\eta,\gamma,\underline{a}) \equiv \left(\underline{a}\frac{\eta}{\eta-1}\right)^{2\gamma}$, we obtain:

$$\frac{\partial\sigma'_{a_0^\gamma}}{\partial\eta'} = \tilde{\chi}(\eta,\gamma,\underline{a})\left(2\gamma\left(\frac{\eta'-1}{\eta'}\right)^{2\gamma-1}\frac{1}{(\eta')^2}\frac{\frac{\eta'}{\gamma}}{\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)}\right)$$
$$+\tilde{\chi}(\eta,\gamma,\underline{a})\left(\frac{\eta'-1}{\eta'}\right)^{2\gamma}\left(\frac{\frac{1}{\gamma}\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)-\frac{\eta'}{\gamma}\left[\frac{2}{\gamma}\left(\frac{\eta'}{\gamma}-1\right)\left(\frac{\eta'}{\gamma}-2\right)+\frac{1}{\gamma}\left(\frac{\eta'}{\gamma}-1\right)^2\right]}{\left(\frac{\eta'}{\gamma}-1\right)^4\left(\frac{\eta'}{\gamma}-2\right)^2}\right) \ .$$

Collecting terms leads to

$$\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'} = \tilde{\chi}(\eta,\gamma,\underline{a}) \left(\frac{\eta'-1}{\eta'}\right)^{2\gamma} \frac{1}{\left(\frac{\eta'}{\gamma}-1\right)^2 \left(\frac{\eta'}{\gamma}-2\right)} \left[\frac{2}{\eta'-1} + \frac{1}{\gamma} - \frac{2\eta'}{\gamma(\eta'-\gamma)} - \frac{\eta'}{\gamma(\eta'-2\gamma)}\right]$$
$$= \frac{1}{\eta'}\sigma'_{a_0^\gamma} \left[1 + \frac{2\gamma}{\eta'-1} - \frac{2\eta'}{(\eta'-\gamma)} - \frac{\eta'}{(\eta'-2\gamma)}\right] .$$

In order to determine the sign of the bracket term, one can simplify to

$$\begin{aligned}\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'} &= \frac{1}{\eta'}\sigma'_{a_0^\gamma} \left[\frac{2\gamma(\eta'-2\gamma) + (\eta'-1)(\eta'-2\gamma) - \eta'(\eta'-1)}{(\eta'-1)(\eta'-2\gamma)} - \frac{2\eta'}{(\eta'-\gamma)}\right] \\ &= \frac{1}{\eta'}\sigma'_{a_0^\gamma} \left[\frac{2\gamma(1-2\gamma)}{(\eta'-1)(\eta'-2\gamma)} - \frac{2\eta'}{(\eta'-\gamma)}\right] \\ &= \frac{2}{\eta'}\sigma'_{a_0^\gamma} \left[\frac{\gamma(1-2\gamma)(\eta'-\gamma) - \eta'(\eta'-1)(\eta'-2\gamma)}{(\eta'-1)(\eta'-\gamma)(\eta'-2\gamma)}\right] .\end{aligned}$$

It is straightforward to show that the previous term is (weakly) negative if

$$\eta' \geq 2\gamma + \gamma\frac{(1-2\gamma)(\eta'-\gamma)}{\eta'(\eta'-1)} .$$

The left hand side of this expression is increasing in η', whereas the right hand side is decreasing if $\gamma \leq \frac{1}{2}$ and increasing if $\gamma > \frac{1}{2}$. Hence, for the case of $\gamma \leq \frac{1}{2}$, we obtain after substituting $\eta' = 2\gamma$ an upper limit of the right hand side given by $\overline{\eta} = \frac{3}{2}\gamma$. Contrary, in the case of $\gamma > \frac{1}{2}$ a straightforward application of L'Hopital's rule results in $\overline{\eta} = 2\gamma$. As a result, we obtain that $\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'} \leq 0 \; \forall \eta' \geq \overline{\eta} = \max\{\frac{3}{2}\gamma, 2\gamma\} = 2\gamma$, which trivially holds due to the implicit assumed finite variance of the $\mathcal{P}a(\underline{a}^\gamma, \frac{\eta}{\gamma})$ distribution. Consequently, the result of Lemma **??** follows (given a small change in the inequality tail). □

A.1.6 Proof of Proposition 1

Proof. The Farlie-Gumbel-Morgenstern (FGM) copula can be written for two arbitrary cumulative distribution functions $\{F(x_1), F(x_2)\}$ as

$$F(x_1,x_2) = C^{FGM}(F(x_1),F(x_2)) = F(x_1)F(x_2) + \varrho F(x_1)F(x_2)(1-F(x_1))(1-F(x_2)) ,$$

where $\varrho \in [-1,1]$. The joint probability density function of $f(x_1,x_2)$ is the obtained by

$$\begin{aligned} f(x_1,x_2) &= (1+\varrho\,(1-2F(x_1))\,(1-2F(x_2)))\,f(x_1)f(x_2) \\ &= (1+\varrho+2\varrho\,(2F(x_1)F(x_2)-F(x_1)-F(x_2)))\,f(x_1)f(x_2) .\end{aligned}$$

Under this assumption that $\vartheta \sim \mathcal{P}a(\underline{\vartheta}, \epsilon)$ and $a_0 \sim \mathcal{P}a(\underline{a}, \eta)$ this provides us with

$$f(\vartheta, a_0) = (1+\varrho) f(\vartheta) f(a_0) + 2\varrho \left[2 \left(\frac{\underline{\vartheta}}{\vartheta} \right)^{\epsilon} \left(\frac{\underline{a}}{a_0} \right)^{\eta} - \left(\frac{\underline{\vartheta}}{\vartheta} \right)^{\epsilon} - \left(\frac{\underline{a}}{a_0} \right)^{\eta} \right] f(\vartheta) f(a_0) \, .$$

where the marginals are given by $f(\vartheta)$ and $f(a_0)$. Given the Pareto assumptions, we have $\mu_\vartheta \equiv \overline{\vartheta} = \underline{\vartheta} \frac{\epsilon}{\epsilon - 1}$ and $\mu_{a_0^\gamma} = \underline{a}^\gamma \frac{\eta}{\eta - \gamma}$. In order to derive $cov(\vartheta, a_0^\gamma)$, we need to compute $\mathbb{E}\left[\vartheta a_0^\gamma\right]$:

$$\mathbb{E}\left[\vartheta a_0^\gamma\right] = \int_{\underline{\vartheta}}^{\infty} \int_{\underline{a}}^{\infty} \vartheta a_0^\gamma f(\vartheta, a_0) d\vartheta da_0 \, .$$

Using the FGM copula, we proceed in four steps:

$$\begin{aligned}
(1+\varrho)\underline{\vartheta}^{\epsilon} \underline{a}^{\eta} \epsilon\eta \int_{\underline{\vartheta}}^{\infty} \int_{\underline{a}}^{\infty} \vartheta^{-\epsilon} a_0^{\gamma-\eta-1} d\vartheta da_0 &= (1+\varrho)\underline{\vartheta} \frac{\epsilon}{\epsilon-1} \underline{a}^\gamma \frac{\eta}{\eta-\gamma} \, , \\
4\varrho \underline{\vartheta}^{2\epsilon} \underline{a}^{2\eta} \epsilon\eta \int_{\underline{\vartheta}}^{\infty} \int_{\underline{a}}^{\infty} \vartheta^{-2\epsilon} a_0^{\gamma-2\eta-1} d\vartheta da_0 &= 4\varrho \underline{\vartheta} \frac{\epsilon}{2\epsilon-1} \underline{a}^\gamma \frac{\eta}{2\eta-\gamma} \, , \\
-2\varrho \underline{\vartheta}^{2\epsilon} \underline{a}^{\eta} \epsilon\eta \int_{\underline{\vartheta}}^{\infty} \int_{\underline{a}}^{\infty} \vartheta^{-2\epsilon} a_0^{\gamma-\eta-1} d\vartheta da_0 &= -2\varrho \underline{\vartheta} \frac{\epsilon}{2\epsilon-1} \underline{a}^\gamma \frac{\eta}{\eta-\gamma} \, , \\
-2\varrho \underline{\vartheta}^{\epsilon} \underline{a}^{2\eta} \epsilon\eta \int_{\underline{\vartheta}}^{\infty} \int_{\underline{a}}^{\infty} \vartheta^{-\epsilon} a_0^{\gamma-2\eta-1} d\vartheta da_0 &= -2\varrho \underline{\vartheta} \frac{\epsilon}{\epsilon-1} \underline{a}^\gamma \frac{\eta}{2\eta-\gamma} \, .
\end{aligned}$$

Combining the previous four equations results in

$$\begin{aligned}
cov(\vartheta, a_0^\gamma) &= \mathbb{E}\left[\vartheta a_0^\gamma\right] - \mathbb{E}\left[\vartheta\right] \mathbb{E}\left[a_0^\gamma\right] \\
&= \underline{\vartheta a}^\gamma \left[\varrho \frac{\epsilon}{\epsilon-1} \frac{\eta}{\eta-\gamma} + 4\varrho \frac{\epsilon}{2\epsilon-1} \frac{\eta}{2\eta-\gamma} - 2\varrho \frac{\epsilon}{2\epsilon-1} \frac{\eta}{\eta-\gamma} - 2\varrho \frac{\epsilon}{\epsilon-1} \frac{\eta}{2\eta-\gamma} \right]
\end{aligned}$$

Further simplifications result in

$$\begin{aligned}
cov(\vartheta, a_0^\gamma) &= \underline{\vartheta a}^\gamma \varrho \left[\frac{\eta}{\eta-\gamma} \left(\frac{\epsilon}{\epsilon-1} - \frac{2\epsilon}{2\epsilon-1} \right) + \frac{2\eta}{2\eta-\gamma} \left(\frac{2\epsilon}{2\epsilon-1} - \frac{\epsilon}{\epsilon-1} \right) \right] \\
&= \underline{\vartheta a}^\gamma \varrho \left[\frac{\eta}{\eta-\gamma} \frac{\epsilon}{(\epsilon-1)(2\epsilon-1)} - \frac{2\eta}{2\eta-\gamma} \frac{\epsilon}{(\epsilon-1)(2\epsilon-1)} \right] \\
&= \underline{\vartheta a}^\gamma \varrho \frac{\epsilon}{(\epsilon-1)(2\epsilon-1)} \frac{\eta\gamma}{(\eta-\gamma)(2\eta-\gamma)} \, .
\end{aligned}$$

As a result, aggregate innovative asset holdings from Lemma 3 are given by

$$K_I = (\widetilde{\omega}/\overline{\vartheta}) \left(1 + \frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)} \right) \underline{\vartheta} \frac{\epsilon}{\epsilon-1} \underline{a}^\gamma \frac{\eta}{\eta-\gamma} \, .$$

Aggregate risk free capital holdings from Lemma 8 are (weakly) positive if the condition $\mu_{a_0} \geq K_I$ holds, which can be rewritten as

$$\frac{\eta-\gamma}{\eta-1} \geq (\widetilde{\omega}/\overline{\vartheta}) \left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right) \frac{\epsilon}{\epsilon-1} \underline{\vartheta a}^{\gamma-1} .$$

