Wealth Inequality and Capital Heterogeneity: A Tale of two Dependencies*

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Abstract

We study the macroeconomic effects of the recently empirically documented heterogeneity in capital investment and associated returns in a unified general equilibrium framework that incorporates *type* and *scale* dependence. The former reflects the selection of agents with heterogeneous preferences and skills across the wealth distribution, while the latter comprises wealth-driven components of investment choices. We first derive a general decomposition of the output response of a wealth inequality shock based on both dependencies, and show how it conditions optimal wealth redistribution. Second, in a quantitative dynamic incomplete markets model, both dependencies are shown to shape (i) wealth inequality, and (ii) the elasticity of aggregate output and investment to a wealth redistribution shock. Under *scale* dependence, there arises a dynamic self-enforcing multiplier as wealth redistribution shocks impact individual investment and thus future wealth and scale dependent effects. We numerically show that optimal top wealth tax rates non-trivially depend on the quantitative distinction between *type* and *scale* dependence, and the degree to which returns reflects their marginal product. When both dependencies are calibrated using novel evidence from US micro data, we find that the top marginal wealth tax is positive, around 3%.

Keywords: Wealth Inequality, Type and Scale Dependence, Capital Supply, Redistribution.

JEL codes: E22, E61, O41.

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1 Introduction

In a renowned dialogue, American novelist F. Scott Fitzgerald is reputed to have said to his fellow, Ernest Hemingway: *"The rich are different from you and me."*. To what, Hemingway is quoted to have replied: *"Yes. They have more money."*. This exchange illustrates the fundamental distinction between the concepts of *type* dependence and *scale* dependence. Following Fitzgerald, *type* dependence, or composition effect, is the notion that the wealth-rich and the wealth-poor are intrinsically different in their innate or persistent characteristics, related among others to education, investment and entrepreneurial skills, and preferences. Following Hemingway, *scale* dependence simply describes the notion that the wealth-rich and the wealth-poor differ because they hold different amounts of wealth that hence condition their behavior.[1] Many contributions in the literature integrate some form of *type* and/or *scale* dependence to analyze the link between the wealth distribution and macroeconomic outcomes, yet surprisingly, their systematic distinction has so far received little attention.

This paper integrates those two prominent modeling devices and derives a decomposition of the joint effects of type and scale dependence on the aggregates when heterogeneity in capital investments and returns – one empirically-relevant mechanism to generate a wealth distribution with a thick right tail (Benhabib et al. (2011, 2019)) – is taken into account. An important result that emerges from our theoretical analysis is that *type* and *scale* dependence, and their combination, can be consistent with key macroeconomic moments, but their relative importance implies a specific aggregate response to a wealth inequality shock, conditioning wealth redistribution. Quantitatively, we unravel and clarify new findings regarding optimal capital taxation. Given the heterogeneity in returns to wealth, we show that the interaction between the degree of *type/scale* dependence as well as how much returns to capital reflect their marginal product condition the way capital stocks and income flows should be taxed. Finally, using US micro evidence, we quantitatively decompose the contributing role of both dependencies in shaping the wealth concentration, the aggregate response to an inequality shock, and derive the US welfare-maximizing wealth tax schedule.

Our paper is motivated by a growing body of recent empirical evidence showing that *type* and *scale* dependence are important components of capital returns heterogeneity across the wealth distribution (Piketty (2018),Bach et al. (2020), Fagereng et al. (2020)) and in portfolio choices (Cal-

[1]Common scale dependence mechanisms include wealth driven risk aversion, convex saving policy function, non-convex investment costs, or non-constant returns to scale.

vet et al. (2019),Meeuwis (2019)). These pieces of evidence are arguably related to the fact that the wealth-rich and the wealth-poor substantially differ in their portfolio allocation. In the US, the 1% wealthiest households owned around 36% of the total wealth in 2012 (Saez and Zucman, 2016) and, according to estimates from the 2010 Survey of Consumer Finances (SCF), they held a substantial share of their wealth in private equity investments. Despite those empirical evidence, many macroeconomic models abstract from returns/investments heterogeneity and none explicitly incorporate *type* and *scale* dependence into a general equilibrium model.

In a tractable two-period model, we first derive a general decomposition of the link between the shape of the wealth distribution and macroeconomic variables. Our setup introduces heterogeneity in portfolio choices due to some form of *type* and *scale* dependence, and embeds two mechanisms to explain why returns may be different across asset classes: because they differ in their marginal product (MPK) or because of some form of rent-extraction coming from market or bargaining power modeled as a zero-sum return wedge EXPLAIN (Cahuc and Challe (2012); Piketty et al. (2014); Rothschild and Scheuer (2016) among others). It first builds on the incomplete markets model of Angeletos and Calvet (2005) and Angeletos and Calvet (2006) in which households allocate their wealth into a risk-free technology and into a riskier technology with high returns, but extend their framework with wealth-dependent risk taking. To achieve this, we introduce *generalized* CARA preferences, that let us capture scale effects on household investment decisions while keeping the model tractable. Second, we introduce innate heterogeneity in risk tolerance, or abilities, that aim to capture capitalist-entrepreneur skills, along the line of Moll (2014). High types, in our setup, are willing to undergo higher capital income risk irrespectively of their wealth level. As a result, the whole economy is populated by heterogeneous households with different degrees of risk taking incentives. Aggregate productivity (TFP) is determined by aggregating efficiency units of household capital investments and the effects of a wealth inequality shock is determined by the reallocation of wealth among households who differ in their marginal investment productivities, which, in our setting, depend on the distribution of risk taking types and wealth holdings.

In this environment, the inequality-efficiency relation is determined by four statistics, (i) the shape of the wealth distribution, (ii) how much returns to capital reflect their marginal product, (iii) the elasticity of risk-taking to wealth, and (iv) a dependence parameter that captures how individuals with different types are selected or sorted along the wealth distribution. The relative strength of *type* versus *scale* dependence determines the inequality-efficiency relation of the economy: for a given level of wealth inequality, this is the level of growth that is achieved with a marginal increase in wealth inequality on the space governing *type* and *scale* dependence. Inequality variations are irrelevant for growth only when both dependencies are inexistent or exactly offset each other. In such a case, the economy is said to be on the *Growth Irrelevance Frontier*, the special *iso-growth* curve that separates the growth-hampering and the growth-enhancing regions. An *iso-growth* curve is decreasing, meaning that positive selection of types along the wealth distribution should be compensated by a lower *scale* dependence to maintain the same level of growth. It also depends on the level of wealth inequality as *scale* dependence triggers a behavioral response which is stronger in a high wealth inequality regime. Therefore, while both dependencies can consistently match the distribution of capital allocation and returns in our static model, they generate substantially different aggregate responses.

Using our analytical framework, we then characterize the welfare-maximizing wealth redistribution when household are valued in a social welfare function. The key trade-offs are as follows. First, equilibrium wage depends on aggregate productivity. Second, whenever returns to capital do not entirely reflect their marginal product, there is a feedback equilibrium effect on overall capital returns such that total capital income distributed in the economy equalizes the total product of capital. Third, there is a standard redistribution motive as households differ in their wealth holdings in the economy. Importantly, the first two effects goes in opposite directions and their magnitude depends on the degree of *type* and *scale* dependence. Our insights generalize to saving decisions that interact with uninsurable idiosyncratic labor productivity shocks. Finally, while we focus on risk-taking, our insights may be reframed with other forms of scale-dependence.

For illustration purposes, the above static analytical decomposition assumed as exogenous the joint distribution of wealth and risk-taking types whereas it arises endogenously in reality. We thus extend the framework to a richer quantitative dynamic environment in which the distribution of wealth is an endogenous object and results from heterogeneity in capital investments and labor productivity of households. The heterogeneity in capital investments is captured in a reduced form portfolio choice that depends on risk-taking types, wealth holdings and an element of *luck*. A government taxes capital (stock and income flows) and labor income to finance an exogenous set of public expenditures. Using the SCF and the PSID, we calibrate the stationary equilibrium to the US economy under three different alternatives. The first features only *scale*-dependence, in which wealth holdings drive portfolio allocations, and the second only *type*-dependence, in which types reflect the amount invested irrespective of household wealth. The third alternative is an *hybrid* model using key facts from the data to calibrate realistic degrees of *type* and *scale* dependence. Specifically, *type* dependence is introduced with two *persistent* (but not permanent) types of households: investors who invest part of their wealth into risky equity, consistent with the data, and non investors, who do not invest in equity. The *scale*-dependent components feature an extensive margin, as the probability to transit from non-equity investor to equity investor increases with wealth, and an intensive margin, as we document that a substantial portions of equity investments are made through diversification of additional private equity business investments at the top of the wealth distribution.

Given the key features highlighted in our analytical setup, it is important that the alternative quantitative models provide a good representation of the wealth distribution, especially at the top. We show that the three models can generate the appropriate thickness of the upper tail, driven by an empirically-consistent distribution of returns to wealth. At the stationary equilibrium and because *types* are persistent, households extracting higher capital returns tend to self-select at the top of the wealth distribution. A more dispersed distribution of types induce a stronger selection and leads to higher wealth inequality. In contrast, *scale* dependence simply implies that wealth-rich households invest differently relative to wealth-poor households. With a positive effect of wealth on the propensity of risky investments, the model is able to generate the top wealth shares. However, while the models are observationally equivalent in the *cross-section*, they produce substantially different output-responses to a wealth inequality shock. In a model with only scale dependence, a 1% tax rate on the top 1% wealthiest households reduces long-run output by 1.9%, against 0.8% in a model with only *type* dependence. The *hybrid* model falls in between, with a long-run elasticity of 1.1%. *Scale* dependence produces an especially high response because it implies a dynamic self-enforcing behavioral response as equity investment (and thus future expected wealth) is a function of wealth itself.

We finally show that the crucial distinction between *type* and *scale* dependence emerges when considering the effects of a tax on wealth. We find that depending on whether returns reflect the marginal product of capital investment, *type* and *scale* dependence lead to rather opposite predictions. Under *scale* dependence, returns are correlated with the level of wealth. Hence, a wealth tax decreases the scale effects in portfolio choices. If returns reflect the marginal product of capital, this translates in rather large efficiency losses, and the optimal top marginal wealth tax is close to zero. If, instead, returns reflect pure rent-extraction and do not improve the product from capital investments, the optimal top marginal wealth tax is found to be very large, at 15%, even if households at the very top are valued in the social welfare function. Under *type-dependence*, a wealth tax improves the selection of high returns households at the top of the wealth distribution. The intuition is given by Guvenen et al. (2019). Suppose that two individuals differ in their wealth returns. By taxing the wealth stock, individuals with lower returns are relatively more affected by the tax, generating a downward wealth mobility for them. Conversely, high capital income earners are even further selected at the top of the distribution. Therefore, the joint distribution of type and wealth is endogenous to the wealth tax and this selection effect turns to be critical. Indeed, if returns reflect the marginal product of capital, it is typically optimal to tax wealth at a relatively high rate (around 3%), to further select high capital earners at the top. On the flip side, as the fraction of rents in capital returns increases, taxing wealth increases the total amount of capital rents in the economy, and the optimal wealth tax tend to decrease, the opposite to what was found under *scale*-dependence. The degree of *type* and *scale* dependence together with the degree of marginal product of capital in returns are thus two critical statistics that conditions the effects of a wealth tax in the economy.

Using our *hybrid* model, we finally compute the welfare maximizing capital tax schedule for the US economy.[2] We find a benchmark (without rent-extraction) optimal top marginal tax rate of 2.8%. For the sake of illustration, we then attribute returns extracted from equity investments made in Finance and Law sectors as rent-extraction following the insights of Lockwood et al. (2017), and find that the optimal wealth tax is slightly lower, at X%, showing that *type-dependence* seem to dominate the direction of the tax in this economy.

After discussing how our paper is integrated into the literature, Section 2 introduces the analytical framework. We present our quantitative dynamic model, its calibration and its properties in Section 3. In Section 4.1.1, we use evidence of *scale* dependence in the PSID and the SCF data to quantitatively decompose the role of both dependencies for the aggregates. Finally, in Section 6, we use our model to study the welfare-maximizing top marginal wealth tax and section 8 concludes. The appendix contains all the proofs as well as empirical and computational details.

Related literature This paper originates from the many mechanisms in the macroeconomic literature that features a *type* or *scale* dependence representation, either in isolation or in combination.[3]

[2] We study optimal capital tax and sidestep the question of whether the taxation of labor income, consumption or bequests are useful redistribution policy tools. We view the taxation of capital as a natural and transparent experiment linking the relationship between capital heterogeneity and macroeconomic outcomes.

[3] Among many others, Kaldor (1956), Stiglitz (1969) and Bourguignon (1981) study the role of the wealth distribution in a neoclassical economy with *scale* dependence in saving behaviors. Other mechanisms include non-convex investment cost and DRS (Banerjee and Newman (1993), Galor and Zeira (1993), Blaum et al. (2013) and Kaplan et al.

Especially relevant to our work are the theoretical papers by Gabaix et al. (2016) who show that both elements are capable of explaining the recent rise of top income inequality, and Moll (2014) who develops a tractable model of types in investment productivity and show that their persistence is key to derive the aggregate TFP. The importance of heterogeneity in capital returns in shaping the observed high wealth concentration is studied theoretically in Benhabib, Bisin and Zhu (2011) and quantitatively in Benhabib, Bisin and Luo (2019), with both papers focusing on *type* dependence. Hubmer, Krusell and Smith Jr (2020) use returns estimates from Bach et al. (2020) and interpret this heterogeneity as only *scale* dependence to study the recent rise of wealth inequality in the US. Xavier (2020) finds increasing returns in the SCF using a method that compares returns for a given quantile across survey waves and shows that a model with earnings heterogeneity and only *type* dependence in returns can account for the top 10% wealth share. To the best of our knowledge, however, this paper is the first to decompose the distinct role of *type* and *scale* dependence in shaping the wealth distribution, the response of aggregate variables to a wealth inequality shock, and optimal wealth taxation in a general equilibrium context.

Market incompleteness and capital income risk link our paper to Angeletos and Calvet (2005, 2006). Their setups, however, features linear investment policy functions which rules out *distributional relevance*, a feature that we precisely aim to study in this paper. By focusing on heterogeneity in efficient investment through risk-taking behaviors, this paper is somewhat related to the risk preference literature (see Peress (2004), Brunnermeier and Nagel (2008), Calvet et al. (2009) and Meeuwis (2019) among others) and to the quantitative framework of Robinson (2012); Herranz et al. (2015). Those papers rationalize a number of mechanisms that could indeed lead to *type* and *scale* dependence, but our insights may prove to be more general.

Ultimately, our paper is related to the recent empirical work shedding light on the presence of both *type* and *scale* dependence in explaining the substantial and persistent heterogeneity in wealth returns (Piketty (2018) (Chapter 12), Bach et al. (2020); Fagereng et al. (2020), Xavier (2020)).[4] We

(2018)), economies of scale in wealth management (Kacperczyk et al., 2019), social status derived from wealth holdings (Roussanov, 2010), investment in financial sophistication (Lusardi et al., 2017), wealth-dependent offshore investments (Alstadsæter et al., 2018) or non-convex investment costs in high return assets (Kaplan et al., 2018). In contrast, Kihlstrom and Laffont (1979), Moll (2014) or Moll and Itskhoki (2019) introduce *type* dependence in which the distribution of types, and their persistence, is crucial to derive aggregate efficiency. Combining effects arises in Quadrini (2000) and Cagetti and De Nardi (2006) through *type* dependence because entrepreneurs/business owners self-select at the top of the wealth distribution, and through *scale* dependence due to a wealth-driven occupational choice and a DRS production technology.

[4]Other related work are worth mentioning. A recent working paper by Halvorsen et al (2021) show that private and public equity investments are responsible for the very high return heterogeneity at the upper end of the wealth distribution using Norwegian administrative data. Fagereng et al. (2019) shows the importance of capital gains and heterogeneity in portfolio composition in explaining differences in gross savings. Also related is Kuhn et al. (2020) who

contribute to the above pieces of evidence for the US derived using detailed portfolio composition in the SCF and the returns to wealth estimated using the panel dimension of the PSID.

Our paper is finally related to those quantifying optimal taxation in models with differing productive assets (Kitao, 2008; Cagetti and De Nardi, 2009; Shourideh et al., 2012; Guvenen et al., 2019; Brüggemann, 2020; Jakobsen et al., 2020; Boar and Midrigan, 2020; Boar and Knowles, 2020; Rotberg and Steinberg, 2021). Despite the rapid growth of this literature, the implications of *type* and *scale* dependence in these models have not yet been explored, and our paper fills this gap.

2 An Analytical Two-Period Model

We begin with an analytical two-period framework with capital heterogeneity to illustrate the key forces at play and introduce the notations. We will later extend the analysis with further results in a rich quantitative model in which the joint distribution of types and wealth arises endogenously.

2.1 Environment

Households A unit mass of heterogeneous households indexed i lives for two periods, $t = 1, 2$, with initial wealth a_0^i and innate risk-taking type ϑ^i drawn from the joint distribution $\mathcal{G}(\vartheta, a_0)$. Preferences over consumption c_1^i and c_2^i are recursive with: $u_1 = U(c_1) + \beta U\Big(G^{-1}\Big(\mathbb{E}\Big[G\Big(U^{-1}(u_2)\Big)\Big)\Big)$, $u_2 = U(c_2)$ and $U = \frac{c^{1-1/\sigma}}{1-1/\sigma}$, with $\beta \in (0,1)$ and $\sigma > 0$. G introduce *generalized* constant absolute risk aversion (CARA) preferences that aggregate consumption across states with:

$$G(c^i; \vartheta^i, a_0^i) = \Big(1/\alpha^i\Big)\Big[1 - e^{-\alpha^i c^i}\Big], \quad \text{with} \quad \alpha^i := \overline{\vartheta} \cdot \Big[\vartheta^i (a_0^i)^\gamma\Big]^{-1}, \tag{1}$$

where the absolute risk aversion α^i depends on the average economy-wide risk tolerance $\overline{\vartheta}$, agent's innate risk-taking type ϑ^i and initial wealth a_0^i. The parameter $\gamma \geq 0$ governs the shape of the wealth-dependence risk-taking. The *generalized* CARA thus captures in a reduced form several mechanisms driving type heterogeneity and wealth dependence in portfolio choices and capital returns mentioned in the related literature.[5]

In $t = 1$, households consume c_1^i and invest optimally a share ω_1^i of their saving a_1^i in *risky* innovative asset with stochastic gross return R_r^i, and a share $(1 - \omega_1^i)$ in *risk-free* asset certain gross return R_f. Agents inelastically supply one labor unit with productivity $h^i \sim \mathcal{N}(1, \sigma_h)$. In $t = 2$,

show the importance of the business equity at the very top.

[5]It extends the utility functions used in Alpanda and Woglom (2007) or Makarov and Schornick (2010) by specifying the wealth normalization with a power function. It is also ultimately linked to Guiso and Paiella (2000) and Gollier (2001), who specify the shape of risk tolerance in terms of consumption rather than wealth.

production, government transfer T, returns $\{R_f, R_r\}$ and wage rate w realize and households consume c_2^i. Household i problem is therefore:

$$v_0^i = \max_{c_1^i, w_i^1, a_1^i} \frac{1}{1-1/\sigma} \left((c_1^i)^{1-1/\sigma} + \beta \left\{ -\frac{1}{\alpha^i} \log \left(\mathbb{E}\left[e^{-\alpha^i c_2^i} \right] \right) \right\}^{1-1/\sigma} \right) \tag{P1}$$

$$s.t. \qquad c_1^i + a_1^i \leq a_0^i\,, \qquad c_2^i \leq wh^i + R_f a_1^i + \omega_1^i a_1^i (R_r^i - R_f) + T + \underline{r} a_1^i\,.$$

where $\underline{r}$ is an equilibrium variable of the total returns to wealth specified below.

Production side A competitive final good producer uses labor n and intermediate goods x^j with technology $Y = \left(\int_j x^j \, dj\right) n^\varphi$ and maximizes $\max_{n,\{x^j\}_j} \left(\int_j x^j \, dj\right) n^\varphi - wn - \int_j p^j x^j \, dj$, where p^j denotes the price of an intermediate good x^j.

An intermediate producer uses household i investments in risk-free and risky capital to run j projects using two linear technologies; a risky *innovative* technology with TFP A_r and a risk-less *traditional* technology with TFP A_f, with $A_f < A_r$, such that: $x^j(a_1^i, \omega_1^i) = \left[A_f(1-\omega_1^i) + A_r \omega_1^i\right] a_1^i$, where the amount of risky/risk-less investment determines the weight of each technology. Project j's profit is $\pi^j(\kappa^i, a_1^i, \omega_1^i) = p^j \left[x^j(a_1^i, \omega_1^i) - (1-\kappa^i)\omega_1^i a_1^i \right]$ with $\kappa \sim \mathcal{N}(1, \sigma_\kappa^2)$ an investment-specific cost shock on risky capital. The intermediate producer redistributes all profits to households. Given the first order conditions, the marginal products to capital are given by $MPK_r^i = (1 - \varphi)(A_r - \kappa^i)$ and $MPK_f = (1-\varphi)A_f$.[6]

We consider the possibility that high return investments, arising mostly because of heterogeneous portfolio composition toward equity investments Bach et al. (2020), may not necessarily reflect high marginal product of capital, due to some form of rent-seeking (monopoly power, rent extraction technology, political connection, bargaining power of investors etc.).[7] To model this parsimoniously, we assume that there is a wedge between R_r on the investor side and MPK_r on the production side, such that $R_r = MPK_r/\mu$, with $\mu \in (0,1)$ where $\mu < 1$ implies some form of investor's rent-extraction, i.e. using Rothschild and Scheuer (2016) terms, "*the pursuit of personal enrichment by extracting a slice of the existing economic pie rather than by increasing the size of that pie*". We view this wedge between the private returns to capital investments and its true productivity as a stylized way to reconcile two existing frameworks. Of course, it posits the assumption

[6]It follows from plugging the solution for n^* in the final producer's profits and using $n^* = \int_i h^i di = 1$.

[7]See notably the discussion in Scheuer and Slemrod (2020) and the work of Piketty et al. (2014) and Rothschild and Scheuer (2016). In a related work, Boar and Midrigan (2019) study a setup with entrepreneurs and workers in which entrepreneurial returns to capital investment reflects partially market power.

that returns from rent-extraction or from the marginal product of capital can not be separated just by observing information on returns to wealth. On one side, some work disentangle returns heterogeneity from the supply side and capital provision, either because of their partial equilibrium structure (Benhabib et al., 2019) or because of full rent extraction (Hubmer et al., 2020). On the other side, models with capitalists and entrepreneurs assume a perfect pass-through between MPK and returns (Cagetti and De Nardi (2006, 2009), Guvenen et al. (2019) among many others). Instead, we study the results for a range of values for the wedge μ between returns to risky assets and its implied MPK. For example, normalizing $A_f = A_r = 1$ (differential returns reflect only rent extraction) and assuming an equity premium $R_r - R_f = 0.15$ and $\varphi = 0.65$, the wedge μ that rationalizes this premium is $\mu = \frac{A_f(1-\varphi)}{A_f(1-\varphi)+R_r-R_f} = 0.7 < 1$. We assume that whenever $\mu < 1$, the equilibrium rate $\underline{r}$ adjusts in order to ensure that the total product of capital on the production side coincide with the total capital income redistributed to the households, such that:

$$\int_i R_f a_1^i(1-\omega_1^i) + R_r a_1^i \omega_1^i + \underline{r} a_1^i di = \int_i A_f(1-\varphi)a_1^i(1-\omega_1^i) + A_r(1-\varphi)a_1^i\omega_1^i di \,.$$

Distributional assumptions (κ, h) follows a multivariate Gaussian distribution with mean $[0\ 1]$, covariance-variance diagonal elements $[\sigma_\kappa\ \ \sigma_h]$, and a correlation between the investment shock and the labor productivity, $\rho_{\kappa,h}$.[8] From this, Lemma 1 characterizes the *terminal wealth* c_2^i.

