Heterogenous Agent New Keynesian Model

Numerical solutions*

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1 Model

In this note I describe the numerical solution of a Heterogenous Agent New Keynesian (HANK) model in the spirit of Kaplan et al. (2016). This note is segmented in two steps:

1. First, we compute the steady-state of the economy.
2. Second, we compute the transition dynamics after a MIT shock. It requires to guess the path of equilibrium prices and to iterate over the policy functions and the distribution of agents.

2 Steady State economy without capital and one asset

2.1 Household

We assume a simple household problem with two state idiosyncratic income shock, $z_j \in \{z_1, z_2\}$. Agent i can save only in one type of asset a_i and works l_t hours. We adopt a continuous-time approach in line with Achdou et al. (2017). The household problem can be summarized by:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t, l_t) dt \tag{1}$$

$$u(c_{i,j,t}, l_{i,j,t}) = \frac{\left[c_{i,j,t} - \Psi z_j \frac{l_{i,j,t}^{1+\frac{1}{\Psi}}}{1+\frac{1}{\Psi}}\right]^{1-\gamma}}{1-\gamma}$$

*Corresponding author: alexandre.gaillard@tse-fr.eu. This draft is based on Achdou et al. (2017) and Benjamin Moll's website. All the mathematics behind this note come from their work.

The budget constraint of the individual with asset level a_i is given by:

$$\dot{a}_i = wz_j l_{i,j} + ra_i - c_{i,j} \tag{2}$$

where w and r are determined by equilibrium conditions. To solve the model, we refer to the method developed in Achdou et al. (2017). That is, our discretized continuous time formulation for numerical solution of this simple household problem is given by

$$(continuous) \qquad \rho v(a_i, z_j) = u(c_{i,j}) + \partial_a v(a_i, z_j)\dot{a}_i + \lambda_j(v(a_i, z_{j'}) - v(a_i, z_j)) + \partial_t v(a_i, z_j) \tag{3}$$

$$(discrete) \qquad \rho v_{i,j}^{n+1} + \frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} = u_{i,j}^n + \frac{v_{i+1,j}^{n+1} - v_{i,j}^{n+1}}{\Delta_a}\dot{a}_{i,j}^+ + \frac{v_{i,j}^{n+1} - v_{i-1,j}^{n+1}}{\Delta_a}\dot{a}_{i,j}^- + \lambda_j(v_{i,j'}^{n+1} - v_{i,j}^{n+1}) \tag{4}$$

where I replace $v(a_i, z_j)$ by $v_{i,j}$ and index n means iteration n. $\dot{a}_{i,j}^-$ and $\dot{a}_{i,j}^+$ can be replaced by:

$$\begin{aligned}
\dot{a}_{i,j}^- &= min\{0, wz_j l_{i,j} + ra_i - c_{i,j}^B\} \\
\dot{a}_{i,j}^+ &= max\{0, wz_j l_{i,j} + ra_i - c_{i,j}^F\} \\
c_{i,j}^B &= u^- 1\Big(\frac{v_{i,j}^{n+1} - v_{i-1,j}^{n+1}}{\Delta_a}\Big) + desutil_l \\
c_{i,j}^F &= u^- 1\Big(\frac{v_{i+1,j}^{n+1} - v_{i,j}^{n+1}}{\Delta_a}\Big) + desutil_l
\end{aligned}$$

This allows us to rewrite:

$$(discrete) \qquad \rho v_{i,j}^{n+1} + \frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} = u_{i,j}^n + v_{i,j}^{n+1} y_{i,j} + v_{i-1,j}^{n+1}\zeta_{i,j} + v_{i+1,j}^{n+1} x_{i,j} + \lambda_j(v_{i,j'}^{n+1} - v_{i,j}^{n+1})$$

$$(matrix\ form) \qquad \rho v^{n+1} + (v^{n+1} - v^n)\frac{1}{\Delta} = u^n + A^n v^{n+1} \qquad A^n = B^n + \Lambda$$

where we have

$$y_{i,j} = \dot{a}_{i,j}^- - \dot{a}_{i,j}^+ \qquad x_{i,j} = \dot{a}_{i,j}^+ \qquad \zeta_{i,j} = \dot{a}_{i,j}^-$$

