Control Principles for Engineered Systems 5SMC0

Networked Control System

Lecture 3

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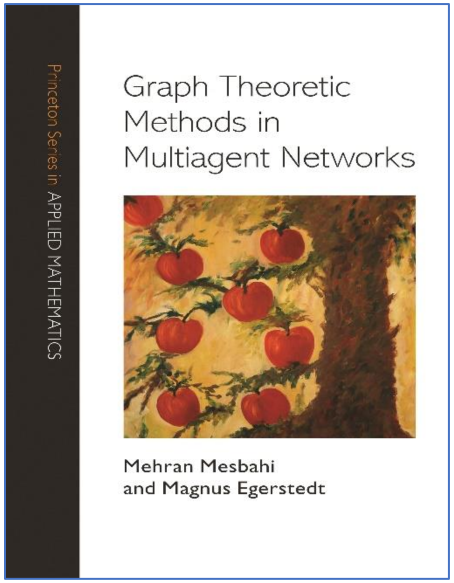
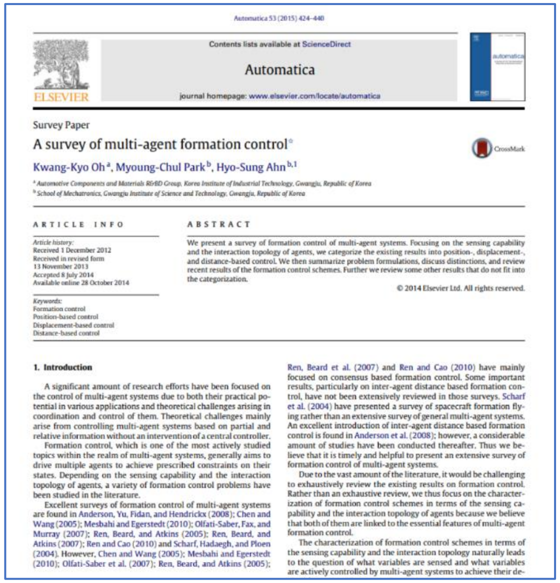
Networked Control System

Lecture 3

Networked System Application: Formation Control / Flocking and Swarm



Chapter 6



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Survey Paper

A survey of multi-agent formation control

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ABSTRACT

We present a survey of formation control of multi-agent systems. Focusing on the sensing capability and the interaction topology of agents, we categorize the existing results into position-, displacement-, and distance-based control. We then summarize problem formulations, discuss distinctions, and review recent results of the formation control schemes. Further we review some other results that do not fit into the categorization.

1. Introduction

A significant amount of research efforts have been focused on the control of multi-agent systems due to both their practical potential in various applications and theoretical challenges arising in coordination and control of them. Theoretical challenges mainly arise from controlling multi-agent systems based on partial and relative information without an intervention of a central controller.

Formation control, which is one of the most actively studied topics within the realm of multi-agent systems, generally aims to drive multiple agents to achieve prescribed constraints on their states. Depending on the sensing capability and the interaction topology of agents, a variety of formation control problems have been studied in the literature.

Excellent surveys of formation control of multi-agent systems are found in Anderson, Yu, Fidan, and Hendrickx (2008); Chen and Wang (2005); Mesbahi and Egerstedt (2010); Olfati-Saber, Fax, and Murray (2007); Ren, Beard, and Atkins (2005); Ren, Beard, and Atkins (2007); Ren and Cao (2010) and Scharf, Hadaegh, and Ploen (2004). However, Chen and Wang (2005); Mesbahi and Egerstedt (2010); Olfati-Saber et al. (2007); Ren, Beard, and Atkins (2005); Ren, Beard et al. (2007) and Ren and Cao (2010) have mainly focused on consensus based formation control. Some important results, particularly on inter-agent distance based formation control, have not been extensively reviewed in those surveys. Scharf et al. (2004) have presented a survey of spacecraft formation flying rather than an extensive survey of general multi-agent systems. An excellent introduction of inter-agent distance based formation control is found in Anderson et al. (2008); however, a considerable amount of studies have been conducted thereafter. Thus we believe that it is timely and helpful to present an extensive survey of formation control of multi-agent systems.

Due to the vast amount of the literature, it would be challenging to exhaustively review the existing results on formation control. Rather than an exhaustive review, we thus focus on the characterization of formation control schemes in terms of the sensing capability and the interaction topology of agents because we believe that both of them are linked to the essential features of multi-agent formation control.

The characterization of formation control schemes in terms of the sensing capability and the interaction topology naturally leads to the question of what variables are sensed and what variables are actively controlled by multi-agent systems to achieve their de-

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Outline

- Gradient system and gradient control law
- Multi-agent formation control
- Distance-based formation control and graph rigidity
- Flocking/swarm dynamics

Introduction: Gradient system

Consider a twice differentiable function $F : U \to \mathbb{R}^n$

A dynamical system

$$\dot{x}(t) = -\nabla F(x(t)) = -\begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{bmatrix}$$

is called a *gradient dynamical system.*

For the gradient system,

- $\dot{F} \leq 0$ for all $x \in U$ (F is strictly decreasing on orbit, except at the equilibrium)
- $\dot{F} = 0$ if and only if x is an equilibrium of the gradient system

$$\begin{aligned} \frac{d}{dt} F(x) &= (\nabla F(x))^T \dot{x} \\ &= -(\nabla F(x))^T \nabla F(x) \\ &\leq 0 \end{aligned}$$

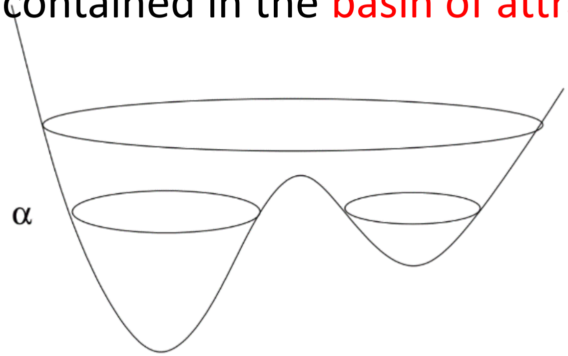
Gradient system

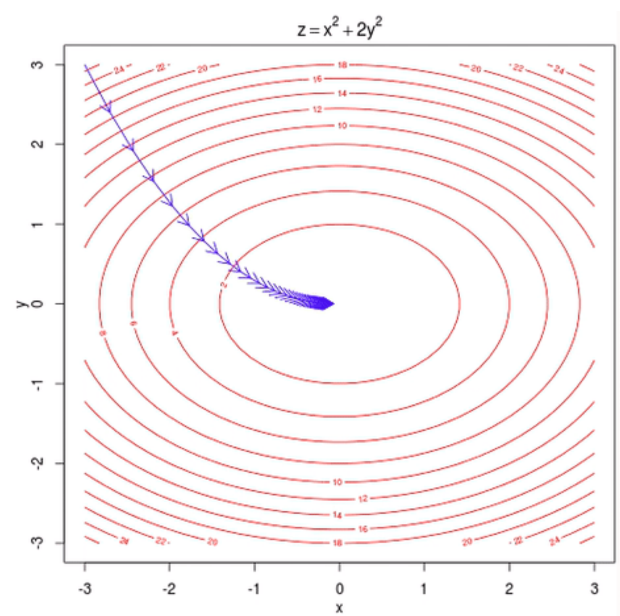
A gradient dynamical system

$$\dot{x}(t) = -\nabla F(x(t))$$

Let $\bar{x}$ be an isolated minimizer of F. Then $\bar{x}$ is an asymptotically stable equilibrium of the gradient system.

- If $\bar{x}$ is a strict local minimum of F with $F < \alpha$ and if the connected component of $\bar{x}$ in $\{x : F(x) \leq \alpha\}$ is compact and contains no equilibria other than $\bar{x}$, then that component is contained in the basin of attraction of $\bar{x}$.





- Isolated minimizer means for the points x around the minimizer, we have $F(x) > F(\bar{x})$.
- And $F(x)$ is a strict Lyapunov function.
- The vector field is perpendicular to the level surfaces.

Gradient system

A gradient dynamical system

$$\dot{x}(t) = -\nabla F(x(t))$$

Let $\bar{x}$ be an isolated minimizer of F. Then $\bar{x}$ is an asymptotically stable equilibrium of the gradient system.

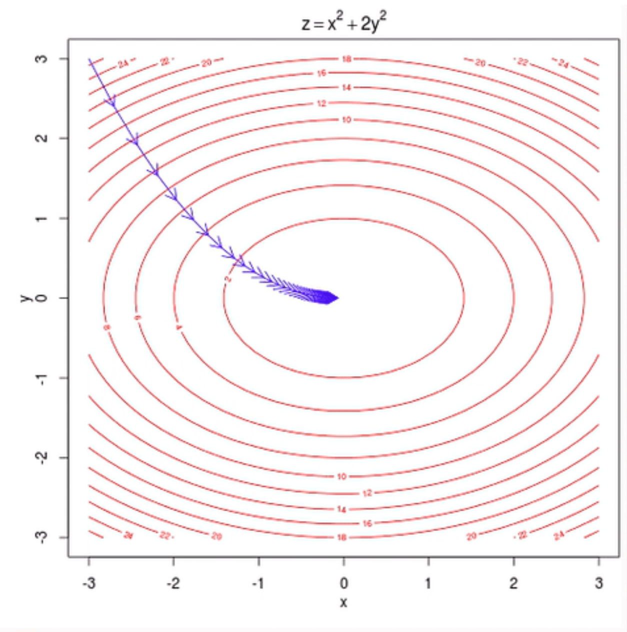
- Isolated minimizer means for the points x around the minimizer, we have $F(x) > F(\bar{x})$.

How about non-*isolated* minimizers?

ω-Limit Set

$\Omega = \{a \in W \mid \exists t_n \to \infty \text{ with } x(t_n) \to a\}$

Let $z \in \Omega$ be an ω-limit point of a trajectory of a gradient flow. Then z is an equilibrium.
(stable)



Gradient system for distributed control: consensus dynamics

Multi-agent consensus dynamics (in *undirected graph*) can be reformulated as a gradient system.

