Free $\dashv$ Forgertful Adjunction between $Mod(\mathbb{T}, \mathbf{Set})$ and $\mathbf{Set}$

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Theorem If $\mathbb{T}$ is an algebraic theory then there is a free/forgetful adjunction from $Mod(\mathbb{T})$ of set-valued models to $\mathbf{Set}$.

First, define the forgetful functor $Fo\colon Mod(\mathbb{T}) \to \mathbf{Set}$ by $Fo(G) = G(1)$, the universe, and for $\alpha\colon F \to G$ we get that $Fo(\alpha) = \alpha_1$. Functoriality is evident.

Now, we define $Fr\colon \mathbf{Set} \to Mod(\mathbb{T})$. If X is a set then $Fr(X)$ is a functor from $\mathbb{T}$ to $\mathbf{Set}$ such that $Fr(X)(1)$ is the set of terms X' modulo interpreted axioms with free variables as elements of X. As we can derive the equality in the strongest context given the equality in the weaker ones via substitution WLOG assume that all axioms are given in the strongest possible context. We build up X' inductively. If $x \in X$ then $x \in Fr(X)(1)$. If $f \in \Sigma_n$, i.e. an n-ary function symbol, and $\overline{t_i}$ terms in $Fr(X)(1)$, then $f(\overline{t_i}) \in Fr(X)(1)$. The equivalence relation we quotient out by is given by $\overline{x_i}|t_1(\overline{x_i}) = t_2(\overline{x_i})$ is in the axioms of $\mathbb{T}$ iff $t_1(\overline{x_i}) \sim t_2(\overline{x_i})]$. Define $Tm_X := X'/\sim$ By a The products are the usual products and the projection maps are the usual projection maps. Then if $f\colon n \to 1$ corresponds to an n-ary function symbol in our signature in $\mathbb{T}$ we have that $Fr(X)(f) : Tm_X^n \to Tm_X$ is defined naturally by $Fr(X)(f)([t_i])_{i \le n} = [f(\overline{t_i})]$. Functoriality is evident and hence well-defined, finite products are preserved by definition.

If $g\colon X \to Y$, then $Fr(g)\colon Fr(X) \to Fr(Y)$ is defined on $Fr(X)(1) \to F(Y)(1)$ by induction. For the base step $[x] \mapsto [g(x)]$ for the inductive step if $f \in \Sigma_n$ and $\overline{t_i}$ for $i \le n$ terms then $Fr(g)([f(\overline{t_i})]) = [f(\overline{Fr(g)(t_i)})]$. This assignment naturally extends to assignment on all n. Check naturality if $g\colon X \to Y$:

$$\begin{tikzcd}
n \arrow[d, "f"'] & Tm_X^n \arrow[r] \arrow[d, "Fr(X)(f)"'] & Tm_Y^n \arrow[d, "Fr(Y)(f)"] \\
1 & Tm_X \arrow[r] & Tm_Y
\end{tikzcd}$$

on the base terms we get

$$\begin{array}{ccccc}
Tm_X^n & \longrightarrow & Tm_X & \longrightarrow & Tm_Y \\
([x_i])_{i \le n} & \mapsto & [f(\overline{x_i})] & \mapsto & [f(\overline{gx_i})]
\end{array}$$

and

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & Tm_Y^n & \longrightarrow & Tm_Y \\ \\ ([x_i])_{i\le n} & \mapsto & ([gx_i])_{i\le n} & \mapsto & [f(\overline{gx_i})] \end{array}$$

so that they agree. On the inductive step, we get

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & Tm_X & \longrightarrow & Tm_Y \\ \\ ([k_i(\overline{t_i})])_{i\le n} & \mapsto & [f(\overline{k_i}(\overline{t_i}))] & \mapsto & [f(\overline{k_i}(Fr(Y)(f)(\overline{t_i})))] \end{array}$$

and

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & Tm_Y^n & \longrightarrow & Tm_Y \\ \\ ([k_i(\overline{t_i})])_{i\le n} & \mapsto & ([k_i(Fr(Y)(f)\overline{t_i}])_{i\le n} & \mapsto & [f(\overline{k_i}(Fr(Y)(f)(\overline{t_i})))] \end{array}$$

and hence naturality holds.

Functoriality is easy to check as suffices to check it for the universe and the base case. If $f\colon X\to Y$ and $g\colon Y\to Z$, then $Fr(gf)_1$ is defined to be $[x]\mapsto[gfx]$ on the base case. Now, $Fr(g)Fr(f)_1$ on the bases will be defined by $[x]\mapsto[fx]\mapsto[gfx]$. Similarly, $Fr(id_X)$ is evidently the identity in the category of models as $Fr(id_X)_1([x])=[x]$ on the base case.

