Group Theory Discussion - 1

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1 Group/Representation

Group G is a set and a binary operation "." $(G_i, .)$ such that:

(a) G is closed under the binary operation:

$$\forall g_1, g_2 \in G \Rightarrow g_1.g_2 = g_3 \in G$$

(b) "." operation is associative:

$$(g_1.g_2).g_3 = g_1.(g_2.g_3)$$

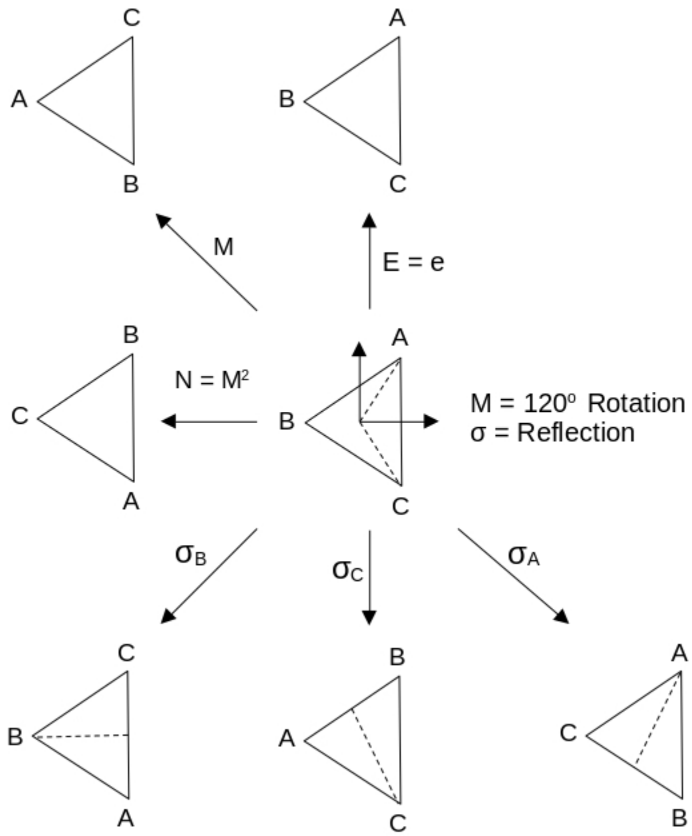
(c) $\forall g \in G$ there exists some elements in G; $\exists e \in G$ such that:

$$e.g = g.e = g$$

(d) There exists an unique inverse $\exists g^{-1}, \forall g$ such that:

$$g^{-1}.g = g.g^{-1} = e$$

It is customary in physics/engineering to think of G as a set of symmetry operations on a set V. Sometimes set V is called "realization of space".



Multiplication or Composition Table:

	E	M	N	σ_A	σ_B	σ_C
E	E	M	N	σ_A	σ_B	σ_C
N	N	E	M	σ_B	σ_C	σ_A
M	M	N	E	σ_C	σ_A	σ_B
σ_A	σ_A	σ_C	σ_B	E	M	N
σ_B	σ_B	σ_A	σ_C	N	E	M
σ_C	σ_C	σ_B	σ_A	M	N	E

Inverse of N is M as from table we can see M.N = N.M = E, similarly inverse of M is N and inverse of E is E itself as it is the identity of our group. Inverse of σ_A is σ_A, σ_B is σ_C and σ_C is σ_C. *[**Theorem:** Inverse elements are unique.]*

2 Subgroups:

If we have a H which is a proper subset of G ($H \subset G$) then, that itself is a group under the same operation, is a subgroup G.

- How to identify a subgroup?

$$\forall h_1, h_2 \in H \subseteq G, \text{ if } h_1.h_2^{-1} \in \text{H, then H is a subgroup.}$$

Every group has a trivial subgroup which is identity element. Identity element itself is a group. Inverse of identity is itself and if we compose identity with itself it will always give back identity itself.

3 Coset:

Pick a subgroup $H \subseteq G$. Consider, an element $g \in G$ but $g \notin H$. Then $g_1H, g_2H, ...$ are called left cosets of H. Likewise Hg_i are right cosets.

* Statements: Cosets are not subgroups.

$$\text{Let, } g \notin H.H = \{h_1, h_2, ...e\}$$

$$gH = \{gh_1, gh_2, ..., h_1\} \not\ni e$$

If there exists an identity in gH then g will be the inverse of an element in H but then that element should have belong to the set G. Hence, there is no identity element in H and it is not a subgroup.

* Cosets form a partition: If $g_1 \neq g_2$ then $(g_1H \cap g_2H) = \emptyset$ or $g_1 = g_2$ Suppose there is an element,

$$g_1h_1 \in g_1H = g_2h_2 \in g_2H$$

$$g_1 = g_2h_2h_1^{-1} = g_2h_3 \text{ [As } h_2h_1^{-1} \in H]$$

Therefore, g_1 is in the coset of $g_2h_3 \in g_2H \Rightarrow g_1 \in g_2H$. But originally g_1 and g_2 are not in the coset they are outside. That contradicts the initial assumption.

