COMPACT MODULI SPACES OF MARKED PLANE CUBIC CURVES

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Abstract. We study compactifications of the moduli space of a plane cubic curve marked by n labeled points up to projective equivalence via Geometric Invariant Theory (GIT). Specifically, we provide a complete description of the GIT walls and show that the moduli-theoretic wall-crossing can be understood through analysis of the singularities of the plane curves and the position of the points.

1. Introduction

Compactifications of the moduli space of smooth curves of genus $g \geq 0$ with n distinct marked points are a central theme in algebraic geometry. We know nowadays that there are many such compactifications whose boundary are controlled by a variety of combinatorial structures [Has03],[Smy13], [BKN23], [Pan95], [Swi08], [GS11], [Jen13], [GJM13]. For example, the boundary of the Deligne-Mumford compactifications $\overline{M}_{g,n}$ are understood via dual graphs and tropical geometry [Cap18] whereas the Hassett compactifications $\overline{M}_{g,\mathbf{w}}$ use weights to determine which points can collide in the boundary [Has03]. There have been recent efforts to understand all modular compactifications, in the sense of [Smy13], and to describe the combinatorial gadgets that govern their boundary [BKN23].

In this work, we focus on plane curves of degree $d \geq 0$ marked by n weighted points. Our work follows the previous work of Giansiracusa and Simpson for genus zero [GS11] and Jensen for the case of one point [Jen13]. We describe our set up next: Let $\mathcal{C}_{n,d}$ be the parameter space whose points correspond to tuples $(C, p_1, \dots, p_n)$ of a degree d curve $C \subset \mathbb{P}^2$ and n closed points of C. When we restrict to the locus $\mathcal{C}_{n,d}^{gen}$ of smooth curves marked with generic points, then there is a quasi-projective variety $\mathcal{C}_{n,d}^{gen} /\!\!/ SL(3)$ whose points correspond to isomorphism classes of pairs $(C, p_1, \dots p_n)$ (Lemma 3.1). We use variation of GIT (VGIT) quotients to construct compactifications of $\mathcal{C}_{n,d}^{gen} /\!\!/ SL(3)$, which we denote by $\overline{M}^{git}_{g(d),\mathbf{w}}$, where $g(d) = \frac{(d-1)(d-2)}{2}$ is the genus of a degree d plane curve. The construction depends on a line bundle L obtained from the Hilbert scheme of curves and the n points, as described in Section 2.3. Given such a line bundle L, if it is nef we can associate to it a tuple of non-negative integers $(\gamma, w_1, \dots, w_n) \in \mathbb{Z}^{n+1}_{\geq 0}$. As in similar settings, we will see that γ can be interpreted as a weight on the curve and the w_i as weights on each point p_i. Our first theorem describes the line bundles that give rise to a well-defined coarse moduli space of marked plane curves. That is, the $SL(3)-$ample cone is the set of classes of line bundles L in $NS^{SL(3)}(\mathcal{C}_{n,d})_{\mathbb{Q}}$ such that the locus of L semi-stable marked curves is non-empty, $\mathcal{C}^{ss}_{n,d}(L) \neq \emptyset$ (Definition 2.2).

Theorem 1.1. *The $SL(3)-$ample cone for $\mathcal{C}_{n,d}$ is*

$$\left\{ (\gamma, w_1, \dots, w_n) \;\middle|\; w_i \leq \frac{W + \gamma(2d-3)}{3},\ w_i \leq \frac{W + \gamma(d-2)}{2},\ w_i + w_j \leq \frac{2W + \gamma d}{3},\ 0 \leq w_i,\ 0 \leq \gamma \right\},$$

where $W = \sum_1^n w_i$ denotes the total weight of the points.

Our results generalize [Jen13, Prop 4.2], which solves the problem for a point and arbitrary degree, and [GS11, Thm 1.1], which solves the case of $d = 2$ and arbitrary n. We prove Theorem 1.1 in Section 3.1.

For our second result, we focus on degree $d = 3$ and classify all GIT quotients for marked cubic curves. This is possible because we can list all degenerations of a plane cubic, whereas a similar result for arbitrary degree is out of reach due to the lack of such a classification. To describe our theorem, we recall the wall and chamber decomposition of the $SL(3)$-ample cone ([DH99], [Tha96]): The $SL(3)-$ample cone is divided by codimension 1, locally polyhedral *GIT walls* into a finite set of polyhedral *chambers* such that the semistable loci $\mathcal{C}^{ss}_{n,d}(L)$ and $\mathcal{C}^{ss}_{n,d}(L')$ are equal if and only if L and L' lie in the same chamber [Tha96, Thm 2.3]. To each wall we associate a closed $SL(3)$ orbit which is strictly semi-stable on the wall. The plane curves with closed orbit associated to each wall have positive dimensional stabilizer [DH99]. In the cubic case $d = 3$, such curves belong to a relatively small list; see Lemma 2.14. The proof of Theorem 1.2 is given in Section 4. We illustrate the chamber decomposition for $n = 2$ in Figure 1