Consider a twice differentiable function $F : U \to \mathbb{R}^n$

$$F(x) = \frac{1}{2} x^T L\, x$$

Its gradient system coincidences with the consensus dynamics

$$\dot{x}(t) = -\nabla F(x(t)) = -\begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{bmatrix}$$

$$= -Lx(t) = -\begin{bmatrix} \vdots \\ \sum_{j \in N_i} w_{ij}(xi\,(t) - xj\,(t)) \\ \vdots \end{bmatrix}$$

The ω-limit set is the consensus subspace!

$$span(\mathbf{1}) = \{x |\; x_i = x_j\}$$

Multi-agent formation control

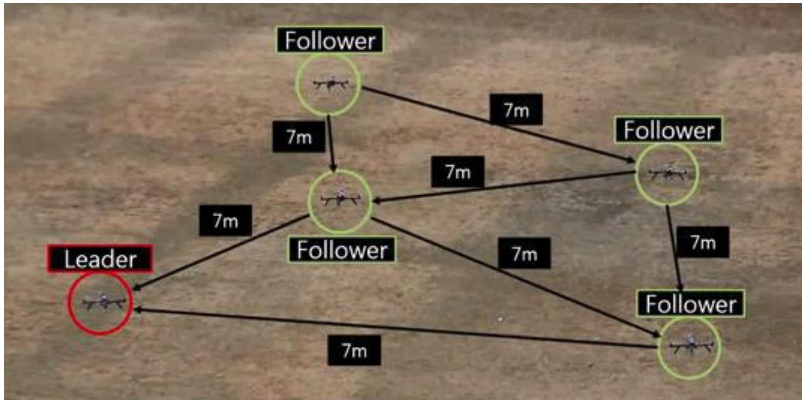
Multi-agent formation shape control

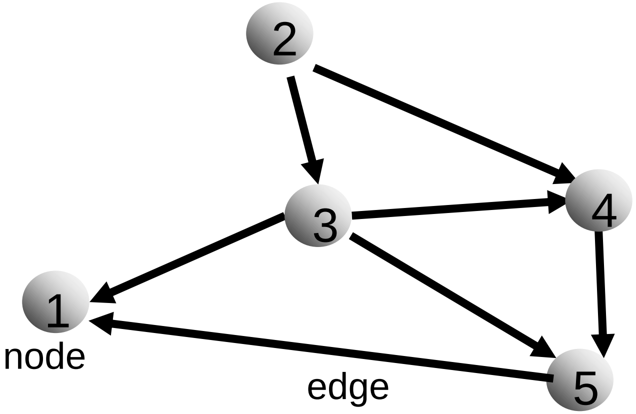
Formulation of the formation shape control problem

- Consider n agents (e.g., robotic vehicles or UAVs) with an internal state $p_i \in R^d$
  - Internal states are often position coordinates (but can be other geometric variables such as bearings or distances).

- Consider an internal dynamics for the state evolution
  - For now, we consider single integrator dynamics $\dot{p}_i = u_i$
  - Double-integrator dynamics are often used for flocking/swarm control.
  - Control tasks may also involve robotics kinematics/dynamics, e.g., non-holonomic motion constraints.

- Consider an interaction graph G
  - Having the agents as nodes, and sensing/communication links as edges.
  - Sensing and/or communication topologies are vital (they can be different).

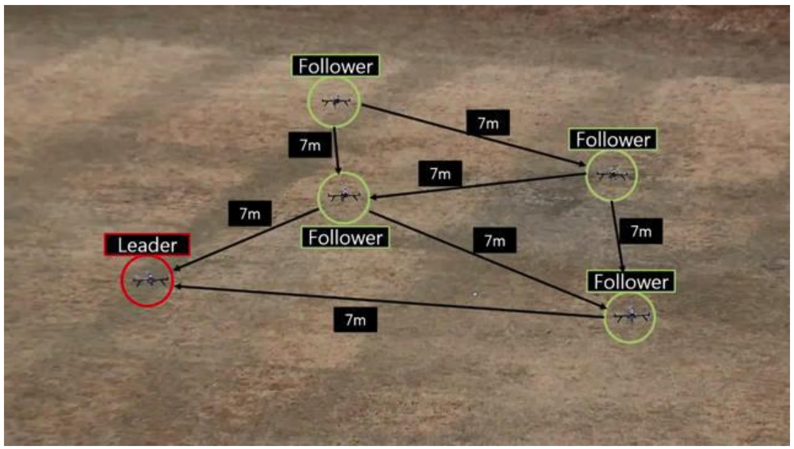


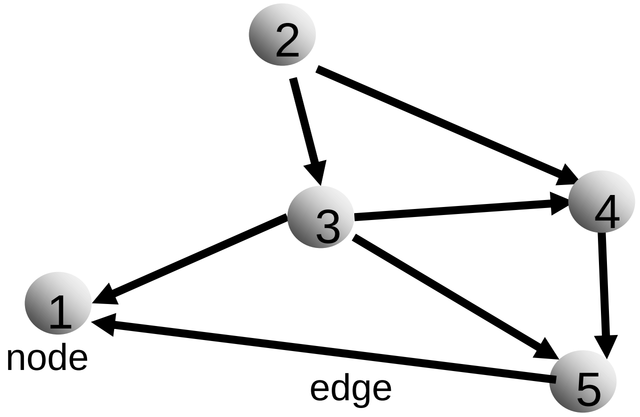


Multi-agent formation shape control

Formulation of the formation shape problem

- Consider n agents with an internal state $p_i \in R^d$, modelled by an internal dynamics for the state evolution $\dot{p}_i = ui$, and an interaction graph G.

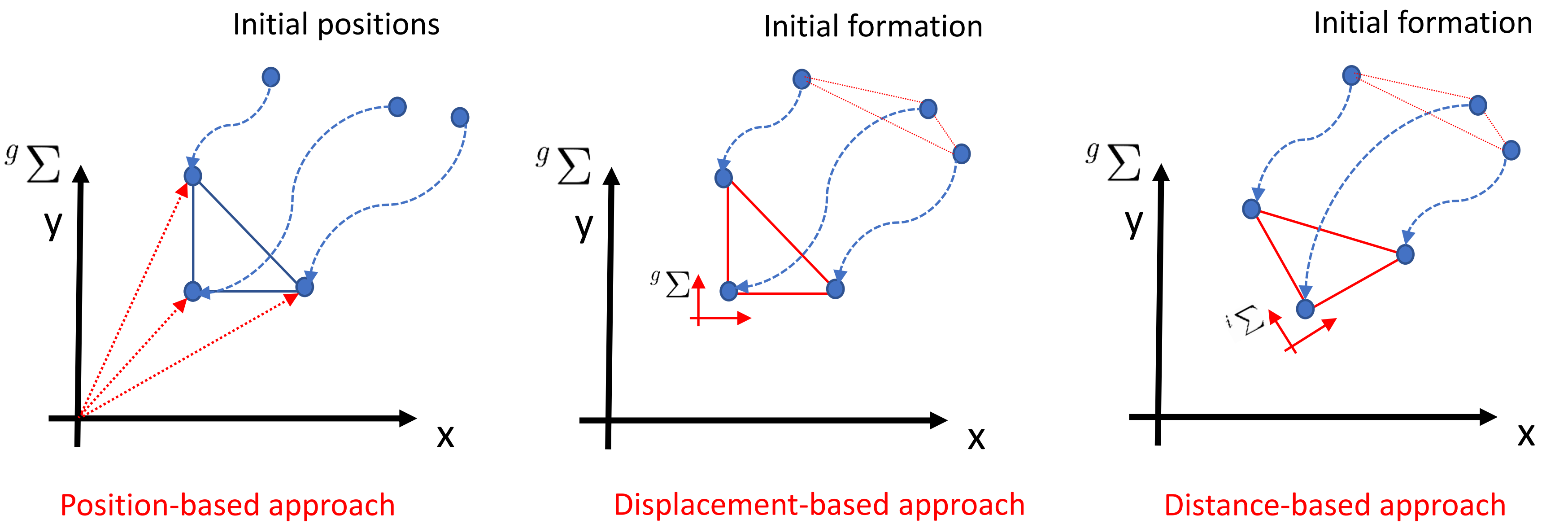




Problem: design the control inputs u_i such that

- All agents reach a target formation shape.
    - Target formation shape may be defined by relative positions/distances/bearings, etc.
- Control inputs use only relative information (relative position etc.) w.r.t. the neighbors' state.

Formation shape control: three common approaches

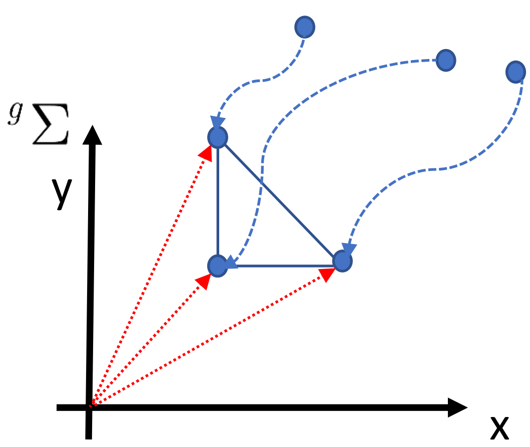


	Position-based
Sensed variables	Positions of agents
Controlled variables	Positions of agents
Coordinate systems	A global coordinate system
Interaction topology	Usually not required

	Displacement-based	Distance-based
Sensed variables	Relative positions of neighbors	Relative positions of neighbors
Controlled variables	Relative positions of neighbors	Inter-agent distances
Coordinate systems	Orientation aligned local coordinate systems	Local coordinate systems
Interaction topology	Connectedness or existence of a spanning tree	Rigidity or persistence

- Oh, K.K., Park, M.C. and Ahn, H.S., 2015. A survey of multi-agent formation control. *Automatica*, *53*, pp.424-440.

Position-based approach

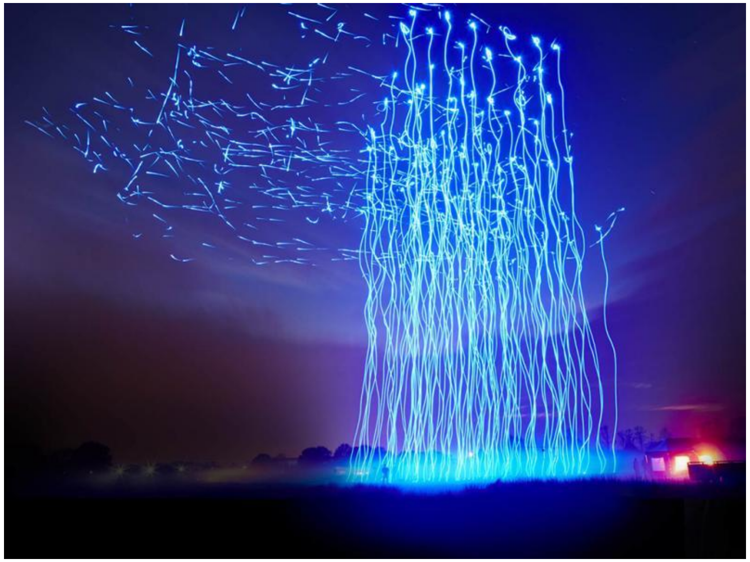


Position-based approach

	Position-based
Sensed variables	Positions of agents
Controlled variables	Positions of agents
Coordinate systems	A global coordinate system
Interaction topology	Usually not required

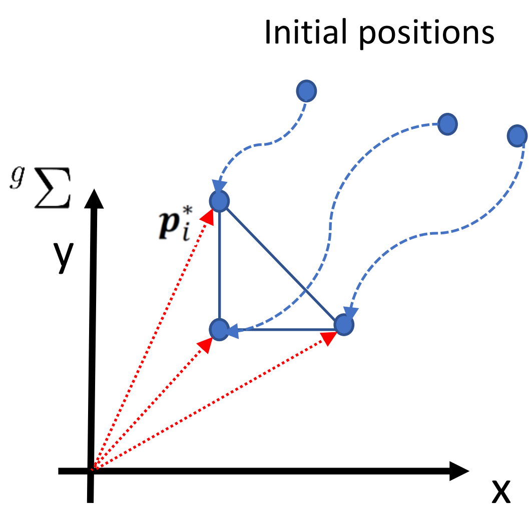
- Target formation shape is defined by *absolute positions*
- A global or common coordinate system is required.
- No interaction among agents
- Require absolute/global position measurement (e.g., GPS).