Example:

Let's take a simple sub-group $\Sigma_A = \{E, \sigma_A\}$.

Multiply Σ_A with any element of the group we have seen before.

$$M\Sigma_A = \{M, \sigma_C\}; \text{ [It is not a sub-group]}$$

Multiply Σ_A with another element from the group.

$$N\Sigma_A = \{N, \sigma_A\}; \text{ [It is also not a sub-group]}$$

We can see there is no intersection between $M\Sigma_A$ and $N\Sigma_A$. Order (numbers of element in each set) of each of these cosets are same as there is no intersection. Order of a particular sub-group is same as the order of its cosets.

Let there is a group G and H is a sub-group of G.

G				
H	g_1H	g_2H	. . .	g_mH

$n_G =$ number of elements in G
$n_H =$ number of elements in H

$n_G = n_H + n_{g_1H} + n_{g_2H} + ... + n_{g_mH}$

As, cosets form a partition.
$n_H = n_{g_1H} = ... = n_{g_mH}$

Hence, $n_G = Nn_H$; where N is an nonzero positive integer.
And $n_G = Nn_H \geq 1$ as each group has an identity element.

Therefore, $\frac{n_G}{n_H} = integer$ [Lagrange's theorem]

4 Normal Subgroup

If left coset is same as right coset ($gH = Hg \Rightarrow gHg^{-1} = H$) then H is a normal subgroup.

For normal subgroups, cosets also form a group, which is called "Quotient Subgroup" (G/H).

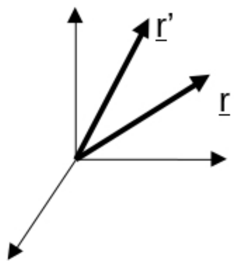
Let assume gH and g'H are cosets. Now if we take any two elements:

$$g_1Hg_2H = g_1g_2HH = g_3H \text{ (As left and right cosets are same)}$$

$HH = \{e, h_1\}.\{e, h_1\} = \{e.e, e.h_1, h_1.e, h_1.h_1\} = \{e, h_1, h_1, e\} = \{e, h_1\}$

Product of two coset for a normal subgroup is also another coset. Therefor cosets forms a group for normal subgroups.

5 Rotation



Let, we have a vector $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Under rotation the length of vector remains same. Which means under rotation $r^T r = (length)^2$ is a liner transformation such-that $(length)^2$ is invariant. Rotation maps origin int origin and the length of the vector.

$$\vec{r'} = R\vec{r}$$

If we want the $(length)^2$ of vector to be invariant then we get:

$$\vec{r'}^T \vec{r'} = \vec{r}^T R^T R\vec{r}$$

Therefore $R^T R = I$, hence rotational matrices are orthogonal matrices.

5.1 Rotation matrices form a group

Let's take product of two rotational matrix R_1 and R_2

$$R_1 R_2 = R_3$$

To prove if R_3 is a rotation matrix, we have to show that $R_3^T R_3 = I$

$$R_3^T = R_2^T R_1 T$$

$$\text{Now, } R_3^T R_3 = R_2^T R_1 T R_1 R_2 = R_2^T I R_2 = R_2^T R_2 = I$$

6 Translation

$\vec{r'} = R\vec{r} + \vec{a}$ this set also forms a group.

Check for closure property:

$$\vec{r"} = R'\vec{r'} + \vec{a'} = R'(R\vec{r} + \vec{a}) + \vec{a'} = R'R\vec{r} + (R'\vec{a} + \vec{a'})$$

We can write the operation $\vec{r'} = R\vec{r} + \vec{a}$ as $\vec{r'} = (R, \vec{a})\vec{r}$

The composition rule is $(R', \vec{a'})(R, \vec{a}) = (R'R, R'\vec{a} + \vec{a'})$

Check for inverse:

$$(R, \vec{a})^{-1} = (R^T, -R^T\vec{a})$$

There are two sub-groups of this:
1. $(R, 0)$ = rotation
2. $(I, \vec{a})$ = translation

*H.W. $\Rightarrow$ Which one is a normal subgroup?

Let, $\vec{r'} = R\vec{r} + \vec{a}$ the first part of this operation $(R\vec{r})$ is multiplicative and second part $(\vec{a})$ is additive. We can do this in a single operation if we do the following:

$$\begin{bmatrix} \vec{r'} \\ 1 \end{bmatrix} = \begin{bmatrix} R & & \vec{a} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{r} \\ 1 \end{bmatrix} = \begin{bmatrix} R\vec{r} + \vec{a} \\ 1 \end{bmatrix}$$

[Rotation in 3D + Translation]