Universidad de los Andes

Department of Physics

# REGULAR BLACK HOLES AND NONLINEAR ELECTRODYNAMICS

*Author:*

Paulina Hoyos Restrepo

*Advisor:*

Pedro Bargueño de Retes, PhD

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## Abstract

In 1998 Eloy Ayón-Beato and Alberto García proposed a regular black hole solution that has generated a lot of discussion until the present day. It was obtained in the scope of general relativity coupled to nonlinear electrodynamics. The purpose of this work is to study in detail such solution and explore its physical characteristics. To do so the Einstein's equations, the energy-momentum tensor for nonlinear electrodynamics, the generalized Maxwell's equations and electric field are derived starting from the least action principle. Also, the FP dual formalism is briefly presented. Then, the main features of the Reissner-Nordströn black hole are covered. Finally, the Ayón-Beato and García regular black hole is analysed; starting from the metric, a calculation of the curvature invariants, the electric field and the structural function is done. In particular, it is shown that the metric has a de Sitter behaviour near the origin of coordinates, which is consistent with the increasing interval of the electric field from $r = 0$ to a certain $r^*$. Therefore, a physical interpretation of the structural function can be given in terms of a cut-off in the electric field that results to be Born-Infeld-like.

## Resumen

En 1998 Eloy Ayón Beato y Alberto García propusieron una solución de agujero negro regular que ha generado múltiples discusiones hasta el día de hoy. Ésta se obtuvo en el marco de la relatividad general acoplada a la electrodinámica no lineal. El propósito de esta trabajo es estudiar en detalle dicha solución y explorar sus características físicas. Para lograr ésto, las ecuaciones de Einstein, el tensor de energía-momento de la electrodinámica no lineal, las ecuaciones de Maxwell generalizadas y el campo eléctrico son derivados a partir del principio de mínima acción. También se presenta el formalismo FP dual. Luego se cubren las características principales del agujero negro de Reissner-Nordström. Finalmente, se analiza el agujero negro regular de Ayón-Beato y García; a partir de la métrica se calculan los invariantes de curvatura, el campo eleéctrico y la función estructural. En particular, se muestra que la métrica tiene un comportamiento tipo de Sitter cerca del origen de coordenadas, lo cual es consistente con el intervalo creciente del campo eléctrico desde $r = 0$ hasta algún $r^*$. Por lo tanto, se puede dar una interpretación física a la función estructural en términos de un máximo en el campo eléctrico que resulta ser de tipo Born-Infeld.

# 1 Introduction

This work was initially motivated by the well-known problem of quantum gravity in theoretical physics. That is, the construction of a consistent theory that describes the gravitational interaction from the principles of quantum physics. There are various reasons that indicate the need of this theory; like divergences in Quantum Field Theory, entropy and evolution of black holes, and singularities in general relativity. A detailed survey about this topic can be found in [1].

In particular, the existence of singularities in spacetime indicate that the theory of general relativity is no longer valid from a particular length scale. Singularity theorems [2] state that under certain general conditions the solutions to Einstein's equations must have a singularity. The hypotheses are three, namely: i) a condition on the energy-matter content of spacetime, ii) a condition about the global and causal structure of spacetime, iii) a condition about the existence of a surface that can trap light. These singularities must be covered by an event horizon, so there is no causal connection between the singularity and the region outside the black hole; as it is asserted by the Penrose weak cosmic censorship conjecture [3].

People have tried to avoid singularities within the framework of general relativity, which requires the violation of at least one of the hypotheses already mentioned. In particular, it is interesting to consider black hole solutions without curvature singularities (they are known as regular black holes [4]); in this case, the natural thing is to relax the conditions about the energy-matter content of the fields that are coupled to gravity. Those fields can be of any type, but this monograph only takes into account the electromagnetic field. Thus, some kind of nonlinear electrodynamics [5] must be considered in order to eliminate the spacetime singularity but keep an event horizon.

With this in mind, the first model of a regular black hole was proposed by Bardeen at the end of the 1960 decade [6]; nevertheless, one of the parameters of this solution had no physical interpretation. Years later, Ayón-Beato and García presented a series of regular black holes with mass $m$ and electric charge $q$ coming from a nonlinear electromagnetic lagrangian [7, 8, 9, 10]. The global structure of these black holes is similar to the Reissner-Nordström solution, except for the singularity at the origin of coordinates; the authors state that the metrics tend asymptotically to the Reissner-Nordströn one and the electric fields tend asymptotically to the Coulomb field with electric charge $q$.

Nevertheless, K.A. Bronnikov published a comment [11] on the first paper by Ayón-Beato and García, where he asserts that all the features of the regular black hole presented by the authors lead to a contradiction. This statement is proved in a no-go theorem which shows that a static, spherically symmetric solution of general relativity coupled to nonlinear electrodynamics with an electric charge $q$ and a Maxwell asymptotic Lagrangian $L(F)$ cannot lead to solutions with a regular center.

