I. Lorentzian geometry

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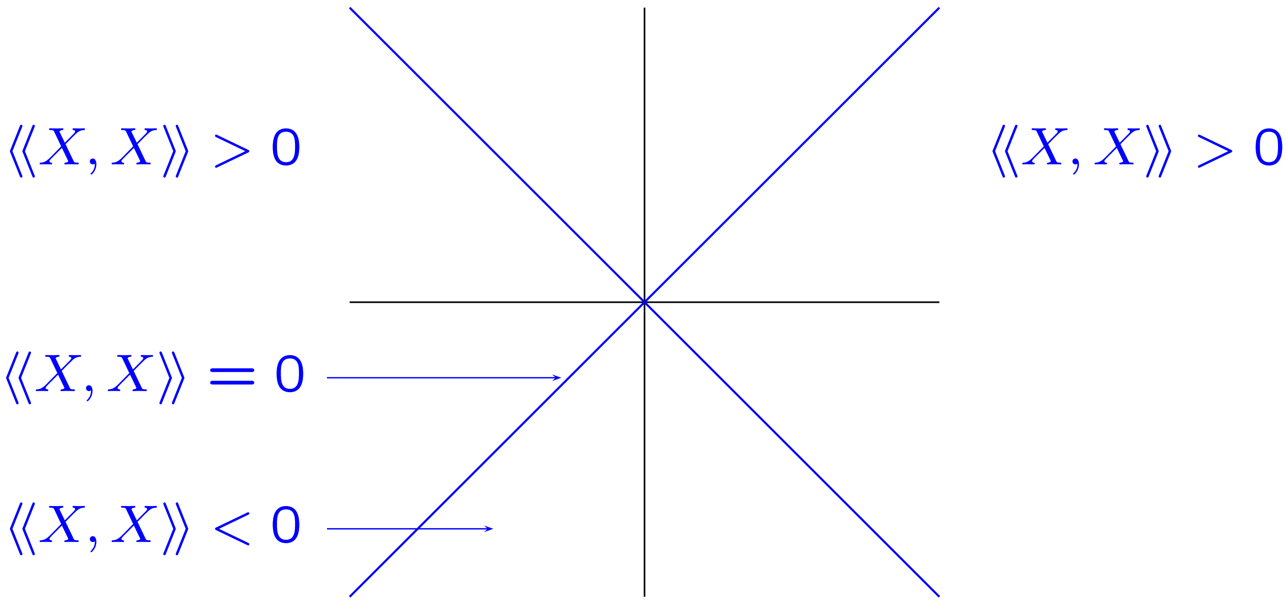
Lorentzian manifold (M, g): the metric g_x on T_xM is of signature $(- + + \ldots +)$ for every $x \in M$, i.e.: $\exists$ basis $(e_0, e_1, \ldots, e_{n-1})$ of T_xM s.t.