Therefore, matrices B^n and C are given by:

$$B^n = \begin{bmatrix}
y_{1,1} & x_{1,1} & 0 & \cdots & & & & \\
\zeta_{2,1} & y_{2,1} & x_{2,1} & 0 & & & & \\
0 & \ddots & \ddots & \ddots & & & & \\
\vdots & 0 & \zeta_{I,1} & y_{I,1} & & & & \\
 & & & & \ddots & & & \\
 & & & & y_{1,J} & x_{1,J} & 0 & \dots \\
 & & & & \zeta_{2,J} & y_{2,J} & x_{2,J} & 0 \\
 & & & & 0 & \ddots & \ddots & \ddots \\
 & & & & \vdots & \ddots & \zeta_{I,J} & y_{I,J}
\end{bmatrix} \qquad \Lambda = \begin{bmatrix}
0 & \dots & 0 & \lambda_1 & 0 & \dots & \dots & \\
0 & \dots & \dots & 0 & \lambda_1 & 0 & \dots & \\
 & & & & \ddots & & & \\
\lambda_2 & 0 & \dots & \dots & & & \dots & 0 \\
0 & \lambda_2 & 0 & \dots & & & \dots & 0
\end{bmatrix}$$

2.2 Firms

Based on https://www3.nd.edu/~esims1/new_keynesian_model.pdf and http://www.princeton.edu/~moll/ECO521_2016/Lecture2_ECO521.pdf.

Final good producers There is a representative final goods producer which aggregates a continuum of intermediate inputs indexed by $k \in [0,1]$, such that:

$$Y = \left(\int_0^1 y_{k,t}^{\frac{\epsilon-1}{\epsilon}} dk \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 0$ is the elasticity of substitution across goods. Cost minimization implies that demand for intermediate good j is

$$y_{k,t}(p_{k,t}) = \left(\frac{p_{k,t}}{P_t}\right)^{-\epsilon} Y_t \qquad \text{where} \qquad P_t = \left(\int_0^1 p_{k,t}^{1-\epsilon} dk \right)^{\frac{1}{1-\epsilon}}$$

Intermediate goods producers Each intermediate good is produced by a monopolistically competitive producer which use only labor $n_{k,t}$ as input, such that:

$$y_{k,t} = Z_t n_{k,t}$$

where Z_t is an aggregate TFP shock. From cost-minization problem, we have

$$w_t = \frac{\epsilon - 1}{\epsilon} Z_t n_t \tag{5}$$

Such that the profit is equal to:

$$\Pi_t = Z_t n_t \left(1 - \frac{\epsilon - 1}{\epsilon}\right) \tag{6}$$

Aggregation

$$Y = \int_0^1 y_{k,t} = \int_0^1 Z_t n_{k,t} = Z_t L_t^d \tag{7}$$

where, according to market clearing condition, we must have:

$$L_t^d = L_t^s = \bar{z} w^{\Psi}; \tag{8}$$

For firm, we assume that Y is equal to total demand, such that

$$Y = C + \Pi + rB_d \tag{9}$$

2.3 Monetary policy

We assume that the monetary policy adopt a simple taylor rule, such that:

$$i_t = \bar{r}_{ss} + \phi \pi_t$$

Inflation in our economy behave according to the following law of motion, using a Rotemberg [1982] and a quadratic price adjustement cost, we have for price setting:

$$\rho\pi = \frac{\epsilon - 1}{\theta}\Big(\frac{\epsilon}{\epsilon - 1}\frac{w_t}{Z_t} - 1\Big) + \dot{\pi}_t$$
$$w_t = \frac{\theta}{\epsilon}(\rho\pi_t - \dot{\pi}_t) + 1$$

2.4 Equilibrium

At the equilibrium, bond demand should be equal to supply.

$$B_d = B_s \tag{10}$$

We assume that $B_d = 0.1$. For B_s, we have

$$B_s = \int_0^1 ag(a) \tag{11}$$

Equilibrium implies that $B_d = B_s$, such that interest rate r adjust. Condition (7) holds by Walras law.

3 Transition Dynamics

In the economy, everything is determined, except inflation rate. Therefore, when studying the path between two steady states, we are looking for the path of π_t for which all markets clear. In order to do so, we have to know the initial and the final condition. These two conditions correspond to two steady-states.