Thus, we see that coupling nonlinear electrodynamics and gravity leads to interesting results and discussion around it. On the other side, it is worth noting that some models of nonlinear electrodynamics appear as effective theories when quantum effects are taken into account, for example in string theory [12, 13] or QED [14]. The better known example is the Born-Infeld Lagrangian [15] which can be understood as the weak field limit of the Euler-Heisenberg Lagrangian, a one loop calculation of vacuum polarization. Therefore, regarding the avoidance of spacetime singularities, the latter suggests a complementary point of view between the nonlinear electrodynamics approach and the quantum gravity one.

Given all the interesting features described above, this works aims to study some regular black hole solutions of the Einstein-Maxwell equations considering nonlinear models for electrodynamics. Section 2 presents the covariant formulation of general relativity and electrodynamics starting from the least action principle, along with the derivation of some useful relations. After that, the Reissner-Nördstrom black hole is briefly studied in section 3 and we continue with an analysis of the regular black hole solution by Ayon-Beato and García in section 4. Finally, section 5 presents some concluding remarks.

We conclude this introduction by pointing out that throughout this document we adopt the signature $(-,+,+,+)$. Also, all the calculations presented here were done with Mathematica 10.

## 2 Derivation of the basic equations

Consider the action of general relativity coupled to nonlinear electrodynamics

$$\tilde{S} = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi}R - \frac{1}{4\pi}L(F,G^2)\right)\,, \tag{1}$$

where the first term is the gravitational (and thus geometrical) part of the action, and it is known as the Einstein-Hilbert action. Here, $g = det(g_{\mu\nu})$ is the natural volume element for spacetime and $R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar. This is the simplest possible choice for a gravitational Lagrangian, since the Ricci scalar is the only independent geometric invariant constructed from the metric which is no higher than second order in its derivatives.

The second term is the action for matter which, in this case, corresponds to nonlinear electrodynamics. The electromagnetic Lagrangian $L(F,G^2)$ is any function of the only two gauge invariant Lorentz scalars:

$$F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{2}$$

$$G^2 = \left(\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}\right)^2 = (\vec{E}\cdot\vec{B})^2\,, \tag{3}$$

where $F^{\mu\nu}$ is the usual field strength tensor and ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual field strength tensor, or Hodge dual. Note that $G$ is a pseudoscalar because it changes its sign under a parity transformation; this explains why $L$ depends on $G^2$ and not only on $G$. A physical requirement for $L(F, G^2)$ is that, in the limit of small field strengths, it must approach the Maxwell Lagrangian $L_M(F) = -F$.

For instance, a widely known nonlinear electrodynamics Lagrangian is the Born-Infeld one [15], given by the function

$$L_{BI} = b^2 \left( \sqrt{1 + \frac{2F}{b^2} - \frac{G^2}{b^4}} - 1 \right) , \tag{4}$$

where $b$ is the maximum electromagnetic field strength. This Lagrangian was proposed with the aim of avoiding infinities in classical electrodynamics, such as the self energy of the electron when it is modeled as a point charge. This theory describes a stable electron with a finite size, a finite self energy and a finite electric field at the origin of coordinates. It is very interesting to note that this Lagrangian was proposed in analogy to the relativistic Lagrangian of a free particle $mc^2(1 - \sqrt{1 - v^2/c^2})$, where the speed of light $c$ is the maximum velocity, which reduces to the Newtonian Lagrangian $\frac{1}{2}mv^2$ for velocities much smaller than the speed of light. This last feature is also true for the Born-Infeld Lagrangian, as it reduces to the Maxwell Lagrangian for fields much weaker than $b$ or, equivalently, in the limit $b \to \infty$. This can be seen with an expansion of the mentioned Lagrangian

$$L_{BI} \xrightarrow[b \to \infty]{} F - \frac{F^2 + G^2}{2b^2} + O\left(\frac{1}{b}\right)^4 , \tag{5}$$

where the following terms can be identified with the weak field limit of the Euler-Heisenberg Lagrangian, a one loop calculation of vacuum polarization. This QED process occurs because of the Heisenberg's uncertainty principle which allows a classical electromagnetic field to create a virtual electron-positron pair from the Dirac field, which in turn changes the Maxwell's equations [14]. The latter is an explicit example of how quantum effects can introduce nonlinear terms in classical theories.

Now we turn the attention back to the action. For simplicity, the magnetic components of the field strength tensor are not considered in this work, therefore $G^2$ is always equal to zero. Thus the action is reduced to

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi} R - \frac{1}{4\pi} L(F) \right) . \tag{6}$$

The well known Einstein's equations come from varying the action (6) with respect to the inverse metric $g^{\mu\nu}$ [16]. First, for the gravitational part of the equations we have

$$\frac{1}{\sqrt{-g}} \frac{\delta S_{EH}}{\delta g^{\mu\nu}} = G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 . \tag{7}$$

To see why this is so, consider a small variation of the Einstein-Hilbert action

$$\delta S_{EH} = \int d^4x \left( \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + R \delta \sqrt{-g} \right) , \tag{8}$$

where the first term is already in the desired form (some expression times $\delta g^{\mu\nu}$); and the other two terms need to be manipulated to write them in this way.