$$g_x = -e_0^* \otimes e_0^* + \sum_{j=1}^{n-1} e_j^* \otimes e_j^*.$$

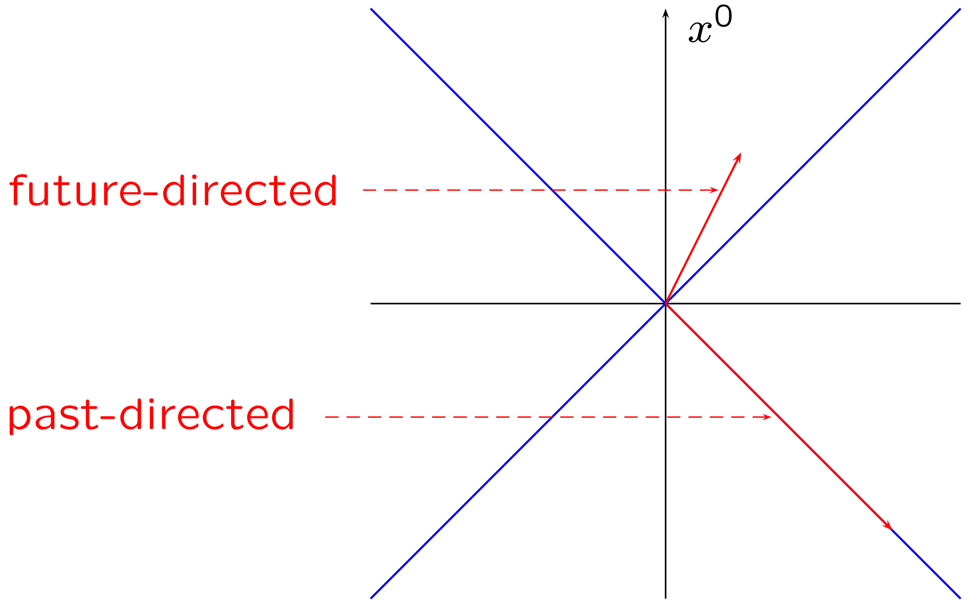
1. Causality relation

- In Minkowski space $(V, \langle\!\langle \cdot\,, \cdot \rangle\!\rangle)$: a vector X is

  - spacelike iff $\langle\!\langle X, X \rangle\!\rangle > 0$,
  - lightlike iff $\langle\!\langle X, X \rangle\!\rangle = 0$,
  - timelike iff $\langle\!\langle X, X \rangle\!\rangle < 0$,
  - causal iff $\langle\!\langle X, X \rangle\!\rangle \leq 0$.



Choice of a time-orientation = choice of one of both connected components of $\{\langle\!\langle X, X \rangle\!\rangle < 0\}$.



- <u>On a Lorentzian manifold</u>:

  - spacelike (resp. timelike,...) tangent vectors, submanifolds (e.g. curves)

  - time-orientation is given by a (C^∞) timelike vector field on M (does not always exist)

From now on all Lorentzian manifolds will be (connected and) time-oriented
("spacetimes"*)*

- Causality relation: $x, y \in M$,

$x \leq y$:⇔ ∃ future-directed causal curve from x to y

(resp. $x < y$:⇔ ∃ ...timelike...)

- Futures/Pasts: $x \in M$

  - causal future (resp. past) of x in M:

$$J_+^M(x) := \{y \in M,\ x \leq y\}$$

(resp. $J_-^M(x) := \{y \in M,\ y \leq x\}$)

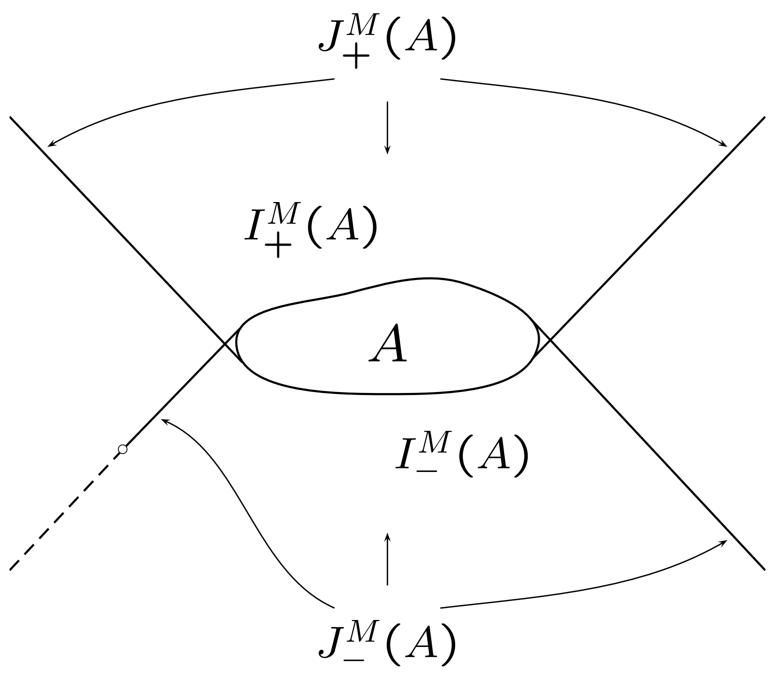
  - chronological future (resp. past) of x in M:

$$I_+^M(x) := \{y \in M,\ x < y\}$$

(resp. $I_-^M(x) := \{y \in M,\ y < x\}$)

