CSE 250A - Hwk 3 Andrew Ghafari A59020215

Prob 1

a) When $t=1$ — Proving $P(X_{t+1}=j \mid X_1=i)$ starting @ $t=1$.

$P(X_2=j \mid X_1=i) = [A^1]_{ij} = A_{ij}$

Assume $t=t$. we have $P(X_{t+1}=j \mid X_1=i) = [A^t]_{ij}$

Now we prove for $t+1$:

$$P(X_{t+2}=j \mid X_1=i) \overset{\text{marginalization}}{=} \sum_x P(X_{t+2}=j, X_{t+1}=x \mid X_1=i)$$

$$= \sum_x \underbrace{P(X_{t+2}=j \mid X_1=i, X_{t+1}=x)}_{① \text{ product rule}} \cdot \underbrace{P(X_{t+1}=x \mid X_1=i)}_{[A^t]_{ij}}$$

for (1): Given X_{t+1}, X_1 will become conditionally (d-sep) indep of X_{t+2} so (1) reduces to

$P(X_{t+2}=j \mid X_{t+1}=x) = A_{ij}$ (Given base formula)

So we get $[A]_{ij} \times [A^t]_{ij} = [A^{t+1}]_{ij}$

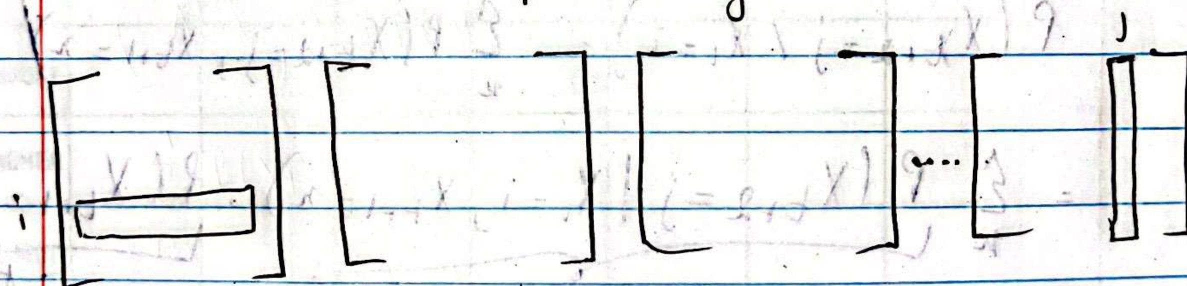
$\Rightarrow$ proved.

While summing over all possible values of k, we get $\sum_k [A^t]_{ik} \cdot [A]_{kj}$ → which is basically $[A^{t+1}]_{ij}$ since I am multiplying.

$A^t_{i1} \cdot A_{1j} + A^t_{i2} \cdot A_{2j} \cdots$ which is

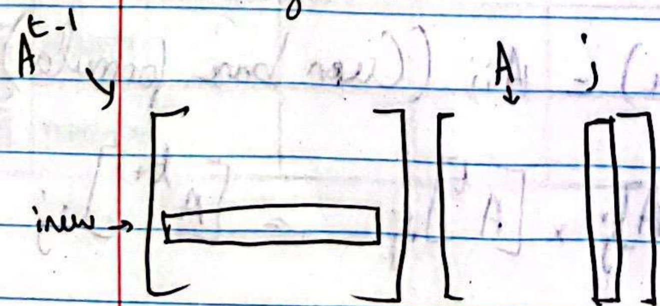
$[A^t \cdot A]_{ij}$ which is $[A^{t+1}]_{ij}$ Done.

b) To get the i,j entry of A^t → what we need to do is the following



so we need to multiply $\underbrace{(1 \times m) \times (m \times m)}_{\text{to get this row again}}$ $(t-2)$ times

will get here.



we multiply → $(1 \times m) \cdot (m \times 1)$ to get $[A^t]_{ij}$

Scanned with CamScanner

So it is

$$(1\times m)\,(m\times m)\cdot(t-2) + (1\times m)\cdot(m\times 1)$$

$$= m^2(t-2)+m$$

$$= O(m^2 t). \text{ Proved.}$$

c) $$[A^t]_{ij} = \overset{A}{[\ \ ]}\cdot\overset{A}{[\ \ ]}\cdot\overset{A}{[\ \ ]}\cdot\overset{A}{[\ \ ]}\cdots\overset{A}{[\ \ ]}$$

$$= {}^{O(m^3)}A^{2}\cdot A^{2\,O(1)}\cdot A^{2\,O(1)}\cdot A^{2\,O(1)}\cdot A^{2\,O(1)} \cdots$$

$$= {}^{O(m^3)}A^{4}\cdot A^{4\,O(1)}\cdot A^{4\,O(1)}\cdot A^{4\,O(1)}$$

$$= {}^{O(m^3)}A^{8}\cdot A^{8\,O(1)} = {}^{(m^3)}A^{16}$$

supposing $t = 16$ we need ($\log_2 16$) rounds of matrix-matrix multiplication ($O(m^3)$ itself) (for $m\times m$ matrices) to get to our answer. (Because we have to do A^2 (once), A^4 (once), A^8 (once) ....)

