Optimal transport and generated Jacobian equations, applications to nonimaging optics

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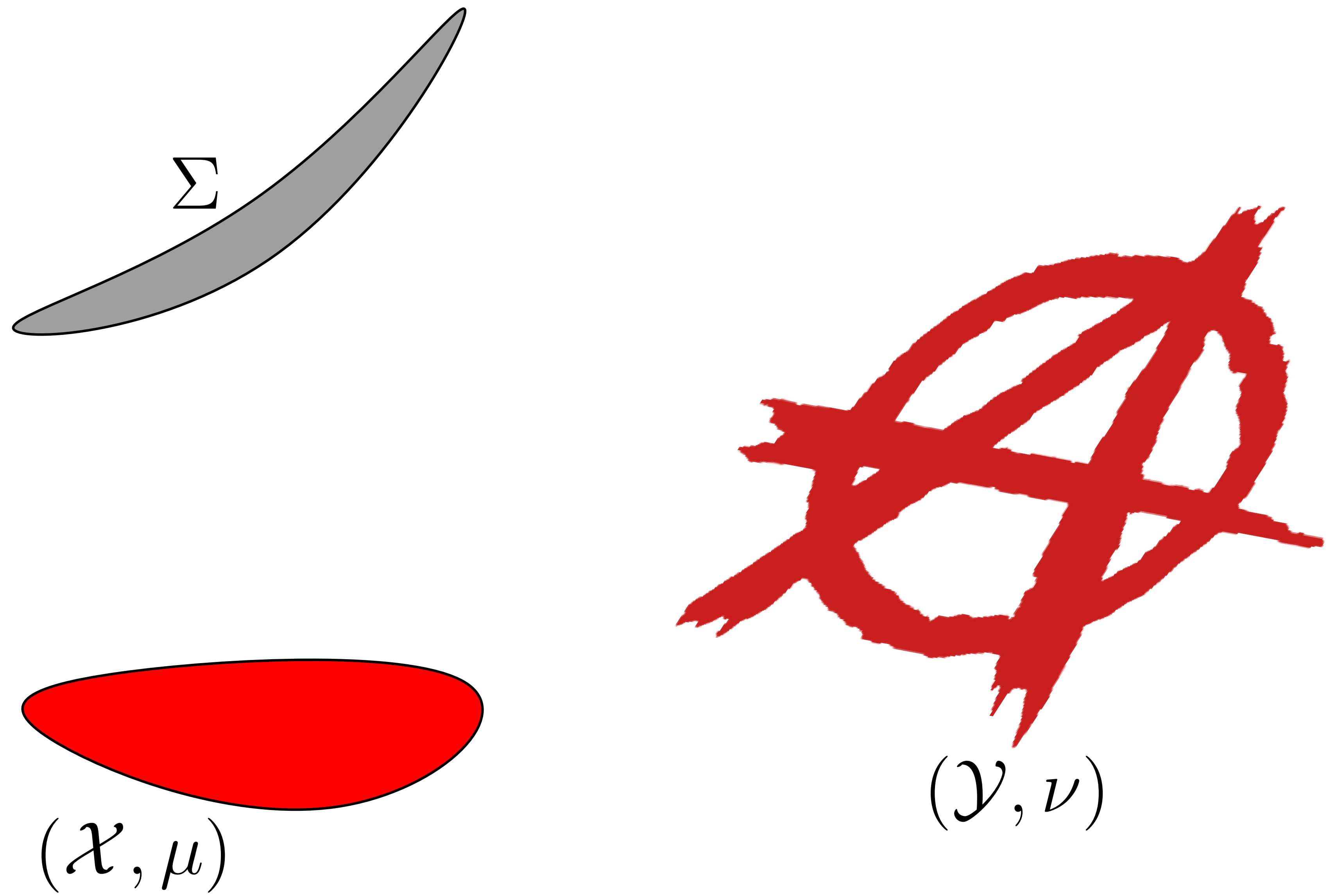
Non-imaging optics

Goal: Construct a mirror that reflects a given light source toward a prescribed target.

Non-imaging optics : Transport of measures



Input: Light source $\mathcal{X}$ with intensity $\mu \in \mathcal{P}(\mathcal{X})$.
Target distribution $\mathcal{Y}$ with intensity $\nu \in \mathcal{P}(\mathcal{Y})$.

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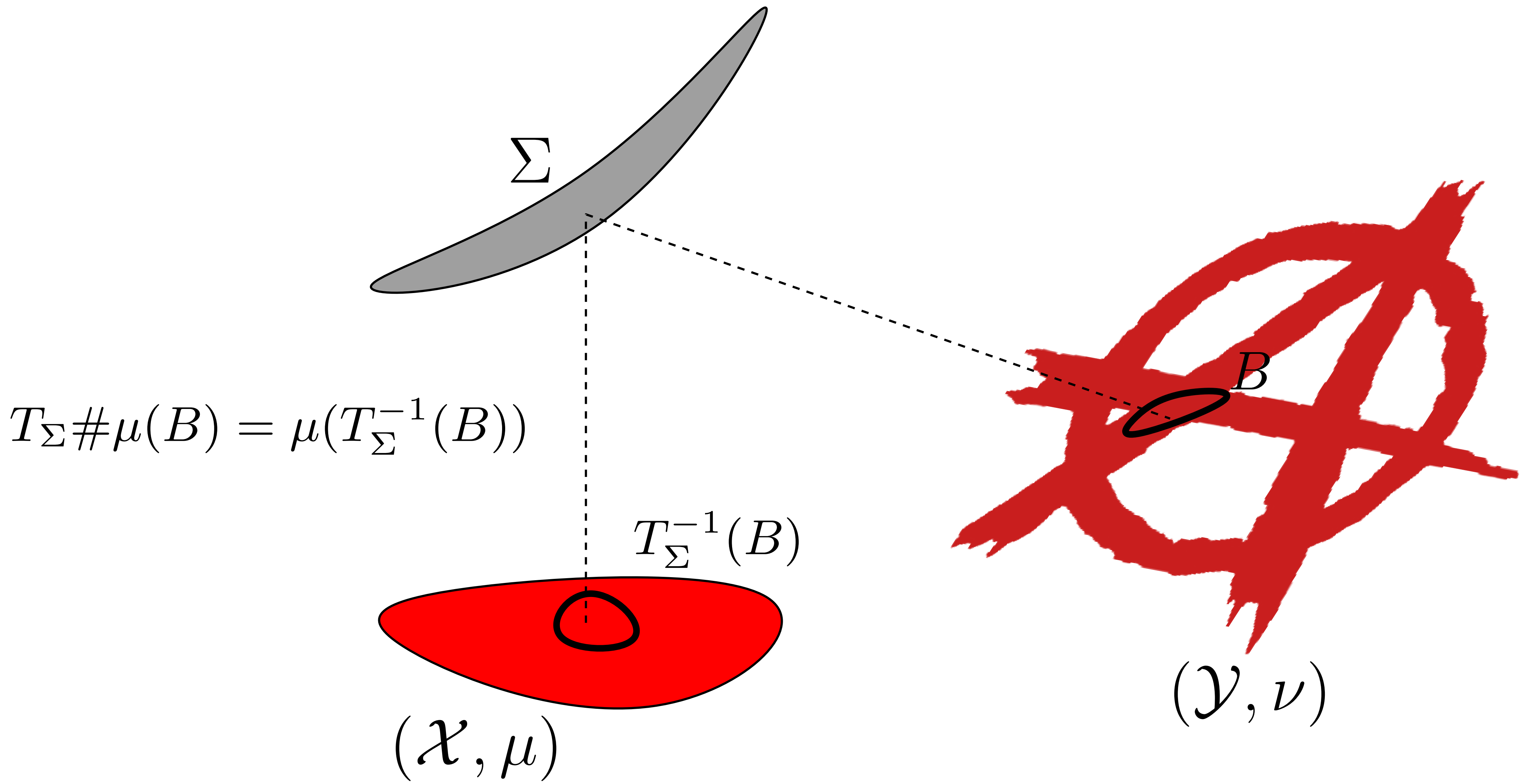


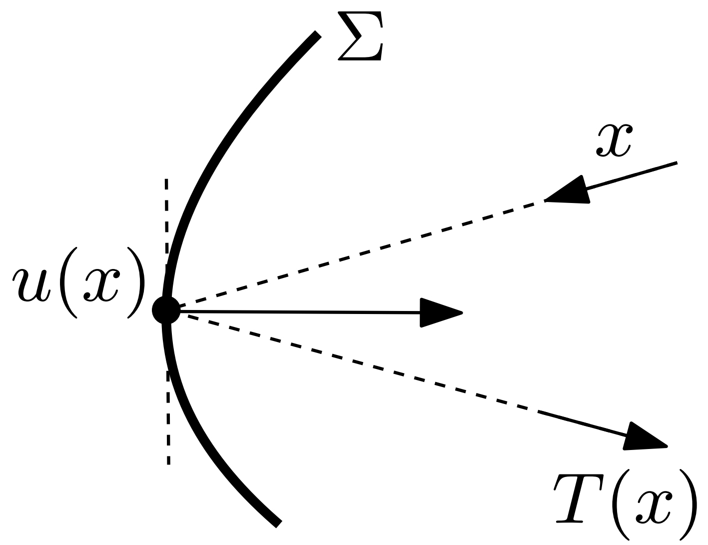
Input: Light source $\mathcal{X}$ with intensity $\mu \in \mathcal{P}(\mathcal{X})$.
Target distribution $\mathcal{Y}$ with intensity $\nu \in \mathcal{P}(\mathcal{Y})$.

Output: A surface Σ such that $T_\Sigma \# \mu = \nu$. Measure prescription equation

Optimal transport & Generated Jacobian eq.

- Assume $\mu(x) = \rho(x)\,\mathrm{d}\,x$ and $\nu(y) = \sigma(y)\,\mathrm{d}\,y$
  then $T_\# \mu = \nu$ amounts to: $\forall x \in \mathcal{X}, \sigma(T(x))\det(DT(x)) = \rho(x)$

- From Snell's law if Σ is parametrized by a function $u : \mathcal{X} \to \mathbb{R}$ then T is a function of x, $u(x)$ and $\nabla u(x)$.



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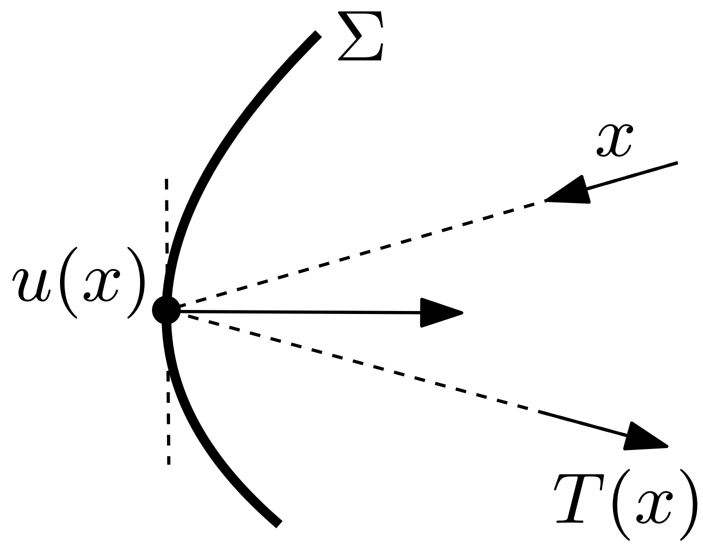


A Monge-Ampère type equation:

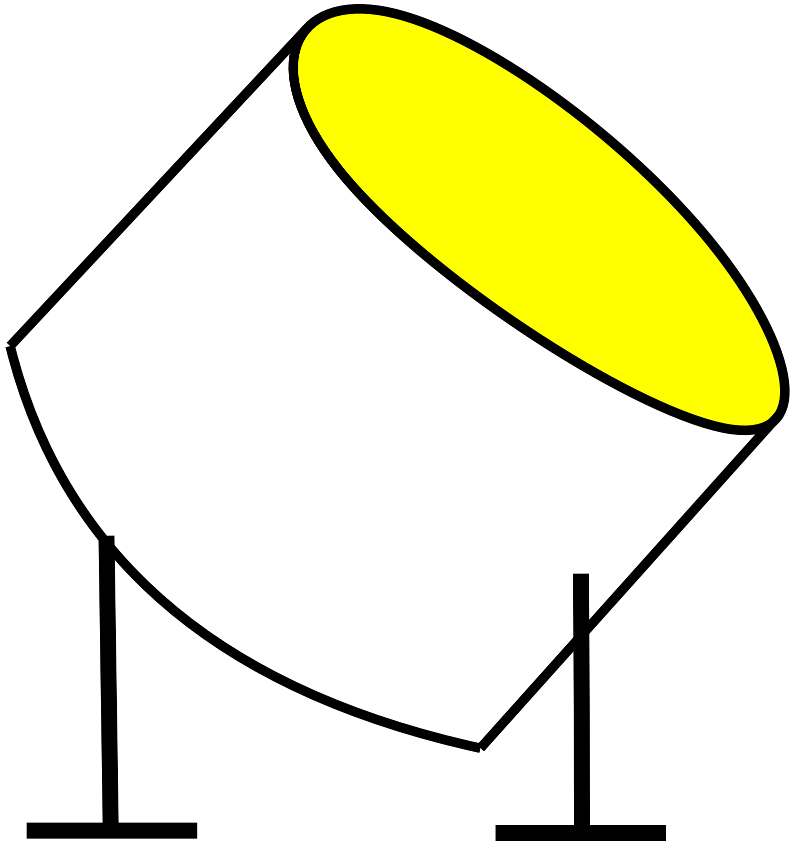
$$\forall x \in \mathcal{X}, \det(DT(x)) = \frac{\rho(x)}{\sigma(T(x))} \text{ with } T(x) = f(x, u(x), \nabla u(x)).$$

- In some particular cases this equation is OT, otherwise it is a GJE.

[Trudinger '14]

$\longrightarrow$ We want to solve this equation numerically

Discretization for numerical purposes



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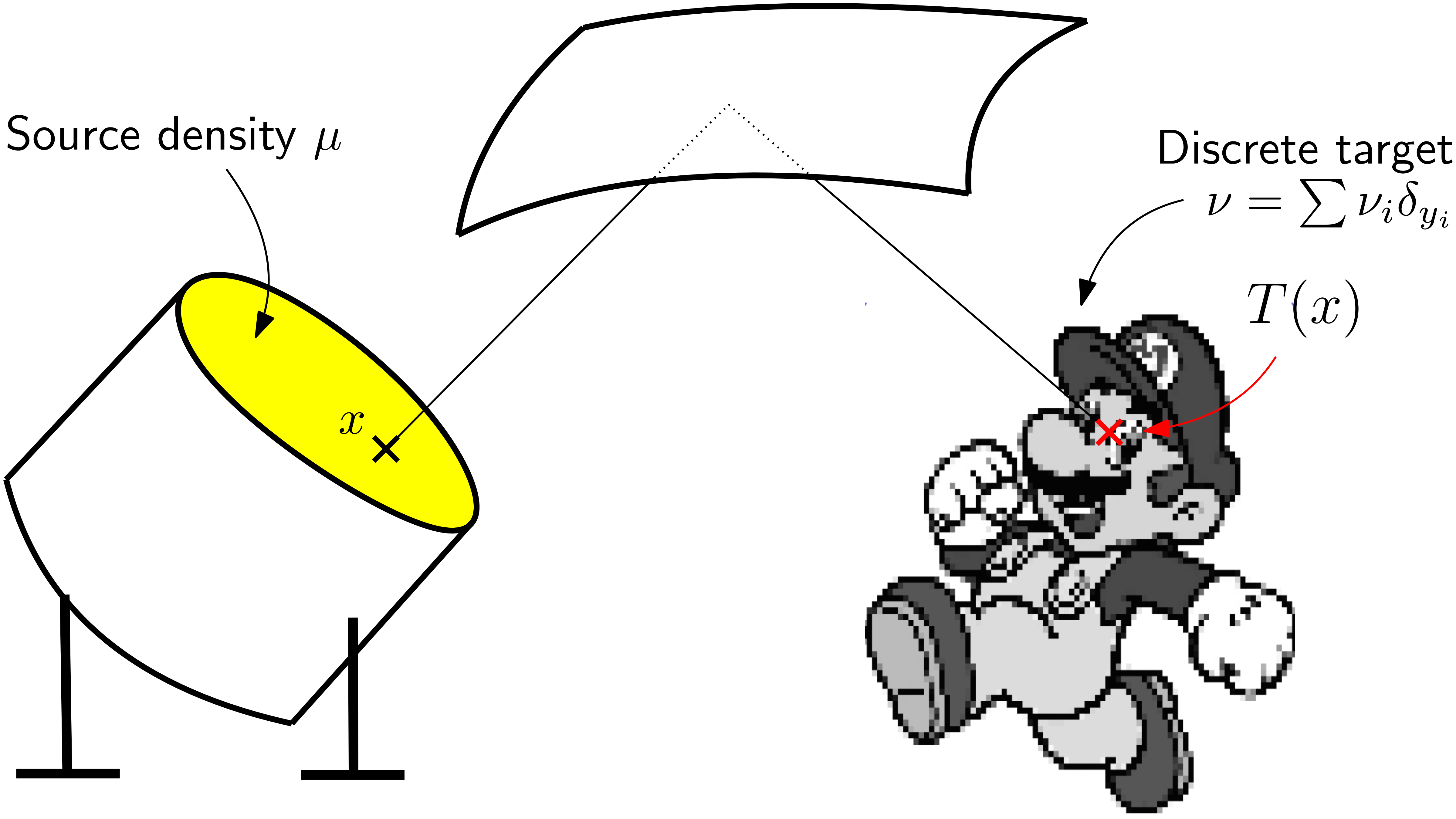

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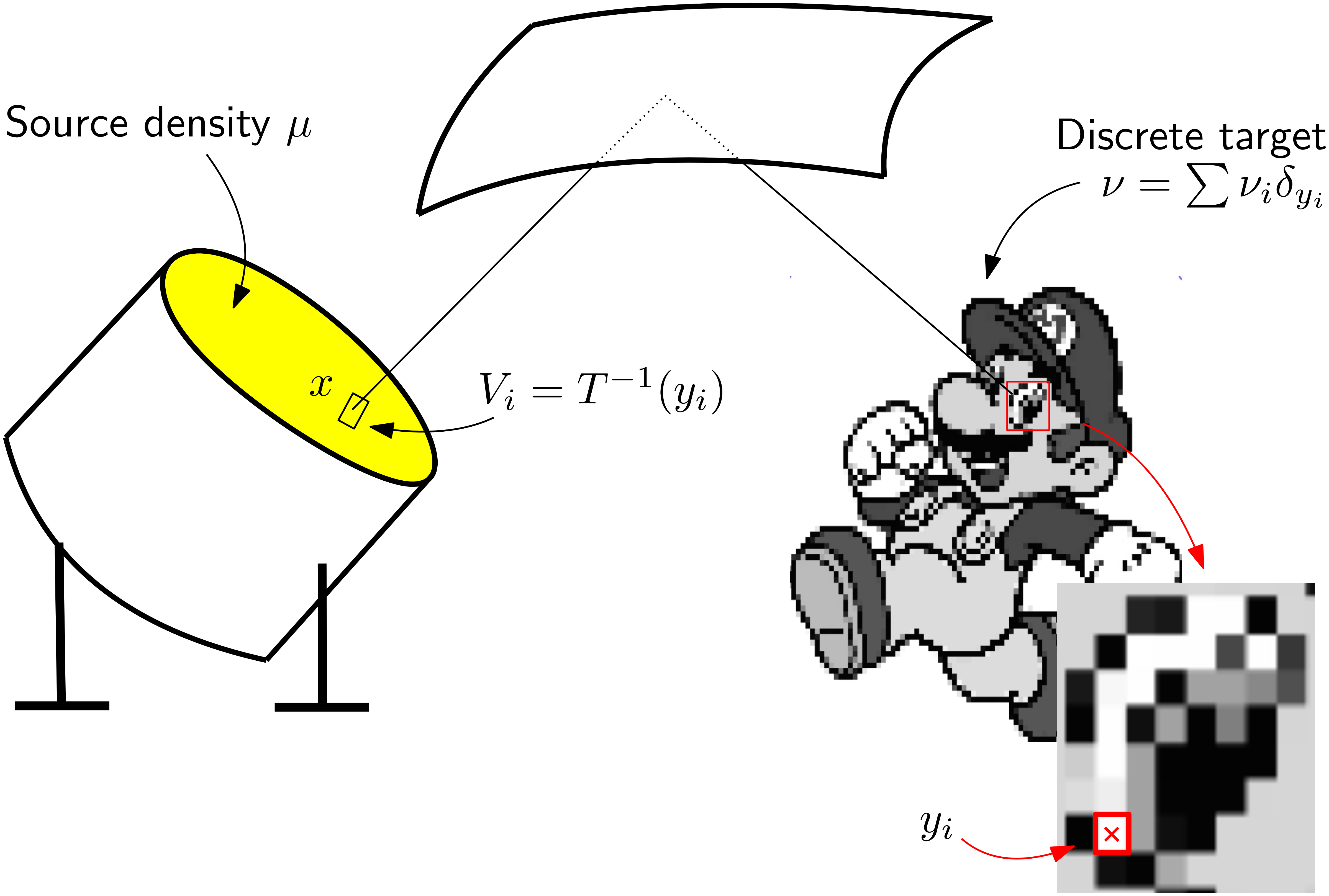