Lemma 1 (TERMINAL WEALTH DISTRIBUTION)**.** *Individual terminal wealth is affine in (κ, h), i.e. its distribution is Gaussian: $c_2^i \sim \mathcal{N}\left(\mu_{c_2}^i, \sigma_{c_2}^i\right)$ with mean $\mu_{c_2}^i = \varphi X + T + (R_f + \underline{r})a_1^i + \omega_1^i a_1^i(1-\varphi)(A_r - A_f)$ and variance $\sigma_{c_2}^i = (\varphi X)^2\sigma_h^2 + (\omega_1^i a_1^i(1-\varphi))^2\sigma_\kappa^2 + 2\rho_{\kappa,h}\varphi X\omega_1^i a_1^i(1-\varphi)\sigma_h\sigma_\kappa$.*

Together with *generalized* CARA preferences, these features deliver tractability of the equilibrium allocation. Finally, initial wealth is assumed to be Pareto distributed.

Assumption 1 (INITIAL WEALTH DISTRIBUTION)**.** *Initial wealth is drawn from a Pareto law with scale $\underline{a}$ and shape $\eta > \max\{\gamma, 1\}$, such that $a_0 \sim \mathcal{P}a\left(\underline{a}, \eta\right)$ with $\mathbb{P}(A_0 \geq a_0) = (\underline{a}/a_0)^\eta$, $\forall a_0 \geq \underline{a}$.*

While theoretically convenient, this assumption is consistent with empirical evidence documenting that the wealth distribution is skewed to the right and display heavy upper tail (Vermeulen, 2016; Klass et al., 2006). The tail index (or shape parameter) η is inversely related to wealth inequality.[9] Because the shape η leads *ceteris paribus* to a change in aggregate wealth, we

[8] σ_κ is assumed small enough to guarantee $c_2^i \geq 0$ in all states. Moreover, the correlation between labor risk and investment risk has implications on households' portfolio choices that we investigate in Section 2.4.

[9] The Pareto tail is also inversely related to the wealth share $q(p)$ of the p wealthiest households with: $q(p) = p^{1-1/\eta}$.

sometimes study the effect of varying η (*redistribution* effect) while preserving the same aggregate wealth level using a change in the scale $\underline{a}$ (*level* effect).

2.2 Efficiency Gains and Redistribution: A Special Case

We first consider a stylized case in which we rule out precautionary motives and focus on the aggregate allocation when household's capital supply is driven by *type* and *scale* dependence.

Assumption 2 (SPECIAL CASE)**.** *There is no consumption in $t = 1$ and no labor income risk, $\sigma_h^2 = 0$.*

Lemma 2 (POLICY FUNCTIONS)**.** *Let $\overline{\omega} = \frac{A_r - A_f}{(1-\varphi)\sigma_z^2}$ denotes the volatility-adjusted excess return on risky assets, under Assumption 2, household i's risky assets and risk-less asset holdings are respectively $\omega_1^i a_0^i$ and $(1-\omega_1^i)a_0^i$ with the share of risky assets given by:*

$$\omega_1^i = \overline{\omega} \cdot (\vartheta^i / \overline{\vartheta}) \cdot (a_0^i)^{\gamma - 1} . \tag{2}$$

Lemma 2 extends the Merton (1969) and Samuelson (1969) result without labor income risk and ($\gamma = 1$, $\vartheta^i = \overline{\vartheta}, \forall i$): the share of risky asset holdings equal the volatility-adjusted excess return over the risk aversion, captured by the term $\overline{\omega}$. Conditional on type ϑ^i, *generalized* CARA preferences mimic IRRA (resp. DRRA) behavior if $\gamma < 1$ (resp. $\gamma > 1$). When $\gamma = 1$, *generalized* CARA preferences nest CRRA behavior (constant risky asset shares) and $\gamma = 0$ implies CARA behavior (constant risky asset holdings). Therefore, the parameter γ pins down the elasticity of risky investments to initial wealth. Using Lemma 2, we derive the aggregate equilibrium quantities.

Lemma 3 (AGGREGATE QUANTITIES)**.** *Given the joint distribution of types and wealth $\mathcal{G}(\vartheta, a_0)$, aggregate risky capital K_I, output Y, productivity Z and the wage rate w satisfy*

$$K_I = \int_{\Theta \times \mathcal{A}} \omega_1(\vartheta, a_0) a_0 \, d\mathcal{G}(\vartheta, a_0) = (\overline{\omega}/\overline{\vartheta}) \Big(Cov(\vartheta, a_0) - \mathbb{E}[\vartheta]\, \mathbb{E}\left[a_0^{\gamma}\right] \Big) \tag{3}$$

$$Y = Z\mathbb{E}[a_0] , \quad \textit{with} \quad Z = A_f \left(1 - \frac{K_I}{\mathbb{E}[a_0]} \right) + A_r \frac{K_I}{\mathbb{E}[a_0]} , \tag{4}$$

$$\textit{and} \quad w = \varphi Y , \quad \underline{r} = \left(\frac{\mu - 1}{\mu} \right) (1-\varphi) A_r \frac{K_I}{\mathbb{E}[a_0]} . \tag{5}$$

Due to the CRS structure of final good production, demand for intermediate goods is perfectly elastic but its supply is bounded as households are risk averse. Consequently, household's risky investment shares together with the joint distribution of wealth and types, $\mathcal{G}(\vartheta, a_0)$, determine the economy wide productivity and output level. Therefore, a wealth inequality shock impacts productivity to the extent that it impacts household i's investment in risky assets. Condition (5) implies that if $\mu < 1$ then $\underline{r} < 0$, i.e. the presence of rent-extraction from risky investments induces a GE feedback effect that decreases wealth returns for all households.

We now study the effect of a wealth inequality change on the aggregates.

Proposition 1 (Inequality and Allocation: Decomposition)**.** *The effect on aggregate risky asset holdings K_I of a mean preserving change in the Pareto tail to $\eta' \neq \eta$ can be decomposed as*

$$\Delta K_I(\eta', \eta) = \underbrace{\Delta^w K_I(\eta', \eta)}_{\substack{\text{scale dependence in} \\ \text{portfolio holdings}}} + \underbrace{\Delta^s K_I(\eta', \eta)}_{\substack{\text{type heterogeneity and} \\ \text{selection}}} ,$$

where $\Delta^w K_I(\eta', \eta)$ is zero in η' if $\gamma \in \{0, 1\}$, increasing in η' if $\gamma \in (0, 1)$ and decreasing in η' if $\gamma > 1$. A sufficient condition for $\Delta^s K_I(\eta', \eta)$ to decrease in η' is $\left(\frac{\partial corr(\vartheta, a_0^\gamma)}{\partial \eta} + \frac{\partial corr(\vartheta, a_0^\gamma)}{\partial \underline{a}} \frac{\underline{a}}{\eta(\eta - 1)} \right) \frac{1}{corr(\vartheta, a_0^\gamma)} \leq 0.$

Proposition 1 decomposes the effect of a change in the shape of the wealth distribution on aggregate risky capital (K_I) into two terms: (i) a *scale* dependence term $\Delta^w K_I(\eta', \eta)$ that depends on the wealth dependent risk taking elasticity γ and a (ii) *type* dependence term $\Delta^s K_I(\eta', \eta)$ which encapsulates the selection of ϑ-types across the wealth distribution. Consider first the case with no type heterogeneity ($\vartheta^i = \overline{\vartheta}, \forall i$). Redistribution from the bottom to the top decreases (resp. increases) K_I if $0 < \gamma < 1$ (resp. $\gamma > 1$). When $\gamma = \{0, 1\}$, distributional *irrelevance* arises because aggregate variables can be determined without information regarding the wealth distribution, either because risky investments are constant ($\gamma = 0$), or because the share invested is constant ($\gamma = 1$). With selection of types, there is a bound on $corr(\vartheta, a_0^\gamma)$ such that the effects of inequality variations on K_I, conditional on γ, is positive. Therefore, even without scale effets (i.e. $\gamma = \{0, 1\}$), the distribution of wealth is *relevant* through *type* dependence.[10]

2.2.1 The Efficiency-Inequality Decomposition: A Closed-form Representation

Although the decomposition in Proposition 1 is general, we now illustrate our case in closed form under additional assumptions to represent the equilibrium and study wealth redistribution.

Assumption 3 (Joint Distribution)**.** *Let $\vartheta \sim \mathcal{P}a(\underline{\vartheta}, \epsilon)$ and $\overline{\vartheta} = E[\vartheta]$, the joint cdf $\mathcal{G}(\vartheta, a_0)$ is constructed based on the Farlie-Gumbel Morgenstern (FGM) copula with dependence parameter $\varrho \in [-1, 1]$.*

Under assumption 3, when $\varrho > 0$ (resp. $\varrho < 0$), there is a positive (resp. negative) correlation of types and wealth, while $\varrho = 0$ induces no correlation.[11] The level of ϱ translates the degree

[10] Interestingly, the economy can however be represented by a representative households, as shown in Appendix **??**.

[11] The dependence parameter ϱ and the spearman's correlation, ϱ^s, under the FGM copula are related by $\varrho^s = \varrho/3$.

of the selection, which, for simplicity, is exogenous in this section. In the quantitative model of section 3 this selection will arise endogenously.[12]

The following result decompose the trade-off between inequality and efficiency into three terms which captures *type* dependence, *scale* dependence and an interaction term.

Proposition 2 (Efficiency-Inequality Relation). *Given assumptions 1-3 and assuming* $\frac{\eta-\gamma}{\eta-1} \geq \overline{\omega}\left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\underline{a}^{\gamma-1}$ *such that* $\frac{K_I}{\mathbb{E}[a_0]} < 1$ *in equilibrium, total output is given by* $Y(\eta) = A_f + (A_r - A_f)\overline{\omega}\left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\frac{\eta-1}{\eta-\gamma}\underline{a}^{\gamma-1}$. *The marginal effects of a change in the shape* η *on* $Y(\eta)$ *is*

$$\frac{\partial Y(\eta)}{\partial \eta} = -\Phi(\eta;\gamma,\varrho)\overline{\omega}(A_r - A_f)\underline{a}^{\gamma-1}\,, \tag{6}$$

where the function $\Phi(\eta;\gamma,\varrho)$ *decompose the type and scale dependence effects such that:*

$$\Phi(\eta;\gamma,\varrho) = \underbrace{\Omega^{w}(\eta,\gamma)\cdot(\gamma-1)}_{\textit{scale dependence}} + \underbrace{\Omega^{s}(\eta,\gamma)\cdot\varrho}_{\textit{type dependence}} + \underbrace{\Omega^{ws}(\eta,\gamma)\cdot\rho(\gamma-1)}_{\textit{interaction term}} \tag{7}$$

with $\Omega^{w}(\eta,\gamma) = \frac{1}{(\eta-\gamma)^2}$, $\Omega^{s}(\eta,\gamma) = \frac{2(\eta-1)}{(2\epsilon-1)(2\eta-\gamma)^2(\eta-\gamma)}$ *and* $\Omega^{ws}(\eta,\gamma) = \frac{(\gamma(2\eta-\gamma)+2(\eta-1)(\eta-\gamma))}{(2\epsilon-1)(2\eta-\gamma)^2(\eta-\gamma)^2}$ *are strickly positive auxiliary parameters .*

The relationship between wealth inequality and output depends on the relative strength of the *type* and *scale* dependence and their interaction, respectively captured by the terms $\gamma - 1$, ϱ and $\varrho(\gamma-1)$. If *generalized* CARA preferences mimic DRRA behavior ($\gamma > 1$) and there is a positive selection of types ($\varrho > 0$), then output unambiguously rises with inequality, i.e. $d\eta < 0$, since it reallocates wealth to agents investing in riskier but more productive capital.

Importantly, variations of the Pareto tail $d\eta$ exhibit highly nonlinear effects on output captured by the terms $\Omega^{w}(\eta,\gamma)$, $\Omega^{s}(\eta,\gamma)$ and $\Omega^{ws}(\eta,\gamma)$. For a complete unequal economy, i.e. $\eta \to \max\{\gamma, 1\}$, small variations of the Pareto tail result in rather large output variations, while in an egalitarian societies, i.e. $\eta \to \infty$, small variations in η result in small output variations. Intuitively, in more unequal economies, the wealth dependent risk taking effect and the selection effect are stronger in magnitude such that even small variations of η lead to strong investment reallocations.

Proposition 2 also implies that, for a given level of inequality η, there is an infinite number of possible combination of *type* and *scale* dependence (ϱ, γ) on an infinite bounded two-dimensional

[12]A crucial condition for this in our context arises if some types extract higher capital returns for multiple periods, and are thus more represented at the top of the wealth distribution. This selection is stronger if types are persistent (Moll (2014)), and if capital returns extracted from different types are more dispersed. The first statement is proven by Moll (2014), the second is obtained by simulations using his replication files.

set consistent with a given marginal effect of a wealth inequality variations on output. In Definition 1, we specify the notion of *iso-growth* which describes all parameter pairs (γ, ϱ) for which a marginal variation of the Pareto shape η generates a given output response $\overline{g}$.

Definition 1 (ISO-GROWTH OF INEQUALITY)**.** *For a given wealth Pareto tail η, the iso-growth with level $\overline{g}$ is described by the pair (γ, ϱ) which satisfies isoG$(\eta, \overline{g}) \equiv \left\{(\gamma, \varrho) \in \mathbb{R}_+ \times [-1, 1] : \frac{\partial Y(\eta)}{\partial \eta} = \overline{g}\right\}$.*

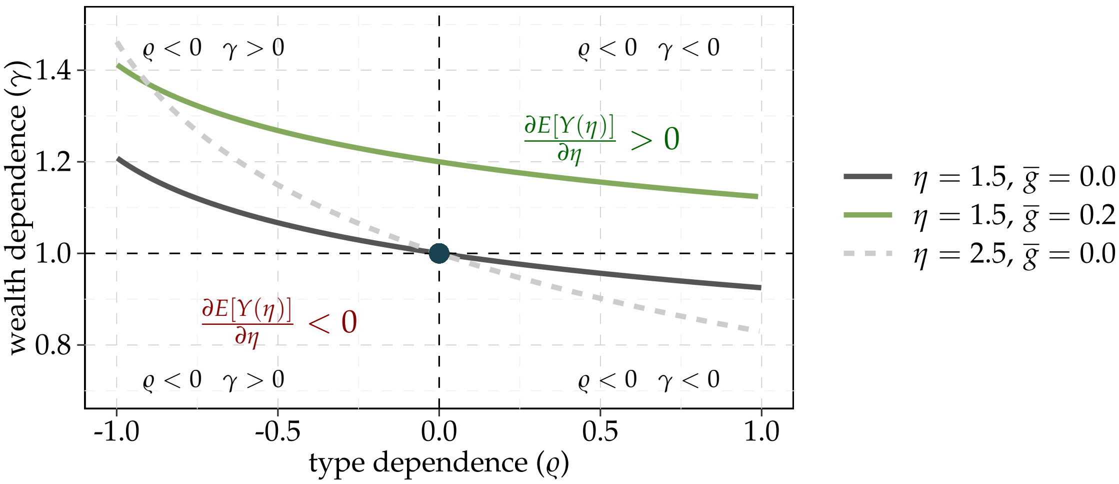
A special case ensues for $\overline{g} = 0$ for which the *iso-growth* separates the *growth enhancing region*, i.e. $\frac{\partial Y(\eta)}{\partial \eta} < 0$, from the *growth dampening region*, i.e. $\frac{\partial Y(\eta)}{\partial \eta} > 0$. For this reason, we label this special *iso-growth* the *Growth Irrelevance Frontier (GIF)*. Lemma **??** in Appendix **??** provides the conditions for its existence given the strength of *type* and *scale* dependence and the wealth Pareto tail η. Outside those restrictions, an infeasible pair (ϱ, γ) would be required to obtain *growth neutrality*.[13] These cases thus appear as a very special case of an economy where *type* and *scale* dependence exactly offset, or are absent. In Lemma 4, we provide key properties of the *iso-growth*: it is decreasing on the space (γ, ϱ) and rotates with inequality η.

Lemma 4 (PROPERTIES OF THE GIF AND ISO-GROWTH)**.** *The GIF is strictly decreasing on the defined set of Lemma* **??**. *A higher tail η shifts the GIF such that $\frac{d\gamma}{d\overline{g}}|_{d\varrho=0} < 0$, $\frac{d\gamma}{d\eta}|_{d\varrho=0} > 0$ for $\gamma > 1$ and $\frac{d\gamma}{d\eta}|_{d\varrho=0} \leq 0$ for $\gamma \leq 1$. Finally, for a higher level $\overline{g}$, isoG$(\eta, \overline{g})$ is the translation of the GIF and $\frac{d\gamma}{d\overline{g}}|_{d\varrho=0} > 0$.*

Figure 1 illustrates the *iso-growth* for two different Pareto tails η and output levels $\overline{g}$. There are four regions which delimit the sign of the *type* and *scale* dependence. In the top-right (resp. bottom-left) region, wealth-dependent risk taking and selection effects move in the same direction. An increase in inequality therefore unambiguously induces more (resp. less) economy wide risk taking and higher (resp. lower) productivity. In the top-left and bottom-right region, wealth-dependent risk taking and selection effects move in opposite directions. In these regions, there is a combination of *type* and *scale* dependence such that both effects may offset each other such that the GIF exists. In the top-left region with $\{\varrho < 0, \gamma > 1\}$, an increase in inequality leads to higher growth only if the positive *scale* dependence is sufficiently strong. In the bottom-right region with $\{\varrho > 0, \gamma < 1\}$ an increase in inequality decreases output if the wealth-dependent risk taking elasticity γ is sufficient low for a given positive selection $\varrho > 0$.

[13] Notice that the *isoG*$(\eta, 0)$ does not exist in a complete unegalitarian economy ($\eta = \max\{1, \gamma\}$), since in such case wealth dependence effects would always dominate type-dependence. Moreoever, under an egalitarian economy, both effects are small in magnitude such that the *isoG*$(\eta, 0)$ exists on a bounded set.

Figure 1. The inequality–efficiency diagram.
Note: numerical values are $\varepsilon = 2.0, \underline{a} = \underline{\vartheta} = 1.0, A_r = 1.1, \sigma_\kappa = 0.2, A = 1.0$.



For a higher effect of wealth inequality on the level of output (an increase in $\overline{g}$), the iso-growth moves upward in the (γ, ϱ) diagram, meaning that higher impact of wealth inequality change on growth can only be rationalized with sufficient degree of positive *type* and/or *scale* dependence. Finally, the level of wealth inequality η^{-1} change the relative strength of the type and wealth dependence effects: higher inequality (lower η) reinforces the *scale* dependence effect relative to the selection effect and, as a result, the GIF (and the translated *iso-growth*) somewhat rotates counterclockwise. Finally, the level of the *iso-growth* decreases with the productivity gap $A_r - A_f$ as less reallocation between the two productive sectors is needed to achieve a given level $\overline{g}$.

The decomposition derived in this static setup carries over the quantitative dynamic model of Section 3 with a realistic shape of the wealth distribution in which the joint distribution of types and wealth arises endogenously. As such, to understand the effects of wealth redistribution on the aggregates, there are four important statistics required: (i) the productivity gap between asset classes, (ii) the shape of the wealth distribution η, and the associated *iso-growth* which is defined by the pair (γ, ϱ) comprising (iii) the dependence between the distribution of types and the distribution of wealth and (iv) the wealth-dependent risk taking elasticity.

Discussion and practical implications A number of frameworks can be unified on the basis of *type* and *scale* dependence mechanisms that ultimately fall within a representation of the inequality-efficiency diagram of Figure 1. In the basic Aiyagari (1994) economy, the distribution of wealth is almost growth irrelevant due to the quasi linearity of the savings policy functions of the wealthy and thus is located close to the center of the inequality-efficiency diagram ($\gamma = 1$ and $\varrho = 0$). Models with entrepreneurs (Cagetti and De Nardi (2006, 2009), Brüggemann (2020), Guvenen et al. (2019)) often display positive *type* dependence at the steady-state, as entrepreneurs self-select at the top of the wealth distribution ($\varrho > 0$), but feature decreasing marginal product of capital investments which can be reinterpreted as negative *scale* dependence ($\gamma < 0$). In those models, redistribution to the upper end of the wealth distribution has an ambiguous effect on output and are positioned in the bottom-right region. Similarly, the seminal paper by Galor and Zeira (1993) with non-convex human capital investment cost can be reinterpreted as a form of DRS ($\gamma < 1$), but without type heterogeneity ($\varrho = 0$).

From the diagram, it is interesting to see that *type* and *scale* dependence can not be simply identified using information on the effect of inequality on growth, as an infinite combination of pair (γ, ϱ) may rationalize the relationship.[14] It is also difficult to identify both dependencies based on micro data as both mechanisms may in principle generate consistent cross-sectional patterns regarding the portfolio allocation and associated returns to capital across the wealth distribution. Yet, both effects are bound to have different effects.

A numerical example Consider two distinct models: the first with only *scale* dependence, and the second with only *type* dependence. We calibrate *type* and *scale* dependence to generate a consistent cross-sectional pattern of risky equity shares, targeting the share of the top 1% wealthiest households of 65% as in the SCF (2010). Note that because the portfolio shape is increasing in wealth in the cross-section, the two models are located respectively in the $\{\varrho = 0, \gamma > 0\}$ and $\{\varrho > 0, \gamma = 0\}$ locuses. We set the shape $\eta = 1.40$ consistent with Vermeulen (2016), and the risk premium to 15% (assuming $\mu = 1$). In the *type* dependence model, we set the Pareto shape of type $\varepsilon = 2$ and vary the correlation between types and wealth to match the moment, such that $corr(\vartheta, a_0) = 0.65$.[15] In the *scale* dependence model, we vary the wealth-dependent risk-taking elasticity γ and obtain $\gamma = 1.39$. In Appendix B.1, we show that both models reproduce well the overall cross-sectional pattern of portfolio shares, and not just at the upper end of the wealth distribution. However, the responses of output to a wealth tax on the top 1% wealthiest households,

[14]There are, however, identified under particular conditions. Figure 1 shows that γ and ϱ are identified with *two* different couples of observation $(\eta_1, \bar{g}_1)$ and $(\eta_2, \bar{g}_2)$, but this requires that type and wealth dependence are constant over time. Moreover, estimating this relationship is somewhat complex as shown by the variety of empirical results in this related literature. See among others Forbes (2000), Voitchovsky (2005) or Barro (2008).

[15]Instead, it is possible to fix the correlation of type but vary the extend that individuals are different by varying the shape ϵ. There is marginal difference in considering one or the other, as long as the cross-sectional distribution of portfolio shares and wealth are well matched.

redistributed through the lump-sum transfers T, differ substantially. The response is a reduction of output by 0.43% under *type* dependence, substantially lower than the 0.70% reduction under *scale* dependence. This difference originates from the behavioral response triggered by the *scale* dependence mechanisms as wealth varies (see Lemma 2). Fortunately, information regarding their relative importance has been recently provided in Bach et al. (2020) and Fagereng et al. (2020) exploiting the long panel dimension of the Norwegian and Swedish administrative datasets. In section 3, we provide evidence for the US economy.

2.3 From Efficiency to Welfare

We now shift focus to welfare through the redistribution of wealth. We consider the case of a log-linear tax similar to Feldstein (1969) and Heathcote et al. (2017) on initial wealth such that $t_a(a_0^i) = a_i^0 - (a_i^0)^{1-p_a}$, where $p_a \in (-\infty, 1)$ captures the progressivity of the wealth tax. We denote with $\sim$ the *post-tax* variables. Aggregate conditions are isomorphic, replacing initial wealth a_i^0 with $\widetilde{a}_i^0 = a_i^0 - t_a(a_i^0)$, which implies an updated wealth Pareto tail $\widetilde{\eta} = \frac{\eta}{1-p_a}$ and scale $\widetilde{\underline{a}} = \underline{a}\left(1 + \frac{p_a}{\eta-1}\right)$. Under this specification, $p_a \to 1$ implies full equality, with $\widetilde{\eta} \to \infty$ and $\widetilde{\underline{a}} = \underline{a}\frac{\eta}{\eta-1}$.

