Free $\dashv$ Forgertful Adjunction between $Mod(\mathbb{T})$ and **Set**

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Theorem If $\mathbb{T}$ is an algebraic theory then there is a free/forgetful adjunction from $Mod(\mathbb{T})$ of set-valued models to **Set**.

First, define the forgetful functor $Fo\colon Mod(\mathbb{T}) \to \mathbf{Set}$ by $Fo(G) = G(1)$, the universe, and for $\alpha\colon F \to G$ we get that $Fo(\alpha) = \alpha_1$. Functoriality is evident.

Now, we define $Fr\colon \mathbf{Set} \to Mod(\mathbb{T})$. If X is a set then $Fr(X)$ is a functor from $\mathbb{T}$ to **Set** such that $Fr(X)(1)$ is the set of terms Tm_X modulo axioms formed from elements of X. We build up this set inductively. If $x \in X$ then $x \in Fr(X)(1)$. If $f \in \Sigma_n$, i.e. an n-ary function symbol, and $t_1, ..., t_n$ terms in $Fr(X)(1)$, then $f(t_1, ..., t_n) \in Fr(X)(1)$. If $\overline{x_i}|t_1(\overline{x_i}) = t_2(\overline{x_i})$ is in the axioms of $\mathbb{T}$, then $t_1(\overline{x_i}) = t_2(\overline{x_i})$ in Tm_X. The products are the usual products and the projection maps are the usual projection maps. Functoriality is evident and hence well-defined, finite products are preserved by definition.

If $f\colon X \to Y$, then $Fr(f)\colon Fr(X) \to Fr(Y)$ is defined on $Fr(X)(1) \to F(Y)(1)$ by induction. For the base step $x \mapsto f(x)$ for the inductive step if $g \in \Sigma_n$ and $t_1, ..., t_n$ terms then $Fr(f)(g(t_1, ..., t_n)) = g(Fr(f)(t_1), ...Fr(f)(t_n))$. This assignment naturally extends to assignment on all n. Then if $f : n \to 1$ corresponds to an n-ary function symbol in our signature in $\mathbb{T}$ we have that $Fr(X)(f) : Tm_X^n \to Tm_X$ is defined naturally by $Fr(X)(f)(t_1, ..., t_n) = f(t_1, ..., t_n)$. Check naturality if $g\colon X \to Y$:

$$\begin{tikzcd} n \arrow[d, "f"] & Tm_X^n \arrow[r] \arrow[d, "Fr(X)(f)"'] & Tm_Y^n \arrow[d, "Fr(Y)(f)"] \\ 1 & Tm_X \arrow[r] & Tm_Y \end{tikzcd}$$

on the base terms we get

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & Tm_X & \longrightarrow & Tm_Y \\ \\ \overline{x_i} & \mapsto & f(\overline{x_i}) & \mapsto & f(\overline{gx_i}) \end{array}$$

and

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & Tm_Y^n & \longrightarrow & Tm_Y \\ \\ \overline{x_i} & \mapsto & \overline{gx_i} & \mapsto & f(\overline{gx_i}) \end{array}$$

so that they agree. On the inductive step, we get

$$
\begin{array}{ccccc}
Tm_X^n & \longrightarrow & Tm_X & \longrightarrow & Tm_Y \\
\\
\overline{k_i}(\overline{t_i}) & \mapsto & f(\overline{k_i}(\overline{t_i})) & \mapsto & f(\overline{k_i}(Fr(Y)(f)(\overline{t_i})))
\end{array}
$$

and

$$
\begin{array}{ccccc}
Tm_X^n & \longrightarrow & Tm_Y^n & \longrightarrow & Tm_Y \\
\\
\overline{k_i}(\overline{t_i}) & \mapsto & \overline{k_i}(Fr(Y)(f)\overline{t_i}) & \mapsto & f(\overline{k_i}(Fr(Y)(f)(\overline{t_i})))
\end{array}
$$

and hence naturality holds.

Functoriality is easy to check as suffices to check it for the universe and the base case. If $f\colon X \to Y$ and $g\colon Y \to Z$, then $Fr(gf)_1$ is defined to be $x \mapsto gfx$ on the base case. Now, $Fr(g)Fr(f)_1$ on the bases will be defined by $x \mapsto fx \mapsto gfx$. Similarly, $Fr(id_X)$ is evidently the identity in the category of models as $Fr(id_X)_1(x) = x$ on the base case.