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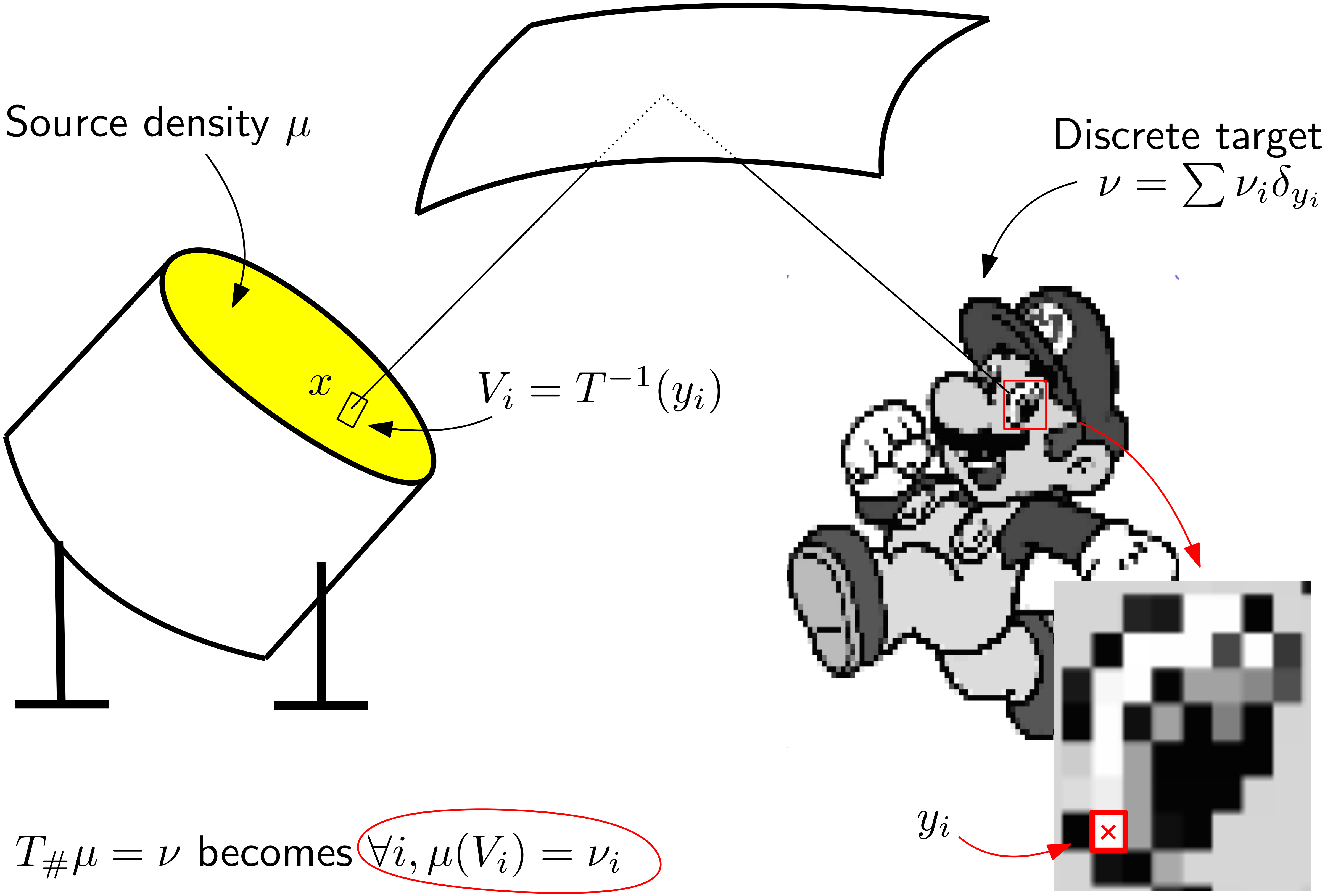

$T_{\#}\mu = \nu$ becomes $\forall i, \mu(V_i) = \nu_i$

Prescribe the mass of each cell

Outline

- Stability in optimal transport
  1) Strongly c-concave functions
  2) Stability under strong c-concavity
  3) Application to the Far-field point reflector

- Numerical resolution of Generated Jacobian equations
  1) Damped Newton algorithm for GJE
  2) Application to the Near Field parallel reflector

Stability in optimal transport

Problem: Numerical optimal transport involves discretization of measure(s).

Examples: Discrete: *[Oliker-Prussner '89]*, *[Galichon-Salanie '09]* *[Cuturi '13]*.
Semi-discrete: *[Kitagawa-Mérigot-Thibert '16]*.

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Stability with respect to the data:

Assume $T : \mu \to \nu$ and $\tilde{T} : \tilde{\mu} \to \tilde{\nu}$ optimal.

$$\boxed{\text{We want } d(T, \tilde{T}) \leq d((\mu, \nu), (\tilde{\mu}, \tilde{\nu}))^{\alpha}.}$$

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Existing results:

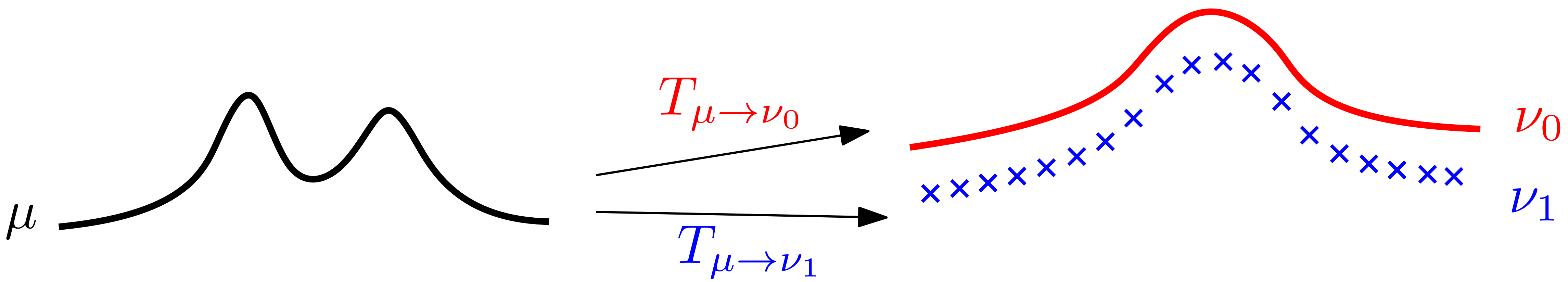
- *[Ambrosio-Gigli '09]* Local stability near Lipschitz transport map.
- *[Berman '18]* Global stability.
- *[Mérigot-Delalande-Chazal '19]* Global stability, independent of the dimension.
- *[Li-Nochetto '20]* Local stability with respect to source and target.

All these results are for the quadratic cost $c(x, y) = \|x - y\|^2$.

Previous stability results

Theorem: *[Ambrosio-Gigli '09]*
Let $\mathcal{X}$ and $\mathcal{Y}$ be compact domains of $\mathbb{R}^d$, $\mu \in \mathcal{P}(\mathcal{X})$, $\nu_0 \in \mathcal{P}(\mathcal{Y})$ be abs. cont. and any $\nu_1 \in \mathcal{P}(\mathcal{Y})$. Let $T_{\mu\to\nu_i}$ be optimal between μ and ν_i for the cost $c(x,y) = \|x-y\|^2$ and assume that $T_{\mu\to\nu_0}$ is K-Lipschitz. Then

$$\|T_{\mu\to\nu_0} - T_{\mu\to\nu_1}\|^2_{L^2(\mu)} \leq 4KM_{\mathcal{X}}\,\mathrm{W}_1(\nu_0,\nu_1).$$



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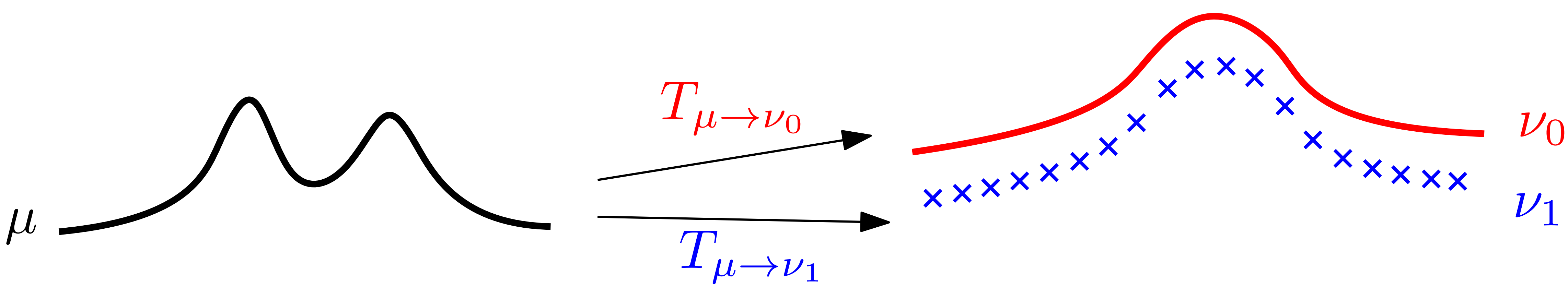


- T_0 is K-Lipschitz iff ψ_0 is $1/K$-strongly convex.
- *[Li-Nochetto '20]* have a similar result with respect to both measures.

Motivation : Generalize stability results to other cost $c : \mathcal{X}\times\mathcal{Y}\to\mathbb{R}$.

$\longrightarrow$ Introduce the notion of strongly c-concave function.

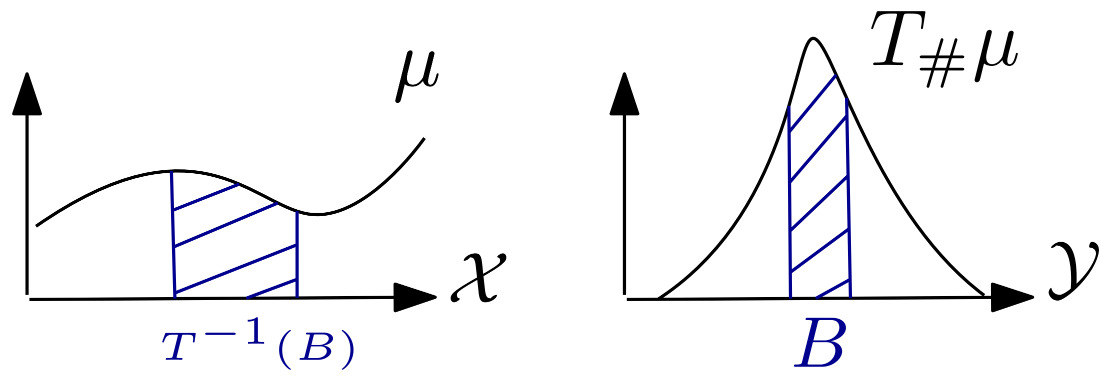
Optimal transport

Pushforward measure:

Let $T : \mathcal{X} \to \mathcal{Y}$, and $\mu \in \mathcal{P}(\mathcal{X})$,

then $T_{\#}\mu \in \mathcal{P}(\mathcal{Y})$ and

$\forall B \subset \mathcal{Y}$, $T_{\#}\mu(B) = \mu(T^{-1}(B))$.



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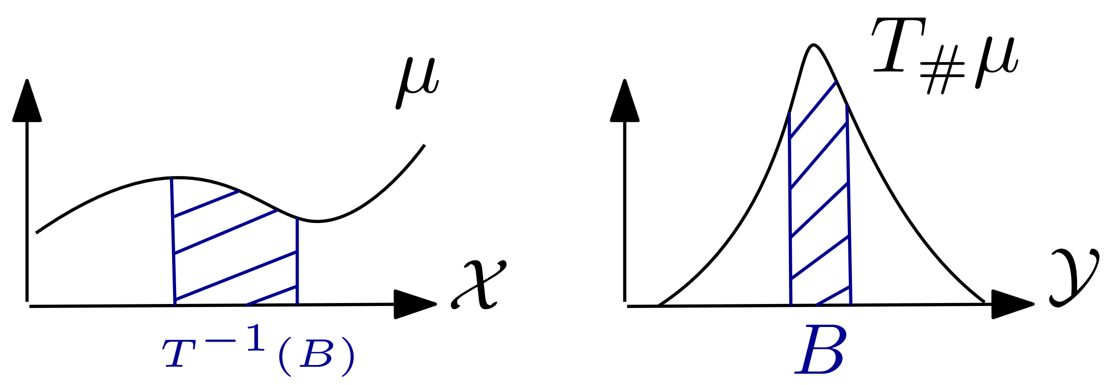

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Given $\mu \in \mathcal{P}(\mathcal{X})$, $\nu \in \mathcal{P}(\mathcal{Y})$ and $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ with $\mathcal{X}$ and $\mathcal{Y}$ submanifolds of $\mathbb{R}^d$.

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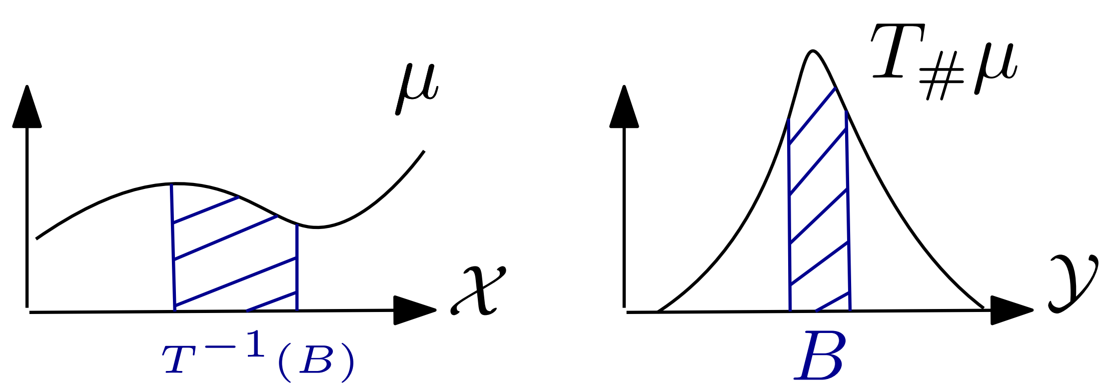

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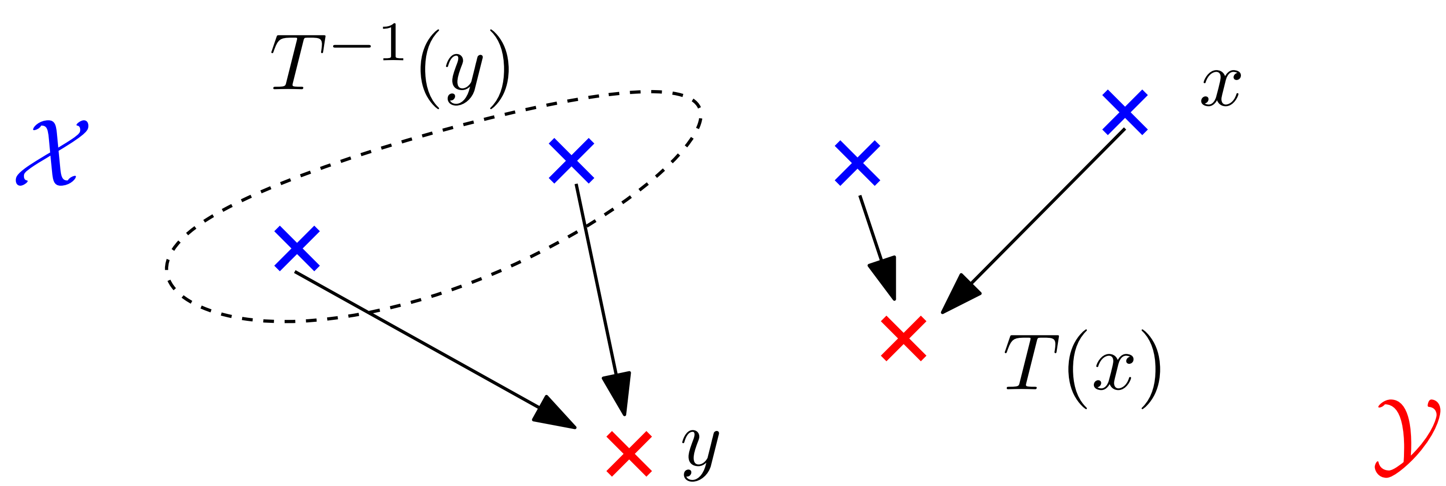

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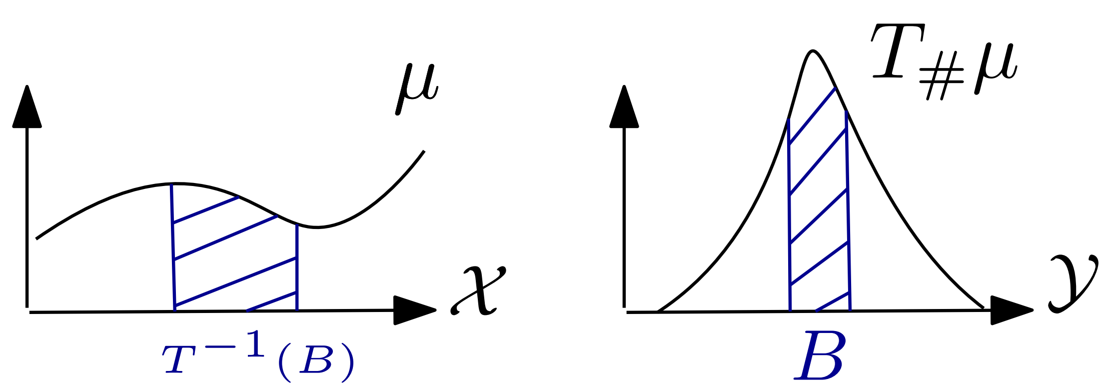