Finally, as $\mu_\vartheta = \mathbb{E}[\vartheta] = \overline{\vartheta}$, let $\widetilde{\omega} = (\widetilde{\omega}/\overline{\vartheta})\mu_\vartheta \underline{a}^{\gamma-1} = \widetilde{\omega}\underline{a}^{\gamma-1}$ and $C = (2\epsilon-1)$, we derive the marginal effect of a change in the Pareto tail η on wealth-normalized output $\widetilde{Y}(\eta) \equiv \frac{Y(\eta)}{\mu_{a_0}} = A + \mu\widetilde{\omega}(\phi - A)\Psi(\eta)$ with $\Psi(\eta) = \left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\frac{\eta-1}{\eta-\gamma}$ as:

$$\begin{aligned}
\frac{\partial Y(\eta)}{\partial\eta} &= \mu\widetilde{\omega}(\phi-A)\frac{\partial\Psi(\eta)}{\partial\eta} \\
&= \mu\widetilde{\omega}(\phi-A)\left[\left(\frac{1}{\eta-\gamma}\right)\left(1+\frac{\varrho\gamma}{C(2\eta-\gamma)}\right) - \left(\frac{\eta-1}{(\eta-\gamma)^2}\right)\left(1+\frac{\varrho\gamma}{C(2\eta-\gamma)}\right) - 2\left(\frac{\varrho\gamma}{C(2\eta-\gamma)^2}\right)\left(\frac{\eta-1}{\eta-\gamma}\right)\right] \\
&= -\mu\widetilde{\omega}(\phi-A)\left[\left(\frac{1}{(\eta-\gamma)^2}\right)(\gamma-1)\left(1+\frac{\varrho\gamma}{C(2\eta-\gamma)}\right) + 2\left(\frac{\varrho\gamma}{C(2\eta-\gamma)^2}\right)\left(\frac{\eta-1}{\eta-\gamma}\right)\right] \\
&= -\mu\widetilde{\omega}(\phi-A)\left[(\gamma-1)\left(\frac{1}{(\eta-\gamma)^2}\right) + \varrho(\gamma-1)\left(\frac{\gamma}{C(2\eta-\gamma)(\eta-\gamma)}\right) + 2\left(\frac{\varrho\gamma}{C(2\eta-\gamma)^2}\right)\left(\frac{\eta-1}{\eta-\gamma}\right)\right] \\
&= -\mu\widetilde{\omega}(\phi-A)\left[(\gamma-1)\underbrace{\left(\frac{1}{(\eta-\gamma)^2}\right)}_{:=\Omega^w} + \varrho(\gamma-1)\underbrace{\left(\frac{(\gamma(2\eta-\gamma)+2(\eta-\gamma)(\eta-1))}{C(2\eta-\gamma)^2(\eta-\gamma)^2}\right)}_{:=\Omega^{ws}} + \varrho\underbrace{\left(\frac{2(\eta-1)}{C(2\eta-\gamma)^2(\eta-\gamma)}\right)}_{:=\Omega^s}\right] \\
&= -\mu\widetilde{\omega}(\phi-A)\left(\frac{1}{C(2\eta-\gamma)^2(\eta-\gamma)^2}\right)\left[(\gamma-1)\cdot C(2\eta-\gamma)^2 + \varrho(\gamma-1)\cdot 2(\eta-1)(\eta-\gamma) + \varrho\cdot(2\eta(\eta-1)+\gamma)\right]
\end{aligned}$$

where the before last line follows from $\varrho\gamma = \varrho + \varrho(\gamma-1)$. In order to determine the sign of the derivatives, we need to know the sign of $\left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)$. To do so, let us assume that

$$1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)} \leq 0 \Leftrightarrow 1 \leq -\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)} \leq \frac{\gamma}{(2\epsilon-1)(2\eta-\gamma)} ,$$

where the last inequality follows from $\varrho \in [-1,1]$. As we have $\epsilon > 1$ and $\eta > \gamma$, an upper bound of $\frac{\gamma}{(2\epsilon-1)(2\eta-\gamma)}$ is given by $\frac{\gamma}{\epsilon\eta}$, which is strictly smaller than one. Hence, we obtain a contradiction and conclude that $1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}$ is strictly positive. This completes the proof of Corollary.

A.1.7 Existence of the Growth Irrelevance Frontier of Wealth Inequality

Lemma 7 (EXISTENCE GIF). *For a tail of wealth η and of type ϵ, type dependence ϱ, and wealth-dependent risk taking $\gamma \in (\underline{\gamma}, \overline{\gamma})$ with $\underline{\gamma} = \frac{2(2\epsilon-1)}{1+2(2\epsilon-1)}$ and $\overline{\gamma} = 2$, there exists a η^* which lies on the GIF.*

(a) For $\gamma \in (1, \overline{\gamma})$ and $-1 \leq \varrho \leq 2\frac{(2\epsilon-1)(1-\gamma)}{\gamma}$, the GIF exists for a unique $\eta^* \in (\gamma, \infty)$.

(b) With $\gamma = 1$ and $\varrho = 0$, any Pareto tail $\eta^* \in (\gamma, \infty)$ lies on the GIF.

(c) For $\gamma \in (\underline{\gamma}, 1)$, and $2^{\frac{(2\epsilon-1)(1-\gamma)}{\gamma}} \leq \varrho \leq 1$, the GIF exists for a unique $\eta^* \in (\underline{\eta}^{dc}, \infty)$, $\underline{\eta}^{dc} > 1$.

item a. proves that a negative *type* dependence ($\varrho < 0$) is needed for the existence of an economy which lies on the GIF if the *scale* dependence is positive ($\gamma > 0$). Conditions in item b. without *type* dependence requires no *scale* dependence for an economy to lie on the GIF, and conditions in item c. with positive *type* dependence shows that an economy with some $\gamma < 0$ can be located on the GIF.

Proof of Lemma 7: Existence of the $isoG(\eta, 0)$

Proof. Using Lemma 1, the *iso-growth* at level $\overline{g}$ can be implicitly defined as

$$(\phi - A)\overline{g} = -(\phi - A)\Big[(\gamma-1)\Omega^{w} + \varrho(\gamma-1)\Omega^{ws} + \varrho\Omega^{s}\Big] \tag{25}$$

Which, for a level $\overline{g} = 0$ can be rewritten as:

$$\left(1 + \frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\frac{1-\gamma}{(\eta-\gamma)^2} - 2\frac{\eta-1}{\eta-\gamma}\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)^2} = 0\,,$$

which we can rearrange to

$$\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)^2(\eta-\gamma)^2}\Big((1-\gamma)(2\eta-\gamma) - 2(\eta-1)(\eta-\gamma)\Big) = -\frac{1-\gamma}{(\eta-\gamma)^2}\,,$$

If $(1-\gamma)(2\eta-\gamma) - 2(\eta-1)(\eta-\gamma) \neq 0$ (which only occurs in the case of $\gamma < 1$) we can state the GIF algebraically as

$$\varrho(\gamma, \eta, \epsilon; \overline{g} = 0) = \frac{(2\epsilon-1)(1-\gamma)}{\gamma}\frac{(2\eta-\gamma)^2}{2(\eta-1)(\eta-\gamma) - (1-\gamma)(2\eta-\gamma)}\,. \tag{26}$$

The sign of the derivative w.r.t. to the Pareto tail is determined by

$$\begin{aligned}\operatorname{sgn}\left(\frac{\partial\varrho}{\partial\eta}\right) &= \operatorname{sgn}\Big((1-\gamma)\Big)\operatorname{sgn}\Big(4(2\eta-\gamma)\Big(2(\eta-1)(\eta-\gamma) - (1-\gamma)(2\eta-\gamma)\Big) - (2\eta-\gamma)^2\Big(4(\eta-1)\Big)\Big)\\ &= \operatorname{sgn}\Big((1-\gamma)\Big)\operatorname{sgn}\Big((\gamma-\eta)(2-\gamma)\Big)\,.\end{aligned}$$

Hence, we obtain due to $\eta > \gamma$

$$\frac{\partial \varrho}{\partial \eta}\begin{cases} < 0 & \text{if } \gamma > 2\,, \\ = 0 & \text{if } \gamma = 2\,, \\ > 0 & \text{if } 1 < \gamma < 2\,, \\ = 0 & \text{if } \gamma = 1\,, \\ < 0 & \text{if } 0 < \gamma < 1\,. \end{cases}$$

Before turning to the existence of η^{GIF}, we first study the limits of (26). Thus, we obtain

$$\lim_{\eta \to \gamma^+} \varrho(\gamma, \eta, \epsilon) = -\frac{(2\epsilon - 1)(1-\gamma)}{\gamma}\frac{(2\eta - \gamma)}{1-\gamma} = -(2\epsilon - 1)\,.$$

Similarly, we have

$$\lim_{\eta \to 1^+} \varrho(\gamma, \eta, \epsilon) = -\frac{(2\epsilon - 1)(1-\gamma)}{\gamma}\frac{(2 - \gamma)}{1-\gamma} = -\frac{(2\epsilon - 1)(2-\gamma)}{\gamma}\,.$$

Finally, by an application of L'Hopitals rule we derive

$$\lim_{\eta \to \infty} \varrho(\gamma, \eta, \epsilon) = \frac{(2\epsilon - 1)(1-\gamma)}{\gamma}\frac{2\eta - \gamma}{\eta - 1}|_{\eta=\infty} = 2\frac{(2\epsilon - 1)(1-\gamma)}{\gamma}\,.$$

As a result, we obtain the following bounds on the copula dependence parameter

$$\begin{aligned} -(2\epsilon - 1) < \varrho < 2\frac{(2\epsilon-1)(1-\gamma)}{\gamma} \qquad & \text{if } \gamma > 1 \\ \varrho = 0 \qquad & \text{if } \gamma = 1 \\ -\frac{(2\epsilon-1)(2-\gamma)}{\gamma} < \varrho < 2\frac{(2\epsilon-1)(1-\gamma)}{\gamma} \qquad & \text{if } \gamma < 1 \end{aligned}$$

It is straightforward to see for $\gamma > 2$ that the required $\varrho \notin \mathcal{R}$ such that the GIF is empty (i.e. $\nexists\ \eta^* \in (\gamma, \infty)$ s.t. $\text{GIF}(\eta^*) = 0$). Contrary, for $1 < \gamma < 2$ there exists by an application of the intermediate value theorem a unique $\eta^* \in (\gamma, \infty)$ such that $\text{GIF}(\eta^*) = 0$ if $-1 < \varrho < \overline{\rho}^{FGM} < 0$, where $\overline{\rho}^{FGM} \equiv 2\frac{(2\epsilon-1)(1-\gamma)}{\gamma}$. Additionally, in the case of $\gamma = 1$, being on the growth irrelevance frontier requires $\varrho = 0$. As a result, for any $\eta \in (1, \infty)$ the GIF goes through the point $\{\gamma = 1, \varrho = 0\}$. Finally, in the case of $\gamma < 1$ we can show that the growth irrelevance frontier is discontinuous at the points (cf. denominator of equation (26))

$$\eta_{dc}^{1,2} = 1 \pm \sqrt{1 - \frac{1}{2}\left(3\gamma - \gamma^2\right)}\,.$$

Recognizing that $3\gamma - \gamma^2$ is strictly increasing in γ on the interval $(0,1)$, we conclude that the expression in the square brackets is strictly positive and lies on the interval $(0,1)$. Due to the imposition of $\eta > 1$, we can hence exclude the smaller solution. Let us subsequently denote by $\eta^*_{dc} \in (1,2)$ the only feasible discontinuity point. As $\frac{\partial \varrho}{\partial \eta} < 0$, the denominator of (26) is strictly increasing in η and $\lim_{\eta \to 1^+} < -1$ for $\gamma \in (0,1)$, we conclude that $\varrho > 0$ is a necessary condition for the existence of a GIF solution. This implies that the feasible set of Pareto tails reduces to $\eta \in (\eta^*_{dc}, \infty)$. Ensuring that $\varrho \leq 1$ gives us finally a lower bound on the wealth dependent risk taking parameter such that $\gamma \geq \underline{\gamma} = \frac{2(2\epsilon-1)}{1+2(2\epsilon-1)}$. As a result, for all $\gamma < \underline{\gamma}$ a GIF solution does not exists. Contrary, by an application of the intermediate value theorem an unique $\eta^* \in (\eta^*_{dc}, \infty)$ exists for $\underline{\gamma} \leq \gamma < 1$. This completes the proof of Lemma **??**. $\square$

<u>Proof Lemma ??</u>

Proof. Using equation (26) from Lemma **??**, we obtain

$$\begin{aligned}\frac{\partial \varrho}{\partial \gamma} =& -(2\epsilon - 1)\frac{1}{\gamma^2}\frac{(2\eta-\gamma)^2}{2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)} \\ &+ (2\epsilon-1)\frac{1-\gamma}{\gamma}\frac{-2(2\eta-\gamma)\left[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\right]-(2\eta-\gamma)^2\left[2(1-\gamma)+1\right]}{\Big(2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big)^2}\,.\end{aligned}$$

Hence, the sign of the previous expression is determined by the sign of

$$\begin{aligned}\operatorname{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big) =& \Big(-(2\eta-\gamma)^2\Big[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big]\Big) \\ &+ \Big((1-\gamma)\gamma(2\eta-\gamma)\Big[-4(\eta-1)(\eta-\gamma)+2(1-\gamma)(2\eta-\gamma)-2(2\eta-\gamma)(1-\gamma)-2(\eta-\gamma)\Big]\Big)\,.\end{aligned}$$

Simplifying terms provides us with

$$\operatorname{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big) = \operatorname{sgn}\Big(\underbrace{-(2\eta-\gamma)\Big[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big]}_{\equiv\mathcal{A}} + \underbrace{(1-\gamma)\gamma\Big[-4(\eta-1)(\eta-\gamma)-2(\eta-\gamma)\Big]}_{\equiv\mathcal{B}}\Big)\,.$$

Let us begin with the case $\gamma = 1$. It is straightforward to see that

$$\operatorname{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big)\Big|_{\gamma=1} = \operatorname{sgn}(\mathcal{A}) = \operatorname{sgn}\Big(-2(2\eta-\gamma)(\eta-1)(\eta-\gamma)\Big) < 0\,,$$

such that the GIF is strictly decreasing in the point $\gamma = 1$. For the case $\underline{\gamma} < \gamma < 1$, the reasoning is slightly more evolved. First, notice that $\mathcal{B} < 0$ in this case. Second, requiring that the inner bracket of $\mathcal{A}$ is weakly positive is equivalent to requiring that

$$2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\geq 0 \Leftrightarrow 2\left(\eta^2-\eta\gamma-\eta+\gamma\right)-2\eta+\gamma+2\eta\gamma-\gamma^2\geq 0$$