Now, we claim that $Fr \dashv Fo$. To see this, we define

$$Mod(\mathbb{T})(Fr(X), G) \cong \mathbf{Set}(X, Fo(G))$$

by sending $\alpha\colon Fr(X) \to G$ to $\alpha_1|_X$ by abuse of notation, looking at the base-level terms as elements of X and from the other direction sending $f\colon X \to Fo(G)$ to a natural transformation α defined by extending $\alpha_1\colon Fr(X)(1) \to G(1)$. That is, $\alpha_1(x) = f(x)$ and $\alpha_1(k(\overline{t_i})) = G(k)(\overline{\alpha_1 t_i})$. This naturally extends to the products. We need to prove naturality. As before, suffices to check on maps to 1.

$$
\begin{array}{ccc}
n & & \\
\big\downarrow{\scriptstyle g} & & \\
1 & &
\end{array}
\qquad
\begin{tikzcd}
Tm_X^n \arrow[r, "\tilde{f}^n"] \arrow[d, "Fr(X)(g)"'] & G(1)^n \arrow[d, "G(g)"] \\
Tm_X \arrow[r, "\tilde{f}"] & G(1)
\end{tikzcd}
$$

On the base case we have

$$
\begin{array}{ccccc}
Tm_X^n & \longrightarrow & Tm_X & \longrightarrow & G(1) \\
\\
\overline{x_i} & \mapsto & g(\overline{x_i}) & \mapsto & G(g)(\overline{fx_i})
\end{array}
$$

in the other

$$
\begin{array}{ccccc}
Tm_X^n & \longrightarrow & G(1)^n & \longrightarrow & G(1) \\
\\
\overline{x_i} & \mapsto & \overline{fx_i} & \mapsto & G(g)(\overline{fx_i})
\end{array}
$$

on the induction step

$$
\begin{array}{ccccc}
Tm_X^n & \longrightarrow & Tm_X & \longrightarrow & G(1) \\
\overline{k_i}(\overline{t_i}) & \mapsto & g(\overline{k_i}(\overline{t_i})) & \mapsto & G(g)(\overline{G(k_i)}(\overline{\alpha_1 t_i}))
\end{array}
$$

$$
\begin{array}{ccccc}
Tm_X^n & \longrightarrow & G(1)^n & \longrightarrow & G(1) \\
\overline{k_i}(\overline{t_i}) & \mapsto & \overline{G(k_i)}(\overline{\alpha_1 t_i}) & \mapsto & G(g)(\overline{G(k_i)}(\overline{\alpha_1 t_i}))
\end{array}
$$

In one direction, if $\alpha\colon Fr(X) \Rightarrow G$ it gets taken to $\alpha_1|_X$ which then gets taken to a natural transformation defined on the base level as $x \mapsto \alpha_1(x)$ which extends to an interpretation on all terms producing α.

In the other direction, of $f\colon X \to Fo(G)$, then it first gets taken to a natural transformation α defined on 1 by applying on the base terms the function f and then gets taken to a map that is $\alpha_1|_X$, i.e. exactly our map f.

Now, let us prove naturality in both variables. Let us consider the diagram

$$
\begin{tikzcd}
X & Mod(\mathbb{T})(Fr(X), G) \arrow[r, "\cong"] \arrow[d, "Fr(f)^*"] & \mathbf{Set}(X, Fo(G)) \arrow[d, "f^*"] \\
Y \arrow[u, "f"] & Mod(\mathbb{T})(Fr(Y), G) \arrow[r, "\cong"] & \mathbf{Set}(Y, Fo(G))
\end{tikzcd}
$$

Suppose $\alpha\colon Fr(X) \Rightarrow G$. Recall that suffices to see what happens at the bottom level. So, by abuse of notation we have that

$$
\begin{array}{ccccc}
Mod(\mathbb{T})(Fr(X), G) & \longrightarrow & Mod(\mathbb{T})(Fr(Y), G) & \longrightarrow & \mathbf{Set}(Y, Fo(G)) \\
\alpha & \mapsto & \alpha(Fr(f)) & \mapsto & \alpha(Fr(f))|_Y
\end{array}
$$

where $\alpha(Fr(f))_1$ is defined on the base level as $y \mapsto f(y) \mapsto \alpha_1 f(y)$ and on the inductive step as $k(\overline{t_i}) \mapsto G(k)(\overline{\alpha_1(t_i)})$ so that $\alpha(Fr(f))|_Y = (\alpha_1|_Y)f$

$$
\begin{array}{ccccc}
Mod(\mathbb{T})(Fr(X), G) & \longrightarrow & \mathbf{Set}(X, Fo(G)) & \longrightarrow & \mathbf{Set}(Y, Fo(G)) \\
\alpha & \mapsto & \alpha_1|_X & \mapsto & (\alpha_1|_Y)f
\end{array}
$$

in the other variable, get

$$
\begin{tikzcd}
F \arrow[d, Rightarrow, "\alpha"] & Mod(\mathbb{T})(Fr(X),F) \arrow[r, "\cong"] \arrow[d, "\alpha_*"] & \mathbf{Set}(X,Fo(F)) \arrow[d, "Fo(\alpha)_*"] \\
G & Mod(\mathbb{T})(Fr(X),G) \arrow[r, "\cong"] & \mathbf{Set}(X,Fo(G))
\end{tikzcd}
$$

so take $\eta\colon Fr(X) \Rightarrow F$. In one direction, following both sides of the square we get $\alpha_1\eta_1|_X$ and hence we have the desired adjunction.

Corollary

The category of set-valued models of essentially algebraic theories has a free-forgetful adjunction formulated by this procedure.