We measure welfare in terms of consumption equivalent, $\bar{c}^i$, i.e. the amount that makes a household i in the *pre-tax* economy as well off as in the situation with progressive tax p_a, such that $E[u(\widetilde{c}_2^i + \bar{c}^i)] \stackrel{!}{=} E[u(c_2^i)]$. Under *generalized* CARA preference, this gives: $\bar{c}^i = \frac{\alpha_i}{\widetilde{\alpha}_i} x_{c_2}^i - \widetilde{x}_{c_2}^i + \Delta_c$, with $x_{c_2}^i$ and $\widetilde{x}_{c_2}^i$ the certain equivalent of the pre-tax and post-tax consumption in period 2 and Δ_c a term that arises because the utility is a positive function of initial wealth through the risk tolerance term α_i. The equivalent variation-based welfare measure is given by:

$$\mathcal{W} = \arg\max_{p_a} \int s(\vartheta, a_0)\, \bar{c}(\vartheta, a_0)\, d\mathcal{G}(\vartheta, a_0) \qquad \text{s.t.} \qquad T = \frac{\eta \underline{a}}{\eta - 1} - \frac{\eta \underline{a}^{1-p_a}}{\eta - 1 + p_a}\,. \tag{8}$$

where $s(\vartheta, a_0)$ defines the social welfare weight, such that $\int s(\vartheta, a_0)\, d\mathcal{G}(\vartheta, a_0) = 1$, and the last equality balances the government budget constraint, such that $\int t_a(a_i^0) di = T$.

Lemma 5 (Optimal wealth redistribution). *Assume $x_r = A_r - A_f - \frac{\sigma_\kappa^2}{2}(\overline{\omega}/\overline{\vartheta}) > 0$ and let $x_{c_2}^i = \mu_{c_2^i} - \frac{\alpha^i}{2}\sigma_{c_2^i}^2$ be the certain equivalent of c_2^i, the optimal progressivity p_a^* is implicitly given by:*

$$\left(\underbrace{\frac{\partial Y(\widetilde{\eta}; \varrho, \gamma)}{\partial \widetilde{\eta}} \varphi}_{\text{GE wage; efficiency}} + \underbrace{\frac{\partial \underline{r}(\widetilde{\eta}; \varrho, \gamma)}{\partial \widetilde{\eta}}}_{\text{rent-extraction if } \mu < 1} \right) \frac{\partial \widetilde{\eta}}{\partial p_a} + \underbrace{\frac{\partial T}{\partial p_a} + \int s(\vartheta, a_0) \frac{\partial \widetilde{a}_0}{\partial p_a} \mathcal{R}(\vartheta, a_0) d\mathcal{G}(\vartheta, a_0)}_{\text{lump-sum transfers} > 0\text{, and direct effects of wealth taxation} < 0} \approx 0,$$

where $\mathcal{R}(\vartheta^i, a_0^i) = A_f + \left(\overline{\omega}(1-\varphi)x_r + e^{\alpha_i x_{c_2}^i} - \alpha_i c_{c_2}^i\right)\gamma\frac{\vartheta}{\overline{\vartheta}}(\widetilde{a}_0)^{\gamma-1}$*. The sign of* $\frac{\partial Y(\widetilde{\eta};\varrho,\gamma)}{\partial\widetilde{\eta}}$ *is characterized in Proposition 2 and sgn* $\left(\frac{\partial \underline{r}(\widetilde{\eta};\varrho,\gamma)}{\partial\widetilde{\eta}}\right) = -sgn\left(\frac{\partial Y(\widetilde{\eta};\varrho,\gamma)}{\partial\widetilde{\eta}}\right)$.

Lemma 5 characterizes the optimal progressivity of the wealth tax. The redistribution channels is composed of welfare gains from lump-sum transfers T and diminishing welfare through the progressive tax, which reduces terminal wealth. With positive *type* dependence ($\varrho > 0$) and *scale* dependence ($\gamma > 1$) general equilibrium feedback effects of a wealth tax comes from efficiency gains (as risky investments increases the wage rate w) and rent-extraction (whenever $\mu < 1$ risky investors experience larger private returns relative to their social value which lowers $\underline{r}$). The optimal proportional wealth tax thus balances efficiency *versus* rent-extraction and equity. Importantly to notice, in this static model with exogenous type dependence, a lower μ implies a higher progressivity of the wealth tax. Section 3 extends the results to a quantitative dynamic model in which the joint distribution of ϑ-types and wealth arises endogenously.

2.4 Generalization

The assumptions made throughout the special case are meant to focus on risk-taking decisions arising from *type* and *scale* dependence. We now discuss the case with savings and uninsurable labor income risk, alternative sources of *scale* dependence and the case with aggregate shocks.

2.4.1 Saving decisions and uninsurable labor income risk

Let $c_1^i > 0$ and $\sigma_h > 0$. In this case, the joint heterogeneity in marginal propensities to save and the portfolio allocation are key to study aggregate allocation, and gives rise to a generalized *iso-growth*.

Lemma 6 (Individual Portfolio Choice)**.** *Denote* $\widetilde{\beta} = (\beta R_f)^\sigma$ *and let us assume an interior solution to* (P1)*, individual portfolio choices of risky and risk-less assets denoted respectively,* $k_1^i = \omega_1^i a_1^i$ *and* $b_1^i = (1-\omega_1^i)a_1^i$ *are given by*

$$k_1^i = \underbrace{\overline{\omega}\left(\frac{\vartheta^i}{\overline{\vartheta}}\right)(a_0^i)^\gamma}_{\text{G-CARA portfolio}} - \underbrace{\left(\frac{\varphi Y}{1-\varphi}\right)\frac{\rho_{\kappa,h}\sigma_h}{\sigma_\kappa}}_{\text{labor income risk}}, \quad b_1^i = \frac{1}{\widetilde{\beta}+R_f}\left(\underbrace{\widetilde{\beta}a_0^i - (\widetilde{\beta}+R_r^i)k_1^i - \varphi Y}_{\text{Intertemporal substitution}} + \underbrace{\frac{1}{2}\alpha^i\sigma_{c_2^i}^2(k_1^i)}_{\text{Prec. savings}}\right).$$

Lemma 6 is a generalized counterpart to Proposition 1 in Angeletos and Calvet (2006) derived under a baseline CARA specification. The optimal decision of k_1^i has now two components: the G-CARA portfolio choice of Lemma 2, and a term that arises due to the possible correlation between labor income and investment risks. If $\rho_{\kappa,h} > 0$, it discourages wealth-poor households from risky asset investments and encourages them simultaneously to risk-free asset investments. Intuitively, in absence of correlated shocks, labor income plays the role of a risk-free assets and encourages risk-taking.[16] The two period structure also leads to an intertemporal substitution effect due to the risky asset holdings and a precautionary savings effect that arises from additional uncertainty about the realization of c_2^i. The effect of wealth increases on precautionary savings is *a priori* ambiguous and linked to the risk tolerance shape. On the one hand, increasing wealth decreases risk prudence. On the other hand, lower risk aversion enhances risk taking such that $\sigma_{c_2}^i$ rises.

In this economy, the effects of a change in the shape of the wealth distribution on output can be characterized in sufficient statistics.

Corollary 1 (Efficiency and inequality). *The output in this economy is:* $Y = \mathbb{E}[a_1(\vartheta, a_0)]A_f + (A_r - A_f)\mathbb{E}[k_1(\vartheta, a_0)]$. *The effect of a change in inequality on wealth-normalized output is given by:*

$$\frac{\partial Y/\mathbb{E}[a_0]}{\partial \eta} = A_f \cdot Cov\left(mps^i, \frac{da_0^i}{\mathbb{E}[a_0]}; \eta, \varrho, \gamma\right) + (A_r - A_f) \cdot Cov\left(mpr^i \times mps^i, \frac{da_0^i}{\mathbb{E}[a_0]}; \eta, \varrho, \gamma\right),$$

with $mps^i = \frac{\partial a_1^i}{\partial a_0^i}$ *and* $mpr^i = \left(\frac{\partial k_1^i}{\partial a_1^i}\right) / \left(\frac{\partial a_1^i}{\partial a_0^i}\right)$ *are the marginal propensity to save and to invest risky.*

Corollary 1 characterizes the overall effect of a wealth inequality change on aggregate efficiency into sufficient statistics in an economy with capital accumulation. In such economy, *type* and *scale* dependence determine the extend that agents invest in risky assets (captured by the distribution of mpr^i), and how they accumulate wealth (captured by the distribution of mps^i). The quantitative setting in Section 3 incorporates both elements in a dynamic framework.

2.4.2 Extensions

Other sources of wealth dependence Appendix **??** extend the results to other forms of *scale* dependence. First, we introduce DRS on intermediate producer's investment, which mirrors the case adopted in most entrepreneurial settings in the literature (see Cagetti and De Nardi (2006) and Guvenen et al. (2019)). Under this specification, aggregate output may negatively depends on wealth inequality due to the decreasing MPK at the top but positively with the selection of entrepreneurs at the top of the distribution. Such models are located in the bottom-right area of the inequality-efficiency diagram displayed in Figure 1. Second, we consider the case of wealth-dependent borrowing constraint (as in Boar and Knowles (2020)) and show that it generates similar results as the

[16] This argument originates from Merton and Samuelson (1992) or Viceira (2001) and is numerically illustrated in Cocco, Gomes and Maenhout (2005). Under some conditions, this effect gives rise to a risky asset investment threshold as shown in Appendix **??**.

one derived above under wealth-dependent risk-aversion.

Aggregate shocks Throughout the paper, we assumed that investment risks are mostly idiosyncratic. Instead, under aggregate risky investment with $A_r + z \sim \mathcal{N}(A_r, \sigma_z^2)$, the main insights are unchanged, but two interesting new features emerge. First, *a growth – variance trade-off* arises because increasing inequality affects expected growth, but also its volatility. A social planner seeking to redistribute wealth has an additional social incentive for stabilizing the wage rate, pushing toward less inequality. Under the special case, this creates a link between inequality and output volatility $\sigma_Y^2(\eta)$, such that: $\frac{\partial \sigma_Y^2(\eta)}{\partial \eta} = -2\sigma_z^2 \Phi(\eta;\gamma,\varrho)\frac{E[Y(\eta)]-A_f}{A_r-A_f}$ where Φ is defined in Proposition 2. Second, *a GE precautionary saving effect* arises on household's portfolio allocation in the case of saving decisions because aggregate risks induces a correlation between the wage rate and returns.

We view those extensions as especially interesting, however, given the focus of this paper, we leave these discussions for future research.

3 A Quantitative Model with Investment Heterogeneity

The previous section shed light on the theoretical decomposition and importance of *type* and *scale* dependence for the macro. We now analyze their role in shaping the observed wealth distribution, the responsiveness of output and how they affect the redistribution of wealth.

3.1 Environment

The distribution of wealth now arises endogenously from two empirically-relevant features: heterogeneity in human capital as in a standard Aiyagari (1994)-type of incomplete markets setup and heterogeneity in capital investments and associated returns. While Benhabib et al. (2011, 2019) show that heterogeneity in capital returns is key to generate the right tail of the wealth distribution, we more generally study the role of *type* and *scale* dependence.

Apart from those key elements, the rest of the model is kept purposely simple and parsimonious: it is in fact a standard incomplete markets model with capital investment heterogeneity.

3.1.1 Preference and endowments

Households are now infinitely lived and derive a flow of utility $\mathcal{V}_t^i$ from consumption c_t^i. Agents differ in their wealth a_t^i, in their permanent component of labor productivity h_t^i, and in their propensity to invest in risky assets (due to investment specific skills or risk-tolerance), denoted

ϑ_t^i. They discount future at rate $\beta \in (0,1)$ and face the probability of death $p_d(h_t^i)$, such that:

$$\mathcal{V}_t^i = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left[1 - p_d(h_t^i)\right]^t u(c_t^i) \right]. \tag{9}$$

Unless necessary, we now drop time and households indexes. We denote $\mathbf{s} = (a, \vartheta, h)$.

Pre-tax labor income is $w\ell(h,y)$, with w the equilibrium wage rate, $\ell(h,y) = \mathcal{H}(h)y$ the agent's inelastic labor supply, and y is the transitory labor productivity component. As in Hubmer et al. (2020), we try to improve the fit of the earnings distribution by assuming that the persistent component $\mathcal{H}(h)$ follows a lognormal AR(1) with persistence ρ_h and variance σ_h^2. However, the values at the top of the income distribution are drawn from a Pareto Law with shape $\eta_h > 1$, such that:

$$\mathcal{H}(h) = \begin{cases} e^h & \text{if} \quad F_h(h) \leq q_h, \\ F^{-1}_{Pareto(\eta^h)}\left(\frac{F_h(h)-q_h}{1-q_h}\right) & \text{otherwise.} \end{cases} \tag{10}$$

where $F_h(.)$ is the cdf of h and $F^{-1}_{Pareto(\eta^h)}(.)$ the inverse cdf for a Pareto distribution with lower bound $F_h^{-1}(q_h)$ with $q_h \in (0,1)$. The persistent component is discretized into bins $h \in \{h_1, ..., h_H\}$ and while the agent is alive, its evolution follows a first order Markov chain with transition matrix $\pi_h(h'|h)$. Upon death, a newborn imperfectly inherits the persistent component of her parents. With probability p_h she draws her parent's persistent labor productivity, and with probability $(1-p_h)$ she draws her productivity from the invariant distribution $F_h(h)$ generated by $\pi_h(h'|h)$.

Following on Sommer and Sullivan (2018), our model features stylized stochastic aging to capture the dynamics of income over the life-cycle (and the associated wealth accumulation) without the need to explicitly incorporate household age in the state space. We assume that there is a probability $\chi_h = \frac{1}{p_h M}$ of transiting from h to h' due to an aging shock, where p_h is the fraction of the population with productivity h, and M is a constant equal to the expected lifetime. For this reason, the probability to die is assumed to increase with income, i.e. $p_d(h)$ increases in h. Appendix **??** provides details of the full transition matrix associated to our assumptions.

Risk taking type follows a S-states Markov chain with state $\vartheta \in \{\vartheta_1, ..., \vartheta_S\} \in \Theta$ and transition matrix $\pi_\vartheta(\vartheta'|\mathbf{s})$. A newborn draws her parent's risk taking type with probability p_ϑ and from the invariant distribution $F_\vartheta(\vartheta)$ otherwise.

Households are heterogeneous in their capital investments. They are assumed to split their savings into risk-free assets (cash, savings, deposits, bonds, housing and other safe assets) and risky productive assets (equity). An agent with risk taking type ϑ and wealth a invests a share

$\omega(a,\vartheta)$ in the risky asset. For the clarity's sake, our portfolio specification should be understood as a simplified version of a more elaborated portfolio choice over private and public equity assets or as an entrepreneurial model where some households invest in private equity businesses. Let r_F and r_R, with $r_R > r_F$, respectively be the risk-free and risky rates determined in equilibrium. The *ex post* pre-tax return on investment is defined as

$$r(a,\vartheta,\kappa) = \underline{r} + r_F \cdot (1-\omega(a,\vartheta)) + (r_R+\kappa)\cdot\omega(a,\vartheta) \tag{11}$$

where $\underline{r}$ is an aggregate return component and $\kappa \sim \mathcal{N}(0,\sigma_\kappa)$ an element of *luck* approximated into m discrete bins $\kappa \in \{\kappa_1, ..., \kappa_m\}$ with corresponding probabilities $\pi_\kappa(\kappa)$. The variance of returns is therefore given by $(\sigma_r(a,\vartheta))^2 = (\omega(a,\vartheta)\sigma_\kappa)^2$ and implies that households with higher equity shares experience higher portfolio risk. Such a feature is supported in the PSID, in Bach, Calvet and Sodini (2020) and in Fagereng, Guiso, Malacrino and Pistaferri (2020). From equation (11), it is clear that returns are correlated over time through wealth itself, and from the process governing the evolution of risk-taking type ϑ.

Agents optimally choose their saving a' and can not borrow. The recursive program is stated in reduced form as

$$v(a,\vartheta,h) = \mathbb{E}_{e,y}\left\{\max_{c>0,\,a'\geq 0}\left\{u(c) + \beta(1-p_d(h))\mathbb{E}_{h',\vartheta'|h,\vartheta}\left[v(a',\vartheta',h')\right]\right\}\right\} \tag{12}$$

$$\text{s.t.} \quad c + a' = w\ell(h,y) - t_w(w\ell(h,y)) + (1+r(a,\vartheta,\kappa))a - t_r(r(a,\vartheta,\kappa)a) - t_a(a), \tag{13}$$

where $t_r(.)$, $t_w(.)$ and $t_a(.)$ are respectively the tax schedule on capital income, labor income and wealth. Upon death, bequests are taxed such that $a^{child} = a^{parents} - t_b(a^{parents})$.

3.1.2 Production, Government and Equilibrium

Production An intermediate producer operates at no cost a continuum of project j. Similar to the two-periods model, each of them uses financial assets supplied by a household (with wealth a and type ϑ) to produce x intermediate goods, such that:

$$x_j(a,\vartheta) = \Big[A_F + \omega(a,\vartheta)(A_R - A_F)\Big]a^\nu, \tag{14}$$

with $A_R \geq A_f$ and $\nu > 0$ are returns to scale on the technology. The intermediate goods producer sells units of intermediate goods to a final good producer at price $p_j(a,\vartheta)$ and, for a given project, obtain profit $\pi_j(a,\vartheta) = (p_j \cdot x_j)(a,\vartheta) - \delta a$, that are entirely redistributed to the shareholders (the households). Recall that intermediate producers do not face any risk, but investors are subject to the investment shock κ (see equation (11)).

A competitive final good producer uses labor L and intermediate goods $X = \int_j x_j dj$ to produce final goods with technology $Y = F(X,L)$, with $F(.)$ satisfying the Inada conditions and profit are given by $Y - \int_j p_j x_j \, dj - wL$, where the depreciation δ is paid within the period by the final producer. Profit maximization yields the following prices: $p_j = \frac{\partial F(X,L)}{\partial X}\frac{\partial X}{\partial x_j}$, and $w = \frac{\partial F(X,L)}{\partial L}$. As each intermediate good enters the final goods production equivalently, $p_j = p, \forall j$.

As stressed above, heterogeneity in capital returns in our setup may not necessarily reflect differential marginal product of capital between risky capital (MPK_R) and risk-less capital (MPK_F) in the presence of rent-extraction. Just as before, we parsimoniously capture this by assuming a wedge between risky returns r_R on the investor side and the marginal product of risky capital (MPK_R) generated by the intermediate producer. Given equations (14) and the profit equation, the returns to assets are thus given by

$$r_F = MPK_F := pA_F a^{\nu-1} - \delta\,, \qquad r_R = \frac{MPK_R}{\mu} := p\frac{A_R}{\mu}a^{\nu-1} - \delta\,, \tag{15}$$

where $\mu < 1$ implies some form of rent-extraction.[17]

Government The government finances an external government expenditure level $\overline{G}$ using four sources of revenue: capital income taxes T_r, estate taxes T_b, wealth taxes T_a, and labor income taxes T_w. Capital income is subject to a proportional tax $t_r(\mathrm{x}) = \mathrm{x}\tau_r$. Labor income is subject to a proportional tax $t_w(\mathrm{x}) = \mathrm{x}\tau_w$. The bequest tax is proportional above an exemption level $\overline{a}_b$ such that: $t_b(x) = \mathbb{1}_{x \geq \overline{a}_b}\tau_b\,(x - \overline{a}_b)$. Finally, we consider a progressive wealth tax in the form of a proportional tax τ_a on wealth above an exemption level $\overline{A}_\tau$, such that $t_a(x) = \mathbb{1}_{x \geq \overline{A}_\tau}\tau_a(x - \overline{A}_\tau)$.[18]

3.2 Equilibrium

Definition 2. *Let $\mathbf{s} = (a,\vartheta,h) \in \mathcal{S} \equiv \mathbb{R}^+ \times \Theta \times \mathcal{H}$ be the state vector. A steady-state equilibrium of this economy is characterized by total intermediate goods X, total labor L, and the aggregate return component $\underline{r}$, such that: (1). $\{p, w\}$ are given by their respective marginal products; (2). given prices, households solve*

[17]Households' returns are: $\pi(a,\vartheta)/a = (1-\omega(a,\vartheta))(pA_F a^{\nu-1}) + \omega(a,\vartheta)(pA_R a^{\nu-1}) = (1-\omega(a,\vartheta))r_F + \omega(a,\vartheta)r_R$.

[18]This kind of wealth taxation applying a positive top marginal wealth tax is used among almost all countries that implements or implemented a wealth tax (i.e. Switzerland, France, Danemark, Norway etc.).

the stationary version of their decision problem (12), giving rise to an invariant distribution $\Gamma(\boldsymbol{s})$.[19] *(3). the government budget constraint balances;*

$$\overline{G} = \int \Big(t_w(\boldsymbol{s})wh + t_r(r(\boldsymbol{s})a(\boldsymbol{s})) + t_a(a(\boldsymbol{s})) + p_d(h)t_b(a(\boldsymbol{s}) - t_a(a(\boldsymbol{s}))\Big)\, d\Gamma(\boldsymbol{s})\,, \tag{16}$$

(4). labor market clear, i.e. $L = \int h(\boldsymbol{s})\, d\Gamma(\boldsymbol{s})$*, and capital markets clear, i.e.,*

$$X = \int \Big(\big[1 - \omega(\boldsymbol{s})\big] A_F + \omega(\boldsymbol{s}) A_R \Big) a(\boldsymbol{s})\, d\Gamma(\boldsymbol{s}), \quad \textit{and} \tag{17}$$

$$0 = \int \sum_{\kappa} \Big[\pi(\boldsymbol{s}) - \Big(\underline{r} + r_F(\boldsymbol{s}) + \omega(\boldsymbol{s}) \big[r_R(\boldsymbol{s}) + \kappa - r_F(\boldsymbol{s}) \big] \Big) a(\boldsymbol{s}) \Big]\, \pi_\kappa(\kappa)\, d\Gamma(\boldsymbol{s}). \tag{18}$$

Whenever $\mu < 1$, capital market clearing requires two conditions. Condition (17) states the standard capital clearing condition such that total capital used in the production sector must equals the total capital supplied by households. Condition (18) states that total household's capital income must equate total intermediate producer's profits. Therefore, besides X/L, the scalar $\underline{r}$ in equation (11) is the second aggregate equilibrium object that adjusts in equilibrium to satisfy (18). When $\mu < 1$, each unit of risky investment yields returns higher than its MPK. As such, the equilibrium base return $\underline{r}$ must be negative to satisfy (18).

The model has no analytical solution and must be solved numerically. We use the endogenous grid method (Carroll, 2006) and discretize value of κ into bins using a Gauss-Hermite quadrature.

4 Taking the Model to the Data

We map the stationary equilibrium of our model into the US data in two steps. We first fix some parameters based on model-exogenous information. We then calibrate the remaining parameters using a method of moments by numerically simulating the model in several dimensions.

4.1 Capital Heterogeneity: A First Glance at the Data

Prior to the calibration exercise, we begin with a discussion of the data used to calibrate and validate the model.

We use the SCF (1989:2019) with detailed information on households' portfolio composition, comprising a number of very wealth-rich households. When computing moments related to wealth inequality, we will sometimes refer to the adjusted SCF. The adjustments are made fol-

[19] At each state (a, ϑ, h), there is a continuum of individuals experiencing the *iid* shock y and κ. The stationary distribution is obtained using non-stochastic simulations.

lowing the method of Vermeulen (2016) for all of our sample period. First, we correct for under-reporting of assets by adjusting survey estimates of real assets, financial assets and liabilities such that they align with national balance sheet. Second, we adjust for under-representation at the top by merging the SCF with households in the Forbes rich lists and use it to compute wealth shares based on estimates of a Pareto Law for wealth. The procedure is detailed in Appendix **??**.

We also use the PSID (1998:2018) as it is a large and nationally representative biennal survey with a long panel dimension, containing details on capital income and costs (by broad classes of assets), assets prices, and inflow/outflow for each asset.[20] We restrict to the population between 20 to 70 years of age. Our definition of wealth is the net worth and we consider risky equity (private and public assets) as "productive risky assets" while other assets (detailed below) are considered as safe productive assets. This classification is not specific to us: Cagetti and De Nardi (2006) separate private equity entrepreneurial assets from other capital invested in a corporate firm and Kaplan et al. (2018)) consider only equity and commercial or business real estate as productive assets.