Collecting terms gives us the following condition

$$2\eta^2-4\eta+3\gamma-\gamma^2\geq 0\ . \tag{27}$$

We know from Lemma **??** that the GIF is only defined in this case if $\eta>\eta^*_{dc}=1+\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)}$. Substituting this expression into the former inequality yields

$$\begin{aligned}\text{LHS} &= 2\left(1+\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)}\right)^2-4\left(1+\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)}\right)+3\gamma-\gamma^2\\ &= 2+4\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)}+2\left(1-\frac{1}{2}\left(3\gamma-\gamma^2\right)\right)-4-4\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)}+3\gamma-\gamma^2\\ &= 0\ .\end{aligned}$$

As the left hand side of the inequality (27) is strictly increasing in η on the set $\eta>\eta^*_{dc}>1$, we know that the above inequality is always satisfied strictly. Hence, we conclude that $\mathcal{A}<0$. Finally, this provides us with

$$\operatorname{sgn}\left(\frac{\partial\varrho}{\partial\gamma}\right)\Big|_{\underline{\gamma}\leq\gamma<1}=\operatorname{sgn}\left(\mathcal{A}+\mathcal{B}\right)<0\ .$$

Let us finally consider the case $1\leq\gamma<\overline{\gamma}$. It is evident that $\operatorname{sgn}(\mathcal{A})<0$ and $\operatorname{sgn}(\mathcal{B})>0$ hold in this case. Hence, the sign of the derivative is *a priori* undetermined. To show our claim, let us first rewrite the claim on the sign of our intitial inequality as

$$\begin{aligned}&-(2\eta-\gamma)\Big[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big]+(1-\gamma)\gamma\Big[-4(\eta-1)(\eta-\gamma)-2(\eta-\gamma)\Big]<0\\ \Leftrightarrow&(2\eta-\gamma)\Big[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big]>\gamma(1-\gamma)\Big[-4(\eta-1)(\eta-\gamma)-2(\eta-\gamma)\Big]\\ \Leftrightarrow&(\gamma-1)(2\eta-\gamma)\Big[2\eta-2\gamma\Big]+2(\eta-1)(\eta-\gamma)\Big[2\eta-\gamma+2\gamma(1-\gamma)\Big]>0\\ \Leftrightarrow&(\gamma-1)(2\eta-\gamma)+(\eta-1)\Big[2\eta-\gamma+2\gamma(1-\gamma)\Big]>0\\ \Leftrightarrow&(2\eta-\gamma)(\eta+\gamma-2)+2(\eta-1)\gamma(1-\gamma)>0\ .\end{aligned}\tag{28}$$

We know that $\eta>\gamma$ has to hold. Substituting $\eta=\gamma$ into the previous inequality yields

$$2\gamma(\gamma-1)>2\gamma(\gamma-1)^2\ ,$$

which is trivially satisfied for $\gamma < 2$. Hence, it suffices to show that the left hand side of (28) is increasing in η. To do so, let us take the derivative of (28) w.r.t. η and let us simultaneously impose that the derivative is positive:

$$\mathcal{Q}(\eta) \equiv 2(\eta+\gamma-2)+2\eta-\gamma+2\gamma(1-\gamma) = 4(\eta-1)+3\gamma-2\gamma^2 > 4(\gamma-1)+3\gamma-2\gamma^2 \equiv \mathcal{Q}(\gamma)\ ,$$

where the second last inequality holds due to $\eta > \gamma$. Hence, $\mathcal{Q}(\gamma) > 0$ implies also $4(\eta-1)+3\gamma-2\gamma^2 > 0$. To show the validity of the previous inequality, let us compute the values of γ of the second order polynomial $\mathcal{Q}(\gamma)$ for which the function equals exactly zero:

$$\tilde{\gamma}^{1,2} = \frac{-7 \pm \sqrt{49-32}}{-4} = \frac{7}{4} \pm \frac{1}{4}\sqrt{17}\ ,$$

As a consequence, we have that $\tilde{\gamma}^1 < 1$ and $\tilde{\gamma}^2 > 2$ such that $\mathcal{Q}(\gamma)$ is due to continuity strictly positive on the interval $\gamma \in [1,2]$. As a result, we know that $\mathcal{Q}(\eta)$ is also strictly positive on the entire interval $\gamma \in [1,2]$ which proves that equation (28) is satisfied. This shows

$$\operatorname{sgn}\left(\frac{\partial \varrho}{\partial \gamma}\right)|_{1<\gamma<\overline{\gamma}} = \operatorname{sgn}\left(\mathcal{A}+\mathcal{B}\right) < 0\ ,$$

such that we overall obtain

$$\operatorname{sgn}\left(\frac{\partial \varrho}{\partial \gamma}\right)|_{\underline{\gamma}\leq\gamma<\overline{\gamma}} = \operatorname{sgn}\left(\mathcal{A}+\mathcal{B}\right) < 0\ .$$

Let us finally define the growth irrelevance equation (26) by $\varrho \equiv \mathcal{G}(\gamma)$, where $\{\epsilon, \eta\}$ enter the $\mathcal{G}$ function as constants. Recognize that $\mathcal{G}$ is strictly decreasing on the interval $\underline{\gamma} \leq \gamma < \overline{\gamma}$ and thus *injective*. Additionally, it is differentiable at $\mathcal{G}^{-1}(\varrho)$ and hence continuous on the interval $\mathbb{G}$. As a result, we can define the inverse function of the growth irrelevance frontier equation (26) by

$$\gamma \equiv \mathcal{G}^{-1}(\varrho)\ .$$

Consequently, it is straightforward to obtain

$$\frac{\partial \gamma}{\partial \varrho} = \frac{\partial \mathcal{G}^{-1}(\varrho)}{\partial \varrho} = \frac{1}{\mathcal{G}'(\mathcal{G}^{-1}(\varrho))} = \frac{1}{\mathcal{G}'(\gamma)} < 0\ ,$$

which completes the first part of the proof of Lemma **??**.

PART 2. Let us consider now the general growth irrelevance frontier at an arbitrary growth level $\overline{g}$, possibly different from zero. Without loss of generality let us further assume that $\psi_{min,0} = 1$.

The GIF is then implicitly characterized by

$$\frac{(1-\gamma)(2\eta-\gamma)-2(\eta-1)(\eta-\gamma)}{(2\epsilon-1)(2\eta-\gamma)^2(\eta-\gamma)^2}\gamma\varrho+\frac{1-\gamma}{(\eta-\gamma)^2}=\chi_{\overline{g}}\overline{g}\ ,$$

where $\chi_{\overline{g}}$ denotes a strictly positive constant which is independent of $\{\varrho,\gamma\}$. Total differentiation of the previous equation yields

$$\chi_{\rho}d\varrho+\chi_{\gamma}d\gamma=\chi_{\overline{g}}d\overline{g}\ .$$

Rearranging the previous condition results in

$$d\gamma=-\frac{\chi_{\rho}}{\chi_{\gamma}}d\varrho+\frac{\chi_{\overline{g}}}{\chi_{\gamma}}d\overline{g}\ .$$

On the restricted set of Lemma 6, we have $\chi_{\rho}<0$. Additionally, we have that $\frac{\chi_{\rho}}{\chi_{\gamma}}=-\frac{1}{\mathcal{G}'(\gamma)}$, which implies $\chi_{\gamma}<0$. Hence, an increase in $\overline{g}$ (i.e. lower growth rate) decreases γ, conditional on ϱ. As a result, the GIF shifts downwards. Similarly, if $\overline{g}$ decreases (i.e. higher growth rate), γ increases conditional on ϱ which shifts the GIF upwards. This concludes the proof. □

A.1.8 Proof of Lemma 5

We first solve for the consumption equivalent variation $\Delta^{CE,i}$, defined as the amount of consumption that makes an individual indifferent between the reformed economy with progressivity p_a and the initial status quo situation, such that $\mathbb{E}[u(\tilde{c}_2^i-\Delta^{CE,i})]=\mathbb{E}[u(c_2^i)]$. We get:

$$E[(1-exp(-\tilde{\alpha}^i(\tilde{c}_2^i-\Delta^{CE,i})))]/\tilde{\alpha}^i=E[(1-exp(-\alpha^i c_2^i))]/\alpha^i$$

which, under G-CARA and $\Delta^{\alpha}=\left(1/\tilde{\alpha}^i-1/\alpha^i\right)$ is equivalent to

$$\begin{aligned}\Delta^{\alpha}-exp\left(-\tilde{\alpha}^i(\tilde{x}_2^i-\Delta^{CE,i})\right)/\tilde{\alpha}^i&=-exp\left(-\alpha^i x_2^i\right)/\alpha^i\\ exp\left(-\tilde{\alpha}^i(x_2^i-\Delta^{CE,i})+\alpha^i x_2^i\right)&=\left[1+\Delta^{\alpha}exp\left(\alpha^i x_2^i\right)\alpha^i\right](\tilde{\alpha}^i/\alpha^i)\\ -\tilde{\alpha}^i(\tilde{x}_2^i-\Delta^{CE,i})+\alpha^i x_2^i&=\Delta^c\end{aligned}$$

where $\tilde{x}_2^i$ and x_2^i the certainty equivalents. Rearranging the terms, this yields:

$$\Delta^{CE,i}=\tilde{x}_2^i-x_2^i+\frac{\Delta^c}{\tilde{\alpha}^i}$$

with $\Delta^c=-\tilde{\alpha}^i\left(\frac{\alpha^i}{\tilde{\alpha}^i}-1\right)x_2^i+\ln\left(1+\left(\frac{\alpha^i}{\tilde{\alpha}^i}-1\right)exp\left(\alpha^i x_2^i\right)\right)+\ln\left(\frac{\tilde{\alpha}^i}{\alpha^i}\right)$, a term that arises because a change in the progressively p_a affects a_0^i which impacts the curvature of the utility function since $\alpha^i\neq\tilde{\alpha}^i$.

We now analyze the effects of introducing a proportional tax on each component. First, let us analyze the effect on $\widetilde{x}_2^i = \widetilde{\mu}_2^i - (\widetilde{\alpha}^i/2)(\widetilde{\sigma}_2^i)^2$, with:

$$\widetilde{\mu}_2^i = \varphi Y + T + \underline{r}\widetilde{a}_0^i + A(1-\varphi)\widetilde{a}_0^i + (\phi - A)(1-\varphi)\omega_1^i\widetilde{a}_0^i$$
$$((\widetilde{\alpha}^i/2)(\widetilde{\sigma}_2^i)^2) = \frac{1}{2}\sigma_\kappa^2(\widetilde{a}_0^i)^\gamma\widetilde{\omega}^2\frac{\vartheta^i}{\overline{\overline{\vartheta}}}(1-\varphi)^2$$

we get:

$$\frac{\partial\widetilde{\mu}_2^i}{\partial p_a} = \varphi\frac{\partial Y}{\partial\widetilde{\eta}}\frac{\partial\widetilde{\eta}}{\partial p_a} + \frac{\partial\underline{r}}{\partial\widetilde{\eta}}\frac{\partial\widetilde{\eta}}{\partial p_a}\widetilde{a}_0^i + \frac{\partial T}{\partial p_a} + \frac{\partial\widetilde{a_0}^i}{\partial p_a}\left[A(1-\varphi) + \underline{r} + \gamma\widetilde{\omega}\frac{\vartheta^i}{\overline{\overline{\vartheta}}}(\widetilde{a}_0^i)^{\gamma-1}(1-\varphi)(\phi - A)\right]$$
$$\frac{\partial((\widetilde{\alpha}^i/2)(\widetilde{\sigma}_2^i)^2)}{\partial p_a} = \frac{\partial\widetilde{a_0}^i}{\partial p_a}\left[\gamma(1/2)\sigma_\kappa^2(\widetilde{a}_0^i)^{\gamma-1}\widetilde{\omega}^2\frac{\vartheta^i}{\overline{\overline{\vartheta}}}(1-\varphi)^2\right]$$

which yields:

$$\frac{\partial\widetilde{x}_2^i}{\partial p_a} = \varphi\frac{\partial Y}{\partial\widetilde{\eta}}\frac{\partial\widetilde{\eta}}{\partial p_a} + \frac{\partial\underline{r}}{\partial\widetilde{\eta}}\frac{\partial\widetilde{\eta}}{\partial p_a}\widetilde{a}_0^i + \frac{\partial T}{\partial p_a} + \frac{\partial\widetilde{a_0}^i}{\partial p_a}\left[A(1-\varphi) + \underline{r} + \gamma\frac{\vartheta^i}{\overline{\overline{\vartheta}}}(\widetilde{a}_0^i)^{\gamma-1}x_r\right]$$

where $x_r = \widetilde{\omega}(\phi - A)(1-\varphi) - \frac{1}{2}\sigma_\kappa^2\widetilde{\omega}^2(1-\varphi)^2$.

