The Bardeen Model as a Nonlinear Magnetic Monopole

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The Bardeen model—the first regular black hole *model* in General Relativity—is reinterpreted as the gravitational field of a nonlinear magnetic monopole, i.e., as a magnetic solution to Einstein equations coupled to a nonlinear electrodynamics.

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The study on the global regularity of black hole solutions is quite important in order to understand the final state of gravitational collapse of initially regular configurations. The Penrose (weak) cosmic censorship conjecture claims that the singularities predicted by the celebrated "singularity theorems" [1,2] must be dressed by event horizons (cf. [3,4] for recent reviews). The first black hole solutions known in General Relativity were all in accordance with such point of view. However, the conjecture does not forbid the existence of regular (singularity–free) black holes. In fact, as it was pointed by Hawking and Ellis [1], and more recently by Borde [5,6], the model proposed by Bardeen [7] at the early stage of investigations on singularities, explicitly shows the impossibility of proving more general singularity theorems, if one avoids the assumption of either the strong energy condition or the existence of global hyperbolicity. The Bardeen model [7] is a regular black hole space–time satisfying the weak energy condition. This model has been recently revived by Borde [5,6], who also clarified the avoidance of singularities in this space–time, and in similar examples, as due to a topology change in the evolution of the space–like slices from open to closed [6].

Other regular black hole models were proposed later [5,8–10]. None of these "Bardeen black holes," as they have been called by Borde [6], is an exact solution to Einstein equations, thus, there is no known physical sources associated with any of them. Regular black hole solutions to Einstein equations with physically reasonable sources has been reported only recently [11–14]. The objective of this contribution is to provide the pioneering Bardeen model [7] with a physical interpretation. The Bardeen model is described by the metric

$$\boldsymbol{g} = -\left(1 - \frac{2mr^2}{(r^2+g^2)^{3/2}}\right)\boldsymbol{dt}^2 + \left(1 - \frac{2mr^2}{(r^2+g^2)^{3/2}}\right)^{-1}\boldsymbol{dr}^2 + r^2\boldsymbol{d\Omega}^2. \tag{1}$$

It can be noted that this metric asymptotically behaves as $-g_{tt} = 1 - 2m/r + 3mg^2/r^3 + O(1/r^5)$; from the $1/r$ term it follows that the parameter m is associated with the mass of the configuration (incidentally, this fact can also be verified from the explicit evaluation of the ADM mass definition), but, the following term goes as $1/r^3$, therefore this does not allow one to associate the parameter g with some kind of "Coulomb" charge as, for instance, in the Reissner–Nordström solution. This fact causes that up–to–date there is no known physical interpretation for the regularizing parameter g. The main objective of this work is to show that g is the monopole charge of a self–gravitating magnetic field described by a nonlinear electrodynamics.

The Bardeen model (1) describes a regular space–time; this can be realized from the analytical expressions of its curvature invariants

$$R = \frac{6mg^2\left(4\,g^2 - r^2\right)}{\left(r^2+g^2\right)^{7/2}}, \tag{2}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{18m^2g^4\left(8\,g^4 - 4\,g^2r^2 + 13\,r^4\right)}{\left(r^2+g^2\right)^{7}}, \tag{3}$$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{12m^2\left(8\,g^8 - 4\,g^6r^2 + 47\,g^4r^4 - 12\,g^2r^6 + 4\,r^8\right)}{\left(r^2+g^2\right)^{7}}, \tag{4}$$

which are all regular everywhere. For certain range of the parameter g the Bardeen metric describes also a black hole. Making the substitutions $y = r/|g|$, $s = |g|/2m$ we rewrite g_{tt} as