3.1 M.I.T shocks

We compute the economy after a one period unanticipated shock that occurs in period $t = 1$. The initial condition ($t = 0$) and final condition ($t = T$ with T large enough) are described by the same steady-state economy. At steady-state, $\pi_t = 0$ and $r = \bar{r}_{ss}$. Our algorithm for transition path is in line with Achdou et al. (2017). The system to be solved is:

$$(Bond\ market) \qquad B_d(t) = \int_{\bar{a}}^{\infty} ag_1(a,t)da + \int_{\bar{a}}^{\infty} ag_2(a,t)da$$
$$(HJB) \qquad \rho v_j(a,t) = max_c u(c) + \partial_a v_j(a,t)\dot{a}(t) + \lambda_j\Big[v_{-j}(a,t) - v_j(a,t)\Big] + \partial_t v_j(a,t)$$
$$(Fokker - Plank) \qquad \partial_t g_j(a,t) = -\partial_a\Big[s_j(a,t)g_j(a,t)\Big]\lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t)$$

The algorithm to solve the transition dynamics is the following

1. for iteration l, guess path of π_t^l given that $\pi_0^l = 0$.

2. given π_t^l, compute the associated prices r_t, w_t and solve the HJB equation. To solve HJB, proceed backward. Start at time $T-1$ where v_T are given by v^{ss}. Then, compute v^{T-1}. Do the same for v^{T-1} given v^{T-1}. You get saving decision using the difference between the current and future value functions. Given v^{t+1}, the system to be solved can be summarized by:

$$\rho v^t = u^{t+1} + A^{t+1} v^t + \frac{1}{\Delta_t}(v^{t+1} - v^t)$$

   where A^{t+1} is the transition matrix computed above.

3. Given saving behavior, solve the Fokker-Plank equation using initial condition $g_j(a, 0) = g_j(a, ss)$. Go forward in time to compute $g_j(a, t)$. To do so, we can use directly the transition matrix A^{t+1}. Such that given the Fokker-plank equation we have:

$$\frac{g^{t+1} - g^t}{\Delta_t} = A^{t+1} g^t \qquad g^{t+1} = (\mathbf{I} - \Delta_t A^{t+1})^{-1} g^t$$

4. Given saving behavior and the distribution g, compute

$$S(t) = \int_{\bar{a}}^{\infty} a g_1(a, t) da + \int_{\bar{a}}^{\infty} a g_2(a, t) da$$

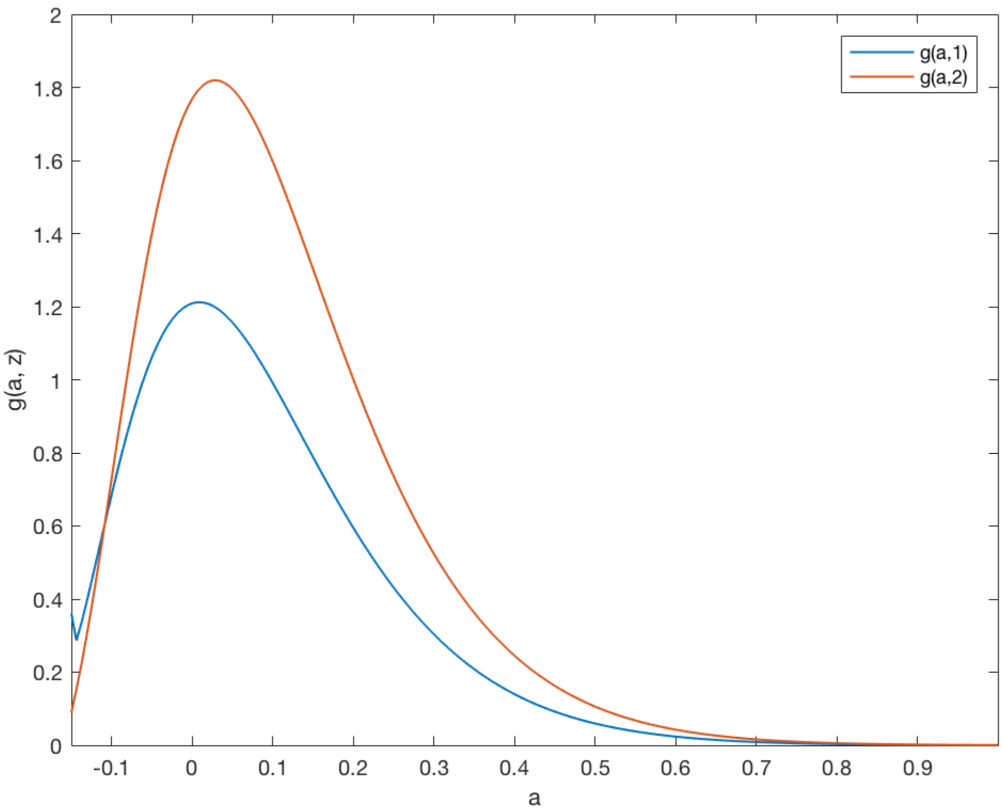
5. update prices $\pi_t^{l+1} = \pi_t^l - \xi \frac{dS(t)}{dt}$, where $\xi > 0$.