Portfolio composition. In Figure 2 we first use the SCF data to get a full *cross-sectional* picture of the average household's portfolio composition across the US wealth distribution for our sample period. As already largely documented in the literature, private and public equity investments are strongly correlated with wealth in the cross-section. The top 1% in the US hold on average 65% of their wealth in risky equity, while the corresponding share for the median household is 7%.[21]

Returns to wealth. We then measure returns heterogeneity in the US using the PSID. The full details regarding the construction of returns and their robustness is provided in Appendix C.[22] We provide a measure for four asset classes: total net worth, risk-free assets (savings, bonds, checking accounts), financial risky assets (public equities) and private equity businesses, with $l \in \{t, f, r, b\}$. Returns of asset l for household i in year t is given by

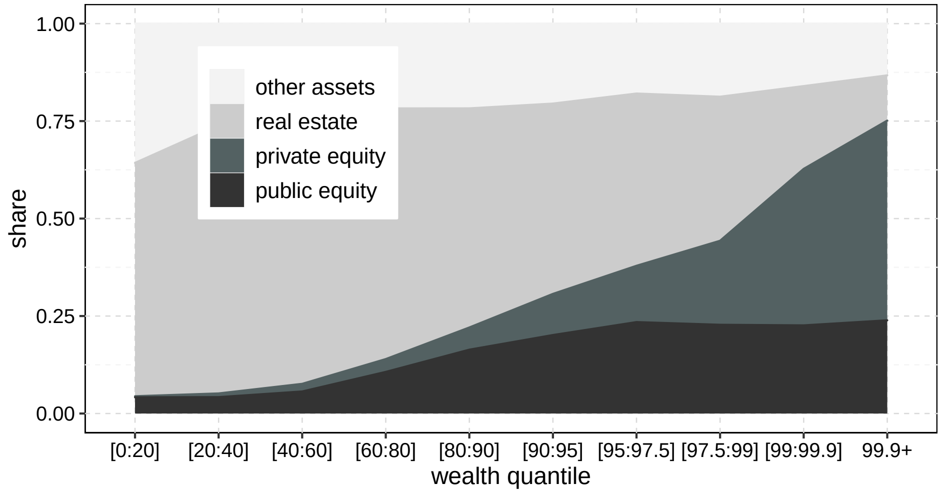
$$r_{i,l,t} = \frac{R^K_{i,l,t} + R^I_{i,l,t} - R^D_{i,l,t}}{(a^g_{i,l,t-2} + a^g_{i,l,t} + F_{i,l,t})/2}, \tag{19}$$

[20]See Pfeffer et al. (2016) for an excellent comparison between those two surveys and Flavin and Yamashita (2002) for a discussion about returns estimated from the PSID. Appendix C provides further details. Interestingly, with its oversampling, the SCF's total capital estimate is close to the one in national accounts. A drawback of the PSID is the presence of only a few households at the very top (within the top 1%) of the wealth distribution.

[21]This is comparable to estimates in Sweden according to Bach et al. (2020), using a similar definition of risky assets.

[22]The robustness includes different measures and specifications, including after-tax returns computed using the NBER TAXSIM program. The qualitative insights are similar to the ones derived in this section.

Figure 2. Average portfolio share of gross assets by wealth percentile.



Source: adjusted SCF from 1989 to 2019, averaged from the different SCF imputations.

where $a^g_{i,l,t-2}$ and $a^g_{i,l,t}$ are the (positive) amount of assets l held in the previous $(t-2)$ and current wave (t) and $F_{i,l,t}$ are inflow minus outflow (net investment), that we divide by two due to the biennial nature of the sample. The values $R^K_{i,l,t}$ and $R^I_{i,l,t}$ and $R^D_{i,l,t}$ correspond respectively to unrealized and realized capital gains, income (dividends and payments) and to the cost of debts (if any). Table 6 provides descriptive statistics regarding returns to wealth in the PSID. Notably, private equity returns display the highest expected returns (15.6%) with substantial heterogeneity and skewness to the right. To a lesser extent, public equity generates also substantial returns (5.8%). Despite the absence of very wealthy individuals in the PSID, our results are comparable to estimates in Fagereng et al. (2020) in Norway and Bach et al. (2020) in Sweden using administrative data. As a direct comparison with US estimates, Xavier (2020) evaluates, using cross-sectional information in the SCF, that aggregate returns are 13.6% for private equity, 6.4% for public equity and between 0.4%-2.1% on the different safe assets.[23] A noticeable difference, however, is that our aggregate estimate of returns on net worth is 2.4%, substantially lower than the one evaluated by Xavier (2020) (6.8%) but closer to the ones estimated by Benhabib et al. (2019) (3.1%) using a quantitative structural model of inequality and to Fagereng et al. (2020) for Norway (3.8%). In comparison to Sweden, Bach et al. (2020) find a median return to net worth of 4.5% with a standard deviation of 13% per year. We attribute this discrepancy to the fact that the PSID do not account for the upper end of the wealth distribution and to different methodology.

In Figure 3, we show the capital returns to net worth (before-tax) by selected wealth quantiles.

[23] Her methodology is very different from ours as we use information of a given household over time, while she computes returns using information on capital income and capital stocks in the cross-section per wealth percentile.

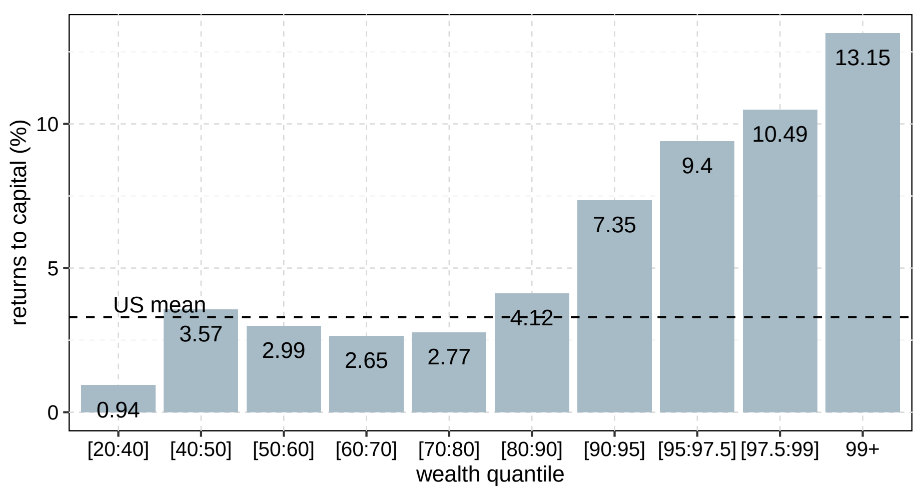
Table 1. Capital returns in the PSID (2000-2018).

Wealth component	Descriptive Statistics					
	Mean	St.Dev.	Skewness	Kurtosis	P20	P80
Net worth (before-tax)	0.024	0.158	0.897	6.243	-0.035	0.089
Private equity	0.156	0.614	2.071	10.967	-0.225	0.500
Public equity	0.058	0.417	-0.122	0.085	-0.248	0.385
Safe assets	0.004	0.009	3.234	10.267	0.000	0.003

Note: we apply a trimming of 1.5% at the top and the bottom for each return class.

Capital returns appear to strongly correlate with wealth in the cross-section. When decomposing by asset class, we find that this increase is not observable within asset class. Therefore, the increase in total returns is likely to be driven by heterogeneity in the household's portfolio composition documented above.[24] This feature is also consistent with existing work establishing the positive correlation between private equity ownership and wealth (Quadrini, 2000; Cagetti and De Nardi, 2006). Of course, those correlations are not informative on whether it is *type* or *scale* dependence that drives the relationship. In practice, there is no obvious way to disentangle the two as both mechanisms are likely to drive the observed pattern.

Figure 3. Average return on net worth by gross wealth quantile



[24]Such observation is shared by Bach et al. (2020) while Fagereng et al. (2020) argue that there is still substantial heterogeneity within broad asset class, which may reflect a key role for heterogeneity in financial knowledge. In fact, by focusing on broad asset class, it is difficult to disentangle whether increased returns are due to financial knowledge or to higher risk-taking within broad asset class. For private equity investments, this may arise due to diversification Penciakova (2018) or due to the interaction between business risk-taking and the borrowing limit (Robinson, 2012).

4.1.1 Scale Dependence in Capital Investments in the US economy

Given the within asset return pattern and the portfolio composition along the wealth distribution, *scale* dependence is likely to be driven by changes in the portfolio composition due to, for example, returns to scale on management costs or to decreasing relative risk aversion. Before detailing our calibration strategy, we start with a brief summary of *scale* dependence in the literature.

Exploiting the panel dimension of the PSID and controlling for individual characteristics, Hurst and Lusardi (2004) find evidence for *scale* dependence in the propensity to select into private equity business ownership among the top 5% wealthiest households. According to their estimates, above the top 5%, the probability to enter business ownership increases by 0.5 percentage point with wealth normalized by average income: reaching around 7% at the very top against 3.5% for most of the wealth distribution.

Using information on returns on the capital endowments of US universities, Piketty (2018) (Chapter 12) finds that returns substantially increase with wealth and argue that it may come from economies of scale in portfolio management.[25] However, the wealth holdings of US universities are substantially larger than most households in the US (above 10 millions dollars), and may be only representative of the behaviors at the upper end of the wealth distribution. As an additional piece of evidence, Bach et al. (2020) use swedish administrative panel data and test for *scale* dependence in returns to wealth. They argue that, even within a sample of twins and controlling for twin-pair fixed effects, there are evidence for strong *scale* dependence, especially increasing at the top of the wealth distribution (Table. 9). Fagereng et al. (2020) use a Norwegian administrative panel on capital returns and regress the average return on wealth on the individual's wealth percentile in the beginning of the period, with individual and year fixed effects. Both scale (wealth) and individual fixed effects are found to be statistically significant. Their estimates imply that *scale* dependence alone explain 48% of the 18 percentage point return difference between the 10% and 90% net worth percentiles.

Finally, Robinson (2012) shows that wealthier business owners tend to start relatively riskier businesses, with higher expected profitability. Penciakova (2018) confirms a similar pattern using a sample of U.S. firms with the Census Bureau's Longitudinal Business Database, patenting data, and Compustat and show that business owners who diversify tend to start riskier additional businesses. This diversification among business owners occur in the data, but mostly among the

[25]In Piketty's argument: "*The most obvious one is that a person with 10 million euros rather than 100,000, or 1 billion euros rather than 10 million, has greater means to employ wealth management consultants and financial advisors.*".

wealthiest households in the US (as shown below).

4.2 Calibration

We consider the calibration of three models. The first refers to the one with only *scale*-dependence (S-model), the second model refers to the one with only type-dependence (T-model), and the third is a *hybrid* model with both *type* and *scale* dependencies (H-model) distinguished using evidence in the PSID and SCF data. Our goal is to illustrate how distinguishing between these two modeling devices affect our results.

4.2.1 Common parameters set exogenously

Preferences and technology The benchmark model is calibrated to the US economy. The model period is one year. Preferences are given by a CRRA utility $u(c) = c^{1-\sigma}/(1-\sigma)$. We set σ to 1.5. The death probability is set to $p_d(h) = 1.5\% + 0.0045\,h$, implying a probability of death of 6% in the last income bracket, which aims to capture the increasing death rate over the life cycle.

We specify $F(X, L) = X^{\alpha}L^{1-\alpha}$ and set $\alpha = 0.33$ and the depreciation rate to $\delta = 5\%$. We normalize $A_F = 1$. In the baseline, we set $\mu = 1.0$ (perfect pass-through between returns and MPK) and study later cases with $\mu < 1$.

As already stressed above, the returns to scale is an other important wealth dependence channel. To estimate this parameter, we use our panel of households in the PSID and estimate the effect of asset holding $a_{i,l,t}$ on asset returns $r_{i,l,t}$, fixing the broad class of asset l, such that: $\log(r_{i,l,t}) = \beta_l \log(a_{i,l,t}) + FE_i + FE_t + \epsilon_{i,l,t}$, where β_l is an estimate of the returns to scale ν and FE_i and FE_t stand for individual and year fixed effects. If β_l is statistically null, returns exhibits CRS. Our results suggest that there is no significance for either IRS nor DRS, even for private equity business holdings.[26] As it is not a standard in the literature with entrepreneurs/capitalists investing in private business equity, we investigate the case with $\nu < 1$ in a sensitive analysis.

Tax system The proportional labor income tax is $\tau_w = 22.5\%$ consistent with Guvenen et al. (2019). The capital income tax is fixed at $\tau_r = 17.5\%$, which is the average top marginal tax rate over our sample period. The estate taxation is fixed to $\tau_b = 40\%$. The baseline economy does not feature any wealth tax and therefore $t_a(\mathrm{x}) = 0$.

[26] This results contrast with most of the literature with entrepreneurs assuming DRS technology on private equity businesses. **?** shows that DRS on the firm side and almost CRS on the households side can be reconciled once we take into account diversification in multiple private equity business investments at the very top of the wealth distribution.

Labor income process The distribution of labor productivity is an important component of the model as it helps generate a realistic earning distribution and contribution to the overall wealth inequality. We set the threshold of the Pareto Law to $q_h = 0.9$ and the shape to $\eta_h = 1.9$, consistent with 1990-2010 estimates for the US (Piketty and Saez, 2003). In line with Storesletten, Telmer and Yaron (2004), parameters of the persistent component of labor productivity are $\rho_h = 0.95$ and $\sigma_h = 0.2$.[27] The correlation between parents' labor productivity with the one of their heir is $p_h = 0.35$, consistent with Chetty et al. (2014). Finally, the expected lifetime is set to $M = 70$.

The transitory process $y \sim log - \mathcal{N}(0, \sigma_v)$ with $\sigma_v = 0.15$, consistent with Heathcote et al. (2010). As Hubmer et al. (2020), we assume an unemployment state $y_0 = \bar{b}$ occurring with probability $\chi = 7.5\%$, independently of (y, h) and over time, with $\bar{b}$ set to 0.35, in the range of the unemployment replacement rate in the US.

4.2.2 Calibrating Capital Heterogeneity and Wealth Returns

We now calibrate the portfolio allocation and variables associated to wealth returns. We set the volatility of risky investments to $\sigma_\kappa = 0.67$, which is our estimate from the PSID displayed in Table 6. Concerning the portfolio share $\omega(\vartheta, a)$, our goal is to match the cross-sectional equity portfolio allocation across the wealth distribution in the three considered models.

***Scale*-dependence model** We begin with the S-model. We direclty use the cross-sectional pattern of the portfolio composition across the wealth distribution as observed in the SCF to pin down the share of equity $\omega(.)$ in the model. While being obviously a naive calibration, it may reasonably represent the most extreme case of *scale* dependence. Specifically, we approximate the portfolio allocations in the S-model, $\omega^{S\text{-model}}(a)$, using a stepwise portfolio function with J wealth brackets A_j and corresponding portfolio shares $\omega^{scf}(A_j)$ for $j = 1, \ldots, J$, such that:

$$\omega^{S\text{-model}}(a) = \begin{cases} \omega^{scf}(A_1) & \text{if} \quad a < A_1 \\ \omega^{scf}(A_1) + (\omega^{scf}(A_2) - \omega^{scf}(A_1)) \left(\frac{a - A_1}{A_2 - A_1}\right) & \text{if} \quad A_1 \geq a < A_2 \\ \vdots & \\ \omega^{scf}(A_{J-1}) + (\omega^{scf}(A_J) - \omega^{scf}(A_{J-1})) \left(\frac{a - A_{J-1}}{A_J - A_{J-1}}\right) & \text{if} \quad A_{J-1} \geq a < A_J \\ \omega^{scf}(A_J) & \text{if} \quad A_J \geq a \end{cases}, \tag{20}$$

[27] For the sake of transparency, we reduce the computational burden by using a reduced transition matrix $\hat{\Pi}_h(h'|h)$ such that: $\hat{\Pi}_h(h'|h) = \begin{cases} \Pi_h(h'|h) & \text{if} \quad \Pi_h(h'|h) \geq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$ with $\epsilon = 10e^{-6}$ and normalizing the matrix $\sum_{h'} \hat{\Pi}_h(h'|h) = 1$. This allows us to exploit the sparsity of the transition matrix.

This stepwise portfolio function is intended to mirror the highly non-linear portfolio allocation across the wealth distribution in the United States as observed in Figure 2. Values $\{A_1, \ldots, A_J\}$ are chosen to correspond to the thresholds of wealth of specific wealth quantile, such that the model implied portfolio shares are in line with the data. We detail the resulting thresholds in Appendix **??**. Finally, notice that in this model the risk-taking type heterogeneity is irrelevant.

***Type*-dependence model** The portfolio shares in the T-model are matched as follows. We assume a two-state Markov chain of ϑ-type $(S = 2)$, such that $\vartheta \in \{\vartheta_1 = 0, \theta_2 > 0\}$. In this model, the state ϑ_2 represent an investor type who invest a positive share in equity:

$$\omega^{\text{T-model}}(\vartheta) = \vartheta \,. \tag{21}$$

Moreover, we calibrate the transition matrix $\pi(\vartheta|\mathbf{s})$ as follows. Consistent with the PSID, we fix the entry into equity investments $\pi(\vartheta' = \vartheta_2|\vartheta = \vartheta_1) = 3.5\%$ while the exit rate is fixed to $\pi(\vartheta' = \vartheta_1|\vartheta = \vartheta_2) = 15\%$. An important aspect of this transition matrix is that it features positive persistence. Due to this, the model will generate a positive selection of ϑ_2-type at the top of the wealth distribution, provided that they earn, on average, higher returns on their wealth. Hence, the T-model will generate a positive correlation between portfolio allocation and wealth as observed in Figure 2. This selection is higher the stronger the persistence of high types, and the higher is the returns extracted from ϑ_2-type (investor). Finally, we set a share of equity investment of $\vartheta_2 = 0.60$, which is roughly the average share invested by investors in the SCF data, conditional on investing at least 1000$ in equities. The resulting share of investors is 16.7%.

***Hybrid* model** We finally move to the H-model. A difficulty that arises when distinguishing *scale* and *type* dependence is that in order to identify carefully the distinction, one would need a large and long datasets on individual behaviors and testing for the dependence of portfolio allocation or returns while controlling for individual characteristics (typically using fixed effects). Following this strategy, Fagereng et al. (2020) and Bach et al. (2020) use long administrative panel on wealth tax records and provide strong support that returns feature *scale* and *type* dependence. However, the results are conditioned by the statistical model used to disentangle *type* and *scale* dependence, which is typically a linear model in wealth percentiles.

We pursue an other strategy that consists in recognizing that there are two common ways to generate *scale* dependence in models featuring private or public equity investments. On one hand, the extensive margin decision to invest might be wealth-dependent. This is the case in many occupational models (Cagetti and De Nardi (2006)) or models with fixed participation cost (Fagereng et al. (2017)). On the other hand, once investing in equity, the portfolio share (intensive margin) can be itself wealth-dependent. In the H-model, we therefore assume two ϑ-types, $\vartheta \in \{\vartheta_1 = 0, \vartheta_2 > 0\}$, where the probability to transit from a non-investor type to an investor type is correlated with wealth. Specifically, we let:

$$\pi^{\mathrm{H}-model}(\vartheta'|\mathbf{s}) \equiv \pi^{\mathrm{H}-model}(\vartheta'|\vartheta, a) = \begin{bmatrix} 1 - \underline{\pi}_\vartheta - \phi(a) & \underline{\pi}_\vartheta + \phi(a) \\ \overline{\pi}_\vartheta & 1 - \overline{\pi}_\vartheta \end{bmatrix} \tag{22}$$

where we fix $\overline{\pi}_\vartheta = 15\%$. The function $\phi(a)$ captures the increasing probability of entering into private equity business investment. We approximate $\phi(a)$ using a stepwise function with N wealth brackets A_k and corresponding excess probability to select into equity investment $\phi^{psid}(A_1)$, with $k = 1, \ldots, N$ such that:

$$\phi(a) = \begin{cases} \phi^{psid}(A_1) & \text{if} \quad a < A_1 \\ \phi^{psid}(A_1) + (\phi^{psid}(A_2) - \phi^{psid}(A_1))g_1(a) & \text{if} \quad A_1 \geq a < A_2 \\ \vdots & \\ \phi^{psid}(A_{N-1}) + (\phi^{psid}(A_N) - \phi^{psid}(A_{N-1}))g_{N-1}(a) & \text{if} \quad A_{N-1} \geq a < A_N \\ \phi^{psid}(A_N) & \text{if} \quad A_N \geq a \end{cases}, \tag{23}$$

where $\phi^{psid}(A_k)$, $\forall k$ are based on estimates of Hurst and Lusardi (2004) from the PSID.[28] This stepwise function aims, again, to capture the highly non-linear probability function of selecting into equity investments in a transparent and parsimonious way. The probability $\underline{\pi}_\vartheta$ is adjusted to generate a share of investor of 16.7% in the economy similar to the T-model.

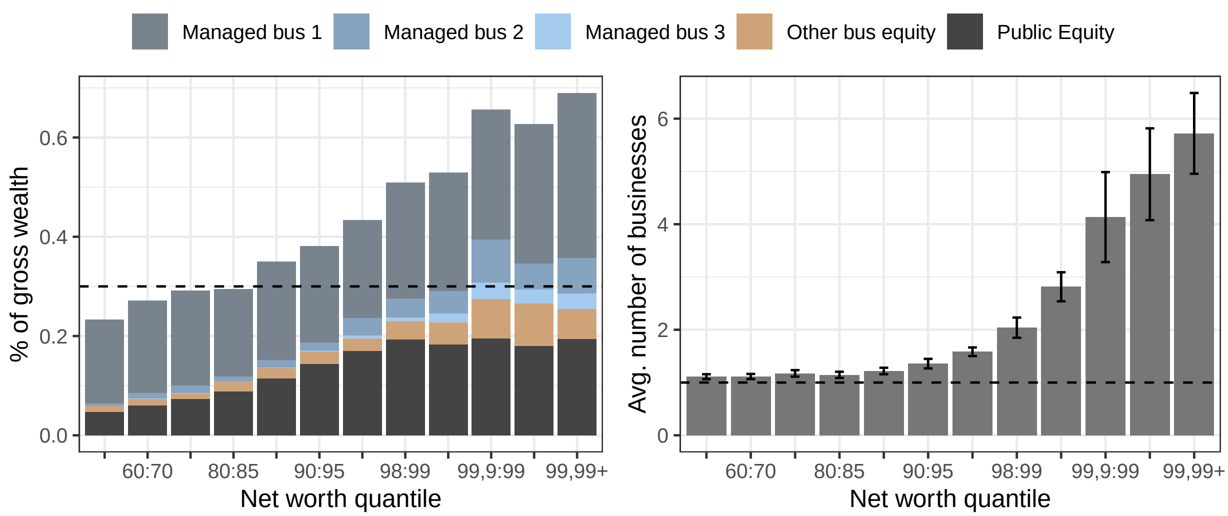
Finally, we calibrate the intensive margin (after becoming an investor) of equity investment as follows. In the SCF, conditioning on business ownership, the share of private equity to net worth increases along the wealth distribution. One practical issue, however, is to distinguish whether the share of private equity held by those owners increase because of systematic capital gains or whether it is the result of additional investments. Moreover, even if it results from additional investments, it is difficult to disentangle whether business owners who invest more are more likely to select at the top of the wealth distribution, or whether wealth itself induces higher investment. To circumvent as much as possible those issues, we exploit the number and the timing of private

[28] Notice that we provide similar results using the recent PSID waves. We relegate the full analysis to Appendix ??.

equity business investments in the SCF, with detailed information on private equity investments, including the share of wealth in each business and their acquisition dates.

Figure 4 shows the average share of private equity investments per net worth quantile, decomposed in different business investments (limited to the biggest three and others). The average share of private equity investment over total gross wealth appears to be strongly correlated with wealth (above the top 5%). This relation is driven by diversification at the top. From our standpoint, this pattern is unlikely to be driven by *type* dependence only. First, borrowing constraints are likely to prevent relatively wealth-poor households from investing in multiple businesses, limiting the concern regarding the possible reverse causality that an owner of multiple businesses is more likely to select over time at the top of the wealth distribution. Second, those additional businesses are generally new founded businesses: 85% (resp. 47%) of the second (resp. first) businesses were created within the preceding decade, and 67% (resp. 42%) within the past five years. Given that the process of wealth accumulation is a slow process, diversification and multiple business ownership is thus likely, among other determinants, to be wealth-driven.[29]

Figure 4. Average share held in private equity, separated into the different owned investments.



