Proof By Induction

Alex Hagopian

May 2025

Definition. *The Fibonacci sequence is defined recursively by* $F_n = F_{n-1} + F_{n-2}$, $n \in N$ *with* $F_1 = F_2 = 1$

Theorem. *Every natural number is either a Fibonacci number or the sum of distinct, nonconsecutive Fibonacci numbers.*

Proof. We will prove statement using strong induction.

Let $P(n)$ be the statement that every natural number, n is either a Fibonacci number or the sum of distinct, nonconsecutive Fibonacci numbers.

Base Case: For $n = 1\, n = 2\, n = 3$:

$$1 = F(1) = F(2) \quad \text{(By definition)}$$

$$2 = F(3) \quad \text{(Is a Fibonacci number)}$$

$$3 = F(4) \quad \text{(Is a Fibonacci number)}$$

$$4 = F(4) + F(2) \quad \text{(Sum of distinct, nonconsecutive Fibonacci numbers)}$$

Inductive Hypothesis: Suppose $P(j)$ is true $\forall j, k \in N : 1 \leq j \leq k$. We will prove that this statement is true for $k + 1$. $k + 1$ is either a Fibonacci number or the sum of distinct, nonconsecutive Fibonacci numbers.

Case 1: $k + 1$ *is a Fibonacci number.* This case is trivial. If $k + 1$ is a Fibonacci number, then $P(k + 1)$ is true.

Case 2: $k + 1$ *is not a Fibonacci number.*

Since $k + 1$ is not a Fibonacci number, we know that there exists some number, $F(m) : F(m) < k + 1$

$$k + 1 - F(m) = l$$

Since $0 < l < F(m) < k+1$, we know by the inductive hypothesis that l is either a Fibonacci number or the sum of disinct, nonconsecutive Fibonacci numbers.

Thus, since $k + 1$ is not a Fibonacci number, l must be nonconsecutive to $F(m)$. This is because if $k + 1$ was a Fibonacci number:

$$k + 1 = F(m) + F(m - 1) \rightarrow k + 1 = F(m) + l$$

Thus, l must be nonconsecutive to $F(m)$.

Therefore, $k + 1$ is the sum of distinct, nonconsecutive Fibonacci numbers.

□