Concerning the term Δ^c, notice that:

$$\ln\left(1 + \left(\frac{\alpha^i}{\widetilde{\alpha}^i} - 1\right)exp\left(\alpha^i x_2^i\right)\right) = \left(\frac{\alpha^i}{\widetilde{\alpha}^i} - 1\right)exp\left(\alpha^i x_2^i\right) + \mathcal{E}(a_0^i)$$

where since $\left(\frac{\alpha^i}{\widetilde{\alpha}^i} - 1\right) < 0$ we have $\mathcal{E}(a_0^i) < 0$ an approximation error to the transformation $\ln(1 + x) \approx x$. Using this, we can rewrite:

$$\begin{aligned}\Delta^c &= -\widetilde{\alpha}^i\left(\frac{\alpha^i}{\widetilde{\alpha}^i} - 1\right)x_2^i + \left(\frac{\alpha^i}{\widetilde{\alpha}^i} - 1\right)exp\left(\alpha^i x_2^i\right) + \mathcal{E}(a_0^i) + \ln(\widetilde{\alpha}^i/\alpha^i)\\ &= \left(\frac{\alpha^i}{\widetilde{\alpha}^i} - 1\right)\left(exp\left(\alpha^i x_2^i\right) - \widetilde{\alpha}^i x_2^i\right) + \mathcal{E}(a_0^i) + \ln(\widetilde{\alpha}^i/\alpha^i)\,.\end{aligned}$$

using $\ln(\widetilde{\alpha}^i/\alpha^i) \approx -\left(\frac{\alpha^i}{\widetilde{\alpha}^i} - 1\right)\frac{\widetilde{\alpha}^i}{\alpha^i}$, we get:

$$\Delta^c \approx \left(\frac{\alpha^i}{\widetilde{\alpha}^i} - 1\right)\left(exp\left(\alpha^i x_2^i\right) - \frac{\widetilde{\alpha}^i}{\alpha^i}(\alpha^i x_2^i + 1)\right) + \mathcal{E}(a_0^i)\,.$$

Using the welfare function, the optimal progressivity p_a solves:

$$\frac{\partial\mathcal{W}}{\partial p_a} = \int s(a_0,\vartheta)\frac{\partial\Delta^{CE}(a_0,\vartheta)}{\partial p_a}\mathcal{G}(a_0,\vartheta) = 0$$

which can be rewritten:

$$\int s(a_0,\vartheta)\left[\varphi\frac{\partial Y}{\partial\widetilde{\eta}}\frac{\partial\widetilde{\eta}}{\partial p_a}+\frac{\partial\underline{r}}{\partial\widetilde{\eta}}\frac{\partial\widetilde{\eta}}{\partial p_a}\widetilde{a}_0^i+\frac{\partial T}{\partial p_a}+\frac{\partial\widetilde{a}_0^{\,i}}{\partial p_a}\left[A(1-\varphi)+\underline{r}+\gamma\frac{\vartheta^i}{\overline{\overline{\vartheta}}}(\widetilde{a}_0^i)^{\gamma-1}x_r\right]+\frac{\partial(\Delta^c/\widetilde{\alpha}^i)}{\partial p_a}\right]\mathcal{G}(a_0,\vartheta)=0$$

or equivalently using $\int s(a_0,\vartheta)\mathcal{G}(a_0,\vartheta)=1$,

$$\underbrace{\varphi\frac{\partial Y}{\partial\widetilde{\eta}}\frac{\partial\widetilde{\eta}}{\partial p_a}}_{\text{efficiency}}+\underbrace{\frac{\partial\underline{r}}{\partial\widetilde{\eta}}\frac{\partial\widetilde{\eta}}{\partial p_a}\int s(a_0,\vartheta)\widetilde{a}_0\mathcal{G}(a_0,\vartheta)}_{\text{rent extraction}}+\underbrace{\frac{\partial T}{\partial p_a}}_{\text{lump-sum transfers}}$$

$$+\int s(a_0,\vartheta)\left(\underbrace{\frac{\partial\widetilde{a}_0^{\,i}}{\partial p_a}\left[A(1-\varphi)+\underline{r}+\gamma\frac{\vartheta^i}{\overline{\overline{\vartheta}}}(\widetilde{a}_0^i)^{\gamma-1}x_r\right]}_{\text{direct effect on level } a_0^i}+\underbrace{\frac{\partial(\Delta^c/\widetilde{\alpha}^i)}{\partial p_a}}_{\text{direct effect on } \alpha^i}\right)\mathcal{G}(a_0,\vartheta)=0$$

A.1.9 Proof of Lemma 6

Proof. Let us first observed that we can rewrite the utility function as:

$$\max_{k_i^1,b_1^i}\left(\frac{1}{1-1/\sigma}\right)\left[\left(a_0^i-k_1^i-b_1^i\right)^{1-1/\sigma}+\beta\left(\mu_{c_2}^i(k_1^i,b_1^i)-\frac{\alpha_i}{2}\sigma_{c_2}^i(k_1^i,b_1^i)\right)^{1-1/\sigma}\right] \tag{29}$$

with $k_1^i=\omega_1^i a_1^i$ and $b_1^i=(1-\omega_1^i)a_1^i$.

To do that, notice that

$$U\Big(G^{-1}\Big(\mathbb{E}\Big[G\Big(U^{-1}(u_2)\Big)\Big)=\Big(G^{-1}\left(\mathbb{E}\left[G(c_2^i)\right]\right)\Big)^{1-1/\sigma}/(1-1/\sigma)$$

using the fact that $x=-(1/\alpha^i)\ln(1-\alpha^i G(x))$ and $\mathbb{E}\left[G(c_2^i)\right]=(1/\alpha^i)(1-exp(-\alpha^i x_{c_2}^i))$ as shown above, we get:

$$\begin{aligned}G^{-1}\left(\mathbb{E}\left[G(c_2^i)\right]\right)&=-(1/\alpha^i)\ln\left(1-\alpha^i(1/\alpha^i)\left(1-exp(-\alpha^i x_{c_2}^i)\right)\right)\\&=x_{c_2}^i\end{aligned}$$

Hence the program of the agent can be rewritten as in (29).

First order condition with respect to k_1^i and b_1^i are

$$-(c_1^i)^{-1/\sigma}+\beta(x_{c_2}^i)^{-1/\sigma}(\underline{r}+A(1-\varphi))=0$$
$$-(c_1^i)^{-1/\sigma}+\beta(x_{c_2}^i)^{-1/\sigma}(\underline{r}+\phi(1-\varphi)-\alpha^i(1-\varphi)^2\sigma_\kappa^2k_1^i-\alpha_i\rho_{\kappa,h}\varphi Y(1-\varphi)\sigma_h\sigma_\kappa)=0$$

Combining both equations yields

$$k_1^i = \frac{(\phi - A)(1-\varphi)}{\alpha^i(1-\varphi)^2\sigma_\kappa^2} - \rho_{\kappa,h}\frac{\varphi Y}{1-\varphi}\frac{\sigma_h}{\sigma_\kappa} = \widetilde{\omega}\frac{\vartheta^i}{\overline{\vartheta}}(a_0^i)^{\gamma-1} - \rho_{\kappa,h}\frac{\varphi Y}{1-\varphi}\frac{\sigma_h}{\sigma_\kappa}$$

Using the condition with respect to b_1^i, we get:

$$(a_0^i - b_1^i - k_1^i) = x_{c_2}^i(\widetilde{R}\beta)^{-\sigma}$$

where $\widetilde{R} = \underline{r} + A(1-\varphi)$. Using the expression of $\mu_{c_2}^i = \varphi Y + (\underline{r} + A(1-\varphi))b_1^i + (\underline{r} + \phi(1-\varphi))k_1^i$ and $x_{c_2}^i = \mu_{c_2}^i - \frac{\alpha^i}{2}\sigma_{c_2}^i$, we obtain:

$$\begin{aligned} b_1^i((\widetilde{R}\beta)^\sigma + \widetilde{R}) &= a_0^i(\widetilde{R}\beta)^\sigma - k_1^i\left[(\widetilde{R}\beta)^\sigma + \underline{r} + \phi(1-\varphi)\right] - \varphi Y + \frac{\alpha^i}{2}\sigma_{c_2}^i \\ b_1^i &= ((\widetilde{R}\beta)^\sigma + \widetilde{R})^{-1}\left(a_0^i(\widetilde{R}\beta)^\sigma - k_1^i\left[(\widetilde{R}\beta)^\sigma + \underline{r} + \phi(1-\varphi)\right] - \varphi Y + \frac{\alpha^i}{2}\sigma_{c_2}^i\right) \end{aligned}$$

□

B Simulation & Computational Details

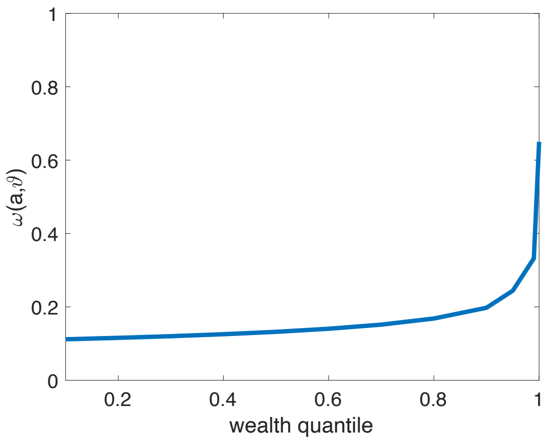
B.1 Fit of the Static Model under pure type/wealth dependence

Figures 7a and 7b show the fit of the type and wealth dependence models relative to the distribution of risky asset shares across the wealth distribution. To fit this shape, we first fixed inequality to $\eta = 1.3$. The parameter $\gamma = 1.41$ is used to match the risky asset share of the top 1% in the wealth dependence model. In the pure type dependence model, we used the parameter controlling the correlation (to 0.55) between the two distribution, fixing the Pareto shape of types to $\varepsilon = 2$. The two models produce an extremely close fit of the observed distribution.

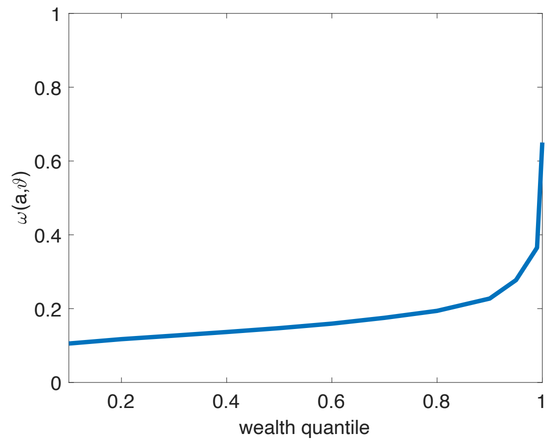
C Empirical Appendix

C.1 Capital Returns Measurement in the PSID

Throughout the paper, we use the PSID to compute returns to wealth across the wealth distribution. Our sample span the period between 1998 to 2019, with the first period dropped when computing the returns to wealth. We follow closely the procedure of Fagereng et al. (2020). Our sample consider households whose the head is aged between 20 and 80. This is to ensure that the financial decision maker is the holder of the assets. We restrict our attention to households with at least 1000$ of net worth. Finally, we trim the distribution of returns in each year at the top and



(a) wealth dependence model



(b) type dependence model

bottom 0.5% and drop observations with trimmed returns.

[DETAIL HERE THE PROCEDURE]

C.2 Adjusted SCF

Throughout the paper, we use the Survey of Consumer Finance (SCF) from 1998 to 2019, representing eight waves. Each survey wave provides cross-sectional data on U.S. households' income and wealth, including detailed information regarding portfolio allocation, as well as families' demographic characteristics. The SCF is a widely used survey to study inequality in the US. The reason is that the SCF oversamples very wealthy individuals, such that the total wealth in the survey is close to the one estimated in the US national account. Specifically, households in the SCF are selected from a double sampling procedure. A first sample is selected from a standard sampling procedure, providing a good representativity of the population. A second sample selects very high income families, with some that are also likely to be very wealthy, from the tax records of the Internal Revenue Service (IRS). The SCF weights are used to combine individual characteristics from the two samples to make estimates for the full U.S. population.

We begin by defining our concept of wealth. Household wealth is defined as the current value of all marketable assets, less the current value of debts. Total assets combines (i) saving, cash and deposits, (ii), directly and indirectly held stocks (such as mutual funds and I.R.A accounts), (iii) net private equity business investments, including commercial real estate, (iv) the value of the primary home and secondary residences, (v) other nonfinancial assets. Total liabilities are the sum of mortgage debt, consumer debt, educational loans, and other debts.