Drone formation light show, at Shenzhen, China
(by Shenzhen DAMODA)



Drone100 light show, at Linz Austria.
(by Intel)

Position-based approach



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$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i, i = 1, 2, \dots, n$$

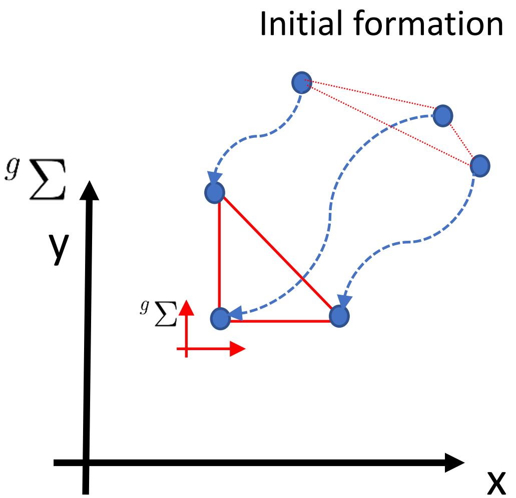
$$\boldsymbol{u}_i = k(\boldsymbol{p}_i^* - \boldsymbol{p}_i), \qquad i = 1, 2, \dots, n$$

Target position (absolute position in a global coordinate frame) that defines a target formation shape.

$$\boldsymbol{p}_i(t) = e^{-kt}\boldsymbol{p}_i(0) + \left(1 - e^{-kt}\right)\boldsymbol{p}_i^* \to \boldsymbol{p}_i^*, \qquad i = 1, 2, \dots, n$$

Displacement-based formation control (consensus-like dynamics)

Displacement-based approach (consensus-like dynamics)



Displacement-based approach

	Displacement-based
Sensed variables	Relative positions of neighbors
Controlled variables	Relative positions of neighbors
Coordinate systems	Orientation aligned local coordinate systems
Interaction topology	Connectedness or existence of a spanning tree

- Target formation shape is defined by *relative positions* (aka displacement vectors)
- A global orientation in a common coordinate system is required
- Control law requires relative position measurement

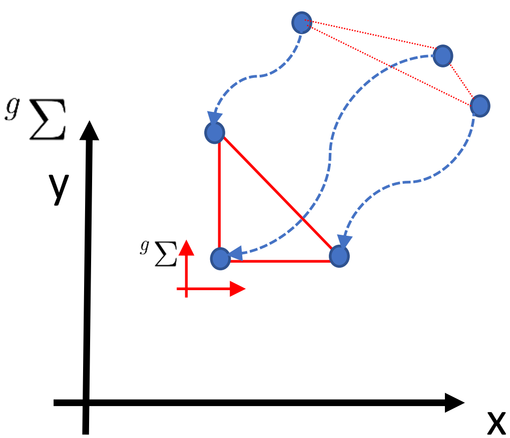
System dynamics: $\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i, i = 1, \dots, n$

Relative position vector: $\boldsymbol{z}_{ij} = \boldsymbol{p}_i - \boldsymbol{p}_j$

Target relative position vector (defining target formation shape): $\boldsymbol{z}_{ij}^* = \boldsymbol{p}_j^* - \boldsymbol{p}_i^* \in \mathbb{R}^d, j \in \mathcal{N}_i$

Formation control law: $\boldsymbol{u}_i = -\sum_{j \in \mathcal{N}_i} k_{ij}(\boldsymbol{z}_{ij}^* - \boldsymbol{z}_{ij}), i = 1, \dots, n$

Displacement-based approach (consensus-like dynamics)



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System dynamics: $$\dot{\boldsymbol{p}}_i = -\sum_{j\in\mathcal{N}_i} k_{ij}\left((\boldsymbol{p}_j^* - \boldsymbol{p}_j) - (\boldsymbol{p}_i^* - \boldsymbol{p}_i)\right), i = 1, \dots, n$$

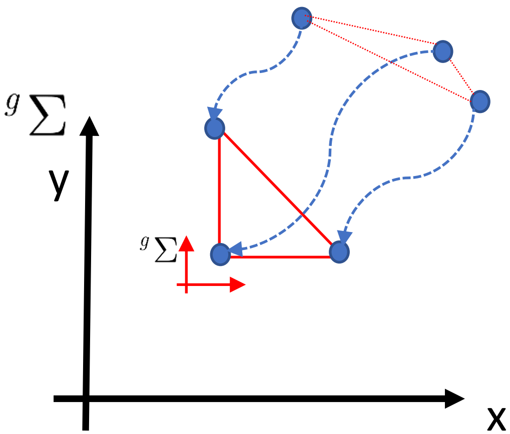
Formation error vector: $$\boldsymbol{r}_i = \boldsymbol{p}_i^* - \boldsymbol{p}_i,$$

$$\dot{\boldsymbol{r}}_i = \sum_{j\in\mathcal{N}_i} k_{ij}(\boldsymbol{r}_j - \boldsymbol{r}_i), i = 1, 2, \dots, n$$

Consensus-like dynamics: $$\dot{\boldsymbol{r}} = -(\boldsymbol{\mathcal{L}} \otimes \boldsymbol{I}_d)\boldsymbol{r} = -\bar{\boldsymbol{\mathcal{L}}}\boldsymbol{r}$$

Displacement-based approach (consensus-like dynamics)

Initial formation



Displacement-based approach

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System dynamics: $$\dot{\boldsymbol{p}}_i = -\sum_{j\in\mathcal{N}_i} k_{ij}\left((\boldsymbol{p}_j^* - \boldsymbol{p}_j) - (\boldsymbol{p}_i^* - \boldsymbol{p}_i)\right), i = 1, \dots, n$$

Consensus-like dynamics: $$\dot{\boldsymbol{r}} = -(\boldsymbol{\mathcal{L}} \otimes \boldsymbol{I}_d)\boldsymbol{r} = -\bar{\boldsymbol{\mathcal{L}}}\boldsymbol{r} \quad \text{with} \quad \boldsymbol{r}_i = \boldsymbol{p}_i^* - \boldsymbol{p}_i,$$

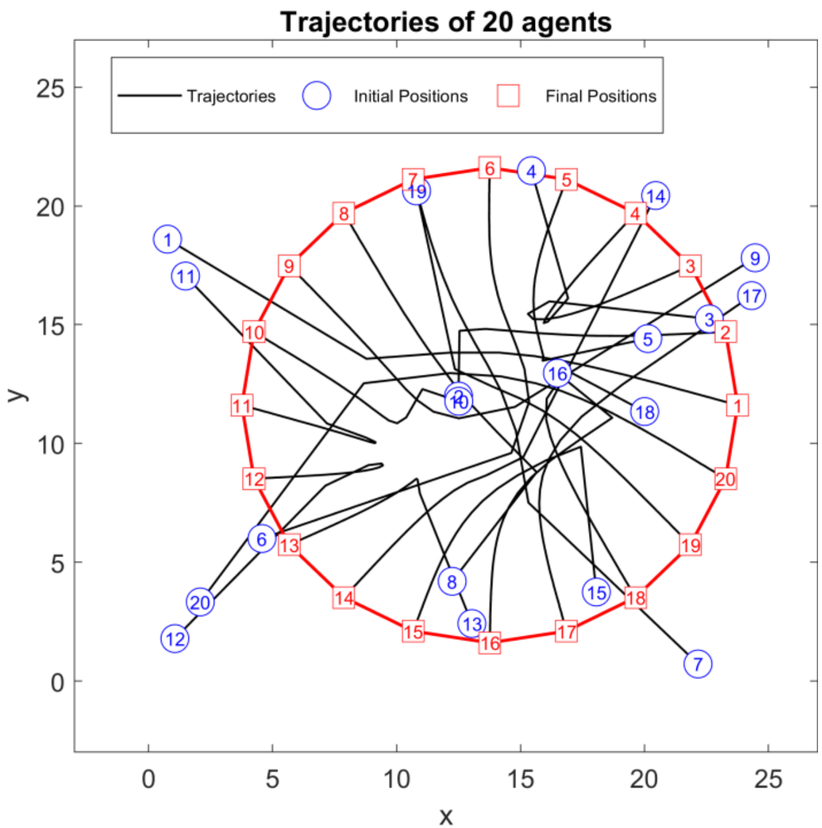
Under connectivity conditions (undirected/directed graphs)

$$\boldsymbol{r}(t) \to \text{span}\{\boldsymbol{1}_n \otimes \boldsymbol{I}_d\} \longrightarrow \boldsymbol{r}_i(t) = \boldsymbol{p}_i^* - \boldsymbol{p}_i(t) \to \boldsymbol{r}^* \quad \text{as} \quad t \to \infty$$

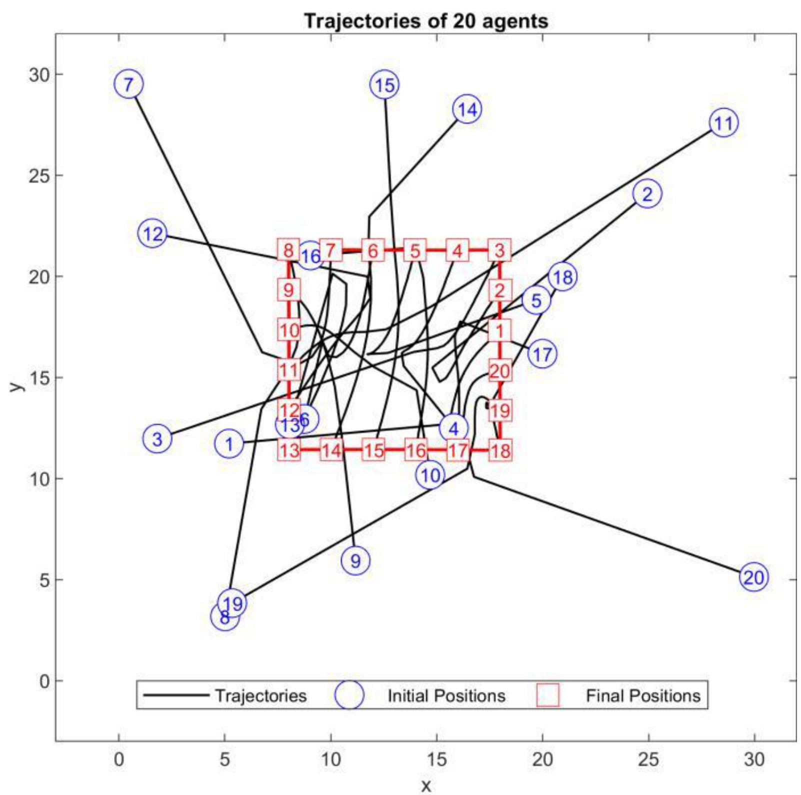
Therefore, $$\boldsymbol{p}_j(\infty) - \boldsymbol{p}_i(\infty) = \boldsymbol{p}_j^* - \boldsymbol{p}_i^* = \boldsymbol{z}_{ij}^*$$