So for t we need $O(m^3\log_2 t)$. Proved.

d) Basing ourselves from answer in (b).
We know that it is in $O(m^2 t)$.
Here, we need $[(1\times m)\cdot(m\times s)]\,(t-2)$ times $+ (1\times m)\cdot(m\times 1)$

$$O((t-2)ms + m)$$ Because all others are zeros.

$$= O(tms). \text{ Proved.}$$

c) ~~$P(X_1 = $~~

$$P(X_1 = i \mid X_{t+1} = j)$$

Bayes $$= \frac{P(X_{t+1} = j \mid X_1 = i) \cdot P(X_1 = i)}{P(X_{t+1} = j)}$$ Bayes

part a

$$A = \frac{[A^t]_{ij} \cdot P(X_1 = i)}{\sum_n P(X_{t+1} = j, X_1 = n)}$$ Marginalization

marginalization

$$= \frac{[A^t]_{ij} \cdot P(X_1 = i)}{\sum_n \underbrace{P(X_{t+1} = j \mid X_1 = n)}_{[A^t]_{nj}} \cdot P(X_1 = n)}$$ Product rule

$$= \frac{[A^t]_{ij} \cdot P(X_1 = i)}{\sum_n [A^t]_{nj} \cdot P(X_1 = n)}$$ **Done!**

Prob 3.2

a) $P(Y_1 / X_1) = \sum_{x_0} P(Y_1, X_0 / X_1)$

Since $E = \{\phi\}$, $P(X_0 | X_1) = P(X_0)$ d_{iii}

$= \sum_{X_0} P(Y_1 | X_0, X_1) \cdot P(X_0 | X_1)$ marginalization, product rule

$= \sum_{X_0} P(Y_1 | X_0, X_1) \cdot P(X_0)$

b) $P(Y_1) = \sum_{X_0} \sum_{X_1} P(Y_1, X_1, X_0)$ marginalization

product rule

$= \sum_{X_0} \sum_{X_1} P(Y_1 | X_0, X_1) \cdot P(X_0, X_1)$ X_0 indep. of X_1

$= \sum_{X_0} \sum_{X_1} P(Y_1 | X_0, X_1) \cdot P(X_0) \cdot P(X_1)$ $E = \{\phi\}$. d_{iii}

c) $P(X_n | Y_1, Y_2, — Y_{n-1}) = P(X_n)$.

all paths from X_n to $\{Y_i\}$ where $i \in [1 : n-1]$ are blocked by Y_n which is not in evidence set

→ case 3 - d_{iii}

d) $P(Y_n | X_n, Y_1 — Y_{n-1}) = \sum_{X_{n-1}} P(Y_n, X_{n-1} / X_n, Y_1 — Y_{n-1})$ marginalization

$= \sum P(Y_n / X_{n-1}, X_n, Y_1 — Y_{n-1}) \cdot P(X_{n-1} / X_n, Y_1 — Y_{n-1})$ product rule

$= \sum P(Y_n / X_n, X_{n-1}) \cdot P(X_{n-1} / Y_1 — Y_{n-1})$

$Y_n \perp\!\!\!\perp \{Y_1 — Y_{n-1}\}$ Given X_{n-1} d_{ii} | d_{iii} $\{X_{n-1} \perp\!\!\!\perp X_n$ when Y_n not in evidence

$$= \sum_{X_{n-1}} \underbrace{P(Y_n \mid X_n, X_{n-1})}_{CPT} \cdot \underbrace{P(X_{n-1} \mid Y_1 — Y_{n-1})}_{\text{required}}.$$

e) $P(Y_n \mid Y_1, — Y_{n-1})$

$$= \sum_{X_n} \sum_{X_{n-1}} P(Y_n, X_n, X_{n-1} \mid Y_1 — Y_{n-1})$$ marginalization & product rule

$$= \sum \sum P(Y_n \overset{①}{\mid} X_n, X_{n-1}, Y_1 — Y_{n-1})$$
$$\overset{②}{\times} P(X_n, X_{n-1} \mid Y_1 — Y_{n-1})$$

①: $= P(Y_n \mid X_n, X_{n-1})$ (explained before)

product rule (2) $= P(X_n \mid \cancel{\quad} Y_1 — Y_{n-1}) \times \underbrace{P(X_{n-1} \mid Y_1 — Y_{n-1})}_{\text{given}}$

