Two-dimensional pattern matching with k mismatches

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Abstract

1 Related Work

Hamming-Threshold PM in 1D

- Early solutions for pattern matching with mismatches use fast Fourier transform and run in $\mathcal{O}(n\sqrt{m\log m})$ time [1]. The first algorithms that explicitly exploit the threshold k run in time roughly $\mathcal{O}(nk)$ [39,52] and use "kangaroo jumping", a technique in which each text position is considered as a potential occurrence, and verified by jumping from mismatch to mismatch using a data structure for longest common extensions.

- Faster solution in $\mathcal{O}(n\sqrt{k\log k})$ or $\tilde{\mathcal{O}}(n + k^3 n/m)$ time [12] by still using convolutions, but also exploiting periodicity and using counting arguments. The latter result was first improved to $\tilde{\mathcal{O}}(n + k^2 n/m)$ time [29], and then further refined to $\tilde{\mathcal{O}}(n + kn/\sqrt{m})$ time [45], which is a smooth trade-off between the previous $\tilde{\mathcal{O}}(n\sqrt{k})$ and $\tilde{\mathcal{O}}(n + k^2 n/m)$ bounds. If Monte-Carlo randomization is allowed, then the time can be slightly improved to $\mathcal{O}(n + kn\sqrt{(\log m)/m})$ [24]. A significantly faster algorithm would imply unexpected consequences for the complexity of boolean matrix multiplication [45].

- The k-mismatch occurrences of the pattern either have a simple and exploitable structure, or the pattern is close to being periodic. This property is a powerful tool for pattern matching with mismatches, and was analyzed in [23,25].

Exact PM in 2D Assuming $n \times n$ text and $m \times m$ pattern, $m \leq n$.

- Two-dimensional pattern matching dates back (at least) to the 1970s, when Richard Bird generalized the Knuth-Morris-Pratt algorithm [50] for one-dimensional pattern matching, achieving $\mathcal{O}(n^2 + m^2)$ time [21].

- By using multiple pattern matching on only $\mathcal{O}(n/m)$ rows of the text, Baeza-Yates and Régnier achieve average time complexity $\mathcal{O}(n^2/m)$, using $\mathcal{O}(m^2)$ space and still running in $\mathcal{O}(n^2)$ time in the worst case [18]. This was followed by more results optimizing the average time complexity [46,62].

- Crochremore at al. [35] solve two-dimensional pattern matching space efficiently, using an $\mathcal{O}(m^2)$ time and $\mathcal{O}(\log m)$ space preprocessing followed by an $\mathcal{O}(n^2)$ time and constant space search phase.

- On a PRAM, the problem can be solved in constant time and $O(n^2)$ work [34], after preprocessing the pattern in $\mathcal{O}(\log \log m)$ time and $\mathcal{O}(m^2)$ work [31].

- Other works studying the problem are, e.g., [5, 19, 40, 49, 63, 64]

Hamming-Threshold PM in 2D   Assuming $n \times n$ text and $m \times m$ pattern, $m \leq n$, and threshold $k < m^2$.

- First result in 1987 by Krithivasan and Sitalakshmi [51], who show that $\tilde{\mathcal{O}}(kmn^2)$ time can be achieved.

- Faster algorithm by Rank and Heywood runs in $\tilde{\mathcal{O}}((k + m)n^2)$ time [60].

- Current state of the art is $\mathcal{O}(kn^2)$ [8]   `J: check`

- More work [17, 46, 58] is aimed at optimizing the average time complexity. This has also been done for the setting in which arbitrary rotations of the pattern are allowed [38].

- Dynamic and online algorithms [30]

- Measures like edit distance are more complicated to define for two-dimensional strings [16].

2D Periodicity

- Two-dimensional periodicity was introduced in [3, 4] and refined in [40]. All periods (and witnesses for all non-periods) can be computed in $\mathcal{O}(n^2)$ time [32]. Witnesses can be used to efficiently eliminate potential occurrences of the pattern. Since witness computation is costly for run-length encoded strings, Amir at al. [6] show additional properties of two-dimensial periodicities that allow the elimination of occurrences without witnesses.

- Famous periodicity theorems have been generalized to two dimensions, for example the Fine-Wilf theorem [37, 56] and the Lyndon-Schützenberger theorem [42, 53].

- Smallest rectangular cover in $\mathcal{O}(n^2)$ time, all aperiodic covers in $\mathcal{O}(n^2 \log n)$ time [27]. Covers of two-dimensional strings were also considered in [36, 41].

- All tile covers in $\mathcal{O}(n^{2+\epsilon})$ time [59].

- There can be $\Omega(n^3 \log n)$ tandem substrings (repeated rectangle, like square in a normal string) [13].

- All tandems can be computed in $\mathcal{O}(n^3 \log n)$ time [14].

- There can be $\Omega(n^2 \log n)$ two-dimensional runs (maximal repetitions) [43].

- There are at most $\mathcal{O}(n^2 \log^2 n)$ two-dimensional runs [26].

- All runs can be computed in $\mathcal{O}(n^2 \log^2 n)$ time [9].

Other 2D Shannanigans

- A survey by Rytter [61] discusses compressed representations of two-dimensional strings and the computation on such representations without explicit decompression. If the text is compressed as a two-dimensional SLP, then merely deciding if a pattern is present is NP-complete with respect to the compressed size [20]. If both text and pattern are compressed using a two-dimensional version of Lempel-Ziv compression, then exact matching can be done in $\mathcal{O}(n^2 + m^3)$ time, using additional working space proportional to the compressed

representation of the pattern [10]. If text and pattern are given in two-dimensional run-length encoding, then the time for exact matching is linear in the size of the compressed text, and the space is linear in the size of the compressed pattern [11]. If multiple patterns are given, then they can be compressed into a space efficient self-index that allows fast dictionary matching [57].

- In parametrized matching, pattern occurrences have to be isomorphic via a bijection on the effective alphabets [2,33]. A key step for efficient parametrized matching in two dimensions is counting the number of distinct symbols in every $m \times m$ substring, which can be done in $\mathcal{O}(n^2)$ time [33].

- More work on structural elements in two-dimensionsal strings, e.g., frames (rectangles with matching outer columns and rows) [22], Lyndon words [55], motif patterns (allowing "don't cares", and satisfying special properties) [15], patterns with at least two occurrences [48], and palindromic substrings [54].

- There is an efficient algorithm for matching a half-rectangular pattern (fixed height, but irregular shape) in a text, allowing a fixed number of mismatches, insertions and deletions [7].

2 Introduction

We consider the one-dimensional all-substring Hamming distance problem (HD1D), where for a given text string T of length n and a string P of length m ($m < n$), we want to calculate the Hamming distance between P and every fragment T of length m.

We consider the two-dimensional all-substring Hamming distance problem (HD2D), where for a given 2D string T of size $n \times n$ and a string P of size $m \times m$ ($m < n$), we want to calculate the Hamming distance between P and every $m \times m$ fragment of T.

We also consider the bounded variants of HD1D and HD2D, where we are only required to calculate the distances which are not greater than k, for some parameter $k \in \mathbb{Z}^+$.

Theorem 1 (Main result). *Bounded* HD2D *can be solved in* $\tilde{\mathcal{O}}((m^2 + mk^{5/4})n^2/m^2)$ *time.*

3 Preliminaries

For our purposes we will not use the standard definition of a two-dimensional string, where we associate it with a two-dimensional array of characters, and instead we will define it more broadly. Although we will occasionally use the array notation, we will do it exclusively for $n \times m$ strings. For any $n \in \mathbb{Z}^+$ we will denote $[n] = \{0, \ldots, n-1\}$. We will use the terms point and vector interchangeably. Our results hold under word-RAM model of computation.

Definition 1 (Two-dimensional string). We define a **string** S as a partial function $\mathbb{Z}^2 \nrightarrow \Sigma$ which maps some arbitrary set of integer points, denoted as $\mathrm{dom}(S)$, to characters. For simplicity we will write $u \in S$ to denote that $u \in \mathrm{dom}(S)$. We say that a string S is **partitioned** into strings $R_1, \ldots, R_\ell$ when the sets $\mathrm{dom}(R_1), \ldots, \mathrm{dom}(R_\ell)$ partition $\mathrm{dom}(S)$ and $R_i(u) = S(u)$ for all $u \in R_i$. We call a string S **monochromatic** when $S(u) = \sigma$ for every $u \in S$ for some $\sigma \in \Sigma$ and we will write $\mathrm{C}(S)$ to denote the value σ. We say that S is $n \times m$ for some $n, m \in \mathbb{Z}^+$ when $\mathrm{dom}(S) = [n] \times [m]$. Physically we represent a string as a list of point-character pairs.

Definition 2 (Shifting). For a set of points $V \subseteq \mathbb{Z}^2$ and a vector $u \in \mathbb{Z}^2$, we denote $V + u$ as $\{v + u : v \in V\}$. For a string S and a vector $u \in \mathbb{Z}^2$ we denote $S + u$ as a string R such that $\mathrm{dom}(R) = \mathrm{dom}(S) + u$ and $R(v) = S(v - u)$ for $v \in \mathrm{dom}(R)$. Intuitively, we shift the set of points while maintaining their character values.

