Regular black hole metrics and the weak energy condition

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Abstract

In this work we construct a family of spherically symmetric, static, charged regular black hole metrics in the context of Einstein-nonlinear electrodynamics theory. The construction of the charged regular black hole metrics is based on three requirements: (a) the weak energy condition should be satisfied, (b) the energy-momentum tensor should have the symmetry $T^0_0 = T^1_1$, and (c) these metrics have to asymptotically behave as the Reissner-Nordström black hole metric. In addition, these charged regular black hole metrics depend on two parameters which for specific values yield regular black hole metrics that already exist in the literature. Furthermore, by relaxing the third requirement, we construct more general regular black hole metrics which do not behave asymptotically as a Reissner-Nordström black hole metric.

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I. INTRODUCTION

Charged regular black hole solutions exist in the framework of Einstein-nonlinear electrodynamics theory and are obtained as solutions of Einstein equations that are characterized by the fact that the metric as well as the curvature invariants R, $R_{\mu\nu}R^{\mu\nu}$, $R_{\kappa\lambda\mu\nu}R^{\kappa\lambda\mu\nu}$ do not present singularities anywhere [1].

The Bardeen black hole is the first of a series of regular black hole solutions obtained [2]. If we write the most general form of a static line element with spherical symmetry

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{1}$$

where the metric function can be written as

$$f(r) = 1 - \frac{2m(r)}{r} \ , \tag{2}$$

then, we can write the mass function of the Bardeen black hole as [2] [2]

$$m(r) = \frac{Mr^3}{(r^2+g^2)^{3/2}} \ . \tag{3}$$

Such a metric has event horizons located at $r_\pm$ if $g^2 \leq (16/27)M^2$, where g can be interpreted as the monopole charge of a self-gravitating magnetic field described by nonlinear electrodynamics [3]. Furthermore, if $r \to \infty$, we get that the metric behaves as

$$f(r) \to 1 - \frac{2M}{r} + \frac{3Mg^2}{r^3} \ . \tag{4}$$

Similarly, when $r \to 0$ the Bardeen metric function behaves as the de Sitter black hole, that is as

$$f(r) \to 1 - \frac{2M}{g^3}r^2 \ . \tag{5}$$

Later on, other solutions [4]-[8] were obtained, using the F-P dual formalism, by considering the action of general relativity coupled to nonlinear electrodynamics, namely

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi}R - \frac{1}{4\pi}L(F)\right) \ . \tag{6}$$

Here, the Lagrangian $L(F)$ is a nonlinear function of the Lorentz invariant $F = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ which, for weak fields, describes the Maxwell theory, and the corresponding regular black hole solution asymptotically behaves as the Reissner-Nordström black hole.

[1] For a recent review see [1].
[2] Henceforth, we use geometrized units, i.e., $G = c = 1$.

The regular black hole may also be characterized by energy conditions [9, 10] that the corresponding energy-momentum tensor should satisfy. In the context of regular black holes, three energy conditions have been utilized by several authors. In particular:

- The strong energy condition (SEC) states that $T_{\mu\nu}t^{\mu}t^{\nu} \geq \frac{1}{2}T^{\mu}_{\ \mu}t^{\nu}t_{\nu}$ for all timelike vector t^{μ}. This condition implies that gravitational force is attractive.

- The dominant energy condition (DEC) states that $T_{\mu\nu}t^{\mu}t^{\nu} \geq 0$ and that $T^{\mu\nu}t_{\mu}$ must be non-spacelike for all timelike vector t^{μ} or, equivalently, $T^{00} \geq |T^{ij}|$ for each $i, j = 1, 2, 3$. This means that the local energy density measured by a given observer must be nonnegative, and that the speed of the energy density flow associated with this observer cannot exceed the speed of light.

- The weak energy condition (WEC) states that the energy density of matter measured by an observer, whose 4-velocity is t^{μ}, satisfies $T_{\mu\nu}t^{\mu}t^{\nu} \geq 0$ for all timelike vector t^{μ}, that is, the local energy density cannot be negative for all observers. The DEC implies the WEC.