6. stop when π^{l+1} is sufficiently close to π^l.

4 Results

4.1 Steady-state distribution

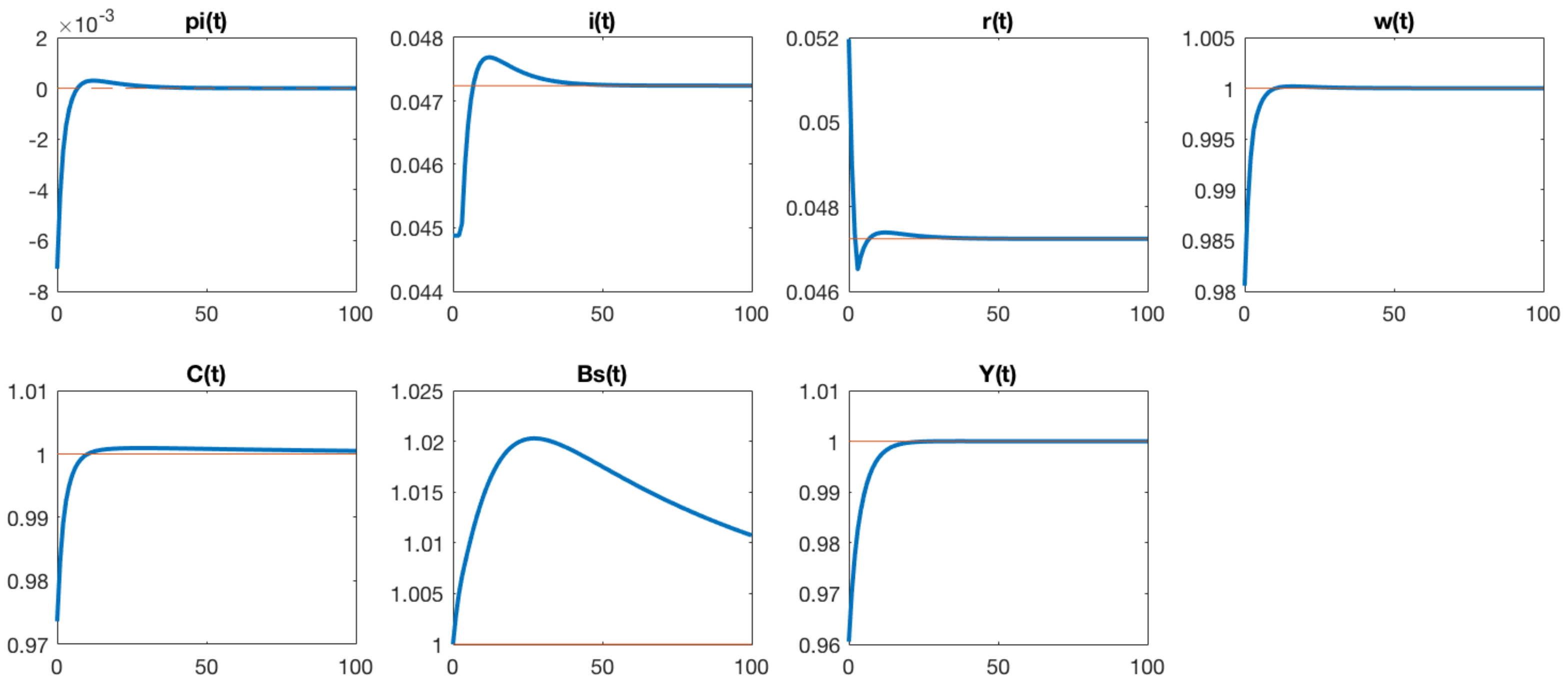
Fig. 1. Steady-state distributions



4.2 MIT shocks

We construct a sequence of TFP shock. At date $t = 1$ we generates a shock which is persistent.