$= \cancel{P(X_n \mid X_{n-1})}$

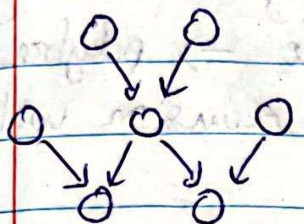
since X_n & X_{n-1} $\perp\!\!\!\perp$ given all Y_i for $1 \to n-1$ (cause Y_n not in evidence. d[illegible])

$$= \sum_{X_n} \sum_{X_{n-1}} \underbrace{P(Y_n \mid X_n, X_{n-1})}_{CPT} \times \underbrace{P(X_n \mid Y_1 — Y_{n-1})}_{(c)} \cdot \underbrace{P(X_{n-1} \mid Y_1 — Y_{n-1})}_{\text{given}}$$

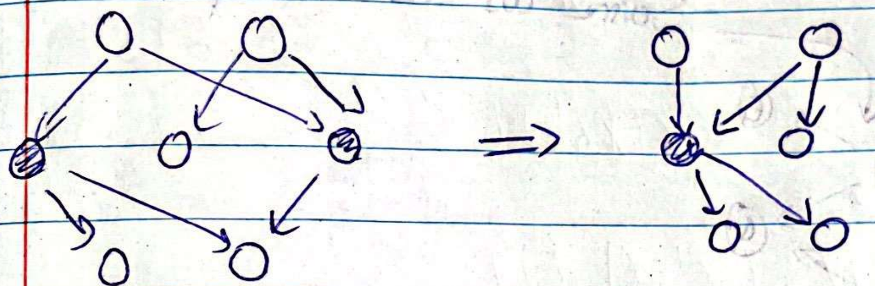
$= P(X_n)$

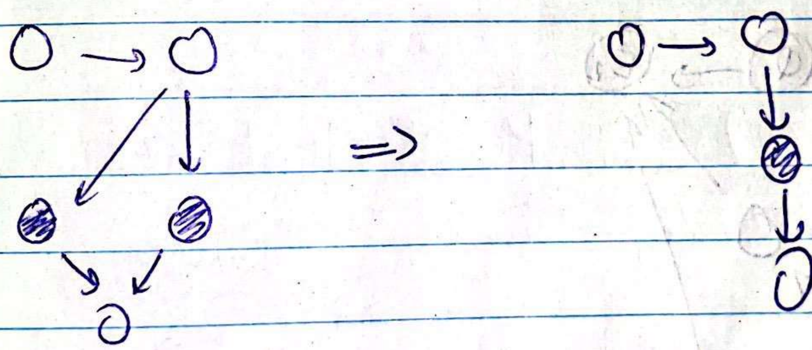
$$= \sum_{X_n} \sum_{X_{n-1}} P(Y_n \mid X_n, X_{n-1}) \cdot P(X_n) \cdot P(X_{n-1} \mid Y_1 — Y_{n-1}).$$

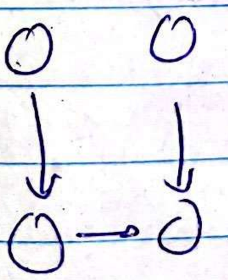
Prob 3.3



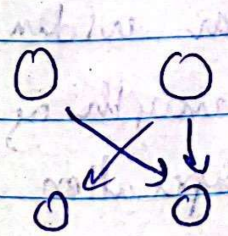
: polytree. no loops/cycles







: polytree : no loops / cycles.

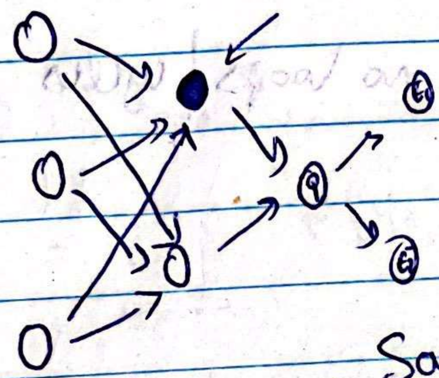


: polytree : no loops / cycles

Problem 3.4

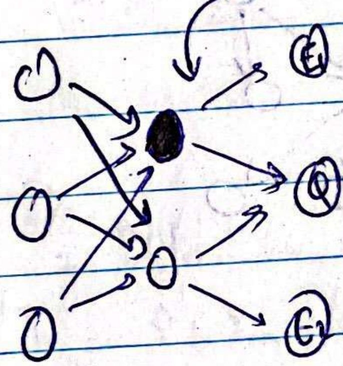
1)

we remove this node by instantiating it as evidence => polytree => recursion will work