Correcting for under-representation and under-reporting Wealth and income concentration measures from survey data face two common issues: (i) under-representation, meaning that wealth-rich households are generally under-represented in survey data, and (ii) underreporting of assets, meaning that individuals tend to under-report wealth, especially financial wealth. We correct for those issues using the procedure described in Vermeulen (2016). The method is iterative and proceed as follows. First, observation at the top of the survey data (above a given threshold of wealth) are supplemented by an external source – such as the Forbes and the Sunday Times rich lists – in order to estimate a Pareto Law. The aim is to generate corrected for missing value data at the top of the wealth distribution using the estimated Pareto distribution. Second, aggregate estimates from the entire wealth distribution, below and above the threshold of wealth, are computed for total, financial and non-financial assets. Households' wealth are then adjusted such that aggregate estimates from the complemented survey data coincide with national households balance sheet. The procedure iterates until the distribution of wealth is fixed.[51]

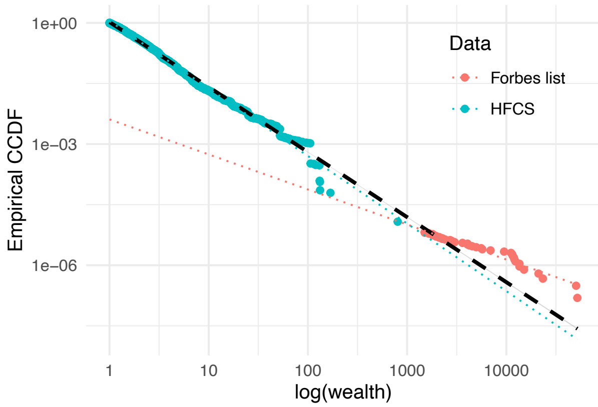
In the Online Appendix E.1, we use adjusted top share to assess the relationship between top wealth shares and subsequent growth. To do so, we estimate our own adjusted survey data estimates using the Household Finance and Consumption Survey (HFCS) for European countries and the Luxembourg Wealth Survey (LWS) for countries in the Commonwealth, following the procedure of Vermeulen (2016). While the Pareto assumption is questionable, it appears that most countries describe a linear log-log relationship between wealth level and its empirical CCDF at the top of the distribution, indicating that a Pareto Law can well describe the distribution. As an example, figure 8 shows the resulting Pareto estimates for three distinct countries: France, Italy and Germany. Table 6 shows the corrected estimates for different countries using the procedure described in Vermeulen (2016).[52]

Table 6. Adjusted and non-adjusted wealth shares for selected countries.

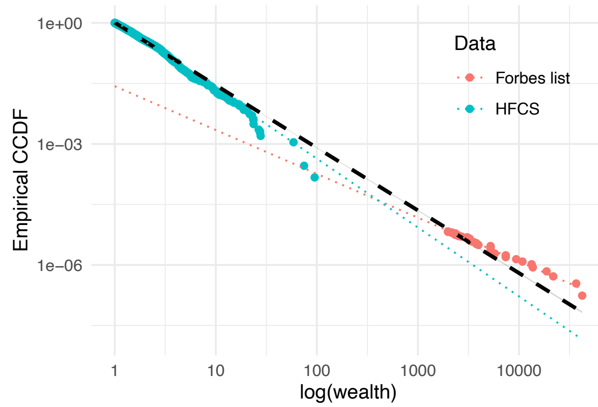
Share of wealth held by	Non-adjusted		Adjusted	
	1%	10%	1%	10%
France	0.189	0.518	0.195	0.511
Germany	0.242	0.596	0.312	0.641
Italy	0.117	0.423	0.185	0.508

[51]While there is no reason to think that this procedure converge to a fixed point, it appears that this is indeed the case.

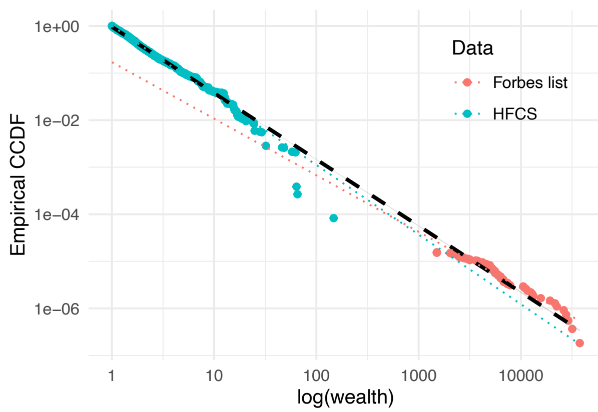
[52]We also worked with non-adjusted data. The results of the papers are quantitatively different, but they are qualitatively similar (the *IG-slope* is positive and significant for top shares). We provide those results in a supplementary appendix available upon request.



(a) France



(b) Italy



(c) Germany

Figure 8. Example: Pareto shape estimation for France, Italy and Germany.

C.3 PSID and investment participation

C.4 PSID and investment diversification

D Quantitative Appendix

D.1 Calibration

D.1.1 death probability

D.1.2 Calibration of the labor income process

Throughout the quantitative section, we made the assumption of stochastic aging. Denote $\pi(h'|h)$ the conditional probability of transiting from h to h' associated with the productivity process h. The overall productivity of transiting from a state h to h', conditional on surviving, is denoted $P_h(h'|h)$, and equals the likelihood of transitioning from h to h' due to an aging shock, plus the likelihood of making this transition from h to h' due to a productivity shock, conditional on not aging, such that

$$P_h = \begin{bmatrix} 0 & \chi_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \chi_{J-1} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} (1-\chi_1) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & (1-\chi_{J-1}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \pi$$

with $\chi_h = \frac{1}{\Pi_h M}$ and Π_h is the fraction of individuals with productivity h.

D.1.3 Type and scale dependence

Throughout the model, we capture *scale* effects in two ways. First, the probability to switch to an investor state is wealth-dependent. To match the data, the thresholds and nodes parameters of the piecewise function $\lambda(a) = \mathcal{S}(a; \mathbf{T}_\lambda, \mathbf{N}_\lambda)$ are

$$\mathbf{T}_\lambda = []$$
$$\mathbf{N}_\lambda = []$$

D.2 Computational appendix

The

Online Appendix (OA): not for publication

E Supplementary Empirical Analysis

E.1 Revisiting the Inequality – Growth Relationship

Our paper is about predicting the role of inequality on output/growth. In our model, the relation is *a priori* ambiguous and depends on the extend that capital heterogeneity translates into higher productivity through an overall reallocation of wealth toward riskier but more productive assets.

In this section, we contribute to the literature by revisiting the cross-country panel data link between inequality and GDP growth (henceforth *IG-slope*) in three ways. First, while previous studies focus mainly on income inequality, we extend the analysis to a larger sample by constructing and using wealth concentration measures. Second, we focus on the heterogeneous impact of inequality along wealth and income distributions on growth, using quantile shares at the top of the distribution. In contrast, previous studies mainly analyze the effect of the income Gini coefficient, a synthetic inequality measure revealing an average effect of inequality.[53] We document

[53] A wide range of empirical papers studies the link between inequality and growth. For instance, Forbes (2000), Barro (2000) and Halter et al. (2014) use the income Gini coefficient as measure of inequality, while Barro (2008) and

that an increase in top wealth and income shares has a significant positive impact on subsequent growth. Third, we decompose the IG-slope and find that top inequality is mainly driving Solow residual and physical capital accumulation, while human capital is of second order.

E.1.1 Assessing the *IG-slope*

To start with, we revise the results in Forbes (2000) by using the same fixed effect baseline specification. Contrary to her analysis, our focus is on wealth and income shares held at different quantiles. According to the World Bank classification, we study the IG-slope using a sample of high-income, upper middle income, and financially developed countries from 1975 to 2020.[54,55] We divide the sample into 5 years non-overlapping time windows such that we retrieve at the maximum nine observations for each country. This ensures we capture the middle-run IG-slope.

Data Description. We use annual data on real GDP at chained PPP from the 9.1^{th} *Penn World Table* (PWT), which we supplement by *IMF* data for the year 2019. Data for inequality concentration measures are collected from various sources. First, we use income concentration measures from the harmonized *World Inequality Database* (WID). At the date of this paper, it constitutes the most complete and harmonized database on wealth and income inequality. It provides aggregate estimates of average wealth and income levels, as well as wealth and income shares held at different quantiles, e.g. top 10%, top 1%, top 0.1%.[56] We complement wealth concentration measures using estimates from recent papers evaluating top shares in a number of countries and our own survey data estimates using the Household Finance and Consumption Survey (HFCS), the Survey of Consumer Finance (SCF), and the Luxembourg Wealth Survey (LWS).[57] The three surveys constitute currently the most reliable data to measure inequality and are for example used by the Global Wealth Report.

Voitchovsky (2005) use quintile and decile shares. None of the previous papers explore the *IG-slope* using wealth concentration measures. To the best of our knowledge, only Voitchovsky (2005) looks at the impact of quantile income concentration at the relatively more aggregated decile level. An exception is Frank (2009), however, his study considers a panel of US states.

[54]The complete list includes: Australia, Austria, Belgium, Canada, Germany, Spain, Finland, France, Greece, Italy, Japan, Luxembourg, Malta, Netherlands, Norway, Portugal, Slovakia, Slovenia, Sweden, the United States, and the United Kingdom. We also use all available periods covered by corresponding survey data.

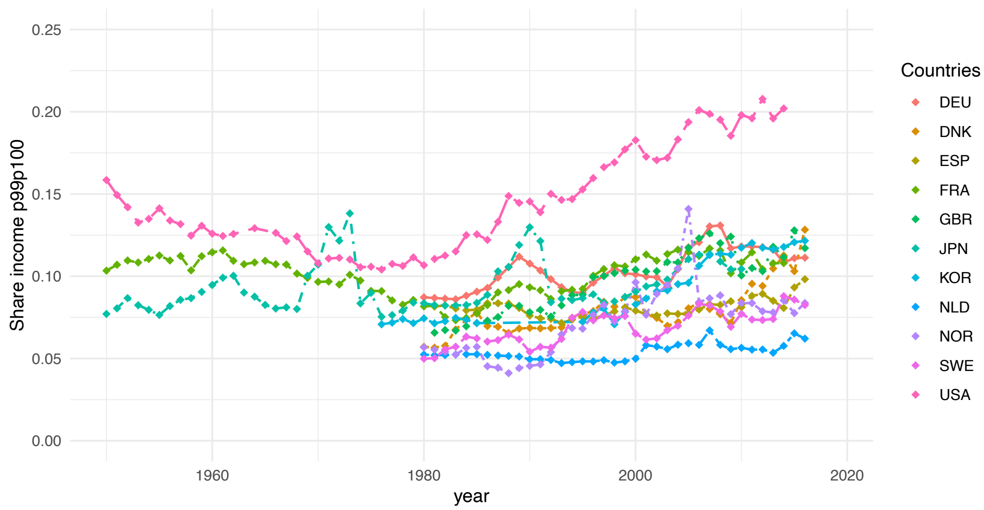
[55]It has been shown that the inequality growth relationship depends substantially on the development stage (Halter et al., 2014). Their analysis indicates a negative relationship between past inequality on GDP growth in lower middle income and low-income countries. Our analysis rather focuses on developed countries.

[56]Previous papers rely mostly on data which neglect the income share held by quantiles at the upper end of the distribution. To further relate to previous papers, we also collect data on income Gini coefficients from the *World Income Inequality Database* (WIID) and Deininger and Squire (1996).

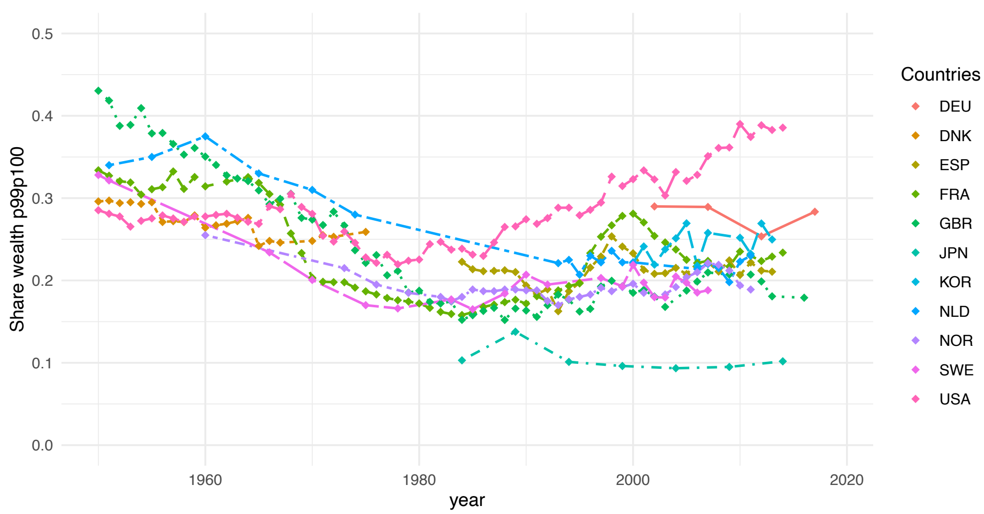
[57]To guarantee a certain quality of wealth concentration measures, data from the literature are taken whenever administrative data, at the individual or household level, are used or survey data accessible only from residents of specific countries. Those measures are often restricted to net worth concentration only, as well as at specific quantiles (top 1%, top 5%, top 10%).