Now, we claim that $Fr\dashv Fo$. To see this, we define

$$Mod(\mathbb{T})(Fr(X),G)\cong \mathbf{Set}(X,Fo(G))$$

by sending $\alpha\colon Fr(X)\to G$ to $\alpha_1|_X$ by abuse of notation, looking at the base-level terms as elements of X and from the other direction sending $f\colon X\to Fo(G)$ to a natural transformation α defined by extending $\alpha_1\colon Fr(X)(1)\to G(1)$. That is, $\alpha_1([x])=f([x])$ and $\alpha_1([k(\overline{t_i})])=G(k)(\overline{\alpha_1[t_i]})$. This naturally extends to the products. We need to prove naturality. As before, suffices to check on maps to 1.

$$\begin{tikzcd} n \arrow[d, "g"] & Tm_X^n \arrow[r, "\tilde{f}^n"] \arrow[d, "Fr(X)(g)"] & G(1)^n \arrow[d, "G(g)"] \\ 1 & Tm_X \arrow[r, "\tilde{f}"] & G(1) \end{tikzcd}$$

On the base case we have

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & Tm_X & \longrightarrow & G(1) \\ \\ ([x_i])_{i\le n} & \mapsto & [g(\overline{x_i})] & \mapsto & G(g)(\overline{f[x_i]}) \end{array}$$

in the other

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & G(1)^n & \longrightarrow & G(1) \\ & & & & \\ ([x_i])_{i\leq n} & \mapsto & (f[x_i])_{i\leq n} & \mapsto & G(g)(\overline{f[x_i]}) \end{array}$$

on the induction step

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & Tm_X & \longrightarrow & G(1) \\ & & & & \\ ([k_i(\overline{t_i})])_{i\leq i} & \mapsto & [g(\overline{k_i}(\overline{t_i}))] & \mapsto & G(g)(\overline{G(k_i)}(\overline{\alpha_1[t_i]})) \end{array}$$

and

$$\begin{array}{ccccc} Tm_X^n & \longrightarrow & G(1)^n & \longrightarrow & G(1) \\ & & & & \\ (k_i(\overline{t_i}))_{i\leq n} & \mapsto & (G(k_i)(\overline{\alpha_1[t_i]}))_{i\leq n} & \mapsto & G(g)(\overline{G(k_i)}(\overline{\alpha_1[t_i]})) \end{array}$$

In one direction, if $\alpha\colon Fr(X) \Rightarrow G$ it gets taken to $\alpha_1|_X$ which then gets taken to a natural transformation defined on the base level as $[x] \mapsto \alpha_1([x])$ which extends to an interpretation on all terms producing α.

In the other direction, of $f\colon X \to Fo(G)$, then it first gets taken to a natural transformation α defined on 1 by applying on the base terms the function f and then gets taken to a map that is $\alpha_1|_X$, i.e. exactly our map f.

Now, let us prove naturality in both variables. Let us consider the diagram

$$\begin{tikzcd} X & Mod(\mathbb{T})(Fr(X),G) \arrow[r, "\cong"] \arrow[d, "Fr(f)^*"] & \mathbf{Set}(X,Fo(G)) \arrow[d, "f^*"] \\ Y \arrow[u, "f"] & Mod(\mathbb{T})(Fr(Y),G) \arrow[r, "\cong"] & \mathbf{Set}(Y,Fo(G)) \end{tikzcd}$$

Suppose $\alpha\colon Fr(X) \Rightarrow G$. Recall that suffices to see what happens at the bottom level. So, by abuse of notation we have that

$$\begin{array}{ccccc} Mod(\mathbb{T})(Fr(X),G) & \longrightarrow & Mod(\mathbb{T})(Fr(Y),G) & \longrightarrow & \mathbf{Set}(Y,Fo(G)) \\ & & & & \\ \alpha & \mapsto & \alpha(Fr(f)) & \mapsto & \alpha(Fr(f))|_Y \end{array}$$

where $\alpha(Fr(f))_1$ is defined on the base level as $[y] \mapsto [f(y)] \mapsto \alpha_1[f(y)]$ and on the inductive step as $[k(\overline{t_i})] \mapsto G(k)(\overline{\alpha_1([t_i])})$ so that $\alpha(Fr(f))|_Y = (\alpha_1|_Y)f$

$$\begin{array}{ccccc} Mod(\mathbb{T})(Fr(X), G) & \longrightarrow & \mathbf{Set}(X, Fo(G)) & \longrightarrow & \mathbf{Set}(Y, Fo(G)) \\ & & & & \\ \alpha & \mapsto & \alpha_1|_X & \mapsto & (\alpha_1|_Y)f \end{array}$$

in the other variable, get

$$\begin{tikzcd} F \arrow[d, Rightarrow, "\alpha"] & Mod(\mathbb{T})(Fr(X), F) \arrow[r, "\cong"] \arrow[d, "\alpha_*"] & \mathbf{Set}(X, Fo(F)) \arrow[d, "Fo(\alpha)_*"] \\ G & Mod(\mathbb{T})(Fr(X), G) \arrow[r, "\cong"] & \mathbf{Set}(X, Fo(G)) \end{tikzcd}$$

so take $\eta \colon Fr(X) \Rightarrow F$. Following both sides of the square we get $\alpha_1\eta_1|_X$ and hence we have the desired adjunction.

Corollary The category of set-valued models of essentially algebraic theories has a free-forgetful adjunction formulated by this procedure.