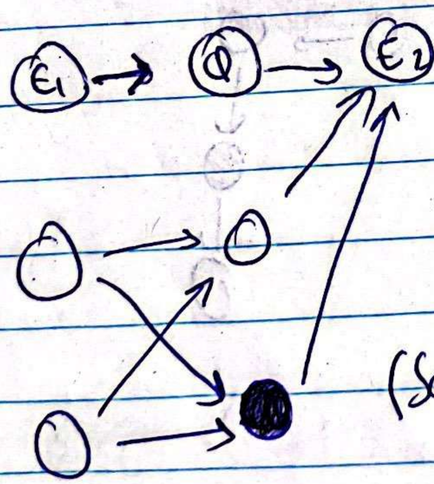


Same as above (explanation)

2)

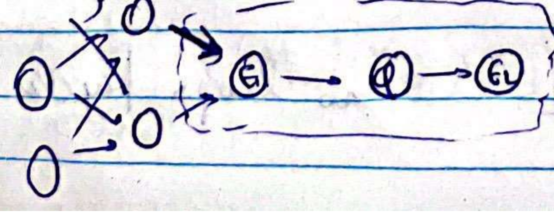


3)



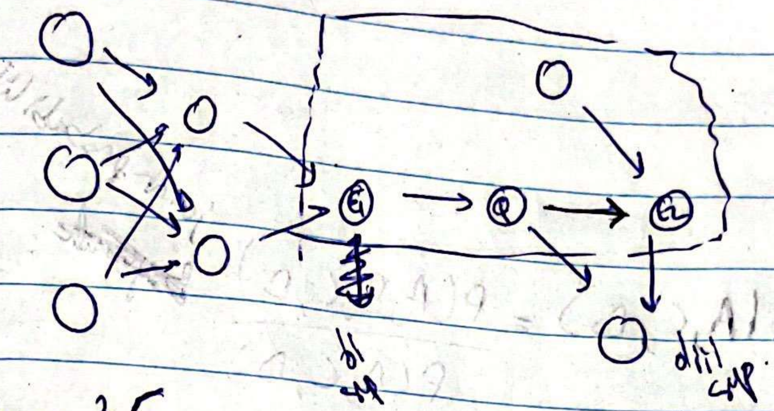
(Same exp as above)

4)



E_1 is an evidence blocks everything by d-separation.

5)



Problem 3.5

$$P(Z \mid B_1, B_2 \ldots B_n) = \left(\frac{1-\alpha}{1+\alpha}\right) \alpha^{|z - f(B)|} \qquad 0 < \alpha < 1$$

$$f(B) = \sum_{i}^{n} 2^{i-1} B_i \qquad P(B_i = 1) = \tfrac{1}{2}.$$

a) $\sum_{z=-\infty}^{\infty} P(Z = z \mid B_1 \ldots B_n)$

$$= \sum_{z=-\infty}^{\infty} \left(\frac{1-\alpha}{1+\alpha}\right) \alpha^{|z - f(B)|}$$

let $t = z - f(B)$, $z = t + f(B)$; if $z \to \infty$, $t \to \infty$

$$= \frac{1-\alpha}{1+\alpha} \sum_{t=-\infty}^{\infty} \alpha^{|t|}$$

$$= \frac{1-\alpha}{1+\alpha} \cdot 2 \sum_{t=0}^{\infty} \alpha^{t}$$

$$= \frac{1-\alpha}{1+\alpha} \cdot \left(2 \sum_{t=1}^{\infty} \alpha^{t} + \underbrace{\alpha^{t=0}}_{1}\right)$$

$$= \frac{1-\alpha}{1+\alpha} \left(\frac{2\alpha + 1 - \alpha}{1-\alpha}\right) = \frac{1-\alpha}{1+\alpha} \times \frac{1+\alpha}{1-\alpha} = \underline{\underline{1}}$$

Geometric series

$$\triangle \quad \sum_{t=0}^{\infty} \alpha^{t} = \frac{1}{1-\alpha}$$

$$\sum_{t=1}^{\infty} \alpha^{t} = \frac{1}{1-\alpha} - 1 = \frac{1 - 1 + \alpha}{1-\alpha} = \frac{\alpha}{1-\alpha}$$

Problem 5:

b) Now $n = 10$ bits & $\alpha = 0.1$.

