Reinforcement Learning: Tutorial 8

Off-policy temporal difference learning with approximation

Week 5
University of Amsterdam

Alejandro Garcia
February 2026

Outline

1. Off-policy TD learning with approximation exercises

2. Ask anything about HW3 or 7.4 (HW4)

Tutorial 8 Overview

1. Off-policy TD learning with approximation exercises
2. Ask anything about HW3 or 7.4 (HW4)

Tutorial 8 Overview

1. Off-policy TD learning with approximation exercises
   - Questions 7.1-7.3
2. Ask anything about HW3 or 7.4 (HW4)

Theory Intermezzo: Everything is called Bellman

1. Bellman equation for the value function v_π
   $v_\pi(s) = \Sigma_a \pi(a|s) \Sigma_{s',r} p(s', r|s, a)[r + \gamma v_\pi(s')]$
2. Bellman operator: plug in $v_{\mathbf{w}}$ instead of v_π
   $B_\pi v_{\mathbf{w}}(s) \doteq \Sigma_a \pi(a|s) \Sigma_{s',r} p(s', r|s, a)[r + \gamma v_{\mathbf{w}}(s')]$
3. Bellman error at state s (expectation of the TD error)
   $\overline{\delta}_{\mathbf{w}}(s) \doteq B_\pi v_{\mathbf{w}}(s) - v_{\mathbf{w}}(s)$
   $\rightarrow \quad = \mathbb{E}_\pi[R_{t+1} + \gamma v_{\mathbf{w}}(S_{t+1}) - v_{\mathbf{w}}(S_t)|S_t = s, A_t \sim \pi]$
4. Projected Bellman error at state s: $PBE_{\mathbf{w}}(s) = \Pi \overline{\delta}_{\mathbf{w}}(s)$
5. Bellman error vector: Bellman errors for all states in a vector: $\overline{\delta}_{\mathbf{w}}$
6. Mean squared Bellman error: weigh the norm of the vector by μ
   $\overline{BE}(\mathbf{w}) = ||\overline{\delta}_{\mathbf{w}}||^2_\mu$