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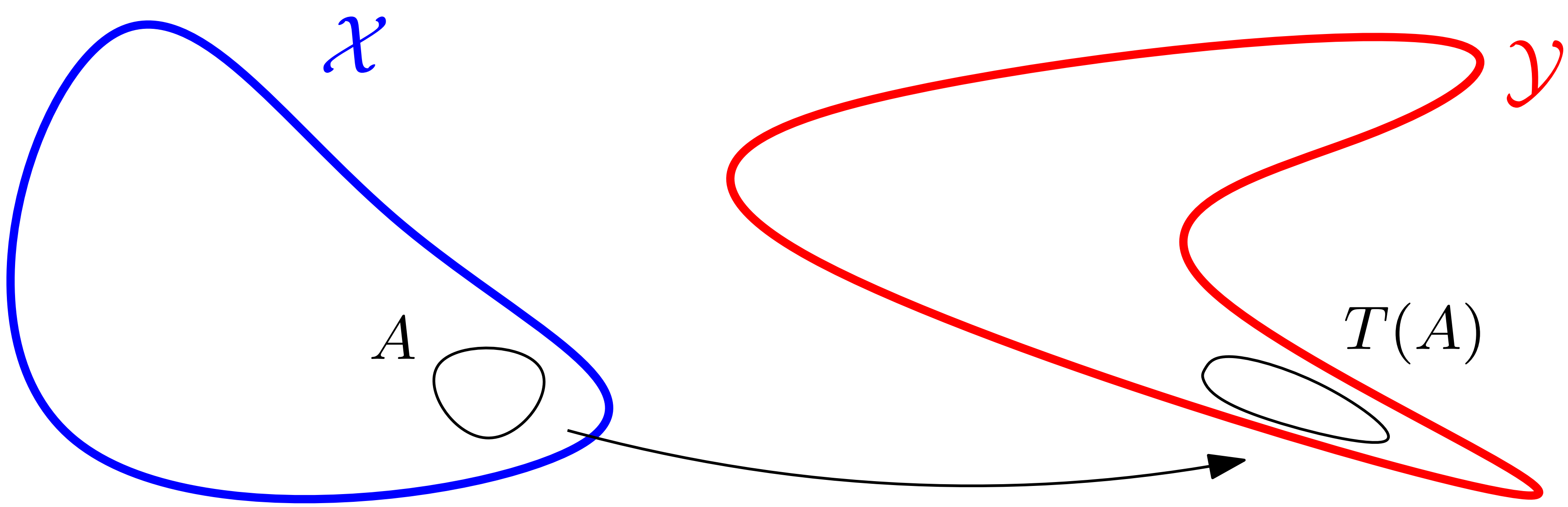

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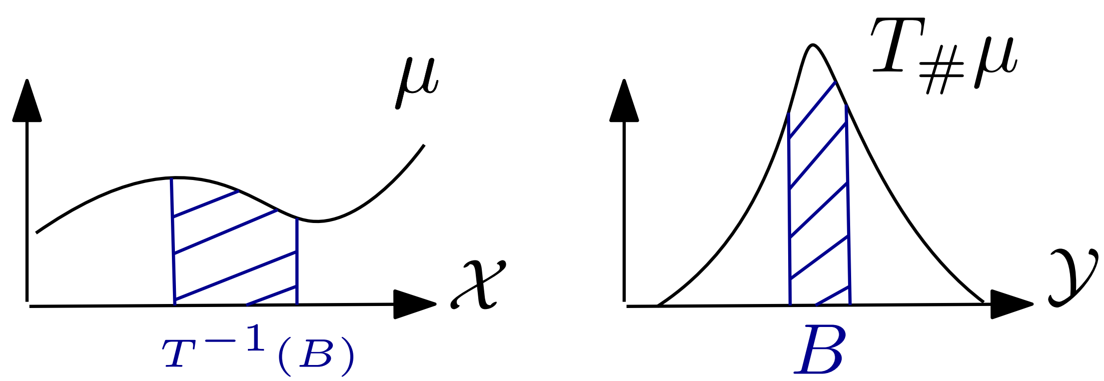

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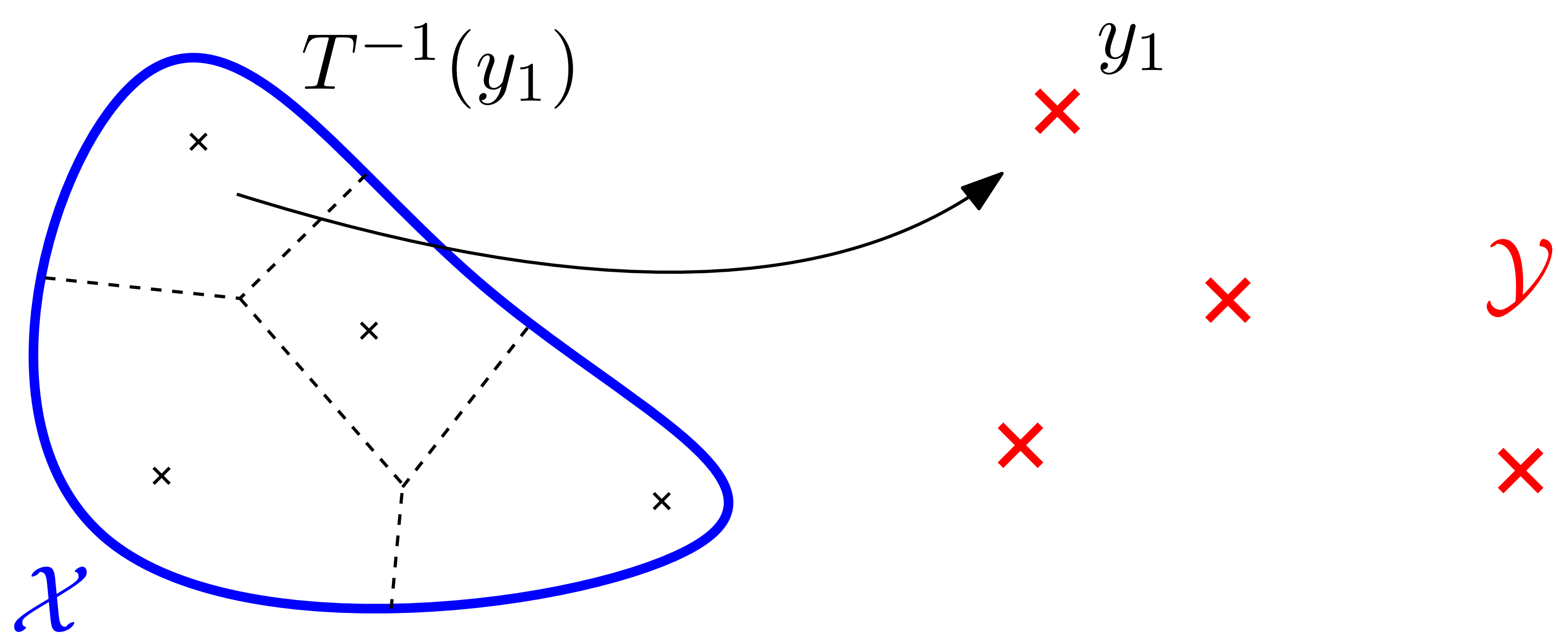

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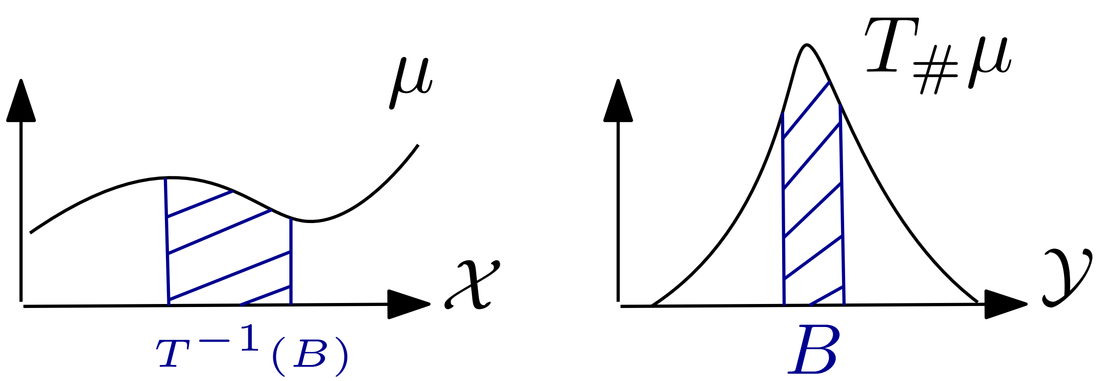

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$$\mathrm{Lag}_i(\psi) = \{x \mid c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j\}$$

Goal: Find ψ such that $\mu(\mathrm{Lag}_i(\psi)) = \nu_i$

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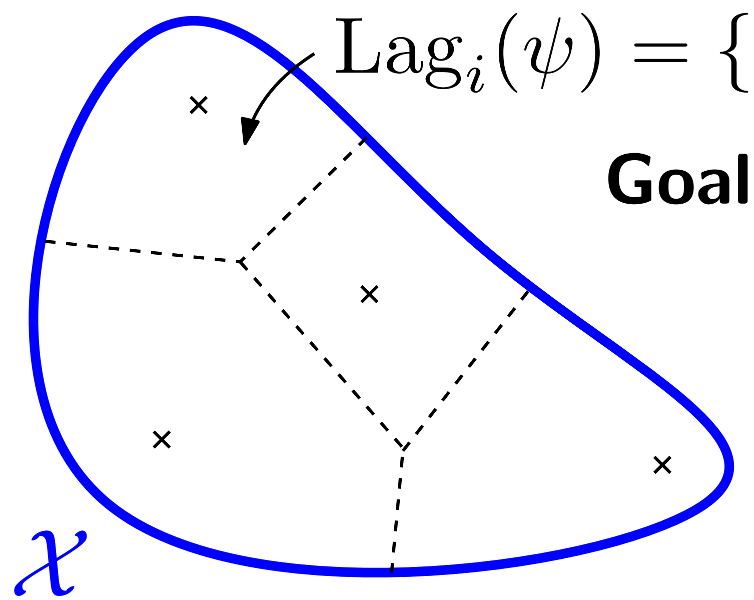

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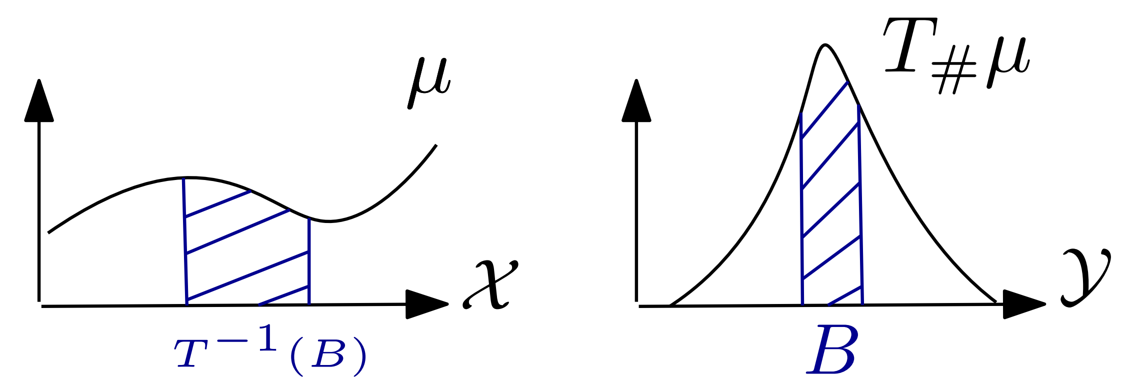

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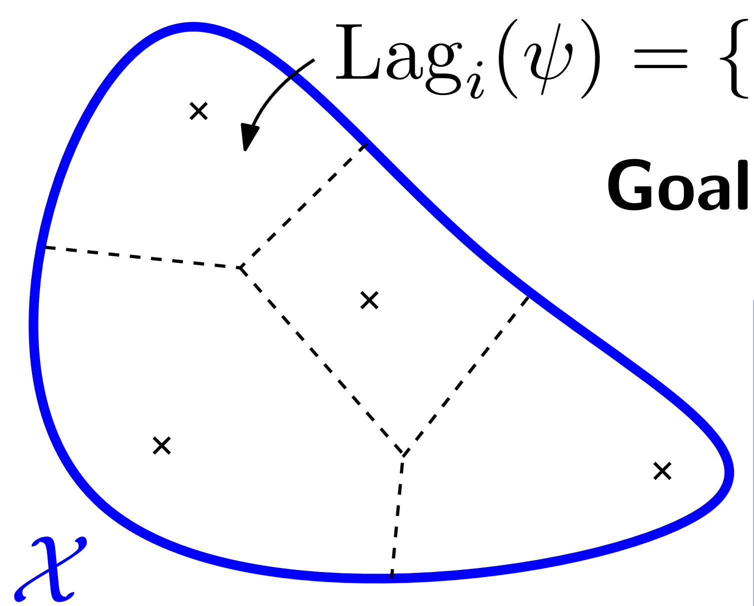

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Theorem: The map $T : \mathcal{X} \to \mathcal{Y}$ defined by

$$T^{-1}(y_i) = \mathrm{Lag}_i(\psi)$$

is an optimal transport map from μ to ν.

Kantorovich relaxation

Definition: (transport plan)
Let $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$, we say that $\gamma \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is a transport plan between μ and ν if it satisfies

$$\forall A \subset \mathcal{X}, \gamma(A \times \mathcal{Y}) = \mu(A) \text{ and } \forall B \subset \mathcal{Y}, \gamma(\mathcal{X} \times B) = \nu(B).$$

We denote by $\Gamma(\mu, \nu)$ the set of tranport plans between μ and ν.

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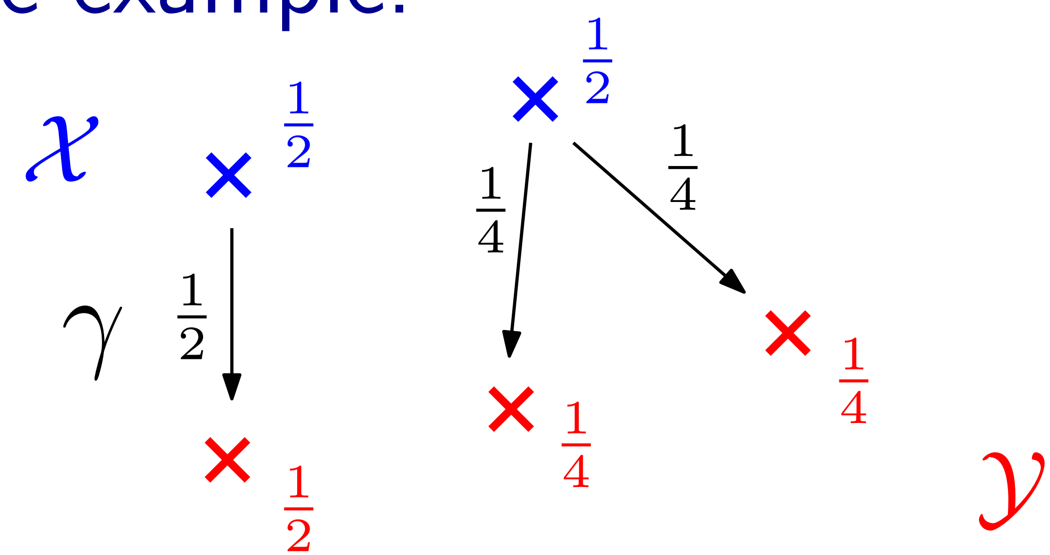
A simple example:



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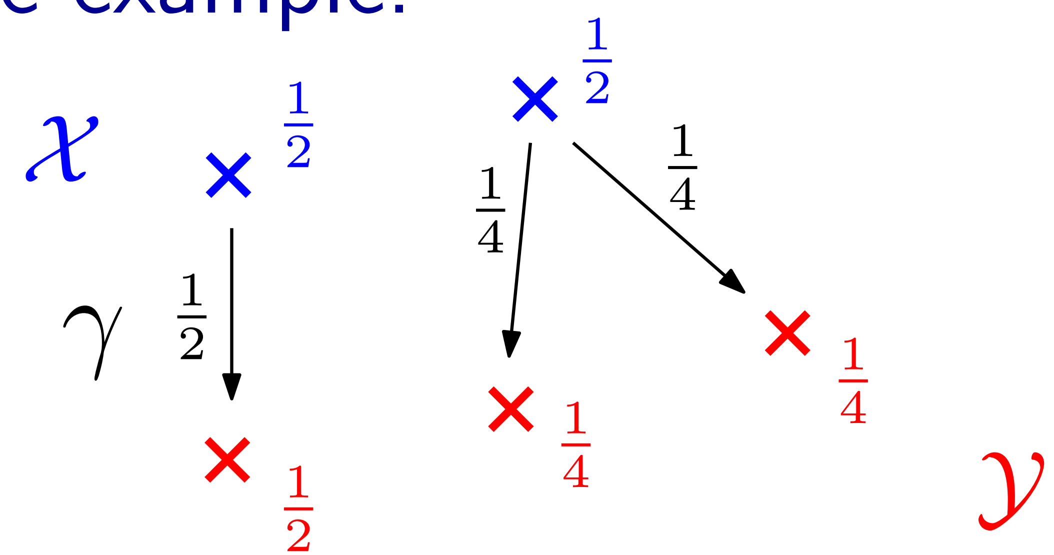

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$$\text{Find } \gamma \in \Gamma(\mu, \nu) \text{ realizing } \min_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathcal{X} \times \mathcal{Y}} c(x, y) \, \mathrm{d}\, \gamma(x, y).$$

- If the cost is twisted and μ has a density, then Monge $\iff$ Kantrovich.

Kantorovich duality & c-concavity

Dual problem: Find $\varphi \in \mathcal{C}^0(\mathcal{X})$ and $\psi \in \mathcal{C}^0(\mathcal{Y})$ realizing

$$(DP) = \sup_{\varphi \oplus \psi \leq c} \int_{\mathcal{X}} \varphi \, \mathrm{d}\, \mu + \int_{\mathcal{Y}} \psi \, \mathrm{d}\, \nu$$

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where $\forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \varphi \oplus \psi(x, y) = \varphi(x) + \psi(y) \leq c(x, y)$

Best choice for a fixed φ $\quad \psi(y) = \inf_{x \in \mathcal{X}} c(x, y) - \varphi(x) := \varphi^c(y)$

c-transform

- c-transform is a generalization of Legendre transform.

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Best choice for a fixed φ

$$\psi(y) = \underbrace{\inf_{x \in \mathcal{X}} c(x,y) - \varphi(x)}_{\text{c-transform}} := \varphi^c(y)$$

- c-transform is a generalization of Legendre transform.
- If (φ, ψ) maximizes (DP) then they are c-conjugate:

$$\psi = \varphi^c \text{ and } \varphi = \psi^c$$

- We recover the optimal transport map by

$$T(x) \in \arg\min_{y \in \mathcal{Y}} c(x,y) - \psi(y)$$

c-concave functions

Definition: (c-concave functions) A function $\psi : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ is c-concave if there exists $\varphi : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$ such that

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- The c-superdifferential of a function $\psi : \mathcal{Y} \to \mathbb{R}$ is defined by
  $$\partial^c \psi(y) = \{x \in \mathcal{X} \mid \forall z \in \mathcal{Y}, c(x, z) - \psi(z) \geq c(x, y) - \psi(y)\}$$
- ψ is c-concave $\iff$ $\partial^c \psi(y) \neq \emptyset$ for any $y \in \mathcal{Y}$.