Displacement-based approach: simulations

System dynamics: $\dot{\boldsymbol{p}}_i = -\sum_{j \in \mathcal{N}_i} k_{ij} \left((\boldsymbol{p}_j^* - \boldsymbol{p}_j) - (\boldsymbol{p}_i^* - \boldsymbol{p}_i) \right), i = 1, \ldots, n$



(a) A circle formation



(b) A square formation

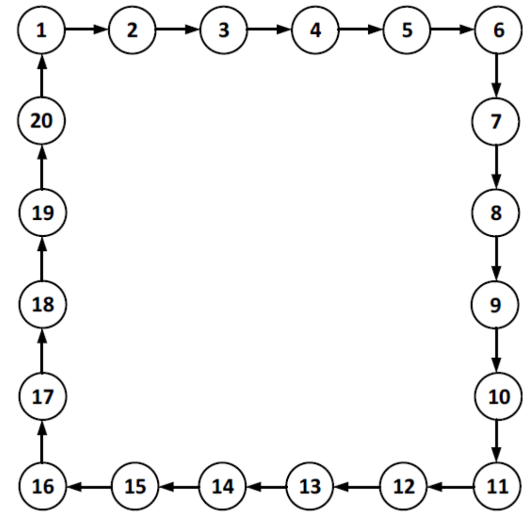
- Control 20 agents to form (a) a circle formation, (b) a square formation
- Target formation shape is defined by displacement vectors (relative positions)

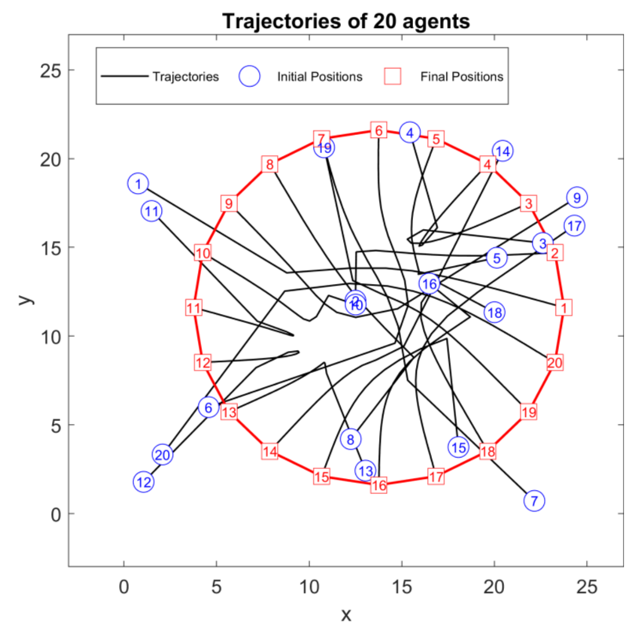
Displacement-based approach: simulations

```
Clear

p0=25*rand(40,1); % random initial position vector
H=eye(20); H(20,1)=-1;

% Incidence matrix for cycle graph
for i=1:1:19
     H(i,i+1)=-1;
end

% Desired relative positions P (Circle)
% and Laplacian matrix L
global Desired_P L;
L=H'*H;
% construct the desired relative positions (circle)
Desired_P=[10;0];
for i=1:1:19
    R=10*[cos(i*pi/10);sin(i*pi/10)];
    Desired_P=[Desired_P;R];
end
[t,p] = ode45(@formation_control_displacement,[0:150],p0);
plot
```

```
function dX = formation_control_displacement(t, X)
global L
global Desired_P

% show the simulation running time
t

dX = -kron(L,eye(2))*(X-Desired_P);
end
```

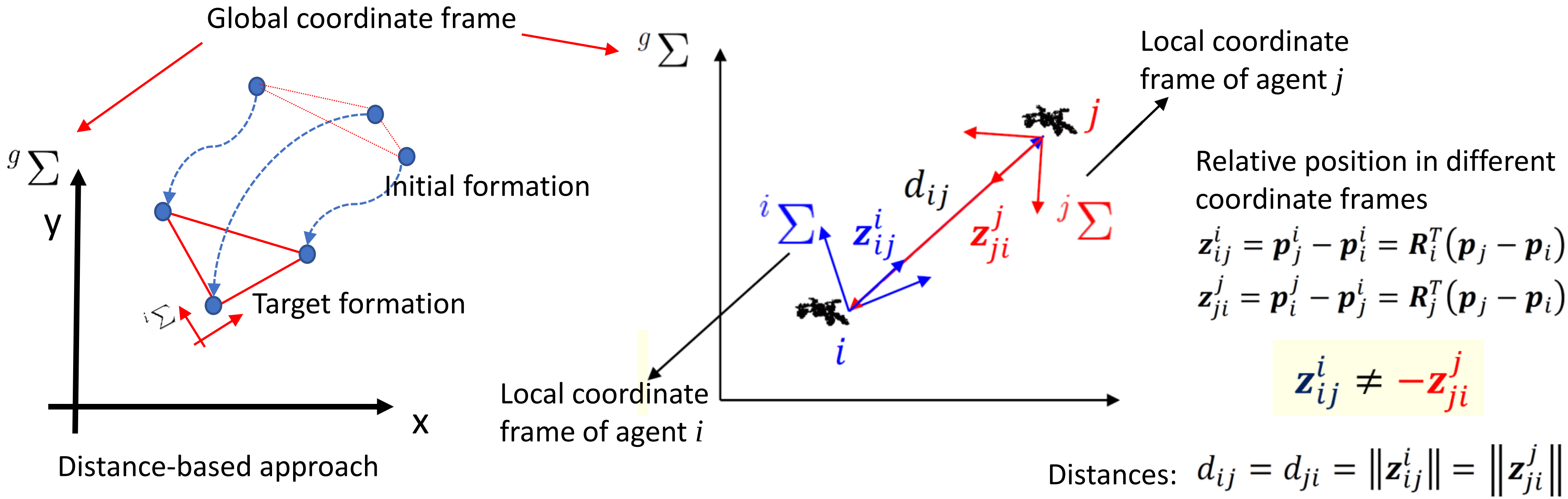
Communication/sensing graph (cycle graph)





Distance-based formation control
Graph rigidity theory

Distance-based formation control



Target formation shape is defined by a set of edge distances

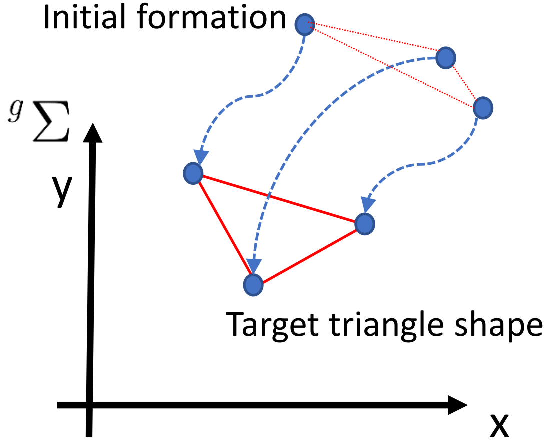
$$\boldsymbol{d}_{ij}^{*} = \|\boldsymbol{p}_j^* - \boldsymbol{p}_i^*\|, j \in \mathcal{N}_i$$

Distance-based formation control aims to achieve the desired formation shape for all edge distances

$$d_{ij} \triangleq \|\boldsymbol{p}_j^i - \boldsymbol{p}_i^i\| = d_{ji} \longrightarrow \boldsymbol{d}_{ij}^{*} = \|\boldsymbol{p}_j^* - \boldsymbol{p}_i^*\|, j \in \mathcal{N}_i$$

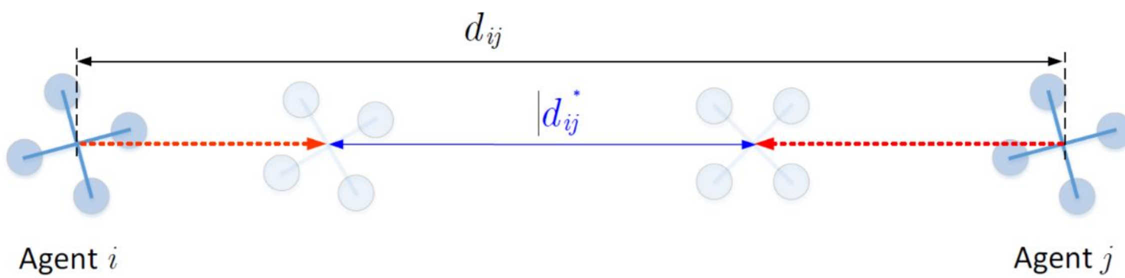
	Distance-based
Sensed variables	Relative positions of neighbors
Controlled variables	Inter-agent distances
Coordinate systems	Local coordinate systems
Interaction topology	Rigidity or persistence

Distance-based approach: gradient control law



Distance-based approach

Two agents to achieve a desired distance:



Current distance: $d_{ij} \triangleq \left\| \boldsymbol{p}_j^i - \boldsymbol{p}_i^i \right\| = d_{ji}$

Target distance: $d_{ij}^* = d_{ji}^*$ squared distance errors: $e_{ij} = \left(d_{ij}^*\right)^2 - d_{ij}^2$

Formation potential function: $V = \frac{1}{4}\left(\left(d_{ij}^*\right)^2 - d_{ij}^2\right)^2 = \frac{1}{4} e_{ij}^2,$