Theory Intermezzo: All types of error

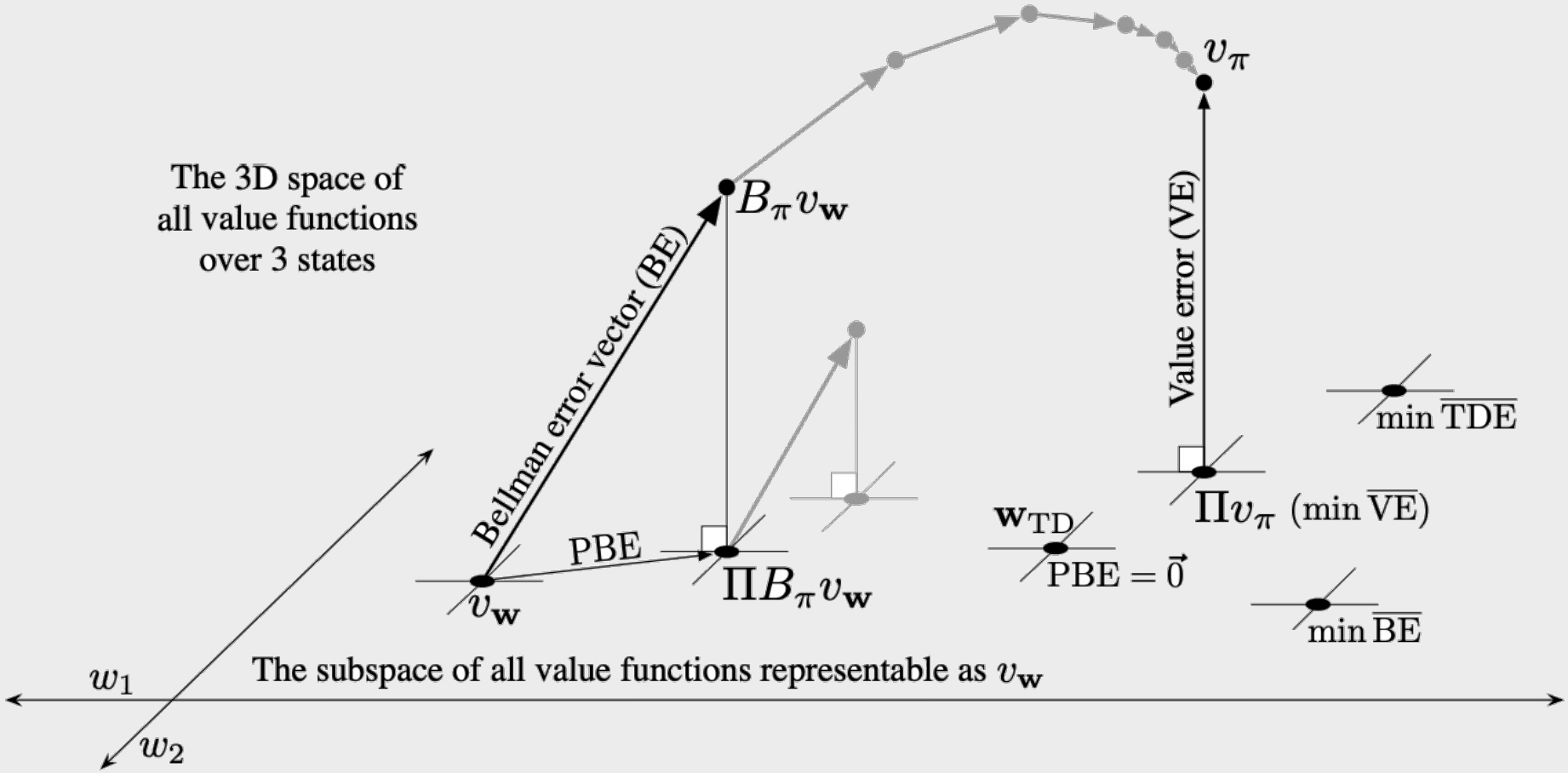
a $\overline{VE}(\mathbf{w}) \doteq ||VE||^2_\mu$; $VE = v_\pi(s) - \hat{v}(s, \mathbf{w})$

b $\overline{TDE}(\mathbf{w}) \doteq \mathbb{E}_b[\rho_t \delta_t^2]$

c $\overline{BE}(\mathbf{w}) \doteq ||\overline{\delta}_\mathbf{w}||^2_\mu$

d $\overline{PBE}(\mathbf{w}) \doteq ||\Pi\overline{\delta}_\mathbf{w}||^2_\mu$

Theory Intermezzo: Geometry of value functions



Q 7.1 Geometry of linear value-function approximation

1. Which error function is minimized by gradient Monte Carlo?

Q 7.1 Geometry of linear value-function approximation

1. Which error function is minimized by gradient Monte Carlo?

   Value error.

Q 7.1 Geometry of linear value-function approximation

2. The Bellman error is zero only when the value error is zero (recall the Bellman equations). Why then does minimizing a TD objective (such as the mean squared (projected) Bellman error) not in general result in minimal mean squared value error ($\overline{VE}$) in the function approximation setting?

Q 7.1 Geometry of linear value-function approximation

2. The Bellman error is zero only when the value error is zero (recall the Bellman equations). Why then does minimizing a TD objective (such as the mean squared (projected) Bellman error) not in general result in minimal mean squared value error ($\overline{VE}$) in the function approximation setting?

   If we could update the value function according to the Bellman error vector, we would indeed find the point where BE=VE=0. However, the Bellman error vector takes us out of the representable subspace (see fig 11.3 in RL:AI), so there is no way to update the weights accordingly. Instead, we can try to take gradients of the mean squared temporal difference error or the mean squared (projected) Bellman error, but these objectives have different solutions in general.

