Control Principles for Engineered Systems 5SMC0

Networked Control System

Lecture 2

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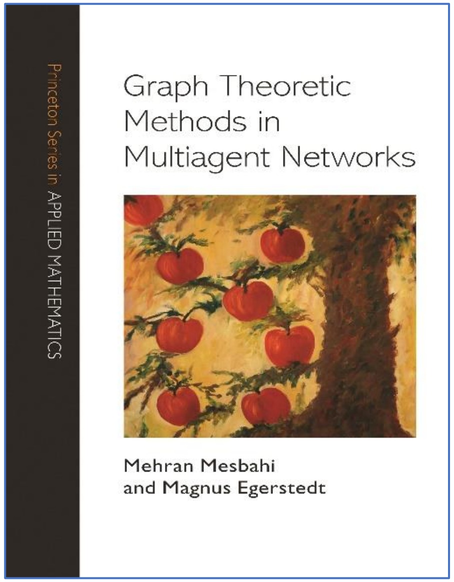
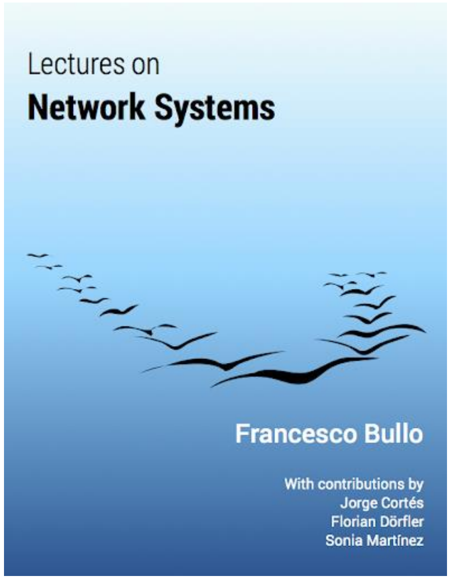
Networked Control System

Lecture 2
Multi-agent Systems & Consensus Seeking



Chapters 3-4



Chapters 6-8

Outline

- Consensus dynamics in *undirected* graphs
- Consensus dynamics in *directed* graphs
- From single-integrator to double-integrator dynamics

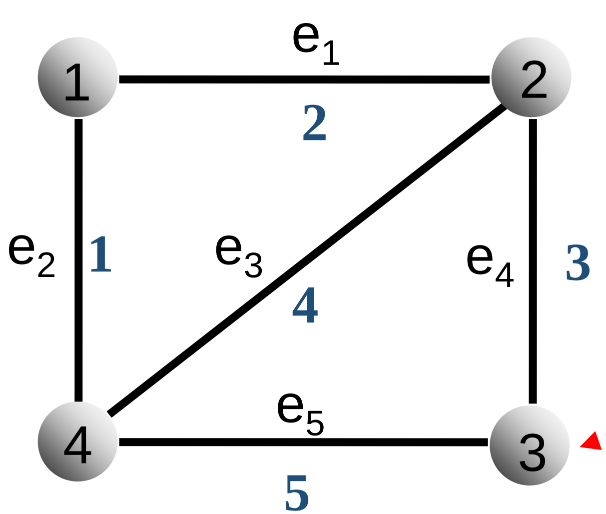
- Application of consensus seeking
- Synchronization in complex networks

Consensus dynamics in undirected graphs

- *Laplacian matrix* of an undirected graph:

$$L = D - A = H^{\mathrm{T}} \, W \, H$$

  - D_{ii} : sum of the weights of all the edges associated with i
  - A : adjacency matrix
  - H : incidence matrix
  - W : diagonal weight matrix
  - W_{ii} : edge weight of e_i



Undirected graph *G*

Node 1 ~ 4

Node 1 ~ 4	1	2	3	4
1	**3**	-2	0	-1
2	-2	**9**	-3	-4
3	0	-3	**8**	-5
4	-1	-4	-5	**10**

L is symmetric,
has zero row sums
and zero column sums

Multi-agent consensus protocol

Formulation of the consensus problem

- Consider *N* agents with an internal state $x_i \in R^d$
  - For now, we consider the scalar state case (d=1). Multi-dimensional state case can be modelled via Kronecker products.

- Consider an internal dynamics for the state evolution
  - For now, we consider single integrator dynamics $\dot{x}_i = u_i$

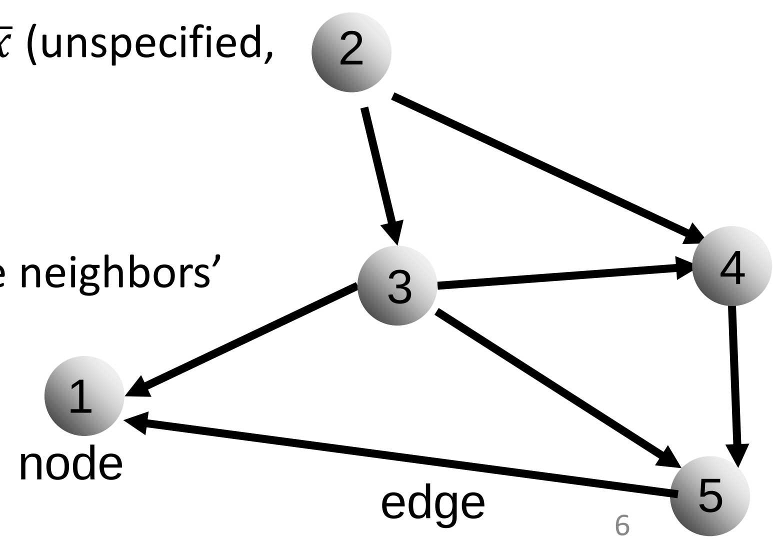
- Consider an interaction graph G,
  - Having the agents as nodes, and sensing/communication links as edges

Problem: design the control inputs u_i so that

- All the states x_i agree on the same common value $\bar{x}$ (unspecified, e.g., average value)

$$\lim_{t \to \infty} x_i(t) = \bar{x}, \ \forall i$$

- By making use of only relative information w.r.t. the neighbors' state (relative sensing and decentralization)



Recap: Graph Laplacian (undirected graphs): Rank, λ_2, and connectivity

Rank condition:

rank L(G) = $N - 1$ if and only if G is connected.

Equivalently (eigenvalue condition)

G is connected if and only if $\lambda_2(L(\mathrm{G})) > 0$.

The eigenvalue λ_2 captures a quantitative notion of graph connectivity: more positive λ_2 means the graph is more connected.

Multi-agent consensus dynamics

$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix} = -\frac{1}{2}\underbrace{\boxed{\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}}}_{L}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}$$

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\\ \dot x_5\end{bmatrix} = -\underbrace{\boxed{\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ -1 & 3 & -1 & 0 & -1\\ 0 & -1 & 3 & -1 & -1\\ 0 & 0 & -1 & 2 & -1\\ 0 & -1 & -1 & -1 & 3\end{bmatrix}}}_{L}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}$$

L: Laplacian matrix of the underlying network

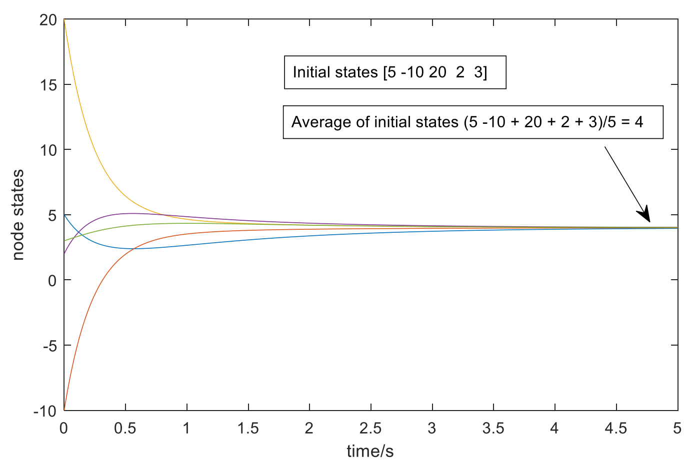
Multi-agent consensus: undirected graphs

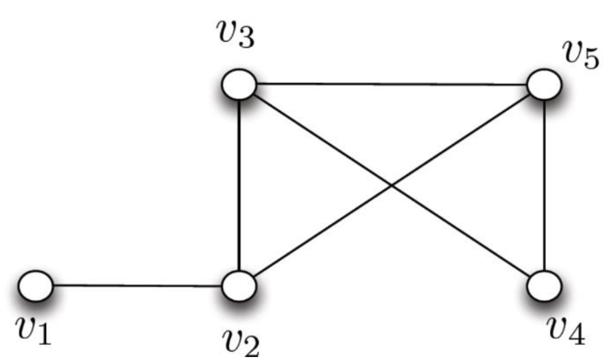
Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$

$$\boxed{\dot{x}_i(t) = -\sum_{j \in N_i} w_{ij}(x_i(t) - x_j(t))}$$

L: the (weighted) Laplacian matrix of an undirected network
w_{ij}: the weight in the edge (i, j)





Analysis of consensus dynamics (undirected graph)

Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$

Main result: All states of the multi-agent Laplacian dynamics converge to consensus if and only if the undirected graph is connected.