Definition 3 (Hamming distance)**.** For a pair of strings S, R we define

$$\mathrm{Ham}(S, R) = |\{\, u : u \in \mathrm{dom}(S) \cap \mathrm{dom}(R), S(u) \neq R(u) \,\}|,$$

which corresponds to the number of mismatches between S and R.

Under such notation, the HD2D problem is equivalent to calculating the (bounded or unbounded) values of $\mathrm{Ham}(P + q, T)$ for all $q \in \mathbb{Z}^2$ such that $\mathrm{dom}(P + q) \subseteq \mathrm{dom}(T)$ (so for $q \in [n - m + 1]^2$).

Definition 4 (Don't care symbol)**.** We define the **don't care** symbol as a special character which matches with every character. We will denote it with **?**. Unless stated otherwise, we assume it is not allowed in Σ and in both HD1D and HD2D every character present in T and P matches only with itself.

Definition 5 (Vector operators)**.** For any $u \in \mathbb{Z}^2$ we refer to its coordinates as $u.x, u.y$. For $u, v \in \mathbb{Z}^2$ we denote $u \cdot v = u.x \cdot v.x + u.y \cdot v.y$ and $u \times v = u.x \cdot v.y - u.y \cdot v.x$. Note that alternatively $u \cdot v = |u||v| \cos \alpha$ and $u \times v = |u||v| \sin \alpha$ where α is the angle between u and v.

4 One-dimensional generalizations

In this section we explore some of the methods used for one-dimensional strings. Specifically, as our goal is to generalize the solution for pattern matching with k mismatches described in [44], we are especially interested in two-dimensional variants of the techniques that were used to solve the one-dimensional case.

Theorem 2 (Instancing)**.** *Consider an algorithm $\mathcal{A}$ which solves* HD2D *(bounded or unbounded), but only when $2|n$ and $n \leq \frac{3}{2}m$. If its running time is $\mathcal{T}(m)$, then the general case can be solved in $\mathcal{O}(\mathcal{T}(m)n^2/m^2)$.*

Proof. Let $r = \lfloor m/2 \rfloor$ and let $n' = r + m - 1$ or $r + m$ if $r + m - 1$ is odd. We see that the set $N = [n']^2$ satisfies the conditions for the text domain. For any vector $q \in [n - m]^2$ we can find a vector u such that $r|u.x, r|u.y$ and $q - u \in [r]^2$, so we have $\mathrm{Ham}(P + q, T) = \mathrm{Ham}(P + q - u, T_u)$ where T_u is the restriction of $T - u$ to N. If $T - u$ is not defined for some $v \in N$, we can pad $T_u(v)$ with any character. We see that $\mathrm{dom}(P + q - u) \subseteq N = \mathrm{dom}(T_u)$. There are $\mathcal{O}(n^2/m^2)$ possible vectors u and we run $\mathcal{A}$ for every pair of T_u and P. $\square$

Theorem 3 (Kangaroo jumps)**.** *Consider an $n \times n$ string T, $m \times m$ string P and set of vectors Q such that $\mathrm{dom}(P + q) \subseteq \mathrm{dom}(T)$ for every $q \in Q$. There exists an algorithm which calculates $d_q = \mathrm{Ham}(P + q, T)$ for every $q \in Q$ in total time $\tilde{\mathcal{O}}(n^2 + \sum_{q \in Q} d_q)$.*

Proof. For the sake of clarity, we will temporarily switch to the classical array notation for strings. Let $T_0, \ldots, T_{n-m}$ denote an array of two-dimensional strings (arrays) such that $T_k[0 \mathbin{..} n - 1, 0 \mathbin{..} m - 1] = T[0 \mathbin{..} n - 1, k \mathbin{..} k + m - 1]$. For every row $P[0], \ldots, P[m - 1]$ of P and every row $T_k[0], \ldots, T_k[n - 1]$ of every T_k we assign an integer identifier so that $\mathrm{Id}(P[i]) = \mathrm{Id}(T_k[j]) \Leftrightarrow P[i] = T_k[j]$ by using the KMR algorithm ([reference]) in $\tilde{\mathcal{O}}(n^2)$.

We use the approach described in [kangaroo reference]. There exists a data structure (suffix array) which for a given one-dimensional array S allows us to detect all mismatches between any given two of its subarrays of equal length. It can be built in $\tilde{\mathcal{O}}(|S|)$ and the query time is $\tilde{\mathcal{O}}(d + 1)$ where d is the number of mismatches. We construct the suffix array for the concatenation of the following arrays:

- the rows $P[i]$ for every i,

- the rows $T[i]$ for every i,

- the array $\mathrm{Id}(P[0])\,\mathrm{Id}(P[1])\ldots\mathrm{Id}(P[m-1])$,

- the arrays $\mathrm{Id}(T_k[0])\,\mathrm{Id}(T_k[1])\ldots\mathrm{Id}(T_k[n-1])$ for every k,

the total length of which is $\mathcal{O}(n^2)$. Let us consider a problem of detecting mismatches between P and some $T' = T[j\mathbin{..}j+m-1,k\mathbin{..}k+m-1]$. We can first find all row indices i for which $P[i] \neq T'[i]$ by finding all mismatches between $\mathrm{Id}(P[0])\ldots\mathrm{Id}(P[m-1])$ and $\mathrm{Id}(T_k[j])\ldots\mathrm{Id}(T_k[j+m-1])$, which we do with query to the data structure. For every such i we can then find all mismatches between $P[i]$ and $T'[i]$ by querying $P[i]$ and $T[i+j][k\mathbin{..}k+m-1]$. If the distance between P and T' is d, the first query takes $\tilde{\mathcal{O}}(d+1)$ operations and all subsequent queries take $\tilde{\mathcal{O}}(d+1)$ operations in total. $\qquad\square$

Lemma 1. HD1D *with don't care symbols can be solved in* $\tilde{\mathcal{O}}(n|\Sigma|)$ *by running* $|\Sigma|$ *instances of FFT.*

Lemma 2. *There exists a* $(1+\varepsilon)$-*approximate algorithm (introduced in [47]) which solves* HD1D *with don't care symbols in* $\tilde{\mathcal{O}}(n)$.

Theorem 4. HD2D *with don't care symbols can be solved in* $\tilde{\mathcal{O}}(n^2|\Sigma|)$.

Proof. We will again use the array notation. We construct one-dimensional strings $\bar{T}$ and $\bar{P}$ by concatenating subsequent rows $T[0],\ldots,T[n-1]$ of T and rows $P[0],\ldots,P[m-1]$ of P padded with don't care symbols:
$$\bar{T} = T[0]\ T[1]\ \ldots\ T[n-1],$$
$$\bar{P} = P[0]\ \texttt{?}^{n-m}\ P[1]\ \texttt{?}^{n-m}\ \ldots\ \texttt{?}^{n-m}\ P[m-1].$$
We run the algorithm from Lemma 1. The distance between $T[i\mathbin{..}i+m-1,j\mathbin{..}j+m-1]$ and P is equal to the distance between $\bar{T}[in+j\mathbin{..}in+j+nm-n+m-1]$ and $\bar{P}$. $\qquad\square$

Theorem 5. *There exists a* $(1+\varepsilon)$-*approximate algorithm which solves* HD2D *with don't care symbols in* $\tilde{\mathcal{O}}(n^2)$.

Proof. Identical to Theorem 4, but we use the algorithm from Lemma 2 instead of Lemma 1. $\qquad\square$

The same reduction as in Theorem 4 can be applied for every HD1D solution which allows don't care symbols. Unfortunately, the most effective known algorithms for bounded HD1D rely on periodicity ([28], [44]) and inherently do not allow don't care symbols, thus, they cannot be easily generalized.

Observation 1 (Don't care padding)**.** *Every* HD2D *solution which allows don't care symbols (eg. the algorithms from Theorem 4 and Theorem 5) can be extended to also calculate the Hamming distance for occurrences of P which are not entirely contained in T. It can be done by padding the text with don't care symbols and it does not change the complexity of the solution.*

5 Proof of Theorem 1

We show an algorithm which works in time $\tilde{\mathcal{O}}(m^2 + mk^{5/4})$, assuming $2|n$ and $m < n \leq \frac{3}{2}m$. By Theorem 2, our main result follows.

We start by running the algorithm from Theorem 5 with $\varepsilon = 1$. We construct the set Q as the set of such vectors $q \in \mathbb{Z}^2$ for which the estimated value of $\mathrm{Ham}(P+q,T)$ is at most $2k$. For every $q \in \{0,\ldots,n-m\}^2 \setminus Q$ we say that $\mathrm{Ham}(P+q,T)$ equals ∞. The next step is to calculate the exact value of $\mathrm{Ham}(P+q,T)$ for every $q \in Q$.

Let us consider the case when $|Q| \leq 2m + m^2/k$. We can run the algorithm from Theorem 3 and by the fact that $\mathrm{Ham}(P+q,T) \leq 4k$ for every $q \in Q$, it will perform $\tilde{\mathcal{O}}(m^2 + mk)$ operations. We are left with the case when $|Q| > 2m + m^2/k$, in which we take advantage of the fact that some strings $P+q$ for $q \in Q$ must have a large overlap and small Hamming distance from each other, and thus P must be periodic.