Fig. 2. Steady-state distributions



5 Steady state with capital and two assets

We consider a new economy in which agents can save in a liquid (cash, deposite account...) and illiquid assets (housing, physical capital...). In line with Kaplan et al. (2016), I use non-convex cost function for deposits.

5.1 Household's problem

We four state idiosyncratic shocks for income, such that $z_k \in \{z_1, z_2, z_3, z_4\}$. Second, we assume that the agent can save in illiquid asset $a_i \in [a_{min}, a_{max}]$ and liquid asset $b_j \in [b_{min}, b_{max}]$ and works l_t hours. Moreover, he is assumed to consume in housing, which enters in his utility function. As before, we adopt a continuous-time approach in line with Achdou et al. (2017). The household's problem can be summarized by:

$$\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t, h_t, l_t) dt$$

$$u(c_t, h_t, l_t) = \frac{\left[(c_t - \Psi z_k \frac{l_t^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}})^{1-\xi} h^{\xi} \right]^{1-\gamma} - 1}{1-\gamma}$$

Asset holdings evolve according to the following law of motions:

$$\dot{b}_t = w_t z_t l_t - \tilde{T}(w_t z_t l_t) + r(b_t)b_t - d_t - \chi(d_t, a_t) - c_t - c_t^h \tag{12}$$

$$\dot{a}_t = r_t^a(1-\omega)a_t + d_t \tag{13}$$

$$h_t = \tilde{r}^h \omega a_t + c_t^h \tag{14}$$

$$b_t \geq -\underline{b} \qquad a_t \geq 0 \tag{15}$$

$$(FOC) \qquad c_t^h = \frac{\xi}{1-\xi}\left[\frac{V_b}{\omega}\right]^{\frac{-1}{\gamma}} - \tilde{r}^h \omega a_t \tag{16}$$

$$(FOC) \qquad c_t = \left[\frac{V_b}{\omega}\right]^{\frac{-1}{\gamma}} + \Psi z_k \frac{l_t^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}} \tag{17}$$

$$(FOC) \qquad l_t = \left(\frac{w_s}{\Psi}\right)^{\psi} \tag{18}$$

$$r(b_t) = \begin{cases} r_b & if \quad b_t > 0 \\ r_b + t & if \quad b_t < 0 \end{cases} \tag{19}$$

We follow the same specification as in Kaplan et al. (2016) for the cost function $\chi(.)$ such that:

$$\chi(d,a) = \chi_0|d| + \chi_1 \frac{d^2}{2\max\{a, \underline{a}\}} \max\{a, \underline{a}\}, \qquad \underline{a} > 0 \tag{20}$$

$$(FOC) \qquad d_t = \left(\frac{\max\{\frac{V_b}{V_a} - \chi_0 - 1, 0\} + \min\{\frac{V_b}{V_a} + \chi_0 - 1, 0\}}{\chi_1}\right)\max\{a, \underline{a}\} \tag{21}$$

where w and r are determined by equilibrium conditions. To solve the model, we refer to the method developped by Achdou et al. (2017).

Assumption 1: $ra < \frac{1}{\chi_1}$. If this assumption were violated, households would accumulate an infinite amount of illiquid wealth.

Using $v(a_i, b_j, z_k) \equiv v_{i,j,k}$, our discretized continuous time formulation for numerical solution of this simple household problem is

$$(continuous) \qquad \rho v_{i,j,k} = u(c,h,l) + \partial_a v_{i,j,k}\dot{a} + \partial_b v_{i,j,k}\dot{b} + \sum_{k' \neq k} \lambda_k (v_{i,j,k'} - v_{i,j,k}) + \partial_t v_{i,j,k}$$

The trick here is to divide the process for b into two drifts: $s^c = w_t z_t l_t - \tilde{T}(w_t z_t l_t) + r(b_t)b_t - c_t - c_t^h$ and $s^d = -d_t - \chi(d_t, a_t)$.