- future (past) of a subset A of M:

$$J_{\pm}^{M}(A) := \underset{x \in A}{\cup} J_{\pm}^{M}(x)$$

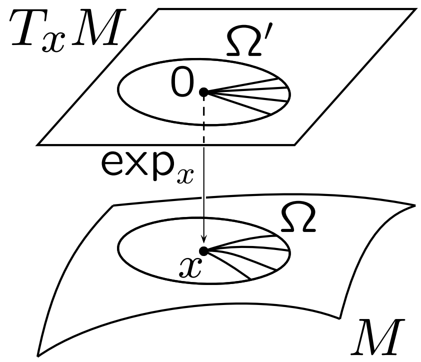


- <u>Topological properties</u>:
  - $I_{\pm}^{M}(A)$ is open $\subset J_{\pm}^{M}(A)$ which is not always closed.

    But: A compact and "$\leq$" closed $\Rightarrow J_{\pm}^{M}(A)$ closed

2. Convex subsets

• $\Omega \subset M$ is geodesically starshaped w.r.t. $x \in \Omega$ $:\Longleftrightarrow$ $\exists\, \Omega' \subset T_xM$ starshaped w.r.t. 0 s.t. $\exp_x : \Omega' \longrightarrow \Omega$ is a diffeomorphism.



• $\Omega \subset M$ is convex $:\Longleftrightarrow$ Ω is geodesically starshaped w.r.t. all of its points.

• Convexity and causality: let $\Omega \subset M$ be geodesically starshaped w.r.t. $x \in \Omega$:

- $\exp_x(J_\pm^{\Omega'}(0)) = J_\pm^{\Omega}(x)$

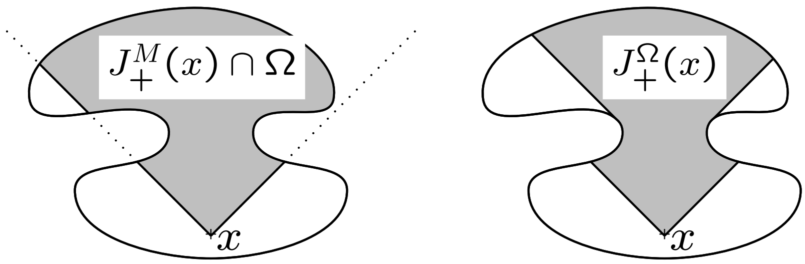
- $dV_g = \mu_x \cdot (\exp_x^{-1})^* d\mathrm{vol}_{g_x}$ ($\mu_x = \sqrt{|\det(g_{ij})|}$ in normal coordinates about x)

3. Subsets and causality

• $\Omega \subset M$ is causally compatible iff

$$J_{\pm}^{\Omega}(x) = J_{\pm}^{M}(x) \cap \Omega$$

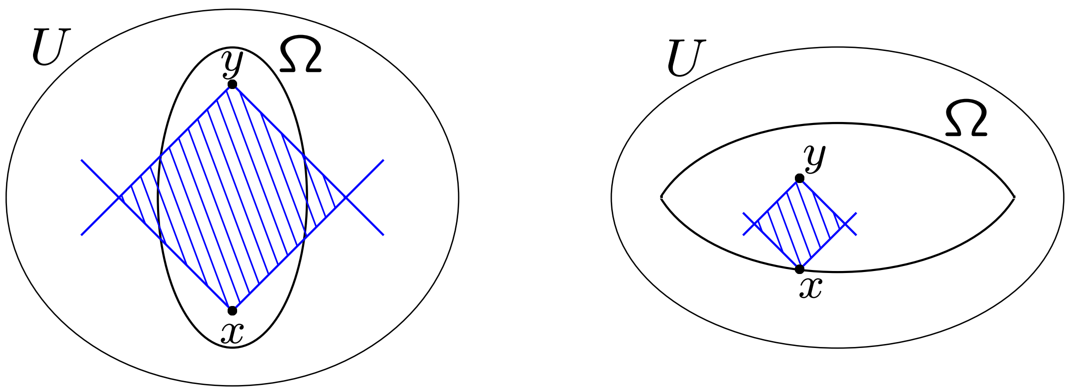
for every $x \in \Omega$.
- ex.: $\Omega \subset M$ convex
- c-ex.:



Ω is *not* causally compatible in M

• $\Omega \subset M$ is causal iff $\exists U \subset M$ convex with
  - $\overline{\Omega} \subset U$
  - $J_+^U(x) \cap J_-^U(y)$ is compact and contained in $\overline{\Omega}$ for all $x, y \in \overline{\Omega}$.

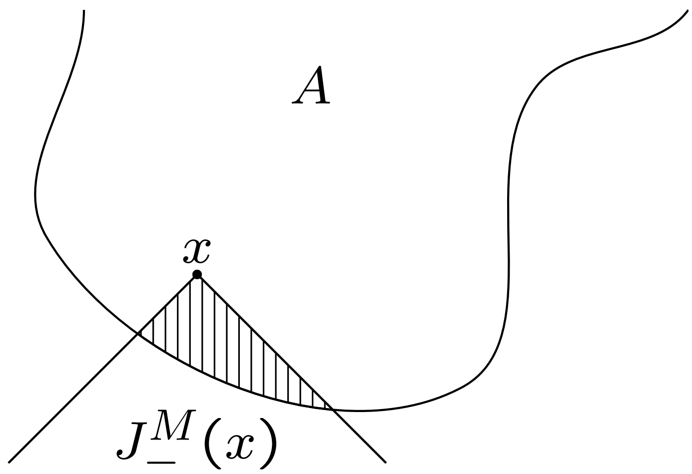
ex. and c-ex.:



convex, but not causal

causal

• $A \subset M$ is past-compact iff $J_-^M(x) \cap A$ is compact for every $x \in M$.

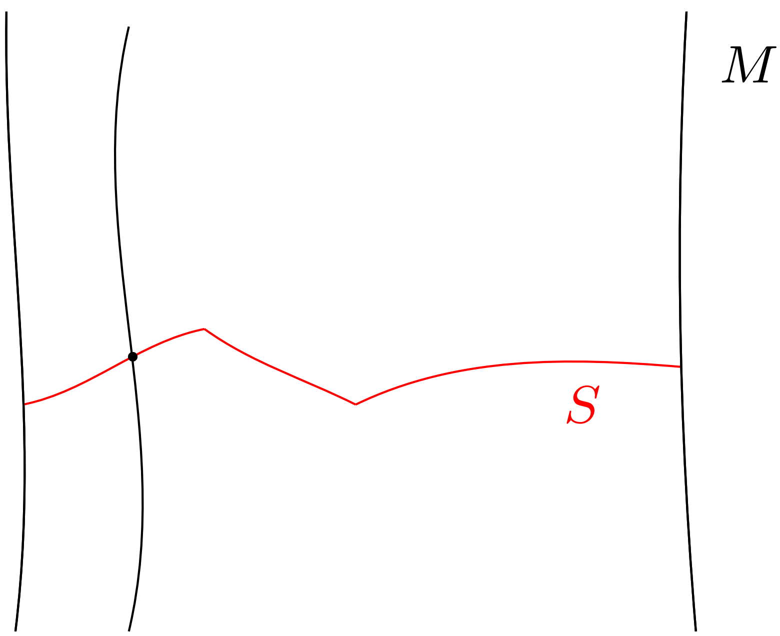


Lemma: A past-compact $\Rightarrow$ $\overline{A \cap J_-^M(K)}$ compact for every $K \subset M$ compact