For the second term consider a small variation of the connection $\delta\Gamma^{\lambda}_{\mu\nu}$. This quantity is a tensor since it is the difference of two connections. Then we can take its covariant derivative

$$\nabla_\rho(\delta\Gamma^{\lambda}_{\mu\nu}) = \partial_\rho\delta\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\rho\alpha}\delta\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\rho\mu}\delta\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\alpha}_{\rho\nu}\delta\Gamma^{\lambda}_{\mu\alpha}\,. \tag{9}$$

Now, recall the definition of the Riemann tensor in terms of the connection

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_\nu\Gamma^{\rho}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}\,, \tag{10}$$

so that a small variation of this tensor is of the form

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda\delta\Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \partial_\nu\delta\Gamma^{\rho}_{\lambda\mu} - \delta\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu}\,. \tag{11}$$

Rearranging these terms and using (9) we get

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda\left(\delta\Gamma^{\rho}_{\nu\mu}\right) + \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \nabla_\nu\left(\delta\Gamma^{\rho}_{\lambda\mu}\right) - \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu}\,. \tag{12}$$

Also, for a torsion free spacetime (the one considered here) the connection is symmetric in the two lower indices, so (12) is simplified to

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda\left(\delta\Gamma^{\rho}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\rho}_{\lambda\mu}\right)\,. \tag{13}$$

Finally, by contracting the indices $\rho$ and $\lambda$ in the last expression we get the variation of the Ricci tensor

$$\delta R_{\mu\nu} = \nabla_\lambda\left(\delta\Gamma^{\lambda}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\lambda}_{\lambda\mu}\right)\,. \tag{14}$$

For the third term consider the following identity, valid for any matrix M

$$\mathrm{Tr}(\ln M) = \ln(\det M)\,. \tag{15}$$

The variation of this identity yields

$$\mathrm{Tr}(M^{-1}\delta M) = \frac{1}{\det M}\delta(\det M) \Leftrightarrow \delta(\det M) = \det M \times \mathrm{Tr}(M^{-1}\delta M)\,. \tag{16}$$

Now, we apply the last expression to $M = g^{\mu\nu}$ and $\det M = \det g^{\mu\nu} = g^{-1}$, so that

$$\delta(g^{-1}) = g^{-1}g_{\mu\nu}\delta g^{\mu\nu}\,; \tag{17}$$

and using the chain rule we finally get the variation of the natural volume element for spacetime, namely

$$\delta\sqrt{-g} = \frac{\partial\sqrt{-g}}{\partial g^{-1}}\delta(g^{-1}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{18}$$

Harkening back to the variation of the Einstein-Hilbert action (8) and replacing the variations given in (14) and (18) we get

$$\delta S_{EH} = \int d^4x \ \left(\sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu} \left[\nabla_\lambda \left(\delta \Gamma^\lambda_{\nu\mu}\right) - \nabla_\nu \left(\delta \Gamma^\lambda_{\lambda\mu}\right)\right] \right.$$
$$\left. - R\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}\right), \quad (19)$$

because of the metric compatibility this is equivalent to

$$\delta S_{EH} = \int d^4x \ \left(\sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} \nabla_\sigma \left[g^{\mu\sigma} \left(\delta \Gamma^\lambda_{\nu\mu}\right) - g^{\mu\nu} \left(\delta \Gamma^\lambda_{\lambda\mu}\right)\right] \right.$$
$$\left. - R\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}\right). \quad (20)$$

Note that the second term corresponds to the integral over the natural volume element of a total covariant derivative, so by the generalized Stoke's theorem in 4 dimensions that is equal to a boundary integral at infinity. In this work the thermodynamics of black holes will not be considered, so we can make the variation vanish at infinity and set this second term to zero. With this in mind $\delta S_{EH}$ becomes

$$\delta S_{EH} = \int d^4x \sqrt{-g} \ (R_{\mu\nu} - R\frac{1}{2} g_{\mu\nu}) \delta g^{\mu\nu} , \quad (21)$$

and now the variational principle can be applied to get the Einstein's equations in vacuum

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 , \quad (22)$$

where $G_{\mu\nu}$ is called the Einstein tensor. This tensor is always convariantly conserved, i.e.

$$\nabla^\mu G_{\mu\nu} = 0 , \quad (23)$$

due to the Bianchi identity. This is a very important property, as we will see later on.