When a black hole is regular, the SEC is necessarily violated somewhere inside the horizon [11]. However, a regular black hole could satisfy the WEC or the DEC everywhere [8]. Considering the line element (1) and the metric function (2), we can write the components of the respective energy-momentum tensor as

$$T^{0}_{\ 0} = T^{1}_{\ 1} = \frac{2}{8\pi r^{2}}\frac{dm(r)}{dr} \ , \ \ T^{2}_{\ 2} = T^{3}_{\ 3} = \frac{1}{8\pi r}\frac{d^{2}m(r)}{dr^{2}} \ , \tag{7}$$

and hence the WEC can be expressed equivalently in terms of the mass function by the following inequalities

$$\frac{1}{r^{2}}\frac{dm(r)}{dr} \geq 0 \ , \tag{8}$$

and

$$\frac{2}{r}\frac{dm(r)}{dr} \geq \frac{d^{2}m(r)}{dr^{2}} \ . \tag{9}$$

From these relations, it is obvious that the Bardeen black hole satisfies the WEC everywhere [2]. The same applies to regular black hole solutions reported in Refs. [4], [7], [8], [12], [13], [14], and [15]. However, there are other regular black hole solutions reported in Refs. [5], [6], [7], [16], and [17], which do not satisfy the WEC.

The regular black hole solutions that satisfy the WEC and their energy-momentum tensor has the symmetry $T^0_{\ 0} = T^1_{\ 1}$, necessarily have de Sitter behavior at $r \to 0$ as was shown in Ref. [13], and illustrated here in Eq. (5) for the Bardeen black hole solution. As we will see later, this leaves an extra condition which allows the WEC to be better exploited compared to the DEC when one is building solutions. However, it should be noted that a de Sitter behavior at the center of a regular black hole is not sufficient by itself to ensure that the solution satisfies the WEC. In addition, it has been shown that for a regular black hole in nonlinear electrodynamics which satisfies the WEC, the non-existence theorems [16] can be circumvented by removing the condition of the Maxwell weak field limit imposed at the center of the black hole. In this way, regular black hole solutions with electric charge do exist [8].

Furthermore, there are other features that characterize regular black holes which are due to the nonlinearities of the field equations. For example, the thermodynamic quantities of the regular black holes do not satisfy the Smarr formula [18], the identity of Bose-Dadhich [19] which refers to the relation between the Brown-York energy and the Komar charge, is not satisfied by regular black holes [20].

In all the above-mentioned solutions, one asymptotically recovers the Schwarzschild black hole metric, and if the condition $r^2 m'(r) \neq 0$ as $r \to \infty$ is satisfied, then one recovers the Reissner-Nordström black hole metric.

In this Letter, in the context of Einstein-nonlinear electrodynamics theory, we will construct a family of spherically symmetric, static, charged regular black hole metrics without utilizing the aforesaid methods but by imposing three conditions: (a) the weak energy condition should be satisfied, (b) the energy-momentum tensor should have the symmetry $T^0_0 = T^1_1$, and (c) these metrics have to asymptotically behave as the Reissner-Nordström black hole metric. In addition, by relaxing the third condition, i.e., condition (c), we construct more general regular black hole metrics which do not behave asymptotically as a Reissner-Nordström black hole metric. The Letter is organized as follows. In Sec. II, we present the equations that we will use in the construction of our metrics. In Sec. III, we obtain a general metric for charged regular black holes that satisfies the WEC and asymptotically behaves as the Reissner-Nordström black hole metric. In addition, we discuss a specific case of the aforesaid general metric function. In Sec. IV, we extend the analysis to obtain metric functions that do not necessarily asymptotically behave as the Reissner-Nordström solution. Finally, in Sec. V, we briefly summarize our results.

II. WEC EQUATIONS

Up to now, regular black holes solutions have been constructed by searching for, or postulating, the Lagrangian function $L(F)$ in the framework of F-P dual formalism [21] [3]. As already mentioned in the Introduction, here we will construct a family of spherically symmetric, static, charged regular black hole metrics by imposing three conditions: (a) the weak energy condition should be satisfied, (b) the energy-momentum tensor should have the symmetry $T_0^0 = T_1^1$, and (c) these metrics have to asymptotically behave as the Reissner-Nordström black hole metric. Therefore, in this section we will derive the equations which will be used in this construction, from the WEC.