5.1 Two-dimensional periodicity

In this section we introduce a range of new tools related to two-dimensional periodicity. We then select some special periods of the pattern and show how to decompose it into some regularly structured monochromatic strings.

Definition 6 (Periodicity). Consider any vector $\delta \in \mathbb{Z}^2$. We say that a string S has an ℓ-period δ when

$$\mathrm{Ham}(S + \delta, S) \leq \ell.$$

Lemma 3. *For every $u, v \in Q$, the vector $u - v$ is an $8k$-period of P.*

Proof. $\mathrm{Ham}(P + u - v, P) = \mathrm{Ham}(P + u, P + v) \leq \mathrm{Ham}(P + u, T) + \mathrm{Ham}(P + v, T) \leq 4k + 4k.$ $\square$

Theorem 6. *For a given $\ell \in \mathbb{Z}^+$ and a set of points $U \subseteq [\ell + 1]^2$ such that $|U| > 4\ell$ there exist $s, t, s', t' \in U$ such that the following conditions hold for $w = t - s$ and $w' = t' - s'$:*

- *$0 < |w||w'| \leq 22\frac{\ell^2}{|U|}$,*

- *$|\sin \alpha| \geq \frac{1}{2}$ where α is the angle between w and w',*

- *$w, w', -w, -w'$ are all contained in different **quadrants**, defined as*

$$\begin{aligned}
\mathcal{Q}_1 &= (0, +\infty) \times [0, +\infty), \\
\mathcal{Q}_2 &= (-\infty, 0] \times (0, +\infty), \\
\mathcal{Q}_3 &= (-\infty, 0) \times (-\infty, 0], \\
\mathcal{Q}_4 &= [0, +\infty) \times (-\infty, 0).
\end{aligned}$$

Such w, w' can be found in $\tilde{\mathcal{O}}(|U|)$ operations.

Proof. See Section 5.4. $\square$

We run the algorithm from Theorem 6 on the set Q (where $\ell = n - m \leq m/2$, thus $|Q| > 2m + m^2/k \geq 4\ell$). We obtain vectors $\varphi \in \mathcal{Q}_4$ and $\psi \in \mathcal{Q}_1$ which by Lemma 3 are $\mathcal{O}(k)$-periods of P. We will refer to those vectors throughout the rest of the description. Note that because $|Q| > 2m + m^2/k$, we have $0 \leq \varphi \times \psi \leq |\varphi||\psi| = \mathcal{O}(\min\{m, k\})$.

Definition 7 (Lattice congruency). We define $\mathcal{L} = \{s\varphi + t\psi : s, t \in \mathbb{Z}\}$. We say that two vectors $u, v \in \mathbb{Z}^2$ are **lattice-congruent** and denote $u \equiv v$ when $u - v \in \mathcal{L}$.

Lemma 4. *There exists a set of points $\Gamma \subseteq \mathbb{Z}^2$ such that $|\Gamma| = \mathcal{O}(\min\{m, k\})$ and every point $u \in \mathbb{Z}^2$ is lattice-congruent to exactly one point $\gamma \in \Gamma$.*

Proof. Let $p = \{s\varphi + t\psi : s \in [0, 1), t \in [0, 1)\}$. We construct $\Gamma = p \cap \mathbb{Z}^2$. It is commonly known, that a simple polygon with integer vertices contains $\mathcal{O}(A)$ integer points in the interior or on the boundary, where A denotes its surface area. Observe that the points in Γ are contained in a parallelogram with vertices $(0, 0), \varphi, \varphi + \psi, \psi$. Since its surface area is $\varphi \times \psi = \mathcal{O}(\min\{m, k\})$, we get $|\Gamma| = \mathcal{O}(\min\{m, k\})$.

Now consider any point $u \in \mathbb{Z}^2$. There exist some unique values $s, t \in [0, 1)$ and $s', t' \in \mathbb{Z}$, such that $u = (s + s')\varphi + (t + t')\psi$. It is easy to see that $u \equiv s\varphi + t\psi$ and $s\varphi + t\psi \in \Gamma$. $\square$

Definition 8 (Parquet). We call a set $U \subseteq \mathbb{Z}^2$ a **parquet** when there exist some values $x_0, x_1, y_0, y_1, \varphi_0, \varphi_1, \psi_0, \psi_1 \in \mathbb{Z}$, which we will call its **signature**, such that

$$U = [x_0, x_1] \times [y_0, y_1] \cap \{u : u \in \mathbb{Z}^2, \varphi \times u \in [\varphi_0, \varphi_1], \psi \times u \in [\psi_0, \psi_1]\}.$$

a) If additionally $x_1 - x_0 + 1 \geq |\varphi.x| + |\psi.x|$ and $y_1 - y_0 + 1 \geq |\varphi.y| + |\psi.y|$, then U is a **spacious** parquet.

b) If additionally $x_0, y_0 = -\infty$ and $x_1, y_1 = +\infty$, then U is a **simple** parquet.

Note that every simple parquet is spacious.

Definition 9 (Subparquet). We call a set $V \subseteq \mathbb{Z}^2$ a **subparquet** when there exists a parquet U and a point $\gamma \in \mathbb{Z}^2$ such that
$$V = \{\, u : u \in U, u \equiv \gamma \,\}.$$

A signature of V consists of a signature of U and the vector γ. We call V a spacious/simple subparquet when there exists U which is (correspondingly) a spacious/simple parquet. We say that V is lattice-congruent to some $v \in \mathbb{Z}^2$ (denoted as $V \equiv v$) when $v \equiv \gamma$. We similarly define the lattice congruency between two subparquets.

Theorem 7 (Subparquet convolution). *For a given list of signatures of simple subparquets $U_0, \ldots, U_{\ell-1}$, list of signatures of subparquets $V_0, \ldots, V_{\ell-1}$ and a set of vectors Q we can calculate*
$$\sum_{i=0}^{\ell-1} |(U_i + q) \cap V_i|$$

for every $q \in Q$ in total time $\tilde{\mathcal{O}}(m^2 + \ell + |Q|)$, assuming that the subparquets only contain vectors of length $\mathcal{O}(m)$.

Proof. See Section 5.5. □

Definition 10 (Parquet string). We call a string S a spacious/simple (sub-)parquet string when $\mathrm{dom}(S)$ is a spacious/simple (sub-)parquet.

Theorem 8 (Periodic string decomposition). *A given spacious/simple parquet string R with $\mathcal{O}(k)$-periods φ and ψ can be partitioned in time $\tilde{\mathcal{O}}(|\mathrm{dom}(R)|)$ into $\mathcal{O}(k)$ monochromatic spacious/simple subparquet strings, correspondingly.*

Proof. See Section 5.6. □

Since $|\varphi.x|, |\varphi.y|, |\psi.x|, |\psi.y| \leq n - m \leq m/2$, the $m \times m$ string P is a spacious parquet string and satisfies the assumptions of Theorem 8. We partition P into a set of strings $\mathcal{V}$. We then group the strings based on the single character they contain. Specifically, we construct $\mathcal{V}_\sigma = \{\, V : V \in \mathcal{V}, \mathrm{C}(V) = \sigma \,\}$ for every character $\sigma \in \Sigma$ present in P.

Theorem 9. *For a given set of monochromatic simple subparquet strings $\mathcal{S}$ we can calculate*
$$\sum_{S \in \mathcal{S}} \mathrm{Ham}(P + q, S)$$

for every $q \in Q$ in total time $\tilde{\mathcal{O}}(m^2 + \sum_{S \in \mathcal{S}} |\mathcal{V}_{\mathrm{C}(S)}|)$, assuming that the sets $\mathrm{dom}(S)$ for $S \in \mathcal{S}$ are some pairwise disjoint subsets of $\mathrm{dom}(T)$.

Proof. Let $U = \bigcup_{S \in \mathcal{S}} \mathrm{dom}(S)$. Observe that
$$\sum_{S \in \mathcal{S}} \mathrm{Ham}(P + q, S) = |(P + q) \cap U| - \sum_{S \in \mathcal{S}} \sum_{V \in \mathcal{V}_{\mathrm{C}(S)}} |\mathrm{dom}(V + q) \cap \mathrm{dom}(S)|.$$

We can calculate $|(P + q) \cap U|$ for every $q \in Q$ with a single instance of FFT (see Theorem 4) or by using prefix sums in time $\tilde{\mathcal{O}}(m^2)$. To calculate the values
$$\sum_{S \in \mathcal{S}} \sum_{V \in \mathcal{V}_{\mathrm{C}(S)}} |\mathrm{dom}(V + q) \cap \mathrm{dom}(S)|$$

we use the algorithm from Theorem 7 (where $\ell = \sum_{S \in \mathcal{S}} |\mathcal{V}_{\mathrm{C}(S)}|$). □

5.2 Text decomposition

Because the text is not necessarily periodic, we unfortunately cannot use the same approach as for the pattern. In this section we show how to decompose T using a similar, but more nuanced approach.