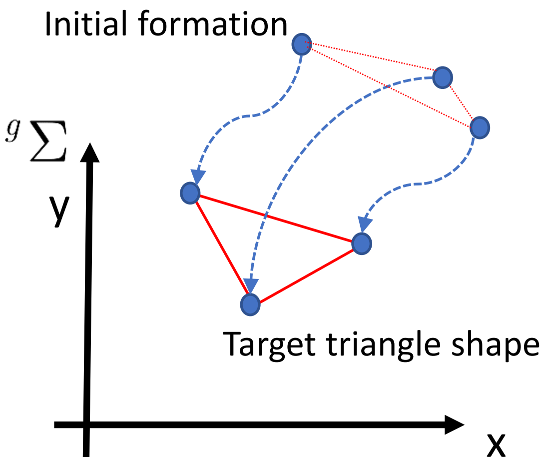
Gradient-based formation system (to minimize the potential function):

$$\dot{\boldsymbol{p}}_i^i = -\left(\left(d_{ij}^*\right)^2 - d_{ij}^2\right)\left(\boldsymbol{p}_j^i - \boldsymbol{p}_i^i\right)$$

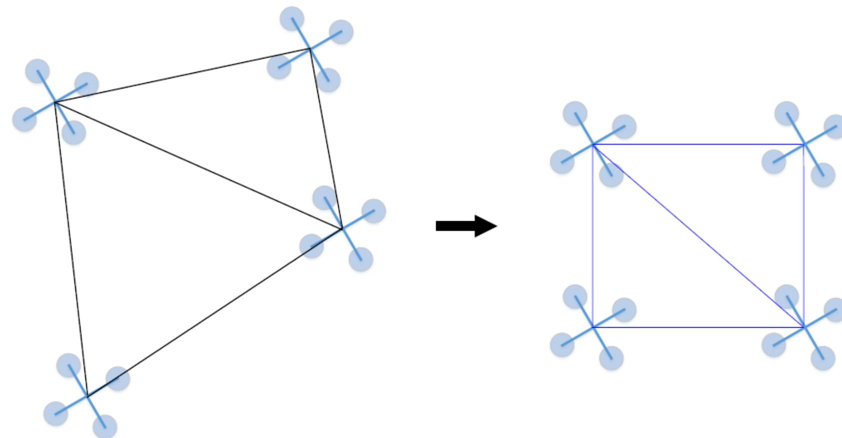
$$\dot{\boldsymbol{p}}_j^j = -\left(\left(d_{ji}^*\right)^2 - d_{ji}^2\right)\left(\boldsymbol{p}_i^j - \boldsymbol{p}_j^j\right)$$

Gradient system of V !

Distance-based approach: gradient control law



Distance-based approach



Initial formation

Desired formation

Current distances: $d_{ij} \triangleq \left\| \boldsymbol{p}_j^i - \boldsymbol{p}_i^i \right\| = d_{ji}$

Target distances: $d_{ij}^* = d_{ji}^*$ squared distance errors: $e_{ij} = \left(d_{ij}^*\right)^2 - d_{ij}^2$

Formation potential function: $V = \frac{1}{4} \sum_{(i,j)\in E} \left(\left(d_{ij}^*\right)^2 - d_{ij}^2 \right)^2 = \frac{1}{4} \sum_{(i,j)\in E} e_{ij}^2$

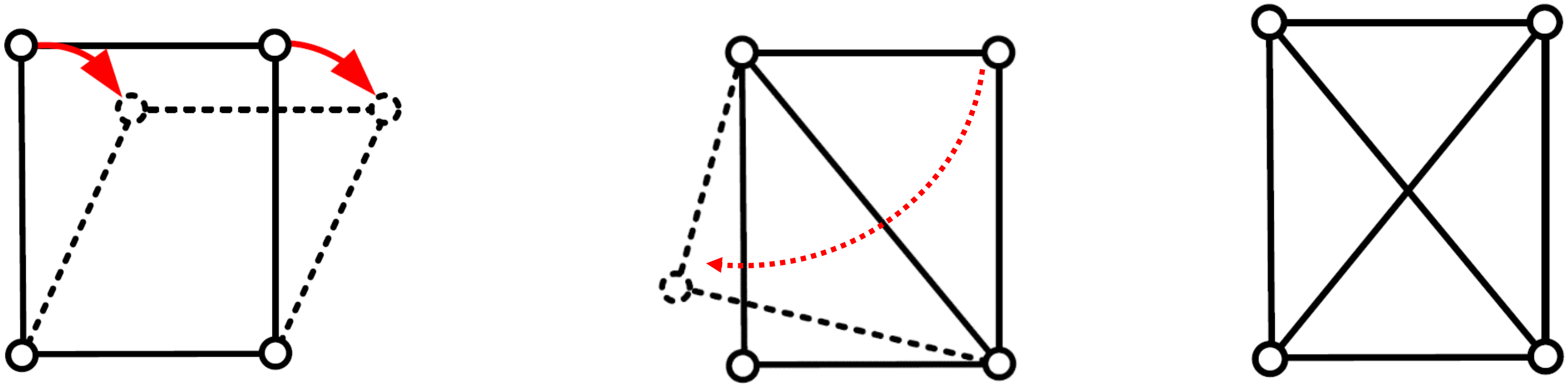
Formation system dynamics (gradient system of *V*):

$$\dot{\boldsymbol{p}}_i^i = -\sum_{j\in\mathcal{N}_i} \left(\left(d_{ij}^*\right)^2 - d_{ij}^2 \right) \left(\boldsymbol{p}_j^i - \boldsymbol{p}_i^i \right) = -\sum_{j\in\mathcal{N}_i} \left(\left(d_{ij}^*\right)^2 - d_{ij}^2 \right) \boldsymbol{z}_{ij}^i$$

Graph rigidity (in 2D): bar and joint framework

Consider a formation where each edge in the graph imposes a desired distance to the incident pair:

- Do the distance constraints determine the shape (spatial arrangement) of the agents?
- Graph (distance) rigidity theory!
- Consider a distance-constrained formation as a bar-and-joint framework. A formation is rigid if the formation shape behaves as a planar rigid body.



Non-rigid (flexible) | Rigid (minimally rigid) | Rigid (globally rigid)

Definition: A framework $(G, \boldsymbol{p})$ is a pair: embedded node position $\boldsymbol{p} = [\boldsymbol{p}_1^T \quad \cdots \quad \boldsymbol{p}_n^T]^T \in \mathbb{R}^{2n}$ in a graph G.

- A framework is minimally rigid if the removal of any edge yields a non-rigid framework.

Graph rigidity (in 2D): bar and joint framework

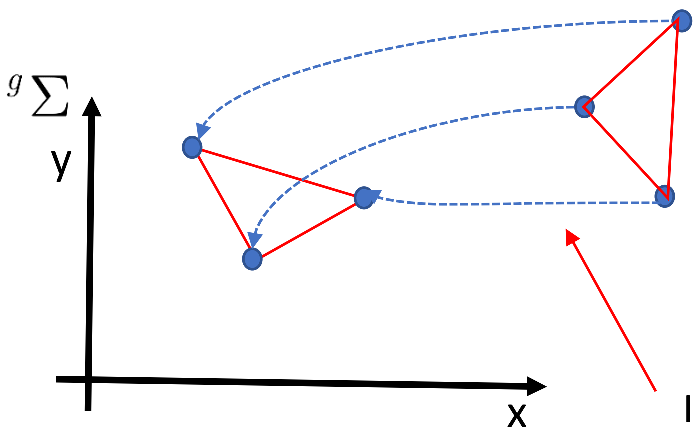
Two frameworks $(G, \boldsymbol{p})$ and $(G, \boldsymbol{q})$ are *equivalent if* $\|\boldsymbol{p}_i - \boldsymbol{p}_j\| = \|\boldsymbol{q}_i - \boldsymbol{q}_j\|, \forall (i,j) \in \mathcal{E}$

Two frameworks $(G, \boldsymbol{p})$ and $(G, \boldsymbol{q})$ are *congruent if* $\|\boldsymbol{p}_i - \boldsymbol{p}_j\| = \|\boldsymbol{q}_i - \boldsymbol{q}_j\|, \forall i, j \in \mathcal{V}, i \neq j$

A framework $(G, \boldsymbol{p})$ is rigid if there exists an ε such that every framework $(G, \overline{\boldsymbol{p}})$ that is equivalent to $(G, \boldsymbol{p})$ and satisfies $\left|\left|\boldsymbol{p} - \overline{\boldsymbol{p}}\right|\right| < \varepsilon$ for all nodes i, is congruent to $(G, \boldsymbol{p})$.

- Intuition: a rigid framework should behave like a rigid body (bar-joint structure), and its shape resembles to a framework with a complete graph.

Infinitesimal trivial motions that preserve all distances



Distance functions

$$\|p_i(t) - p_j(t)\|^2 = d_{ij}^2, (i,j) \in E$$

To preserve the distances

$$\frac{d}{dt}\|p_i - p_j\|^2 = 0, (i,j) \in E$$

Infinitesimal motions

$$(v_i(t) - v_j(t))^T (p_i(t) - p_j(t)) = 0, (i,j) \in E$$

Graph rigidity: infinitesimal rigidity from rigidity matrix

Distance function vector for all edges:

$$\boldsymbol{f}_G(\boldsymbol{p}) = \left[\cdots \quad \left\|\boldsymbol{p}_i - \boldsymbol{p}_j\right\|^2 \quad \cdots\right]^T$$
$$= \left[\cdots \quad \left\|\boldsymbol{z}_{ij}\right\|^2 \quad \cdots\right]^T = [\cdots \quad \|\boldsymbol{z}_k\|^2 \quad \cdots]^T$$

Derivative of $\boldsymbol{f}_G(\boldsymbol{p})$ w.r.t. $\boldsymbol{p}$ gives the rigidity matrix

$$\boldsymbol{R}(\boldsymbol{p}) = \frac{1}{2}\frac{\partial \boldsymbol{f}_G(\boldsymbol{p})}{\partial \boldsymbol{p}} = \boldsymbol{D}(\boldsymbol{z})^T \overline{\boldsymbol{H}}$$

$\overline{\boldsymbol{H}} := H \otimes I_2$, incidence matrix (with Kronecker product)

Block diagonal matrix with relative position vector

$$\boldsymbol{D}(\boldsymbol{z}) = \begin{bmatrix} \boldsymbol{z}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \boldsymbol{z}_m \end{bmatrix}$$

Online quiz: for a 2-D framework (aka 2-D formation) $(G, \boldsymbol{p})$ with *n* agents and *m* edges, what is the size of its rigidity matrix?

(a) n x m; (b) m x n; (c) 2n x m; (d) m x 2n.