We use the method developed in Vermeulen (2016) to correct for *under-reporting*, i.e. individuals tend to under-report financial wealth, and *over-sampling* according to which wealth-rich individuals are under-represented in survey data. The method of Vermeulen (2016) uses (i) national balance sheet data to correct for under-reporting and to harmonize estimates across countries and time, and (ii) a list of billionaires, e.g. the Forbes list or the Sunday Times list, to complement for missing observations at the top of the distribution. It is well-known that the wealth distribution follows a Pareto Law at the top. The method employs this property to estimate a country-specific Pareto tail and uses the billionaire list to extrapolate estimates for the remaining top shares. While this method is subject to measurement error, we believe that these errors spread uniformly over different survey waves. Our final data set contains concentrations measures for 45 countries for income and 29 for wealth. We show in Figure 9 (panel (a) and (b)) the share of income and wealth held by the top 1% in a number of countries. While income concentration measures are fairly stable over a long period for most countries except the US, it displays sizeable variations over time. Turning to the wealth measures, it displays a global decrease between 1950 to 1980 and then a global increase since 1980. By the use of time window fixed effect, we aim to control for global changes in wealth and income concentration, focusing on within country variations. Finally, the panel (c) of Figure 9 shows the positive correlation between wealth and income measures. The R^2 is 0.65, indicating that concentration of wealth can only be partially proxied by the concentration of income. Controlling for country fixed effect and essentially focusing on within country variations might thus limit the measurement-error bias and helps to combine different sources. A complete description of the method is provided in Appendix C.2. Overall, our restricted sample with wealth concentration measures contains 29 developed countries for which we have at least two time window observations. To the best of our knowledge, it constitutes the most complete wealth concentration database.

We include a set of control variables similar to Perotti (1996) and Forbes (2000) which may interact with current concentration measures and subsequent GDP growth. The benchmark control variables include the PWT human capital index, the PWT investment prices as a proxy for market power, the log GDP, and log physical capital per capita. This procedure has two advantages. First, we can directly compare our results to the aforementioned studies since any discrepancy cannot be explained by the specification itself. Second, panel estimation requires a large number of observations. As this is barely satisfied when dealing with inequality data, a parsimonious specification is desirable. Within a robustness analysis, we additionally control for demand effects by including bottom income shares, the degree of financial market deepness captured by the credit volume to GDP ratio, top marginal income tax rates obtained from *World Tax Indicators* (WTI), wealth tax



(a) Top 1% income share in selected countries



(b) Top 1% wealth share in selected countries

(c) Correlation between wealth and income measures

Figure 9. Wealth and income share in selected countries, and correlation between measures.

rates from the OECD, or the population share of more than 65 years old individuals. Of second-order importance, we additionally include the export to GDP ratio, the aggregate college rate, the GDP share of government expenditures in education, the ratio of investment over total GDP, and the population growth rate.

The extent to which wealth and income concentration measures affect subsequent GDP growth might be impacted by several factors beyond the previously discussed controls. For example, increasing globalization and financial interdependence since the 1980s may have boosted simultaneously inequality and GDP growth. We exploit the panel data dimension and control for country and time window fixed effects. Therefore, by isolating within country variations, we aim to minimize confounding effects arising from mismeasurement, country-specific effects, time-invariant unobservable characteristics, or global trends which may bias our estimates.[58]

Empirical Specification & Results. We build on Perotti (1996) and Forbes (2000). As stated previously, instead of using Gini coefficients of income, we use our wealth and income concentration measures. The empirical specification is as follows

$$\left(\frac{\Delta^T GDP}{GDP}\right)_{i,t} = \beta_{1,j}\text{top share}(j)_{i,t-1} + \delta \mathbf{X}_{i,t-1} + \text{TW}_t + \text{C}_i + \epsilon_{i,t} \ , \tag{30}$$

where TW_t and C_i denote respectively time window and country fixed effects. Similarly, $\mathbf{X}$ represents a vector of on period lagged controls. $\left(\frac{\Delta^T GDP}{GDP}\right)_{i,t}$ denotes the average *annual* growth rate in the subsequent T periods.[59] Our benchmark specification uses $T = 5$, which we also compare to $T = 3$ in the Appendix. $s(j)$ denotes the lagged wealth or income share held by the j^{th} quantile of the distribution.

Table 7 summarizes the results using both top income and wealth shares, as well as the benchmark Gini coefficient of income. In line with Forbes (2000), we find that the Gini coefficient of income is a positive and statistically significant predictor of subsequent GDP growth. Highlighting the pronounced role of top of the distribution, we find a significant effect of wealth and income shares held by the top 1% on subsequent GDP growth.

In Figure 10, we dissect the *IG-slope* using the wealth and income shares held in different quantiles. Three main results arise. First, bottom shares are positively correlated with subsequent GDP growth, while not being significant. Second, top shares are positively and significantly correlated

[58] In practice, country fixed effects are also convenient, as they control for time-invariant country-specific measurement methods (households versus individuals, and normalization choices) of income and wealth measures.

[59] The average annual growth rate is computed based on a geometric average according to $\left(\left(1 + \frac{\Delta^T GDP}{GDP}\right)^{\frac{1}{T}} - 1\right)_{i,t}$. Notice, that we vary the length and amount of selected time windows in the sensitive analysis in Appendix C.2.

Table 7. GDP growth and Inequality measures

	GDP growth T=3			GDP growth T=5		
	(1)	(2)	(3)	(4)	(5)	(6)
Gini inc. coeff	0.0012**			0.0012**		
	(0.0005)			(0.0005)		
top 1% income		0.3976***			0.2976***	
		(0.0781)			(0.0816)	
top 1% wealth			0.1463**			0.2133***
			(0.0669)			(0.0644)
CP + TW FE	Yes	Yes	Yes	Yes	Yes	Yes
Observations	383	472	163	242	288	113
R-squared	0.1376	0.1713	0.1115	0.1744	0.2010	0.2820
Notes:				*p<0.1; **p<0.05; ***p<0.01.		

with subsequent GDP growth, whereas middle-class quantiles exhibit a negative effect. The *U-shape* emerges when using both wealth and income concentration measures. Unfortunately, due to missing data, we cannot properly estimate effects at the bottom using wealth measures. Third, the inequality growth relationship substantially increases as we move to the very top of the distribution, i.e. top 0.1%, top 0.01%, and top 0.001%.

The results shown in Figure 10 are robust to an inclusion of a variety of controls and different sample selection. We present the sensitivity analysis in Appendix E.2. As an additional exercise, we also study the relationship between inequality and GDP growth at the state level in the US using the Frank (2009) database, updated in 2015 with top share income inequality. The IG-slope is also found significant and positive. Broadly speaking, our results highlight a significant positive relationship between income and wealth concentration at the upper end of the distribution and subsequent GDP growth. The forces behind this feature are not yet well studied.

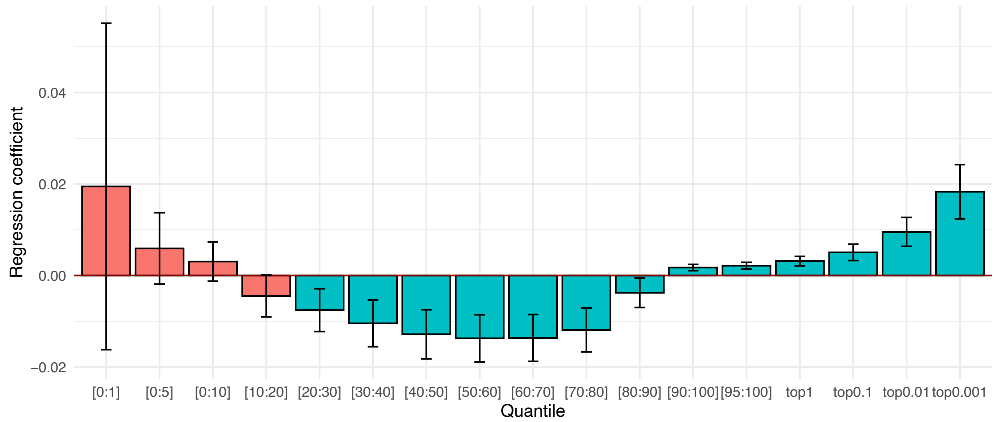
E.1.2 Decomposing the Inequality - GDP Growth Relationship (*IG-slope*)

While the previous section highlighted a positive link between inequality and subsequent growth, in particular at the upper end of the distribution, we now decompose the IG-slope into three main production components: physical capital, human capital, and the Solow residual (SR), i.e. part of growth which is unexplained by movements in capital and human capital and traditionally associated with technology changes. We start from the standard growth accounting model

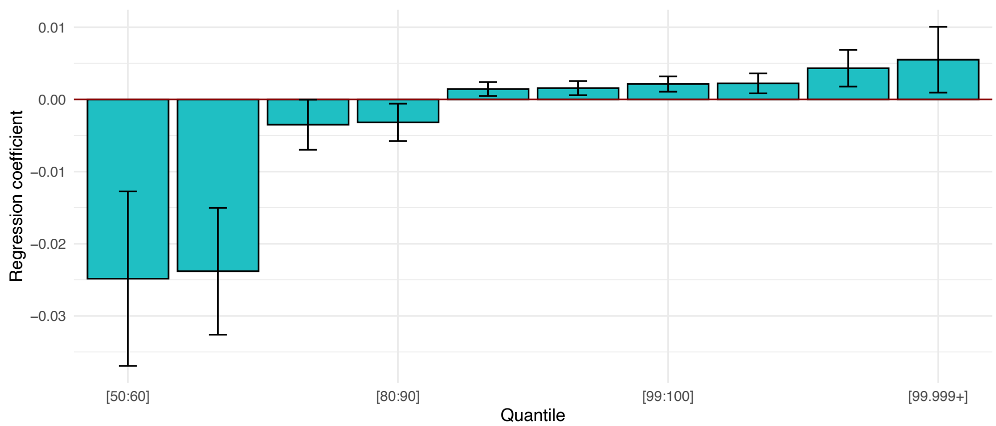
$$g_{y,t} = \alpha g_{k,t} + (1-\alpha) g_{hc,t} + SR_t \tag{31}$$

where $g_{y,t}$, $g_{k,t}$ and $g_{hc,t}$ represent respectively the GDP per capita growth rate, the capital per capita growth rate and the growth rate of human capital. Using the same specification (30) as in

Figure 10. *IG-slope* using 5-years time windows.



(a) Share of income held in different quantile and GDP growth.



(b) Share of wealth held in different quantile and GDP growth.

the previous panel regression, we test

$$\text{Growth component } \mathcal{C}_{i,t} = \beta_{1,j} s(j)_{i,t-1} + \delta \mathbf{X}_{i,t-1} + \mathrm{TW}_t + \mathrm{C}_i + \epsilon_{i,t}, \quad \forall \mathcal{C} \in \{k, hc, SR\} \,. \tag{32}$$

Table 8 displays the result for the top 1% income and wealth shares. Interestingly, capital growth and the Solow residual are significantly positively correlated with higher wealth and income concentration, while human capital growth turns out to be negatively correlated. The latter observation is consistent with the theory developed by Galor and Zeira (1993) stating in the presence of financial frictions a negative relationship between inequality and human capital formation. However, for developed countries, this channel seems to be of second order. Concerning physical capital growth, we find a positive effect of top inequality, which is consistent with a variety of models in which income-rich households tend to save a higher fraction of their income or wealth.

Most importantly, the growth accounting decomposition suggests a key role of the Solow residual in understanding the IG-slope. We point out that the results are robust to using different quantile shares, and mirror the ones found in the previous section.

Table 8. Growth Accounting: Decomposition of *IG-slope*.

	g_y	SR	αg_k	$(1-\alpha)g_{hc}$	g_y	SR	αg_k	$(1-\alpha)g_{hc}$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
top 1p income	0.47**	**0.38****	**0.08*****	−0.001				
	(0.19)	(0.18)	(0.03)	(0.02)				
top 1p wealth					0.27**	**0.25****	**0.04*****	−0.02**
					(0.11)	(0.10)	(0.01)	(0.01)
C + TW FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Observations	206	206	206	206	107	107	107	107
R-squared	0.27	0.29	0.21	0.14	0.36	0.39	0.35	0.33

Notes: *p<0.1; **p<0.05; ***p<0.01.

It is well understood that the Solow residual captures a variety of relevant factors: Among others, technical progress, imperfect competition or misallocation have been intensively discussed in the literature. Empirically, the relationship between inequality and TFP growth is consistent with Samila and Sorenson (2011) who document a significantly positive link between past inequality and the provision of venture capital. As argued by Greenwood et al. (2018), this type of capital has been shown to play a key role in fostering technology and GDP growth in the US.