Definition 11 (Active text)**.** We define the **active text** $T_{\mathbf{a}}$ as the restriction of T to $\bigcup_{q \in Q} \mathrm{dom}(P) + q$. For a point $u \in \mathbb{Z}^2$ we define its **border distance** as $\min \{ \ | u - v| : v \in \mathbb{Z}^2 \setminus \mathrm{dom}(T_{\mathbf{a}}) \ \}$. We say that a set of points $U \subseteq \mathbb{Z}^2$ is d-**peripheral** for some $d \geq 0$, if the border distance of every $u \in U$ is not greater than d. Similarly, we say that a string S is d-peripheral when $\mathrm{dom}(S)$ is d-peripheral.

Observation 2. $\mathrm{Ham}(P + q, T) = \mathrm{Ham}(P + q, T_{\mathbf{a}})$ *for every* $q \in Q$.

Theorem 10 (Active text decomposition)**.** *For a given positive integer $\ell \leq m$, we can partition the active text in time $\tilde{\mathcal{O}}(m^2 + \ell k)$ into a set of $\mathcal{O}(\ell k)$ monochromatic simple subparquet strings and a $\mathcal{O}(m/\ell)$-peripheral string.*

Proof. See Section 5.7. □

We partition $T_{\mathbf{a}}$ using the algorithm from Theorem 10 with $\ell = mk^{-3/4}$ into a set of simple subparquet strings $\mathcal{S}$ and a $\mathcal{O}(k^{3/4})$-peripheral string F. For every $q \in Q$ we then have

$$\mathrm{Ham}(P + q, T_{\mathbf{a}}) = \mathrm{Ham}(P + q, F) + \sum_{S \in \mathcal{S}} \mathrm{Ham}(P + q, S).$$

By Theorem 9, we can calculate $\sum_{S \in \mathcal{S}} \mathrm{Ham}(P + q, S)$ for every $q \in Q$ in time $\tilde{\mathcal{O}}(m^2 + mk^{5/4})$, since $\sum_{S \in \mathcal{S}} |\mathcal{V}_{C(S)}| \leq |\mathcal{S}||\mathcal{V}| = \mathcal{O}(mk^{5/4})$. In the next section we will introduce Theorem 12, which states that for a d-peripheral string F, we can calculate $\mathrm{Ham}(P + q, F)$ for every $q \in Q$ in total time $\tilde{\mathcal{O}}(m^2 + mdk^{1/2})$. By substituting $d = \mathcal{O}(k^{3/4})$, we get the complexity of $\tilde{\mathcal{O}}(m^2 + mk^{5/4})$, which ends the main proof.

5.3 The text periphery

In this section we explore the properties of peripheral strings. We consider any $d > 0$ and a non-empty d-peripheral string S, such that $\mathrm{dom}(S) \subseteq \mathrm{dom}(T_{\mathbf{a}})$. We define a partitioning of S into strings $S_1, \ldots, S_4$, by splitting it through the middle with a horizontal and vertical line. Specifically

- S_1 is the restriction of S to $\{ n/2, \ldots, n - 1 \} \times \{ n/2, \ldots, n - 1 \}$ (upper right quarter),

- S_2 is the restriction of S to $\{ 0, \ldots, n/2 - 1 \} \times \{ n/2, \ldots, n - 1 \}$ (upper left quarter),

- S_3 is the restriction of S to $\{ 0, \ldots, n/2 - 1 \} \times \{ 0, \ldots, n/2 - 1 \}$ (lower left quarter),

- S_4 is the restriction of S to $\{ n/2, \ldots, n - 1 \} \times \{ 0, \ldots, n/2 - 1 \}$ (lower right quarter).

We will demonstrate some characteristics of S_1, and by symmetry, generalize them to S.

Lemma 5. *Assuming $d \leq m/4$, there does not exist $u \in S_1$ and $v \in T_a$ such that $v.x - u.x \geq d$ and $v.y - u.y \geq d$.*

Proof. Assume the contrary. Since $u \in S_1$, the border distance of u is at most d, so there exists $w \in \mathbb{Z}^2 \setminus \mathrm{dom}(T_{\mathbf{a}})$, such that $u.x - d \leq w.x \leq u.x + d$ and $u.y - d \leq w.y \leq u.y + d$. Since $v \in T_a$, there exists $q \in Q$ such that $v \in [m]^2 + q$. We have

$$w.x \geq u.x - d \geq n/2 - m/4 \geq n - m > q.x$$

and

$$w.x \leq u.x + d \leq v.x \leq q.x + m - 1.$$

Similarly we can show that $q.y \leq w.y \leq q.y + m - 1$, and thus $w \in [m]^2 + q$. Since $[m]^2 + q \subseteq \mathrm{dom}(T_{\mathbf{a}})$ and $w \notin T_{\mathbf{a}}$, we get a contradiction. $\qquad \square$

Theorem 11. *We can calculate* $\mathrm{Ham}(P + q, S)$ *for every* $q \in Q$ *in total time* $\tilde{\mathcal{O}}(m^2 + md|\Sigma|)$, *where* $|\Sigma|$ *is the number of different characters present in both* P *and* S.

Proof. See Section 5.3.1. $\qquad \square$

Theorem 12. *We can calculate* $\mathrm{Ham}(P + q, S)$ *for every* $q \in Q$ *in total time* $\tilde{\mathcal{O}}(m^2 + mdk^{1/2})$.

Proof. See Section 5.3.2. $\qquad \square$

5.3.1   Proof of Theorem 11

We base our approach on the simple method of calculating the Hamming distance by running an instance of FFT for each unique character. We will again utilize partitioning to reduce the problem to some smaller ones and then solve them naively.

Definition 12. (width & height) For a non-empty set $U \subseteq \mathbb{Z}^2$ we define its **width** as $\max\{u.x - v.x + 1 : u, v \in U\}$ and its **height** as $\max\{u.y - v.y + 1 : u, v \in U\}$. For a non-empty string R we define the width and height as the width and height of $\mathrm{dom}(R)$.

Theorem 13. *Given two non-empty strings* P *and* T *of widths* w_P, w_T *and heights* h_P, h_T, *we can calculate* $\mathrm{Ham}(P + q, T)$ *for every* $q \in \mathbb{Z}^2$, *for which the result is non-zero, in total time* $\tilde{\mathcal{O}}((|\Sigma| + 1)(w_P + w_T)(h_P + h_T))$, *where* $|\Sigma|$ *denotes the number of different characters present in both* P *and* T.

Proof. We can prove it by slightly generalizing Theorem 4, although following the same method, and utilizing Observation 1. $\qquad \square$

We will assume that $d \leq m/4$, since for $d = \Omega(m)$ we can, by Theorem 13, calculate the results in time $\tilde{\mathcal{O}}(m^2 + m^2|\Sigma|)$, which is sufficient.

Recall that $\mathrm{Ham}(P + q, S) = \mathrm{Ham}(P + q, S_1) + \cdots + \mathrm{Ham}(P + q, S_4)$. We will only show how to calculate $\mathrm{Ham}(P + q, S_1)$ for every $q \in Q$, since the other cases are symmetric.

We will take advantage of the fact that the points close to the border can overlap only with a small subset of points from the pattern when considering the occurrences fully contained in the active text. Specifically, let us consider a string P_0, defined as the restriction of P to $[m - d]^2$ and a string P_1, defined as the restriction of P to $\mathrm{dom}(P) \setminus \mathrm{dom}(P_0)$. Since the strings P_0 and P_1 partition P, we have

$$\mathrm{Ham}(P + q, S_1) = \mathrm{Ham}(P_0 + q, S_1) + \mathrm{Ham}(P_1 + q, S_1).$$

Lemma 6. $\mathrm{dom}(P_0 + q) \cap \mathrm{dom}(S_1) = \emptyset$ *for every* $q \in Q$.

Proof. Let us assume the contrary. Select any $q \in Q$ such that $\mathrm{dom}(P_0 + q) \cap \mathrm{dom}(S_1)$ contains some point u and consider the point $v = (u.x + d, u.y + d)$. Since $u \in [m - d]^2 + q$, we have $v \in [m]^2 + q \subseteq \mathrm{dom}(T_{\mathbf{a}})$, thus the points $u \in S_1$ and $v \in T_{\mathbf{a}}$ contradict Lemma 5. $\qquad \square$

Observation 3. P_1 *can be partitioned into two strings* P_2 *and* P_3 *such that the width of* P_2 *and the height of* P_3 *are equal to* d.

By Lemma 6, $\mathrm{Ham}(P_0 + q, S_1) = 0$ for every $q \in Q$ and by Observation 3 we then have

$$\mathrm{Ham}(P + q, S_1) = \mathrm{Ham}(P_1 + q, S_1) = \mathrm{Ham}(P_2 + q, S_1) + \mathrm{Ham}(P_3 + q, S_1).$$

for some newly constructed strings P_2 and P_3, such that the width of P_2 and the height of P_3 equals d. We calculate $\mathrm{Ham}(P_2 + q, S_1)$ and $\mathrm{Ham}(P_3 + q, S_1)$ for every Q independently and sum the results. We only show how to calculate $\mathrm{Ham}(P_2 + q, S_1)$, since the other case is symmetric.