Theorem 1.2. *For degree $d = 3$, there are four types of inner walls of the $SL(3)-$ample cone, described below. The walls are segments of hyperplanes defined in Table 1. If the hyperplane segment intersects the interior of the $SL(3)-$ample cone then it is a GIT wall and all inner GIT walls are of this form.*

- *(i) For each nonempty, proper subset $I \subset [n]$ there is a hyperplane segment $W(3A_1, I)$ associated to the union of three non-concurrent lines $C(3A_1)$ with points $p_i, i \in I$ supported at a node.*
- *(ii) For each proper subset $I \subset [n]$ there is a hyperplane segment $W(A_2, I)$ associated to the cuspidal curve $C(A_2)$ with points $p_i, i \in I$ supported at the cusp.*
- *(iii) For each ordered pair of disjoint subsets $I, J \subset [n]$ there is a hyperplane segment $W(A_3, I, J)$ associated to the union of a conic with a tangent line $C(A_3)$ with points $p_i, i \in I$ supported at the tacnode, and $p_j, j \in J$ supported at the unique linear component of the curve.*
- *(iv) For each subset $I \subset [n]$ with $|I| \leq n - 3$ there is a hyperplane segment $W(D_4, I)$ associated to the cone over three points $C(D_4)$ with points $p_i, i \in I$ supported at the singularity.*

Hyperplane Segment	Hyperplane	Boundary Conditions
$W(3A_1, I)$	$\sum_{i\in I} w_i - \frac{1}{2}\sum_{j\notin I} w_j = 0$	$w_m \leq \frac{1}{2}\sum_{j\notin I} w_j + \gamma,$ for each $m \notin I$.
$W(A_2, I)$	$\sum_{i\in I} w_i - \frac{4}{5}\sum_{j\notin I} w_j + \frac{3}{5}\gamma = 0$	$\sum_{i\in I} w_i \leq \frac{1}{2}\sum_{j\notin I} w_j$.
$W(A_3, I, J)$	$\sum_{i\in I} w_i - \sum_{k\notin I\cup J} w_k + \gamma = 0$	$\sum_{i\in I} w_i - \gamma \leq \sum_{j\in J} w_j \leq \sum_{i\in I} w_i + 2\gamma$.
$W(D_4, I)$	$\sum_{i\in I} w_i - \frac{1}{2}\sum_{j\notin I} w_j + \frac{3}{2}\gamma = 0$	There exists a partition $B_1 \sqcup B_2 \sqcup B_3$ of $\{1, \ldots, n\} \setminus I$ such that $\sum_{j\in B_k} w_j \leq \sum_{i\in I} w_i + 2\gamma$ for each $k \in \{1, 2, 3\}$.

Table 1. The walls in Theorem 1.2 are given by intersecting the vanishing locus of a hyperplane with a set of points that cut out the boundary of the wall. The above hyperplane segments are GIT walls if they intersect $SL(3)-$ample cone.

We illustrate Theorem 1.2 for plane cubics with two marked points in Corollary 6.2 and Figure 1.

Next, we discuss the wall-crossing phenomenon for our problem. We recall that given an inner wall, there exists a linearization L_0 at the wall, linearizations L_+ and L_- at each chambers adjacent to the wall, and configurations of marked curves that are stable for L_+, strictly semistable for L_0, and unstable for L_-. We want to describe the generic configurations of marked curves that transition from stable to unstable when crossing each GIT wall from Theorem 1.2. For this purpose, we recall that each of our walls is associated to a curve $C(T)$ and one (or two) subsets of $[n]$. Let $S(T, I, +)$ be the generic curve that satisfies two conditions: it is stable when the value of the linear function defining $W(T, I)$ (Table 1) is $0 < \epsilon \ll 1$ and the boundary conditions are satisfied, and it is unstable whenever the value of the linear function is negative. Conversely, $S(T, I, -)$ denotes the generic curve that is stable when the value of the linear function defining $W(T, I)$ is $-1 \ll \epsilon < 0$ and the boundary conditions are satisfied, and is unstable whenever the value of the linear function is positive.