Graph rigidity: infinitesimal rigidity from rigidity matrix

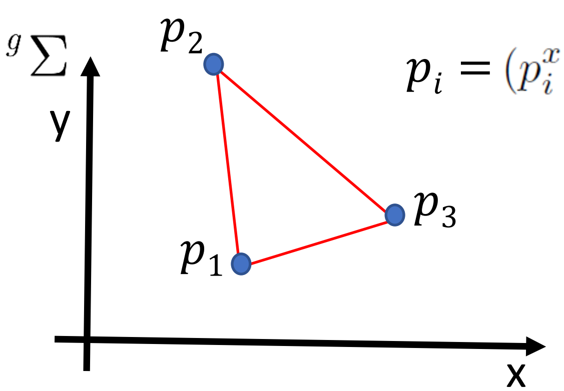
Distance function vector for all edges:

$$\boldsymbol{f}_G(\boldsymbol{p}) = \begin{bmatrix} \cdots & \|\boldsymbol{p}_i - \boldsymbol{p}_j\|^2 & \cdots \end{bmatrix}^T$$
$$= \begin{bmatrix} \cdots & \|\boldsymbol{z}_{ij}\|^2 & \cdots \end{bmatrix}^T = [\cdots \quad \|\boldsymbol{z}_k\|^2 \quad \cdots]^T$$

Derivative of $\boldsymbol{f}_G(\boldsymbol{p})$ w.r.t. $\boldsymbol{p}$ gives the rigidity matrix

$$\boldsymbol{R}(\boldsymbol{p}) = \frac{1}{2}\frac{\partial \boldsymbol{f}_G(\boldsymbol{p})}{\partial \boldsymbol{p}} = \boldsymbol{D}(\boldsymbol{z})^T \overline{\boldsymbol{H}}$$

$\overline{\boldsymbol{H}} := H \otimes I_2$, incidence matrix (with Kronecker product)



$p_i = (p_i^x, p_i^y)$

Block diagonal matrix with relative position vector
$$\boldsymbol{D}(\boldsymbol{z}) = \begin{bmatrix} \boldsymbol{z}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \boldsymbol{z}_m \end{bmatrix}$$

$$R(p) = \begin{bmatrix} p_1^x - p_2^x & p_1^y - p_2^y & p_2^x - p_1^x & p_2^y - p_1^y & 0 & 0 \\ p_1^x - p_3^x & p_1^y - p_3^y & 0 & 0 & p_3^x - p_1^x & p_3^y - p_1^y \\ 0 & 0 & p_2^x - p_3^x & p_2^y - p_3^y & p_3^x - p_2^x & p_3^y - p_2^y \end{bmatrix}$$

Rank(R) = 3

Graph rigidity: infinitesimal rigidity from rigidity matrix

Distance function vector for all edges:

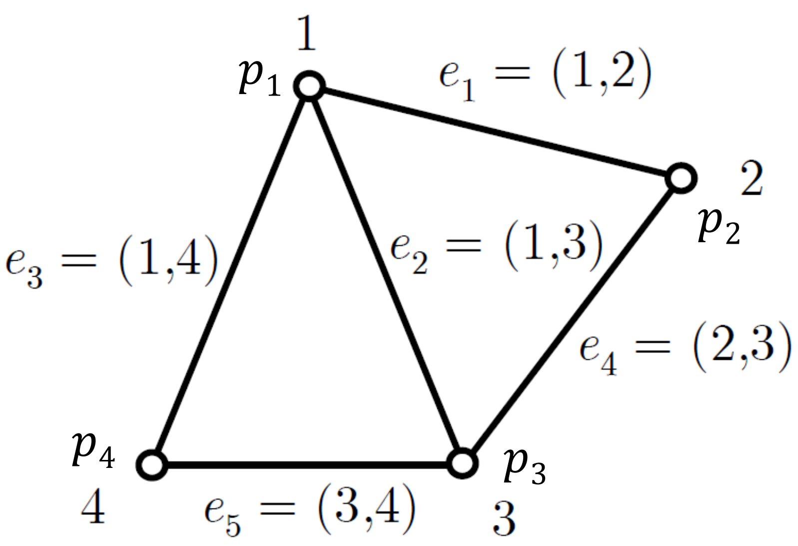
$$\boldsymbol{f}_G(\boldsymbol{p}) = \left[\cdots \quad \left\|\boldsymbol{p}_i - \boldsymbol{p}_j\right\|^2 \quad \cdots\right]^T$$
$$= \left[\cdots \quad \left\|\boldsymbol{z}_{ij}\right\|^2 \quad \cdots\right]^T = [\cdots \quad \|\boldsymbol{z}_k\|^2 \quad \cdots]^T$$

Derivative of $\boldsymbol{f}_G(\boldsymbol{p})$ w.r.t. $\boldsymbol{p}$ gives the rigidity matrix

$$\boldsymbol{R}(\boldsymbol{p}) = \frac{1}{2}\frac{\partial \boldsymbol{f}_G(\boldsymbol{p})}{\partial \boldsymbol{p}} = \boldsymbol{D}(\boldsymbol{z})^T \overline{\boldsymbol{H}}$$

$\overline{\boldsymbol{H}} := H \otimes I_2$, incidence matrix (with Kronecker product)

Block diagonal matrix with relative position vector $\boldsymbol{D}(\boldsymbol{z}) = \begin{bmatrix} \boldsymbol{z}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \boldsymbol{z}_m \end{bmatrix}$



$$\boldsymbol{R} = \begin{bmatrix} (\boldsymbol{p}_1 - \boldsymbol{p}_2)^T & (\boldsymbol{p}_2 - \boldsymbol{p}_1)^T & \mathbf{0} & \mathbf{0} \\ (\boldsymbol{p}_1 - \boldsymbol{p}_3)^T & \mathbf{0} & (\boldsymbol{p}_3 - \boldsymbol{p}_1)^T & \mathbf{0} \\ (\boldsymbol{p}_1 - \boldsymbol{p}_4)^T & \mathbf{0} & \mathbf{0} & (\boldsymbol{p}_4 - \boldsymbol{p}_1)^T \\ \mathbf{0} & (\boldsymbol{p}_2 - \boldsymbol{p}_3)^T & (\boldsymbol{p}_3 - \boldsymbol{p}_2)^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\boldsymbol{p}_3 - \boldsymbol{p}_4)^T & (\boldsymbol{p}_4 - \boldsymbol{p}_3)^T \end{bmatrix}$$

Infinitesimal rigidity from rigidity matrix (2D)

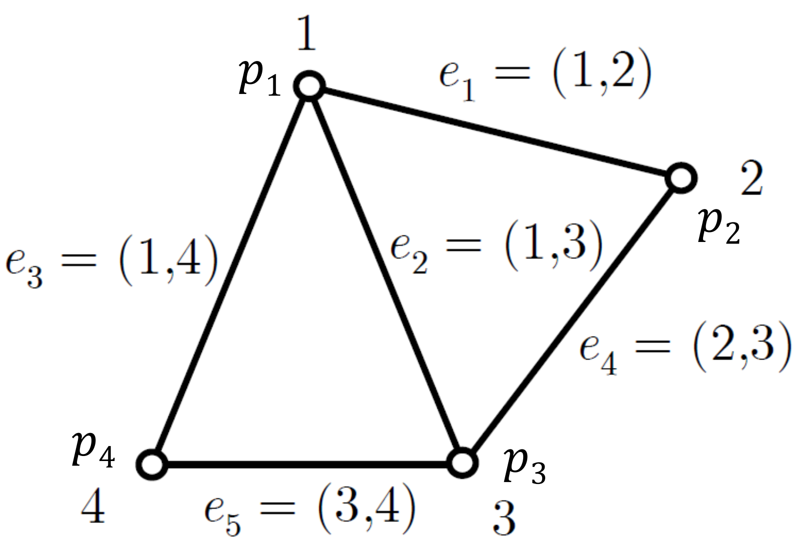
Rigidity matrix

$$\boldsymbol{R}(\boldsymbol{p}) = \frac{1}{2}\frac{\partial \boldsymbol{f}_G(\boldsymbol{p})}{\partial \boldsymbol{p}} = \boldsymbol{D}(\boldsymbol{z})^T \overline{\boldsymbol{H}}$$

A 2-D framework is infinitesimally rigid if and only if

$$\operatorname{rank}\big(\boldsymbol{R}(\boldsymbol{p})\big) = 2n - 3$$

- Intuition: For a rigid framework, the null space of *R* consists of only translation motion (2 DOFs) and a rotation (1 DOF).
- If the number of edges is 2n-3 (the same to rank(*R*)), then the framework is minimally rigid.



$$\boldsymbol{R} = \begin{bmatrix} (\boldsymbol{p}_1 - \boldsymbol{p}_2)^T & (\boldsymbol{p}_2 - \boldsymbol{p}_1)^T & \boldsymbol{0} & \boldsymbol{0} \\ (\boldsymbol{p}_1 - \boldsymbol{p}_3)^T & \boldsymbol{0} & (\boldsymbol{p}_3 - \boldsymbol{p}_1)^T & \boldsymbol{0} \\ (\boldsymbol{p}_1 - \boldsymbol{p}_4)^T & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{p}_4 - \boldsymbol{p}_1)^T \\ \boldsymbol{0} & (\boldsymbol{p}_2 - \boldsymbol{p}_3)^T & (\boldsymbol{p}_3 - \boldsymbol{p}_2)^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{p}_3 - \boldsymbol{p}_4)^T & (\boldsymbol{p}_4 - \boldsymbol{p}_3)^T \end{bmatrix}$$

Choose a set of *generic* position vectors *p*:

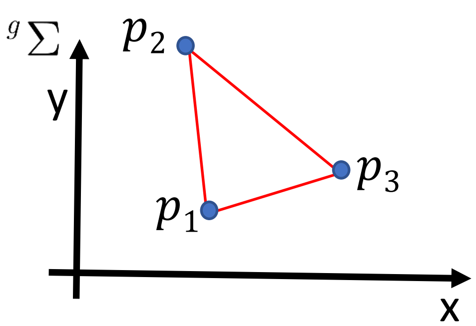
$$rank(\boldsymbol{R}) = 5$$

$\longrightarrow$ The framework is rigid.

Remarks on graph rigidity

- Rigidity is a property for graph, while infinitesimal rigidity is a property for framework (with both graph topology and embedded vertex positions).
  - The rank of rigidity matrix rank(***R***(***p***)) depends on both graph topology (*G*) and framework embedding (position ***p***).
  - For *generic* position ***p***, ***R***(***p***) attains its maximum rank and rank(***R***(***p***)) is used to evaluate infinitesimal rigidity of a framework.
  - There exist special positions ***p*** for a rigid framework that the rigidity matrix ***R*** loses rank, but these positions are non-generic.

- Infinitesimal rigidity implies rigidity.