We will now partition S_1. Consider an array of strings $U_0, \ldots, U_{\lceil n/d \rceil - 1}$, where U_i is the restriction of S_1 to $\{ id, \ldots, id + d - 1 \} \times [n] \cap \mathrm{dom}(S_1)$. For the sake of formality (since the maximum/minimum of an empty set is undefined), let $V_0, \ldots, V_{\ell-1}$ consist of all non-empty strings U_i, given in the increasing order of i. Observe that $V_0, \ldots, V_{\ell-1}$ partition S_1 and their width is not greater than d.

For each $i \in [\ell]$ we find $h_i \in \mathbb{Z}^+$, which we define as the minimal number such that $(u.x, u.y + h_i) \notin T_{\mathbf{a}}$ for every $u \in V_i$.

Lemma 7. *The sum of all h_i is $\mathcal{O}(m)$.*

Proof. Since for $\ell < 2$, the proof is trivial, we assume $\ell \geq 2$. For every i (since h_i is minimal) there exists a point $u_i \in V_i$, such that $(u_i.x, u_i.y + h_i - 1) \in T_{\mathbf{a}}$. It can be shown that for all $i \geq 2$ we have

$$h_i \leq u_{i-2}.y - u_i.y + d,$$

since if that was not the case for some i, then the points u_{i-2} and $v = (u_i.x, u_i.y + h_i - 1)$ would contradict Lemma 5. We can conclude that

$$\sum_{i=0}^{\ell-1} h_i \leq h_0 + h_1 + \sum_{i=2}^{\ell-1}(u_{i-2}.y - u_i.y + d) = h_0 + h_1 + u_0.y + u_1.y - u_{\ell-2}.y - u_{\ell-1}.y + (\ell-2)d = \mathcal{O}(m).$$

$\square$

Observation 4. *For every i, the height of V_i is not greater than h_i. By Lemma 7 we have $|S_1| = \mathcal{O}(md)$ and by extension $|S| = \mathcal{O}(md)$.*

For every $i \in [l]$ we construct the string L_i as the restriction of P_2 to $[m] \times [m - h_i] \cap \mathrm{dom}(P_2)$ and the string H_i as the restriction of P_2 to $\mathrm{dom}(P_2) \setminus \mathrm{dom}(L_i)$. Since L_i and H_i partition V_i, we have

$$\mathrm{Ham}(P_2 + q, S_1) = \sum_{i=0}^{\ell-1} \mathrm{Ham}(P_2 + q, V_i) = \sum_{i=0}^{\ell-1} \mathrm{Ham}(L_i + q, V_i) + \sum_{i=0}^{\ell-1} \mathrm{Ham}(H_i + q, V_i).$$

Lemma 8. $\mathrm{dom}(L_i + q) \cap \mathrm{dom}(V_i) = \emptyset$ *for every $q \in Q$ and $i \in [\ell]$.*

Proof. Let us assume the contrary. Select any $q \in Q$ and $i \in [\ell]$, such that $\mathrm{dom}(L_i + q) \cap \mathrm{dom}(V_i)$ contains some point u and consider the point $v = (u.x, u.y + h_i)$. Since $u \in [m] \times [m - h_i] + q$, we have $v \in [m]^2 + q \subseteq \mathrm{dom}(T_{\mathbf{a}})$, thus $v \in T_{\mathbf{a}}$, which contradicts the definition of h_i. $\square$

By Lemma 8, for every $q \in Q$ we have $\sum_{i=0}^{\ell-1} \mathrm{Ham}(L_i + q, V_i) = 0$, thus our result is equal to $\sum_{i=0}^{\ell-1} \mathrm{Ham}(H_i + q, V_i)$. We run the algorithm from Theorem 13 for every pair of H_i and V_i and, since both H_i and V_i have widths not greater than d and heights not greater than h_i, we obtain the total complexity of $\tilde{\mathcal{O}}(\sum_{i=0}^{\ell-1}(|\Sigma| + 1)dh_i)$, which, by Lemma 7, is $\tilde{\mathcal{O}}(m^2 + md|\Sigma|)$.

5.3.2   Proof of Theorem 12

Recall the construction of the sets $\mathcal{V}_\sigma$ described in Section 5.1. We define $\sigma \in \Sigma$ to be a **frequent** character if $|\mathcal{V}_\sigma| \geq \sqrt{k}$ and if $|\mathcal{V}_\sigma| < \sqrt{k}$, we call it an **infrequent** character.

Observation 5. *The number of different frequent characters is $\mathcal{O}(\sqrt{k})$.*

We partition S into two strings F and I, based on character frequency, so that F consists of only the frequent characters and I consists of only the infrequent ones. For every $q \in Q$ we then have

$$\mathrm{Ham}(P + q, S) = \mathrm{Ham}(P + q, F) + \mathrm{Ham}(P + q, I).$$

By Observation 5 and Theorem 11, we can calculate $\mathrm{Ham}(P + q, F)$ for every $q \in Q$ in total time $\tilde{\mathcal{O}}(m^2 + mdk^{1/2})$, since F is d-peripheral.

We partition I into $|\mathrm{dom}(I)|$ strings, one per every $u \in I$. Specifically, let I_u be the restriction of I to $\{u\}$ for every $u \in I$. We have $\mathrm{Ham}(P + q, I) = \sum_{u \in I} \mathrm{Ham}(P + q, I_u)$ for every $q \in Q$. By Definition 9, I_u are simple subarquet strings, and thus, we can by Theorem 9 calculate the results in $\tilde{\mathcal{O}}(m^2 + \sum_{u \in I} |\mathcal{V}_{I(u)}|)$. Since $I(u)$ is an infrequent character for every $u \in I$, we have $|\mathcal{V}_{I(u)}| < k^{1/2}$ for every $u \in I$. By Observation 4 we have $|\mathrm{dom}(I)| = \mathcal{O}(md)$, and thus the total complexity is $\tilde{\mathcal{O}}(m^2 + mdk^{1/2})$.

5.4   Proof of Theorem 6

First, we find any closest pair of vectors $s, t \in U$ by running the standard $\tilde{\mathcal{O}}(|U|)$ time algorithm and denote $w = t - s$. We define a partial order $\leq_w$ where $v \leq_w u$ for some $u, v \in U$ when at least one condition holds:

(a) $u = v$,

(b) $u - v$ and w belong to the same quadrant,

(c) $\alpha \in (-\frac{\pi}{6}, \frac{\pi}{6})$ where α is the angle between w and $u - v$.

We find the longest chain C and the longest antichain A using dynamic programming in $\tilde{\mathcal{O}}(|U|)$ operations. We then find any closest pair of vectors $s', t' \in A$ and denote $w' = t' - s'$. We have the following inequalities:

(i) $|U| \leq |C||A|$ (by Dilworth's theorem),

(ii) $(|C| - 1)|w| \leq (1 + \sqrt{3})\ell$ (roughly by the fact that vectors in C must be increasing in a certain direction),

(iii) $(|A| - 1)|w'| \leq 2\ell$ (by using a similar argument for vectors in A).

By considering the assumption $|U| > 4\ell$ it can be proven that $|w||w'| \leq 22\frac{\ell^2}{|U|}$ and the other conditions also hold.

5.5   Proof of Theorem 7

Throughout this section we will denote $D = \{u : u \in \mathcal{L}, \varphi \times u \geq 0, \psi \times u \geq 0\}$, where $\mathcal{L}$ is the set introduced in Definition 7.

Lemma 9. *Given a set of subparquets $\mathcal{V}$ and a set of points Q, we can calculate*

$$\sum_{V \in \mathcal{V}} |(D + q) \cap V|$$

for every $q \in Q$ in total time $\tilde{\mathcal{O}}(n^2 + |Q| + |\mathcal{V}|)$, assuming that every $V \in \mathcal{V}$ consists of vectors of length $\mathcal{O}(n)$.

Proof. For every $u \in \mathbb{Z}^2$ let us define $\text{score}(u) = |\{V : V \in \mathcal{V}, u \in V\}|$. Observe that

$$\sum_{V \in \mathcal{V}} |(D+q) \cap V| = \sum_{u \in D+q} \text{score}(u).$$

We start by explicitly calculating the scores. We find the maximum length of a vector that some $V \in \mathcal{V}$ is defined for, which we denote ℓ. We construct the set $U \subseteq \mathbb{Z}^2$ of all vectors of length at most ℓ. By the assumption, we have $\ell = \mathcal{O}(n)$, and thus $|U| = \mathcal{O}(l^2) = \mathcal{O}(n^2)$. We observe that since all the scores are zero for points outside of U, we can only calculate them for $u \in U$.