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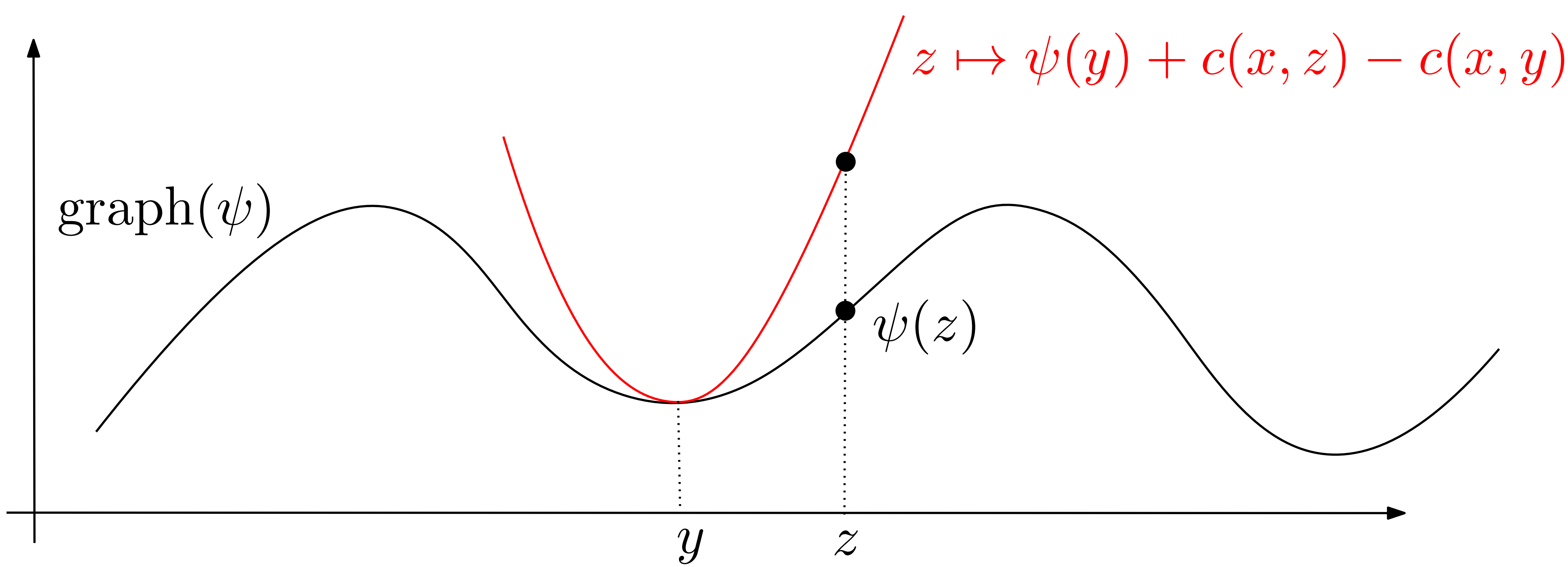

c-concave functions

Definition: (c-concave functions) A function $\psi : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ is c-concave if there exists $\varphi : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$ such that

$$\psi(y) = \varphi^c(y) = \inf_{x \in \mathcal{X}} c(x, y) - \varphi(x)$$

- The c-superdifferential of a function $\psi : \mathcal{Y} \to \mathbb{R}$ is defined by
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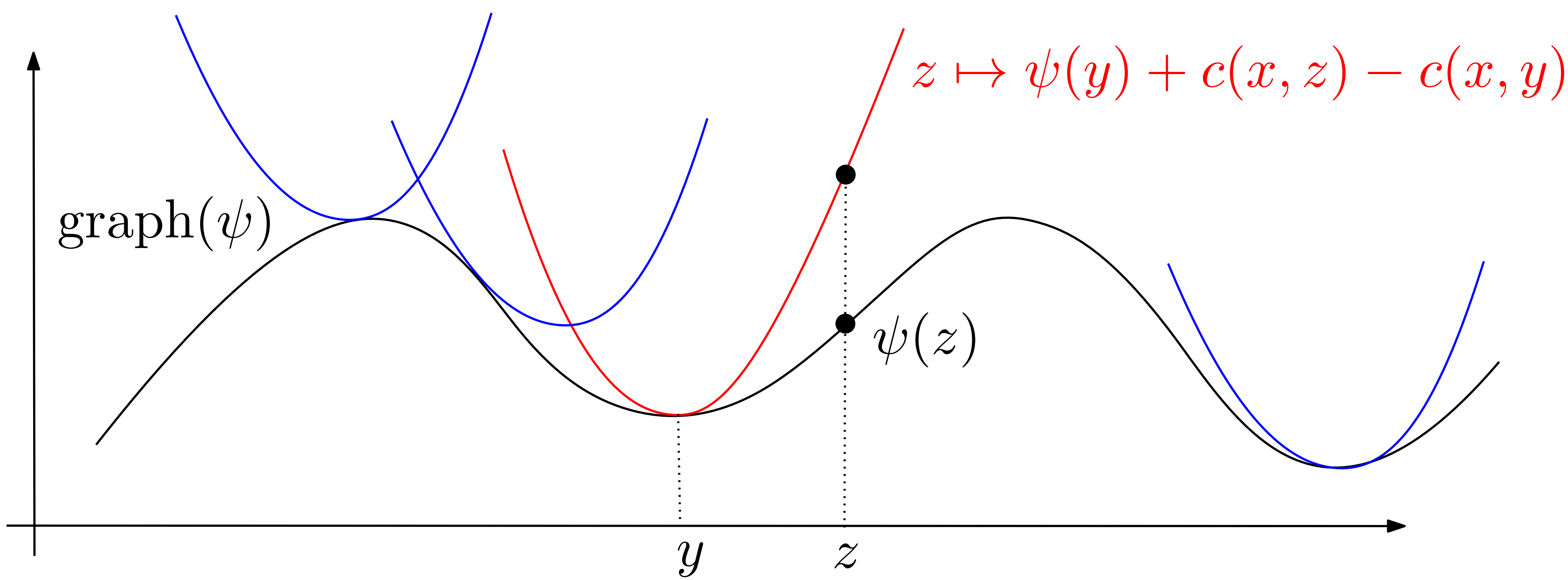

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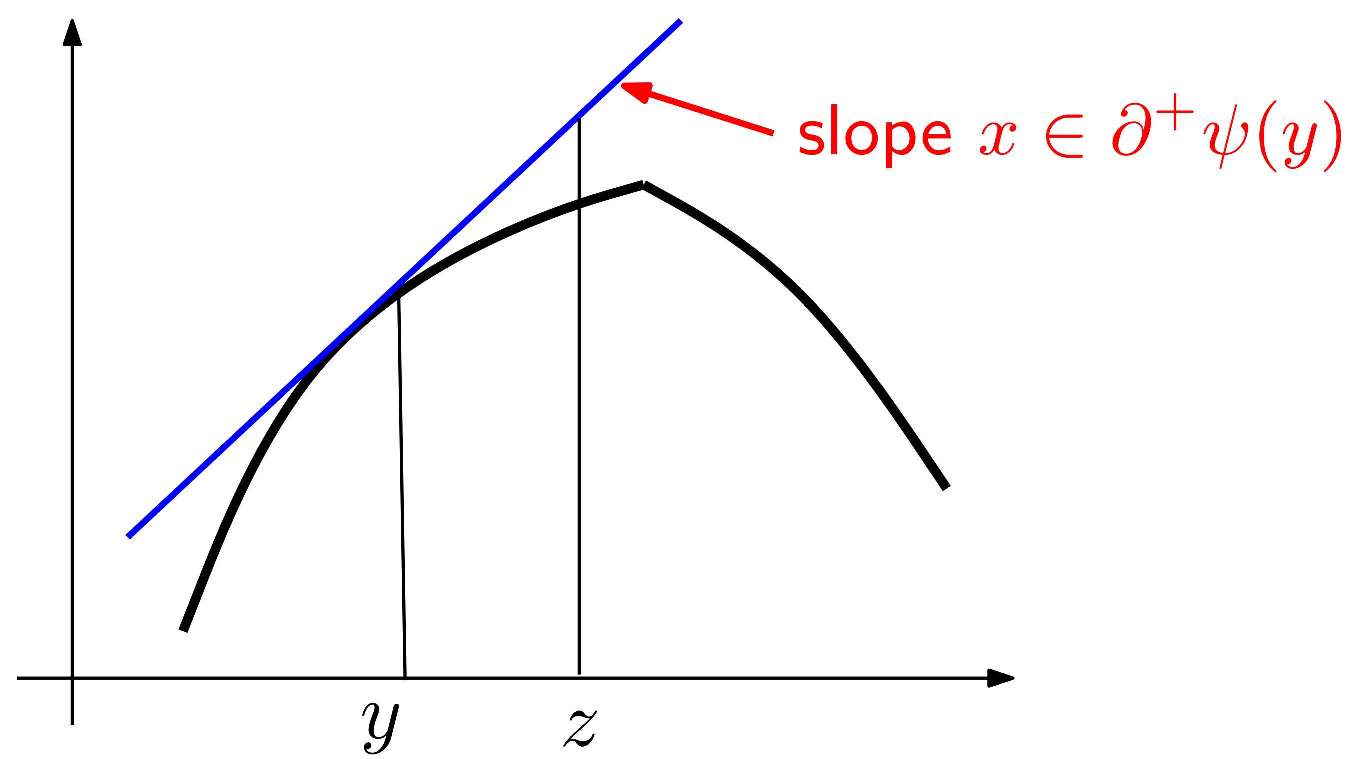
- ψ is c-concave $\iff$ $\partial^c \psi(y) \neq \emptyset$ for any $y \in \mathcal{Y}$.

Particular case:

- $c(x, y) = \langle x | y \rangle$
- c-concavity matches regular concavity
- $\partial^c \psi(y) = \partial^+ \psi(y)$



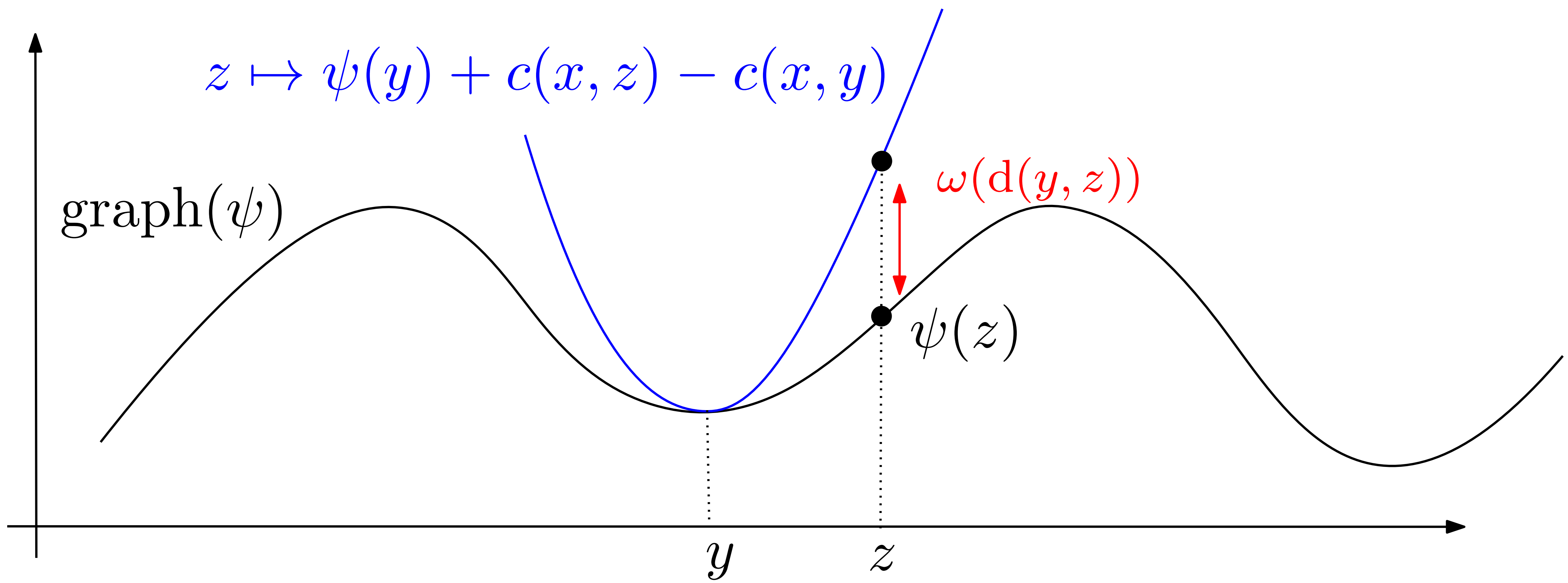
Strong c-concavity

Definition: *[G.-Mérigot-Thibert '22]* (Strong c-concavity)

A c-concave function $\psi : \mathcal{Y} \to \mathbb{R}$ is said to be strongly c-concave with modulus $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ if for any $y, z \in \mathcal{Y}$ and $x \in \partial^c \psi(y)$,

$$c(x,z) - \psi(z) \geq c(x,y) - \psi(y) + \omega(\mathrm{d}(y,z))$$

- In practice we have $\omega(\mathrm{d}(y,z)) = C\,\mathrm{d}(y,z)^2$.



Stability w.r.t target measure

Theorem 1: *[G.-Mérigot-Thibert '22]* Let $D \subseteq \mathcal{X} \times \mathcal{Y}$ be a compact set and $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ be C^1 on D. Let $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu_0, \nu_1 \in \mathcal{P}(\mathcal{Y})$. We assume $T_{\mu \to \nu_i}$ is an optimal transport map from μ to ν_i with associated potential $\psi_i : N \to \mathbb{R}$ $(i = 0, 1)$ such that:

- ψ_0 is Lipschitz on $\mathcal{Y}$ and c-concave on D.
- ψ_1 is Lipschitz on $\mathcal{Y}$ and strongly c-concave with $\omega(r) = Cr^2$ on D.
- The maps T_i satisfies for any $x \in \mathcal{X}$, $(x, T_i(x)) \in D$.

Then $\boxed{C \| \mathrm{d}_{\mathcal{Y}}(T_{\mu \to \nu_0}, T_{\mu \to \nu_1}) \|^2_{L^2(\mu)} \leq (\mathrm{Lip}(\psi_0) + \mathrm{Lip}(\psi_1)) W_1(\nu_0, \nu_1)}$

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- Justifies the semi-discrete approach on continuous problems
- Generalizes Ambrosio: replaces T_1 Lipschitz by ψ_1 strongly c-concave.

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- Justifies the semi-discrete approach on continuous problems
- Generalizes Ambrosio: replaces T_1 Lipschitz by ψ_1 strongly c-concave.
- If $D = \mathcal{X} \times \mathcal{Y}$ blue assumptions disappear (but necessary in some appl.)

Proof of Theorem 1

By Kantorovich-Rubinstein Theorem:

$$\int_{\mathcal{Y}} (\psi_1 - \psi_0)\,\mathrm{d}(\nu_1 - \nu_0) \leq \mathrm{Lip}(\psi_0 + \psi_1)\,\mathrm{W}_1(\nu_0, \nu_1).$$

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Finally $\quad \boxed{\int_{\mathcal{X}} \omega(\mathrm{d}_{\mathcal{Y}}(T_1(x), T_0(x)))\,\mathrm{d}\,\mu(x) \leq \mathrm{Lip}(\psi_0 + \psi_1)\,\mathrm{W}_1(\nu_0, \nu_1).}$

Error bound for transport plans

Theorem 2: *[G.-Mérigot-Thibert '22]* Let $D \subseteq \mathcal{X} \times \mathcal{Y}$ be a compact set, $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu_0 \in \mathcal{P}(\mathcal{Y})$. We assume T is an optimal transport map from from μ to ν with associated potential ψ such that

- ψ is strongly c-concave with modulus $\omega(r) = Cr^2$ on D.
- For any $x \in \mathcal{X}$, $(x, T(x)) \in D$.

Then for any transport plan $\gamma \in \Gamma(\mu, \nu)$:

$$\int_{\mathcal{X}\times\mathcal{Y}} C \, \mathrm{d}_{\mathcal{Y}}(T(x), y)^2 \, \mathrm{d}\,\gamma(x,y) \leq \int_{\mathcal{X}\times\mathcal{Y}} c(x,y) \, \mathrm{d}\,\gamma(x,y) - \int_{\mathcal{X}} c(x, T(x)) \, \mathrm{d}\,\mu(x)$$

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$\geq C\,\mathrm{W}_1^2(\gamma_T, \gamma)$

cost of γ $-$ optimal cost

$=$ suboptimality gap

- Generalizes a result of (Li Nochetto '20)
- Kind of strong convexity of the total transport cost for transport plans.

Stability w.r.t to both measures

Theorem 3: *[G.-Mérigot-Thibert '22]*
Let $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be Lipschitz, $\mu, \tilde{\mu} \in \mathcal{P}(\mathcal{X})$ and $\nu, \tilde{\nu} \in \mathcal{P}(\mathcal{Y})$.
Let $T_{\mu\to\nu}$ be an optimal transport map from μ to ν induced by a c-concave potential ψ and $\tilde{\gamma} \in \Gamma(\tilde{\mu}, \tilde{\nu})$ be an optimal transport plan between $\tilde{\mu}$ and $\tilde{\nu}$.
Assume that ψ is strongly c-concave with modulus $\omega(r) = Cr^2$.

Then $\mathrm{W}_1(\gamma_{T_{\mu\to\nu}}, \tilde{\gamma}) \leq \varepsilon + \sqrt{\frac{2\mathrm{Lip}(c)}{C}}\sqrt{\varepsilon}$, where $\varepsilon = \mathrm{W}_1(\mu, \tilde{\mu}) + \mathrm{W}_1(\nu, \tilde{\nu})$.

- Assume that the cost c to be Lipschitz on the whole product space $\mathcal{X} \times \mathcal{Y}$.

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Idea of proof.

- Gluing of measure

$$\begin{tikzcd}
\mu \arrow[r, "\gamma_{T_{\mu \to \nu}}"] \arrow[d, "\alpha"] \arrow[r, bend right=90, looseness=2, "\gamma"'] & \nu \\
\tilde{\mu} \arrow[r, "\tilde{\gamma}"] & \tilde{\nu} \arrow[u, "\beta"']
\end{tikzcd}$$

- Bound the gap:
$$\int_{\mathcal{X} \times \mathcal{Y}} c \, \mathrm{d}\gamma - \int_{\mathcal{X} \times \mathcal{Y}} c \, \mathrm{d}\gamma_T \leq 2\mathrm{Lip}(c)\varepsilon$$

- Apply Theorem 2

Ma-Trudinger-Wang tensor

Let $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ of class $\mathcal{C}^4$ and that satisfies (Stwist) on $D \subseteq \mathcal{X} \times \mathcal{Y}$, meaning that $\nabla_x c(x, \cdot)$ is injective and D^2_{xy} is non-singular.

Definition: The Ma-Trudinger-Wang tensor is defined for $(x_0, y_0) \in D$ and $(\eta, \zeta) \in T_{x_0}\mathcal{X} \times T_{y_0}\mathcal{Y}$ by

$$\mathfrak{S}_c(x_0, y_0)(\eta, \zeta) = -\frac{3}{2} \frac{\partial^2}{\partial q^2_{\tilde{\eta}}} \frac{\partial^2}{\partial y^2_{\zeta}} \left(c(\mathrm{cexp}_{y_0}(q), y) \right) \Big|_{y=y_0, q=-\nabla_y c(x_0, y_0)}$$

with $\tilde{\eta} = -\nabla^2_{xy} c(x_0, y_0)\eta \in T_{y_0}\mathcal{Y}$

Here $-\nabla^2_{xy} c(x_0, y_0) : T_{x_0}\mathcal{X} \times T_{y_0}\mathcal{Y} \to \mathbb{R}$ is a non singular bilinear form.
the linear form $\tilde{\eta} : T_{y_0}\mathcal{Y} \to \mathbb{R}$ is identified with a vector.

Definition: The weak MTW condition (MTWw) is satisfied on a compact set $D \subseteq \mathcal{X} \times \mathcal{Y}$ if there exists a constant $C > 0$ such that for any $(x_0, y_0) \in D$ and $(\eta, \zeta) \in T_{x_0}\mathcal{X} \times T_{y_0}\mathcal{Y}$ we have

$$\mathfrak{S}_c(x_0, y_0)(\eta, \zeta) \geq -C |\langle \zeta | \tilde{\eta} \rangle| \|\zeta\| \|\eta\|$$

4th order condition that appears in the regularity theory [MTW 2005]

Sufficient condition for strong c-concavity

Theorem 4: *[G.-Mérigot-Thibert '22]*
We consider $D \subseteq \mathcal{X} \times \mathcal{Y}$ a symmetrically c-convex compact set. We assume that $c \in \mathcal{C}^4(D, \mathbb{R})$, that c and $\check{c}$ satisfy (STwist) on D where $c(x, y) = \check{c}(y, x)$. We also assume that the weak MTW condition is satisfied on D. Let $\psi \in \mathcal{C}^2(\mathcal{Y}, \mathbb{R})$ be a c-concave function on D and such that there exists $\lambda > 0$ satisfying for any $x \in \partial^c \psi(y)$

$$D^2_{yy} c(x, y) - D^2 \psi(y) \geq \lambda Id \qquad (*)$$

Then ψ is strongly c-concave on D with modulus $\omega(\mathrm{d}_{\mathcal{Y}}(\bar{y}, y)) = C\, \mathrm{d}_{\mathcal{Y}}(\bar{y}, y)^2$ where $C > 0$ depends on λ, c, $\mathcal{X}$ and $\mathcal{Y}$.