Source: SCF (1989:2019). The dashed line indicates the average private equity share.

We further elaborate on this point using two additional piece of evidence. First, the average

[29]The result is not due to household composition. Even when focusing on single households, private equity investments increase with wealth through diversification. Moreover, additional businesses are different than the first established business: 70% of additional private equity investments are made in a different sector. Moreover, they are only slightly more in finance-related industry, limiting the concern that it constitutes a financial affiliate company of the main business. Those facts challenge the widely held view that business owners are poorly diversified (Moskowitz and Vissing-Jørgensen (2002)) and are consistent with recent findings in Penciakova (2018). Diversification occurs, but only at the very top. In numbers, 12% of business owners own multiple managed businesses in the US. In the top 1%, they are 40%.

number of private equity business investments substantially rise at the upper end of the wealth distribution, consistent with our previous findings. Again, zooming on the timing of those additional businesses, we find that the last acquired business is particularly recent relative to the main business. Finally, despite the lack of observations at the very top in the PSID, we use its panel dimension and confirm that investing in additional businesses, conditional on already being a business owner and on individual characteristics, is statistically positively correlated with net worth. We elaborate on those additional pieces of evidence in Appendix **??**.

To take into account this intensive margin of equity investments, conditional on being an investor, we take a similar stepwise portfolio function as in the S-model but recalibrating the wealth brackets and the corresponding portfolio allocation to account for the increase of portfolio share due to diversification as in Figure 4. That is, we account for part of the observed increase in equity investments at the upper end of the wealth distribution. Denoting by $\omega^{scf-conditional}(a)$ the stepwise portfolio function that accunt for this increase, we assume that:

$$\omega^{H\text{-model}}(\vartheta, a) = \vartheta + \mathbb{1}_{\vartheta=\vartheta_2}\omega^{scf-conditional}(a)\psi \tag{24}$$

where ψ is a coefficient of correction that we introduce to not over-represent the importance of *scale*-dependence based on this strategy. In the baseline, we set $\psi = 0.5$, meaning that 50% of the observed increase in additional private equity investments is attributed to scale dependence, and study the sensitivity of the model to this parameter later on. Moreover, $\vartheta_2 = 0.35$ consistent with the average share invested in equity conditional on being investor as in Figure 4.[30]

4.2.3 Remaining endogeneously picked parameters

The discount factor β replicate a capital-output ratio $\frac{K}{Y}$ of 2.6. The productivity A_r match a top 1% wealth share of 34% by generating dispersion in returns to wealth across the wealth distribution in the three considered models. Let M_g denote the vector of the empirical moments. We find $\boldsymbol{\theta} = [\beta \; A_r]$ by numerically solving:

$$\boldsymbol{\theta} = \arg\min_{\boldsymbol{\theta}} \; \left[M(\boldsymbol{\theta}) - \widehat{M}\right]' \left[M(\boldsymbol{\theta}) - \widehat{M}\right] \; . \tag{25}$$

[30] In an alternative calibration strategy, we used the longitudinal information of returns to wealth in the PSID to calibrate portfolio shares $\omega(\vartheta, a)$ consistent with the shape of returns to net worth. However, there are important limitations of doing this. First, it is hard, given the few number of observations in the PSID, to test for the non-linearity of *scale*-dependence at the very top of the distribution. Second, this restrict ourself to a specification in which we rule out any interaction between *type* and *scale* dependence. However, evidence suggest that *scale* dependence occurs in both the extensive and the intensive margins of equity investments, thus introducing important interactions.

The procedure is standard.[31] Our model is exactly identified and match exactly the targets.

Finally, it is worth noting that we also explored versions with heterogeneity in discount factors correlated with the persistent labor income component, and a calibration allowing for some amount of debt. Those alternatives leave the conclusion of this paper materially unchanged.

4.3 Properties of the Models

We now discuss key properties of the calibrated models regarding: (i) the overall wealth inequality, portfolio choice and wealth returns (section 4.3.1), (ii) mobility and persistence of wealth-rank (section 4.3.2), and (iii) the response of output to a wealth inequality shock which translates the inequality-efficiency trade-off (section 4.4).

4.3.1 Wealth inequality and wealth returns

As demonstrated in section 2, it is important that the model captures well the top shares of the wealth distribution, as it conditions the relative strength of the *scale* and *type* dependence effects. In Table 2, we first assess the models' ability to generate an overall wealth distribution relative to its SCF counterpart, with a focus at the top. A striking result is that, outside the top 1% wealth share which is targeted, the three models account remarkably well for the observed top wealth shares. This may explain why Benhabib, Bisin and Luo (2019) (Table. 9) find that including *scale* dependence in a model already featuring *type* dependence does not provide further explanatory power and is thus badly identified.

Table 2. Wealth distribution: data (1998:2019) and models.

	Gini wealth	Share of wealth (in %) held by the top x%						
		40	20	10	5	1	0.1	0.01
US data (World Inequality Database)	–	97.6	85.1	70.8	57.8	35.7	18.2	8.1
US data (adjuted SCF) [b]	0.80	97.2	86.4	72.7	59.7	37.2	17.8	7.3
S-model	0.81	93.1	85.1	75.5	62.4	35.2	16.1	7.3
T-model	0.75	90.4	78.5	66.5	55.1	35.1	19.2	10.2
H-model	0.74	89.5	78.6	66.5	54.6	35.0	21.1	12.4
– no type, $\vartheta_2 = 0$	0.61	83.8	65.8	50.1	35.0	13.1	2.7	0.7
– no scale, $\psi = 0$	0.69	87.0	73.2	59.3	46.4	24.3	10.8	5.0
– no scale, $\phi(a) = 0$	0.72	88.9	76.2	62.6	50.8	30.7	16.7	9.2
– no *luck*, $\sigma_\kappa = 0$	0.74	89.4	78.6	66.5	54.6	35.0	21.0	12.4
– no portfolio heterogeneity	0.56	79.9	60.5	42.7	29.3	9.5	1.5	0.2

[a] Adjusted for under-representation and underreporting following the procedure in Vermeulen (2016).

[31] We solve the SMM as follows. We start with a Sobol sequence of parameters, and then use a CRS algorithm to iterately choose the best candidate solution.

The high concentration of wealth can be traced back to the large heterogeneity in wealth returns implied by the different equity portfolio allocation between households. Table 3 shows that consistent with the data, the three models produce an average return to wealth which is increasing along the wealth distribution. In the T-model, this is driven by selection only. Because risk-taking ϑ-type are persistent, households with high propensity to invest in equities have higher expected returns for several periods and are thus more likely to be selected at the top of the distribution, hence driving the observed cross-sectional relationship.[32] Under the S-model, the relationship is generated by construction, such that higher level of wealth is associated with higher risk-taking and higher returns. In the H-model, both *scale* and *type* dependence drive the observed pattern. To see this, we compute in the last two columns the contributing role of *scale* and *type* dependence in the baseline H-model by reporting the average returns to wealth across the wealth distribution assuming homogeneity in types (i.e. $\vartheta_2 = 0$) and assuming no scale dependence in the intensive margin (i.e. $\psi = 0$). We find that both forces contribute significantly to the observed increase in average returns across the wealth distribution. This result is supported by empirical evidence, both Fagereng et al. (2020) and Bach et al. (2020) find evidence that wealth returns are driven by *scale* dependence and *type* dependence.[33] Finally, and consistent with Bach et al. (2020) and Fagereng et al. (2020), the cross-sectional standard deviation of returns to wealth sharply increases with net worth because of high idiosyncratic risk in private equity. This implies that in the H-model, the resulting standard deviation of returns to wealth is 12.5%, against 15% in the PSID.

To further relate to the literature, we also compare the result to a model of entrepreneurship along the line of Cagetti and De Nardi (2006) or Guvenen et al. (2019), assuming that $\vartheta_2 = 1.0$ and adding decreasing returns to scale (DRS) with scale $\nu_r = 0.9$ on risky productive investment.[34] We refer to this model as a T-model with DRS and calibrate it to reproduce the top 1% wealth share and the K/Y ratio. It turns out that the *scale* dependence shifts sign due to the DRS specification, and the overall shape of returns across the wealth distribution is hump-shaped. Such a hump-shaped curve is not observed in the data. Focusing on private-equity business returns, there is no evidence of DRS in Fagereng et al. (2020) nor in Bach et al. (2020) (but returns are slightly decreasing due to leverage effects). Notice also that in the tradition of models of entrepreneurship,

[32]See Benhabib et al. (2011) and Moll (2014) for an theoretical illumination on the role of persistence in returns.

[33]Moreover, notice that the role of private equity is also very important in Fagereng et al. (2020): *"All in all, heterogeneity in our most comprehensive measure of returns to wealth can be traced in the first place to heterogeneity in returns to private equity and the cost of debt and only partially to heterogeneity in returns to financial wealth."*.

[34]This requires to add a maximum on the level of risky assets a household would like to invest. This maximum is never reached under our parameterization.

DRS is often assumed as it is the relevant assumption in the firm dynamics literature. In Gaillard (2021), we show that multiple business investments and diversification as observed in Figure 4 can rationalize CRS on returns on the investor side, but DRS on the firm side.

Table 3. Returns to wealth across the wealth distribution: data and model

Wealth group	Data				S-model	T-model			H-model[d]		
	PSID	SCF[a]	Norway[b]	Sweden[c]		CRS	DRS base.	DRS scale	base.	scale $\vartheta_2 = 0$	type $\psi = 0$
P40-P50	REF.	REF.	REF.	REF.	REF.	REF.	REF.	REF.	REF.	REF.	REF.
P50-P60	-0.6	|	1.5	0.2	0.0	0.4	0.9	-0.1	0.3	0.0	0.3
P60-P70	-0.9	-0.4		0.3	0.0	1.0	2.4	-0.2	0.6	0.0	0.6
P70-P80	-0.8	0.0	2.6	0.3	0.0	2.0	4.4	-0.4	1.0	0.0	1.0
P80-P90	0.5	0.2		0.5	0.7	3.1	6.2	-0.6	1.6	0.0	1.6
P90-P95	3.8	1.4	3.6	0.8	2.6	3.2	5.6	-0.9	1.6	0.7	1.5
P95-P97.5	5.8	2.6	5.2	1.1	3.8	3.1	8.1	-1.0	1.7	2.1	1.3
P97.5-P99	6.9	3.8		1.5	5.4	5.2	9.2	-1.2	3.2	2.9	2.5
Top 1%	9.6	4.6	8.3	2.5	7.3	6.7	8.1	-1.5	5.7	4.2	4.3
Top 0.1%	–	–	–	3.7	8.6	6.1	6.4	-1.8	6.0	5.0	4.2

[a] Estimates are taken from Xavier (2020).
[b] Estimates are taken from Bach et al. (2020).
[c] Estimates are taken from two separate papers: Fagereng et al. (2020) and Halvorsen et al. (2021).
[d] The no type model refers to the case with $\vartheta_2 = 0$ and the no scale model refers to the case with $\psi = 0$.

Finally, as stressed by Gabaix et al. (2016), both *scale* and *type* dependence can generate the speed of the dynamics of wealth inequality while stochastic returns (*luck*) can not. Focusing on the heterogeneity in returns due to heterogeneous portfolio allocation in risky equity, we inquire into the respective contribution of *scale* dependence, *type* dependence and *luck* in the observed steady-state wealth inequality. To answer this question, we decompose their role by shutting down each component in the H-model and recomputing the steady-state stationary distribution. A first counterfactual assumes no *type* dependence, such that only investment driven by wealth are taken into account, i.e. $\vartheta_2 = 0$. Second, we investigate the effects of *scale* dependence by shutting down the intensive margin $\psi = 0$ or the extensive margin $\phi(a) = 0$. Third, we assume no risk on equity investments, such that $\sigma_\kappa = 0$. Fourth, we remove all components linked to heterogeneous investments, such that the wealth distribution is almost entirely driven by heterogeneity in labor income. We report the results in the last rows of Table 2. We find that *scale* and *type* dependence are both important components of wealth inequality, as evidenced by the lower wealth shares generated under those counterfactuals, while *luck* explain very little of the wealth concentration. As expected, a model without heterogeneity in investment fail to account for the observed wealth concentration. As shown by Benhabib et al. (2011), this is because in such a case the tail of the wealth distribution inherits the tail of the labor income distribution, which is $\eta_h = 1.9$, much higher than the one estimated using the adjusted SCF (1.4).

4.3.2 Wealth Mobility

We now investigate how the intra-generational wealth mobility matrix in the models compares with Klevmarken et al. (2003), who estimate a five-state (quintiles) five-year transition matrix from 1994–1999 PSID data. We report the results in Table 4. We find that the three models overstate the persistence of wealth-rank in the top quintiles, while being broadly consistent with the U-shaped diagonal transition rate. This may be due to measurement issues in the data or to the absence of deterministic age states in the models. Interestingly, adding portfolio heterogeneity helps in generating an empirical-consistent wealth mobility. Overall, we conclude that the three models are hardly distinguished from the resulting wealth mobility matrix.

Table 4. Wealth mobility: data and model

5-years transition	Diagonal element (quintile – quintile)				
	Q1 – Q1	Q2 – Q2	Q3 – Q3	Q4 – Q4	Q5 – Q5
PSID (1994-1999), Klevmarken et al. (2003)	0.58	0.44	0.42	0.48	0.71
S-model	0.59	0.32	0.32	0.55	0.89
T-model	0.63	0.37	0.38	0.56	0.87
H-model	0.54	0.38	0.40	0.60	0.88
No portfolio heterogeneity	0.71	0.51	0.60	0.73	0.97

4.4 Wealth inequality shock: the role of *type* and *scale* dependence

The three considered models are consistent with key facts regarding the distribution of wealth and returns and the implied wealth mobility transition matrix. As identified as a key variable in our analysis of section 2, we fit very well the thin tail of the wealth distribution. The main mechanism at play, heterogeneity in portfolio choice, act as a key driver.

We now turn to the response of the model to a wealth inequality shock. The analysis we perform is to compute the long-run steady-state adjustment of the economy following a 1% top marginal wealth tax on the top 1% wealthiest households. That is, we define $\overline{A}_\tau$ the wealth threshold above which wealth is taxed to the one corresponding to the wealth held by the top 1% wealthiest household, and the marginal tax rate $\tau_a = 1\%$. The goal is to get an overall sense on the location of those models relative to *the Growth Irrelevance Frontier* represented in Figure 1, i.e. the deviation from the combinations of *type* and *scale* dependence such that inequality is neutral for the macro.

Table 5 shows the results. While each model produces consistent wealth inequality patterns, they differ greatly in terms of output response to a wealth inequality shock. Moving from an only *scale* dependence model to an only *type* dependence model reduce the output response from -1.92% to -0.89%, a division by more than two. The *hybrid* model falls in between, with a response of -1.12%, and shows that accounting for a correct degree of *scale* dependence is quantitatively important. We also report several other models. The model of entrepreneurship with *type*-dependence and negative *scale* dependence through DRS (column (3)) produces, consistent with previous results, an output-response of only -0.58%. Therefore, models with optimal taxation such as Brüggemann (2017), Guvenen et al. (2019), or Macnamara et al. (2021) are likely to be sensitive to the degree of *scale* and *type* dependence. Column (5) and (6) provide the result in a recalibrated version of the *hybrid* case where the intensive and extensive margins in portfolio choices are respectively removed, setting $\vartheta_2 = 0$. As before, higher degree of scale dependence is related to stronger output-response. Finally, a model without portfolio heterogeneity (column (7)) produces a slight reduction in output of -0.18% and is much closer to growth neutrality (in fact, the response is only related to change in overall wealth accumulation).[35]

Table 5. Response to a 1% top marginal wealth tax

	S-model	T-model		H-model [a]			no portfolio heterogeneity, $\omega(\vartheta, a) = 0$
		$\nu = 1.0$	$\nu = 0.9$	base.	no scale on portfolio, $\psi = 0$	no type on portfolio, $\vartheta_2 = 0$	
ΔGDP (in %)	-1.92	-0.89	-0.58	-1.12	-0.99	-1.53	-0.18
Δtop1% (in %)	-21.9	-11.9	-7.2	-13.6	-12.4	-16.3	-8.99

[a] model without scale or type dependence are recalibrated to match the top 1% and the K/Y ratio.

The results above are reported for the long-run steady-state adjustments. In Appendix **??**, we show that the contemporaneous response is also very sensible to the degree of *scale* dependence. As wealth accumulation is lower through time, it amplifies the negative response coming from *scale* dependence which then translates into an overall large output loss in the long-run.

[35]Strong empirical evidence on the effect of wealth tax on wealth accumulation is found for example in Jakobsen et al. (2020).

5 Introducing a Wealth Tax on the Wealth-rich Households

relate to boar

This section simulate the effect of introducing a wealth tax on the top 5% household ($q_a(\bar{a}_w) = 0.05$) from multiple angles. The focus on wealth taxation is natural, as both type and wealth dependence are related to the wealth distribution, and so to such taxes.[36] The relative simplicity of our experiment also makes it appealing from a policy perspective, compared to a full resolution of an optimal tax problem that requires changes to various instruments simultaneously.

We restrict attention to once-and-for-all tax reforms. Any increase (decrease) in tax revenue following the change in the wealth tax is balanced with an adjustment of the labor income tax τ_w to keep the same level of government spending $\overline{G}$ as in the baseline. Alternatively, we tried with adjustments in the lump-sum transfers, and the qualitative results remain valid.

In order to compute the effect of a wealth tax on overall welfare, it is necessary to define what a planner is maximizing over. We consider a planner who maximizes aggregate welfare $\mathcal{W}$ in the economy, that is, the average consumption equivalent variation $\Delta^{CE}(\mathbf{s})$ defined as the variation of consumption that makes a newborn in the initial steady-state distribution indifferent between the initial steady-state and the post-tax reform economy, formally:

$$\mathcal{W} = \int g(\mathbf{s})\Delta^{CE}(\mathbf{s})\, d\Gamma(\mathbf{s}) \qquad \text{such that,} \tag{26}$$

$$\mathbb{E}_0 \sum_{t=0}^{\infty} u\left((1+\Delta^{CE}(\mathbf{s}))c_t^{\text{pre-reform}}(\mathbf{s})\right) = \mathbb{E}_0 \sum_{t=0}^{\infty} u\left(c_t^{\text{post-reform}}(\mathbf{s})\right), \qquad \forall \mathbf{s} = (a, \vartheta, e, \kappa) \in \mathcal{S} \tag{27}$$

Optimal policy in our setting therefore balances the desire to reduce the initial wealth inequality (through wealth tax and labor income transfers) against the efficient cost coming from a reduction in accumulated capital (level effect) and productivity (reallocation effect).

5.1 Results

The left panel of Figure **??** reports the welfare gains/losses associated with a level of top marginal wealth tax under different combination of type and wealth dependence. The right panel reports the optimal top marginal tax rate τ_a^* in function of the degree of wealth dependence $\omega_a(a)$. It is worth noting that under the case of type dependence, the optimal wealth tax is substantial (3.3%) and higher than in the case with wealth dependence only. Again, this comes from the

[36]Recently, Guvenen et al. (2019) and Boar and Midrigan (2020) have studied the role of wealth tax in general equilibrium with entrepreneurship, with almost no role for wealth-dependence (apart from DRS).

high elasticity of capital allocation under wealth dependence. Under wealth-dependence, the wealth tax is positive but small (about 0.5%). The positive tax comes from the strong redistribution motive induced by the model economy, in which agents can not freely smooth their consumption due to the borrowing constraint.[37] It is interesting to note that the optimal wealth tax under type-dependence is close to the one found in Guvenen et al. (2019) who relies mostly on a type-dependence economy with entrepreneurs investing all their wealth into their business up to an exogenous borrowing constraint.

We also report the level of tax (and its confidence bounds) implied by the estimated degree of wealth dependence in the US economy. We find that given our estimates, an optimal marginal top wealth tax would be 2.5% applied on the top 5% wealthiest household. This shows that understanding the force behind the allocation of capital across the wealth distribution is of first order importance in understanding the effects of taxation. Notice that because we focus on the stationary distribution, types are sorted across the wealth distribution. Moreover, our estimate of wealth dependence is positive. Our model therefore locate on the top-right area of the inequality-efficiency diagram. In such case, this is the degree of type *versus* wealth dependence which implicitly provide on which *iso-growth* our economy is located. In Section **??**, we investigate the model with a negative wealth dependence arising from the use of DRS technology.

For illustration, we also report the results if we were using estimates in Fagereng et al. (2020) and the case in which we would use the cross-sectional estimates using the SCF estimate.[38]

Finally, notice that in absence of type and wealth dependence (i.e. only stochastic returns), it is optimal to tax wealth at a very high rate. This is a reminiscence of the results formulated in Stiglitz (1969) and Shourideh et al. (2012) which we quantitatively illustrate here. This is also related to the findings in Heathcote et al. (2020)

5.2 Elasticity of output to wealth inequality shock

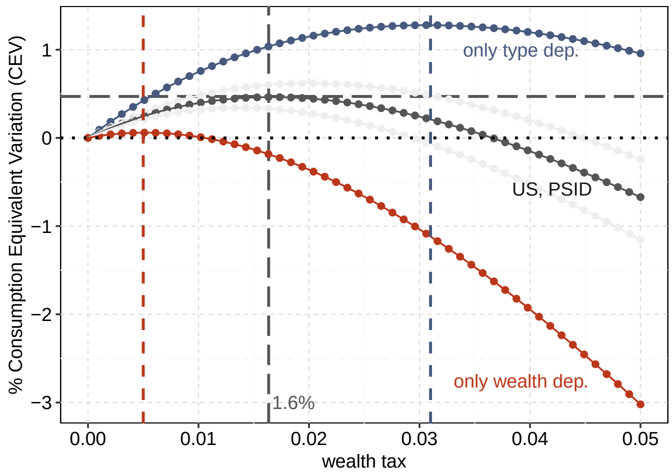
Table 6. Elasticity of Output to Top Wealth Tax.

Scale-dependence	Hybrid model	Type-dependence
-1.7%	-1.2%	-0.7%

[37] In that sense, in our economy, the Atkinson-Stiglitz theorem does not apply.

[38] We derive the results using Fagereng et al. (2020) by using the fact that they report that 50% of the returns heterogeneity between the bottom 10% and the top 10% is accounted for by wealth dependence.

Figure 5. Optimal top marginal tax rate and degree of type/wealth dependence [ADD OTHER PLOT].



Notes: the US economy is derived from the decomposition studied in Section **??** and is based on the degree of wealth dependence estimated in the PSID and the SCF.

Capital and return heterogeneity Our model account for two important channels generating heterogeneity in household's capital investments that are well appreciated in the literature: (i) heterogeneity in type (ϑ), and (ii) heterogeneity in wealth (a). Specifically, we assume that the level of risky investment is determined by household's wealth rank (wealth-dependence) and risk-tolerance (type), such that

$$\omega(a,\vartheta) = \omega_\vartheta(\vartheta) + \omega_a^1 \times q_a(a) + \omega_a^2 \times q_a(a) \times D(q_a(a) > \overline{q_a}) \tag{28}$$

with $q_a(a) \in (0,1)$ the wealth percentile of the household.[39] Consistent with the data, this implies that portfolio composition is a major component, if not the most important, that affect the returns. To focus on individuals that are especially prompt to be subject to wealth-dependence effects at the top, we include a dummy, $D(q_a(a) > \overline{q_a}) = 1$, for households in the top $\overline{q}_a$%. We choose $\overline{q}_a = 0.1$, consistent with the increased risky-asset share in the SCF and the increased returns to wealth (and associated risk) in the PSID at the very top.[40]

The parametric restrictions chosen is selected parsimoniously so that they can be recovered from the available data and compared to estimates using similar specifications in the literature

[39] In Appendix, we discuss frameworks that micro-found similar functional forms. We consider a model in which equities itself enters the utility function and a model in which there are management cost in managing risky assets.