We find the set Γ introduced in Lemma 4 and for every $\gamma \in \Gamma$ we construct $U_\gamma = U \cap (\mathcal{L} + \gamma)$. Consider any $u \in U_\gamma$ for some fixed $\gamma \in \Gamma$ and any $V \in \mathcal{V}$. We observe that if $V \not\equiv \gamma$, then $u \notin V$ and thus V does not contribute to $\text{score}(u)$. If $V \equiv \gamma$, then we can find a parquet W such that $V = W \cap (\mathcal{L} + \gamma)$ and we have $u \in V \Leftrightarrow u \in W \cap (\mathcal{L} + \gamma) \Leftrightarrow u \in W$. Thus, if we denote $\mathcal{W}_\gamma$ as the set of parquets W obtained for every $V \in \mathcal{V}$ such that $V \equiv \gamma$, then $\text{score}(u)$ for $u \in U_\gamma$ is the number of parquets $W \in \mathcal{W}_\gamma$ such that $u \in W$. We calculate $\text{score}(u)$ for every $u \in U_\gamma$ by sweeping U_γ and $\mathcal{W}_\gamma$ in time $\tilde{\mathcal{O}}(|U_\gamma| + |\mathcal{W}_\gamma|)$. We do it independently for every $\gamma \in \Gamma$, performing $\tilde{\mathcal{O}}(|U| + |\mathcal{V}|) = \tilde{\mathcal{O}}(n^2 + |\mathcal{V}|)$ operations in total.

Now consider a query vector $q \in Q$. Let $\gamma \in \Gamma$ be such that $q \equiv \gamma$. We have already showed that the sum of scores for $u \in D + q$ is equal to the sum of scores for $u \in (D+q) \cap U$. Since $(D+q) \cap U = (D+q) \cap U_\gamma$, we see that the result is the sum of scores for such $u \in U_\gamma$, for which $\varphi \times u \geq \varphi \times q$ and $\psi \times u \geq \psi \times q$. If we denote $Q_\gamma = Q \cap (\mathcal{L} + \gamma)$, we see that we can calculate the results for all $q \in Q_\gamma$ by sweeping Q_γ and U_γ in time $\tilde{\mathcal{O}}(|Q_\gamma| + |U_\gamma|)$. We do it independently for every $\gamma \in \Gamma$, performing $\tilde{\mathcal{O}}(|Q| + |U|) = \tilde{\mathcal{O}}(n^2 + |Q|)$ operations in total. $\square$

Lemma 10. *For any simple subparquet U we can find $w_0, \ldots, w_3 \in \mathbb{Z}^2$, such that*

$$|U \cap X| = \sum_{j=0}^{3} (-1)^j |(D + w_j) \cap X|$$

for every $X \subseteq \mathbb{Z}^2$. If U consists of vectors of length $\mathcal{O}(n)$, then $w_0, \ldots, w_3$ are of length $\mathcal{O}(n)$.

Proof. Let

$$\varphi_0 = \min\{\varphi \times u : u \in U\}, \quad \varphi_1 = \max\{\varphi \times u : u \in U\},$$
$$\psi_0 = \min\{\psi \times u : u \in U\}, \quad \psi_1 = \max\{\psi \times u : u \in U\}.$$

Note that these values can be extracted from the signature. Since U is a parquet, there exist unique points $u_0, \ldots, u_3 \in U$, such that

- $\varphi \times u_0 = \varphi_0$ and $\psi \times u_0 = \psi_0$,

- $\varphi \times u_1 = \varphi_1$ and $\psi \times u_1 = \psi_0$,

- $\varphi \times u_2 = \varphi_1$ and $\psi \times u_2 = \psi_1$,

- $\varphi \times u_3 = \varphi_0$ and $\psi \times u_3 = \psi_1$.

We construct

$$w_0 = u_0, \quad w_1 = u_1 + \psi, \quad w_2 = u_2 + \varphi + \psi, \quad w_3 = u_3 + \varphi.$$

It can be proven that the condition is satisfied. $\square$

We apply Lemma 10 to every U_i and find $w_{i,0}, \ldots, w_{i,3}$, so that we have

$$\sum_{i=0}^{\ell-1} |(U_i + q) \cap V_i| = \sum_{i=0}^{\ell-1} |U_i \cap (V_i - q)| = \sum_{i=0}^{\ell-1} \sum_{j=0}^{3} (-1)^j |(D + w_{i,j}) \cap (V_i - q)| =$$
$$= \sum_{j=0}^{3} (-1)^j \sum_{i=0}^{\ell-1} |(D + q) \cap (V_i - w_{i,j})|.$$

By Lemma 9 we can independently calculate the values $\sum_{i=0}^{\ell-1} |(D + q) \cap (V_i - w_{i,j})|$ for every j by running the algorithm for $\mathcal{V}_j = \{ V_i - w_{i,j} : i \in [\ell] \}$ and Q.

5.6 Proof of Theorem 8

Definition 13 (Lattice graph)**.** For a set $U \subseteq \mathbb{Z}^2$ we define its **lattice graph** $G_U = (U, E_U)$ where

$$E_U = \big\{ \{ u, u + \delta \} : \delta \in \{ \varphi, \psi \}, u \in U, u + \delta \in U \big\}$$

so every vector is connected with its translations by $\varphi, \psi, -\varphi, -\psi$.

Lemma 11. *If U is a spacious subparquet, then G_U is connected.*

Proof. TODO $\qquad\square$

Firstly, we partition R into a set of subparquet strings $\mathcal{S}$. For every non-empty $S \in \mathcal{S}$ we consider a lattice graph $G_{\mathrm{dom}(S)}$. If S is not monochromatic, then since $G_{\mathrm{dom}(S)}$ is connected, there must exist a pair of neighboring vectors v, w such that $S(v) \neq S(w)$. We select any such pair and partition S into spacious (or simple if S is simple) subparquet strings S' and S'' such that $v \in S'$ and $w \in S''$. For example if $v = w + \varphi$, then $S' = \{ u : u \in S, \psi \times u \leq \psi \times v \}$ and $S'' = \{ u : u \in S, \psi \times u > \psi \times v \}$. In the cases when $v = w + \delta$ for $\delta \in \{ -\varphi, \psi, -\psi \}$ the construction in similar.

We can recursively partition S' and S'' further until we obtain monochromatic strings. Because R has $\mathcal{O}(k)$-periods φ and ψ, the total number of neighbor pairs v, w such that $S(v) \neq S(w)$ is $\mathcal{O}(k)$ throughout all $S \in \mathcal{S}$. Thus the total number of recursive calls is $\mathcal{O}(k)$ and because $|\mathcal{S}| = \mathcal{O}(k)$, the total number of constructed strings is $\mathcal{O}(k)$. The algorithm can be implemented to work in time $\tilde{\mathcal{O}}(|\mathrm{dom}(R)|)$.

5.7 Proof of Theorem 10

In this section we will consider an empty set to be a valid parallelogram.

Definition 14. For a set of points $U \subseteq \mathbb{R}^2$ we will denote

$$\mathrm{X}(U) = \{ u.x : u \in U \}, \quad \mathrm{Y}(U) = \{ u.y : u \in U \}.$$

Observation 6. *For any given $\ell \in \mathbb{Z}^+$ and $v \in \mathbb{Z}^2$ we can find an array of parallel lines $f_0, f_1, \ldots, f_\ell$, where $f_i = \{ u : u \in \mathbb{R}^2, v \times u = c_i \}$ for some $c_i \in \mathbb{R} \setminus \mathbb{Q}$, such that*

- *$c_0 < v \times u < c_\ell$ for every $u \in [n]^2$, or namely, the set $[n]^2$ is between f_0 and f_ℓ,*

- *$0 < c_{i+1} - c_i = \mathcal{O}(n|v|/\ell)$ for every $i \in [\ell]$, or namely, the distance between every two consecutive lines is $\mathcal{O}(n/\ell)$.*

We use Observation 6 with $v = \varphi$ to construct the lines $h_0, \ldots, h_\ell$ and with $v = \psi$ to construct the lines $s_0, \ldots, s_\ell$. For every $i, j \in [\ell + 1]$ we construct a point $w_{i,j}$ as the intersection of h_i and s_j (it is easy to see that since φ and ψ are not collinear, h_i and s_j are not parallel). For every $i, j \in [\ell]$ we construct a parallelogram $p_{i,j}$ defined as the area between s_i and s_{i+1} intersected with the area between h_j and h_{j+1}. Specifically,

$$p_{i,j} = \{ u : u \in \mathbb{R}^2, \varphi \times u \in [\varphi \times w_{i,j}, \varphi \times w_{i+1,j+1}], \psi \times u \in [\psi \times w_{i,j}, \psi \times w_{i+1,j+1}] \}.$$

For a better reference, the vertices of $p_{i,j}$ are $w_{i,j}, w_{i+1,j}, w_{i+1,j+1}, w_{i,j+1}$. Observe that every $u \in [n]^2$ is contained strictly inside exactly one parallelogram $p_{i,j}$.