$$-\,g_{tt} = A(x,s) \equiv 1 - \frac{1}{s}\frac{x^2}{(1+x^2)^{3/2}}. \tag{5}$$

As it can be noted from the derivative of the last function, it has a single minimum at $x_{\rm m} = \sqrt{2}$, independently of the nonvanishing value of s. The equation $A(x_{\rm m}, s) = 0$ is solved by the single root $s_{\rm c} = 2/\sqrt{27}$. At $x_{\rm m}$, for $s < s_{\rm c}$ the quoted minimum of $A(x, s)$ is negative, for $s = s_{\rm c}$ the minimum vanishes and for $s > s_{\rm c}$ the minimum is positive. From the regularity of the curvature invariants (2–4), it follows that for $s \le s_{\rm c}$ the singularities appearing in (1), due to the vanishing of A, are only coordinate–singularities describing the existence of event horizons. Consequently, we are in the presence of black hole space–times for $g^2 \le 4s_{\rm c}^2 m^2 = (16/27)m^2$. For the strict inequality $g^2 < (16/27)m^2$, there are inner and event horizons for the Killing field $\boldsymbol{k} = \boldsymbol{\partial/\partial t}$, defined by $k_\mu k^\mu = g_{tt} = 0$. For the equality $g^2 = (16/27)m^2$, the horizons shrink into a single one, where also $\nabla_\nu(k_\mu k^\mu) = 0$, *i.e.*, this case corresponds to an extreme black hole as in the Reissner–Nordström solution. The extension of the Bardeen metric beyond the horizons, up to $r = 0$, becomes apparent by passing to the standard advanced and retarded Eddington–Finkelstein coordinates, in terms of which the metric is well–behaved everywhere, even in the extreme case. The maximal extension of the Bardeen metric can be achieved by following the main lines used for the Reissner–Nordström solution, taking care of course, of the more involved integration in the present case of the tortoise coordinate $r^* \equiv \int A^{-1} dr$. The global structure of the Bardeen space–time is similar to the structure of the Reissner–Nordström black hole, as it has been pointed out by previous authors [1,6], except that the usual singularity of the Reissner–Nordström solution, at $r = 0$, is smoothed out and now it simply corresponds to the origin of the spherical coordinates. This fact implies that the topology of the slices change from $S^2 \times \mathbb{R}$ outside the black hole to S^3 inside of it, as it has been pointed by Borde [6].

In what follows we will show that it is possible to provide a physical interpretation to the quoted parameter g, and to think of the Bardeen model as a regular black hole solution of General Relativity simply by coupling to the Einstein equations an appropriate nonlinear electrodynamics. In this context g is the monopole charge of a magnetic field ruled by the quoted nonlinear electrodynamics. The dynamics of the proposed theory is governed by the action

$$\mathcal{S} = \int dv \left(\frac{1}{16\pi} R - \frac{1}{4\pi} \mathcal{L}(F) \right), \tag{6}$$

where R is scalar curvature, and $\mathcal{L}$ is a function of $F \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, where $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$ is the electromagnetic strength. We would like to recall that there are more general Lagrangians depending also on the second invariant, $F^*_{\mu\nu} F^{\mu\nu}$, but for the objective of this work it is enough to consider only an action as the one given in (6). The Einstein–nonlinear–electrodynamics field equations resulting from action (6) are

$$G_\mu^{\ \nu} = 2(\mathcal{L}_F F_{\mu\lambda} F^{\nu\lambda} - \delta_\mu^{\ \nu} \mathcal{L}), \tag{7}$$

$$\nabla_\mu \left(\mathcal{L}_F F^{\alpha\mu} \right) = 0. \tag{8}$$

where, $\mathcal{L}_F \equiv \partial\mathcal{L}/\partial F$. The particular non–linear electrodynamics source used to derive the Bardeen black hole is determined by the following function $\mathcal{L}$:

$$\mathcal{L}(F) = \frac{3}{2sg^2} \left(\frac{\sqrt{2\, g^2 F}}{1 + \sqrt{2\, g^2 F}} \right)^{5/2}, \tag{9}$$

where s stands for $s \equiv |g|/2m$; g and m are free parameters which we anticipate to be associated with magnetic charge and mass respectively. To obtain a solution compatible with (1), we consider a static and spherically symmetric configuration

$$\boldsymbol{g} = -\left(1 - \frac{2M(r)}{r}\right) \boldsymbol{dt}^2 + \left(1 - \frac{2M(r)}{r}\right)^{-1} \boldsymbol{dr}^2 + r^2 \boldsymbol{d\Omega}^2, \tag{10}$$

and assume the following magnetic ansatz for the Maxwell field

$$F_{\mu\nu} = 2\delta^\theta_{[\mu} \delta^\varphi_{\nu]} B(r, \theta). \tag{11}$$

With these choices, equations (8) are easily integrated,

$$F_{\mu\nu} = 2\delta^\theta_{[\mu} \delta^\varphi_{\nu]} f(r) \sin(\theta). \tag{12}$$

Using now that

$$0 = \boldsymbol{dF} = f'(r)\sin(\theta)\boldsymbol{dr} \wedge \boldsymbol{d\theta} \wedge \boldsymbol{d\varphi}, \tag{13}$$

we conclude that $f(r) = \text{const.} = g$, where the integration constant has been chosen as g. As it was anticipated above, g is the magnetic monopole charge of the configuration:

$$\frac{1}{4\pi}\int_{S^\infty} \boldsymbol{F} = \frac{g}{4\pi}\int_0^\pi \int_0^{2\pi} \sin(\theta) d\theta d\varphi = g, \tag{14}$$

where S^∞ is a sphere at infinity.