Having derived the left-hand-side of the equations, we now focus on the action for matter $S_M$ in order to get the missing right-hand-side. From the variation of this action with respect to the inverse metric one can define the following

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} , \quad (24)$$

called the energy-momentum tensor or stress-energy tensor. It contains all the information regarding the energy-like aspects of a physical system, such as energy density, pressure and stress. A macroscopic definition of $T_{\mu\nu}$ is "the flux of $\mu$ momentum ($p_\mu$) across a surface of constant $x_\nu$" [17]. More specifically, its different components contain the following information

$\mathbf{T_{00}} \equiv \rho =$ energy density.

$\mathbf{T_{0i}}$ = energy flux across the $x_i$ surface. A heat-conduction term in a momentarily comoving reference frame (MCRF).

$\mathbf{T_{i0}}$ = i momentum density.

$\mathbf{T_{ij}}$ = flux of the i momentum across the $x_j$ surface. An stress-related term in a MCRF.

From the definition given in (24) it follows immediately that the energy-momentum tensor is gauge invariant, covariantly conserved ($\nabla^\mu T_{\mu\nu} = 0$) and symmetric ($T_{\mu\nu} = T_{\nu\mu}$). In this context, the first property means invariance under general coordinate transformations. The second property is evident from the Einstein's equations and the Bianchi identity (23), whereas the third one is a consequence of the symmetry of the inverse metric. All these are desired and nice features on $T_{\mu\nu}$.

At this point we already have both sides of Einstein's equations. In short, the action $S = S_{EH} + S_M$ is varied with respect to the inverse metric $g^{\mu\nu}$, so from (21) and (24) we get

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \frac{1}{2}T_{\mu\nu} = 0\,, \tag{25}$$

and by organizing these terms we are led to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}\,. \tag{26}$$

These equations describe how energy and momentum affect the geometry of spacetime and create a curvature on it. The metric $g_{\mu\nu}$ encodes the spacetime geometry, as all curvature tensors and invariants are defined in terms of the connection coefficients, which in turn are defined in terms of the components of the metric and its derivatives. Therefore, once the theory for matter is specified via $T_{\mu\nu}$ one is left with a set of second order differential equations for the metric. The Bianchi identity and the symmetry of $g_{\mu\nu}$ put some constraints, so that there are six independent equations in (26) that must be solved.

One can also think of this problem the other way around. That is, impose any metric for spacetime and then find the energy-momentum tensor related to it. This can be done by calculating $G_{\mu\nu}$ and setting it equal to $T_{\mu\nu}$. The important thing is to know if the resulting stress-energy tensor is physically reasonable. In this sense, there are some inequalities called energy conditions [18] that give a criteria about this, namely

**Null** energy condition (NEC): $T_{\mu\nu}k^\mu k^\nu \geq 0$ for all null vectors $k^\mu$.

**Weak** energy condition (WEC): $T_{\mu\nu}t^\mu t^\nu \geq 0$ for all timelike vectors $t^\mu$. This means that the local energy density must be non-negative for all observers.

**Dominant** energy condition (DEC): $T_{\mu\nu}t^\mu t^\nu \geq 0$ and $g^{\nu\sigma}\left(T_{\mu\nu}t^\mu\right)\left(T_{\rho\sigma}t^\rho\right) \leq 0$ for all timelike vectors $t^\mu$. This states that the local energy density must be non-negative and that the speed of the energy density flow cannot exceed the speed of light for all observers.

**Strong** energy condition (SEC): $T_{\mu\nu}t^\mu t^\nu - \frac{1}{2}T^\mu{}_\mu t^\nu t_\nu \geq 0$ for all timelike vectors $t^\mu$. Via the congruence of geodesics in the Raychaudhuri equation this condition implies that the gravitational force is atractive.

All these seem like physically reasonable requirements. However, there are classical and quantum field theories that violate some of these conditions.

So far all we have said about the energy-momentum tensor is valid for any theory of matter. For the specific case of nonlinear electrodynamics, $T_{\mu\nu}$ can be calculated explicitly. Replacing the matter action from (6) into (24) we get

$$\begin{aligned} T_{\mu\nu} &= \frac{-2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(-\sqrt{-g}\frac{1}{4\pi}L(F)\right) = \frac{1}{2\pi\sqrt{-g}}\left(\sqrt{-g}\frac{\delta L(F)}{\delta g^{\mu\nu}} + L(F)\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right) \\ &= \frac{1}{2\pi}\left(\frac{\delta L(F)}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-g}}L(F)\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right), \end{aligned} \tag{27}$$

where the variation of $\sqrt{-g}$ was given in (18) and for $L(F)$ we can apply the chain rule, so

$$T_{\mu\nu} = \frac{1}{2\pi}\left(\frac{dL}{dF}\frac{\delta F}{\delta g^{\mu\nu}} - \frac{1}{\sqrt{-g}}L(F)\frac{\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}}{2\delta g^{\mu\nu}}\right) = \frac{1}{2\pi}\left(\frac{dL}{dF}\frac{\delta F}{\delta g^{\mu\nu}} - \frac{1}{2}L(F)g_{\mu\nu}\right). \tag{28}$$