$$\begin{aligned}(discrete) \qquad \rho v_{i,j,k}^{n+1} + \frac{v_{i,j,k}^{n+1} - v_{i,j,k}^{n}}{\Delta} = u_{i,j,k}^{n} &+ \frac{v_{i+1,j,k}^{n+1} - v_{i,j,k}^{n+1}}{\Delta_a}\dot{a}_{i,j,k}^{+} \frac{v_{i,j,k}^{n+1} - v_{i-1,j,k}^{n+1}}{\Delta_a}\dot{a}_{i,j,k}^{-} \\ &+ \frac{v_{i,j+1,k}^{n+1} - v_{i,j,k}^{n+1}}{\Delta_b}(s^{c^+} + s^{d^+}) + \frac{v_{i,j,k}^{n+1} - v_{i,j-1,k}^{n+1}}{\Delta_b}(s^{c^-} + s^{d^-}) \\ &+ \sum_{k' \neq k} \lambda_{k'|k}(v_{i,j,k'}^{n+1} - v_{i,j,k}^{n+1})\end{aligned}$$

This allows us to rewrite:

$$(discrete) \qquad \rho v_{i,j,k}^{n+1} + \frac{v_{i,j,k}^{n+1} - v_{i,j,k}^{n}}{\Delta} = u_{i,j,k}^{n} + v_{i,j,k}^{n+1}(y_{i,j,k}^{b} + y_{i,j,k}^{b}) + v_{i,j-1,k}^{n+1} m_{i,j,k}^{b} + v_{i,j+1,k}^{n+1} x_{i,j,k}^{b}$$
$$+ v_{i-1,j,k}^{n+1} m_{i,j,k}^{a} + v_{i+1,j}^{n+1} x_{i,j,k}^{a} + \sum_{k' \neq k} \lambda_{k|k'}(v_{i,j,k'}^{n+1} - v_{i,j,k}^{n+1})$$

$$(matrix\ form) \qquad \rho v^{n+1} + (v^{n+1} - v^{n})\frac{1}{\Delta} = u^{n} + G^{n} v^{n+1} \qquad G^{n} = A^{n} + B^{n} + \Lambda$$

we now have to define the matrices corresponding to our problem. Let us define:

$$y_{i,j,k}^{b} = -s^{c^{+}} - s^{d^{+}} + s^{c^{-}} + s^{d^{-}} \qquad x_{i,j,k}^{b} = s^{c^{+}} + s^{d^{+}} \qquad m_{i,j,k}^{b} = -s^{c^{-}} - s^{d^{-}}$$
$$y_{i,j,k}^{a} = -\dot{a}_{i,j,k}^{+} + \dot{a}_{i,j,k}^{-} \qquad x_{i,j,k}^{a} = \dot{a}_{i,j,k}^{+} \qquad m_{i,j,k}^{a} = -\dot{a}_{i,j,k}^{-}$$

we can write matrices A^n, B^n for $z_k \equiv z_1$ as:

$$B^{n} = \begin{bmatrix} y_{1,1,1}^{b} & x_{1,1,1}^{b} & 0 & \cdots & & & & & \\ m_{2,1,1}^{b} & y_{2,1,1}^{b} & x_{2,1,1}^{b} & 0 & & & & & \\ 0 & \ddots & \ddots & \ddots & & & & & \\ \vdots & 0 & \zeta_{I,1} & y_{I,1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & y_{1,J,1}^{b} & x_{1,J,1}^{b} & 0 & \ldots \\ & & & & & m_{2,J,1}^{b} & y_{2,J,1}^{b} & x_{2,J,1}^{b} & 0 \\ & & & & & 0 & \ddots & \ddots & \ddots \\ & & & & & \vdots & \ddots & m_{I,J,1}^{b} & y_{I,J,1}^{b} \end{bmatrix}$$

$$A^{n} = \begin{bmatrix} y_{1,1,1}^{a} & x_{1,1,1}^{a} & 0 & \cdots & & & & & \\ m_{2,1,1}^{a} & y_{2,1,1}^{a} & x_{2,1,1}^{a} & 0 & & & & & \\ 0 & \ddots & \ddots & \ddots & & & & & \\ \vdots & 0 & \zeta_{I,1} & y_{I,1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & y_{1,J,1}^{a} & x_{1,J,1}^{a} & 0 & \ldots \\ & & & & & m_{2,J,1}^{a} & y_{2,J,1}^{a} & x_{2,J,1}^{a} & 0 \\ & & & & & 0 & \ddots & \ddots & \ddots \\ & & & & & \vdots & \ddots & m_{I,J,1}^{a} & y_{I,J,1}^{a} \end{bmatrix}$$

References

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Kaplan, G., Moll, B., Violante, G.L., 2016. Monetary Policy According to HANK. CEPR Discussion Papers 11068. C.E.P.R. Discussion Papers.