The ${}_t{}^t$ component of Einstein equations (7) yields

$$M'(r) = r^2 \mathcal{L}(F). \tag{15}$$

Substituting $\mathcal{L}$ from (9) with $F = g^2/2r^4$, and using that $m = \lim_{r\to\infty} M(r)$, one can write the integral of (15) as

$$M(r) = m - 3mg^2 \int_r^\infty dy \frac{y^2}{(y^2 + g^2)^{5/2}}. \tag{16}$$

The last integral can be easily accomplished using hyperbolic functions, yielding

$$M(r) = \frac{mr^3}{(r^2 + g^2)^{3/2}}. \tag{17}$$

Substituting $M(r)$ into (10) one finally obtains the Bardeen metric (1).

It can be noted that the proposed source (9) satisfies the weak energy condition. Let $\boldsymbol{X}$ be a time–like field, without loss of generality it can be chosen normal ($X_\mu X^\mu = -1$). Using the right hand side of (7), one can write the local energy density along $\boldsymbol{X}$ as

$$4\pi T_{\mu\nu} X^\mu X^\nu = \mathcal{L} + E_\lambda E^\lambda \mathcal{L}_F, \tag{18}$$

where $E_\lambda \equiv F_{\lambda\mu} X^\mu$ is by definition orthogonal to $\boldsymbol{X}$, so it is an space–like vector ($E_\lambda E^\lambda > 0$). It follows from (18) that if $\mathcal{L} \geq 0$ and $\mathcal{L}_F \geq 0$ the local energy density along any time–like field is non–negative everywhere, which is the requirement of the weak energy condition. For the proposed non–linear electrodynamics the non–negativeness of these quantities follows from the definition (9), hence, the matter proposed by us as source of the Bardeen model satisfies the weak energy condition.

Summarizing, for the Bardeen metric (1) we found a field source related to a nonlinear electrodynamics given by the Lagrangian (9). The solution to the Einstein equations coupled with the energy–momentum tensor associated to the magnetic strength (12) corresponds to a self–gravitating magnetic monopole charge g (14). It is well known the Bardeen model, now climbed to the status of an exact solution of Einstein equations, fulfills the weak energy condition and is regular everywhere, although the invariant of the associated electromagnetic field exhibits the usual singular behavior $F = g^2/2r^4$ of magnetic monopoles. Finally, we would like to point out that the nonlinear electrodynamics used is stronger than the Maxwell one, in the weak field limit, as it can be seen from the expansion of the Lagrangian (9); $\mathcal{L}(F \ll 1) \sim F F^{1/4}$, while for Maxwell electrodynamics $\mathcal{L}_{\mathrm{M}}(F) = F$.

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[1] Hawking, S.W., Ellis, G.F., *The Large Scale Structure of Space–Time* (Cambridge Univ. Press 1973).
[2] Senovilla, J.M.M., *Gen. Relat. Gravit.*, **30**, 701 (1998).
[3] Wald, R.M., "Gravitational Collapse and Cosmic Censorship," `gr-qc/9710068` (1997).
[4] Jhingan, J., Magli, G., "Gravitational Collapse of Fluid Bodies and Cosmic Censorship: Analytic Insights," `gr-qc/9903103` (1999).

[5] Borde, A., *Phys. Rev.*, **D50**, 3392 (1994).
[6] Borde, A., *Phys. Rev.*, **D55**, 7615 (1997).
[7] Bardeen, J., presented at GR5, Tiflis, U.S.S.R., and published in the conference proceedings in the U.S.S.R. (1968).
[8] Barrabès, C., Frolov, V.P., *Phys. Rev.*, **D53**, 3215 (1996).
[9] Mars, M., Martín–Prats, M.M., Senovilla, J.M.M., *Class. Quant. Grav.*, **13**, L51 (1996).
[10] Cabo, A., Ayón–Beato, E., *Int. J. Mod. Phys.*, **A14**, 2013 (1999).
[11] Ayón–Beato, E., García, A., *Phys. Rev. Lett.*, **80**, 5056 (1998).
[12] Ayón–Beato, E., García, A., *Gen. Rel. Gravit.*, **31**, 629 (1999).
[13] Ayón–Beato, E., García, A., *Phys. Lett.*, **B464**, 25 (1999).
[14] Magli, G., *Rept. Math. Phys.*, **44**, 407 (1999).