For simplicity and future convenience, we replace the variable r that appears in Eqs. (8) and (9) with a new variable x which is defined as $r = 1/x$ and obviously $x \in [0, \infty)$. It is evident that by employing the new variable x, the WEC inequalities, namely Eqs. (8) and (9), now read

$$x^4 \frac{dm(x)}{dx} \leq 0 \tag{10}$$

and

$$4\, x^3 \frac{dm(x)}{dx} + x^4 \frac{d^2 m(x)}{dx^2} \leq 0 \ . \tag{11}$$

It should be noted that these conditions plus regularity imply that the regular black hole metric function must satisfy the following limit

$$-\frac{dm(x)}{dx} x^4 \rightarrow b \qquad \text{when} \qquad x \rightarrow \infty \tag{12}$$

where b is a positive constant.

At this point, we should stress that if we want the regular black hole metric to behave asymptotically as the Schwarzschild black hole, then we have to demand our metric function to satisfy the following condition

$$m(x) \neq 0 \qquad \text{when} \qquad x \rightarrow 0 \ . \tag{13}$$

[3] The F-P dual formalism is briefly presented in Appendix A.

However, if we want the regular black hole metric to behave asymptotically as the Reissner-Nordström black hole, then we must also require our metric function to satisfy the condition

$$-\frac{dm(x)}{dx} \neq 0 \qquad \text{when} \qquad x \to 0 \ . \tag{14}$$

It is easily seen that two mass functions which satisfy the aforesaid conditions, i.e. Eqs. (10)-(14), are defined by the following WEC equations

$$-\frac{dm(x)}{dx} = \frac{c_1}{(1 + c_2\, x^{\alpha})^{4/\alpha}} \tag{15}$$

and

$$-\frac{dm(x)}{dx} = \frac{c_3}{(1 + c_4\, x^{1/\beta})^{4\beta}} \tag{16}$$

where α is a positive integer, β is a positive constant, c_1, c_2, c_3, and c_4 are arbitrary but also positive constants related by $c_1 = (c_2)^{4/\alpha} b$ and $c_3 = (c_4)^{4/\beta} b$. However, the latter mass function has to be discarded since its expansion leaves fractional powers (with the exception of the case with $\beta = 1$).

III. REGULAR BLACK HOLE METRICS

In this section, we will construct a family of spherically symmetric, static, charged regular black hole metrics by imposing the following three conditions on them: (a) the weak energy condition should be satisfied, (b) the energy-momentum tensor should have the symmetry $T_0^0 = T_1^1$, and (c) the metrics have to asymptotically behave as the Reissner-Nordström black hole metric.

First we transform the WEC equation given by Eq. (15) into the integral form

$$m(x) = \int_x^{\infty} \frac{c_1}{(1 + c_2\, y^{\alpha})^{4/\alpha}} dy \tag{17}$$

and then we compute the above integral. The mass function is, thus, given by the expression

$$m(x) = \frac{c_1}{3(c_2)^{4/\alpha}} \frac{1}{x^3}\ {}_2F_1\left(\frac{3}{\alpha}, \frac{4}{\alpha}; \frac{3+\alpha}{\alpha}; -\frac{1}{c_2\, x^{\alpha}}\right) \tag{18}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function .

At this point, we demand the metric function given in Eq. (2), i.e.,

$$f(r) = 1 - \frac{2m(r)}{r} \ , \tag{19}$$

to behave asymptotically as the Reissner-Nordström black hole metric, i.e.,

$$f(r)=1-\frac{2M}{r}+\frac{q^2}{r^2}\ . \tag{20}$$

Now, we substitute Eq. (18) in Eq. (19) and Taylor expand it around $r=0$. By comparing the coefficients of the series expansion at the asymptotic limit, i.e., $r\to\infty$, with the corresponding ones in Eq. (20), we define the constants c_1 and c_2 as follows

$$c_1 = \frac{q^2}{2} \tag{21}$$

$$c_2 = \left[\frac{q^2\Gamma(\frac{1}{\alpha})\Gamma(\frac{\alpha+3}{\alpha})}{6M\Gamma(\frac{4}{\alpha})}\right]^{\alpha}\ . \tag{22}$$

Therefore, the mass function given by Eq. (18) becomes

$$m(r)=\frac{r^3q^2}{6}\left(\frac{6\,\Gamma(\frac{4}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(\frac{\alpha+3}{\alpha})}\frac{M}{q^2}\right)^4 {}_2F_1\left(\frac{3}{\alpha},\frac{4}{\alpha};\frac{3+\alpha}{\alpha};-\left(\frac{6\,\Gamma(\frac{4}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(\frac{\alpha+3}{\alpha})}\frac{M}{q^2}r\right)^{\alpha}\right)\ . \tag{23}$$