$P(B_i = 1 \mid Z = 128) =$

$$\frac{\sum_{k=1}^{N} \underbrace{I(b_i^k, 1)}_{\text{indicator function}} \times \overbrace{P(Z=128 \mid B_1 = b_1^k, B_2 = b_2^k \ldots B_{10} = b_{10}^k)}^{\text{given by formula.}}}{\sum_{k=1}^{N} P(Z=128 \mid B_1 = b_1^k, B_2 = b_2^k \ldots B_{10} = b_{10}^k)}$$

$$= \frac{\sum_{k=1}^{N} I(b_i^k, 1) \times \left(\frac{1-\alpha}{1+\alpha}\right) \alpha^{\left|128 - \sum_{i=1}^{n=10} 2^{i-1} B_{ik}\right|}}{\sum_{k=1}^{N} \left(\frac{1-\alpha}{1+\alpha}\right) \alpha^{\left|128 - \sum_{i=1}^{10} 2^{i-1} B_{ik}\right|}}$$

$\frac{1-\alpha}{1+\alpha}$ can be taken out of sum in numerator & denominator.

$\Rightarrow$ we get:

$$\frac{\sum_{k=1}^{N} I(b_i^k, 1) \cdot \alpha^{\left|128 - \sum_{i=1}^{10} 2^{i-1} B_{ik}\right|}}{\sum_{k=1}^{N} \alpha^{\left|128 - \sum_{i=1}^{10} 2^{i-1} B_{ik}\right|}}$$

$\alpha = 0.1$

$$\Rightarrow \frac{\sum_{k=1}^{N} I(b_i^k, 1) \times 0.1^{|128 - \sum_{i=1}^{10} 2^{i-1} B_{ik}|}}{\sum_{k=1}^{N} 0.1^{|128 - \sum_{i=1}^{10} 2^{i-1} B_{ik}|}}$$

After running code we get:

for

	$P(B_i = 1 \mid Z = 128)$
$i = 2$	0.096599
$i = 5$	0.0394734.
$i = 8$	0.90845
$i = 10$.	0.0

~~b)~~

3.6)

a) $P(B|A,C,D) = \dfrac{P(A,B,C,D)}{P(A,C,D)}$ joint probability ~~[illegible]~~

$= \dfrac{P(D|B,C,A) \cdot P(B,C,A)}{P(A,C,D)}$ product rule

d; sep $D \perp A \; E = \{B\}$ $\leftarrow$ $= \dfrac{P(D|B,C) \times P(B,C|A) \cdot \cancel{P(A)}}{P(C,D|A) \cdot \cancel{P(A)}}$

$= \dfrac{\overset{(1)}{P(D|BC)} \cdot \overset{(2)}{P(B,C|A)}}{{}^{(3)}P(C,D|A)}$

$B \perp C \cdot E = \{A\}$
d iii $\leftarrow$ $= \dfrac{P(D|B,C) \cdot P(B|A) \cdot P(C|A)}{P(C,D|A)}$

$P(C|A) = P(C)$ $= \dfrac{\overset{(1)}{P(D|B,C)} \cdot \overset{(2)}{P(B|A)} \cdot \overset{(3)}{P(C)}}{{}^{(4)}P(C,D|A)}$

(1): CPT
(2) CPT
(3): Given.

4) $P(C,D|A) = \sum_{B} P(B,C,D|A)$ marginalization

$$= \sum_B P(D|B,C,A) \cdot P(B,C|A) \quad \text{product rule}$$

$$= \sum_B P(D|BC) \times P(B|A) \cdot P(C)$$

$$\Rightarrow \underline{(a)} \Rightarrow P(B|A,C,D) = \frac{P(D|B,C) \cdot P(B|A) \times \cancel{P(C)}}{\cancel{P(C)} \cdot \sum_B P(D/B,C) \cdot P(B/A)}$$

$$= \frac{P(D/B,C) \cdot P(B/A)}{\sum_B P(D/B,C) \cdot P(B/A)}$$

b) $P(B|A,C,D,E,F) = P(B|A,C,D)$ $\quad B \perp\!\!\!\perp \{E,F\}$ given $\{A,C,D\}$ d.i.

$$= \frac{P(D|B,C) \times P(B|A)}{\sum_B P(D|B,C) \cdot P(B|A)}$$

c) $P(B,E,F|A,C,D) = \underbrace{P(B|A,C,D,E,F)}_{(1)} \times \underbrace{P(E,F|A,C,D)}_{(2)}$

(1) is (b)

(2) $P(E,F|A,C,D) = P(F|E,A,C,D) \times P(E|A,C,D)$

$$= P(F|A) \times P(E|C)$$

$\{E,C,D\} \perp\!\!\!\perp F$ given A d.i.

$\{A,D\} \perp\!\!\!\perp E$ given C (d.i.)