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- Generalizes a result from *[Trudinger-Wang '06]* for c-concavity using techniques from *[Kim-McCann '07]*
- We need to work on a set $D \subseteq \mathcal{X} \times \mathcal{Y}$ where the cost is regular.

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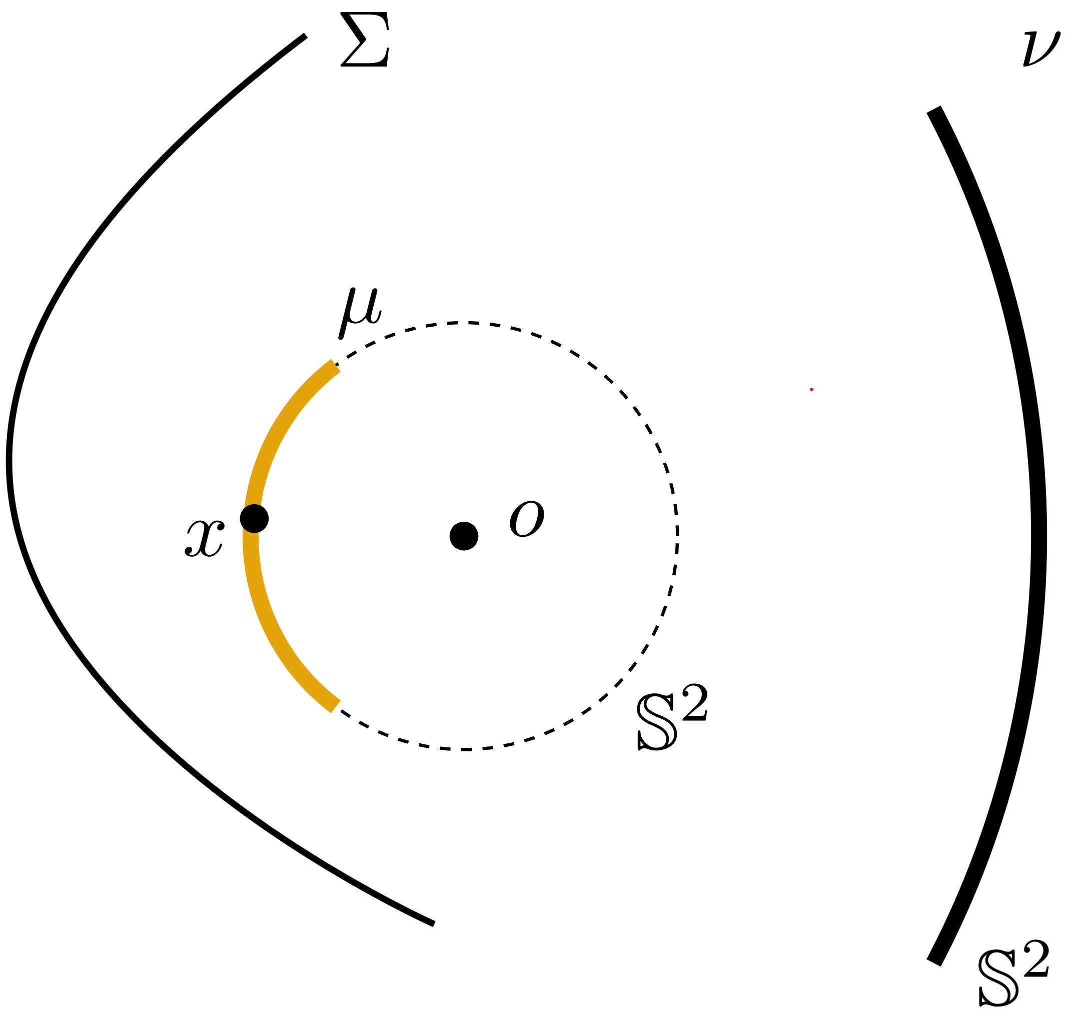
- Generalizes a result from *[Trudinger-Wang '06]* for c-concavity using techniques from *[Kim-McCann '07]*
- We need to work on a set $D \subseteq \mathcal{X} \times \mathcal{Y}$ where the cost is regular.
- Hypothesis $(*)$ can be replaced by:
  - The map $T(x) = \arg\min_y c(x,y) - \psi(y)$ is $\mathcal{C}^1$.
  - $\forall x \in \mathcal{X}$, $(x, T(x)) \in D$.

Applies to O.T.

The Far-field point reflector



- Punctual light source at origin o $\mu \in \mathcal{P}(\mathbb{S}^2)$.
- Target light at infinity, $\nu \in \mathcal{P}(\mathbb{S}^2)$.

Problem : Find a mirror surface Σ that sends $(\mathbb{S}^2, \mu)$ to $(\mathbb{S}^2, \nu)$.

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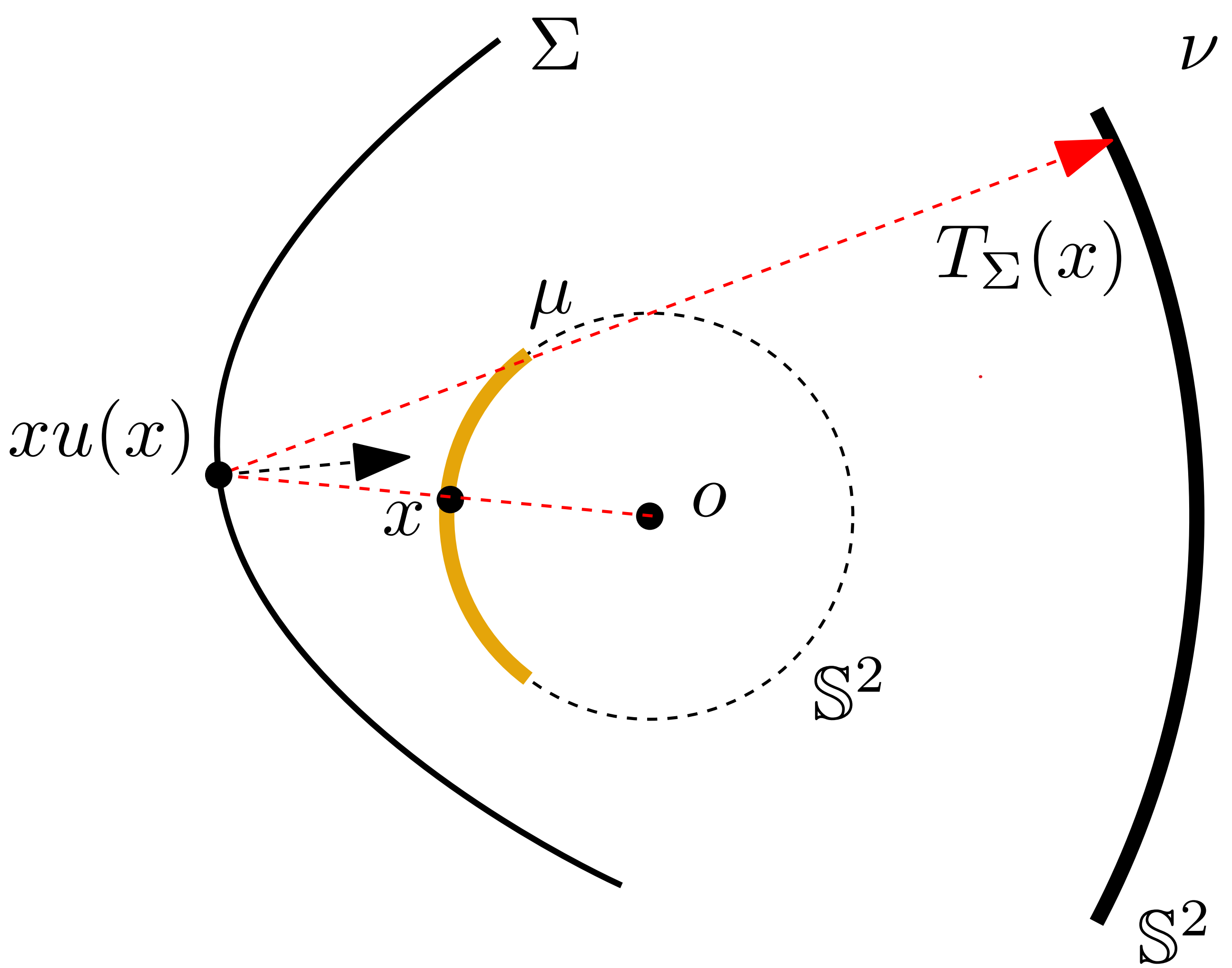

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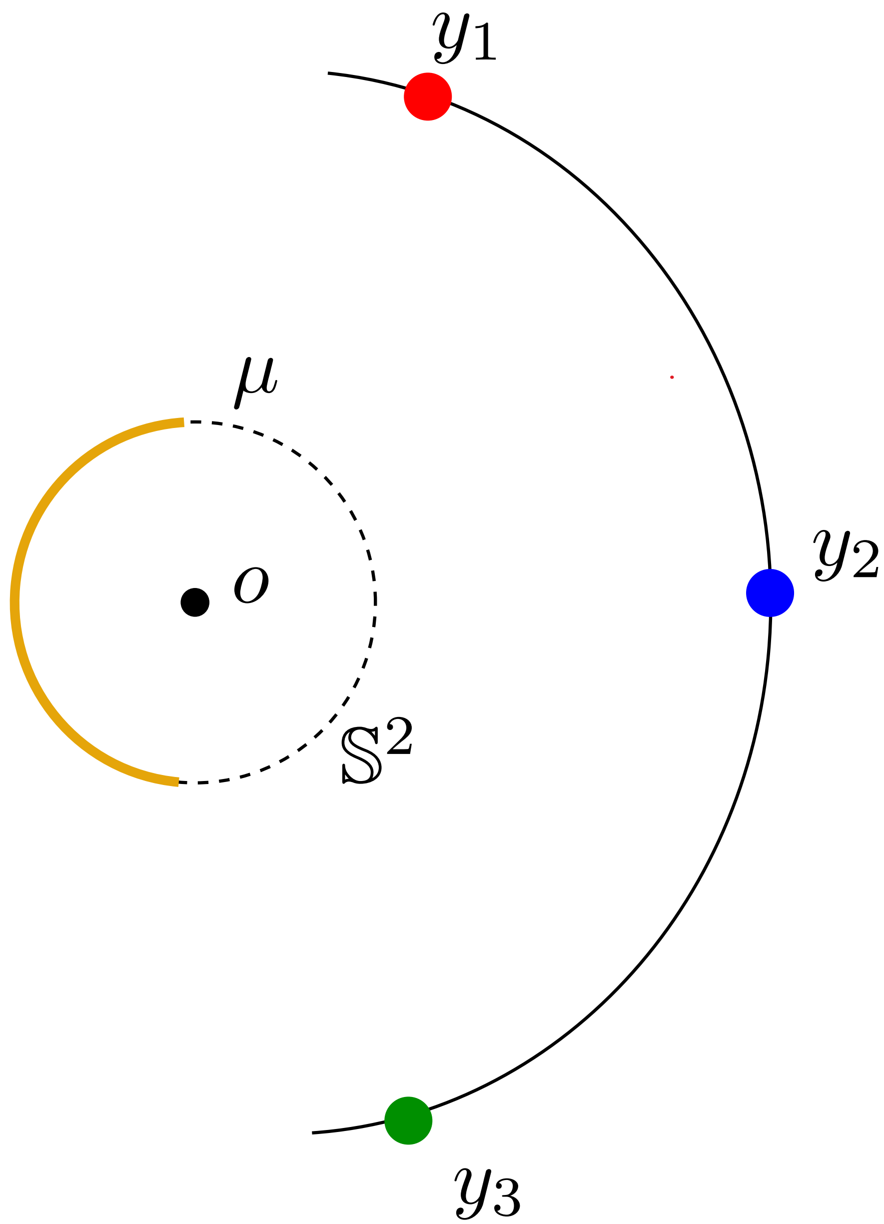
Problem : Find a mirror surface Σ that sends $(\mathbb{S}^2, \mu)$ to $(\mathbb{S}^2, \nu)$.

- Parametrize Σ by $x \in \mathbb{S}^2 \mapsto xu(x)$ with $u : \mathbb{S}^2 \to \mathbb{R}^+$ radial distance.

- Snell's law: $T : x \in \mathbb{S}^2 \mapsto y = x - 2\langle x|n\rangle n$.

FF-point as semi-discrete optimal transport



Semi-discrete :

- μ absolutely continuous
- $\nu = \sum_{1 \leq i \leq N} \nu_i \delta_{y_i}$ discrete

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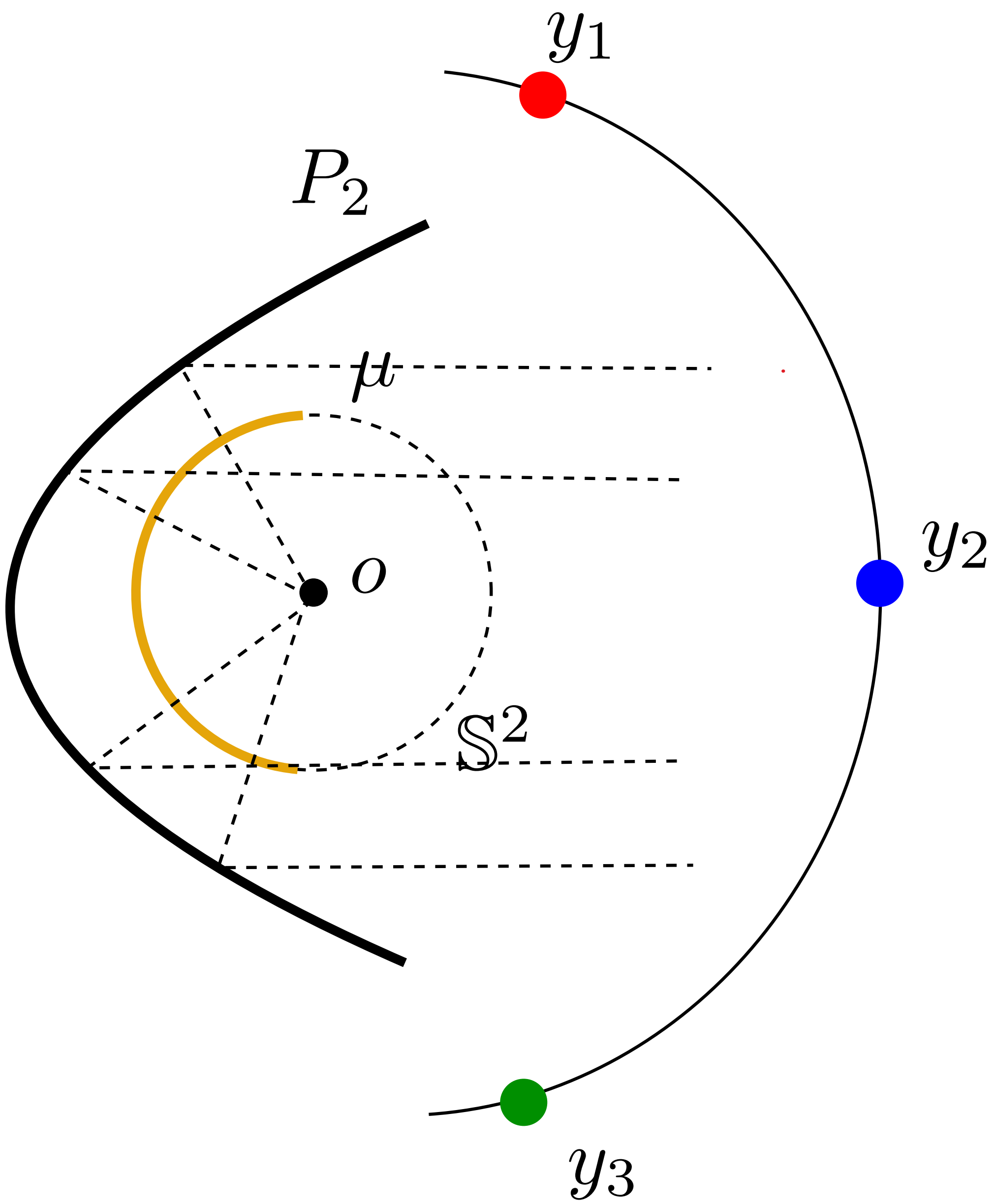

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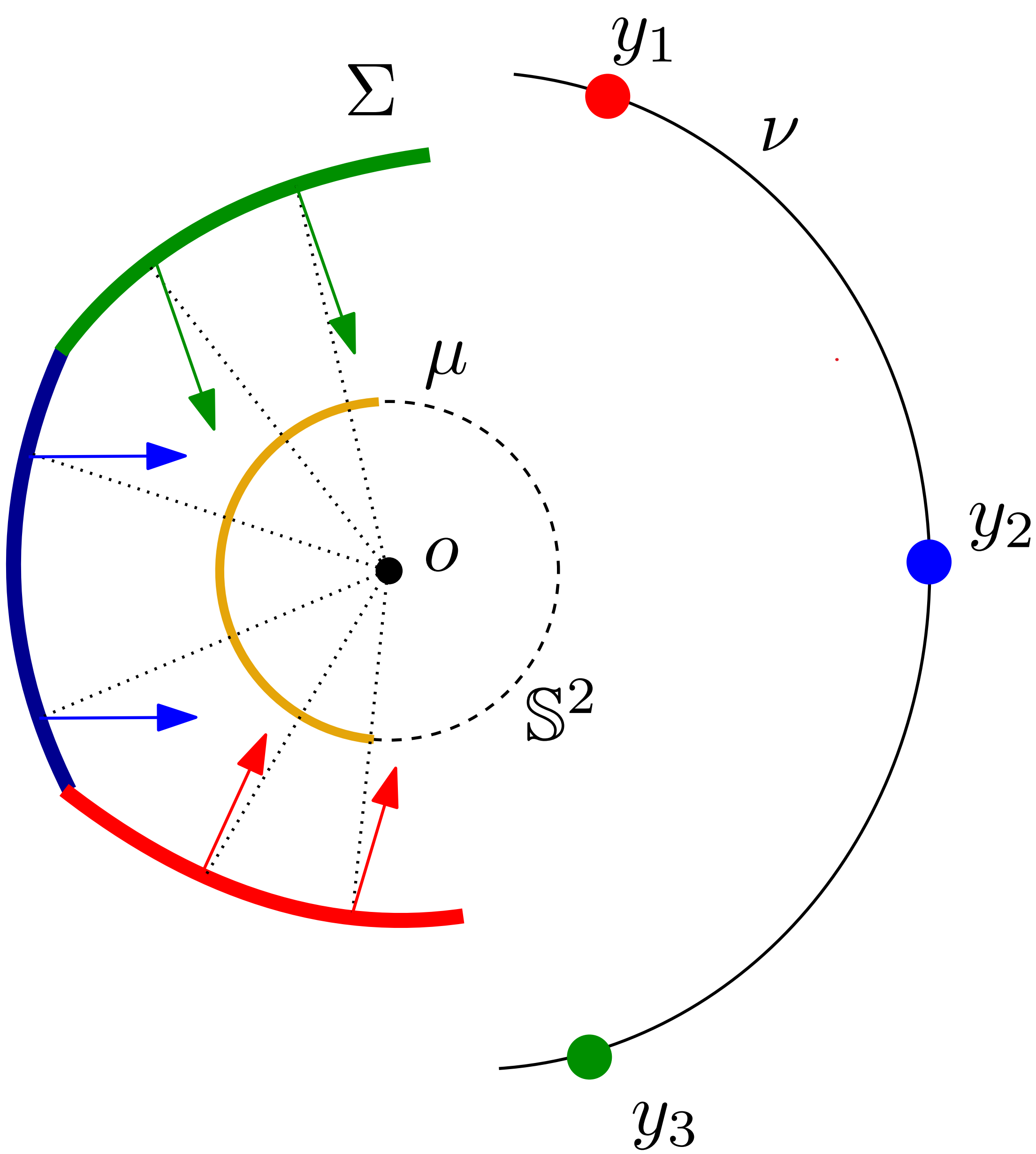
[Caffarelli-Kochengin-Oliker '99]

P_i paraboloid of focus o and axis y_i redirects all the light in direction y_i.