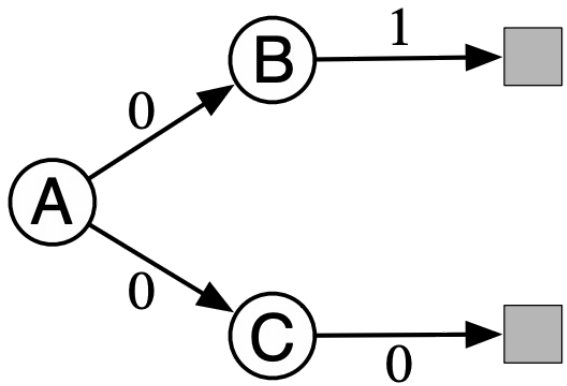
Q 7.1 Geometry of linear value-function approximation

3. Is applying (full) gradient descent on the TD error a good approach to approximate the value function? Motivate your answer.

Q 7.1 Geometry of linear value-function approximation

3. Is applying (full) gradient descent on the TD error a good approach to approximate the value function? Motivate your answer.

   If we do full gradient descent on the TD error, we update the weights considering the target value function. This means we also backpropagate the error of the target value to the weights. This can lead to strange situations, where the estimated value of a state depends on how you got there, rather than possible future trajectories from that state. (see example 11.2, p.271 RL:AI).



Q 7.2 *Exam question: Errors and function approximation

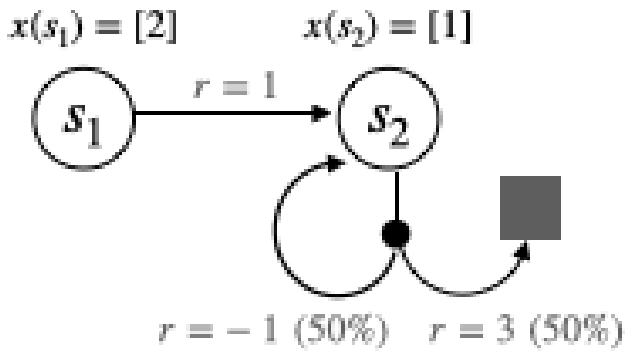
1. Indicate which of the following statements are true:
   - If the mean-squared value error is zero, the mean-squared Bellman error is zero.
   - If the mean-squared Bellman error is zero, then the mean-squared value error is zero.
   - Regardless of features, with linear function approximation there is always a $\hat{v}_w$ such that the MSBE is zero.
   - If the mean squared value error is zero, the mean squared TD error is zero.

Q 7.2 *Exam question: Errors and function approximation

1. Indicate which of the following statements are true:
   - If the mean-squared value error is zero, the mean-squared Bellman error is zero.
   - If the mean-squared Bellman error is zero, then the mean-squared value error is zero.
   - Regardless of features, with linear function approximation there is always a $\hat{v}_w$ such that the MSBE is zero.
   - If the mean squared value error is zero, the mean squared TD error is zero.

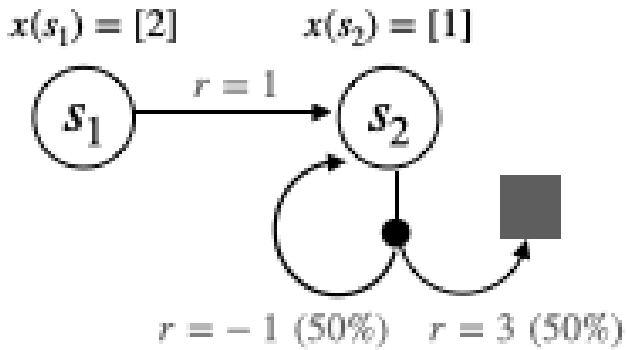
Items 1 and 2 are correct, 3 and 4 are incorrect.

Q 7.2 *Exam question: Errors and function approximation



2. Consider the MDP in the Figure above. With all trajectories starting in s_1, the on-policy distribution μ for the MDP shown is given by $\mu(s_1) = 1/3, \mu(s_2) = 2/3$. Briefly explain what this means and why these values are correct.