Let us examine the variation of $F$

$$\begin{aligned} \delta F &= \delta\left(-\frac{1}{4}F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}\right) = -\frac{1}{4}F_{\mu\nu}F_{\alpha\beta}(g^{\nu\beta}\delta g^{\mu\alpha} + g^{\mu\alpha}\delta g^{\nu\beta}) \\ &= -\frac{1}{4}F_{\mu\nu}F_{\alpha\beta}\left(g^{\nu\beta}\delta^{\alpha}_{\nu}\delta g^{\mu\nu} + g^{\mu\alpha}\delta^{\beta}_{\mu}\delta g^{\nu\mu}\right) = -\frac{1}{4}\left(F_{\mu}{}^{\beta}F_{\nu\beta} + F^{\alpha}{}_{\nu}F_{\alpha\mu}\right)\delta g^{\mu\nu} \\ &= -\frac{1}{2}F_{\mu}{}^{\beta}F_{\nu\beta}\delta g^{\mu\nu}. \end{aligned} \tag{29}$$

Replacing this into (28) we obtain the general expression for the energy-momentum tensor of any nonlinear electrodynamics theory as specified in (6), i.e.

$$T_{\mu\nu} = -\frac{1}{4\pi}\left(\frac{dL}{dF}F_{\mu}{}^{\beta}F_{\nu\beta} + L(F)g_{\mu\nu}\right). \tag{30}$$

With (30) at hand, now we would like to know the equations of motion for nonlinear electrodynamics. These follow immediately from the conservation property of $T_{\mu\nu}$, that is,

$$\begin{aligned} 0 = \nabla^{\mu}T_{\mu\nu} &= -\frac{1}{4\pi}\nabla^{\mu}\left(\frac{dL}{dF}F_{\nu}{}^{\beta}F_{\mu\beta} + L(F)g_{\mu\nu}\right) \\ &= -\frac{1}{4\pi}\left[\left(\nabla^{\mu}F_{\nu}{}^{\beta}\right)\frac{dL}{dF}F_{\mu\beta} + F_{\nu}{}^{\beta}\nabla^{\mu}\left(\frac{dL}{dF}F_{\mu\beta}\right) + \frac{dL}{dF}g_{\mu\nu}\nabla^{\mu}\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right)\right] \\ &= -\frac{1}{4\pi}\left[-\frac{dL}{dF}\left(\nabla^{\mu}F_{\nu}{}^{\beta}\right)F_{\beta\mu} + F_{\nu}{}^{\beta}\nabla^{\mu}\left(\frac{dL}{dF}F_{\mu\beta}\right) - \frac{1}{2}\frac{dL}{dF}F_{\alpha\beta}\nabla_{\nu}F^{\alpha\beta}\right]. \end{aligned} \tag{31}$$

As it is shown in [19], the first and last terms above cancel each other if $F$ is derivable from a potential $A$. Then we are left with

$$0 = -4\pi \nabla^\mu T_{\mu\nu} = F_\nu{}^\beta \nabla^\mu \left( \frac{dL}{dF} F_{\mu\beta} \right) = 0 \tag{32}$$

which can be considered the electromagnetic field equations in the most general case, since $F_\nu{}^\beta$ might not be invertible. Now, if one considers only a radial electric field as the source, i.e.

$$F_{\mu\nu} = E(r) \left( \delta^1_\mu \delta^0_\nu - \delta^1_\nu \delta^0_\mu \right) , \tag{33}$$

then (32) can be simplified to

$$\nabla^\mu \left( \frac{dL}{dF} F_{\mu\beta} \right) = 0 \, . \tag{34}$$

Furthermore, from (33) and (34) one can obtain an explicit expression for the electric field. Here we merely present the result, in Gaussian units that is

$$E(r) = -\frac{q}{r^2} \left( \frac{dL}{dF} \right)^{-1} , \tag{35}$$

where q is a constant without any physical interpretation in general.

The symmetries of the field strength tensor in (33) are only compatible with a static, spherically symmetric geometry on spacetime. The most general asymptotically flat metric for such spacetime is

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{36}$$

where $f(r)$ locally determines the geometry. Also, the curvature invariants and the components of the Einstein tensor are given by

$$\begin{aligned}
R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} &= \frac{4\left[(-1+f(r))^2 + r^2 f'(r)^2\right]}{r^4} + f''(r)^2 \\
R^{\mu\nu} R_{\mu\nu} &= \frac{4 + 4f(r)[-2 + f(r) + 2rf'(r)] + 4rf'(r)[-2 + 2rf'(r) + r^2 f''(r)]}{2r^4} \\
&\quad + \frac{r^4 f''(r)^2}{2r^4} \\
R = g^{\mu\nu} R_{\mu\nu} &= \frac{-2[-1 + f(r) + 2rf'(r)]}{r^2} - f''(r)
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
G_{00} &= -\frac{f(r)\left[-1 + f(r) + rf'(r)\right]}{r^2} \\
G_{11} &= \frac{-1 + f(r) + rf'(r)}{r^2 f(r)} \\
G_{22} &= \frac{1}{2} r \left[2f'(r) + rf''(r)\right] \\
G_{33} &= \frac{1}{2} r \sin^2\theta \left[2f'(r) + rf''(r)\right] = \sin^2\theta G_{22} \, ,
\end{aligned} \tag{38}$$

respectively.