Given an initial condition x_0, the explicit solution of the consensus dynamics (which is a linear time-invariant system) is

$$x(t) = e^{-Lt}x_0$$

Since L is symmetric, we perform diagonal decomposition for L by an orthonormal matrix *U*

$$L = U\Lambda U^T \qquad \text{with} \qquad \Lambda = diag(\lambda_i)$$

Also note

$$U = [u_1, u_2, \cdots, u_N]$$

$$e^{-U\Lambda U^T t} = Ue^{-\Lambda t}U^T \longrightarrow x(t) = Ue^{-\Lambda t}U^T x_0$$

Analysis of consensus dynamics (undirected graph)

Consensus dynamics

$$[u_1, u_2, \cdots, u_N] \begin{bmatrix} e^{-\lambda_1 t} & & & \\ & e^{-\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{-\lambda_N t} \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_N^T \end{bmatrix}$$

$$\dot{x}(t) = -L(G)x(t)$$

$$x(t) = Ue^{-\Lambda t}U^T x_0 \longrightarrow x(t) = u_1 u_1^T e^{-\lambda_1 t} x_0 + \sum_{i=2}^{N} u_i u_i^T e^{-\lambda_i t} x_0$$

- We already know that

The zero eigenvalue $\lambda_1 = 0$

Its normalized eigenvector $u_1 = \frac{\mathbf{1}}{\sqrt{N}}$

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$x(t) = \boxed{\frac{(\mathbf{1}^T x_0)\mathbf{1}}{N}} + \boxed{\sum_{i=2}^{N} u_i u_i^T e^{-\lambda_i t} x_0}$$

If the graph is connected, then $\lambda_N \geq \ldots \geq \lambda_2 > 0$

$$\longrightarrow \lim_{t\to\infty} x(t) = \frac{(\mathbf{1}^T x_0)\mathbf{1}}{N}$$

Analysis of consensus dynamics (undirected graph)

Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$

$$\lim_{t\to\infty} x(t) = \frac{(\mathbf{1}^T x_0)\mathbf{1}}{N}$$

$$\mathbf{1}^T x_0 = x_1(0) + x_2(0) + \cdots + x_N(0) = \sum_{i=1}^{N} x_i(0)$$

$\dfrac{\mathbf{1}^T x_0}{N}$: the average of the initial state

$$x_i \to \frac{\mathbf{1}^T x_0}{N}, \quad \forall i$$

- All agents' states converge towards a common value, that is, the average of the initial state.

Definition: the consensus (agreement) subset

$$\mathcal{A} \subseteq \mathbb{R}^N = span(\mathbf{1}) = \{x |\ x_i = x_j\}$$

Analysis of consensus dynamics (undirected graph): Lyapunov proof

Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$

Second proof: exploit Lyapunov arguments

- Define the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^T x$$

- Its evolution (along the system trajectories) is

$$\dot{V}(x) = x^T \dot{x} = -x^T L x$$

- This shows that the state trajectories are bounded since *V(x)* does not increase over time (note $\dot{V}(x) \leq 0$ since L is positive semi-definite).
- To draw additional conclusions, we must resort to LaSalle's Invariance theorem

We analyze the largest invariant set contained in

$$\dot{V}(x) = 0$$

Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$

- If the undirected graph is connected, the invariant set is the null-space of the Laplacian

$$\dot{V}(x) = x^T L x = 0$$

$$\mathcal{A} \subseteq \mathbb{R}^N = span(\mathbf{1})$$

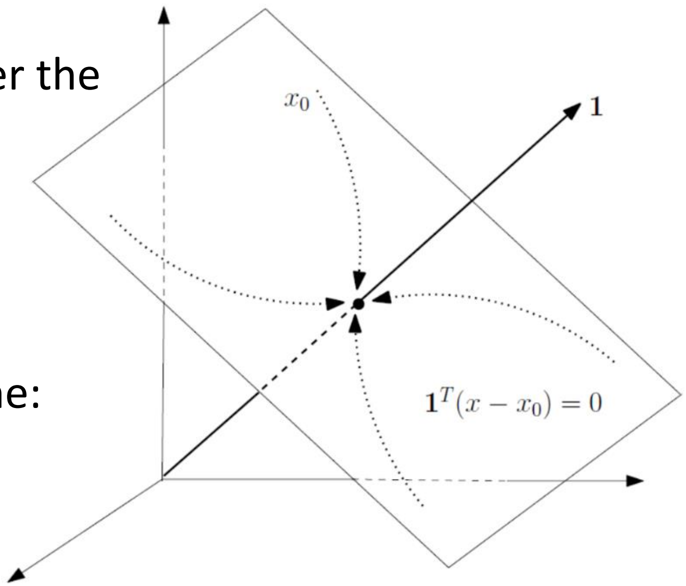
$$x(t) \rightarrow \mathcal{A} = span(\mathbf{1})$$

- The average state: what is its time evolution under the consensus protocol?

$$d(1^T x(t))/dt = \mathbf{1}^T \dot{x} = -\mathbf{1}^T L x = 0$$

$$\longrightarrow \mathbf{1}^T x(t) \equiv \mathbf{1}^T x_0 = const$$

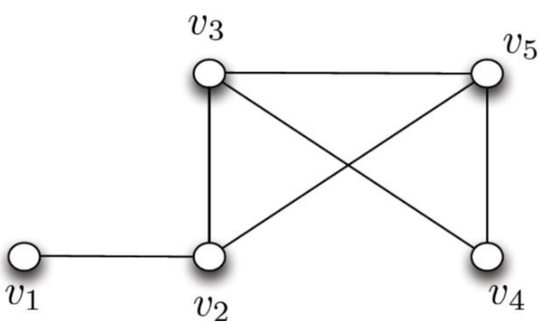
- The centroid of the states never changes over time: the *centroid* of the system states is a constant of motion for the agreement protocol.

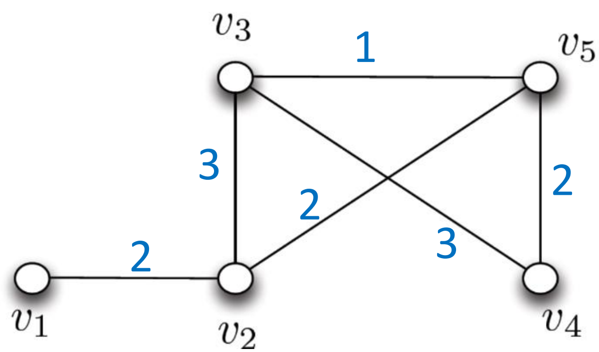


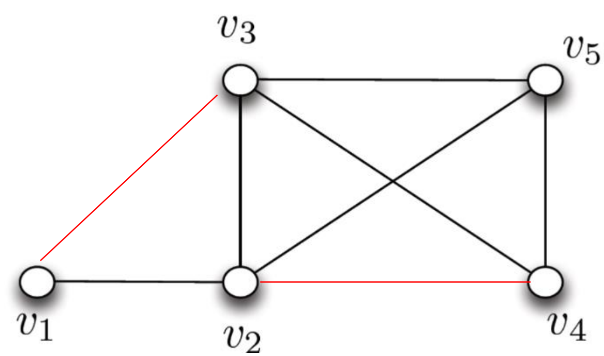
Consensus dynamics (undirected graph)

Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$





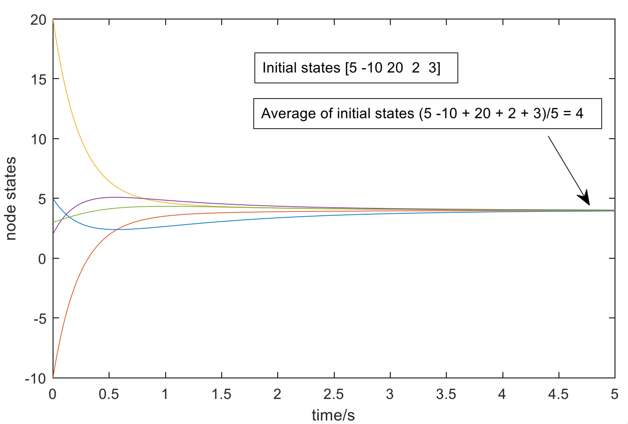


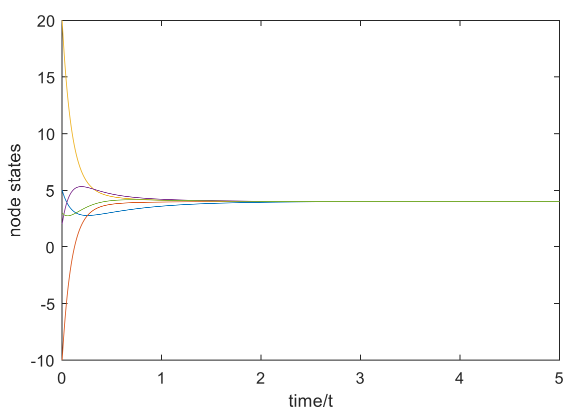
$\lambda_2(G)$ =0.8299

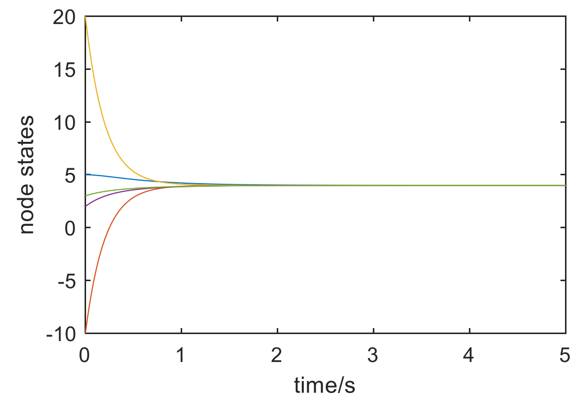
Weighted graph

$\lambda_2(G)$ = 1.8142

$\lambda_2(G)$ = 2







Multi-agent consensus: undirected graphs

Rate of convergence is directly related to the value of $\lambda_2(L)$ (i.e., to the connectivity level of the graph).

- Sparse graph ---> slow convergence
- Dense graph ---> fast convergence
- Large values of the edge weights ---> fast convergence

$$x(t) = u_1 u_1^T e^{-\lambda_1 t} x_0 + \sum_{i=2}^{N} u_i u_i^T e^{-\lambda_i t} x_0$$

- The value of $\lambda_2(G)$ (smallest positive eigenvalue in the sum) dictates the rate of the asymptotic decay of the sum of exponential functions

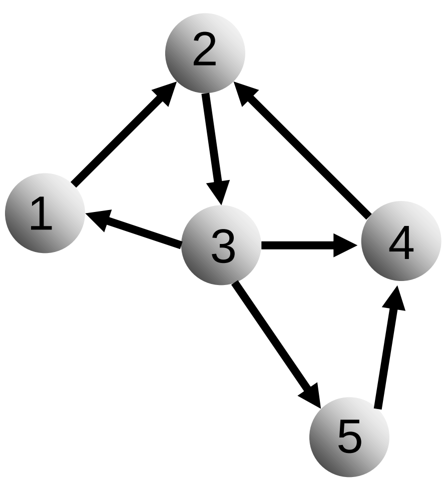
- If $\lambda_2(G)$ is large, the exponential sum will decay faster.
  - Therefore: the more connected the graph, the faster the consensus convergence.