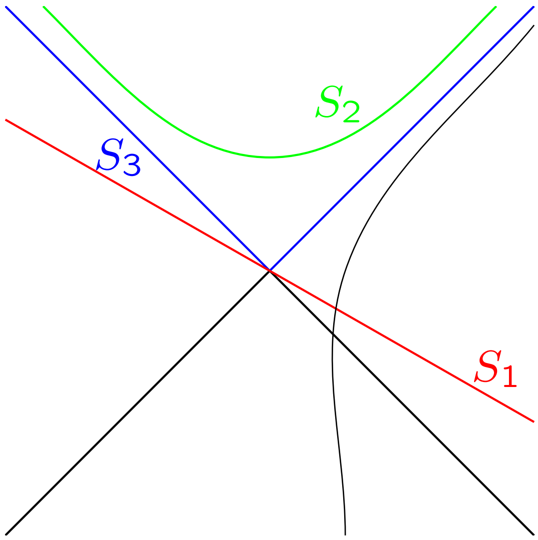
4. Cauchy hypersurfaces and globally hyperbolic manifolds

• $S \subset M$ is achronal (resp. acausal) iff every inextendible timelike (resp. causal) curve hits S *at most once*.
Note: S acausal $\Rightarrow$ S achronal ("$\Leftarrow$" true if e.g. S is a *spacelike hypersurface*)

• $S \subset M$ is a Cauchy-hypersurface iff every inextendible timelike curve hits S *exactly once*.



- Ex. and c-ex. in Minkowski space:



- <u>Properties of Cauchy-hypersurfaces</u>: let S be a Cauchy-hypersurface of M, then:

  - S is an achronal topological hypersurface of M and is hit by any inextendible causal curve,

  - any further Cauchy-hypersurface is homeomorphic to S,

  - for every $K \subset M$ compact, $J^M_{\pm}(S) \cap J^M_{\mp}(K)$ is compact.

- M is globally hyperbolic iff
  - $J_{+}^{M}(x) \cap J_{-}^{M}(y)$ is compact for all $x, y \in M$,
  - there is no almost closed causal curve in M: $\forall\, x \in M$ and $\forall$ neighbourhood V of x, $\exists$ neighbourhood $U \subset V$ of x s.t. every *causal* curve starting *and* ending in U lies in V.

- Equivalently: M is globally hyperbolic iff one of the following holds:

  i) M admits a Cauchy-hypersurface,

  ii) $(M, g) \overset{\text{isom.}}{\simeq} (\mathbb{R} \times S, -\beta dt^2 + g_t)$ where β is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth space-like Cauchy hypersurface in M,

  iii) there exists a time-function on M.

• <u>Important property</u>: on any globally hyperbolic manifold M the causality relation $\leq$ is closed (in particular $J_+^M(K) \cap J_-^M(K')$ is compact for all $K, K' \subset M$ compact).

• <u>Examples</u>:

- the Minkowski space is globally hyperbolic,

- every causal domain is globally hyperbolic,

- the *anti-deSitter space* is *not* globally hyperbolic.

5. Normally hyperbolic operators

Let E be a complex vector bundle over M.

• A normally hyperbolic operator is a second-order differential operator P on E of which symbol is given by minus the metric.

In local coordinates:

$$P = -\sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x)\frac{\partial}{\partial x^j} + B_1(x)$$

where A_j and B_1 are matrix-valued coefficients depending smoothly on x.

• Example: the d'Alembert operator is defined on smooth functions by

$$\Box f := -\mathrm{tr}(\mathrm{Hess}(f)),$$

where $\mathrm{Hess}(f)(X,Y) := \langle \nabla_X \mathrm{grad} f, Y\rangle$.
In normal coordinates:

$$\Box f = -\mu_x^{-1} \sum_{i=1}^{n} \frac{\partial}{\partial x^i}(\mu_x(\mathrm{grad} f)_i)$$

• *Lemma*: for every normally hyperbolic operator P on E there exists a unique connection ∇ (called the P-compatible one) on E and a unique endomorphism field B of E such that

$$P = \Box^\nabla + B,$$

where

$$\begin{aligned} \Box^\nabla &:= -\mathrm{tr}(\nabla^2) \\ &= -\sum_{i,j=1}^{n} g^{ij}\left(\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}}\right). \end{aligned}$$

• <u>Aim in the following</u>: solve, for some given $f \in C^\infty(M,E)$,

$$Pu = f$$

with "adapted" initial conditions.