[40] Our specification rule out, apart from the dummy, non-linearity in percentile of wealth. However, notice that the mapping between percentile and wealth is non-linear (and concave in fact). For $\omega_a^1 > 0$ and $\omega_a^2 = 0$, wealth-dependence increases with wealth, but less than proportionally. Finally, notice that the presence of the dummy could capture occupational choice threshold in entrepreneurial setup (Cagetti and De Nardi, 2006) or fixed cost Fagereng et al. (2017). In Appendix X, we use a logit-Verhulst specification to replicate the likelihood of participation in risky assets, which depends on wealth.

(Bach et al. (2020), Fagereng et al. (2020) and Meeuwis (2019)). Under this specification, equation (28) is testable in the PSID data using information on wealth returns and equation (11). Substituting the two equations yield:

$$r(a,\vartheta,\kappa) = \widetilde{\omega_\vartheta}(\vartheta) + \widetilde{\omega_a^1} q_a(a) + \widetilde{\omega_a^2} q_a(a) D(q_a(a) > \overline{q_a}) + \widetilde{\sigma_\kappa}(a,\vartheta)\kappa \tag{29}$$

with $\widetilde{\omega_\vartheta} = \underline{r} + r_F + (r_R - r_F)\omega_\vartheta(\vartheta)$, $\widetilde{\omega_a^1}(a) = (r_R - r_F)\omega_a^1$, $\widetilde{\omega_a^2}(a) = (r_R - r_F)\omega_a^2$ and $\widetilde{\sigma_\kappa}(a,\vartheta) = \omega_\vartheta(\vartheta) + \omega_a^1 q_a(q) + \omega_a^2 q_a(a) D(q_a(a) > \overline{q_a})$. An advantage of equation (29) is that capital returns have a continuous support in the riskiness dimension which is not observable when looking at portfolio share of broad class of assets in the data. This is especially important since recent work highlight heterogeneous investments even within broad asset class (see Robinson (2012) for private equities). An assumption, however, is that returns heterogeneity reflects heterogeneity in the riskiness of the underlying portfolio choice.[41]

One challenge with a model with type and wealth dependence is to select a process which characterize the "types". We parsimonious specify the process using a two-state Markov chain with $\omega_\vartheta(\vartheta) \in \{\omega_{\vartheta_1} = 0, \omega_{\vartheta_2}\}$, where ω_{ϑ_2} is chosen endogeneously. As we will show later, this process is able to capture the large wealth heterogeneity observed in the data, as well as the large heterogeneity in returns.[42] At the stationary equilibrium, selection of type along the wealth distribution occurs as long as the process for ϑ displays some persistence (Moll (2014)). Therefore, on top of the value $\omega_\vartheta(\vartheta_2)$, it is important to control the diagonal of the two-by-two transition matrix, $p_1 = \pi_\vartheta(\vartheta_1|\vartheta_2)$ and $p_2 = \pi_\vartheta(\vartheta_2|\vartheta_1)$, which directly controls the persistence of risk-tolerance types. We choose p_1 internally and $p_2 = 0.10$, corresponding to a 10% exit rate of equity investor.[43] Finally, we set the inter-generational correlation of the risk taking type, p_ϑ, to 0.15 following estimates from Fagereng et al. (2020).

6 Taxation, Endogenous Selection and Rents

The foregoing discussion highlighted the interplay between wealth inequality, the degree of *type* versus *scale* dependence, and the aggregate economy. Recently, Guvenen et al. (2019) and Boar and Midrigan (2020) have studied the role of wealth tax in general equilibrium with entrepreneurs

[41] This formulation also implies a standard deviation for returns $r(a,\vartheta,\kappa)$ which is increasing in wealth, as documented by Fagereng et al. (2020) and Bach et al. (2020) for Norwegian and Swedish data.

[42] Cagetti and De Nardi (2006) adopt a similar parsimonious process in a model with entrepreneurs, and show that it generates a good fit of the key distributional aspect of the data.

[43] We define an equity investor as an individual investing at least $\omega(a,\vartheta) > 0.1$ in equity.

with almost no role for *scale* dependence (apart from DRS). In this section we further show that the degree of *type* versus *scale* dependence, together with the fraction of MPK in returns, is key to assess the effects of policy reforms. In what follow, we base our experiment to Guvenen et al. (2019) and consider the case of capital taxation (either through the stock of wealth or through the flow of capital income).

Tax experiments In a first experiment, we remove taxes on capital income $\tau_w = 0$ and let the government chooses the labor income tax τ_w and the top marginal wealth tax τ_a on the top 10% wealthiest households in the economy. We refer to this experiment as the optimal wealth tax (OWT) reform. The second experiment consists in finding the optimal capital income tax τ_r while letting τ_w to adjust, and constraining $\tau_a = 0$. We refer to this experiment as the optimal capital income tax (OCIT) reform. We perform the analysis under three scenarios regarding the importance of rent-extractions in the returns to capital. Relative to the benchmark economy of each model, we adjust μ such that returns are driven entirely by MPK (our benchmark cases), half of the returns is driven by MPK, and the case where returns reflect only rent-extractions.

Measuring welfare The measure we adopt is standard in the quantitative macro literature (see Brüggemann (2020) or Guvenen et al. (2019) among many others). It consists in finding the consumption-equivalent variation Δ^{CE} that makes every newborn households in state $\mathbf{s} \in \mathcal{S}$ in the post-reform economy (identified with a $\sim$) as well-off as in the pre-reform economy:[44]

$$\int_{\mathcal{S}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \widehat{\beta}^t u\Big((1+\Delta^{CE})\widetilde{c}_t(\mathbf{s}) \Big) \right] \widetilde{\Gamma}(\mathbf{s}) = \int_{\mathcal{S}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \widehat{\beta}^t u\Big(c_t(\mathbf{s}) \Big) \right] \Gamma(\mathbf{s})$$

6.1 Results

The left panel of Figure **??** reports the welfare gains/losses associated with a level of top marginal wealth tax under different combination of type and wealth dependence. The right panel reports the optimal top marginal tax rate τ_a^* in function of the degree of wealth dependence $\omega_a(a)$. It is worth noting that under the case of type dependence, the optimal wealth tax is substantial (3.3%) and higher than in the case with wealth dependence only. Again, this comes from the high elasticity of capital allocation under wealth dependence. Under wealth-dependence, the wealth tax is positive but small (about 0.5%). The positive tax comes from the strong redistribution

[44] Alternatively, we tried with aggregating the micro consumption-equivalent variation $\Delta^{CE}(\mathbf{s})$ of each individuals. The results are qualitatively similar.

motive induced by the model economy, in which agents can not freely smooth their consumption due to the borrowing constraint.[45] It is interesting to note that the optimal wealth tax under type-dependence is close to the one found in Guvenen et al. (2019) who relies mostly on a type-dependence economy with entrepreneurs investing all their wealth into their business up to an exogenous borrowing constraint.

We also report the level of tax (and its confidence bounds) implied by the estimated degree of wealth dependence in the US economy. We find that given our estimates, an optimal marginal top wealth tax would be 2.5% applied on the top 5% wealthiest household. This shows that understanding the force behind the allocation of capital across the wealth distribution is of first order importance in understanding the effects of taxation. Notice that because we focus on the stationary distribution, types are sorted across the wealth distribution. Moreover, our estimate of wealth dependence is positive. Our model therefore locate on the top-right area of the inequality-efficiency diagram. In such case, this is the degree of type *versus* wealth dependence which implicitly provide on which *iso-growth* our economy is located. In Section **??**, we investigate the model with a negative wealth dependence arising from the use of DRS technology.

For illustration, we also report the results if we were using estimates in Fagereng et al. (2020) and the case in which we would use the cross-sectional estimates using the SCF estimate.[46]

Finally, notice that in absence of type and wealth dependence (i.e. only stochastic returns), it is optimal to tax wealth at a very high rate. This is a reminiscence of the results formulated in Stiglitz (1969) and Shourideh et al. (2012) which we quantitatively illustrate here. This is also related to the findings in Heathcote et al. (2020)

7 Robustness

7.1 The role of wealth inequality

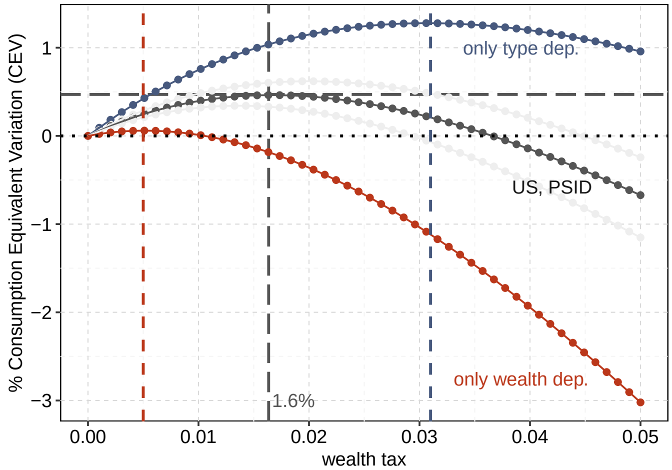
8 Conclusion

A satisfactory model of household's capital investments is essential for understanding the concentration of wealth, the response of aggregate output to wealth shock and ultimately the effects of wealth redistribution. This paper has developed a comprehensive framework to understand and decompose the inequality-efficiency trade-off into two terms: *type-dependence* coming from

[45] In that sense, in our economy, the Atkinson-Stiglitz theorem does not apply.

[46] We derive the results using Fagereng et al. (2020) by using the fact that they report that 50% of the returns heterogeneity between the bottom 10% and the top 10% is accounted for by wealth dependence.

Figure 6. Optimal top marginal tax rate and degree of type/wealth dependence [ADD OTHER PLOT].



Notes: the US economy is derived from the decomposition studied in Section **??** and is based on the degree of wealth dependence estimated in the PSID and the SCF.

heterogeneity in preferences and skills of households across the wealth distribution, and *wealth-dependence* arising from wealth-driven effects and decisions that affect households' investments.

In a two-period setup, we first showed that the combination of type and wealth dependence, for a given level of inequality, is essential to evaluate the inequality-efficiency profile of an economy, locating it on a unique *iso-growth*. In particular, the *Growth Irrelevance Frontier*, the *iso-growth* on which wealth inequality variations are growth neutral, separates the growth enhancing region from the growth dampening region. As such, to characterize the growth profile of a given economy, three statistics are key: a dependence between the distribution of types and the distribution of wealth, the wealth-dependent risk-taking elasticity and the level of inequality *per-se*.

We then evaluate quantitatively the importance of type and wealth dependence in a general equilibrium multi-periods incomplete markets economy. We calibrated the model for all *feasible* combination of type and wealth dependence that produce a correct representation of the US economy, in particular the level of wealth inequality that our simplified model as identified as a key statistic. Each combination is also found to produce a realistic cross-sectional distribution of returns to wealth. We find that the degree of type versus wealth dependence is critical for the inequality-efficiency profile, and therefore is key for evaluating optimal wealth redistribution.

To select the US economy on a particular combination of type and wealth dependence, we finally estimate wealth dependence in returns to wealth using the PSID data. Given our estimate, the optimal wealth tax on the top 10% wealthiest households is 2.5%. The result depends on the pass-through between returns and MPK and to which redistributive instrument is used.

This paper restricted attention to type and wealth dependencies that occur on the propensity to invest in risky assets. Much more research is needed to estimate carefully general type and wealth dependence through multiple channels: for example through offshore investments, tax avoidance behavior or human capital investments. Moreover, future research should attempt to empirically evaluate the pass-through between returns and MPK.

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A Appendix

A.1 Appendix for Section 2

A.1.1 Final producer maximisation

For clarity, we detail the steps of the final good producer who maximizes the use of labor n and intermediate goods x_s^j, with

$$\max_{\{n,x_s^j\}} \left(\sum_s \int_j x_s^j \, dj \right) n^{\varphi} - wn - \sum_s \int_j q_s^j \left(\frac{x_s^j}{\kappa_s^j} \right) dj$$

Taking the first order condition with respect to labor gives $w = \varphi \left(\sum_s \int_j x_s^j \, dj \right) n^{\varphi - 1}$. Plugging this condition together with the assumption that $n = \int_i e^i di = 1$ into the profit function yield:

$$\begin{aligned}
\Pi^f &= (1-\varphi) \left(\sum_s \int_j x_s^j \, dj \right) n^{\varphi} - \sum_s \int_j q_s^j \left(\frac{x_s^j}{\kappa_s^j} \right) \\
&= (1-\varphi) \left(\sum_s \int_j x_s^j \, dj \right) - \sum_s \int_j q_s^j \left(\frac{x_s^j}{\kappa_s^j} \right) dj
\end{aligned}$$

A.1.2 Proof of Lemma 1

Proof. Because of $u'(c_2^i) > 0$, we know that the second period budget constraint holds in equilibrium with equality. Thus, second period consumption is given by $c_2^i = wh_i + R_f a_1^i + \omega_1^i a_1^i (R_f - R_r^i)$. Substituting from the wage rate $w = \varphi Y$ and returns R_f and R_r^i, we get

$$c_2^i = \varphi Y h^i + A_f (1-\varphi)(1-\omega_1^i) a_1^i + (A_r + \kappa)(1-\varphi)\omega_1^i a_1^i \, .$$

We obtain that $c_2^i \sim \mathcal{N}\left(\mu_{c_2}^i, \sigma_{c_2}^i\right)$, with

$$\begin{aligned}
\mu_{c_2}^i &= \varphi Y + A_f (1-\varphi)(1-\omega_1^i) a_1^i + A_r (1-\varphi) \omega_1^i a_1^i, \\
\sigma_{c_2}^i &= \varphi Y \sigma_h^2 + \sigma_\kappa^2 (1-\varphi)^2 (\omega_1^i a_0^i)^2 + 2\rho_{\kappa,h} \sigma_\kappa \sigma_h \varphi Y (1-\varphi) \omega_1^i a_0^i,
\end{aligned}$$

which completes the proof of the result in the main body of the text. □

A.1.3 Proof of Lemma 2

Proof. We first need to derive $\mathbb{E}\left[u(c_2^i)|\mathcal{I}_1\right]$ analytically. To do so, we use an arbitrary Gaussian distribution with mean $\mu_{c_2}^i$ and variance $(\sigma_{c_2}^i)^2$. Using Lemma 1, we obtain

$$\begin{aligned}
\mathbb{E}\left[u(c_2^i)|\mathcal{I}_1\right] &= \frac{1}{\alpha_i}\int\left(1-e^{-\alpha_i c_2^i}\right)\times\frac{1}{\sqrt{2\pi(\sigma_{c_2}^i)^2}}e^{-\frac{1}{2(\sigma_{c_2}^i)^2}\left(c_2^i-\mu_{c_2}^i\right)^2}dc_2^i \\
&= \frac{1}{\alpha_i}-\frac{1}{\alpha_i}\int\frac{1}{\sqrt{2\pi(\sigma_{c_2}^i)^2}}e^{-\frac{1}{2(\sigma_{c_2}^i)^2}\left[\left(c_2^i-\mu_{c_2}^i\right)^2+2\alpha_i(\sigma_{c_2}^i)^2c_2^i\right]}dc_2^i \\
&= \frac{1}{\alpha_i}-\frac{1}{\alpha_i}e^{-\alpha_i\mu_{c_2}^i+\frac{1}{2}\alpha_i^2(\sigma_{c_2}^i)^2}\int\frac{1}{\sqrt{2\pi(\sigma_{c_2}^i)^2}}e^{-\frac{1}{2(\sigma_{c_2}^i)^2}\left(c_2^i-(\mu_{c_2}^i-\alpha_i(\sigma_{c_2}^i)^2)\right)^2}dc_2^i
\end{aligned}$$

Recognizing that the term in the integral is the pdf of a normally distributed random variable with mean $\mu_{c_2}^i-\alpha_i(\sigma_{c_2}^i)^2$ and variance $(\sigma_{c_2}^i)^2$, we finally obtain

$$\mathbb{E}\left[u(c_2^i)|\mathcal{I}_1\right] = \frac{1-e^{-\alpha_i\mu_{c_2}^i+\frac{1}{2}\alpha_i^2(\sigma_{c_2}^i)^2}}{\alpha_i}\ .$$

Under the additional set of assumptions within the special case section, we have $a_1^i \equiv a_0^i$ and $c_2^i = \varphi Y + R_f(1-\omega_1^i)a_0^i + R_r^i\omega_1^i a_0^i$ such that $\mu_{c_2}^i = \varphi Y + A_f(1-\varphi)(1-\omega_1^i)a_0^i + A_r(1-\varphi)\omega_1^i a_0^i$ and $\sigma_{c_2}^i = \omega_1^i a_0^i\sigma_\kappa(1-\varphi)$. We solve for the maximization problem given by

$$\max_{\{\omega_1^i\}}\left[1-exp\left\{-\alpha_i\left(\mu_{c_2}^i-\frac{\alpha_i}{2}\sigma_{c_2}^i\right)\right\}\right]\ ,$$

Denoting $\mathbb{V} = exp\left\{-\alpha_i\left(\mu_{c_2}^i-\frac{\alpha_i}{2}\sigma_{c_2}^i\right)\right\}$, the corresponding first order condition is given by

$$-\frac{1}{\alpha_i}\mathbb{V}\left[-\alpha_i(A_f-A_r)(1-\varphi)a_0^i+\alpha_i^2(a_0^i)^2\omega_1^i\sigma_\kappa^2(1-\varphi)^2\right]=0\ ,$$

which results after rearranging

$$\omega_1^i = \frac{A_r-A_f}{(1-\varphi)\alpha_i\sigma_\kappa^2}(a_0^i)^{-1} = \frac{A_r-A_f}{(1-\varphi)\sigma_\kappa^2}\frac{\vartheta^i}{\vartheta}(a_0^i)^{\gamma-1}\ .$$

To ensure that the solution is indeed a maximum, we derive the second order condition as

$$-\frac{1}{\alpha_i}\mathbb{V}\left[-\alpha_i(A_r-A_f)(1-\varphi)a_0^i+\alpha_i^2(a_0^i)^2\omega_1^i\sigma_\kappa^2(1-\varphi)^2\right]-\frac{1}{\alpha_i}\mathbb{V}\alpha_i^2\sigma_\kappa^2(1-\varphi)^2(a_0^i)^2<0\ .$$

which complete the proof. □

A.1.4 Proof of Lemma 3

Proof. It follows from the aggregation of agent's policy function regarding the share of risky assets invested. The expression for K_N follows from the aggregate resource constraint:

$$K_N + K_I = \mathbb{E}[a_0^i]$$

The expression for aggregate output follows from the $n = \int_i e^i \, di = 1$ and

$$Y = A_r K_I + A_f K_N = \left(A_r \frac{K_I}{\mathbb{E}[a_0]} + A_f \frac{K_N}{\mathbb{E}[a_0]}\right)\mathbb{E}[a_0] = \underbrace{\left[A_r \chi + A_f (1-\chi)\right]}_{:=Z} \mathbb{E}[a_0]$$

□

A.1.5 Proof of Corollary ??

Proof. The expression for aggregate innovative asset holdings follows straightforward from integrating over household dynamics while applying the covariance formula:

$$\begin{aligned} K_I = (\overline{\omega}/\overline{\vartheta})\mathbb{E}\left[\vartheta a_0^\gamma\right] &= \frac{A_r - A_f}{\alpha(\sigma_\kappa)^2}\left(cov(\vartheta, a_0^\gamma) + \mathbb{E}\left[\vartheta\right]\mathbb{E}\left[a_0^\gamma\right]\right) \\ &= \frac{\mu_z - R}{\alpha(\sigma_\kappa)^2}\left(\rho_{\vartheta, a_0^\gamma}\sigma_\vartheta \sigma_{a_0^\gamma} + \mu_\vartheta \mu_{a_0^\gamma}\right) . \end{aligned}$$

To prove the result regarding the effect of a mean preserving change in wealth inequality on K_I, we compare the aggregate innovate asset holdings for two economies with different Pareto tails $\eta' \neq \eta$ while keeping aggregate wealth μ_{a_0} constant. We then proceed by case distinction.

Case 1: $\rho_{\vartheta, a_0^\gamma} = 0$

The difference in aggregate innovative asset holdings between the two economies is written as

$$\begin{aligned} \Delta^w K_I(\eta', \eta) &= (\overline{\omega}/\overline{\vartheta})\mu_\vartheta \left(\frac{\eta'}{\eta' - \gamma}(\underline{a}')^\gamma - \frac{\eta}{\eta - \gamma}\underline{a}^\gamma\right) \\ &= \overline{\omega}\frac{\eta}{\eta - \gamma}\underline{a}^\gamma \left(\frac{\eta'}{\eta' - \gamma}\frac{\eta - \gamma}{\eta}\left(\frac{\underline{a}'}{\underline{a}}\right)^\gamma - 1\right) . \end{aligned}$$

Making use of the mean preserving assumption, i.e. $\underline{a}\frac{\eta}{\eta - 1} = \underline{a}'\frac{\eta'}{\eta' - 1}$, we obtain

$$\Delta^w K_I(\eta', \eta) = \overline{\omega}\frac{\eta}{\eta - \gamma}\underline{a}^\gamma \left(\frac{\eta'}{\eta' - \gamma}\frac{\eta - \gamma}{\eta}\left(\frac{\eta' - 1}{\eta'}\frac{\eta}{\eta - 1}\right)^\gamma - 1\right) .$$

Defining $\chi(\eta,\gamma) \equiv \frac{\eta-\gamma}{\eta}\left(\frac{\eta}{\eta-1}\right)^{\gamma}$ and taking the derivative of the inner expression w.r.t. η', we get

$$\chi(\eta,\gamma)\left(\frac{\eta'-1}{\eta'}\right)^{\gamma}\left[-\frac{\gamma}{(\eta'-\gamma)^2}+\frac{\gamma}{(\eta'-\gamma)(\eta'-1)}\right] = \chi(\eta,\gamma)\left(\frac{\eta'-1}{\eta'}\right)^{\gamma}\frac{\gamma(1-\gamma)}{(\eta'-\gamma)^2(\eta'-1)} .$$

As a result, we obtain finally

$$\frac{\partial\Delta^w K_I(\eta',\eta)}{\partial\eta'} = \overline{\omega}\mu_{a_0^\gamma}\chi(\eta,\gamma)\left(\frac{\eta'-1}{\eta'}\right)^{\gamma}\frac{\gamma(1-\gamma)}{(\eta'-\gamma)^2(\eta'-1)} .$$

The Lemma then follows by recognizing that $\frac{\partial\Delta^w K_I(\eta',\eta)}{\partial\eta'} = 0$ if $\gamma\in\{0,1\}$. Similarly, we obtain $\frac{\partial\Delta^w K_I(\eta',\eta)}{\partial\eta'} > 0$ if $\gamma\in(0,1)$ and $\frac{\partial\Delta^w K_I(\eta',\eta)}{\partial\eta'} < 0$ if $\gamma>1$.