- The centroid of the states (i.e., the average of initial states) remains invariant.
  - All node dynamics reach an average consensus (the average of initial states).

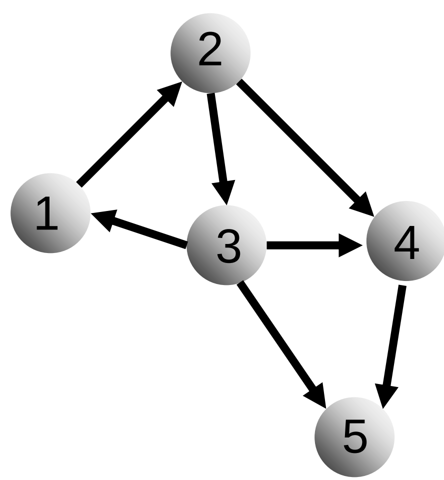
Consensus dynamics in directed graphs

Recap: Connectivity in directed graphs

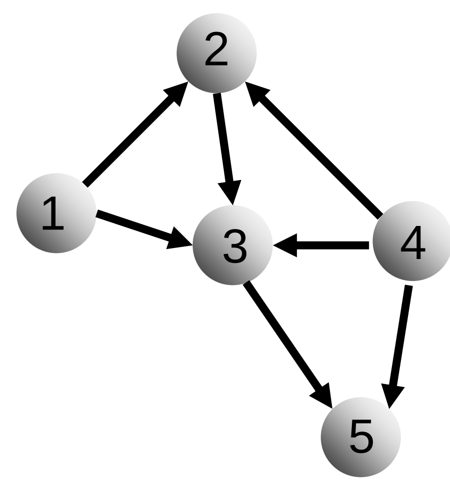
- A directed graph is
  - strongly connected, if each node can reach all the other nodes via directed paths.
  - quasi-strongly connected, if there is a node that can reach all the other nodes via directed paths.
  - weakly connected, if the underlying undirected graph is connected.



Strongly connected



Quasi-strongly connected
(it contains a directed spanning tree)
(it contains a rooted out-branching)



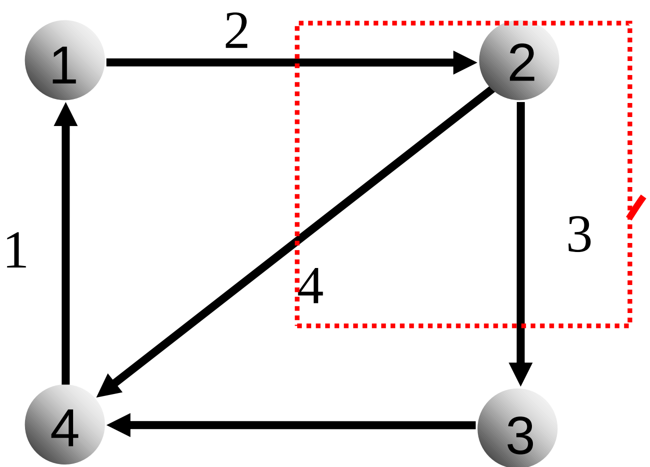
Weakly connected

Recap: Laplacian matrix for *directed* graph

- *Laplacian matrix* for a directed graph: L

$$L = D - A$$

  - D_{ii} : sum of the weights (out-degrees) of all the out-going edges from i
  - A : adjacency matrix



Directed graph G

Node 1 ~ 4

Node 1 ~ 4	1	2	3	4
1	**2**	-2	0	0
2	0	**7**	-3	-4
3	0	0	**5**	-5
4	-1	0	0	**1**

L is asymmetric,
and has zero row sums

Location of Laplacian spectrum: Gershgorin disk theorem

Gershgorin Disk Theorem: For any square matrix $A \in \mathbb{R}^{n \times n}$

$$\mathrm{spec}(A) \subset \bigcup_{i \in \{1,\dots,n\}} \underbrace{\left\{ z \in \mathbb{C} \mid |z - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}| \right\}}_{\text{disk in the complex plane centered at } a_{ii} \text{ with radius } \sum_{j=1, j\neq i}^{n} |a_{ij}|}$$

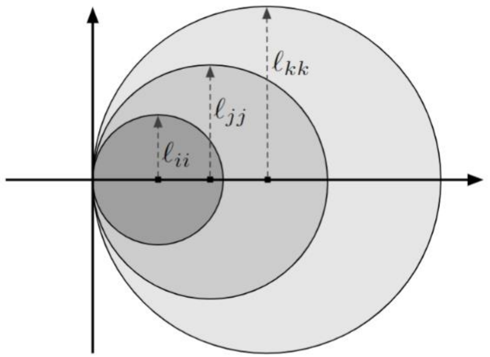
Example:

$$\begin{bmatrix} 1 & -1 & 0 \\ -3 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

All eigenvalues are located in the union of sets

$$\{|z-1| \leq 1\} \cup \{|z-2| \leq 7\} \cup \{|z-3| \leq 0\}$$

Theorem: Given a weighted directed graph G with Laplacian L, the eigenvalues of L different from zero have strictly-positive real part.



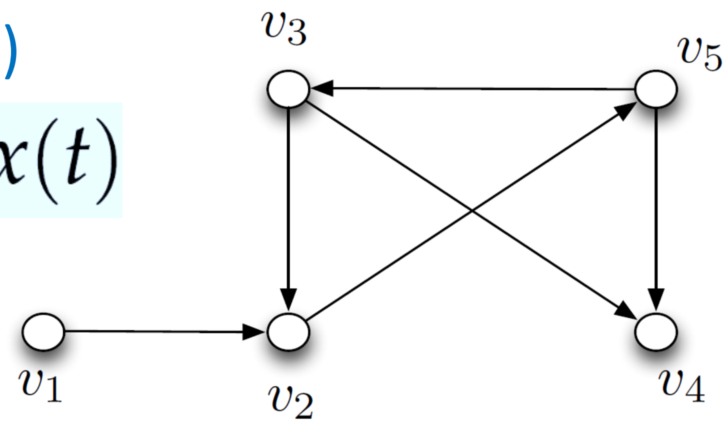
Proof. Recall $\ell_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij} \geq 0$ and $\ell_{ij} = -a_{ij} \leq 0$ for $i \neq j$. By the Geršgorin Disks Theorem we know that each eigenvalue of L belongs to at least one of the disks

$$\left\{ z \in \mathbb{C} \mid |z - \ell_{ii}| \leq \sum_{j=1, j\neq i}^{n} |\ell_{ij}| \right\} = \{ z \in \mathbb{C} \mid |z - \ell_{ii}| \leq \ell_{ii} \}.$$

Multi-agent consensus: directed graphs

Consensus dynamics (in directed graphs)

$$\dot{x}(t) = -L(G)x(t)$$



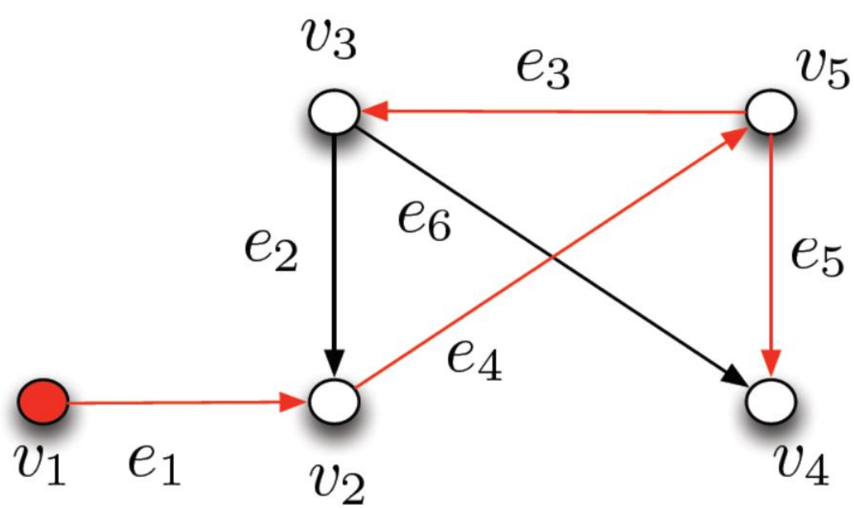
Fact I:

If and only if the graph contains a rooted out-branching (i.e., it contains a directed spanning tree; equivalently, it is quasi-strongly connected)

$$\mathrm{rank}(L) = N - 1$$

A rooted out-branching is a directed graph such that

- it contains no cycles.
- it has a vertex (root) with a directed path to all the other vertices.



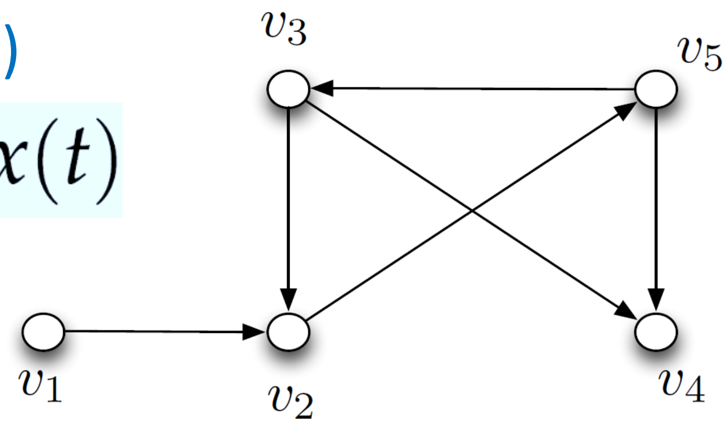
If $\mathrm{rank}(L) = N - 1$

then $\mathbf{1}$ is the only vector spanning its right null-space.

Multi-agent consensus: directed graphs

Consensus dynamics (in directed graphs)

$$\dot{x}(t) = -L(G)x(t)$$



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If $\mathrm{rank}(L) = N - 1$

then $\mathbf{1}$ is the only vector spanning its right null-space.