This is the mass function of a charged regular black hole metric given by Eq. (19) which asymptotically behaves as the Reissner-Nordström black hole if α is a positive constant. It is noteworthy that for $\alpha=2$, we retrieve the regular black hole metric given in Ref. [8]. Furthermore, in the context of Einstein-nonlinear electrodynamics theory, the electric field associated with the above regular black hole metric is given as [22]

$$E=\frac{q}{r^2}\left(1+\left(\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{3+\alpha}{\alpha})}{6\,\Gamma(\frac{4}{\alpha})}\frac{q^2}{Mr}\right)^{\alpha}\right)^{-(4+\alpha)/\alpha}\ , \tag{24}$$

which behaves as $E=\frac{q}{r^2}$ when $r\to\infty$.

Finally, as an example, we choose $\alpha=3$ in which case the metric function is of the form

$$f(r)=1-\frac{2M}{r}\left(1-\frac{1}{\left(1+\left(\frac{2Mr}{q^2}\right)^3\right)^{1/3}}\right)\ . \tag{25}$$

This regular black hole metric has event horizons if the electric charge satisfies the condition $q\leq 1.0257M$. Moreover, in the context of Einstein-nonlinear electrodynamics theory, the associated electric field is given as [22]

$$E=\frac{q}{r^2}\left(1+\left(\frac{q^2}{2Mr}\right)^3\right)^{-7/3}\ . \tag{26}$$

IV. MORE GENERAL METRICS

In this section, we will construct more general regular black hole metrics. The same analysis as in the previous section will be adopted here but we will not demand the metric to behave asymptotically as the Reissner-Nordström black hole metric. For this reason, we will relax the condition given in Eq. (14). The mass function $m(x)$ will now satisfy the following WEC equation

$$-\frac{dm(x)}{dx}=\frac{c_1 x^{\mu-4}}{(1+c_2 x^{\alpha})^{\mu/\alpha}} \tag{27}$$

where α and μ are integers and also $\alpha \geq 1$ and $\mu \geq 4$ and, thus, it is given by the expression

$$m(x)=\frac{c_1}{3(c_2)^{\mu/\alpha}}\frac{1}{x^3}\,{}_2F_1\left(\frac{3}{\alpha},\frac{\mu}{\alpha};\frac{3+\alpha}{\alpha};-\frac{1}{c_2\, x^{\alpha}}\right) \quad . \tag{28}$$

The coefficients c_1 and c_2 are now given by

$$c_1 \;=\; \frac{q^2}{2}\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{\mu}{\alpha})}{\Gamma(\frac{4}{\alpha})\Gamma(\frac{\mu-3}{\alpha})}\left(\frac{6\Gamma(\frac{4}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(\frac{\alpha+3}{\alpha})}\frac{M}{q^2}\right)^{4-\mu} \tag{29}$$

$$c_2 \;=\; \left[\frac{6\Gamma(\frac{4}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(\frac{\alpha+3}{\alpha})})\frac{M}{q^2}\right]^{-\alpha} \quad . \tag{30}$$

Thus, the mass function of the regular black hole metric becomes

$$m(r) \;=\; \frac{r^3 q^2}{6}\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{\mu}{\alpha})}{\Gamma(\frac{4}{\alpha})\Gamma(\frac{\mu-3}{\alpha})}\left(\frac{6\,\Gamma(\frac{4}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(\frac{\alpha+3}{\alpha})}\frac{M}{q^2}\right)^4 \\ {}_2F_1\left(\frac{3}{\alpha},\frac{\mu}{\alpha};\frac{3+\alpha}{\alpha};-\left(\frac{6\,\Gamma(\frac{4}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(\frac{\alpha+3}{\alpha})}\frac{M}{q^2}r\right)^{\alpha}\right) \quad . \tag{31}$$

This is the mass function of a charged regular black metric given by Eq. (19), but which does not asymptotically behave as the Reissner-Nordström black hole metric except for the case with $\mu=4$.