(C) becomes:

CPTs

$$P(B,E,F \mid A,C,D) = \frac{\overbrace{P(D \mid B,C) \cdot P(B \mid A)}^{(a)} \times P(F \mid A) \times P(E \mid C)}{\sum_{B} P(D \mid B,C) \cdot P(B \mid A)}$$

Problem 3.7

indicator functions

a) $$P(Q=q \mid E=e) = \frac{\sum_{i=1}^{N} I(Q,q) \times P(E=e \mid Y=y_i, Z=z_i)}{\underbrace{\sum_{i=1}^{N} P(E=e \mid Y=y_i, Z=z_i)}_{\text{CPTs}}}$$

b) $$P(Q_1=q_1, Q_2=q_2 \mid E_1=e_1, E_2=e_2)$$

$$= \frac{\sum_{t=1}^{N} I(Q_1,q_1) \cdot I(Q_2,q_2) \cdot \overbrace{P(E_1=e_1, E_2=e_2 \mid Q_1=q_1, X=x_t, Z=z_t)}^{(a)}}{\sum_{i=1}^{N} \underbrace{P(E_1=e_1, E_2=e_2 \mid Q_1=q_1, X=x_t, Z=z_t)}_{(a)}}$$

$$(a) = P(E_2=e_2 \mid E_1=e_1, Q_1=q_1, X=x_t, Z=z_t)$$

$$\times P(E_1=e_1 \mid Q_1=q_1, X=x_t, Z=z_t)$$

- $E_2 \perp\!\!\!\perp \{Q_1, X\}$ Given $\{E_1, Z\}$ by case d_i on E_1 and Z.

- $E_1 \perp\!\!\!\perp Z$ Given $\{X, Q_1\}$ by case d_{ii} on X
d_{iii} on E_2

$\Rightarrow$ we get:

$$P(Q_1 = q_1, Q_2 = q_2 \mid E_1 = e_1, E_2 = e_2)$$

$$= \sum_{t=1}^{N} I(Q_1, q_{1t}) \cdot I(Q_2, q_{2t}) \cdot P(E_2 = e_2 \mid E_1 = e_1, Z = z_t)$$

this also in sum $\rightarrow$ $\times P(E_1 = e_1 \mid X = x_t, Q_1 = q_1)$

$$\sum_{t=1}^{N} P(E_2 = e_2 \mid E_1 = e_1, Z = z_t) \cdot P(E_1 = e_1 \mid X = x_t, Q_1 = q_1)$$

In cleaner fashion :

$\Rightarrow$ $P(Q_1 = q_1, Q_2 = q_2 \mid E_1 = e_1, E_2 = e_2) =$

$$\sum_{t} I(Q_1, q_{1t}) \cdot I(Q_2, q_{2t}) \cdot P(E_2 = e_2 \mid E_1 = e_1, Z = z_t) \cdot P(E_1 = e_1 \mid X = x_t, Q_1 = q_1)$$

$$\sum_{t} P(E_2 = e_2 \mid E_1 = e_1, Z = z_t) \cdot P(E_1 = e_1 \mid X = x_t, Q_1 = q_1)$$

Scanned with CamScanner

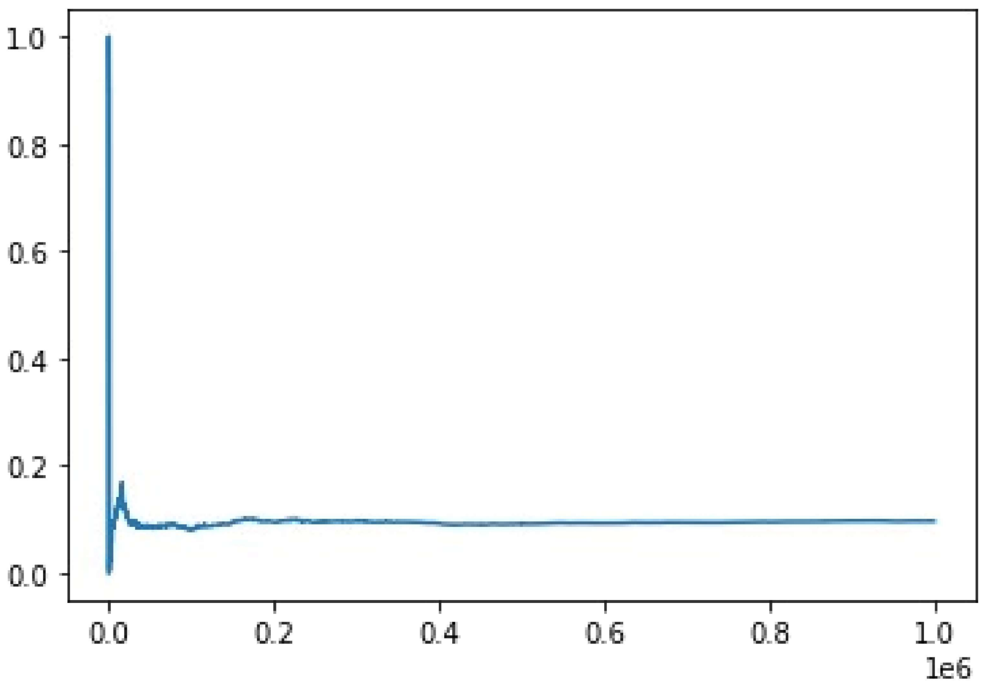
```
alpha = 0.1 # alpha
N = 1000000 #number of samples
K  = 10 #
import random as rd

i2 = []
i5 = []
i8= []
i10 = []
Nar = []
for i in range(1,N+1):
  Nar.append(i)
print(Nar)
i = [2,5,8,10]
```

```
IOPub data rate exceeded.
The notebook server will temporarily stop sending output
to the client in order to avoid crashing it.
To change this limit, set the config variable
`--NotebookApp.iopub_data_rate_limit`.

Current values:
NotebookApp.iopub_data_rate_limit=1000000.0 (bytes/sec)
NotebookApp.rate_limit_window=3.0 (secs)
```