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

L is not symmetric

L has zero row sums

$$L\mathbf{1} = 0$$

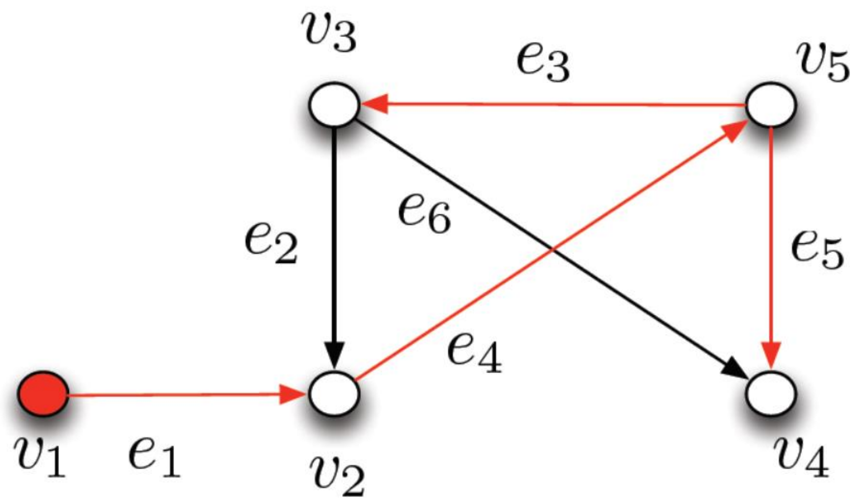
but not necessarily zero column sums

In general $\mathbf{1}^T L \neq 0$

Multi-agent consensus: directed graphs

Fact I:

$\mathrm{rank}(L) = N - 1$ if and only if the graph contains a rooted out-branching.



Fact II:

A Laplacian matrix L for directed graphs has all the eigenvalues with non-negative real part. (Application of Gershgorin Circle Theorem)

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Combining Fact I and Fact II:

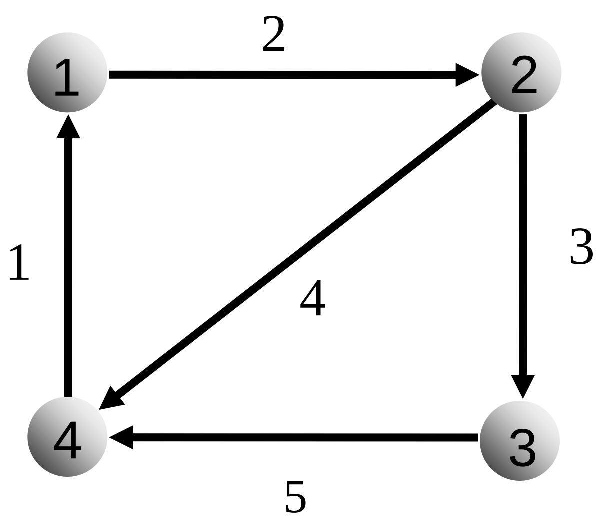
The Laplacian L has a zero eigenvalue, and all other eigenvalues have positive real part, if and only if the directed graph contains a rooted out-branching (i.e., it is quasi-strongly connected)

$\lambda_1 = 0$ and $0 < \Re(\lambda_2) \leq \ldots \leq \Re(\lambda_N)$

eig(L) : 0 0.2451 1.8774 – 0.7449i, 1.8774 + 0.7449i, 2.000

Properties of Laplacian matrix: directed graph

A *strongly connected directed* graph



Strongly connected graph *G*

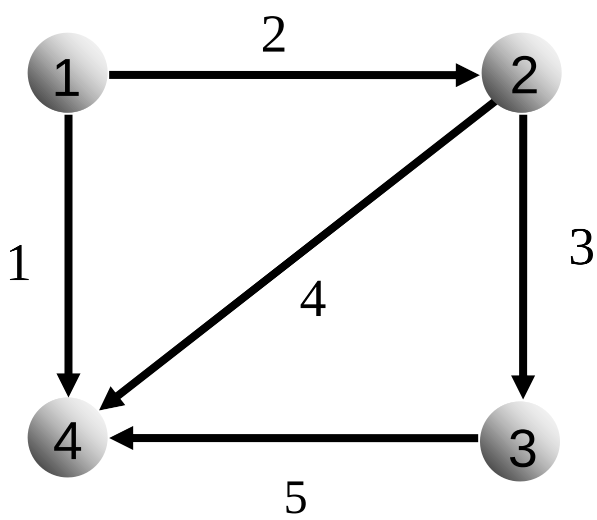
$$L = \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 7 & -3 & -4 \\ 0 & 0 & 5 & -5 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

- $L\mathbb{1} = 0$ i.e., the row sum is zero
- rank $L = N - 1$
- L has only one zero eigenvalue, and all the other eigenvalues have positive real parts

eig(L) : 0 3.9100 + 1.2507i 3.9100 - 1.2507i 7.1800

Properties of Laplacian matrix: **directed graph**

A *quasi-strongly connected* directed graph



Quasi-strongly connected graph G

$$L = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 7 & -3 & -4 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- $L\mathbb{1} = 0$, i.e., the row sum is zero
- rank L = N − 1, and L has only one zero eigenvalue, and all the other eigenvalues have positive real parts

$$\boxed{\mathrm{eig}(L): \quad 0 \quad 3 \quad 5 \quad 7}$$

Weakly connected directed graphs may have multiple zero eigenvalues

Multi-agent consensus: directed graphs

Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$

$(p_1 q_1^T)x_0$

$= p_1(q_1^T x_0)$

$= (q_1^T x_0)p_1$

$q_1^T x_0$ is scalar

- We perform the Jordan decomposition of L

$$L = PJ(\Lambda)P^{-1}$$

With the Jordan block $J(\Lambda) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & J(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & J(\lambda_N) \end{bmatrix}$

- The explicit solution of the consensus dynamics

$$x(t) = e^{-Lt}x_0 = (p_1 q_1^T)x_0 + \sum_{i=2}^{N} P_i(e^{-J(\lambda_i)t})(P^{-1})_i x_0$$

$p_1 = \mathbf{1}$: the right eigenvector associated to $\lambda_1 = 0$

q_1 : the left eigenvector associated to $\lambda_1 = 0$ (normalizing $q_1^T \mathbf{1} = 1$)

$0 < \Re(\lambda_2) \leq \ldots \leq \Re(\lambda_N) \longrightarrow \lim_{t\to\infty} x(t) = (q_1^T x_0)p_1 = (q_1^T x_0)\mathbf{1}$

Multi-agent consensus: directed graphs

Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$

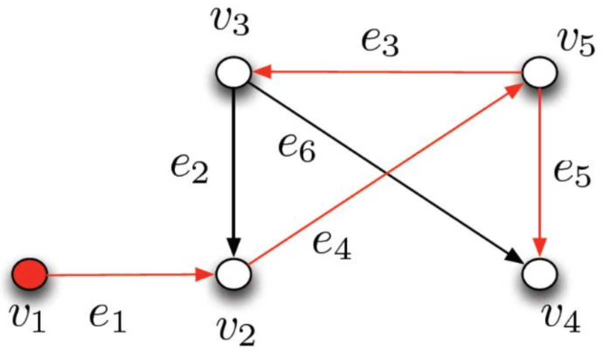
all node states reach consensus, if and only if the directed graph contains a rooted out-branching (i.e., it is quasi-strongly connected)

$$\lim_{t\to\infty} x(t) = (q_1^T x_0)p_1 = (q_1^T x_0)\mathbf{1}$$

Note that in general $q_1 \notin span(\mathbf{1})$

The converged consensus value is not necessarily the average consensus.

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$



$p_1 = \mathbf{1} \quad q_1 = span([1\ 0\ 0\ 0\ 0]^T)$

All node states reach consensus to the state of v_1 (the leader node)

Multi-agent consensus: directed graphs

Consensus dynamics

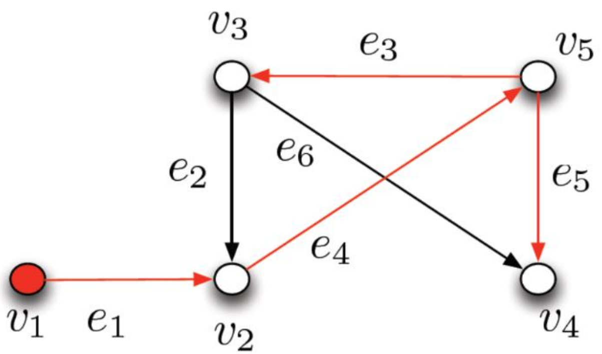
$$\dot{x}(t) = -L(G)x(t)$$

Is it possible to achieve **average consensus** in directed graphs?

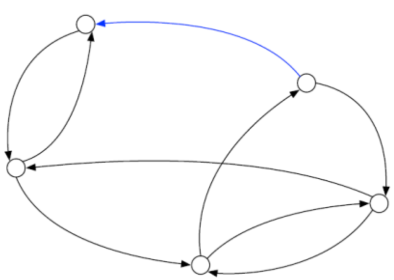
$$\lim_{t\to\infty} x(t) = (q_1^T x_0)p_1 = (q_1^T x_0)\mathbf{1} \longrightarrow \lim_{t\to\infty} x(t) = \frac{(\mathbf{1}^T x_0)\mathbf{1}}{N}$$

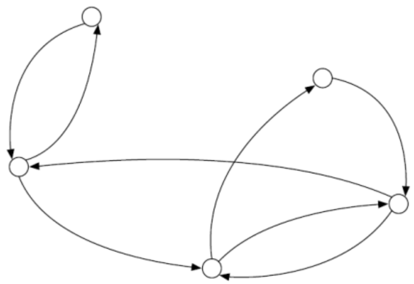
Equivalently, when does $q_1 \in span(\mathbf{1})$ **?**

Balanced directed graph: a directed graph is called balanced if, for every vertex, the in-degree equals the out-degree.



unbalanced


unbalanced


balanced

Multi-agent consensus: directed graphs

Consensus dynamics

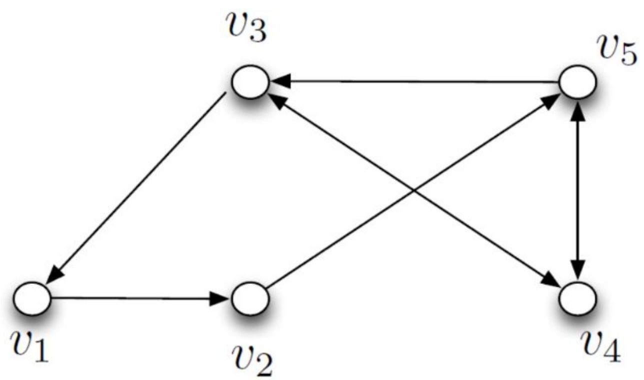
$$\dot{x}(t) = -L(G)x(t)$$

Is it possible to achieve **average consensus** in directed graphs?

$$\lim_{t\to\infty} x(t) = (q_1^T x_0)p_1 = (q_1^T x_0)\mathbf{1} \longrightarrow \lim_{t\to\infty} x(t) = \frac{(\mathbf{1}^T x_0)\mathbf{1}}{N}$$

Equivalently, when does $q_1 \in span(\mathbf{1})$ **?**

Balanced directed graph: a directed graph is called balanced if, for every vertex, the in-degree equals the out-degree.