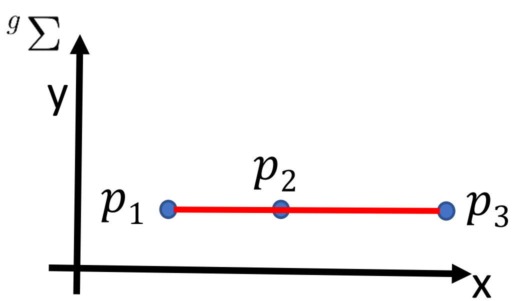


$$R(p) = \begin{bmatrix} p_1^x - p_2^x & p_1^y - p_2^y & p_2^x - p_1^x & p_2^y - p_1^y & 0 & 0 \\ p_1^x - p_3^x & p_1^y - p_3^y & 0 & 0 & p_3^x - p_1^x & p_3^y - p_1^y \\ 0 & 0 & p_2^x - p_3^x & p_2^y - p_3^y & p_3^x - p_2^x & p_3^y - p_2^y \end{bmatrix}$$

generic position ***p*** (a generic triangle)

```
% p1 = [2, 1]';
% p2 = [3, 2]';
% p3 = [1, 4]';
```

Rank(R) = 3



Non-generic position ***p*** (triangle reduces to a line shape)

```
% p1 = [2, 1]';
% p2 = [5, 1]';
% p3 = [8, 1]';
```

Rank(R) = 2

Stability of distance-based formation system

Potential function

$$V = \frac{1}{4}\sum_{(i,j)\in E}\left((d_{ij}^*)^2 - d_{ij}^2\right)^2 = \frac{1}{4}\|\boldsymbol{f}_G(\boldsymbol{p}^*) - \boldsymbol{f}_G(\boldsymbol{p})\|^2$$

Formation control system

Agent i: $\dot{\boldsymbol{p}}_i^i = -\sum_{j\in\mathcal{N}_i}\left((d_{ij}^*)^2 - d_{ij}^2\right)(\boldsymbol{p}_j^i - \boldsymbol{p}_i^i) = -\boldsymbol{\nabla}_{\boldsymbol{p}_i}V$

Compact form: $\dot{\boldsymbol{p}} = -\boldsymbol{\nabla}_{\boldsymbol{p}}V = \frac{1}{2}\left(\frac{\partial \boldsymbol{f}_G(\boldsymbol{p})}{\partial \boldsymbol{p}}\right)^T\left(\boldsymbol{f}_G(\boldsymbol{p}^*) - \boldsymbol{f}_G(\boldsymbol{p})\right) = \boldsymbol{R}(\boldsymbol{p})^T\boldsymbol{e}$

Distance error vector $\boldsymbol{e} = [\dots, e_{ij}, \dots]^{\mathrm{T}} = [e_1, \dots, e_m]^{\mathrm{T}}$,

$$e_{ij} = (d_{ij}^*)^2 - d_{ij}^2$$

Time derivative of *V*:

$$V = \frac{1}{4}\sum_{(i,j)\in E}\|\boldsymbol{f}_G(\boldsymbol{p}^*) - \boldsymbol{f}_G(\boldsymbol{p})\|^2 = \frac{1}{4}\boldsymbol{e}^T\boldsymbol{e} \longrightarrow \dot{V} = -\boldsymbol{e}^T\boldsymbol{R}(\boldsymbol{p})\boldsymbol{R}(\boldsymbol{p})^T\boldsymbol{e} \le 0$$

Stability of distance-based formation system

Potential function

$$V = \frac{1}{4} \sum_{(i,j)\in E} \|\boldsymbol{f}_G(\boldsymbol{p}^*) - \boldsymbol{f}_G(\boldsymbol{p})\|^2 = \frac{1}{4}\boldsymbol{e}^T\boldsymbol{e}$$

Formation control system

$$\dot{\boldsymbol{p}} = -\boldsymbol{\nabla}_{\boldsymbol{p}} V = \frac{1}{2}\left(\frac{\partial \boldsymbol{f}_G(\boldsymbol{p})}{\partial \boldsymbol{p}}\right)^T \left(\boldsymbol{f}_G(\boldsymbol{p}^*) - \boldsymbol{f}_G(\boldsymbol{p})\right) = \boldsymbol{R}(\boldsymbol{p})^T\boldsymbol{e}$$

Time derivative of V: $\dot{V} = -\boldsymbol{e}^T\boldsymbol{R}(\boldsymbol{p})\boldsymbol{R}(\boldsymbol{p})^T\boldsymbol{e} \leq 0$

If $\boldsymbol{R}(\boldsymbol{p})$ is of full row rank $\longrightarrow$ $\boldsymbol{R}(\boldsymbol{p})\boldsymbol{R}(\boldsymbol{p})^T$ is positive definite

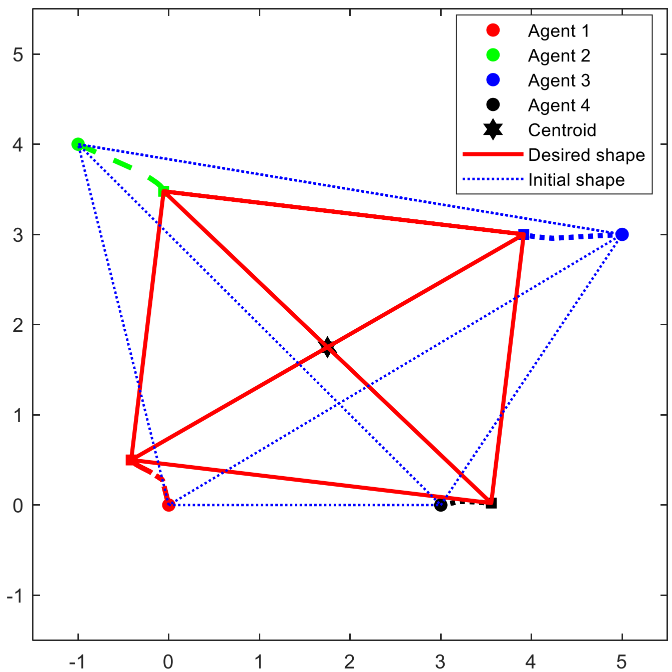
Condition: the formation is infinitesimally and minimally rigid.

The convergence of the formation system is (locally) exponentially fast

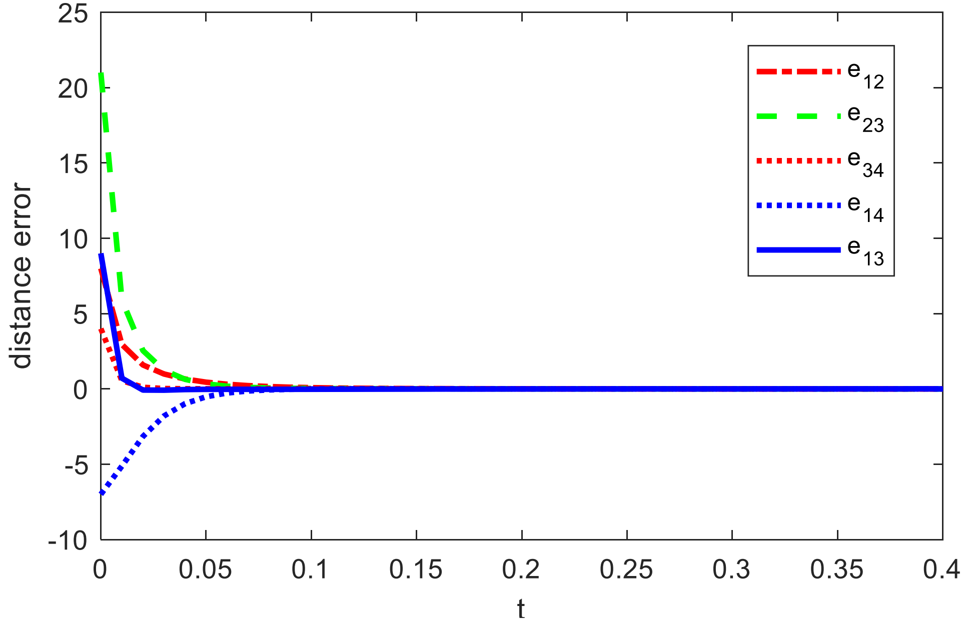
$$\begin{aligned}\dot{V} &= -\frac{\boldsymbol{e}^T\boldsymbol{R}(\boldsymbol{p})\boldsymbol{R}(\boldsymbol{p})^T\boldsymbol{e}}{V}V \\ &= -4\frac{\boldsymbol{e}^T\boldsymbol{R}(\boldsymbol{p})\boldsymbol{R}(\boldsymbol{p})^T\boldsymbol{e}}{\boldsymbol{e}^T\boldsymbol{e}}V \\ &\leq -4\lambda_{\min}(\boldsymbol{R}(\boldsymbol{p})\boldsymbol{R}(\boldsymbol{p})^T)V\end{aligned}$$

Distance-based formation: simulations

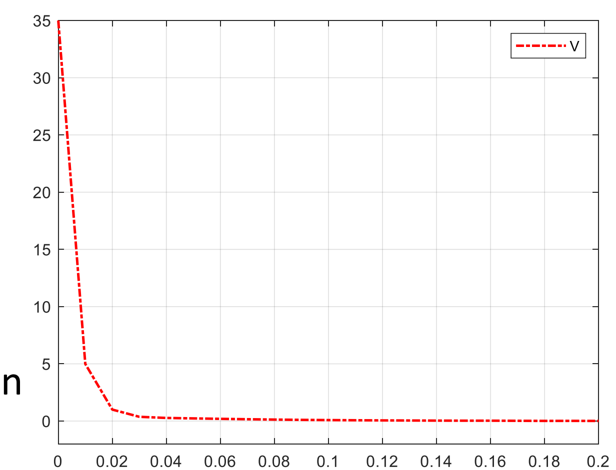
Control 4 agents to reach a rectangular formation shape



Convergence of the formation shape



Convergence of distance errors



Exponential decay of the potential function

Multi-agent flocking/swarm (with formation constraints)

Flocking dynamics of double integrators

Consider the double-integrator model

$$\dot{p}_i = v_i$$
$$\dot{v}_i = u_i$$

p_i, v_i: agent i's position and velocity;
u_i: agent i's control input (for acceleration)

Control input with both *velocity consensus and formation shape control* (flocking dynamics)

$$u_i = f_{velocity\ consensus} + f_{formation}$$

$$f_{velocity\ consensus} = \sum_{j \in \mathcal{N}_i} (v_j - v_i)$$

$f_{formation}$ *Displacement-based or distance-based control law*

Flocking dynamics of double integrators: displacement-based formation

Consider the double-integrator model

$$\dot{p}_i = v_i$$
$$\dot{v}_i = u_i$$

Control input with both *velocity consensus and formation shape control* (displacement-based formation control)

$$u_i = \gamma \sum_{j \in N_i} (v_j - v_i) + \sum_{j \in N_i} \Big((p_j - p_i) - (p_j^* - p_i^*) \Big)$$

Define $p = [p_1^T, \cdots, p_n^T]^T$, $v = [v_1^T, \cdots, v_n^T]^T$, $\bar{L} = L \otimes I_d$

Compact form (i.e., double-integrator flocking dynamics)

$$\dot{p} = v$$
$$\dot{v} = -\gamma L v - L(p - p*)$$

Flocking dynamics of double integrators: displacement-based formation

Compact form (i.e., double-integrator flocking dynamics)

$$\dot{p} = v$$
$$\dot{v} = -\gamma L v - L(p - p*)$$

Theorem

Consider the double-integrator flocking system coupled in a connected undirected graph.