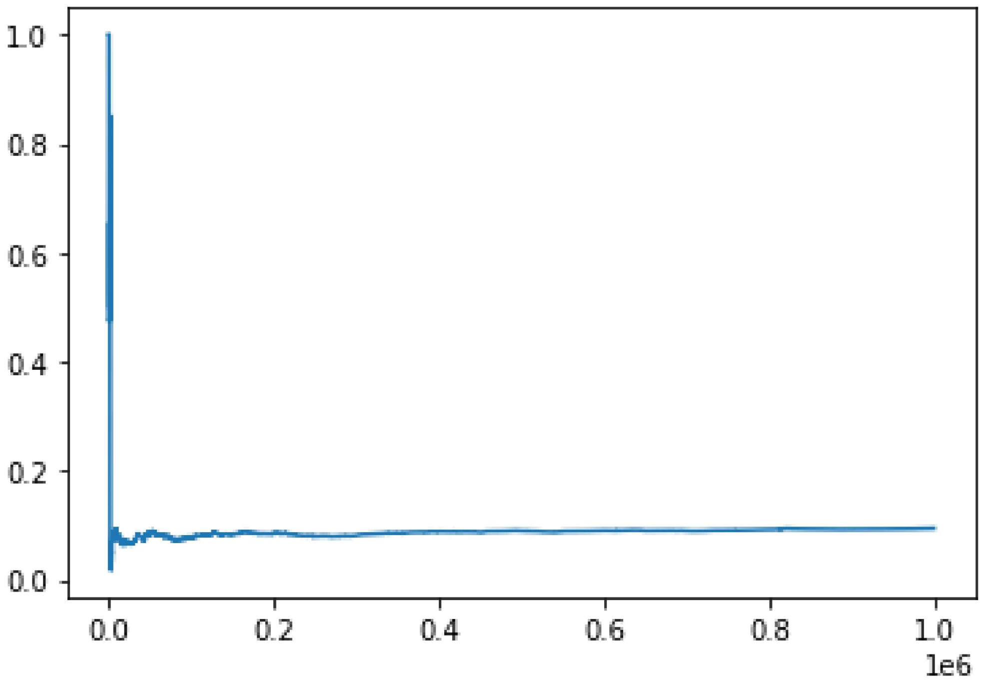
```
i2 = []
i5 = []
i8= []
i10 = []
for elem in i:
  num = 0.0
  denum = 0.0
  for k in range(1,N+1):
    randombit = []
    for j in range(K):
      randombit.append(rd.randint(0,1))
    Ind = randombit[elem-1]
    total = 0
    for j in range(K):
      tmp = pow(2,(j))*randombit[j]
      total += tmp
    total = 128-total
    total = pow(0.1,abs(total))
    denum += total
    total2 = Ind*total
    num += total2
    if(denum == 0):
      if (k !=1 ):
```

```
        eval("i"+str(elem)).append(eval("i"+str(elem))[-1])
      else:
        eval("i"+str(elem)).append(0)
    else:
      eval("i"+str(elem)).append(num/denum)

from matplotlib import pyplot as plt
for k in i:
  print("i = " + str(k))
  plt.plot(Nar,eval("i"+str(k)))
  print("Probability is" , eval("i"+str(k))[-1])
  plt.show()
```