We want to relate the above geometry with the electric field. To do so, consider the components of the energy-momentum tensor for a radial electric field in a static spherically symmetric spacetime. From (30), (33) and (36) we get

$$
\begin{aligned}
T_{00} &= -\frac{1}{4\pi}\left[\frac{dL}{dF}g^{11}F_{01}F_{01} + L(F)g_{00}\right] = -\frac{1}{4\pi}f(r)\left[\frac{dL}{dF}E(r)^2 - L(F)\right] \\
T_{11} &= -\frac{1}{4\pi}\left[\frac{dL}{dF}g^{00}F_{10}F_{10} + L(F)g_{11}\right] = \frac{1}{4\pi}f(r)^{-1}\left[\frac{dL}{dF}E(r)^2 - L(F)\right] \\
T_{22} &= -\frac{1}{4\pi}L(F)g_{22} = -\frac{1}{4\pi}r^2 L(F) \\
T_{33} &= -\frac{1}{4\pi}L(F)g_{33} = -\frac{1}{4\pi}r^2\sin^2\theta L(F) = \sin^2\theta T_{22}\,.
\end{aligned}
\tag{39}
$$

Then, take the following combination

$$
\begin{aligned}
\frac{r^2}{f(r)}T_{00} + T_{22} &= -\frac{1}{4\pi}r^2\left[\frac{dL}{dF}E(r)^2 - L(F)\right] - \frac{1}{4\pi}r^2 L(F) = -\frac{1}{4\pi}r^2\frac{dL}{dF}E(r)^2 \\
&= \frac{1}{4\pi}r^2\frac{q}{r^2E(r)}E(r)^2 = \frac{q}{4\pi}E(r)\,,
\end{aligned}
\tag{40}
$$

where we have replaced (35) in the second line above. Rearranging (40) and applying the Einstein's equations we are led to

$$
E(r) = \frac{4\pi}{q}\left(\frac{r^2}{f(r)}\frac{G_{00}}{8\pi} + \frac{G_{22}}{8\pi}\right) = \frac{1}{4q}\left(2 - 2f(r) + r^2 f''(r)\right)\,. \tag{41}
$$

On the other hand, a direct calculation gives

$$
4R^{\mu\nu}R_{\mu\nu} - R^2 = \frac{\left(2 - 2f(r) + r^2 f''(r)\right)^2}{r^4}\,. \tag{42}
$$

Thus, from (41) and (42) we finally conclude that

$$
E(r) = \frac{r^2}{4q}\sqrt{4R^{\mu\nu}R_{\mu\nu} - R^2}\,. \tag{43}
$$

This relation will be very useful in section 4, because it allows one to directly calculate the electric field from the metric proposed by Ayón-Beato and García in [10]. Note that (43) can also be derived from the following identity

$$
4R^{\mu\nu}R_{\mu\nu} - R^2 = (16\pi)^2\left(T^{\mu\nu}T_{\mu\nu} - \frac{T^2}{4}\right)\,, \tag{44}
$$

which can be directly deduced from Einstein's equations.

We conclude this section with a brief description of the FP dual formalism, an alternative formulation of nonlinear electrodynamics [20]. First, an auxiliary field and its invariant are defined as

$$P_{\mu\nu} = \frac{dL}{dF} F_{\mu\nu} \qquad P = -\frac{1}{4} P_{\mu\nu} P^{\mu\nu} = \left(\frac{dL}{dF}\right)^2 F\,; \tag{45}$$

Then, a canonical description of the system can be obtained by means of a Legendre transformation

$$H = 2F\frac{dL}{dF} - L(F)\,, \tag{46}$$

where $H = H(P)$ is called the structural function. As it was required for $L(F)$, in the limit of small field strengths $H(P)$ must approach the linear theory $H_M = -P$. Also, the electromagnetic tensor and the lagrangian can be obtained back with an inverse Legendre transformation

$$F_{\mu\nu} = \frac{dH}{dP} P_{\mu\nu} \qquad L = 2P\frac{dH}{dP} - H(P)\,. \tag{47}$$

Thus, from (30), (45) and (47) the energy-momentum tensor reads

$$T_{\mu\nu} = -\frac{1}{4\pi}\left[\frac{dH}{dP} P_\mu{}^\beta P_{\nu\beta} + \left(2P\frac{dH}{dP} - H(P)\right) g_{\mu\nu}\right]\,. \tag{48}$$