The key takeaway from our analysis is that the upper end of the distribution is especially important in understanding the IG-slope. By decomposing the IG-slope into the classical growth components, the Solow residual appears to be a key driver of the relationship. This is consistent with the reallocation mechanisms between low productive and high productive risky capital outlined analytically within the tractable model. Finally, we point out that several other studies find a negative IG-slope, especially when considering low-income countries. In light of our theoretical framework, we rationalize these findings by using the GIF concept from section 2. The stage of development of a county goes along with different relative strength of type- and wealth-dependence. As a result, developed countries sort themselves more likely into the growth-enhancing region with a positive correlation between types and wealth. On the contrary, less developed countries fall into the growth dampening region.

E.2 Robustness of the Inequality – GDP growth (*IG slope*) relationship

We show in Table 9 and Table 10 sensitive analysis regarding the IG-slope. Broadly speaking, the results are consistent with the fact that an increase in the top wealth share is associated with higher subsequent growth.

Table 9. GDP growth and Inequality measures

	GDP growth T=3			GDP growth T=5		
	(1)	(2)	(3)	(4)	(5)	(6)
Gini inc. coeff	0.0012**			0.0006		
	(0.0005)			(0.0007)		
top 1p income		0.3976***				
		(0.0781)				
top 1p wealth			0.1463**			0.2470***
			(0.0669)			(0.0704)
Market Power					0.7660**	
					(0.3018)	
Log GDP per capita	−0.0171	−0.0066	−0.0219	0.0098	0.0007	−0.0122
	(0.0149)	(0.0143)	(0.0297)	(0.0262)	(0.0308)	(0.0289)
Human capital	−0.0599***	−0.0653***	−0.0402**	−0.1144***	−0.0599***	−0.0619***
	(0.0097)	(0.0087)	(0.0163)	(0.0181)	(0.0169)	(0.0180)
HC	0.0133	0.0603***	0.0358	0.0160	0.0845**	0.0423
	(0.0184)	(0.0169)	(0.0469)	(0.0447)	(0.0383)	(0.0472)
Country FE	Yes	Yes	Yes	Yes	Yes	Yes
Time Window FE	Yes	Yes	Yes	Yes	Yes	Yes
Observations	383	472	163	96	120	105
R-squared	0.1376	0.1713	0.1115	0.4655	0.2927	0.3079

Notes: *p<0.1; **p<0.05; ***p<0.01.

F Supplementary Material for the Analytical Model

F.1 Idiosyncratic productivity shock

We derive the solution of the analytical model using the assumption that idiosyncratic risks are in the form of productivity shocks on the risky technology, such that:

$$x_I^j = \kappa^i(\mu\phi + (1-\mu)A)\omega_1^i a_0^i \qquad \kappa^i \sim \mathcal{N}(1, \sigma_\kappa^2).$$

The returns to wealth on the household side is given by $R_r^i = (1-\varphi)\kappa^i\phi$, where again μ controls the pass-through between MPK on the production side and the realized return on the household side. Under this specification, the solution to the portfolio choice problem is isomorphic to the one derived in the baseline model.

Aggregation of x_I^i, however, follows from the assumption that there is a sub-continuum of households in each state (a_0, ϑ), such that

$$\begin{aligned}
X^I &= \int_\kappa \int_{\mathcal{A}\times\Theta} \kappa(\mu\phi + (1-\mu)A)\omega_1(a_0,\vartheta)a_0\, d\mathcal{G}(a_0,\vartheta)\, dF(\kappa) \\
&= \int_\kappa \kappa \int_{\mathcal{A}\times\Theta} (\mu\phi + (1-\mu)A)\omega_1(a_0,\vartheta)a_0\, d\mathcal{G}(a_0,\vartheta)\, dF(\kappa) \\
&= (\mu\phi + (1-\mu)A) \int_{\mathcal{A}\times\Theta} \omega_1(a_0,\vartheta)a_0\, d\mathcal{G}(a_0,\vartheta)
\end{aligned}$$

Table 10. Sensitive analysis: IG-slope

	Inequality measure	Coeff.	length of $\Delta^T GDP$	$n\|n\ TW$	Period	TW-C	R^2
Sample selection							
Period selected	share(99) income	0.230***	5y	40 | 232	1995-2018	Yes	.25
Period selected	share(99) wealth	0.304***	5y	27 | 86	1995-2018	Yes	.39
With all countries	share(99) income	0.109**	5y	100 | 675	1975-2018	Yes	.29
With all countries	share(99) wealth	0.079**	5y	36 | 175	1975-2018	Yes	.19
No bench. countries	share(99) income	0.160**	5y	60 | 353	1975-2018	Yes	.33
No bench. countries	share(99) wealth	-0.120	5y	5 | 18	1975-2018	Yes	.69
Excluding US	share(99) income	0.340***	5y	39 | 279	1975-2018	Yes	.21
Excluding US	share(99) wealth	0.281***	5y	30 | 104	1975-2018	Yes	.32
Additional controls & alternative measures							
Incl. $\log(\frac{credit}{GDP})$	share(99) income	0.105	5y	40 | 209	1975-2018	Yes	.20
Incl. $\log(\frac{credit}{GDP})$	share(99) wealth	0.178**	5y	28 | 94	1975-2018	Yes	.30
Incl. avg level [a]	level(99) income	0.016	5y	40 | 270	1975-2018	Yes	.25
Incl. avg level [a]	level(99) wealth	0.065***	5y	25 | 77	1975-2018	Yes	.60
Incl. avg level [a]	share(99) income	0.156	5y	40 | 270	1975-2018	Yes	.25
Incl. avg level [a]	share(99) wealth	0.190***	5y	25 | 77	1975-2018	Yes	.60
Incl. avg level alone	level(99) income	0.028***	5y	40 | 270	1975-2018	Yes	.24
Incl. avg level alone	level(99) wealth	0.079***	5y	25 | 77	1975-2018	Yes	.55
top tax rate	share(99) income	0.260**	5y	39 | 156	1975-2018	Yes	.55
top tax rate	share(99) wealth	0.222	5y	17 | 49	1975-2018	Yes	.33
Alt. measure	ratio(1/80) income	0.023	5y	34 | 235	1975-2018	Yes	.21
Alt. measure	ratio(1/80) wealth	0.054***	5y	30 | 104	1975-2018	Yes	.29
Asset pricing	share(99) income	0.174**	5y	38 | 243	1975-2018	Yes	.19
Asset pricing	share(99) wealth	0.157**	5y	27 | 103	1975-2018	Yes	.31
Δ share$_{t-1}$	share(90) income	0.281	5y	21 | 120	1975-2018	Yes	.25
Δ share$_{t-1}$	share(95) income	0.391*	5y	20 | 119	1975-2018	Yes	.26
Δ share$_{t-1}$	share(99) income	0.766**	5y	21 | 120	1975-2018	Yes	.29
Δ share$_{t-1}$	share(99.9) income	1.308**	5y	21 | 114	1975-2018	Yes	.37
Concentration of R2 [b]	share(90) R2	0.043	5y	24 | 53	1975-2018	Yes	.58
Concentration of R2 [b]	share(99) R2	0.119	5y	24 | 53	1975-2018	Yes	.62
Concentration of R2 [b]	share(99.9) R2	0.257*	5y	24 | 53	1975-2018	Yes	.64
Concentration of R2 [b]	share(99.99) R2	0.562**	5y	24 | 53	1975-2018	Yes	.65

Notes: *p<0.1; **p<0.05; ***p<0.01. In parenthesis: std. deviation.
[a] the regressions include both the level (in log) and the share of wealth and income held in the given quantile. We report both the coefficient for the level and the share.
[b] we use data using 4 years time windows in order to increase the number of observations.

F.2 Other form of wealth dependence

F.2.1 Decreasing returns to scale

We now show that some usual assumptions in the literature of the capitalist/entrepreneur framework can be nested within a *scale* dependence representation. Our example is build upon the seminal paper of Cagetti and De Nardi (2006). Suppose that individuals can now invest in their own private equity business investment, and that the latter generates decreasing returns as the size of the investment increases. Moreover, assume that there is a borrowing limit that prevents individuals investing in private equity investments, such that $k_1^i \leq \lambda a_0^i$.

The budget constraint is given by

$$c_2^i = wh^i + z^i (k_1^i)^\delta + R(a_0^i - k_1^i)$$

where z^i defines the entrepreneur's productivity (the types in this setting) and $(a_0^i - k_1^i)$ is the capital invested in a technology used by a corporate sector with technology: $Y^c = A\left(\int_i (a_0^i - k_1^i)di\right) n^\alpha$. There is no risk in this economy. The optimal choice of k_1^i is given by

$$k_1^i = \begin{cases} \lambda a_0^i & \text{if} \quad \lambda a_0^i < \left(\frac{\nu z^i}{R}\right)^{\frac{1}{1-\nu}} \\ \left(\frac{\nu z^i}{R}\right)^{\frac{1}{1-\nu}} & \text{if} \quad \lambda a_0^i \geq \left(\frac{\nu z^i}{R}\right)^{\frac{1}{1-\nu}} \end{cases}.$$

We make the simplifying assumption that $z \in \{0, z^h\}$, with z^h large enough such that investors are always reaching the borrowing constraint. This is the case in the Cagetti and De Nardi (2006) framework. In this economy, aggregate output is given by:

$$\begin{aligned} Y = Y^e + Y^c &= \int_i z^i (k_1^i)^\delta di + A\left(\int_i (a_0^i - k_1^i)di\right) n^\alpha \\ &= \lambda^\delta \int_i z^h \omega_z^h(a_0^i)(a_0^i)^\delta di + A\int_i a_0^i di - A\lambda \int_i \omega_z^h(a_0^i) a_0^i di \end{aligned}$$

where we use $\int_i h^i di = n = 1$ and $\omega_z^h(a_0^i)$ captures the proportion of individuals with state $z^i = z^h$ at the level of wealth a_0^i. For simplicity, we assume that there is no type dependence, such that $\omega_z^h(a_0^i) = \theta \, \forall a_0^i$. Then we obtain:

$$\begin{aligned} Y = Y^e + Y^c &= z^h \lambda^\delta \theta \int_i (a_0^i)^\delta di + A\int_i a_0^i di - A\lambda\theta \int_i a_0^i di \\ &= z^h \lambda^\delta \theta \frac{\underline{a}^\delta \eta}{\eta - \delta} + A(1-\lambda)\frac{\underline{a}\eta}{\eta - 1} \end{aligned}$$

such that wealth-normalized output is $\widetilde{Y} = A(1-\lambda) + \lambda^\delta \theta \underline{a}^{\delta - 1}\frac{\eta-1}{\eta-\delta}$. Computing the effect of a marginal change in the wealth pareto tail η on wealth-normalized output gives

$$\frac{\partial \widetilde{Y}(\eta)}{\partial \eta} = -\frac{z^h \lambda^\delta \theta \underline{a}^{\delta-1}}{(\eta-\delta)^2}(\delta - 1)\,.$$

That is, an increase in inequality (a decrease in η) decreases aggregate output as long as $\delta < 1$. In other words, decreasing returns to scale on private equity investments map into a framework with negative scale dependence.

With type dependence, we can make the simplifying assumption that $\omega_z^h(a_0^i) = \theta + (a_0^i)^\varrho$, where ϱ controls the selection of entrepreneurs across the wealth distribution. Under this specification, we can show that the effect of a change of η on wealth normalized output is given by:

$$\frac{\partial \widetilde{Y}(\eta)}{\partial \eta} = -z^h \lambda^\delta \left[(\delta - 1) \cdot \left(\frac{\theta \underline{a}^{\delta-1}}{(\eta-\delta)^2} + \frac{\xi \underline{a}^{\delta-\varrho-1}}{(\eta-\varrho-\delta)^2} \right) + \varrho \cdot \frac{\xi \underline{a}^{\delta-\varrho-1}}{(\eta-\varrho-\delta)^2} \right].$$

With positive sorting of entrepreneur, i.e. $\varrho > 0$, and negative scale dependence through decreasing returns to scale, i.e. $\delta < 1$, the effect of a change in the Pareto shape η is ambiguous.

F.3 Aggregate shocks

We augment the setup assuming that risky investment productivity is now given by $z(\phi\mu + A(1-\mu))$ with $\mathcal{N}(1, \sigma_z)$. On the investor side, the risky returns are now given by $R_r^i = (1-\varphi) z\kappa^i$. We assume that the product $z\kappa^i$ follows a Gaussian distribution with mean ϕ and variance $\sigma_{z\kappa}^2$.