Lemma 12. *For every $i \in [\ell - 1]$ and $j \in [\ell]$ we have*

$$\min \mathrm{X}(p_{i,j}) < \min \mathrm{X}(p_{i+1,j}), \quad \min \mathrm{Y}(p_{i,j}) \le \min \mathrm{Y}(p_{i+1,j}),$$
$$\max \mathrm{X}(p_{i,j}) < \max \mathrm{X}(p_{i+1,j}), \quad \max \mathrm{Y}(p_{i,j}) \le \max \mathrm{Y}(p_{i+1,j})$$

and for every $i \in [\ell]$ and $j \in [\ell - 1]$ we have

$$\min \mathrm{X}(p_{i,j}) \ge \min \mathrm{X}(p_{i,j+1}), \quad \min \mathrm{Y}(p_{i,j}) < \min \mathrm{Y}(p_{i,j+1}),$$
$$\max \mathrm{X}(p_{i,j}) \ge \max \mathrm{X}(p_{i,j+1}), \quad \max \mathrm{Y}(p_{i,j}) < \max \mathrm{Y}(p_{i,j+1}).$$

Proof. It follows from the fact that we selected $\varphi \in [0, +\infty) \times (-\infty, 0)$ and $\psi \in (0, +\infty) \times [0, +\infty)$. For example, to prove the first inequality, we can consider a point $u \in p_{i+1,j}$, such that $u.x = \min \mathrm{X}(p_{i+1,j})$ and then construct a point $v \in p_{i,j}$, such that $v = u - t\psi$ for some $t > 0$, and thus $\min \mathrm{X}(p_{i,j}) \le v.x \le u.x = \min \mathrm{X}(p_{i+1,j})$. The other inequalities can be proven analogously. $\square$

Lemma 13. *For every $i, j \in [\ell]$ and every $u, v \in \mathrm{X}(p_{i,j}) \times \mathrm{Y}(p_{i,j})$, we have $|u - v| = \mathcal{O}(n/\ell)$.*

Proof. See Section 5.7.2. $\square$

Consider the case when $\max \mathrm{X}(p_{i,j}) - \min \mathrm{X}(p_{i,j}) \ge m/4$ for some $i, j \in [\ell]$. By Lemma 13, we would have $m/4 \le \max \mathrm{X}(p_{i,j}) - \min \mathrm{X}(p_{i,j}) = \mathcal{O}(n/\ell)$, and thus $\ell = \mathcal{O}(1)$. In that case we can return a trivial partitioning where $F = T_{\mathbf{a}}$ and the set of monochromatic strings is empty, since $T_{\mathbf{a}}$ is $\mathcal{O}(m)$-peripheral. We can use the same argument if we have $\max \mathrm{Y}(p_{i,j}) - \min \mathrm{Y}(p_{i,j}) \ge m/4$ for some $i, j \in [\ell]$. Thus, from now on we will assume that $\max X_{i,j} - \min X_{i,j} < m/4$ and $\max Y_{i,j} - \min Y_{i,j} < m/4$ for every $i, j \in [\ell]$.

Let $z = \frac{n-1}{2}$. We split the plain with two lines $x = z$ and $y = z$ into four quarters:

1) $K_1 = (z, +\infty) \times (z, +\infty)$,

2) $K_2 = (-\infty, z) \times (z, +\infty)$,

3) $K_3 = (-\infty, z) \times (-\infty, z)$,

4) $K_4 = (z, +\infty) \times (-\infty, z)$.

Let us denote by $\mathcal{I}$ the set of all parallelograms $p_{i,j}$, such that they intersect with the line $x = z$ or with the line $y = z$ (or both). Observe that every parallelogram $p_{i,j} \notin \mathcal{I}$ must be fully contained in one of the quarters, meaning $p_{i,j} \subseteq K_d$ for some $d \in \{1, \ldots, 4\}$.

Lemma 14. $|\mathcal{I}| = \mathcal{O}(\ell)$.

Proof. Consider the line $x = z$, denoted f. It intersects with every line $h_0, \ldots, h_\ell$ at most once (and does not overlap with any of them). Similarly, it intersects with every line $s_0, \ldots, s_\ell$ at most once. Denote the set of such intersections as U. For every parallelogram $p \in \mathcal{I}$, there must exist $u \in U$, such that $u \in p$. For every $u \in U$, there are at most four parallelograms $p \in \mathcal{I}$, such that $u \in p$, thus $|\mathcal{I}| \le 4|U| = \mathcal{O}(\ell)$. $\square$

Now consider any $j \in [\ell]$. By Lemma 12, we can find $s, t \in [\ell + 1]$, such that the array $p_{0,j}, \ldots, p_{\ell-1,j}$ is split into three groups:

a) $p_{0,j}, \ldots, p_{s-1,j}$, which includes only parallelograms fully contained in K_3,

b) $p_{s,j}, \ldots, p_{t-1,j}$, which does not include any parallelogram fully contained in K_1 or K_3,

c) $p_{t,j}, \ldots, p_{\ell-1,j}$, which includes only parallelograms fully contained in K_1.

We now "merge together" the parallelograms from group (a) and from group (c). Specifically, we construct

$$g_j^3 = \bigcup_{i=0}^{s-1} p_{i,j}, \quad g_j^1 = \bigcup_{i=t}^{\ell-1} p_{i,j}.$$

We do it for every $j \in [\ell]$. Observe that the sets $g_0^1, \ldots, g_{\ell-1}^1$ are are parallelograms (possibly empty) and that they cover the same area as all the fully contained in K_1 parallelograms $p_{i,j}$. The same is true for $g_0^3, \ldots, g_{\ell-1}^3$ and the parallelograms in K_3.

Now for every $i \in [\ell]$ we similarly find $s, t \in [\ell + 1]$, such that the array $p_{i,0}, \ldots, p_{i,\ell-1}$ is split into three groups:

a) $p_{i,0}, \ldots, p_{i,s-1}$, which includes only parallelograms fully contained in K_4,

b) $p_{i,s}, \ldots, p_{i,t-1}$, which does not include any parallelogram fully contained in K_2 or K_4,

c) $p_{i,t}, \ldots, p_{i,\ell-1}$, which includes only parallelograms fully contained in K_2,

and then construct

$$g_i^4 = \bigcup_{j=0}^{s-1} p_{i,j}, \quad g_i^2 = \bigcup_{j=t}^{\ell-1} p_{i,j}.$$

We denote

$$\mathcal{G} = \{\, g_0^1, \ldots, g_{\ell-1}^1 \,\} \cup \{\, g_0^2, \ldots, g_{\ell-1}^2 \,\} \cup \{\, g_0^3, \ldots, g_{\ell-1}^3 \,\} \cup \{\, g_0^4, \ldots, g_{\ell-1}^4 \,\}.$$

Again observe that for every $u \in [n]^2$ there exists exactly one parallelogram $p \in \mathcal{G} \cup \mathcal{I}$, such that $u \in p$, and since the sides of p do not contain integer points, u lays strictly inside p.

Definition 15 (Coverability)**.** We say that a set $U \subseteq \mathbb{Z}^2$ is **coverable** if $U \subseteq \mathrm{dom}(P + q)$ for some $q \in Q$.

Lemma 15. *For every $p \in \mathcal{I}$, the set $p \cap \mathbb{Z}^2$ is either coverable or $\mathcal{O}(n/\ell)$-peripheral.*

Proof. Consider any $p \in \mathcal{I}$. Since the other cases are rotationally symmetric, assume that it intersects with some point $s \in \mathbb{R}^2$, such that $s.x = z$ and $s.y \geq z$. Let $v = (\lfloor \max \mathrm{X}(p), \max \mathrm{Y}(p) \rfloor)$. We have $v.x \geq \lfloor z \rfloor = n/2 - 1$ and $v.y \geq \lfloor z \rfloor = n/2 - 1$. If $v \notin T_\mathbf{a}$, we can see that by Lemma 13, $|u - v| = \mathcal{O}(n/\ell)$ for every $u \in p \cap \mathbb{Z}^2$, thus p is $\mathcal{O}(n/\ell)$-peripheral. If $v \in T_\mathbf{a}$, there exists $q \in Q$, such that $v \in [m]^2 + q$. Consider any $u \in p \cap \mathbb{Z}^2$. By the assumption that $\max \mathrm{X}(p) - \min \mathrm{X}(p) < m/4$ and $\max \mathrm{Y}(p) - \min \mathrm{Y}(p) < m/4$ we get

$$u.x \geq n/2 - m/4 \geq n - m \geq q.x,$$
$$u.y \geq n/2 - m/4 \geq n - m \geq q.y,$$

and since $u.x \leq v.x \leq q.x + m - 1$ and $u.y \leq v.y \leq q.y + m - 1$, we get $u \in [m]^2 + q$, thus $U \subseteq [m]^2 + q$. □

Lemma 16. *For every $g \in \mathcal{G}$ we can construct two parallelograms c and b, such that*

- $c \cap \mathbb{Z}^2$ *is coverable,*

- $b \cap \mathbb{Z}^2$ *is $\mathcal{O}(n/\ell)$-peripheral,*

- $g \cap \mathbb{Z}^2$ *is partitioned into $b \cap \mathbb{Z}^2$ and $c \cap \mathbb{Z}^2$.*

Proof. See Section 5.7.1. □

Lemma 17. *The restriction of T to a coverable set has $\mathcal{O}(k)$-periods φ and ψ.*

Proof. Let R denote the restriction. For $q \in Q$, such that $\mathrm{dom}(R) \subseteq \mathrm{dom}(P+q)$ we have

$$\mathrm{Ham}(R+\varphi, R) \leq \mathrm{Ham}(R+\varphi, P+q+\varphi) + \mathrm{Ham}(P+q+\varphi, P+q) + \mathrm{Ham}(P+q, R) \leq$$
$$\leq \mathrm{Ham}(T, P+q) + \mathrm{Ham}(P+\varphi, P) + \mathrm{Ham}(P+q, T) = \mathcal{O}(k)$$

and identically $\mathrm{Ham}(R+\psi, R) = \mathcal{O}(k)$. □

We split every non-empty parallelogram $g \in \mathcal{G}$ (by Lemma 16) into parallelograms c and b. We construct a set $\mathcal{C}$ consisting of all the obtained parallelograms c and a set $\mathcal{B}$ consisting of all the obtained parallelograms b.