Once again, we want to relate the geometry of an static, spherically symmetric spacetime with the canonical description of the source. To do so, without loss of generality we can write the $f(r)$ function in (36) in analogy to the Schwarzchild metric, that is

$$f(r) = 1 - \frac{2M(r)}{r}\,, \tag{49}$$

where $M(r)$ is called the mass function. Therefore the $\binom{0}{0}$ component of the Einstein tensor is

$$G^0{}_0 = g^{00} G_{00} = -\frac{2M'(r)}{r^2}\,, \tag{50}$$

which follows from a direct calculation after replacing (49) into (38). Now let us examine the matter content. From (33) and (45) we get

$$P = \left(\frac{dL}{dF}\right)^2 F = -\left(\frac{dL}{dF}\right)^2 \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\left(\frac{dL}{dF}\right)^2 \frac{1}{2} F^{01} F_{01} = -\frac{1}{2} P^{01} P_{01}\,, \tag{51}$$

so that the $\binom{0}{0}$ component of the energy-momentum tensor is

$$T^0{}_0 = g^{00} T_{00} = -\frac{1}{4\pi}\left[\frac{dH}{dP} P^{01} P_{01} + \left(2P\frac{dH}{dP} - H(P)\right)\right] = \frac{1}{4\pi} H(P)\,. \tag{52}$$

Finally, the Einstein's equations, (50) and (52) yield to

$$M'(r) = -r^2 H(P)\,, \tag{53}$$

where $r$ and $P$ are connected by

$$P = \left(\frac{dL}{dF}\right)^2 F = \left(\frac{dL}{dF}\right)^2 \frac{1}{2} E(r)^2 = \frac{q^2}{2r^4}\,. \tag{54}$$

The equation (53) is an exceptional one, because it allows one to easily obtain the canonical description of the system from the geometry and vice versa. Also note that it is simpler and more direct than (35). Thereby, one is tempted to prefer the canonical formalism over the Lagrangian one, like Ayón-Beato and García do. However, in section 4 we will see that one should be cautious about that, as the Bronnikov's theorem indicates.

All the relations that we have set up in this section will allow us to easily study the black holes in the next two sections.

# 3 Reissner-Nordström black hole

The gravitational field of a spherically symmetric body with mass $m$, charge $q$ and no angular momentum is described by the Reissner-Nordström metric. This metric is a spherically symmetric solution of the coupled Einstein-Maxwell equations. By Birkhoff's theorem, which reads

*Any electrovacuum spherically symmetric solution to the coupled Einstein-Maxwell equations is necessarily static and it coincides with a piece of the Reissner-Nordström geometry.*

it is also static and unique. The equations are usually solved as in [21]; but in this section we will do differently.

If we consider the classical electrodynamics Lagrangian $L_M(F) = -F$ , then we can use all the equations derived in the previous section by replacing $\frac{dL_M}{dF} = -1$ into them. Therefore, the aforementioned system is described by the action

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi} R + \frac{1}{4\pi} F\right)\,, \tag{55}$$

which implies the coupled Einstein-Maxwell equations, i.e.

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 8\pi T_{\mu\nu} \\ T_{\mu\nu} &= \frac{1}{4\pi} \left(F_\mu{}^\beta F_{\nu\beta} + F g_{\mu\nu}\right)\,. \end{aligned} \tag{56}$$

In order to solve the above equations, let us take the trace of the Einstein's equations

$$\begin{aligned} 0 &= -g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 8\pi T_{\mu\nu}\right) = -g^{\mu\nu}R_{\mu\nu} + \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} + 8\pi g^{\mu\nu}T_{\mu\nu} \\ &= -R + \frac{4}{2}R + 8\pi T = R + 8\pi T = 0\,. \end{aligned} \tag{57}$$

Note that the energy-momentum tensor is traceless, that is

$$\begin{aligned} T = g^{\mu\nu}T_{\mu\nu} &= \frac{1}{4\pi}\left(g^{\mu\nu}F_{\mu}{}^{\beta}F_{\nu\beta} + Fg^{\mu\nu}g_{\mu\nu}\right) = \frac{1}{4\pi}\left(F^{\mu\beta}F_{\nu\beta} + 4F\right) \\ &= \frac{1}{4\pi}\left(F^{\mu\beta}F_{\nu\beta} - 4\frac{1}{4}F^{\mu\beta}F_{\nu\beta}\right) = 0\,, \end{aligned} \tag{58}$$

so we conclude that the scalar curvature vanishes, i.e.

$$R = 0\,. \tag{59}$$

Now, since we are considering a static and spherically symmetric geometry, from (37) we get a homogeneous second order differential equation for $f(r)$

$$\begin{aligned} &0 = R = \frac{-2[-1 + f(r) + 2rf'(r)]}{r^2} - f''(r) \\ &\Longrightarrow \\ &0 = r^2 f''(r) + 4rf'(r) + 2f(r) - 2\,, \end{aligned} \tag{60}$$

which is trivially solved, yielding

$$f(r) = 1 + \frac{A}{r} + \frac{B}{r^2} \tag{61}$$

with $A$ and $B$ constants of integration that must be specified in terms of the parameters of the system.