FF-point as semi-discrete optimal transport



Semi-discrete :

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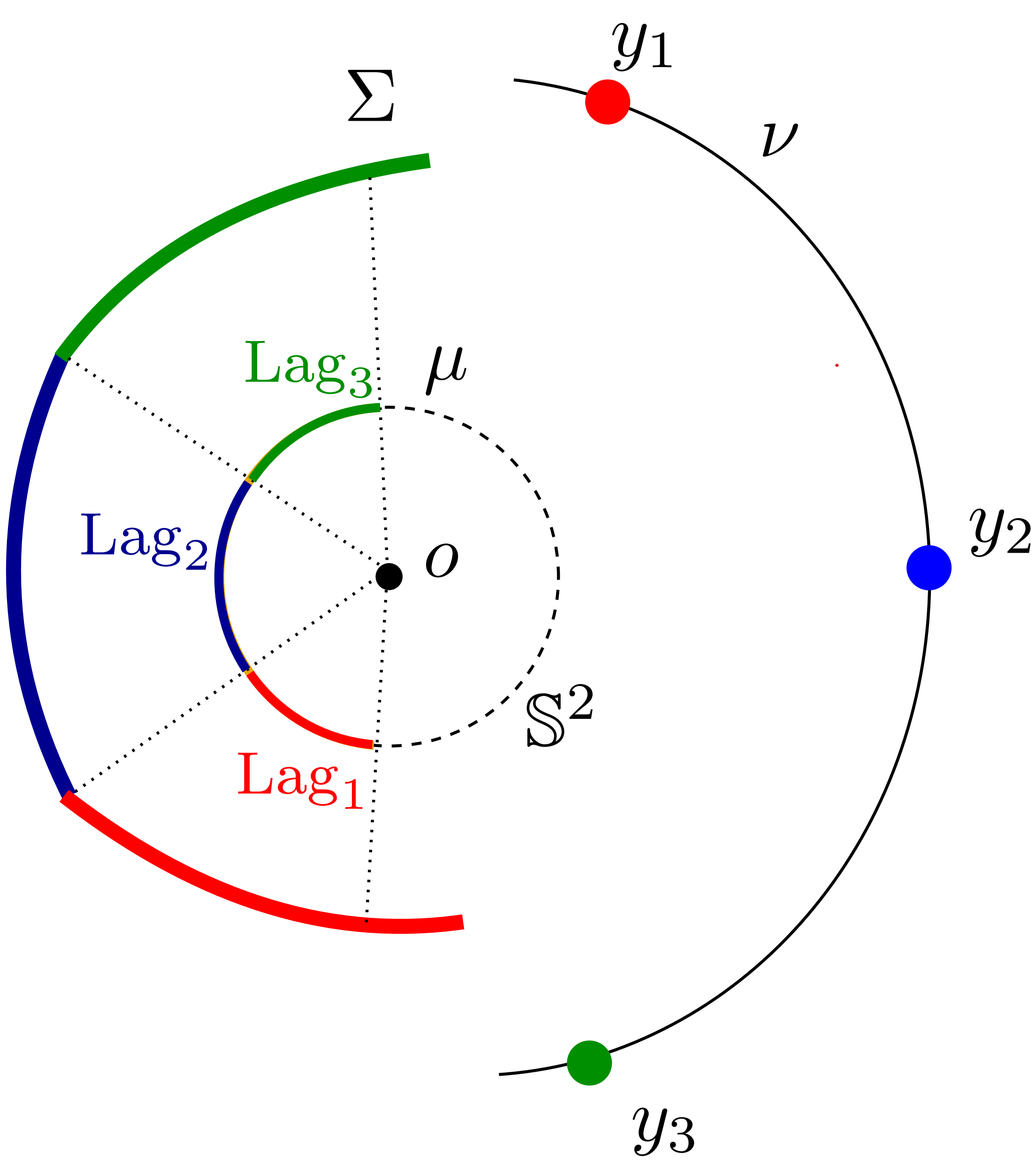
- Mirror Σ is a minimum of paraboloids of focus o and direction $(y_i)_{1 \le i \le N}$.

- Parametrization:

$$u(x) = \min_{1 \le i \le N} \frac{1}{v_i} \frac{1}{1 - \langle x | y_i \rangle}$$

FF-point as semi-discrete optimal transport



Semi-discrete :

- μ absolutely continuous
- $\nu = \sum_{1 \leq i \leq N} \nu_i \delta_{y_i}$ discrete

- Mirror Σ is a minimum of paraboloids of focus o and direction $(y_i)_{1 \leq i \leq N}$.

- Parametrization:

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$$V_i = \left\{ x \in \mathbb{S}^2 \mid \forall j \in \{1, \cdots, N\}, \frac{1}{v_i} \frac{1}{1 - \langle x | y_i \rangle} \leq \frac{1}{v_j} \frac{1}{1 - \langle x | y_j \rangle} \right\}$$

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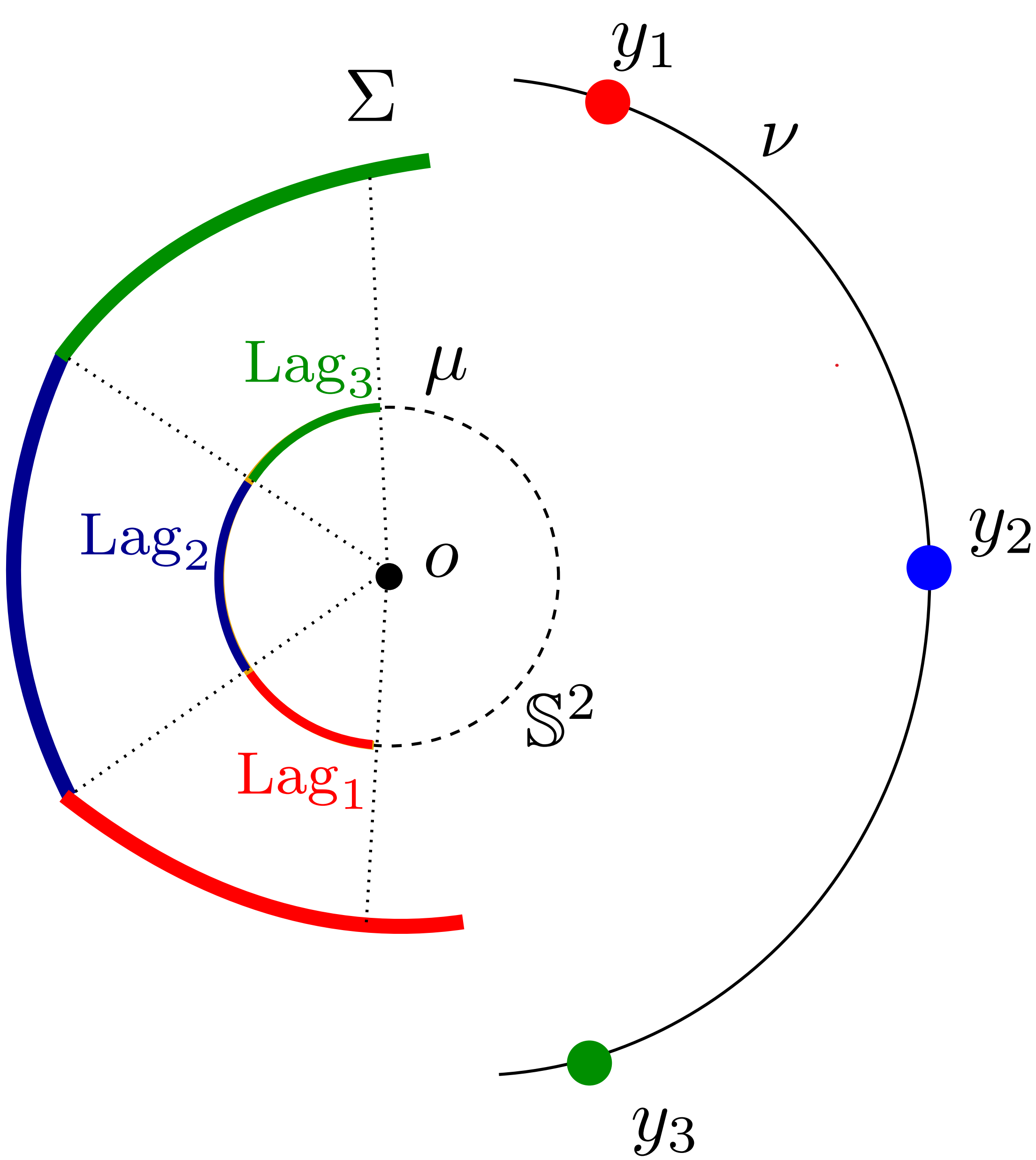

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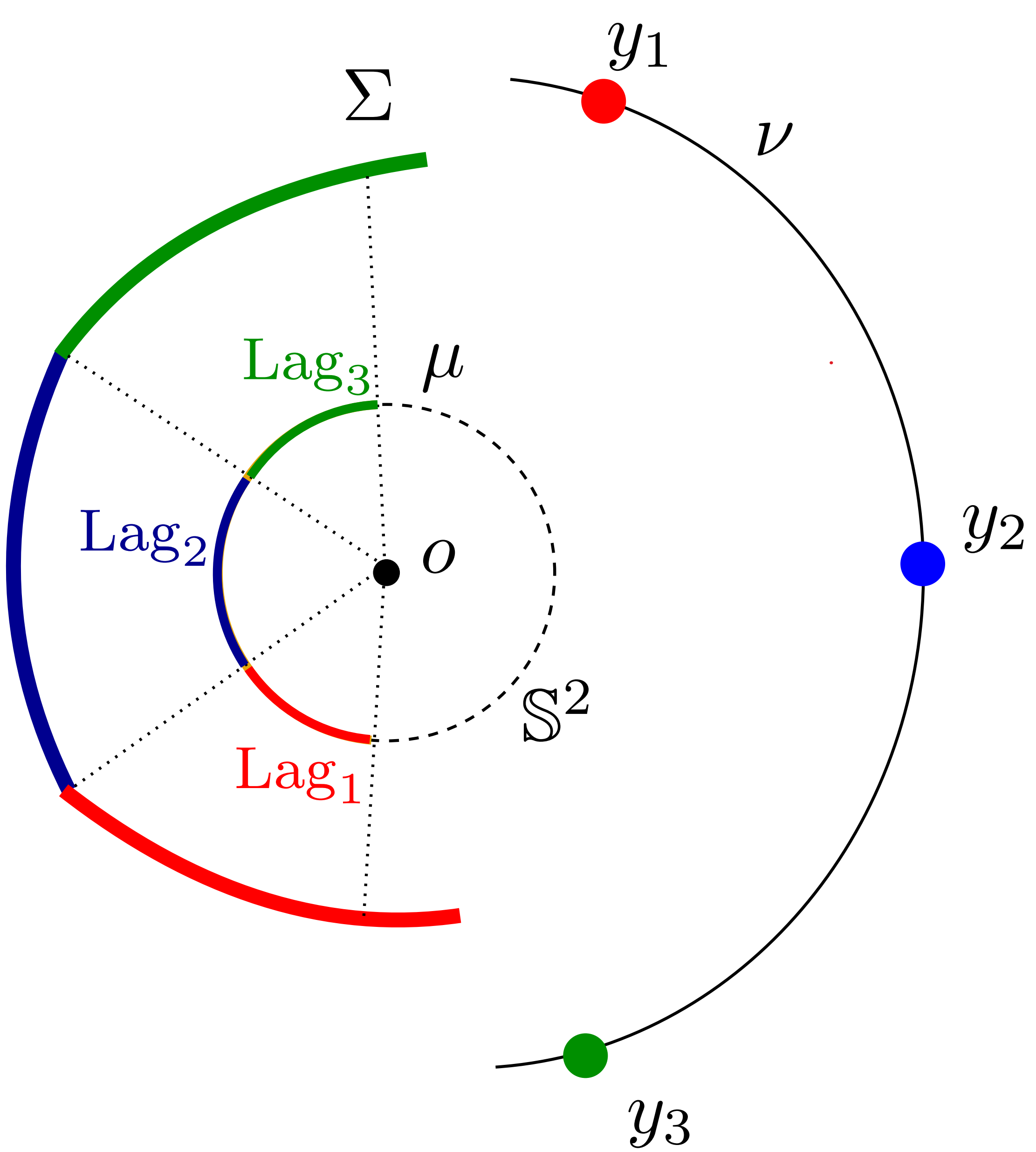
$= -\ln(v_i)$

FF-point as semi-discrete optimal transport



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- μ absolutely continuous
- $\nu = \sum_{1 \leq i \leq N} \nu_i \delta_{y_i}$ discrete

Cost function:

- $c(x, y) = -\ln(1 - \langle x | y \rangle)$

Semi-discrete optimal transport problem

Problem: Find $\psi \in \mathbb{R}^N$ such that for all $i \in \{1, \cdots, N\}$, $\mu(\mathrm{Lag}_i(\psi)) = \nu_i$

with $\mathrm{Lag}_i(\psi) = \{x \in \mathbb{S}^2 \mid \forall j \in \{1, \cdots, N\}, c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j\}$

Application to the Far-field point reflector

Theorem 5: *[G-Mérigot-Thibert '22]* Let $\mu, \nu_0 \in \mathcal{P}(\mathbb{S}^2)$ be two abs. cont. strictly positive $\mathcal{C}^{1,1}$ densities. Then for any $\beta > 0$ there exists $C > 0$ s.t

$$\forall \nu_1 \in \mathcal{P}(\mathbb{S}^2) \text{ s.t } M_{\nu_1}(\beta) < 1/8, \| \mathrm{d}_{\mathbb{S}}(T_1, T_0)\|^2_{L^2(\mu)} \leq C W_1(\nu_0, \nu_1)$$

where $M_\nu(\beta) = \sup_{x \in \mathbb{S}^2} \nu(\mathrm{B}(x, \beta))$, and C depends on β, μ and ν_0.

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Applies to discrete measures

Main difficulty:

- Show that the transport is supported on $D_\varepsilon = \{(x, y) \mid \mathrm{d}(x, y) > \varepsilon\}$

Then:

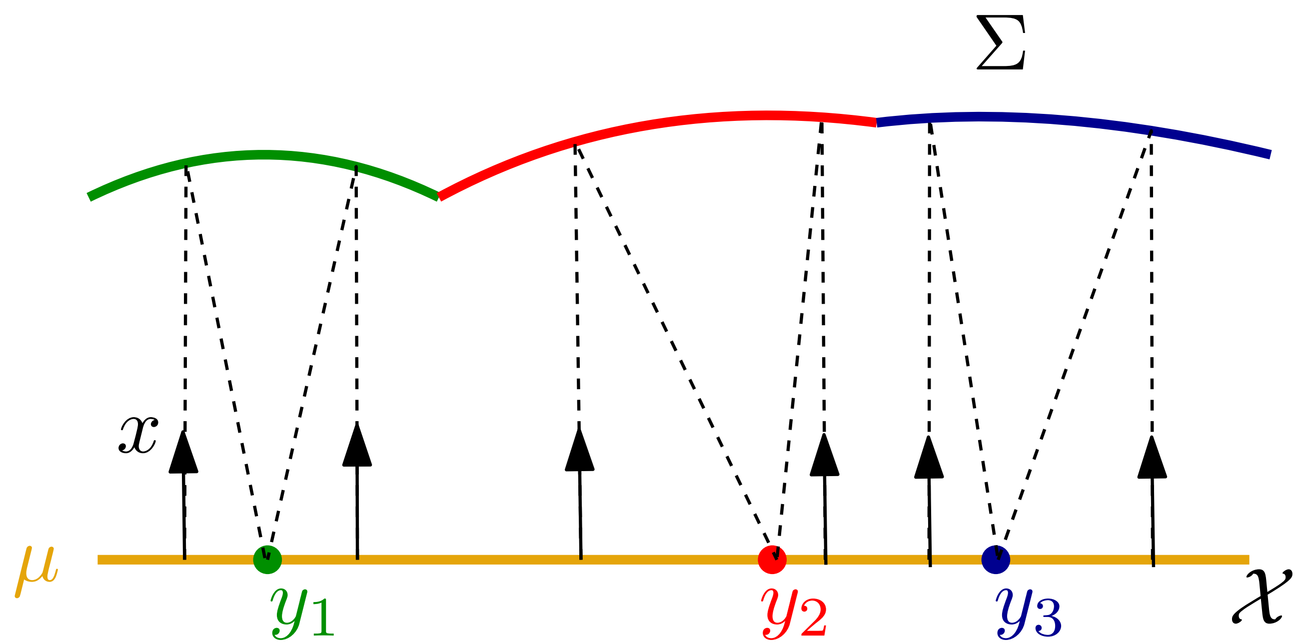
- The cost $c(x, y) = -\ln(1 - \langle x | y \rangle)$ is regular on D_ε, in particular it satisfies the (MTW) hypothesis.

- Regularity of the map T_0 comes from the strictly positive $\mathcal{C}^{1,1}$ densities.

Outline

- Stability in optimal transport
  - 1) Strongly c-concave functions
  - 2) Stability under strong c-concavity
  - 3) Application to the Far-field point reflector
- Numerical resolution of Generated Jacobian equations
  - 1) Damped Newton algorithm for GJE
  - 2) Application to the Near Field parallel reflector

The Near-Field Parallel reflector



- Collimated light source:

  $\mu \in \mathcal{P}(\mathcal{X})$ abs. cont.

- Near field target:

  $\nu = \sum_{1 \leq i \leq N} \nu_i \delta_{y_i}$ with $y_i \in \mathbb{R}^2$.