- All agents' positions reach a target formation shape (defined by $(p_j^* - p_i^*)$);
- All agents' velocities reach consensus.

```
function dX = Flocking_formation_double_integrator(t,X)

p = X(1:10);
v = X(11:20);
% the gain to the velocity consensus part
k_v = 2;
Desired_p = [0, 0, 1, 0, 1, 1, 2, 1, 2, 0]';
dX = zeros(20,1);

% the Laplacian matrix
L = [1   -1   0    0    0;
    -1   3   -1   0    -1;
    0    -1   3    -1   -1;
    0    0   -1    2   -1;
    0   -1   -1   -1    3];
% flocking dynamics
dX(1:10)  = v;
dX(11:20) = - k_v*kron(L,eye(2))*v -kron(L,eye(2))*(p - Desired_p);
```

Flocking dynamics of double integrators: distance-based formation

Consider the double-integrator model

$$\dot{p}_i = v_i$$
$$\dot{v}_i = u_i$$

Control input with both *velocity consensus and formation shape control* (distance-based formation control)

$$u_i = \gamma \sum_{j \in N_i} (v_j - v_i) + \sum_{j \in N_i} \Big((\|p_j - p_i\|^2 - d_{ij}^{*2})(p_j - p_i) \Big)$$

Define $p = [p_1^T, \cdots, p_n^T]^T$, $v = [v_1^T, \cdots, v_n^T]^T$, $\bar{L} = L \otimes I_d$, $e_{ij} = (d_{ij}^*)^2 - d_{ij}^2$

$$\boldsymbol{e} = [\dots, e_{ij}, \dots]^{\mathrm{T}} = [e_1, \dots, e_m]^{\mathrm{T}}$$

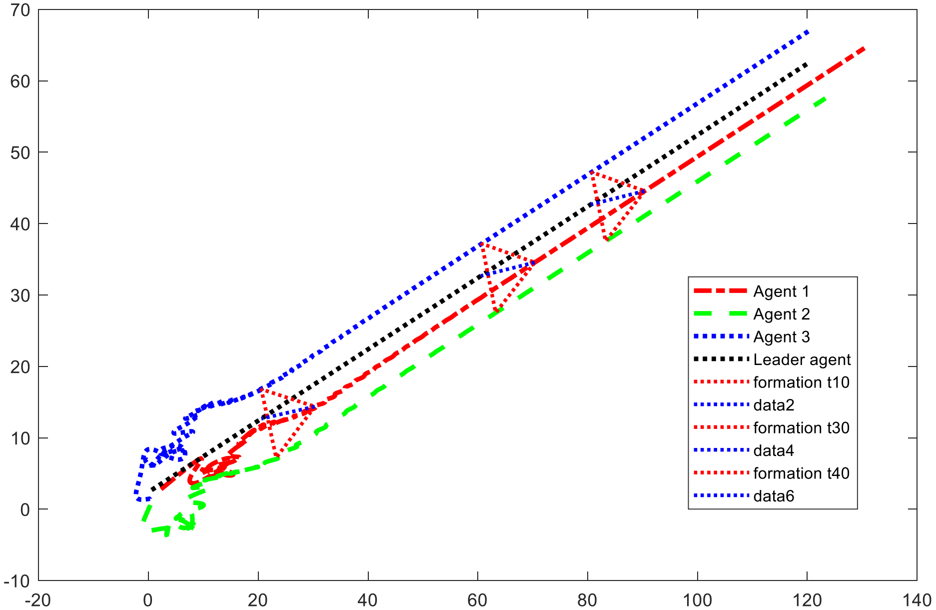
Compact form (i.e., double-integrator flocking dynamics)

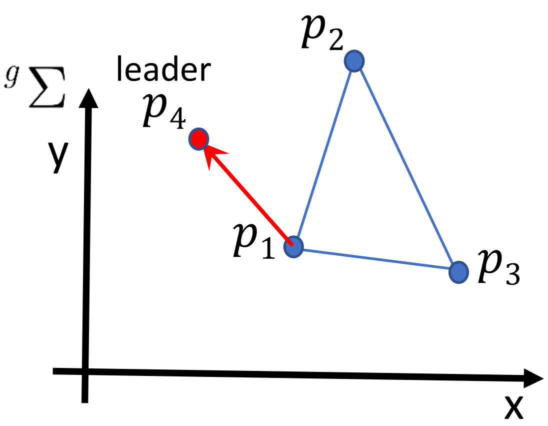
$$\dot{p} = v$$
$$\dot{v} = -\gamma \bar{L} v + R^T e$$

- All agents' positions locally converge to a target formation shape;
- All agents' velocities reach consensus.

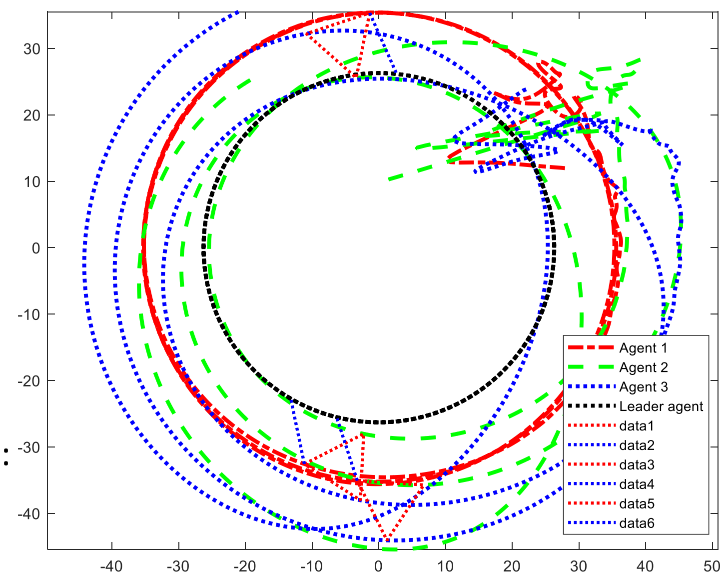
Flocking/swarm dynamics (formation + velocity consensus): simulation

Double-integrator flocking dynamics with both formation control and velocity consensus





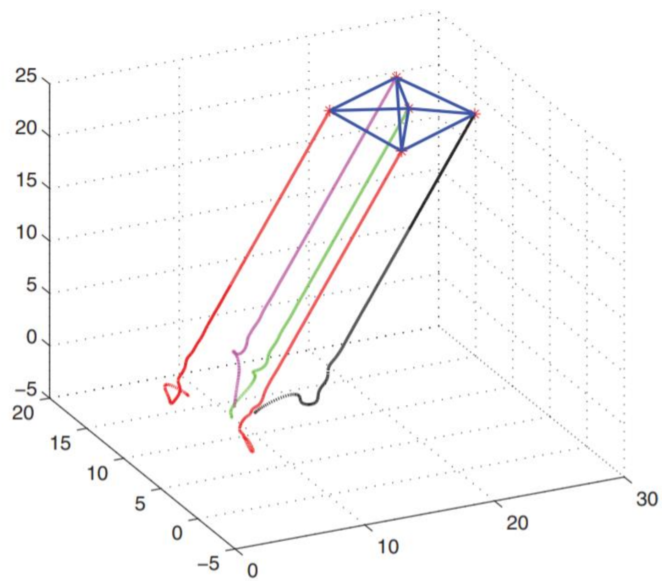
Flocking of four agents (leader-follower structure): three followers maintaining a triangle shape and translating in a line motion



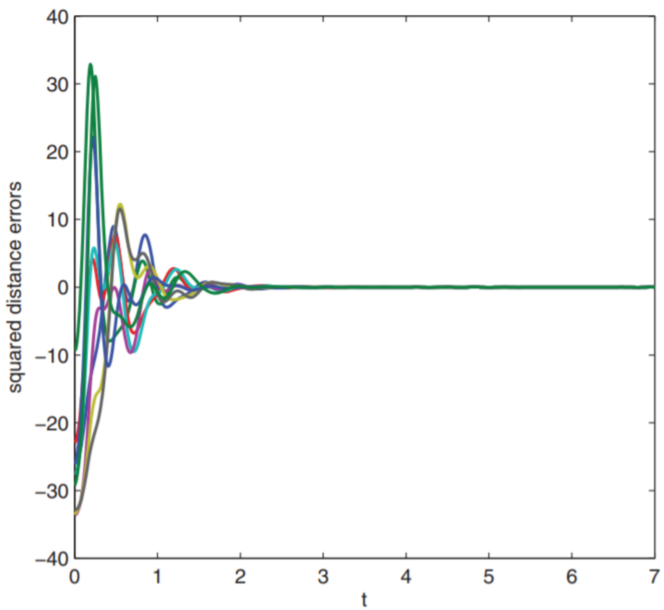
Flocking of four agents (leader-follower structure): three followers maintaining a triangle shape and performing a circular motion

Flocking/swarm dynamics (formation + velocity consensus): simulation

Double-integrator flocking dynamics with both formation control and velocity consensus



Flocking of five agents in 3-D (leaderless structure):
Five agents maintaining a double tetrahedron shape and translating in straight lines



Convergence of the distance errors (distance-based formation shape control)

Review of formation control approaches

Brief history

- Before consensus (2004): virtual structure, behavior-based approach, etc.
- After consensus.

Define target formations

- Inter-agent relative position
- Inter-agent distance
- Inter-agent bearing
- Complex Laplacian
- Barycentric coordinate
- Stress matrix
- Angles
- …

Key issues in implementation

- Local or global/common coordinate frames?
- Communication or onboard measurements?
- Distributed or local information?
- …

Graph topology

- Undirected/directed
- Distance rigidity
- Bearing rigidity
- Angle rigidity
- Stress matrix
- Time-varying
- Weighted
- …

Required measurements

- Relative position
- Inter-agent bearing
- Distance
- Angles
- Mixed measurements
- …

Agent dynamics

- Single integrator
- Double integrator
- General linear systems
- Passive systems
- Continuous time
- Discrete time
- Non-linear systems
- Hybrid dynamics
- Stochastic systems
- Quadrotor UAVs
- Fixed-wing UAVs
- Unicycle robotics
- Heterogeneous robots
- …

Summary/take home messages

- Multi-agent formation control
  - Position-based approach
  - Displacement-based approach (consensus-like dynamics)
  - Distance-based approach

- Graph rigidity theory
  - Formation shape control, rigidity condition (rigidity matrix)

- Flocking/swarm control
  - Double-integrator dynamics, with both velocity consensus and formation control (displacement-based/distance-based method)

Instructions/discussions

Time for instructions

Time for discussions and simulations