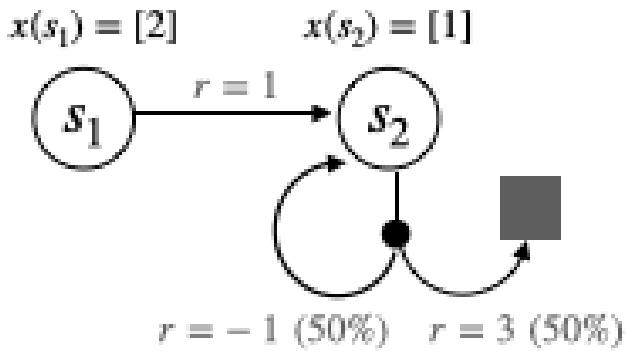
Q 7.2 *Exam question: Errors and function approximation



2. Consider the MDP in the Figure above. With all trajectories starting in s_1, the on-policy distribution μ for the MDP shown is given by $\mu(s_1) = 1/3, \mu(s_2) = 2/3$. Briefly explain what this means and why these values are correct.

In expectation, 1/3 of the time is spend in s1 and 2/3 of the time is spend in s2. Each episodes starts in s1 and stays there for 1 step. s2 will have at least 1 visit, or ≥ 2 visits with probability 1/2, or ≥ 3 visits with probability 1/4. We know that this sequence adds up to 2 visits on average. So that makes 1/3 steps in s1 and 2/3 steps in s2.

Q 7.2 *Exam question: Errors and function approximation



3. Consider the same MDP. What is the mean squared temporal difference error ($\overline{\text{TDE}}$) in the above example when $\mathbf{w} = [1]$?

Q 7.2 *Exam question: Errors and function approximation

The MSTD error is given by
$\sum_s \mu(s) \sum_a \pi(a|s) \sum_{s'} p(s'|s,a)\delta(s,a,s')^2$.
So, in this case we get (adding over the three kinds of transitions):

$$1/3 * \delta(s_1, a_1, s_2)^2 + 2/3 * 1/2 * \delta(s_2, a_1, s_2)^2 + 2/3 * 1/2 * \delta(s_2, a_1, T)^2$$

$\rightarrow$

$$\delta(s_1, a_1, s_2) = 1 + 1 - 2 = 0$$

$$\delta(s_2, a_1, s_2) = -1 + 1 - 1 = -1$$

$$\delta(s_2, a_1, T) = 3 + 0 - 1 = 2$$

Which gives a total MSTDE of: $1/3 * 0 + 1/3 * 1 + 1/3 * 4 = 5/3$

Q 7.3 *Exam Question: Function approximation

1. Consider two types of function approximation for scalar s:
   a) Using "Gaussian" radial basis features $\left(\phi_j(s) = \exp(-\frac{(s-\mu_j)^2}{2\lambda^2})\right)$ with λ the width of the kernel.
   b) Using polynomial features $\left(\phi_j(s) = s^j\right)$.

   Name one advantage of a) compared to b), and one advantage of b) compared to a). Assume the same number of features is used in both cases.

Q 7.3 *Exam Question: Function approximation

1. Consider two types of function approximation for scalar s:
   a) Using "Gaussian" radial basis features $\left(\phi_j(s) = \exp(-\frac{(s-\mu_j)^2}{2\lambda^2})\right)$ with λ the width of the kernel.
   b) Using polynomial features $\left(\phi_j(s) = s^j\right)$.

   Name one advantage of a) compared to b), and one advantage of b) compared to a). Assume the same number of features is used in both cases.

   Valid advantage RBF: it is local, no extrapolation.
   Valid advantage polynomial: it is global, so extrapolates more, does not require specification of RBF means or knowing the range of inputs.

Q 7.3 *Exam Question: Function approximation

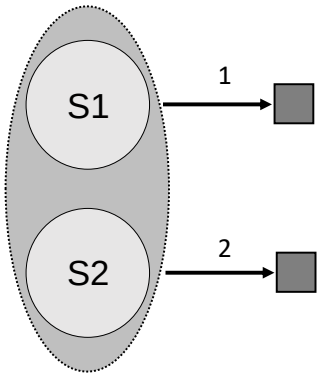
2. With linear function approximation, does gradient Monte Carlo (gradient MC) **always, sometimes, or never** converge to the same solution as semi-gradient TD(0)? Explain your answer.