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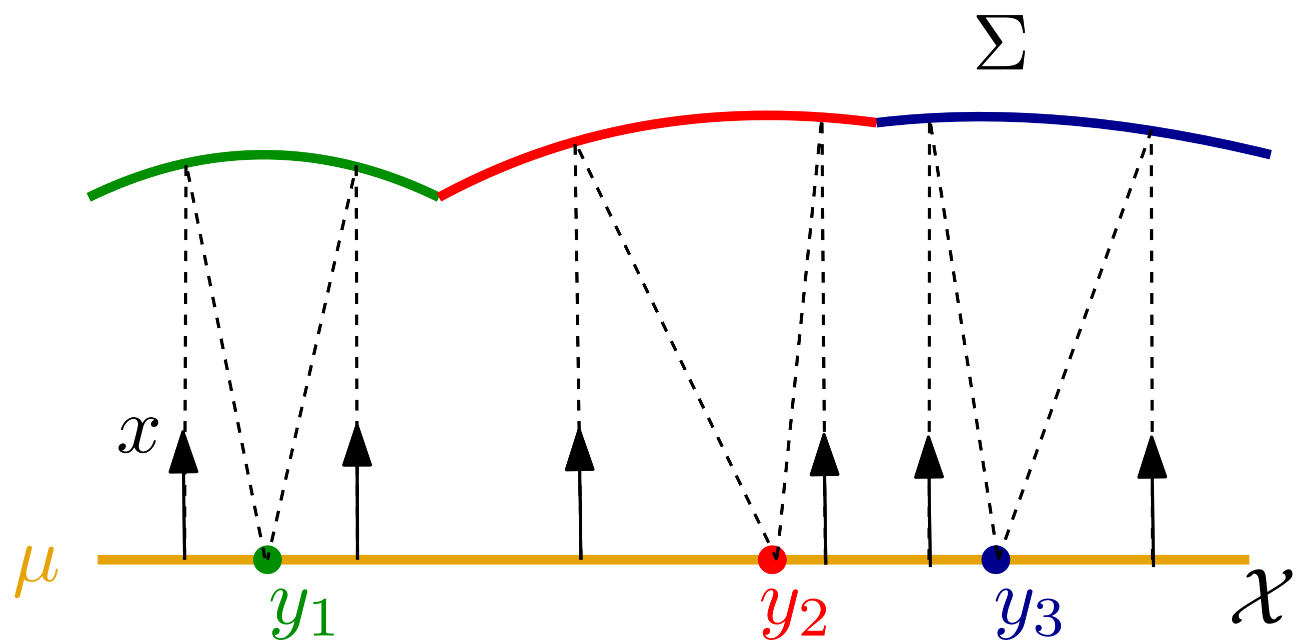

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$$\mathrm{Lag}_i(\psi) = \left\{ x \in \mathcal{X} \mid \forall j, \frac{1}{2\psi_i} - \frac{\psi_i}{2}\|x - y_i\|^2 \ge \frac{1}{2\psi_j} - \frac{\psi_j}{2}\|x - y_j\|^2 \right\}$$

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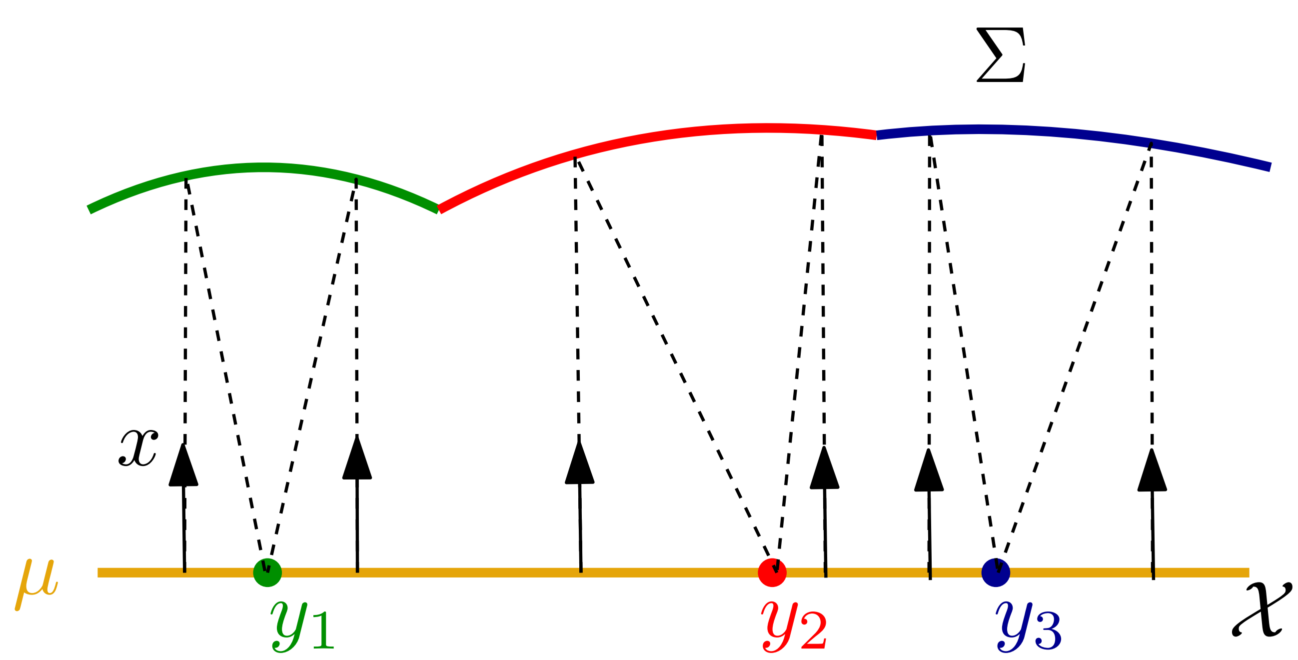

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Problem: Find $\psi \in \mathbb{R}^N$ such that for all $i \in \{1, \cdots, N\}$, $\mu(\mathrm{Lag}_i(\psi)) = \nu_i$

Not linear in $\psi \to$ not optimal transport.

A Generated Jacobian equation

Definition: (Generating function & generalized Laguerre cells)
A function $G : \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ is said to be a generating function if it satisfies (Reg), (Twist), (UC) and (Mono).
The generalized Laguerre cells form a partition of the set $\mathcal{X}$ defined for $\psi \in \mathbb{R}^N$ and $y_i \in \mathcal{Y}$ by

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Generated Jacobian equation: Find $\psi \in \mathbb{R}^N$ such that

$$H(\psi) = \nu \tag{GJE}$$

where the mass function H is defined by $H(\psi) = \big(\mu(\mathrm{Lag}_i(\psi))\big)_{1 \leq i \leq N}$.

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- Here we identified $\mathbb{R}^{\mathcal{Y}}$ with $\mathbb{R}^N$ (semi-discrete).
- Examples of generating functions:

Optimal transport:

$$G(x, y, v) = -c(x, y) - v$$

Near field parallel reflector:

$$G(x, y, v) = \frac{1}{2v} - \frac{v}{2}\|x - y\|^2$$

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Solve using Newton alg.

- Examples of generating functions:

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$$G(x, y, v) = -c(x, y) - v$$

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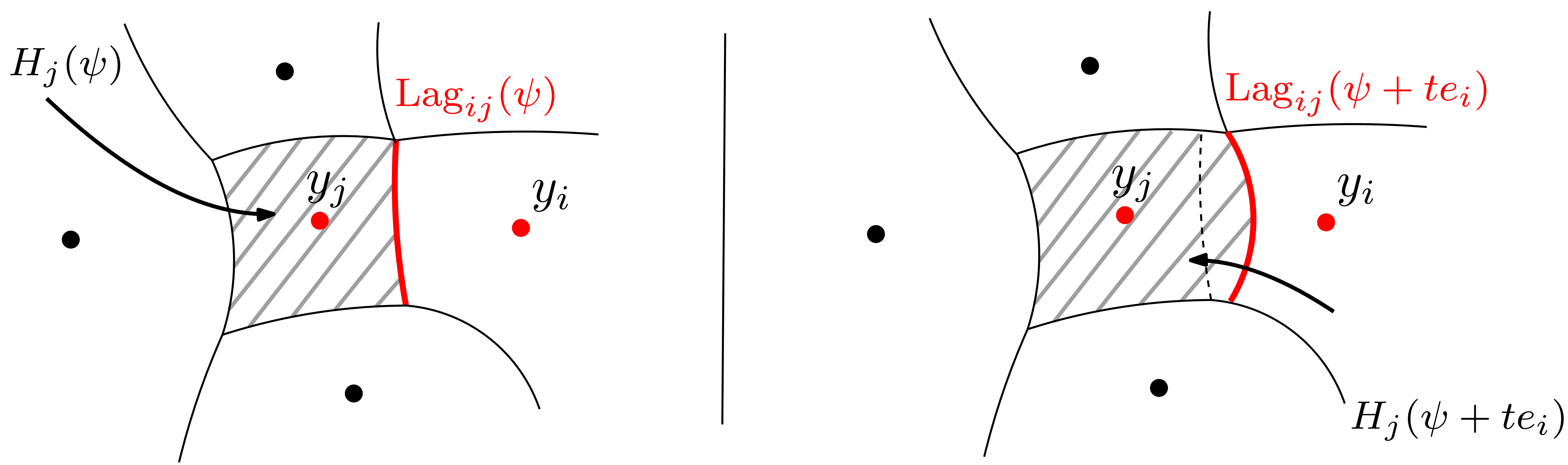
Differential of the mass function H

Proposition 1: *[G-M-T '21]* (Formula for DH) For any $i \neq j$ we have

- $$\frac{\partial H_j}{\partial \psi_i}(\psi) = \int_{\mathrm{Lag}_{ij}(\psi)} \rho(x) \frac{|\partial_v G(x, y_i, \psi_i)|}{\|\nabla_x G(x, y_j, \psi_j) - \nabla_x G(x, y_i, \psi_i)\|} d\mathcal{H}^{d-1}(x) \geq 0$$

$\mathrm{Lag}_{ij} = \mathrm{Lag}_i \cap \mathrm{Lag}_j$

- $$\sum_i H_i(\psi) = 1 \implies \frac{\partial H_i}{\partial \psi_i}(\psi) = -\sum_{j \neq i} \frac{\partial H_j}{\partial \psi_i}(\psi)$$



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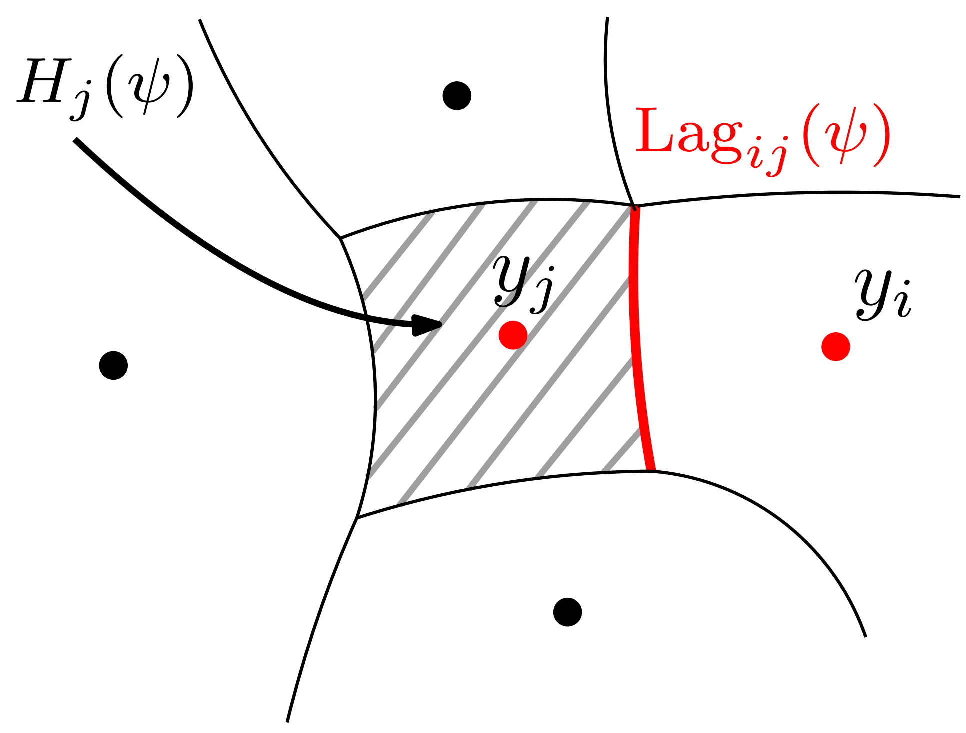
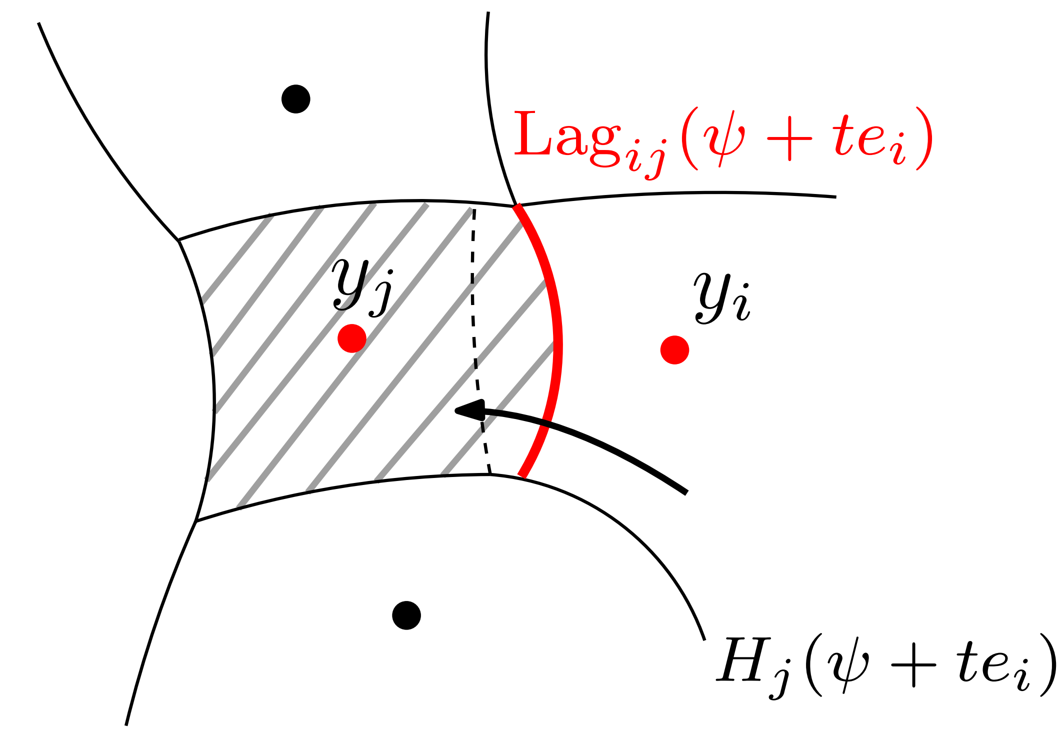
$\neq 0$ by (Twist)

- $$\sum_i H_i(\psi) = 1 \implies \frac{\partial H_i}{\partial \psi_i}(\psi) = -\sum_{j \neq i} \frac{\partial H_j}{\partial \psi_i}(\psi)$$ < 0 if $H_i(\psi) > 0$





Descent direction for Newton

Proposition 2: *[G-M-T '21]* Let $\psi \in \mathcal{S}^+ = \{\psi \in \mathbb{R}^N \mid \forall i, H_i(\psi) > 0\}$, then

- The differential $DH(\psi)$ is of rank $N-1$.
- It image is $\mathrm{Im}(DH(\psi)) = \mathbf{1}^\perp$ where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^N$.
- Its kernel is $\ker(DH(\psi)) = \mathrm{span}(w)$ with $w_i > 0$ for $1 \leq i \leq N$.

Remark: The differential DH has no reason to be symmetric, while it is the case for optimal transport problems.

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Remark: The differential DH has no reason to be symmetric, while it is the case for optimal transport problems.

Corollary: (Descent direction) Let $\psi \in \mathcal{S}^+$, the system

$$\begin{cases} DH(\psi)u = H(\psi) - \nu \\ u_1 = 0 \end{cases}$$

has a unique solution $u \in \mathbb{R}^N$.

Idea of proof:

- We have $H(\psi) - \nu \in \mathbf{1}^\perp = \mathrm{Im}(DH(\psi))$.
- Fixing $u_1 = 0$ is possible because of the structure of $\ker(DH(\psi))$.
- Uniqueness comes from the rank of $DH(\psi)$.

Damped Newton algorithm

Newton algorithm for solving $H(\psi) = \nu$

Initialize $\psi^0 \in \mathcal{S}^\delta = \{\psi \in \mathbb{R}^N \mid \forall i, H_i(\psi) > \delta\}$ and $\varepsilon > 0$.

While $\|H(\psi) - \nu\| \geq \varepsilon$:

ψ_1^k is fixed

$\longrightarrow$ Compute u^k solution of $\begin{cases} DH(\psi)u^k = H(\psi) - \nu \\ u_1^k = 0 \end{cases}$

$\longrightarrow$ Define for $\tau \in [0,1]$, $\psi^{k,\tau} = \psi^k - \tau u^k$.

Damping Parameter

$\longrightarrow$ Compute $\tau^k = \sup\left\{\tau \in [0,1] \;\middle|\; \begin{array}{r} \|H(\psi^{k,\tau}) - \nu\| \leq (1 - \frac{\tau}{2})\|H(\psi^k) - \nu\| \\ \text{and } \underline{\psi^{k,\tau} \in \mathcal{S}^\delta} \end{array}\right\}$

Iterate stays in admissible set

$\longrightarrow$ Put $\psi^{k+1} \leftarrow \psi^{k,\tau^k}$ and $k \leftarrow k+1$

Return ψ^k.

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Linear convergence

$\longrightarrow$ Define for $\tau \in [0,1]$, $\psi^{k,\tau} = \psi^k - \tau u^k$.

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Iterate stays in admissible set

Return ψ^k.

Theorem : *[G-M-T '21]* (Convergence) Assume that the support of μ is connected and compact and that the set $\mathcal{Y}$ is generic. Then $\exists \tau^* > 0$ s.t

$$\|H(\psi^k) - \nu\| \leq \left(1 - \frac{\tau^*}{2}\right)^k \|H(\psi^0) - \nu\|$$

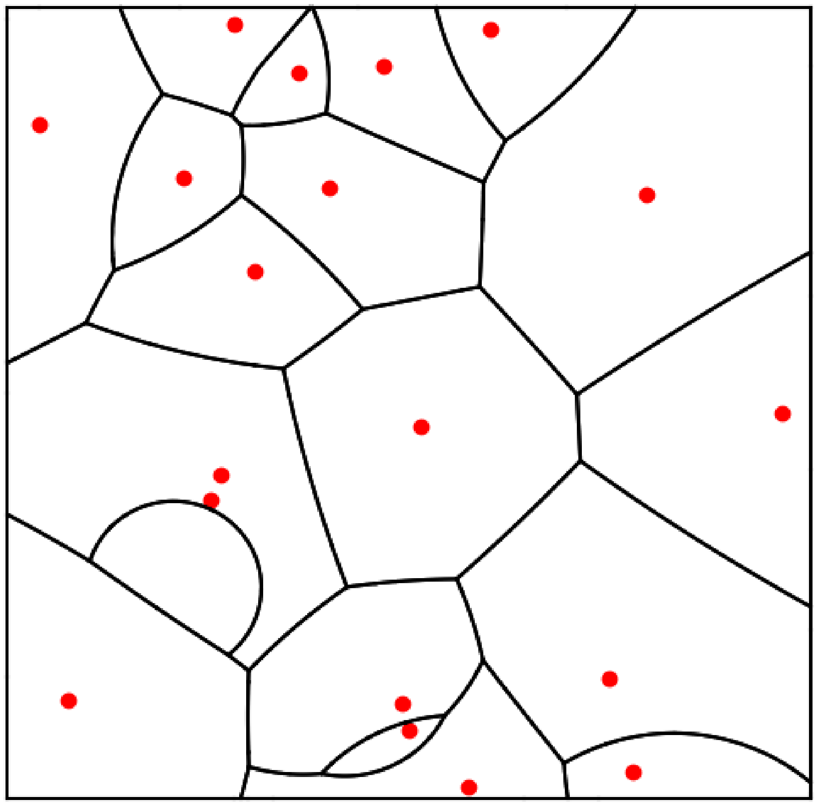
Proof: Bound τ^k below for any k by compactness of the set

$$K = \{\psi \in \mathcal{S}^\delta \mid \psi_1 = \psi_1^0 \text{ and } \|H(\psi) - \nu\| \leq \|H(\psi^0) - \nu\|\}.$$

Implementation for the Near field reflector

- Laguerre diagram for (NF-par):

$$\mathrm{Lag}_i(\psi) = \left\{ x \in \mathbb{S}^2 \mid \forall j, \frac{1}{2\psi_i} - \frac{\psi_i}{2}\|x-y\|^2 \geq \frac{1}{2\psi_j} - \frac{\psi_j}{2}\|x-y\|^2 \right\}$$