Theorem 1.3. *With notation as above, the changes of stability and semi-stability for marked curves when crossing the walls from Theorem 1.2 are described as follow:*

- *(i) Case of plane cubics with a nodal singularity*
  - *$S(3A_1, I, -)$ is an irreducible nodal curve with marked points p_i coinciding at the A_1 singularity for $i \in I$ and the remaining marked points in general position.*
  - *$S(3A_1, I, +)$ is the union of a conic and a transverse line, with marked points p_j lying in general position on the linear component for $j \notin I$.*
- *(ii) Case of plane curves with a cuspidal singualrity*

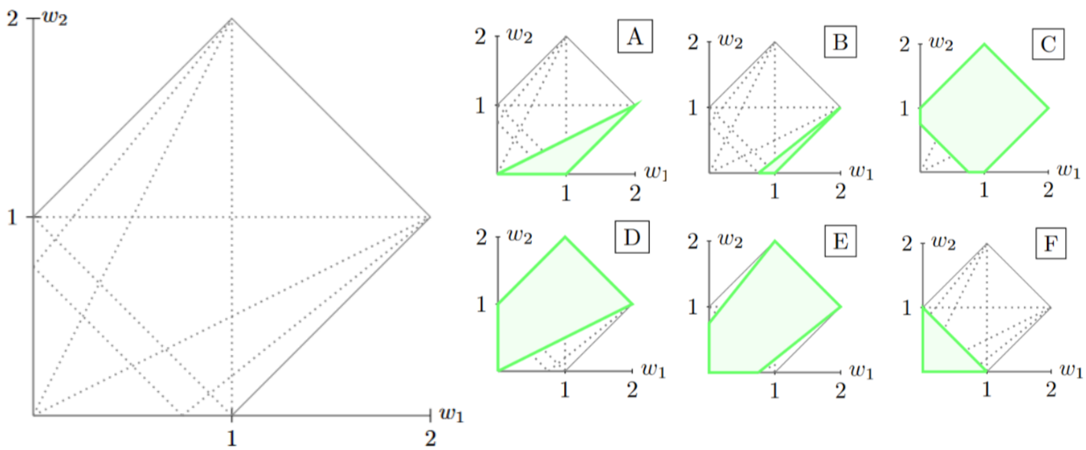

FIGURE 1. (Left) The linearization polytope $\Delta(\mathcal{C}_{2,3})$. (Right) The locus of line bundles at which marked curves (C, p_1, p_2) with the following pathologies are stable: A) p_2 supported at a node. B) p_2 supported at a cusp. C) C is cuspidal. D) p_1 is supported on a linear component of C. E) p_1 and p_2 are both supported at inflection points. F) p_1 and p_2 collide.

- *$S(A_2, I, -)$ is an irreducible cuspidal cubic curve with marked points p_i coinciding at the A_2 singularity for $i \in I$.*
- *$S(A_2, I, +)$ is a smooth cubic curve with marked points p_j coinciding at an inflection point for $j \notin I$.*

(iii) Case of plane cubic with a tacnode
- *$S(A_3, I, J, -)$ is the union of a conic with a tangent line with marked points p_i coinciding at the A_3 singularity for $i \in I$ and p_j lying on the tangent line for $j \in J$.*
- *$S(A_3, I, J, +)$ is a smooth cubic curve C with marked points p_k coinciding for $k \notin I \cup J$ and marked points p_j coinciding at the transversal intersection of C with the line tangent to C at p_k for $j \in J$.*

(iv) Case of a plane cubic with a D_4 singularity
- *$S(D_4, I, -)$ is three concurrent lines with marked points p_i coinciding at the D_4 singularity for $i \in I$ and the points indexed by B_1, B_2, and B_3 lying on the three lines, respectively.*
- *$S(D_4, I, +)$ is a smooth cubic curve with the points indexed by B_1 coinciding at a point q_1, the points indexed by B_2 coinciding at a different point q_2, and the points indexed by B_3 coinciding at a third point on the line $\overline{q_1 q_2}$.*

See Figure 2 for an illustration of the walls.

Next, we discuss some applications of our results. Their proofs are given in Section 6. In his thesis [Laz09], Radu Laza constructs a series of compactifications of the moduli space of *degree* 3 *pairs* consisting of a cubic curve and a line in $\mathbb{P}^2$ by taking the VGIT quotients of $\mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(3))) \times \mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(1)))$ by $SL(3)$, denoted $M^{1,3}_{pairs}(t)$. He finds a VGIT chamber corresponding to the line bundle parameter $t = \frac{3}{2} - \epsilon$ such that $M^{1,3}_{pairs}(\frac{3}{2} - \epsilon)$ is the moduli space of pairs (C, L) where C has at worst isolated singularities of type A_k and L is a line intersecting C transversely. Laza then considers the moduli spaces $M^{(1,3)lab}_{pairs}(t)$ of pairs of a plane cubic and a line, with labeled intersection. He uses a classical construction to prove that $M^{(1,3)lab}_{pairs}(\frac{3}{2} - \epsilon)$ is the coarse moduli space for cubic surfaces containing a marked Eckardt point and at worst A_k singularities. In our case, there is a chamber represented by $\mathbf{w}$ giving rise to a compact moduli space of plane cubics with two marked points $\overline{M}^{git}_{1,\mathbf{w}}$ and a map from $\overline{M}^{git}_{1,\mathbf{w}}$ to $M^{(1,3)lab}_{pairs}(\frac{3}{2} - \epsilon)$ given by taking (C, p_1, p_2) to $(C, \overline{p_1 p_2})$ and marking the intersection points of $C \cap \overline{p_1 p_2}$ with respect to the ordering of p_1 and p_2.