Q 7.3 *Exam Question: Function approximation

2. With linear function approximation, does gradient Monte Carlo (gradient MC) **always, sometimes, or never** converge to the same solution as semi-gradient TD(0)? Explain your answer.

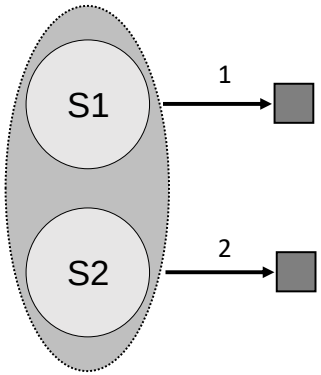
   Sometimes it converges to the same solution as semi-gradient TD(0). In tabular setting, they will find the same solution. But with function approximation, a different objective is optimized that generally has a different optimum.

Q 7.3 *Exam Question: Function approximation



3. Consider the simple MDP shown in Figure above. States S1 and S2 are indistinguishable (have the same features). Only a single action can be applied, that always ends the episode. The reward obtained is 1 or 2, respectively. Episodes start in S1 or S2 with equal probability.
   a) For the shown MDP, what is the minimal mean squared Bellman error? Why?

Q 7.3 *Exam Question: Function approximation



Both states need to be assigned the same value. The best compromise is average: V(S1) = V(S2) = 1.5.
Using linear value function approximation with a constant feature c for both states:

$$v_w = \mathbf{w}\, c$$

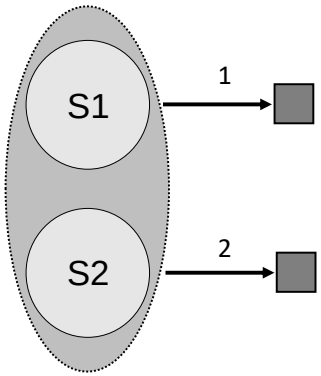
OLS solution $\hat{\mathbf{w}}$:

$$\hat{\mathbf{w}} = \frac{1}{c}\begin{bmatrix}1.5\\1.5\end{bmatrix} \Rightarrow v_{\hat{\mathbf{w}}} = \frac{1}{c}c\begin{bmatrix}1.5\\1.5\end{bmatrix} = \begin{bmatrix}1.5\\1.5\end{bmatrix}$$

s.t. minimal MSBE:

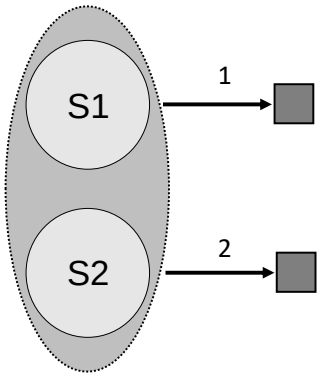
$$\overline{BE}(\hat{\mathbf{w}}) = ||\bar{\delta}_{\hat{\mathbf{w}}}||^2_\mu = (1+\gamma 0 - 1.5)^2 \cdot 0.5 + (2+\gamma 0 - 1.5)^2 \cdot 0.5 = 0.25$$

Q 7.3 *Exam Question: Function approximation



b) For the shown MDP, what is the minimal mean squared projected Bellman error? Why?

Q 7.3 *Exam Question: Function approximation



We are in the case of linear function approximation. Therefore we know that MSPBE = 0 at the TD fixed point, so the answer must be 0. Alternative: If we project the bellman errors [-0.5, 0.5] on a basis consisting of just the constant feature, we get projected errors of [0,0].


WELL DONE!

Tutorial 8 Overview

1. On-policy TD learning with approximation exercises
2. Ask anything about HW3 or 7.4 (HW4)
   - Questions 6.5, 7.4

That's it!

Good luck with the HW and see you the next day!