CASE 2: $\rho_{\vartheta,a_0^\gamma}\neq 0$

In the case of an arbitrary correlation between innate risk aversion types and wealth, we obtain

$$\Delta K_I(\eta',\eta) = \Delta^s K_I(\eta',\eta) + \Delta^w K_I(\eta',\eta) ,$$

where the first term denotes distributional relevance arising from the selection effect, whereas the second term resembles distributional relevance arising from wealth dependent risk taking. Notice that the latter is equivalent to Case 1. Contrary, the first effect can be written as

$$\Delta^s K_I(\eta',\eta) = (\overline{\omega}/\overline{\vartheta})\rho_{\vartheta,a_0^\gamma}\sigma_\vartheta\sigma_{a_0^\gamma}\left(\frac{\rho_{\vartheta,a_0^\gamma}(\eta',\underline{a}',\cdot)}{\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}\frac{\sigma_{a_0^\gamma}(\eta',\underline{a}')}{\sigma_{a_0^\gamma}(\eta,\underline{a})}-1\right) .$$

Using the relation $\underline{a}\frac{\eta}{\eta-1} = \underline{a}'\frac{\eta'}{\eta'-1}$, a change in the Pareto tail η' preserves the mean wealth if $\underline{a}'(\eta') = \underline{a}(\frac{\eta}{\eta-1})\left(\frac{\eta'-1}{\eta'}\right)$. Using a first order Taylor approximation of $\rho_{\vartheta,a_0^\gamma}(\eta',\underline{a}'(\eta'),\cdot)$ around η, we obtain:

$$\begin{aligned}\rho_{\vartheta,a_0^\gamma}(\eta',\underline{a}'(\eta'),\cdot) &\approx \rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot) + \left|\frac{\partial\rho^1_{\vartheta,a_0^\gamma}(\eta',\underline{a}'(\eta'),\cdot)}{\partial\eta} + \frac{\partial\rho^2_{\vartheta,a_0^\gamma}(\eta',\underline{a}'(\eta'),\cdot)}{\partial\underline{a}'(\eta')}\frac{\partial\underline{a}'(\eta')}{\partial\eta'}\right|_{\eta'=\eta}(\eta'-\eta)\\ &\approx \rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot) + \left(\frac{\partial\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}{\partial\eta} + \frac{\partial\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}{\partial\underline{a}}\frac{\underline{a}}{\eta(\eta-1)}\right)(\eta'-\eta) .\end{aligned}$$

Substituting the previous expression into the one for $\Delta^s K_I(\eta',\eta)$ we arrive at

$$\Delta^s K_I(\eta',\eta) \approx (\overline{\omega}/\overline{\vartheta})\rho_{\vartheta,a_0^\gamma}\sigma_\vartheta\sigma_{a_0^\gamma}\left(\left[1+\frac{\left(\frac{\partial\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}{\partial\eta}+\frac{\partial\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}{\partial\underline{a}}\frac{\underline{a}}{\eta(\eta-1)}\right)}{\rho_{\vartheta,a_0^\gamma}(\eta,\underline{a},\cdot)}(\eta'-\eta)\right]\frac{\sigma_{a_0^\gamma}(\eta',\underline{a}')}{\sigma_{a_0^\gamma}(\eta,\underline{a})}-1\right) .$$

With a slight abuse of notation, we can take the derivative w.r.t. η' to obtain

$$\frac{\partial \Delta^s K_I(\eta',\eta)}{\partial \eta'} \approx (\overline{\omega}/\overline{\vartheta})\rho_{\vartheta,a_0^\gamma}\sigma_\vartheta\sigma_{a_0^\gamma}\left(\frac{1}{\sigma_{a_0^\gamma}}\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'} + \frac{\frac{\partial \rho_{\vartheta,a_0^\gamma}}{\partial \eta} + \frac{\partial \rho_{\vartheta,a_0^\gamma}}{\partial \underline{a}}\frac{\underline{a}}{\eta(\eta-1)}}{\rho_{\vartheta,a_0^\gamma}}\left(\frac{\sigma'_{a_0^\gamma}}{\sigma_{a_0^\gamma}} + (\eta'-\eta)\frac{1}{\sigma_{a_0^\gamma}}\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'}\right)\right) .$$

To get an impression about the sign of the previous derivative, let us first analyze the sign of $\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'}$. Notice that it is straightforward to show that a_0^γ follows a $\mathcal{P}a(\underline{a}^\gamma, \frac{\eta}{\gamma})$ distribution. As a result, the variance is given by

$$\sigma'_{a_0^\gamma} = (\underline{a}')^{2\gamma}\frac{\frac{\eta'}{\gamma}}{\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)} = \left(\underline{a}\frac{\eta}{\eta-1}\right)^{2\gamma}\left(\frac{\eta'-1}{\eta'}\right)^{2\gamma}\frac{\frac{\eta'}{\gamma}}{\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)} ,$$

where again the last equality follows from using $\underline{a}\frac{\eta}{\eta-1} = \underline{a}'\frac{\eta'}{\eta'-1}$. Defining the auxiliary variable $\tilde{\chi}(\eta,\gamma,\underline{a}) \equiv \left(\underline{a}\frac{\eta}{\eta-1}\right)^{2\gamma}$, we obtain:

$$\begin{aligned}\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'} =& \tilde{\chi}(\eta,\gamma,\underline{a})\left(2\gamma\left(\frac{\eta'-1}{\eta'}\right)^{2\gamma-1}\frac{1}{(\eta')^2}\frac{\frac{\eta'}{\gamma}}{\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)}\right)\\ &+\tilde{\chi}(\eta,\gamma,\underline{a})\left(\frac{\eta'-1}{\eta'}\right)^{2\gamma}\left(\frac{\frac{1}{\gamma}\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)-\frac{\eta'}{\gamma}\left[\frac{2}{\gamma}\left(\frac{\eta'}{\gamma}-1\right)\left(\frac{\eta'}{\gamma}-2\right)+\frac{1}{\gamma}\left(\frac{\eta'}{\gamma}-1\right)^2\right]}{\left(\frac{\eta'}{\gamma}-1\right)^4\left(\frac{\eta'}{\gamma}-2\right)^2}\right) .\end{aligned}$$

Collecting terms leads to

$$\begin{aligned}\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'} &= \tilde{\chi}(\eta,\gamma,\underline{a})\left(\frac{\eta'-1}{\eta'}\right)^{2\gamma}\frac{1}{\left(\frac{\eta'}{\gamma}-1\right)^2\left(\frac{\eta'}{\gamma}-2\right)}\left[\frac{2}{\eta'-1}+\frac{1}{\gamma}-\frac{2\eta'}{\gamma(\eta'-\gamma)}-\frac{\eta'}{\gamma(\eta'-2\gamma)}\right]\\ &= \frac{1}{\eta'}\sigma'_{a_0^\gamma}\left[1+\frac{2\gamma}{\eta'-1}-\frac{2\eta'}{(\eta'-\gamma)}-\frac{\eta'}{(\eta'-2\gamma)}\right] .\end{aligned}$$

In order to determine the sign of the bracket term, one can simplify to

$$\begin{aligned}\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'} &= \frac{1}{\eta'}\sigma'_{a_0^\gamma}\left[\frac{2\gamma(\eta'-2\gamma)+(\eta'-1)(\eta'-2\gamma)-\eta'(\eta'-1)}{(\eta'-1)(\eta'-2\gamma)}-\frac{2\eta'}{(\eta'-\gamma)}\right]\\ &= \frac{1}{\eta'}\sigma'_{a_0^\gamma}\left[\frac{2\gamma(1-2\gamma)}{(\eta'-1)(\eta'-2\gamma)}-\frac{2\eta'}{(\eta'-\gamma)}\right]\\ &= \frac{2}{\eta'}\sigma'_{a_0^\gamma}\left[\frac{\gamma(1-2\gamma)(\eta'-\gamma)-\eta'(\eta'-1)(\eta'-2\gamma)}{(\eta'-1)(\eta'-\gamma)(\eta'-2\gamma)}\right] .\end{aligned}$$

It is straightforward to show that the previous term is (weakly) negative if

$$\eta' \geq 2\gamma + \gamma \frac{(1-2\gamma)(\eta'-\gamma)}{\eta'(\eta'-1)} \, .$$

The left hand side of this expression is increasing in η', whereas the right hand side is decreasing if $\gamma \leq \frac{1}{2}$ and increasing if $\gamma > \frac{1}{2}$. Hence, for the case of $\gamma \leq \frac{1}{2}$, we obtain after substituting $\eta' = 2\gamma$ an upper limit of the right hand side given by $\overline{\eta} = \frac{3}{2}\gamma$. Contrary, in the case of $\gamma > \frac{1}{2}$ a straightforward application of L'Hopital's rule results in $\overline{\eta} = 2\gamma$. As a result, we obtain that $\frac{\partial \sigma'_{a_0^\gamma}}{\partial \eta'} \leq 0 \; \forall \eta' \geq \overline{\eta} = \max\{\frac{3}{2}\gamma, 2\gamma\} = 2\gamma$, which trivially holds due to the implicit assumed finite variance of the $\mathcal{P}a(\underline{a}^\gamma, \frac{\eta}{\gamma})$ distribution. Consequently, the result of Lemma **??** follows (given a small change in the inequality tail). □

A.1.6 Proof of Proposition 1

Proof. The Farlie-Gumbel-Morgenstern (FGM) copula can be written for two arbitrary cumulative distribution functions $\{F(x_1), F(x_2)\}$ as

$$F(x_1, x_2) = C^{FGM}(F(x_1), F(x_2)) = F(x_1)F(x_2) + \varrho F(x_1)F(x_2)(1-F(x_1))(1-F(x_2)) \, ,$$

where $\varrho \in [-1, 1]$. The joint probability density function of $f(x_1, x_2)$ is the obtained by

$$\begin{aligned} f(x_1, x_2) &= (1 + \varrho\,(1-2F(x_1))\,(1-2F(x_2)))\, f(x_1)f(x_2) \\ &= (1 + \varrho + 2\varrho\,(2F(x_1)F(x_2) - F(x_1) - F(x_2)))\, f(x_1)f(x_2) \, . \end{aligned}$$

Under this assumption that $\vartheta \sim \mathcal{P}a(\underline{\vartheta}, \epsilon)$ and $a_0 \sim \mathcal{P}a(\underline{a}, \eta)$ this provides us with

$$f(\vartheta, a_0) = (1+\varrho) f(\vartheta) f(a_0) + 2\varrho \left[2 \left(\frac{\underline{\vartheta}}{\vartheta}\right)^{\epsilon} \left(\frac{\underline{a}}{a_0}\right)^{\eta} - \left(\frac{\underline{\vartheta}}{\vartheta}\right)^{\epsilon} - \left(\frac{\underline{a}}{a_0}\right)^{\eta} \right] f(\vartheta) f(a_0) \, .$$

where the marginals are given by $f(\vartheta)$ and $f(a_0)$. Given the Pareto assumptions, we have $\mu_\vartheta \equiv \overline{\vartheta} = \underline{\vartheta}\frac{\epsilon}{\epsilon-1}$ and $\mu_{a_0^\gamma} = \underline{a}^\gamma \frac{\eta}{\eta-\gamma}$. In order to derive $cov(\vartheta, a_0^\gamma)$, we need to compute $\mathbb{E}\left[\vartheta a_0^\gamma\right]$:

$$\mathbb{E}\left[\vartheta a_0^\gamma\right] = \int_{\underline{\vartheta}}^{\infty} \int_{\underline{a}}^{\infty} \vartheta a_0^\gamma f(\vartheta, a_0) d\vartheta da_0 \, .$$

Using the FGM copula, we proceed in four steps:

$$(1+\varrho)\underline{\vartheta}^{\epsilon}\underline{a}^{\eta}\epsilon\eta\int_{\underline{\vartheta}}^{\infty}\int_{\underline{a}}^{\infty}\vartheta^{-\epsilon}a_0^{\gamma-\eta-1}d\vartheta da_0=(1+\varrho)\underline{\vartheta}\frac{\epsilon}{\epsilon-1}\underline{a}^{\gamma}\frac{\eta}{\eta-\gamma}\,,$$

$$4\varrho\underline{\vartheta}^{2\epsilon}\underline{a}^{2\eta}\epsilon\eta\int_{\underline{\vartheta}}^{\infty}\int_{\underline{a}}^{\infty}\vartheta^{-2\epsilon}a_0^{\gamma-2\eta-1}d\vartheta da_0=4\varrho\underline{\vartheta}\frac{\epsilon}{2\epsilon-1}\underline{a}^{\gamma}\frac{\eta}{2\eta-\gamma}\,,$$

$$-2\varrho\underline{\vartheta}^{2\epsilon}\underline{a}^{\eta}\epsilon\eta\int_{\underline{\vartheta}}^{\infty}\int_{\underline{a}}^{\infty}\vartheta^{-2\epsilon}a_0^{\gamma-\eta-1}d\vartheta da_0=-2\varrho\underline{\vartheta}\frac{\epsilon}{2\epsilon-1}\underline{a}^{\gamma}\frac{\eta}{\eta-\gamma}\,,$$

$$-2\varrho\underline{\vartheta}^{\epsilon}\underline{a}^{2\eta}\epsilon\eta\int_{\underline{\vartheta}}^{\infty}\int_{\underline{a}}^{\infty}\vartheta^{-\epsilon}a_0^{\gamma-2\eta-1}d\vartheta da_0=-2\varrho\underline{\vartheta}\frac{\epsilon}{\epsilon-1}\underline{a}^{\gamma}\frac{\eta}{2\eta-\gamma}\,.$$

Combining the previous four equations results in

$$\begin{aligned}cov(\vartheta,a_0^{\gamma})&=\mathbb{E}\left[\vartheta a_0^{\gamma}\right]-\mathbb{E}\left[\vartheta\right]\mathbb{E}\left[a_0^{\gamma}\right]\\&=\underline{\vartheta a}^{\gamma}\left[\varrho\frac{\epsilon}{\epsilon-1}\frac{\eta}{\eta-\gamma}+4\varrho\frac{\epsilon}{2\epsilon-1}\frac{\eta}{2\eta-\gamma}-2\varrho\frac{\epsilon}{2\epsilon-1}\frac{\eta}{\eta-\gamma}-2\varrho\frac{\epsilon}{\epsilon-1}\frac{\eta}{2\eta-\gamma}\right]\end{aligned}$$

Further simplifications result in

$$\begin{aligned}cov(\vartheta,a_0^{\gamma})&=\underline{\vartheta a}^{\gamma}\varrho\left[\frac{\eta}{\eta-\gamma}\left(\frac{\epsilon}{\epsilon-1}-\frac{2\epsilon}{2\epsilon-1}\right)+\frac{2\eta}{2\eta-\gamma}\left(\frac{2\epsilon}{2\epsilon-1}-\frac{\epsilon}{\epsilon-1}\right)\right]\\&=\underline{\vartheta a}^{\gamma}\varrho\left[\frac{\eta}{\eta-\gamma}\frac{\epsilon}{(\epsilon-1)(2\epsilon-1)}-\frac{2\eta}{2\eta-\gamma}\frac{\epsilon}{(\epsilon-1)(2\epsilon-1)}\right]\\&=\underline{\vartheta a}^{\gamma}\varrho\frac{\epsilon}{(\epsilon-1)(2\epsilon-1)}\frac{\eta\gamma}{(\eta-\gamma)(2\eta-\gamma)}\,.\end{aligned}$$

As a result, aggregate innovative asset holdings from Lemma 3 are given by

$$K_I=(\overline{\omega}/\overline{\vartheta})\left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\underline{\vartheta}\frac{\epsilon}{\epsilon-1}\underline{a}^{\gamma}\frac{\eta}{\eta-\gamma}\,.$$

Aggregate risk free capital holdings from Lemma 7 are (weakly) positive if the condition $\mu_{a_0}\geq K_I$ holds, which can be rewritten as

$$\frac{\eta-\gamma}{\eta-1}\geq(\overline{\omega}/\overline{\vartheta})\left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\frac{\epsilon}{\epsilon-1}\underline{\vartheta a}^{\gamma-1}\,.$$

Finally, let $\widetilde{\omega}=(\overline{\omega}/\overline{\vartheta})\mu_{\vartheta}\underline{a}^{\gamma-1}=\overline{\omega}\underline{a}^{\gamma-1}$ and $C=(2\epsilon-1)$, we derive the marginal effect of a change in the Pareto tail η on expected output $Y(\eta)=A_f+\widetilde{\omega}(A_r-Af)\Psi(\eta)$ with $\Psi(\eta)=$

$\left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)\frac{\eta-1}{\eta-\gamma}$ as:

$$
\begin{aligned}
\frac{\partial Y(\eta)}{\partial \eta} &= \widetilde{\omega}(A_r-A_f)\frac{\partial \Psi(\eta)}{\partial \eta} \\
&= \widetilde{\omega}(A_r-A_f)\left[\left(\frac{1}{\eta-\gamma}\right)\left(1+\frac{\varrho\gamma}{C(2\eta-\gamma)}\right)-\left(\frac{\eta-1}{(\eta-\gamma)^2}\right)\left(1+\frac{\varrho\gamma}{C(2\eta-\gamma)}\right)-2\left(\frac{\varrho\gamma}{C(2\eta-\gamma)^2}\right)\left(\frac{\eta-1}{\eta-\gamma}\right)\right] \\
&= -\widetilde{\omega}(A_r-A_f)\left[\left(\frac{1}{(\eta-\gamma)^2}\right)(\gamma-1)\left(1+\frac{\varrho\gamma}{C(2\eta-\gamma)}\right)+2\left(\frac{\varrho\gamma}{C(2\eta-\gamma)^2}\right)\left(\frac{\eta-1}{\eta-\gamma}\right)\right] \\
&= -\widetilde{\omega}(A_r-A_f)\left[(\gamma-1)\left(\frac{1}{(\eta-\gamma)^2}\right)+\varrho(\gamma-1)\left(\frac{\gamma}{C(2\eta-\gamma)(\eta-\gamma)}\right)+2\left(\frac{\varrho\gamma}{C(2\eta-\gamma)^2}\right)\left(\frac{\eta-1}{\eta-\gamma}\right)\right] \\
&= -\widetilde{\omega}(A_r-A_f)\left[(\gamma-1)\underbrace{\left(\frac{1}{(\eta-\gamma)^2}\right)}_{:=\Omega^w}+\varrho(\gamma-1)\underbrace{\left(\frac{(\gamma(2\eta-\gamma)+2(\eta-\gamma)(\eta-1))}{C(2\eta-\gamma)^2(\eta-\gamma)^2}\right)}_{:=\Omega^{ws}}+\varrho\underbrace{\left(\frac{2(\eta-1)}{C(2\eta-\gamma)^2(\eta-\gamma)}\right)}_{:=\Omega^s}\right]
\end{aligned}
$$

where the last line follows from $\varrho\gamma=\varrho+\varrho(\gamma-1)$. In order to determine the sign of the derivatives, we need to know the sign of $\left(1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\right)$. To do so, let us assume that

$$
1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\leq 0 \Leftrightarrow 1\leq -\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}\leq\frac{\gamma}{(2\epsilon-1)(2\eta-\gamma)},
$$

where the last inequality follows from $\varrho\in[-1,1]$. As we have $\epsilon>1$ and $\eta>\gamma$, an upper bound of $\frac{\gamma}{(2\epsilon-1)(2\eta-\gamma)}$ is given by $\frac{\gamma}{\epsilon\eta}$, which is strictly smaller than one. Hence, we obtain a contradiction and conclude that $1+\frac{\varrho\gamma}{(2\epsilon-1)(2\eta-\gamma)}$ is strictly positive. This completes the proof of Corollary.

Case with aggregate shock on risky investment (extension) TO REDO With a TFP A_r+z with $z\sim\mathcal{N}(0,\sigma_z^2)$, the proof for the variance of output from the fact that $\sigma_Y^2(\eta)=\sigma_z^2\widetilde{\omega}^2\Psi(\eta)^2$ which yield:

$$
\begin{aligned}
\frac{\partial\sigma_Y^2(\eta)}{\partial\eta} = -2\sigma_z^2\widetilde{\omega}^2\left(\frac{\partial\Psi(\eta)}{\partial\eta}\right)\Psi(\eta) &= -2\sigma_z^2\widetilde{\omega}\Phi(\eta;\gamma,\varrho)\Psi(\eta) \\
&= -2\sigma_z^2\widetilde{\omega}\Phi(\eta;\gamma,\varrho)\frac{E[Y(\eta)]-A_f}{\widetilde{\omega}(A_r-A_f)} \\
&= -2\sigma_z^2\Phi(\eta;\gamma,\varrho)\frac{E[Y(\eta)]-A_f}{(A_r-A_f)}
\end{aligned}
$$

where $\Phi(\eta;\gamma,\varrho)$ is defined in the core paper. □

A.1.7 Proof of Lemma ??: Existence of the $isoG(\eta, 0)$

Proof. Using Lemma 1, the *iso-growth* at level $\overline{g}$ can be implicitly defined as

$$(A_r - A_f)\overline{g} = -(A_r - A_f)\Big[(\gamma - 1)\Omega^w + \varrho(\gamma - 1)\Omega^{ws} + \varrho\Omega^s\Big] \tag{30}$$

Which, for a level $\overline{g} = 0$ can be rewritten as:

$$\left(1 + \frac{\varrho\gamma}{(2\epsilon - 1)(2\eta - \gamma)}\right)\frac{1-\gamma}{(\eta-\gamma)^2} - 2\frac{\eta - 1}{\eta - \gamma}\frac{\varrho\gamma}{(2\epsilon - 1)(2\eta - \gamma)^2} = 0\,,$$

which we can rearrange to

$$\frac{\varrho\gamma}{(2\epsilon - 1)(2\eta - \gamma)^2(\eta - \gamma)^2}\Big((1-\gamma)(2\eta - \gamma) - 2(\eta - 1)(\eta - \gamma)\Big) = -\frac{1-\gamma}{(\eta-\gamma)^2}\,,$$

If $(1-\gamma)(2\eta - \gamma) - 2(\eta - 1)(\eta - \gamma) \neq 0$ (which only occurs in the case of $\gamma < 1$) we can state the GIF algebraically as

$$\varrho(\gamma, \eta, \epsilon; \overline{g} = 0) = \frac{(2\epsilon - 1)(1-\gamma)}{\gamma}\frac{(2\eta - \gamma)^2}{2(\eta - 1)(\eta - \gamma) - (1-\gamma)(2\eta - \gamma)}\,. \tag{31}$$

The sign of the derivative w.r.t. to the Pareto tail is determined by

$$\begin{aligned}
\operatorname{sgn}\left(\frac{\partial\varrho}{\partial\eta}\right) &= \operatorname{sgn}\Big((1-\gamma)\Big)\operatorname{sgn}\Big(4(2\eta-\gamma)\Big(2(\eta-1)(\eta-\gamma) - (1-\gamma)(2\eta-\gamma)\Big) - (2\eta-\gamma)^2\Big(4(\eta-1)\Big)\Big) \\
&= \operatorname{sgn}\Big((1-\gamma)\Big)\operatorname{sgn}\Big((\gamma-\eta)(2-\gamma)\Big)\,.
\end{aligned}$$

Hence, we obtain due to $\eta > \gamma$

$$\frac{\partial\varrho}{\partial\eta}\begin{cases} < 0 & \text{if } \gamma > 2\,, \\ = 0 & \text{if } \gamma = 2\,, \\ > 0 & \text{if } 1 < \gamma < 2\,, \\ = 0 & \text{if } \gamma = 1\,, \\ < 0 & \text{if } 0 < \gamma < 1\,. \end{cases}$$

Before turning to the existence of η^{GIF}, we first study the limits of (31). Thus, we obtain

$$\lim_{\eta \to \gamma^+}\varrho(\gamma, \eta, \epsilon) = -\frac{(2\epsilon - 1)(1-\gamma)}{\gamma}\frac{(2\eta - \gamma)}{1-\gamma} = -(2\epsilon - 1)\,.$$

Similarly, we have

$$\lim_{\eta \to 1^+} \varrho(\gamma, \eta, \epsilon) = -\frac{(2\epsilon - 1)(1-\gamma)}{\gamma} \frac{(2-\gamma)}{1-\gamma} = -\frac{(2\epsilon-1)(2-\gamma)}{\gamma} .$$

Finally, by an application of L'Hopitals rule we derive

$$\lim_{\eta \to \infty} \varrho(\gamma, \eta, \epsilon) = \frac{(2\epsilon - 1)(1-\gamma)}{\gamma} \frac{2\eta - \gamma}{\eta - 1}|_{\eta = \infty} = 2\frac{(2\epsilon-1)(1-\gamma)}{\gamma} .$$

As a result, we obtain the following bounds on the copula dependence parameter

$$\begin{aligned}
-(2\epsilon - 1) < \varrho < 2\frac{(2\epsilon-1)(1-\gamma)}{\gamma} \qquad & \text{if } \gamma > 1 \\
\varrho = 0 \qquad & \text{if } \gamma = 1 \\
-\frac{(2\epsilon-1)(2-\gamma)}{\gamma} < \varrho < 2\frac{(2\epsilon-1)(1-\gamma)}{\gamma} \qquad & \text{if } \gamma < 1
\end{aligned}$$

It is straightforward to see for $\gamma > 2$ that the required $\varrho \notin \mathcal{R}$ such that the GIF is empty (i.e. $\nexists\, \eta^* \in (\gamma, \infty)$ s.t. $\text{GIF}(\eta^*) = 0$). Contrary, for $1 < \gamma < 2$ there exists by an application of the intermediate value theorem a unique $\eta^* \in (\gamma, \infty)$ such that $\text{GIF}(\eta^*) = 0$ if $-1 < \varrho < \overline{\rho}^{FGM} < 0$, where $\overline{\rho}^{FGM} \equiv 2\frac{(2\epsilon-1)(1-\gamma)}{\gamma}$. Additionally, in the case of $\gamma = 1$, being on the growth irrelevance frontier requires $\varrho = 0$. As a result, for any $\eta \in (1, \infty)$ the GIF goes through the point $\{\gamma = 1, \varrho = 0\}$. Finally, in the case of $\gamma < 1$ we can show that the growth irrelevance frontier is discontinuous at the points (cf. denominator of equation (31))

$$\eta_{dc}^{1,2} = 1 \pm \sqrt{1 - \frac{1}{2}(3\gamma - \gamma^2)} .$$

Recognizing that $3\gamma - \gamma^2$ is strictly increasing in γ on the interval $(0, 1)$, we conclude that the expression in the square brackets is strictly positive and lies on the interval $(0, 1)$. Due to the imposition of $\eta > 1$, we can hence exclude the smaller solution. Let us subsequently denote by $\eta_{dc}^* \in (1, 2)$ the only feasible discontinuity point. As $\frac{\partial \varrho}{\partial \eta} < 0$, the denominator of (31) is strictly increasing in η and $\lim\limits_{\eta \to 1^+} < -1$ for $\gamma \in (0, 1)$, we conclude that $\varrho > 0$ is a necessary condition for the existence of a GIF solution. This implies that the feasible set of Pareto tails reduces to $\eta \in (\eta_{dc}^*, \infty)$. Ensuring that $\varrho \leq 1$ gives us finally a lower bound on the wealth dependent risk taking parameter such that $\gamma \geq \underline{\gamma} = \frac{2(2\epsilon-1)}{1+2(2\epsilon-1)}$. As a result, for all $\gamma < \underline{\gamma}$ a GIF solution does not exists. Contrary, by an application of the intermediate value theorem an unique $\eta^* \in (\eta_{dc}^*, \infty)$ exists for $\underline{\gamma} \leq \gamma < 1$. This completes the proof of Lemma **??**. □

Proof Lemma ??