- Particular case of Möbius diagram

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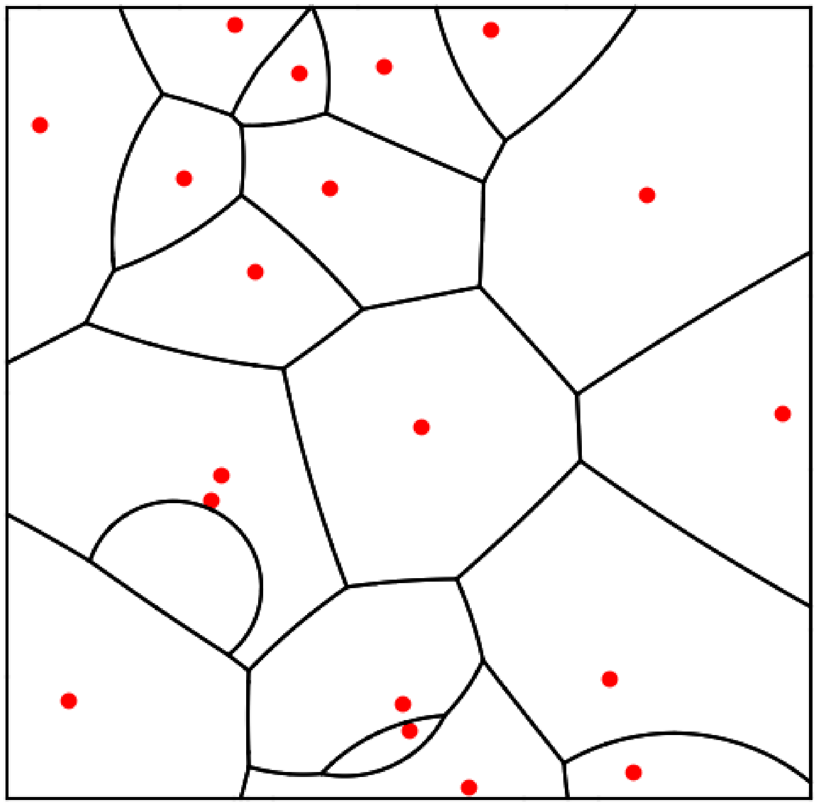

Definition: (Power and Möbius diagram)

- Power diagram : $\mathrm{Pow}_i(c, r) = \{x \mid \forall j, \|x - c_i\|^2 + r_i \leq \|x - c_j\|^2 + r_j\}$
- Möbius diagram : $\mathrm{Mob}_i(y, \lambda, \mu) = \{x \mid \forall j, \lambda_i\|x - y_i\|^2 + \mu_i \leq \lambda_j\|x - y_j\|^2 + \mu_j\}$

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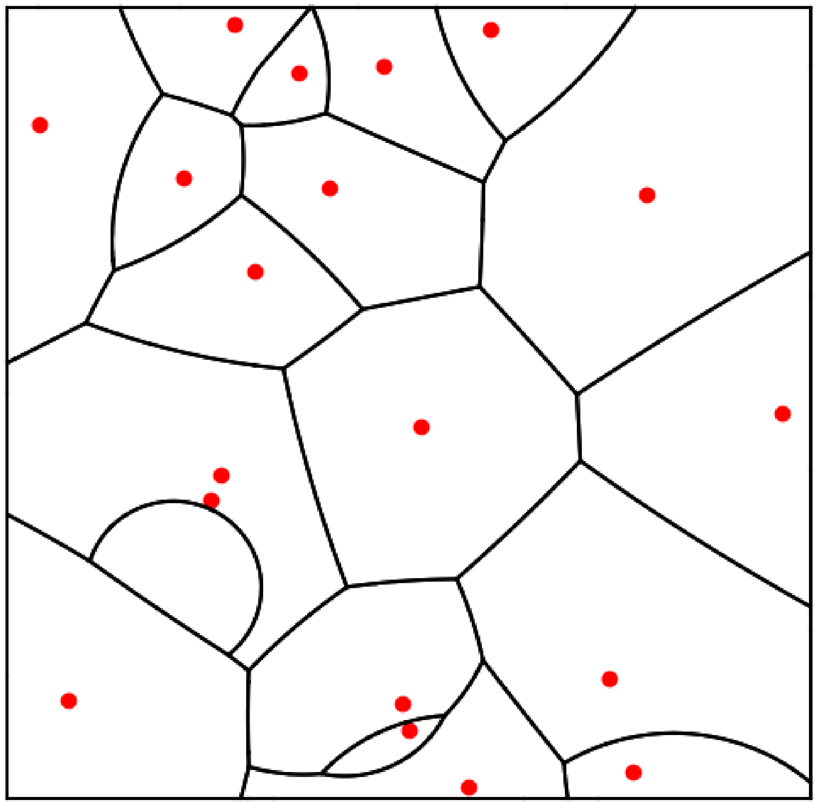

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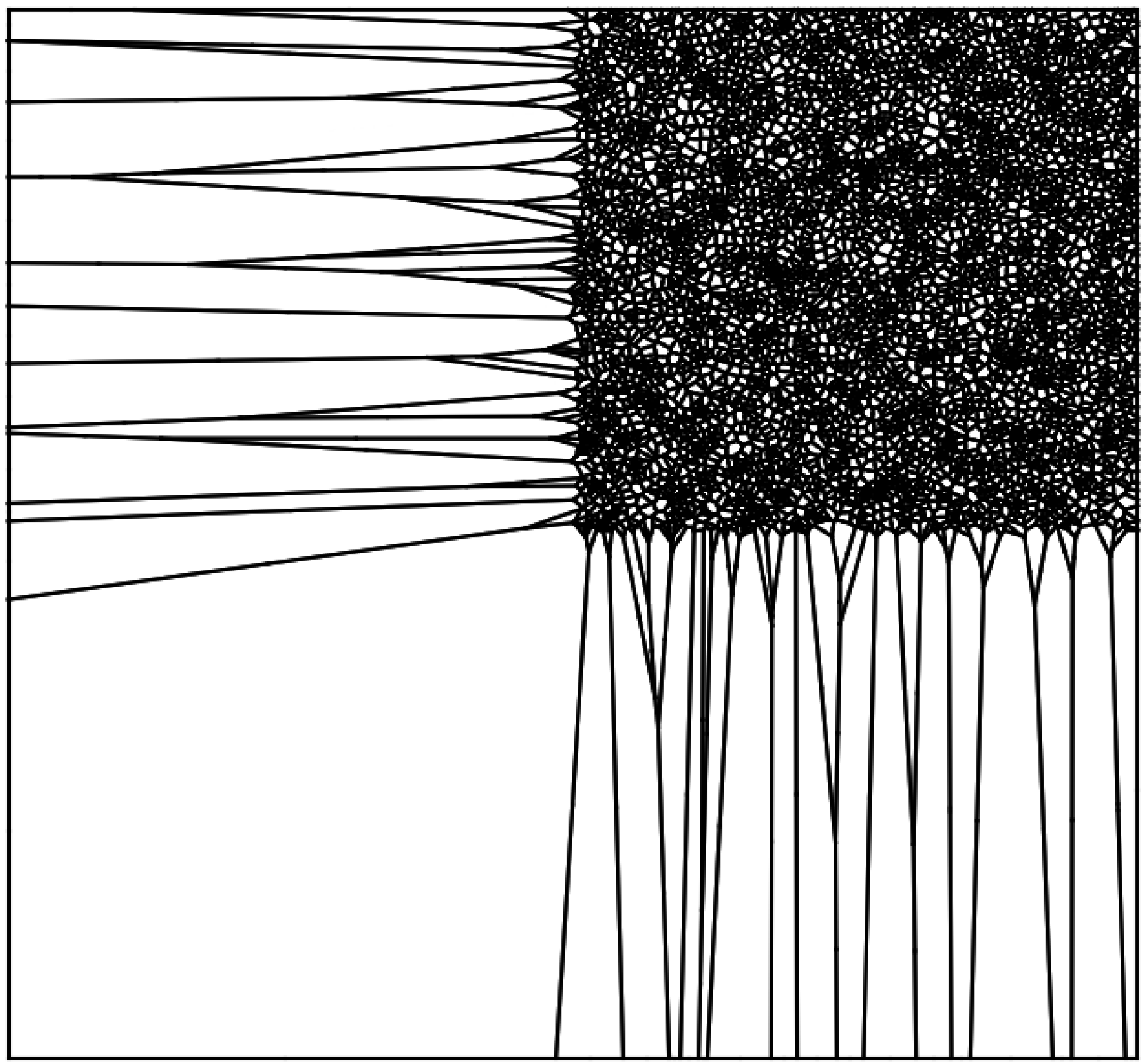
Theorem: *[Boissonnat-Wormser-Yvinec '07].* For any family $(\lambda_i, \mu_i)_i \subset \mathbb{R}$, and $(y_i)_i \subset \mathbb{R}^d$ there exists $(r_i)_i \subset \mathbb{R}$ and $(c_i)_i \subset \mathbb{R}^{d+1}$ such that

$$\mathrm{Mob}_i(y, \lambda, \mu) = \Pi(\mathrm{Pow}_i(c, r) \cap P)$$

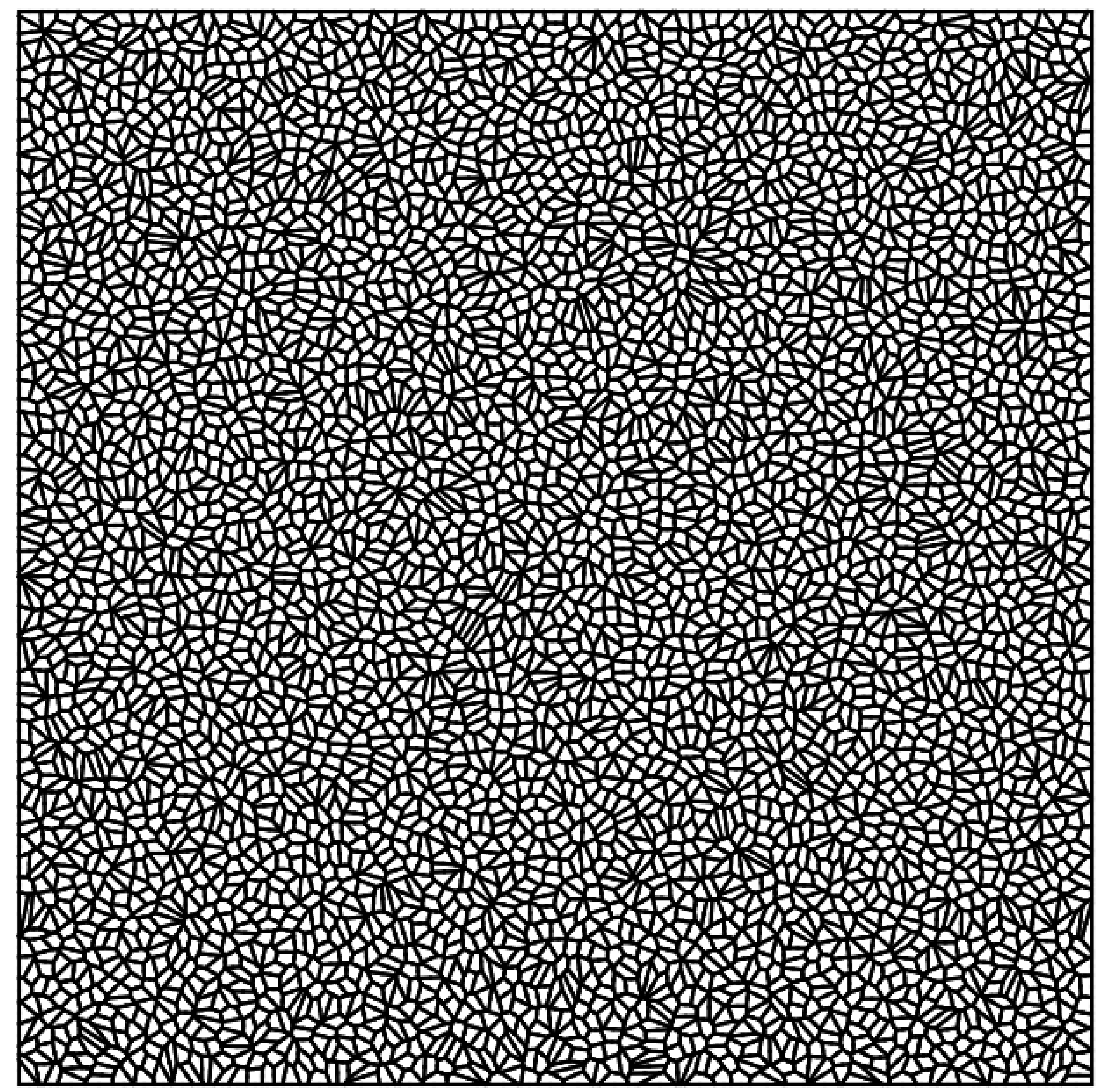
where $P = \{(x, \|x\|^2), x \in \mathbb{R}^d\}$ and Π is the orthogonal projection of $\mathbb{R}^{d+1}$ on $\mathbb{R}^d \times \{0\}$.

Numerical experiments

- $\mathcal{X} = [-1,1]^2$ with μ uniform
- $\mathcal{Y} \subset [0,1]^2$, with ν uniform and $N = 5000$.



Initial diagram



Final diagram

Laguerre diagram before and after convergence of the Newton algorithm

Numerical experiments

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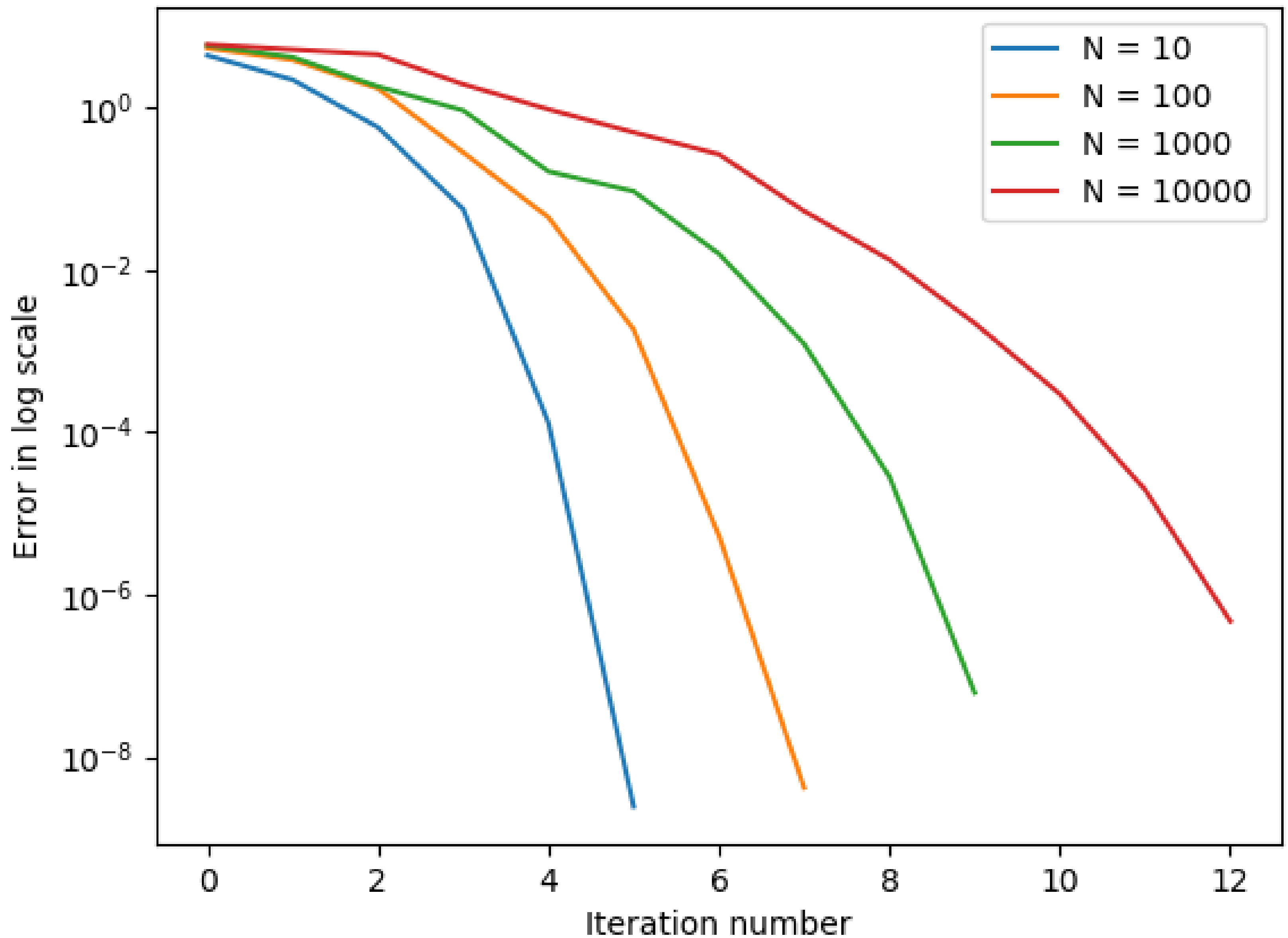
- $\mathcal{Y} \subset [0, 1]^2$, with ν uniform



Convergence rate for different values of N.

Perspectives

- Apply stability results to other optimal transport problems
- Uniqueness of solutions to GJE
- Local stability for GJE
- Detailled study of a stochastic algorithm for (GJE)

Thank you for your attention

Appendix

Generating Function:

- $G(x,y,v)$ is $\mathcal{C}^1$ in x and v and $\sup_{\mathcal{X}\times\mathcal{Y}\times[\alpha,\beta]} |\nabla_x G(x,y,v)| < +\infty$ (Reg)
- $\forall (x,y,v) : \partial_v G(x,y,v) < 0$ (Mono)
- $\forall x \in \mathcal{X}, (y,v) \mapsto (G(x,y,v), \nabla_x G(x,y,v))$ is injective on $\mathcal{Y} \times \mathbb{R}$ (Twist)
- $\forall y \in Y, \lim_{v\to-\infty} \inf_{x\in\mathcal{X}} G(x,y,v) = +\infty$ (UC)

A stochastic algorithm for GJE

Entropic regularization:

- Regularized cells: $\mathcal{L}_{\varepsilon,i}[\psi](x) = \dfrac{e^{G(x,y_i,\psi_i)/\varepsilon}}{\sum_k e^{G(x,y_k,\psi_k)/\varepsilon}} \xrightarrow[\varepsilon \to 0]{} \begin{cases} 1 \text{ if } x \in \mathrm{Lag}_i(\psi) \\ 0 \text{ otherwise} \end{cases}$

- Regularized mass function: $H_i^\varepsilon(\psi) = \displaystyle\int_X \mathcal{L}_{\varepsilon,i}[\psi](x) \, \mathrm{d}\,\mu(x) \xrightarrow[\varepsilon \to 0]{} H_i(\psi)$

Regularized GJE: Find $\psi \in \mathbb{R}^N$ such that $H^\varepsilon(\psi) = \nu$

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- Regularized mass function: $H_i^\varepsilon(\psi) = \displaystyle\int_X \mathcal{L}_{\varepsilon,i}[\psi](x) \,\mathrm{d}\, \mu(x) \xrightarrow[\varepsilon \to 0]{} H_i(\psi)$

Regularized GJE: Find $\psi \in \mathbb{R}^N$ such that $H^\varepsilon(\psi) = \nu$

Fixed point iterate: $\psi^{k+1} = \psi^k + \tau^k (H^\varepsilon(\psi) - \nu)$

A stochastic algorithm for GJE

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Regularized GJE: Find $\psi \in \mathbb{R}^N$ such that $H^\varepsilon(\psi) = \nu$

Stochastic fixed point iterate: $\psi^{k+1} = \psi^k + \tau^k(\mathcal{L}_\varepsilon[\psi](x_k) - \nu)$

where $x_k \sim \mu$ so that $\mathbb{E}(\mathcal{L}_\varepsilon[\psi](x_k)) = H^\varepsilon(\psi)$

- Stochastic gradient descent in the case of optimal transport.
- Numerical experiments converge for $\tau^k = \frac{1}{\sqrt{k}}$
- Proof of convergence is an open problem.