A balanced directed graph that contains a rooted out-branching

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

L is not symmetric, $L\mathbf{1} = 0$

$q_1 \in span(\mathbf{1})$

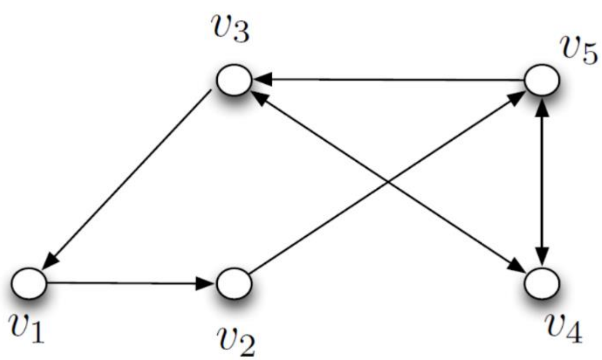
Multi-agent consensus: directed graphs

Consensus dynamics

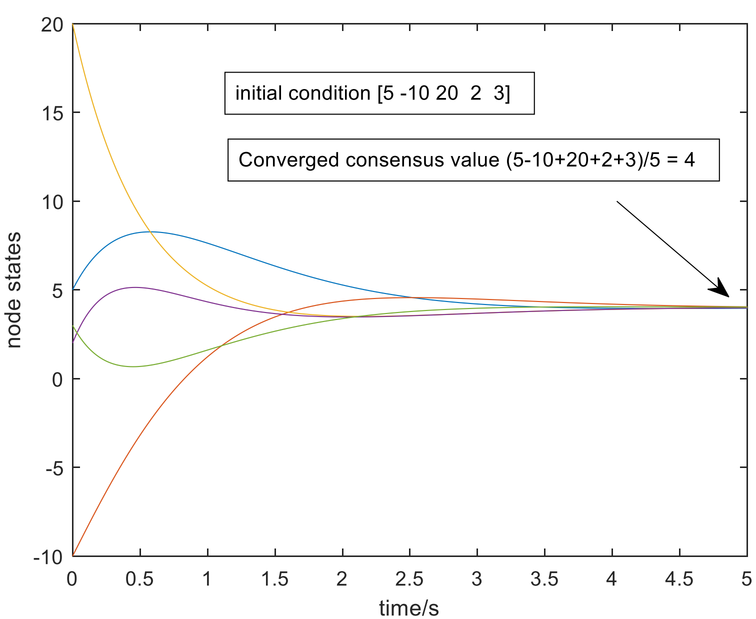
$$\dot{x}(t) = -L(G)x(t)$$

Suppose the directed graph is balanced and contains a rooted out-branching. Then the consensus dynamics converge to an averaged agreement

$$\lim_{t\to\infty} x(t) = \frac{(\mathbf{1}^T x_0)\mathbf{1}}{N}$$



$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$



Consensus dynamics

$$\dot{x}(t) = -L(G)x(t)$$

in directed graphs can also be analyzed by **Lyapunov function approach**. (see Paper [*])

Many applications of the consensus protocol

- rendezvous: meet at a common point (uniform the positions)
- alignment: point in the same direction (uniform the angles)
- distributed estimation: agree on the estimation of some distributed quantity (e.g., average temperature)
- distributed filtering: multiple sensors collaboratively filter a joint uncertain signal
- synchronization: agree on the same time or state (regardless of phase shifts or different rates in the clocks)
- ……

Other fields:

- Social opinion dynamics: forming an agreement of social opinions
- Power system network: stability and synchronization via couplings
- ……

❑ [*] Zhang, H., Lewis, F.L. and Qu, Z., 2011. Lyapunov, adaptive, and optimal design techniques for cooperative systems on directed communication graphs. *IEEE Transactions on Industrial Electronics*, *59*(7), pp.3026-3041.

Consensus dynamics of double-integrator systems

Consensus dynamics of double integrators

Consider the double-integrator model

$$\dot{p}_i = v_i$$
$$\dot{v}_i = u_i$$

p_i, v_i: agent i's position and velocity;
u_i: agent i's control input (for acceleration)

Control input with both position consensus *and velocity consensus* (flocking dynamics)

$$u_i = -\sum_{j \in N_i} (p_i - p_j) - \gamma \sum_{j \in N_i} (v_i - v_j)$$

Define $p = [p_1, \cdots, p_n]^T, v = [v_1, \cdots, v_n]^T$

Compact form (i.e., double-integrator flocking dynamics)

$$\dot{p} = v$$
$$\dot{v} = -Lp - \gamma Lv$$

Consensus dynamics of double integrators

Compact form (i.e., double-integrator flocking dynamics)

$$\dot{p} = v$$
$$\dot{v} = -Lp - \gamma L v$$

Theorem

Consider the double-integrator systems coupled in a connected undirected graph.

- All agents' positions reach consensus (and are moving);
- All agents' velocities reach consensus.

Proof (sketch): Lyapunov function

$$V = \frac{1}{2} v^T v + \frac{1}{2} p^T L p$$

The derivative along system trajectories

$$\begin{aligned} \dot{V} &= v^T \dot{v} + p^T L \dot{p} \\ &= v^T(-Lp - \gamma L v) + p^T L v \\ &= -\gamma v^T L v \leq 0 \end{aligned}$$

LaSalle invariance principle implies $v_i \to v_j$ as $t \to \infty$

At the steady state $\dot{v} \triangleq -Lp \to 0$

$\longrightarrow$ $p_i \to p_j$ as $t \to \infty$

Consensus dynamics of double integrators

Compact form (i.e., double-integrator flocking dynamics)

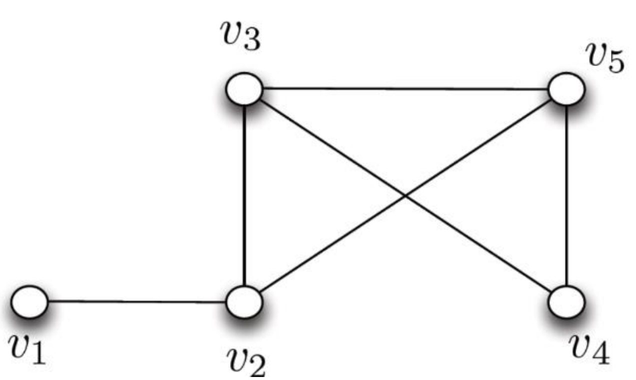
$$\dot{p} = v$$
$$\dot{v} = -Lp - \gamma L v$$

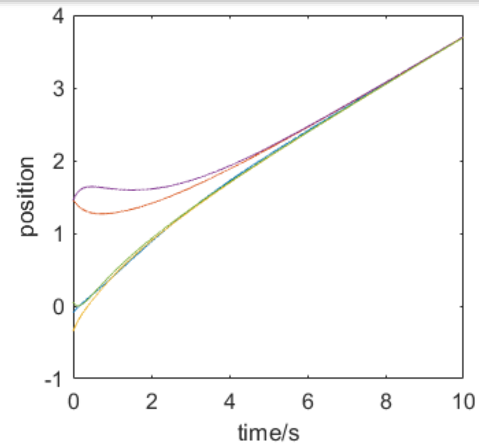
(position consensus and *velocity consensus*)

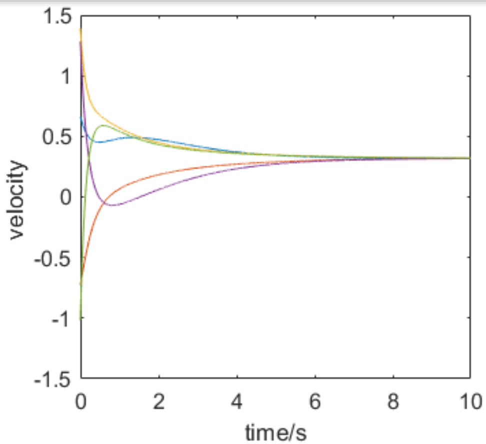
Theorem

Consider the double-integrator systems coupled in a connected *undirected* graph.

- All agents' positions reach consensus (and are moving);
- All agents' velocities reach consensus.







The theorem holds true in directed graph under the rooted out-branching condition, if the gain γ is sufficiently large.

❑ Ren, W. and Beard, R.W., 2008. Consensus algorithms for double-integrator dynamics. *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*, pp.77-104.

Consensus dynamics of double integrators

Consider the double-integrator model

$$\dot{p}_i = v_i$$
$$\dot{v}_i = u_i$$

Control input with both position consensus and *velocity damping*

$$u_i = -\sum_{j \in N_i} (p_i - p_j) - \gamma v_i$$

Compact form (i.e., double-integrator consensus dynamics)

$$\dot{p} = v$$
$$\dot{v} = -Lp - \gamma v$$

Suppose the underlying graph is undirected and connected.