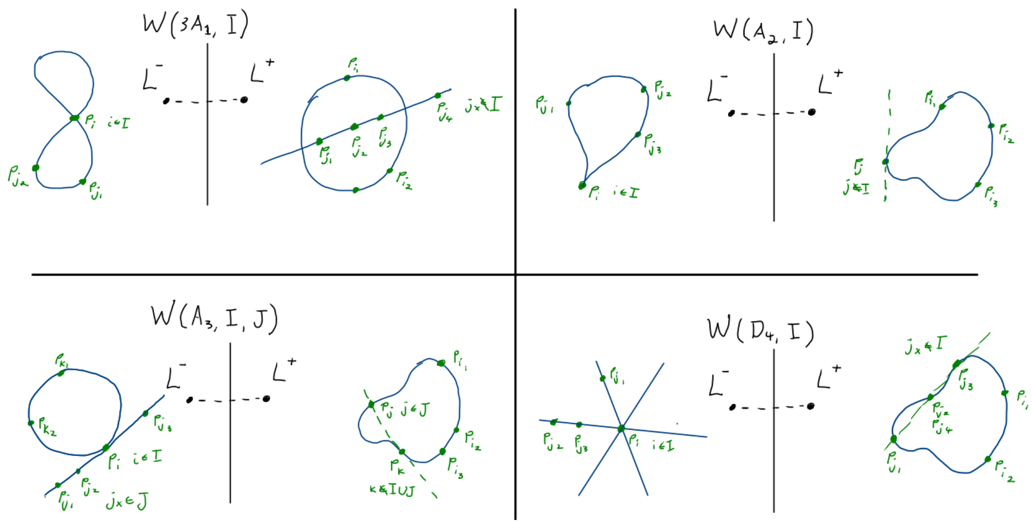

FIGURE 2. Illustration of the wall crossing behavior found in Theorem 1.3. The curves $S(T, I, \pm)$ are stable in the chamber containing $L^{\pm}$ and unstable in the chamber containing $L^{\mp}$.

Corollary 1.4. *There exists an isomorphism*

$$\phi^{lab} : \mathcal{C}_{2,3}^{ss} /\!\!/_L SL(3) \to M_{pairs}^{(1,3)lab}\left(\frac{3}{2} - \epsilon\right) \cong \mathbb{P}(1,2,2,3)$$

where L is a line bundle corresponding to a vector $\mathbf{w}$ in the GIT chamber $\{w_1 > 1, w_2 > 1, w_1 + w_2 < 3\} \subset \Delta(\mathcal{C}_{2,3})$.

Next, we discuss the relation between our compactifications and the moduli space of marked elliptic curves. Recall that an elliptic curve is a curve of genus 1 with a marked point [Dol12]. If (E, p) is an elliptic curve then the linear series $|3p|$ embeds E in $\mathbb{P}^2$ as a cubic curve such that p is an inflection point. For this reason, we are interested in marked cubics for which one of the marked points is specifically an inflection point. In Section 6.2, we construct a space $\mathcal{C}'_{n,3}$ which parametrizes tuples consisting of a plane cubic, n marked points, and an $(n+1)^{th}$ marked inflection point. Forgetting the last $(n+1)^{st}$ point gives a $9:1$ cover $\mathcal{C}'_{n,3} \to \mathcal{C}_{n,3}$. We construct GIT quotients $\mathcal{C}'^{ss}_{n,3} /\!\!/_L SL(3)$ and show that they compactify $M_{1,n+1}$.

Corollary 1.5. *$M_{1,n+1}$ is isomorphic to an open subset of $\mathcal{C}'^{ss}_{n,3} /\!\!/_L SL(3)$ for some $SL(3)-$linearized line bundle L.*

To study the birational map from $\overline{M}_{1,n}$ to $\overline{M}^{git}_{1,\mathbf{w}+}$ is the focus of the author's ongoing work.

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