We similarly divide the parallelograms in $\mathcal{I}$ (by Lemma 15) and construct the sets $\mathcal{C}' = \{p : p \in \mathcal{I}, p \cap \mathbb{Z}^2 \text{ is coverable}\}$ and $\mathcal{B}' = \mathcal{I} \setminus \mathcal{C}'$.

Now construct $\mathcal{U} = \{c \cap \mathbb{Z}^2 : c \in \mathcal{C} \cup \mathcal{C}'\}$ and $V = \bigcup_{b \in \mathcal{B} \cup \mathcal{B}'} b \cap \mathrm{dom}(T_\mathbf{a})$. Observe that all sets $U \in \mathcal{U}$ are coverable simple parquets, the set V is $\mathcal{O}(n/\ell)$-peripheral, and $\mathrm{dom}(T_\mathbf{a})$ is partitioned into sets $\mathcal{U} \cup \{V\}$.

For each $U \in \mathcal{U}$ we construct the restriction of T to U. By Lemma 17, it has $\mathcal{O}(k)$-periods φ and ψ, thus, by Theorem 8, it can be partitioned into $\mathcal{O}(k)$ monochromatic simple subparquet strings. Since $|\mathcal{C}'| \leq |\mathcal{I}| = \mathcal{O}(\ell)$ (by Lemma 14) and $|\mathcal{C}| \leq |\mathcal{G}| = \ell$, we have $|\mathcal{U}| = |\mathcal{C}| + |\mathcal{C}'| = \mathcal{O}(\ell)$, thus the total number of constructed strings is $\mathcal{O}(\ell k)$.

Finally, we construct the restriction of T to V, which is a $\mathcal{O}(n/\ell)$-peripheral string.

5.7.1 Proof of Lemma 16

For an empty parallelogram, the proof is trivial. Consider a non-empty set g_j^1 for some $j \in [\ell]$.

Lemma 18. *Every set $U \subseteq K_1 \cap \mathbb{Z}^2$, such that $(\max \mathrm{X}(U), \max \mathrm{Y}(U)) \in T_\mathbf{a}$ is coverable.*

Proof. Let $v = (\max \mathrm{X}(U), \max \mathrm{Y}(U))$. By assumption, there exists $q \in Q$, such that $v \in [m]^2 + q$. For every $u \in U$ we have $q.x \leq n - m \leq n/2 \leq u.x \leq v.x < q.x + m$ and $q.y \leq n - m \leq n/2 \leq u.y \leq v.y < q.y + m$, thus $u \in [m]^2 + q$. □

Observation 7. *By Lemma 18, there does not exist a pair of points $u \in \mathbb{Z}^2 \cap K_1 \setminus \mathrm{dom}(T_\mathbf{a})$ and $v \in T_\mathbf{a}$, such that $u.x \leq v.x$ and $u.y \leq v.y$.*

Recall that there exists $t \in [\ell + 1]$, such that $g_j^1 = \bigcup_{i=t}^{\ell-1} p_{i,j}$. We find

$$f = \min\{i : i \in \{t, \ldots, \ell-1\}, (\lfloor \max \mathrm{X}(p_{i,j}) \rfloor, \lfloor \max \mathrm{Y}(p_{i,j}) \rfloor) \in \mathbb{Z}^2 \setminus \mathrm{dom}(T_\mathbf{a})\}.$$

If the minimum does not exist, we consider $f = \ell$. We then construct the parallelograms

$$c = \bigcup_{i=0}^{f-1} p_{i,j}, \quad b = \bigcup_{i=f}^{\ell-1} p_{i,j}.$$

We now show that the set $c \cap \mathbb{Z}^2$ is coverable. If $c \cap \mathbb{Z}^2$ is empty, then it is coverable, so let us assume it is not. In that case $f > 0$. It is clear that $c \cap \mathbb{Z}^2 \subseteq K_1$. By Lemma 12, we have

$$\max \mathrm{X}(c) = \max \mathrm{X}(p_{f-1,j}),$$
$$\max \mathrm{X}(c) = \max \mathrm{Y}(p_{f-1,j}),$$

and thus

$$\max \mathrm{X}(c \cap \mathbb{Z}^2) \leq \lfloor \max \mathrm{X}(c) \rfloor = \lfloor \max \mathrm{X}(p_{f-1,j}) \rfloor,$$
$$\max \mathrm{Y}(c \cap \mathbb{Z}^2) \leq \lfloor \max \mathrm{Y}(c) \rfloor = \lfloor \max \mathrm{Y}(p_{f-1,j}) \rfloor.$$

We see that $(\max \mathrm{X}(c \cap \mathbb{Z}^2), \max \mathrm{Y}(c \cap \mathbb{Z}^2)) \in T_{\mathbf{a}}$, since it would otherwise contradict Observation 7, considering that $(\lfloor \max \mathrm{X}(p_{f-1,j}) \rfloor, \lfloor \max \mathrm{Y}(p_{f-1,j}) \rfloor) \in T_{\mathbf{a}}$. We see that $c \cap \mathbb{Z}^2$ satisfies the conditions of Lemma 18, thus $c \cap \mathbb{Z}^2$ is coverable.

We now show that the set $b \cap \mathbb{Z}^2$ is $\mathcal{O}(n/\ell)$-peripheral. If it is empty, then the proof is trivial, so let us assume it is not. In that case $f < \ell$. Denote $v = (\lfloor \max \mathrm{X}(p_{f,j}) \rfloor, \lfloor \max \mathrm{Y}(p_{f,j}) \rfloor)$. By definition, $v \in \mathbb{Z}^2 \setminus \mathrm{dom}(T_{\mathbf{a}})$. Consider any point $u \in b \cap \mathbb{Z}^2$. There exists exactly one $i \in \{f, \ldots, \ell-1\}$, such that u lays strictly inside $p_{i,j}$. Let $w = (\lfloor \max \mathrm{X}(p_{i,j}) \rfloor, \lfloor \max \mathrm{Y}(p_{i,j}) \rfloor)$. By Lemma 12, we have $w.x \geq v.x$ and $w.y \geq v.y$, and by considering Observation 7, we get $w \in \mathbb{Z}^2 \setminus \mathrm{dom}(T_{\mathbf{a}})$. Finally, by Lemma 13, we get $|u - w| = \mathcal{O}(n/\ell)$.

The constructions for g_i^2, g_j^3, g_i^4 are rotationally symmetric.

5.7.2 Proof of Lemma 13

Consider a parallelogram $p_{i,j}$ for some $i, j \in [\ell]$.

Lemma 19. *For every* $u, v \in p_{i,j}$, *we have* $|u - v| = \mathcal{O}(n/\ell)$.

Proof. Consider any $u, v \in p_{i,j}$ and denote $w = u - v$. By definition of $p_{i,j}$, we have

$$|\varphi \times w| = |\varphi \times (u - v)| = |\varphi \times u - \varphi \times v| = \mathcal{O}(n|\varphi|/\ell)$$

and similarly $|\psi \times w| = \mathcal{O}(n|\psi|/\ell)$. Since φ and ψ are not collinear, there exist $s, t \in \mathbb{R}$, such that $w = s\varphi + t\psi$. Recall that by Theorem 6 we have $|\varphi \times \psi| \geq \frac{1}{2}|\varphi||\psi|$ (since $|\sin \alpha| \geq 1/2$), thus

$$\frac{1}{2}|t||\varphi||\psi| \leq |t||\varphi \times \psi| = |\varphi \times (s\varphi + t\psi)| = |\varphi \times w| = \mathcal{O}(n|\varphi|/\ell),$$

which gives us $|t\varphi| = \mathcal{O}(n/\ell)$. We can similarly prove that $|s\psi| = \mathcal{O}(n/\ell)$ and finally

$$|w| = |s\varphi + t\psi| \leq |s\varphi| + |t\psi| = \mathcal{O}(n/\ell).$$

$\square$

Lemma 20. *For every point* $u \in \mathrm{X}(p_{i,j}) \times \mathrm{Y}(p_{i,j})$ *there exists a point* $v \in p_{i,j}$, *such that* $|u - v| = \mathcal{O}(n/\ell)$.

Proof. There exists a point $w \in p_{i,j}$, such that $u.x = w.x$, and a point $v \in p_{i,j}$, such that $u.y = v.y$. By Lemma 19, we have

$$|u - v| = |u.x - v.x| = |w.x - v.x| \leq |w - v| = \mathcal{O}(n/\ell).$$

$\square$

Consider any $u, v \in p_{i,j}$. By Lemma 20, there exist $u', v' \in p_{i,j}$ such that $|u - u'| = \mathcal{O}(n/\ell)$ and $|v - v'| = \mathcal{O}(n/\ell)$. By Lemma 19, we have

$$|u - v| \leq |u - u'| + |u' - v'| + |v' - v| = \mathcal{O}(n/\ell).$$

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