- Then all positions reach consensus and all velocities converge to zero.

$$v_i \to 0,\ p_i \to p_j, \text{ as } t \to \infty$$

Consensus dynamics of double integrators

Compact form (i.e., double-integrator consensus dynamics)

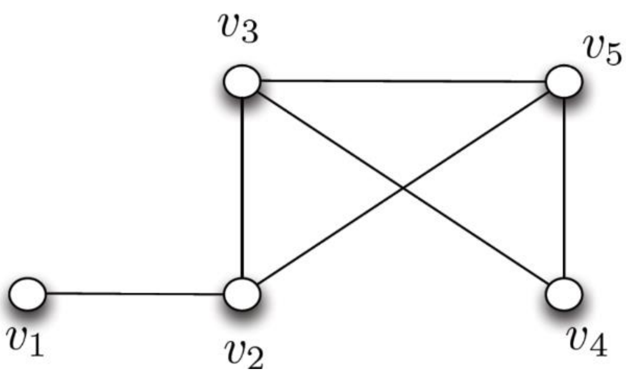
$$\dot{p} = v$$
$$\dot{v} = -Lp - \gamma v$$

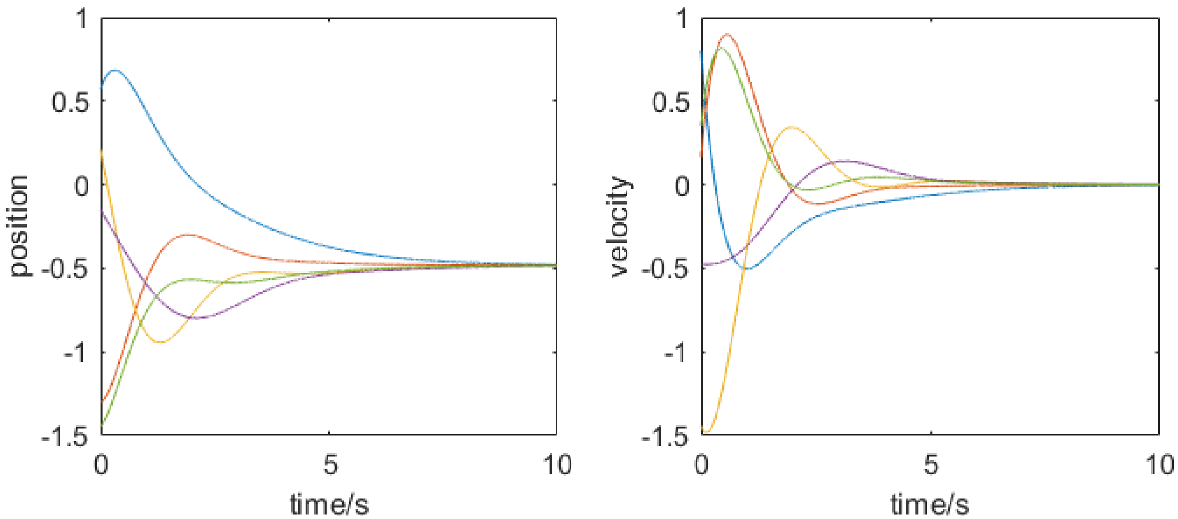
(position consensus and *velocity damping*)

Suppose the underlying graph is undirected and connected.

- Then all positions reach consensus and all velocities converge to zero.

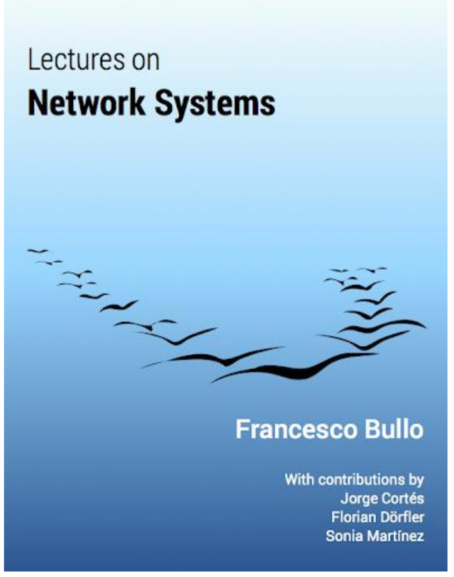
$v_i \to 0,\ p_i \to p_j$, as $t \to \infty$





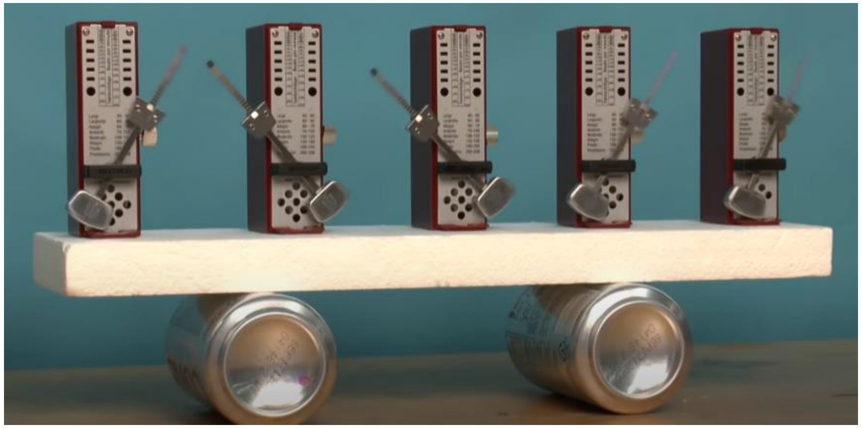
Application of consensus seeking

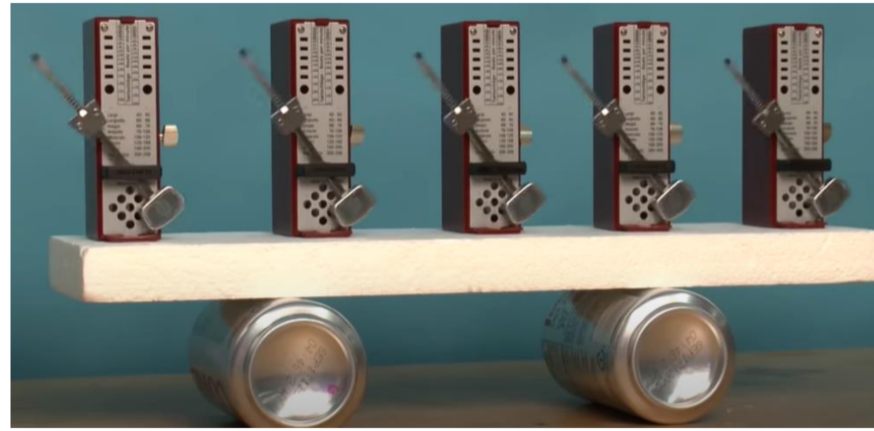
- *Phase synchronization of Kuramoto oscillators*



Chapter 17

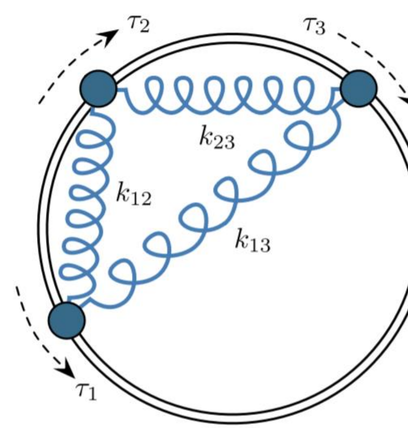
Networked Kuramoto oscillators





Kuramoto model: (assuming zero natural frequency)

$$\dot{\theta}_i = -\sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \quad i \in \{1, \ldots, n\}$$



- Oscillator state: $\theta_i \in \mathbb{S}^1$ (angle in the circle space)
- Coupling function: $\sin(\theta_i - \theta_j)$
- Coupling graph topology by the adjacency matrix: a_{ij}

❑ Dörfler, F. and Bullo, F., 2014. Synchronization in complex networks of phase oscillators: A survey. *Automatica*, *50*(6), pp.1539-1564.

Synchronization of networked Kuramoto oscillators

$$\dot{\theta}_i = -\sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \quad i \in \{1, \ldots, n\}.$$

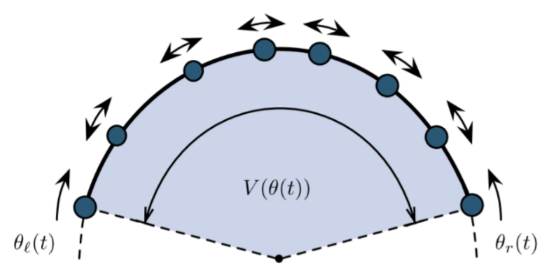
- Networked Kuramoto oscillators as consensus-like dynamics

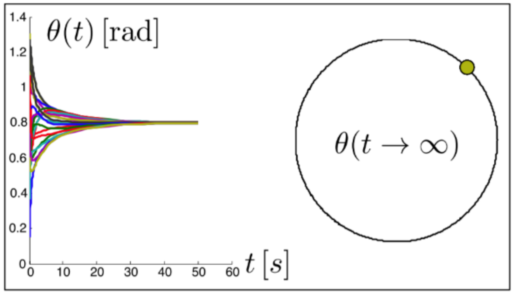
$$\dot{\theta}_i = -\sum_{j=1}^{n} c_{ij}(\theta)(\theta_i - \theta_j) \qquad c_{ij}(\theta) = a_{ij}\,\mathrm{sinc}(\theta_i - \theta_j) \geq 0$$

$$\mathrm{sinc} = \frac{\sin(x)}{x}$$

- Lyapunov function for synchronization

$$V(\theta(t)) = \max_{i,j\in\{1,\ldots,n\}} |\theta_i(t) - \theta_j(t)|$$





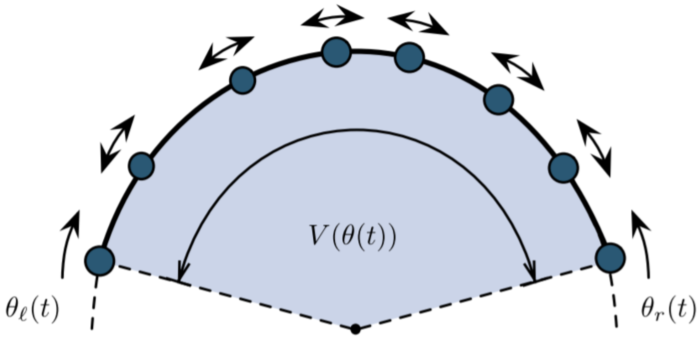
- If all angles are contained in an arc of length strictly less than π, then the arc length $V(\theta(t))$ is a Lyapunov function candidate for phase synchronization.

Synchronization of networked Kuramoto oscillators

$$\dot{\theta}_i = -\sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \quad i \in \{1, \ldots . n\}.$$

- Lyapunov function for synchronization

$$V(\theta(t)) = \max_{i,j \in \{1,\ldots,n\}} |\theta_i(t) - \theta_j(t)|$$



Theorem

Consider a network of Kuramoto oscillators coupled in a connected undirected graph. Suppose initial angles are contained in an arc of length strictly less than π (i.e., in a half circle).