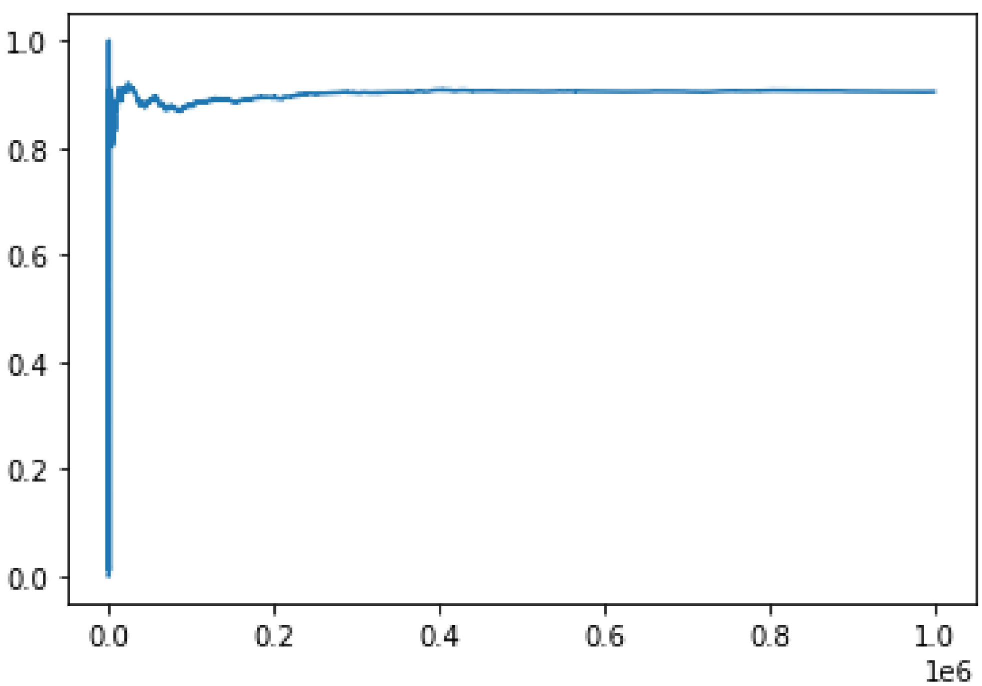
```
i = 2
Probability is 0.09659953725899915
```



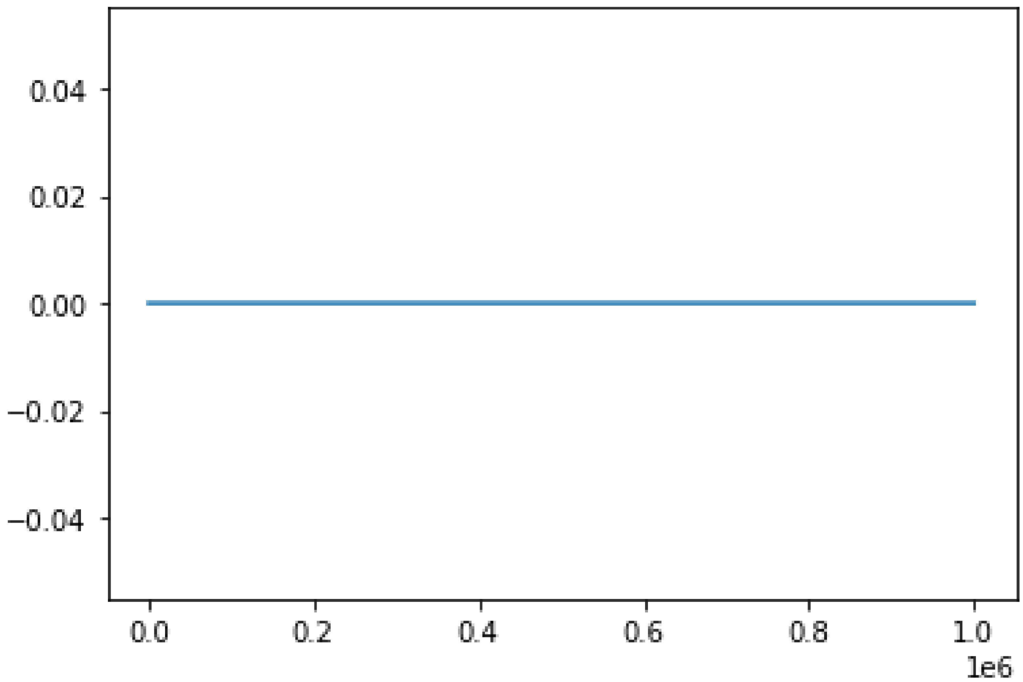

```
i = 5
Probability is 0.09473495765444159
```




```
i = 8
Probability is 0.9045313112953837
```




```
i = 10
Probability is 0.0
```


0.04
0.02
0.00
−0.02
−0.04
0.0
0.2
0.4
0.6
0.8
1.0
1e6