It is worthy of notice that, with an appropriate choice of parameters, we can derive from Eq. (31) several regular black hole metrics which already exist in the literature. For instance, if we choose $\mu=5$ and $\alpha=2$, we obtain

$$m(r)=\frac{Mr^3}{(r^2+\frac{\pi^2 q^4}{64M^2})^{3/2}} \quad . \tag{32}$$

By replacing $\pi^2 q^4/(64M^2)$ with g^2 in Eq. (32) , we recover the Bardeen metric whose mass function is given by Eq. (3).

Now, if we take $\mu = 6$ and $\alpha = 3$, and we replace the factor $\frac{q^6}{8M^3}$ with $2l^2M$, we obtain the regular black hole metric given in Ref. [15] with mass function

$$m(r) = \frac{Mr^3}{r^3 + 2l^2M} \quad . \tag{33}$$

Finally, if we choose $\mu = 3$ and let α be arbitrary, then we obtain the following mass function

$$m(r) = M\left(1 - \frac{1}{\left(1 + \left(\frac{2M}{q^2}r\right)^3\right)^{(\alpha-3)/3}}\right) \quad . \tag{34}$$

V. CONCLUSIONS

In this Letter, we have constructed a family of spherically symmetric, static, charged regular black hole metrics in the context of Einstein-nonlinear electrodynamics theory. Our analysis is based on the fact that we impose three conditions on the black hole metrics: (a) the weak energy condition should be satisfied, (b) the energy-momentum tensor should have the symmetry $T_0^0 = T_1^1$, and (c) these metrics have to asymptotically behave as the Reissner-Nordström black hole metric. Moreover, by relaxing the third requirement, we construct more general regular black hole metrics which do not behave asymptotically as a Reissner-Nordström black hole metric. In addition, we discuss as examples several special cases of the more general regular black hole metrics. These special cases have been obtained by choosing specific values for the parameters that characterize the mass function of the more general regular black hole metric. Some of these regular black hole metrics already exist in the literature but they are obtained in the context of F-P dual formalism. All the above regular black hole metrics also satisfy the DEC, although it was not imposed as a condition.

Acknowledgments

L.B. would like to thank Dirección de Investigación y Postgrado de la Universidad de La Frontera (DIUFRO) for the financial support.

Appendix A: Dual P Formalism

An equivalent method for deriving the regular black hole metrics obtained here is the F-P dual formalism. For this reason, we briefly present here the description based on the F-P dual representation of nonlinear electrodynamics obtained by a Legendre transformation [21] and reproduce the results of Sec. III.

The regular black hole metrics can be described by the metric function and its corresponding electromagnetic field which arise as a solution of Einstein field equations coupled to a nonlinear electrodynamics, that is of the action given by Eq. (6). One can also describe the considered system in terms of an auxiliary field defined by $P_{\mu\nu} = (dL/dF)F_{\mu\nu}$. The dual representation is obtained by means of a Legendre transformation

$$H = 2F\frac{dL}{dF} - L \tag{35}$$

which is a function of the invariant $P = \frac{1}{4}P_{\mu\nu}P^{\mu\nu}$. Thus, we can express the Lagrangian L depending on $P_{\mu\nu}$ as

$$L = 2P\frac{dH}{dP} - H \ , \tag{36}$$

and the electromagnetic field as

$$F_{\mu\nu} = \frac{dH}{dP}P_{\mu\nu} \ . \tag{37}$$

The energy-momentum tensor in the F-P dual representation is given by

$$T_{\mu\nu} = \frac{1}{4\pi}\frac{dH}{dP}P_{\mu\alpha}P^{\alpha}_{\nu} - \frac{1}{4\pi}g_{\mu\nu}\left(2P\frac{dH}{dP} - H\right) \ . \tag{38}$$

It follows from the components of $T_{\mu\nu}$ that $M'(r) = -r^2 H(P)$. Hence, we can obtain the corresponding mass function.

As an example, we give the function $H(P)$ for the regular black hole metrics of Sec. III

$$H(P) = \frac{P}{(1 + \Omega\,(-P)^{\alpha/4})^{4/\alpha}} \tag{39}$$

where

$$P = -\frac{q^2}{2r^4} \tag{40}$$

and Ω is defined as

$$\Omega = \left(\frac{q^{3/2}}{6M}\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{3+\alpha}{\alpha})}{\Gamma(\frac{4}{\alpha})}\right)^{\alpha} \ . \tag{41}$$

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