Proof. Using equation (31) from Lemma **??**, we obtain

$$\begin{aligned}\frac{\partial \varrho}{\partial \gamma} =& - (2\epsilon - 1)\frac{1}{\gamma^2}\frac{(2\eta-\gamma)^2}{2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)} \\ &+ (2\epsilon-1)\frac{1-\gamma}{\gamma}\frac{-2(2\eta-\gamma)\left[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\right]-(2\eta-\gamma)^2\left[2(1-\gamma)+1\right]}{\Big(2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big)^2} \ .\end{aligned}$$

Hence, the sign of the previous expression is determined by the sign of

$$\begin{aligned}\mathrm{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big) =& \Big(-(2\eta-\gamma)^2\Big[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big]\Big) \\ &+ \Big((1-\gamma)\gamma(2\eta-\gamma)\Big[-4(\eta-1)(\eta-\gamma)+2(1-\gamma)(2\eta-\gamma)-2(2\eta-\gamma)(1-\gamma)-2(\eta-\gamma)\Big]\Big) \ .\end{aligned}$$

Simplifying terms provides us with

$$\mathrm{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big) = \mathrm{sgn}\Big(\underbrace{-(2\eta-\gamma)\Big[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big]}_{\equiv \mathcal{A}} + \underbrace{(1-\gamma)\gamma\Big[-4(\eta-1)(\eta-\gamma)-2(\eta-\gamma)\Big]}_{\equiv \mathcal{B}}\Big) \ .$$

Let us begin with the case $\gamma = 1$. It is straightforward to see that

$$\mathrm{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big)|_{\gamma=1} = \mathrm{sgn}(\mathcal{A}) = \mathrm{sgn}\Big(-2(2\eta-\gamma)(\eta-1)(\eta-\gamma)\Big) < 0 \ ,$$

such that the GIF is strictly decreasing in the point $\gamma = 1$. For the case $\underline{\gamma} < \gamma < 1$, the reasoning is slightly more evolved. First, notice that $\mathcal{B} < 0$ in this case. Second, requiring that the inner bracket of $\mathcal{A}$ is weakly positive is equivalent to requiring that

$$2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma) \geq 0 \Leftrightarrow 2\left(\eta^2-\eta\gamma-\eta+\gamma\right)-2\eta+\gamma+2\eta\gamma-\gamma^2 \geq 0$$

Collecting terms gives us the following condition

$$2\eta^2 - 4\eta + 3\gamma - \gamma^2 \geq 0 \ . \tag{32}$$

We know from Lemma **??** that the GIF is only defined in this case if $\eta > \eta^*_{dc} = 1 + \sqrt{1 - \frac{1}{2}\left(3\gamma - \gamma^2\right)}$.

Substituting this expression into the former inequality yields

$$\begin{aligned} \text{LHS} &= 2\left(1+\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)}\right)^2 - 4\left(1+\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)}\right) + 3\gamma-\gamma^2 \\ &= 2+4\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)} + 2\left(1-\frac{1}{2}\left(3\gamma-\gamma^2\right)\right) - 4 - 4\sqrt{1-\frac{1}{2}\left(3\gamma-\gamma^2\right)} + 3\gamma-\gamma^2 \\ &= 0\,. \end{aligned}$$

As the left hand side of the inequality (32) is strictly increasing in η on the set $\eta > \eta^*_{dc} > 1$, we know that the above inequality is always satisfied strictly. Hence, we conclude that $\mathcal{A} < 0$. Finally, this provides us with

$$\operatorname{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big)|_{\underline{\gamma}\leq\gamma<1} = \operatorname{sgn}\left(\mathcal{A}+\mathcal{B}\right) < 0\,.$$

Let us finally consider the case $1 \leq \gamma < \overline{\gamma}$. It is evident that $\operatorname{sgn}(\mathcal{A}) < 0$ and $\operatorname{sgn}(\mathcal{B}) > 0$ hold in this case. Hence, the sign of the derivative is *a priori* undetermined. To show our claim, let us first rewrite the claim on the sign of our intitial inequality as

$$\begin{aligned} &-(2\eta-\gamma)\Big[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big] + (1-\gamma)\gamma\Big[-4(\eta-1)(\eta-\gamma)-2(\eta-\gamma)\Big] < 0 \\ \Leftrightarrow &(2\eta-\gamma)\Big[2(\eta-1)(\eta-\gamma)-(1-\gamma)(2\eta-\gamma)\Big] > \gamma(1-\gamma)\Big[-4(\eta-1)(\eta-\gamma)-2(\eta-\gamma)\Big] \\ \Leftrightarrow &(\gamma-1)(2\eta-\gamma)\Big[2\eta-2\gamma\Big] + 2(\eta-1)(\eta-\gamma)\Big[2\eta-\gamma+2\gamma(1-\gamma)\Big] > 0 \\ \Leftrightarrow &(\gamma-1)(2\eta-\gamma) + (\eta-1)\Big[2\eta-\gamma+2\gamma(1-\gamma)\Big] > 0 \\ \Leftrightarrow &(2\eta-\gamma)(\eta+\gamma-2) + 2(\eta-1)\gamma(1-\gamma) > 0\,. \end{aligned} \tag{33}$$

We know that $\eta > \gamma$ has to hold. Substituting $\eta = \gamma$ into the previous inequality yields

$$2\gamma(\gamma-1) > 2\gamma(\gamma-1)^2\,,$$

which is trivially satisfied for $\gamma < 2$. Hence, it suffices to show that the left hand side of (33) is increasing in η. To do so, let us take the derivative of (33) w.r.t. η and let us simultaneously impose that the derivative is positive:

$$\mathcal{Q}(\eta) \equiv 2(\eta+\gamma-2) + 2\eta-\gamma+2\gamma(1-\gamma) = 4(\eta-1)+3\gamma-2\gamma^2 > 4(\gamma-1)+3\gamma-2\gamma^2 \equiv \mathcal{Q}(\gamma)\,,$$

where the second last inequality holds due to $\eta > \gamma$. Hence, $\mathcal{Q}(\gamma) > 0$ implies also $4(\eta - 1) + 3\gamma - 2\gamma^2 > 0$. To show the validity of the previous inequality, let us compute the values of γ of the second order polynomial $\mathcal{Q}(\gamma)$ for which the function equals exactly zero:

$$\tilde{\gamma}^{1,2} = \frac{-7 \pm \sqrt{49 - 32}}{-4} = \frac{7}{4} \pm \frac{1}{4}\sqrt{17}\,,$$

As a consequence, we have that $\tilde{\gamma}^1 < 1$ and $\tilde{\gamma}^2 > 2$ such that $\mathcal{Q}(\gamma)$ is due to continuity strictly positive on the interval $\gamma \in [1, 2]$. As a result, we know that $\mathcal{Q}(\eta)$ is also strictly positive on the entire interval $\gamma \in [1, 2]$ which proves that equation (33) is satisfied. This shows

$$\operatorname{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big)|_{1<\gamma<\overline{\gamma}} = \operatorname{sgn}\left(\mathcal{A} + \mathcal{B}\right) < 0\,,$$

such that we overall obtain

$$\operatorname{sgn}\Big(\frac{\partial \varrho}{\partial \gamma}\Big)|_{\underline{\gamma}\leq\gamma<\overline{\gamma}} = \operatorname{sgn}\left(\mathcal{A} + \mathcal{B}\right) < 0\,.$$

Let us finally define the growth irrelevance equation (31) by $\varrho \equiv \mathcal{G}(\gamma)$, where $\{\epsilon, \eta\}$ enter the $\mathcal{G}$ function as constants. Recognize that $\mathcal{G}$ is strictly decreasing on the interval $\underline{\gamma} \leq \gamma < \overline{\gamma}$ and thus *injective*. Additionally, it is differentiable at $\mathcal{G}^{-1}(\varrho)$ and hence continuous on the interval $\mathbb{G}$. As a result, we can define the inverse function of the growth irrelevance frontier equation (31) by

$$\gamma \equiv \mathcal{G}^{-1}(\varrho)\,.$$

Consequently, it is straightforward to obtain

$$\frac{\partial \gamma}{\partial \varrho} = \frac{\partial \mathcal{G}^{-1}(\varrho)}{\partial \varrho} = \frac{1}{\mathcal{G}'(\mathcal{G}^{-1}(\varrho))} = \frac{1}{\mathcal{G}'(\gamma)} < 0\,,$$

which completes the first part of the proof of Lemma **??**.

PART 2. Let us consider now the general growth irrelevance frontier at an arbitrary growth level $\overline{g}$, possibly different from zero. Without loss of generality (??) let us further assume that $\psi_{min,0} = 1$. The GIF is then implicitly characterized by

$$\frac{(1-\gamma)(2\eta-\gamma) - 2(\eta-1)(\eta-\gamma)}{(2\epsilon-1)(2\eta-\gamma)^2(\eta-\gamma)^2}\gamma\varrho + \frac{1-\gamma}{(\eta-\gamma)^2} = \chi_{\overline{g}}\overline{g}\,,$$

where $\chi_{\overline{g}}$ denotes a strictly positive constant which is independent of $\{\varrho, \gamma\}$. Total differentiation of the previous equation yields

$$\chi_\rho d\varrho + \chi_\gamma d\gamma = \chi_{\overline{g}} d\overline{g} \,.$$

Rearranging the previous condition results in

$$d\gamma = -\frac{\chi_\rho}{\chi_\gamma} d\varrho + \frac{\chi_{\overline{g}}}{\chi_\gamma} d\overline{g} \,.$$

On the restricted set of Lemma 6, we have $\chi_\rho < 0$. Additionally, we have that $\frac{\chi_\rho}{\chi_\gamma} = -\frac{1}{\mathcal{G}'(\gamma)}$, which implies $\chi_\gamma < 0$. Hence, an increase in $\overline{g}$ (i.e. lower growth rate) decreases γ, conditional on ϱ. As a result, the GIF shifts downwards. Similarly, if $\overline{g}$ decreases (i.e. higher growth rate), γ increases conditional on ϱ which shifts the GIF upwards. This concludes the proof. □

A.1.8 Proof of Proposition 5

A.1.9 Proof of Lemma 6

Proof. Let us first observed that we can rewrite the utility function as:

$$\max_{k_i^1, b_1^i} \left(\frac{1}{1-1/\sigma}\right) \left[\left(a_0^i - k_1^i - b_1^i\right)^{1-1/\sigma} + \beta \left(\mu_{c_2}^i(k_1^i, b_1^i) - \frac{\alpha_i}{2}\sigma_{c_2}^i(k_1^i, b_1^i)\right)^{1-1/\sigma} \right] \tag{34}$$

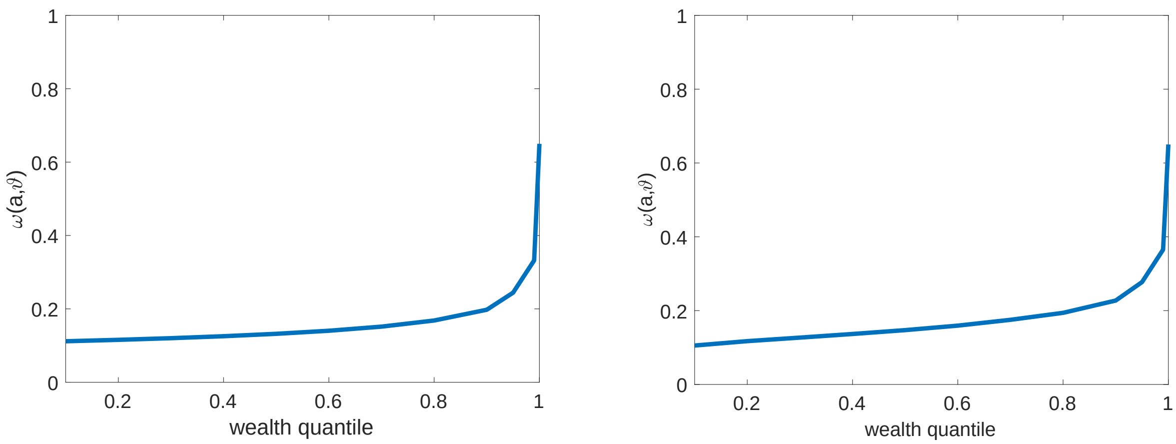
with $k_1^i = \omega_1^i a_1^i$ and $b_1^i = (1-\omega_1^i)a_1^i$. □

A.1.10 Proof of results with rent-extraction

B Simulation & Computational Details

B.1 Fit of the Static Model under pure type/wealth dependence

Figure ?? shows the fit of the type and wealth dependence models relative to the distribution of risky asset shares across the wealth distribution. To fit this shape, we first fixed inequality to $\eta = 1.3$. The parameter $\gamma = 1.41$ is used to match the risky asset share of the top 1% in the wealth dependence model. In the pure type dependence model, we used the parameter controlling the correlation (to 0.55) between the two distribution, fixing the Pareto shape of types to $\varepsilon = 2$. The two models produce an extremely close fit of the observed distribution.



(a) wealth dependence model

(b) type dependence model

C Empirical Appendix

Supplementary Material for Online Appendix

C.1 Additional results: Existence of a representative households

Lemma 7. *Let us define aggregate output of the economy as $Y \equiv A_f K_N + A_r K_I$. It is given by*

$$Y = A_f \left(\mu_{a_0} - K_I\right) + A_r K_I \ .$$

(a) *Suppose that $\vartheta_i = \vartheta \ \forall i$. There exists a representative household (RH) whose aggregate dynamics are equivalent to the ones obtained from the aggregation of individual dynamics if and only if $\gamma \in \{0, 1\}$. In both cases the RH is characterized by $\alpha_{RH} = \frac{\alpha}{\vartheta \frac{\eta}{\eta-\gamma} \underline{a}^{\gamma}}$ and $a_{RH,0} = \mu_{a_0}$.*
(b) *Contrary, let us suppose that $\vartheta_i \in \vartheta$ with $\vartheta \subset \mathbb{R}_+$. Furthermore, let us assume that*

$$\tilde{\vartheta} \equiv \mu_{a_0}^{-\gamma} \left(corr(\vartheta, a_0^{\gamma}) \sigma_{\vartheta} \sigma_{a_0} + \mu_{\vartheta} \mu_{a_0^{\gamma}} \right) \in \vartheta \quad \text{ and } \quad h(\tilde{\vartheta}, \mu_{a_0}) > 0 \, ,$$

where $h(\vartheta, a_0)$ denotes the joint probability density function of types and initial wealth. Then, there exists a RH $\forall \ \gamma \geq 0$, who is characterized by $\vartheta_{RH} = \tilde{\vartheta}$ and $a_{RH,0} = \mu_{a_0}$.

Proof. Let us first consider the more general case with $\vartheta_i \in \Theta$. To establish part (b) of Lemma 7, we first compute aggregate innovative asset holdings by integrating over the individual ones which we have derived in Corollary 1. This gives us,

$$K_I^* = (\overline{\omega}/\overline{\vartheta}) \left(\rho_{\vartheta, a_0^{\gamma}} \sigma_{\vartheta} \sigma_{a_0^{\gamma}} + \mu_{\vartheta} \mu_{a_0^{\gamma}} \right) \ .$$

Due to $u'(c_{i2}) > 0$ we know that the budget constraint binds and all initial wealth is invested due to the existence of a risk-free asset, such that $K_N = \mu_{a_0} - K_I^*$.

Resubstitution into the definition of y gives us then

$$Y = A_f \left(\mu_{a_0} - (\overline{\omega}/\overline{\vartheta}) \left(\rho_{\vartheta, a_0^\gamma} \sigma_\vartheta \sigma_{a_0^\gamma} + \mu_\vartheta \mu_{a_0^\gamma} \right) \right) + A_r (\overline{\omega}/\overline{\vartheta}) \left(\rho_{\vartheta, a_0^\gamma} \sigma_\vartheta \sigma_{a_0^\gamma} + \mu_\vartheta \mu_{a_0^\gamma} \right) \ .$$

To show the existence of a representative household, we consider the production of a fictitious representative household with innate risk aversion type ϑ_{RH} and initial wealth holdings $a_{RH,0}$. Her terminal production is given by

$$y_{RH} = A_f \left(a_{RH,0} - (\overline{\omega}/\overline{\vartheta}) \vartheta_{RH} a_{RH,0}^\gamma \right) + A_r (\overline{\omega}/\overline{\vartheta}) \vartheta_{RH} a_{RH,0}^\gamma \ .$$

We need to show fthat $y = y_{RH}$ holds. Comparing the previous two equations, we easily see that the only candidate for the existence of a representative household is characterized by

$$\vartheta_{RH} a_{RH,0}^\gamma = \left(\rho_{\vartheta, a_0^\gamma} \sigma_\vartheta \sigma_{a_0^\gamma} + \mu_\vartheta \mu_{a_0^\gamma} \right) \ , \tag{ES1}$$

$$a_{RH,0} = \mu_{a_0} \ . \tag{ES2}$$

The previous two equations describe a system of nonlinear equations in two unknowns. Its solution is found by substitution and exists if $\vartheta_{RH} \in \Theta$ and $h(\vartheta_{RH}, a_{RH,0}) > 0$. As a result, the RH is characterized by

$$\vartheta_{RH} = \mu_{a_0}^{-\gamma} \left(\rho_{\vartheta, a_0^\gamma} \sigma_\vartheta \sigma_{a_0^\gamma} + \mu_\vartheta \mu_{a_0^\gamma} \right) \ , \quad \text{and} \quad a_{RH,0} = \mu_{a_0} \ .$$

In order to establish part (a) of Lemma 7, it is straightforward to simplify the previous system of equations ES1-ES2 by recognizing that $\vartheta_i = \vartheta = \vartheta_{RH} = \mu_\vartheta$ such that we obtain

$$a_{RH,0} = (\mu_{a_0^\gamma})^{\frac{1}{\gamma}} \Leftrightarrow \mu_{a_0}^\gamma = \mu_{a_0^\gamma} \Leftrightarrow \left(\frac{\eta}{\eta - 1} \right)^\gamma \underline{a}^\gamma = \frac{\eta}{\eta - \gamma} \underline{a}^\gamma \ .$$

Hence, to show the unique existence of a representative household in this case, we need to verify that for each $\eta > \gamma$ the intersection $\left(\frac{\eta}{\eta - 1} \right)^\gamma = \frac{\eta}{\eta - \gamma}$ exists. For $\gamma \in \{0, 1\}$ this is obviously the case. Hence, we are left to show that there does not exist any other $\gamma > 0$ for which this is valid. To do so, let us first rewrite $\left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}} = \frac{\eta}{\eta-1}$. Taking the derivative of the LHS w.r.t. γ gives us

$$\frac{\partial LHS}{\partial\gamma} = \left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}}\left(-\frac{\ln\left(\frac{\eta}{\eta-\gamma}\right)}{\gamma^2} + \frac{1}{\gamma}\frac{1}{\frac{\eta}{\eta-\gamma}}\frac{\eta}{(\eta-\gamma)^2}\right) = \frac{1}{\gamma^2}\left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{\gamma}{\eta-\gamma} - \ln\left(\frac{\eta}{\eta-\gamma}\right)\right) .$$

The derivative is (strictly) positive if $\frac{\gamma}{\eta-\gamma} > \ln\left(\frac{\eta}{\eta-\gamma}\right) = \ln\left(1+\frac{\gamma}{\eta-\gamma}\right)$ holds $\forall\ \gamma > 0$. Redefining $x \equiv \frac{\gamma}{\eta-\gamma} \in (0,\infty)$, we can equivalently rewrite the previous inequality $x > \ln(1+x)$. To proceed the execution, let us state a property of the natural logarithm.

Result 1. *$\forall$ x>0, the following inequalities hold:* $\frac{x}{1+x} < \ln(1+x) < x$.

The proof of the postulated result is straightforward: Recognizing that the natural logarithm is a strictly concave function $\forall\ x > 0$ and using the auxiliary variable $\tilde{x} \neq 1$, an application of the mean value theorem provides us with $\ln(\tilde{x}) - \ln(1) < \tilde{x} - 1$. Substituting in $\tilde{x} = 1 + x$ yields the upper bound, substituting in $\tilde{x} = \frac{1}{1+x}$ gives us respectively the lower bound.
As a result, we obtain that $\frac{\gamma}{\eta-\gamma} > \ln(\frac{\eta}{\eta-\gamma})$ such that the left hand side of $\left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}} = \frac{\eta}{\eta-1}$ is strictly increasing in γ $\forall\gamma < \eta$. It remains to show that the left hand side limit for $\gamma \to 0$ is lower than $\frac{\eta}{\eta-1}$. An application of L'Hospital's rule gives

$$\lim_{\gamma\to 0}\left(\frac{\eta}{\eta-\gamma}\right)^{\frac{1}{\gamma}} = \lim_{\gamma\to 0} e^{\frac{1}{\gamma}\ln\left(\frac{\eta}{\eta-\gamma}\right)} = e^{\lim_{\gamma\to 0}\frac{1}{\gamma}\ln\left(\frac{\eta}{\eta-\gamma}\right)} = e^{\frac{1}{\eta}} ,$$

where the second equality follows from the continuity of the exponential function. It remains to show that $e^{\frac{1}{\eta}} < \frac{\eta}{\eta-1}$. Taking the natural logarithm on both sides this is equivalent to showing $\frac{1}{\eta} < \ln\left(\frac{\eta}{\eta-1}\right) = \ln\left(1+\frac{1}{\eta-1}\right)$. By making use of the lower bound of Result 1, one can verify that the inequality holds indeed. This concludes the proof that an intersection for an arbitrary η occurs if $\gamma \in \{0,1\}$. In both cases initial wealth is given by $a_{RH,0} = \mu_{a_0}$ and absolute risk aversion by $\alpha_{RH} = \frac{\alpha}{\mu_\vartheta \mu_{a_0}^{\gamma}}$. This shows that $\gamma \in \{0,1\}$ is necessary for the existence of a representative household. Finally, sufficiency follows from substituting $\gamma \in \{0,1\}$ into the system ES1-ES2 and showing that a RH indeed exists in this case. $\square$