- Then all oscillators achieve phase synchronization.

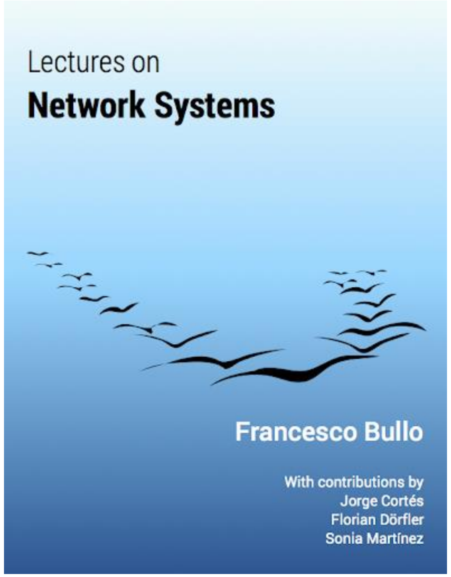
The phase synchronization result also extends to Kuramoto models with a natural frequency ω_0

$$\dot{\theta}_i = \omega_0 - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

- ❑ Dörfler, F. and Bullo, F., 2014. Synchronization in complex networks of phase oscillators: A survey. *Automatica*, *50*(6), pp.1539-1564.

Multi-agent systems with general dynamics

- *Diffusively-coupled linear networked systems*
- *Synchronization behavior in complex networks*



Chapter 8

Diffusively-coupled linear systems

- Consider diffusive interconnections among identical linear systems.
- Each node (agent) is a continuous-time linear dynamical system described by

$$\mathbf{\Sigma}_i : \begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases}$$

with $x_i \in \mathbb{R}^n, u_i, y_i \in \mathbb{R}^m$ the states, inputs and outputs, respectively.

- The agents are interconnected through a weighted graph *G* with adjacency matrix $\mathcal{A}$.
- Consider the output-dependent diffusive coupling law

$$u_i(t) = \sum_{j \in \mathcal{N}_i^-} \mathcal{A}_{ij}[y_j(t) - y_i(t)]$$

- What is the compact expression for the state matrix of a diffusively-coupled network of linear systems?

Modelling via Kronecker products

- The Kronecker product of $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$ is given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np\times mq}$$

$$\boxed{\underset{R^{n\times m}}{A}\,\underset{R^{m\times p}}{B} = \underset{R^{n\times p}}{C}}$$

- For example, two simple cases:

$$I \otimes A = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix} \qquad A \otimes I = \begin{bmatrix} a_{11}I & \cdots & a_{1m}I \\ \vdots & \ddots & \vdots \\ a_{n1}I & \cdots & a_{nm}I \end{bmatrix}$$

- Properties:

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$
$$A \otimes B + A \otimes C = A \otimes (B + C)$$
$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

Diffusively-coupled linear systems

Lemma

Consider a network of diffusive-coupled *N* linear systems described by the system (A, B, C) and the graph G. Then the closed-loop system obeys

$$\dot{X} = (I_N \otimes A - L \otimes BC)X$$

where L is the Laplacian matrix of G, and we adopt the notation $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$

Proof. We write the *N* coupled systems in a single vector-valued equation using the Kronecker product. First, we stack the *N* dynamical systems to write

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B \end{bmatrix} \underbrace{\boxed{\begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}}}_{U}$$

$$\Rightarrow \quad \dot{X} = (I_N \otimes A)X + (I_N \otimes B)U$$

Diffusively-coupled linear systems

Proof (continue). Next, recalling the definition of Laplacian, we have

$$u_i(t) = \sum_{j \in \mathcal{N}_i^-} \mathcal{A}_{ij}[y_j(t) - y_i(t)] \quad \Longrightarrow \quad U = -(L \otimes I_m)Y, \;\; Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

Moreover, we note $Y = (I_N \otimes C)X$ and, following Kronecker product properties , we obtain

$$\begin{aligned} \dot{X} &= (I_N \otimes A)X + (I_N \otimes B)U \\ &= (I_N \otimes A)X + (I_N \otimes B)(-L \otimes I_m)(I_N \otimes C)X \\ &= (I_N \otimes A - L \otimes BC)X \end{aligned}$$

That completes the proof.

Diffusively-coupled linear systems

$$\boldsymbol{\Sigma}_i : \begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \qquad u_i(t) = \sum_{j \in \mathcal{N}_i^-} \mathcal{A}_{ij}[y_j(t) - y_i(t)]$$

Theorem (stability and consensus)

Consider a network of diffusive-coupled *N* linear systems described by the system (A, B, C) and the *undirected* graph G with a symmetric Laplacian matrix L. Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ denote the eigenvalues of L. Then the closed-loop system $\dot{X} = (I_N \otimes A - L \otimes BC)X$

1) is asymptotically stable
   if and only if each matrix $A - \lambda_i BC$ is Hurwitz, for all $i = \{1, 2, \cdots, N\}$

2) achieves asymptotic consensus, i.e., $\lim_{t\to\infty}[x_i(t) - x_j(t)] = 0$
   if each matrix $A - \lambda_i BC$ is Hurwitz, for all $i = \{2, 3, \cdots, N\}$

Diffusively-coupled linear systems: proof

Consider the eigenvalue decomposition of L:

$$L = T\Lambda T^{-1}, \ \Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$$

where T is an orthonormal matrix, with $T^{-1} = T^\top$, and the first column of T is the eigenvector associated with the zero eigenvalue λ_1: $T_1 = \frac{1}{\sqrt{N}}\mathbb{1}$. Consider the change of variable

$$Z = (T^\top \otimes I_n)X \qquad X = (T \otimes I_n)Z$$

Compute

$$\begin{aligned}
\dot{Z} &= (T^\top \otimes I_n)\dot{X} \\
&= (T^\top \otimes I_n)(I \otimes A - L \otimes BC)(T \otimes I_n)Z \\
&= (I \otimes A - T^\top LT \otimes BC)Z \\
&= (I \otimes A - \Lambda \otimes BC)Z \\
&= \begin{bmatrix} A & & & \\ & A - \lambda_2 BC & & \\ & & \ddots & \\ & & & A - \lambda_N BC \end{bmatrix} Z
\end{aligned}$$

This block diagonal form immediately implies the first statement.

Diffusively-coupled linear systems: proof (continued)

$$\dot{Z} = \begin{bmatrix} A & & & \\ & A-\lambda_2 BC & & \\ & & \ddots & \\ & & & A-\lambda_N BC \end{bmatrix} Z$$

Assume that only the *N−1* matrices $A-\lambda_i BC$, $i=\{2,3,\cdots,N\}$ are Hurwitz. Thus,

$$\lim_{t\to\infty} Z(t) = \left(\mathbf{e}_1\mathbf{e}_1^\top \otimes \lim_{t\to\infty} e^{At}\right) Z(0), \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using the relation $X=(T\otimes I_n)Z$, we obtain

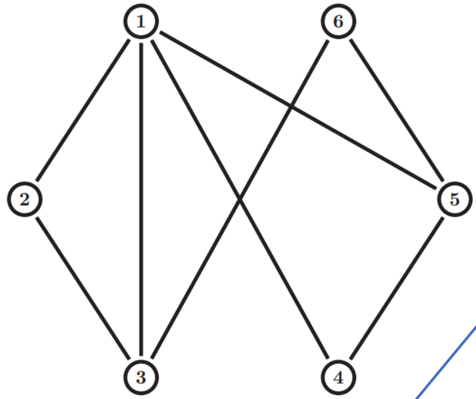
$$\begin{aligned}
\lim_{t\to\infty} X(t) &= (T\otimes I_n)\left(\mathbf{e}_1\mathbf{e}_1^\top \otimes \lim_{t\to\infty} e^{At}\right)(T^\top\otimes I_n)X(0) \\
&= \left(T\mathbf{e}_1\mathbf{e}_1^\top T^\top \otimes \lim_{t\to\infty} e^{At}\right) X(0) \\
&= \left(\frac{1}{N}\mathbb{1}\mathbb{1}^\top \otimes \lim_{t\to\infty} e^{At}\right) X(0)
\end{aligned}$$

Clearly,

$$\lim_{t\to\infty}[x_i(t)-x_j(t)] = \left((\mathbf{e}_i-\mathbf{e}_j)\frac{1}{N}\mathbb{1}\mathbb{1}^\top \otimes \lim_{t\to\infty} e^{At}\right) X(0) = 0$$

Synchronization in complex networks (with nonlinear dynamics)

- Consider *N* node dynamics interconnected via an undirected graph. (Complex network systems)

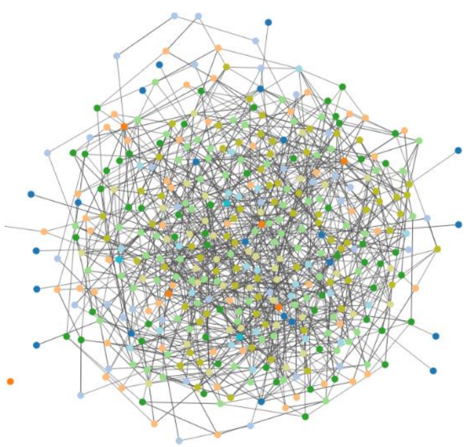


$$\dot{x}_i(t) = f(x_i(t)) + k \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t))$$

Node dynamics
Vector function

Local coupling strength
(node *i*)

Diffusive coupling
(to neighbour nodes)



Synchronization in complex networks:

$$x_i(t) \rightarrow x_j(t), \forall i, j, t \rightarrow \infty$$

- Boccaletti, S., Pisarchik, A.N., Del Genio, C.I. and Amann, A., 2018. *Synchronization: from coupled systems to complex networks*. Cambridge University Press.

Synchronization in complex networks

- Consider *N* node dynamics interconnected via an undirected graph (Complex network systems)

$$\dot{x}_i(t) = f(x_i(t)) + k \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t))$$

- Suppose the node dynamics vector functions satisfy the QUADratic inequality (QUAD condition)

$$(x-y)^T[f(x,t) - f(y,t)] - (x-y)^T \Delta (x-y) \le -\epsilon (x-y)^T (x-y),$$

where Δ is an $n \times n$ diagonal matrix and ϵ is a real scalar.