Doing this under the special case yields an updated portfolio share with $\widetilde{\omega} = \frac{(\phi - A)}{(1-\varphi)\sigma_{z\kappa}^2}$. The aggregate output is now given by

$$\begin{aligned} Y &= z(\phi\mu + A(1-\mu))K_I + AK_N \\ &= z(\phi\mu + A(1-\mu))K_I + A(\mathbb{E}[a_0] - K_I) \\ &= [z(\phi\mu + A(1-\mu)) - A]K_I + A\mathbb{E}[a_0] \end{aligned}$$

The variance of the wealth-normalized output is given by

$$\widetilde{Y} = [z(\phi\mu + A(1-\mu)) - A](K_I/\mathbb{E}[a_0]) + A$$

which after plugging the solution for $(K_I/\mathbb{E}[a_0])$ yields

$$\widetilde{Y}(\eta) = \widetilde{\omega}[z(\phi\mu + A(1-\mu)) - A]\Psi(\eta) + A$$

where the variance of the wealth-normalized output is

$$\sigma_{\widetilde{Y}}(\eta) = \sigma_z^2 \widetilde{\omega}^2 (\phi\mu + A(1-\mu))^2 \Psi(\eta)^2$$

The effect of a marginal change in η on the wealth-normalized output volatility is therefore

$$\frac{\partial \sigma^2_{\tilde{Y}}(\eta)}{\partial \eta} = -2\sigma_z^2 \widetilde{\omega}^2(\phi\mu + A(1-\mu))^2 \left(\frac{\partial \Psi(\eta)}{\partial \eta}\right) \Psi(\eta) \tag{33}$$

where $\Psi(\eta) = \left(1 + \frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right) \frac{\eta-1}{\eta-\gamma}$ and $\left(\frac{\partial \Psi(\eta)}{\partial \eta}\right)$ is given in the proof of Proposition 1 □

F.4 Aggregate DRS

In the core paper, we made the assumption that the demand for capital was perfectly elastic. We now augment the setup with aggregate decreasing return to scale such that: $Y = \left(\int_i x_i di\right)^{1-\varphi} n^{\varphi}$. Under this specification, the price $p = (1-\varphi)^2 X^{-\varphi}$. The portfolio choice of individuals, under the special case, is given by:

$$\omega_1^i = \frac{\phi - A}{(1-\varphi)^2 X^{-\varphi} \sigma_\kappa^2} \frac{\vartheta_i}{\overline{\vartheta}} (a_0^i)^{\gamma-1} \tag{34}$$

An increase in X generates a decrease in the return premium, but also a decrease in the dispersion of returns. The latter effect dominates and the share increases with X. Aggregating the policy functions yields the following relation for output

$$\begin{aligned} Y &= \Big(AK_N + (\mu\phi + (1-\mu)A)K_I\Big)^{1-\varphi} \\ &= \Big(A(1-\chi) + (\mu\phi + (1-\mu)A)\chi\Big)^{1-\varphi} \mathbb{E}[a_0]^{1-\varphi} \\ &= \Big(A + \mu(\phi - A)\chi\Big)^{1-\varphi} \mathbb{E}[a_0]^{1-\varphi} \end{aligned}$$

where $\chi = \frac{K_I}{\mathbb{E}[a_0]} = C(\eta).X^{\varphi}$, where the dependence to X^{φ} comes from aggregating risky assets using the policy function (34).

This yields:

$$Y = \Big(A + \mu(\phi - A)C(\eta)X^{\varphi}\Big)^{1-\varphi} \mathbb{E}[a_0]^{1-\varphi}.$$

Notice that $Y = X^{1-\varphi}$. Therefore we get:

$$X^{1-\varphi} = \Big(A + \mu(\phi - A)C(\eta)X^{\varphi}\Big)^{1-\varphi} \mathbb{E}[a_0]^{1-\varphi}.$$

Which amounts to solve a fixed point problem. Under the assumption that $A = 0$, this yields:

$$X^{1-\varphi-\varphi(1-\varphi)} = \left(\mu\phi C(\eta)\right)^{1-\varphi} \mathbb{E}[a_0]^{1-\varphi}$$
$$X = \left[\left(\mu\phi C(\eta)\right)^{1-\varphi} \mathbb{E}[a_0]^{1-\varphi}\right]^{\frac{1}{(\phi-1)(\phi+1)}} .$$

F.5 Additional results for optimal wealth taxation

F.6 Existence of a representative households

Lemma 8. *Let us define aggregate output of the economy as $Y \equiv AK_N + \phi K_I$. It is given by*

$$Y = A\left(\mu_{a_0} - K_I\right) + \phi K_I .$$

(a) *Suppose that $\vartheta_i = \vartheta\ \forall i$. There exists a representative household (RH) whose aggregate dynamics are equivalent to the ones obtained from the aggregation of individual dynamics if and only if $\gamma \in \{0, 1\}$. In both cases the RH is characterized by $\alpha_{RH} = \frac{\alpha}{\vartheta \frac{\eta}{\eta-\gamma} \underline{a}^{\gamma}}$ and $a_{RH,0} = \mu_{a_0}$.*
(b) *Contrary, let us suppose that $\vartheta_i \in \vartheta$ with $\vartheta \subset \mathbb{R}_+$. Furthermore, let us assume that*

$$\tilde{\vartheta} \equiv \mu_{a_0}^{-\gamma}\left(corr(\vartheta, a_0^{\gamma})\sigma_{\vartheta}\sigma_{a_0} + \mu_{\vartheta}\mu_{a_0^{\gamma}}\right) \in \vartheta \quad \text{and} \quad h(\tilde{\vartheta}, \mu_{a_0}) > 0 ,$$

where $h(\vartheta, a_0)$ denotes the joint probability density function of types and initial wealth. Then, there exists a RH $\forall\ \gamma \geq 0$, who is characterized by $\vartheta_{RH} = \tilde{\vartheta}$ and $a_{RH,0} = \mu_{a_0}$.

Proof. Let us first consider the more general case with $\vartheta_i \in \Theta$. To establish part (b) of Lemma 8, we first compute aggregate innovative asset holdings by integrating over the individual ones which we have derived in Corollary 1. This gives us,

$$K_I^* = (\widetilde{\omega}/\overline{\vartheta})\left(\rho_{\vartheta, a_0^{\gamma}}\sigma_{\vartheta}\sigma_{a_0^{\gamma}} + \mu_{\vartheta}\mu_{a_0^{\gamma}}\right) .$$

Due to $u'(c_{i2}) > 0$ we know that the budget constraint binds and all initial wealth is invested due to the existence of a risk-free asset, such that $K_N = \mu_{a_0} - K_I^*$.

Resubstitution into the definition of y gives us then

$$Y = A\left(\mu_{a_0} - (\widetilde{\omega}/\overline{\vartheta})\left(\rho_{\vartheta, a_0^{\gamma}}\sigma_{\vartheta}\sigma_{a_0^{\gamma}} + \mu_{\vartheta}\mu_{a_0^{\gamma}}\right)\right) + \phi(\widetilde{\omega}/\overline{\vartheta})\left(\rho_{\vartheta, a_0^{\gamma}}\sigma_{\vartheta}\sigma_{a_0^{\gamma}} + \mu_{\vartheta}\mu_{a_0^{\gamma}}\right) .$$

To show the existence of a representative household, we consider the production of a fictitious representative household with innate risk aversion type ϑ_{RH} and initial wealth holdings $a_{RH,0}$.

Her terminal production is given by

$$y_{RH} = A\left(a_{RH,0} - (\widetilde{\omega}/\overline{\vartheta})\vartheta_{RH}a_{RH,0}^{\gamma}\right) + \phi(\widetilde{\omega}/\overline{\vartheta})\vartheta_{RH}a_{RH,0}^{\gamma} \,.$$

We need to show fthat $y = y_{RH}$ holds. Comparing the previous two equations, we easily see that the only candidate for the existence of a representative household is characterized by

$$\vartheta_{RH}a_{RH,0}^{\gamma} = \left(\rho_{\vartheta,a_0^{\gamma}}\sigma_{\vartheta}\sigma_{a_0^{\gamma}} + \mu_{\vartheta}\mu_{a_0^{\gamma}}\right) \,, \tag{ES1}$$

$$a_{RH,0} = \mu_{a_0} \,. \tag{ES2}$$

The previous two equations describe a system of nonlinear equations in two unknowns. Its solution is found by substitution and exists if $\vartheta_{RH} \in \Theta$ and $h(\vartheta_{RH}, a_{RH,0}) > 0$. As a result, the RH is characterized by

$$\vartheta_{RH} = \mu_{a_0}^{-\gamma}\left(\rho_{\vartheta,a_0^{\gamma}}\sigma_{\vartheta}\sigma_{a_0^{\gamma}} + \mu_{\vartheta}\mu_{a_0^{\gamma}}\right) \,, \quad \text{and} \quad a_{RH,0} = \mu_{a_0} \,.$$

In order to establish part (a) of Lemma 8, it is straightforward to simplify the previous system of equations ES1-ES2 by recognizing that $\vartheta_i = \vartheta = \vartheta_{RH} = \mu_{\vartheta}$ such that we obtain

$$a_{RH,0} = (\mu_{a_0^{\gamma}})^{\frac{1}{\gamma}} \Leftrightarrow \mu_{a_0}^{\gamma} = \mu_{a_0^{\gamma}} \Leftrightarrow \left(\frac{\eta}{\eta-1}\right)^{\gamma}\underline{a}^{\gamma} = \frac{\eta}{\eta-\gamma}\underline{a}^{\gamma} \,.$$

Hence, to show the unique existence of a representative household in this case, we need to verify that for each $\eta > \gamma$ the intersection $\left(\frac{\eta}{\eta-1}\right)^{\gamma} = \frac{\eta}{\eta-\gamma}$ exists. For $\gamma \in \{0,1\}$ this is obviously the case. Hence, we are left to show that there does not exist any other $\gamma > 0$ for which this is valid. To do so, let us first rewrite $\left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}} = \frac{\eta}{\eta-1}$. Taking the derivative of the LHS w.r.t. γ gives us

$$\frac{\partial LHS}{\partial \gamma} = \left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}}\left(-\frac{\ln\left(\frac{\eta}{\eta-\gamma}\right)}{\gamma^2} + \frac{1}{\gamma}\frac{1}{\frac{\eta}{\eta-\gamma}}\frac{\eta}{(\eta-\gamma)^2}\right) = \frac{1}{\gamma^2}\left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{\gamma}{\eta-\gamma} - \ln\left(\frac{\eta}{\eta-\gamma}\right)\right) \,.$$

The derivative is (strictly) positive if $\frac{\gamma}{\eta-\gamma} > \ln\left(\frac{\eta}{\eta-\gamma}\right) = \ln\left(1 + \frac{\gamma}{\eta-\gamma}\right)$ holds $\forall\ \gamma > 0$. Redefining $x \equiv \frac{\gamma}{\eta-\gamma} \in (0,\infty)$, we can equivalently rewrite the previous inequality $x > \ln(1+x)$. To proceed the execution, let us state a property of the natural logarithm.

Result 1. *$\forall$ x>0, the following inequalities hold:* $\frac{x}{1+x} < \ln(1+x) < x$.

The proof of the postulated result is straightforward: Recognizing that the natural logarithm is a strictly concave function $\forall\ x > 0$ and using the auxiliary variable $\tilde{x} \neq 1$, an application of the mean value theorem provides us with $\ln(\tilde{x}) - \ln(1) < \tilde{x} - 1$. Substituting in $\tilde{x} = 1 + x$ yields the upper bound, substituting in $\tilde{x} = \frac{1}{1+x}$ gives us respectively the lower bound.

As a result, we obtain that $\frac{\gamma}{\eta-\gamma} > \ln\left(\frac{\eta}{\eta-\gamma}\right)$ such that the left hand side of $\left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}} = \frac{\eta}{\eta-1}$ is strictly increasing in γ $\forall \gamma < \eta$. It remains to show that the left hand side limit for $\gamma \to 0$ is lower than $\frac{\eta}{\eta-1}$. An application of L'Hospital's rule gives

$$\lim_{\gamma \to 0} \left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}} = \lim_{\gamma \to 0} e^{\frac{1}{\gamma} \ln\left(\frac{\eta}{\eta-\gamma}\right)} = e^{\lim_{\gamma \to 0} \frac{1}{\gamma} \ln\left(\frac{\eta}{\eta-\gamma}\right)} = e^{\frac{1}{\eta}} \ ,$$

where the second equality follows from the continuity of the exponential function. It remains to show that $e^{\frac{1}{\eta}} < \frac{\eta}{\eta-1}$. Taking the natural logarithm on both sides this is equivalent to showing $\frac{1}{\eta} < \ln\left(\frac{\eta}{\eta-1}\right) = \ln\left(1 + \frac{1}{\eta-1}\right)$. By making use of the lower bound of Result 1, one can verify that the inequality holds indeed. This concludes the proof that an intersection for an arbitrary η occurs if $\gamma \in \{0, 1\}$. In both cases initial wealth is given by $a_{RH,0} = \mu_{a_0}$ and absolute risk aversion by $\alpha_{RH} = \frac{\alpha}{\mu_\vartheta \mu_{a_0}^{\gamma}}$. This shows that $\gamma \in \{0, 1\}$ is necessary for the existence of a representative household. Finally, sufficiency follows from substituting $\gamma \in \{0, 1\}$ into the system ES1-ES2 and showing that a RH indeed exists in this case. $\square$