DISCRETE-TIME LQR

$$\min_{\{u(k)\}_{k=0,\dots,N-1}} \left(x_N^T Q_N x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) \right)$$

subject to $\quad x_{k+1} = A x_k + B u_k \qquad x(0) = x_0$

Solve with Dynamic PROGRAMMING APPROACH

1. Solve $\quad \displaystyle\min_{u_{N-1}} \; x_N^T Q_N x_N + x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1}$

$$\text{s.t.} \quad x_N = A x_{N-1} + B u_{N-1}$$

$$\min_{u_{N-1}} \; x_N^T Q_N x_N + x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1}$$

TAKE $x_N = A x_{N-1} + B u_{N-1}$

$$= \min_{u_{N-1}} \; (A x_{N-1} + B u_{N-1})^T Q_N (A x_{N-1} + B u_{N-1}) + x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1}$$

$$= \min_{u_{N-1}} \; x_{N-1} A^T Q_N A x_{N-1} + x_{N-1} A^T Q_N B u_{N-1} + u_{N-1}^T B^T Q_N A x_{N-1} + u_{N-1}^T B^T Q_N B u_{N-1} + x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1}$$

$$= \min_{u_{N-1}} \; f(u_{N-1}) + C$$

WITH TERMS NOT DEPENDING ON U_{N-1}

$$C = x_{N-1} A^T Q_N A x_{N-1} + x_{N-1}^T Q x_{N-1}$$

SOLVING for U_{N-1} APPROACH #1

TERMS DEPENDING ON u_N

$$f(u_{N-1}) = x_{N-1} A^T Q_N B u_{N-1} + u_{N-1}^T B^T Q_N A x_{N-1} + u_{N-1}^T B^T Q_N B u_{N-1} + u_{N-1}^T R u_{N-1}$$

$$\nabla f(u_{N-1}) = 2 B^T Q_N A x_{N-1} + 2 (B^T Q_N B + R) u_{N-1}$$

$$u^*_{N-1} = -(B^T Q_N B + R)^{-1} B^T Q_N A x_{N-1}$$

1^{st} ORDER NECESARY CONDITION for OPTIMALITY

$$\nabla^2 f(u_{N-1}) = 2 (B^T Q B + R) \succ 0$$

2^{nd} ORDER CONDITION

$$\min_{u_{N-1}} \quad x_N^T Q_N x_N + x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1}$$

$$= \min_{u_{N-1}} x_{N-1}^T A^T Q_N A x_{N-1} + x_{N-1}^T A^T Q_N B u_{N-1} + u_{N-1}^T B^T Q_N A x_{N-1} + u_{N-1}^T B^T Q_N B u_{N-1}$$
$$+ x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1}$$

APPROACH 2

$$= \min_{u_{N-1}} x_{N-1}^T A^T Q_N A x_{N-1} + x_{N-1}^T Q x_{N-1} - x_{N-1}^T A^T Q_N B (B^T Q B + R)^{-1} B^T Q_N A x_{N-1}$$
$$+ \underbrace{\left(u_{N-1} + (B^T Q B + R)^{-1} B^T Q_N A x_{N-1}\right)^T (B^T Q B + R) \left(u_{N-1} + (B^T Q B + R)^{-1} B^T Q_N A x_{N-1}\right)}_{\text{COMPLETION OF SQUARES ARGUMENT}}$$

SINCE $(B^T Q B + R) > 0$

THE MINIMIZER $u_{N-1}^* = -(B^T Q B + R)^{-1} B^T Q_N A x_{N-1}$

$$= x_{N-1}^T A^T Q_N A x_{N-1} + x_{N-1}^T Q x_{N-1} - x_{N-1}^T A^T Q_N B (B^T Q B + R)^{-1} B^T Q_N A x_{N-1}$$

$$\boxed{\text{✗ DEFINE } M_{N-1} = A^T Q_N A - A^T Q_N B (B^T Q B + R)^{-1} B^T Q_N A + x_{N-1}^T Q x_{N-1}}$$

$$= x_{N-1}^T M_{N-1} x_{N-1}$$

= This is THE OPTIMAL COST-TO-GO AT TIME INSTANT $N-1$ GIVEN x_{N-1}.

THE OPTIMAL ACTION IS

$$u_{N-1}^* = -K_{N-1} x_{N-1} \quad \text{with}$$
$$K_{N-1} = (B^T Q B + R)^{-1} B^T Q_N A$$

2. GIVEN A COST-TO-GO J_{k+1} SOLVE THE OPTIMAL CONTROL AT k.

$$J_{k+1}(x_{k+1}) = x_{k+1}^T M_{k+1} x_{k+1} = \min_{\{u(\ell)\}_{k+1,\dots,N}} \left(x_N^T Q_N x_N + \sum_{\ell=k+1}^{N-1} x_\ell^T Q x_\ell + u_\ell^T R u_\ell \right)$$

$$J_k(x_k) = \min_{u_k} x_{k+1}^T M_{k+1} x_{k+1} + x_k^T Q x_k + u_k^T R u_k$$

↓ REPEAT STEPS IN 1.

$$J_k(x_k) = x_k^T M_k x_k$$

WITH $M_k = A^T M_{k+1} A - A^T M_{k+1} B (B^T M_{k+1} B + R)^{-1} B^T M_{k+1} A + Q$

$$u_{k+1} = -(B^T M_{k+1} B + R)^{-1} B^T M_{k+1} A x_{k+1}$$