Theorem (synchronization in complex networks)

Consider a network of nonlinear systems interconnected via an undirected graph and the node vector function satisfies the above condition.

Then the complex network achieves state synchronization if the coupling strength *k* is large enough.

$x_i(t) \to x_j(t), \forall i, j, t \to \infty$ for large enough coupling strength *k*

Synchronization in complex networks (proof)

Proof (sketch)

- Compact form

$$\dot{x}(t) = f(x(t)) - k(L \otimes I_n)x(t)$$

- Consider the Lyapunov function candidate

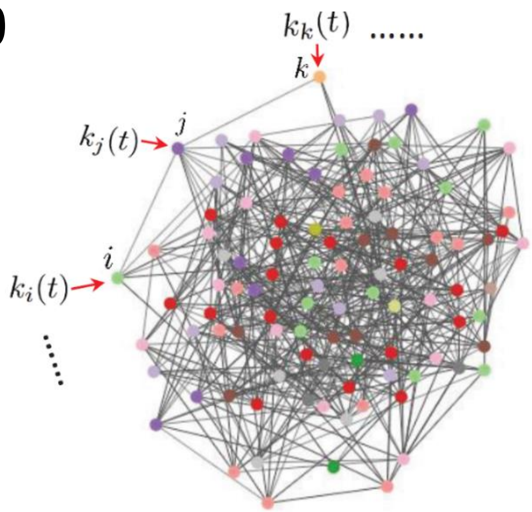
$$V(x(t)) = \frac{1}{2}(\bar{H}x)^T(\bar{H}x) = \frac{1}{2}x^T\bar{L}x,$$

where $\bar{H} = H \otimes I_n$, $\bar{L} = L \otimes I_n$

- The derivative along the network system trajectory

$$\begin{aligned}
\dot{V}(x(t)) =& (\bar{H}x)^T(\bar{H}\dot{x}) \\
=& (\bar{H}x)^T\left(\bar{H}(f(x) - k(L \otimes I_n)x(t))\right) \\
=& (\bar{H}x)^T\bar{H}f(x) - k(\bar{H}x)^T\bar{H}\bar{H}^T\bar{H}x \\
\leq& (\bar{H}x)^T(I_N \otimes (\Delta - \epsilon I_n))\bar{H}x - k(\bar{H}x)^T\bar{H}\bar{H}^T\bar{H}x \\
\leq& (\bar{H}x)^T(I_N \otimes (\Delta - \epsilon I_n))\bar{H}x - k\lambda_2(L)(\bar{H}x)^T\bar{H}x \\
\leq& (\bar{H}x)^T \underbrace{\left(I_N \otimes (\Delta - \epsilon I_n) - k\lambda_2(L)I_{Nn}\right)}_{<\,0 \text{ for large enough } k} \bar{H}x
\end{aligned}$$

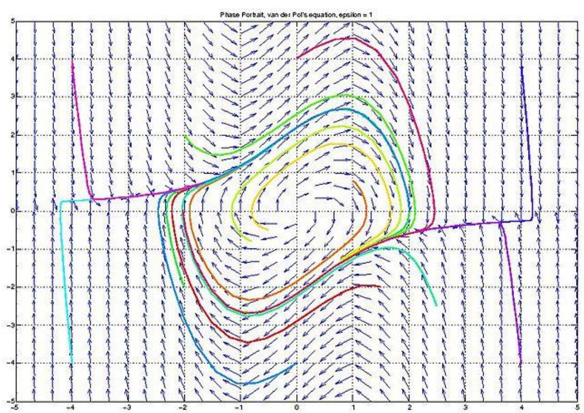
Complex network by Erdos-Renyi models: 100 nodes

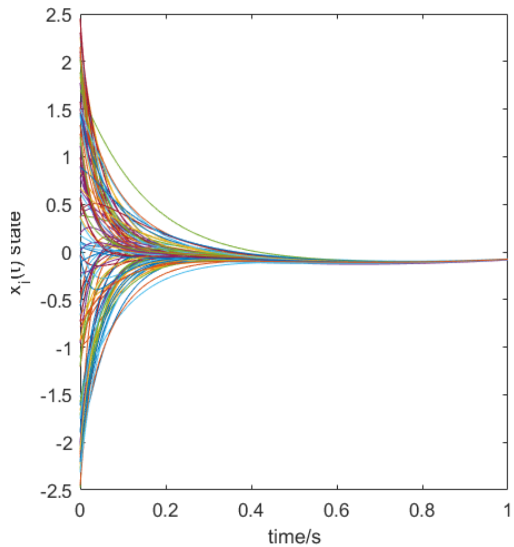


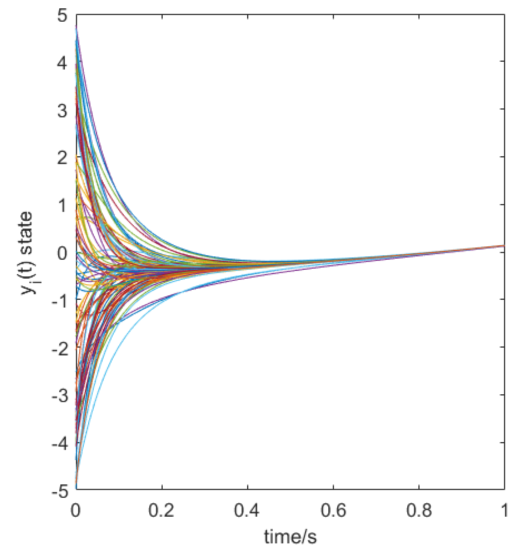
- Node dynamics: Van der Pol oscillators

$$\dot{x}_i = wy_i - \frac{a}{3}x_i^3 - bx_i + k_i(t)\sum_{j\in\mathcal{N}_i}(x_j - x_i),$$

$$\dot{y}_i = -wx_i + \frac{\mu(t)}{w} + k_i(t)\sum_{j\in\mathcal{N}_i}(y_j - y_i),$$

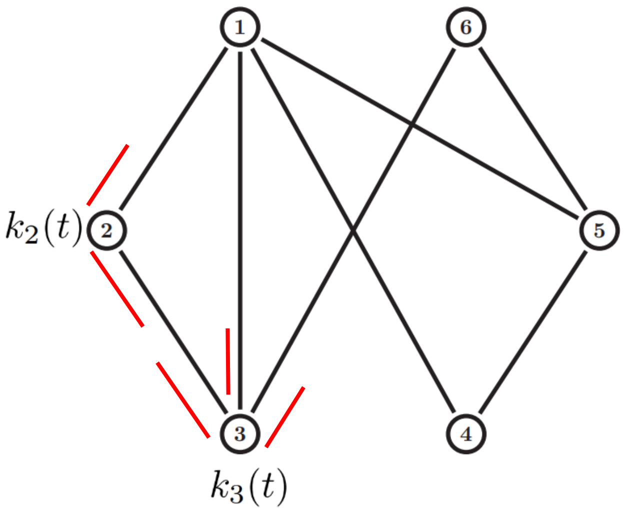






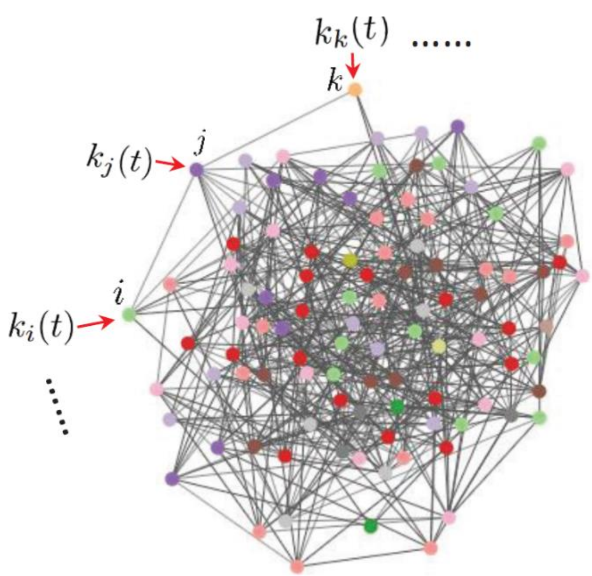
- Coupling strength $k_i = 1$.

Let each node/edge ***locally learn*** coupling weights for network synchronization



$$\dot{x}_i(t) = f(x_i(t)) + k_i(t) \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)),$$

Locally adjust node coupling weight



The adaptive tuning/learning of local coupling weight should involve only local relative state information to neighbor nodes.

Distributed adaptive control/learning-based control over networks solve this problem!

Multi-agent consensus results in the literature

Graph topology

- Undirected
- Directed
- Weighted (static)
- Weighted (dynamic)
- Switching
- Time-varying
- State-dependent
- Random
- Multi-layer
- Hyper-graphs
- Product graphs
- ……

Agent dynamics

- Single integrator
- Double integrator
- General linear systems
- Passive systems
- Continuous time
- Discrete time
- General non-linear systems
- Hybrid dynamics
- Stochastic systems
- Kuramoto oscillators
- Heterogeneous dynamics
- Unicycle robotics
- ……

Coupling

- Linear
- Non-linear
- Passive
- Delays
- Switching
- Saturation
- Static/dynamic
- Quantization
- Synchronous
- Asynchronous
- Random
- ……

Applications

- Distributed filtering (distributed *Kalman* filter)
- Distributed optimization
- Distributed solving equations
- Robotics rendezvous
- Flocking/Swarm/Formation
- Network localization
- ……

❑ Ren, W. and Beard, R.W., 2008. *Distributed consensus in multi-vehicle cooperative control.* London: Springer London.

❑ Lewis, F.L., Zhang, H., Hengster-Movric, K. and Das, A., 2013. *Cooperative control of multi-agent systems: optimal and adaptive design approaches*. Springer Science & Business Media.

Summary/take home messages

- Graph Laplacian spectrum and properties
- Consensus dynamics in undirected/directed graphs
- Consensus seeking from single-integrator to double-integrator systems
- Application of consensus seeking: Kuramoto oscillators
- Networked systems with linear/nonlinear dynamics: diffusive coupling and synchronization behavior.

The next lecture

- Application of multi-agent system; formation control.

- Notice: Instruction (problem set) IV is available in Canvas.