So far, the geometry we are looking for is of the form

$$ds^2 = -\left(1 + \frac{A}{r} + \frac{B}{r^2}\right)dt^2 + \left(1 + \frac{A}{r} + \frac{B}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2\,. \tag{62}$$

A series expansion around zero up to first order in $\frac{1}{r}$ of the above yields

$$ds^2 = -\left(1 + \frac{A}{r}\right)dt^2 + \left(1 - \frac{A}{r}\right)dr^2 + r^2 d\Omega^2\,, \tag{63}$$

which coincides with the Newtonian or weak field limit if we set

$$A = -2m\,. \tag{64}$$

On the other hand, after replacing (61) into (37), a direct calculation gives

$$R^{\mu\nu}R_{\mu\nu} = \frac{4B^2}{r^8}\,. \tag{65}$$

Then, from (43), (60) and (65) one gets the electric field in terms of B

$$E(r) = \frac{r^2}{4q}\sqrt{\frac{16B^2}{r^8}} = \frac{B}{qr^2}\,, \tag{66}$$

which can also be calculated from Gauss's law in Gaussian units

$$\oint_S \vec{E}\cdot d\vec{A} = 4\pi q \Longrightarrow E(r) = \frac{q}{r^2}\,; \tag{67}$$

and from the last two equations we conclude that

$$B = q^2\,. \tag{68}$$

Finally from (62), (64) and (68) we obtain the well-known Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2\,. \tag{69}$$

Now, let us investigate the existence of singularities in the Reissner-Nordström spacetime. In this sense, note that the curvature invariant in (65) diverges at $r = 0$, so there is a physical singularity. Indeed, such singularity is unavoidable for this system, since the energy-momentum tensor of classical electrodynamics satisfies all the energy conditions. In particular the SEC is met, thus geodesics focus in the Raychaudhuri equation and the theorems predict a spacetime singularity.

To prove that the SEC is satisfied, without loss of generality we express the components of a timelike vector as

$$t^\mu = (\cosh\psi, \sinh\psi, 0, 0) = (\gamma, \gamma\beta, 0, 0)\,. \tag{70}$$

And then calculate the following

$$T_{\mu\nu}t^\mu t^\nu = \frac{1}{4\pi}\left(F_\mu{}^\beta F_{\nu\beta} + F g_{\mu\nu}\right)t^\mu t^\nu = \frac{1}{4\pi}\left(F^{\alpha\beta}F_{\nu\beta}t_\alpha t^\nu - F\right)\,. \tag{71}$$

From (33) we get that the only nonzero components of $F^{\mu\nu}$ and $F_{\mu\nu}$ are $F^{10} = -E(r)$ and $F_{10} = E(r)$, therefore

$$\begin{aligned} T_{\mu\nu}t^\mu t^\nu &= \frac{1}{4\pi}\left(F^{10}F_{10}t_1t^1 + F^{01}F_{01}t_0t^0 - F\right) = \frac{1}{4\pi}\left(F^{10}F_{10}(t_1t^1 + t_0t^0) - F\right) \\ &= \frac{1}{4\pi}\left(F^{10}F_{10}(-\cosh^2\psi + \sinh^2\psi) + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \\ &= \frac{1}{4\pi}\left(-F^{10}F_{10} + \frac{2}{4}F^{10}F_{10}\right) = -\frac{1}{8\pi}F^{10}F_{10} = \frac{1}{8\pi}E(r)^2\,. \end{aligned} \tag{72}$$

We conclude that $T_{\mu\nu}t^\mu t^\nu \geq 0$, which amounts to the fulfillment of both the WEC and SEC, recalling that $T = 0$. By a similar procedure, one can verify that the remaining energy conditions are satisfied as well.

Finally, we analyse the structure of the event horizons. These are located at the radius where $g^{rr} = -g_{tt}$ vanishes, so

$$0 = f(r_H) = 1 - \frac{2m}{r_H} + \frac{q^2}{{r_H}^2} \Longrightarrow r_{H\pm} = m \pm \sqrt{m^2 - q^2}\,. \tag{73}$$

There are 3 possible solutions for $r_{H\pm}$, depending on the relative values of $m^2$ and $q^2$. Let us analyse these cases separately.

$\boldsymbol{(m^2 - q^2) < 0}$: Here $r_{H\pm}$ are complex numbers; thereby there are no event horizons and the metric is regular everywhere in the coordinates $(t, r, \theta, \phi)$ except for $r = 0$. Therefore, the singularity at the origin of coordinates is a naked one, which violates the Penrose cosmic censorship conjecture. Nonetheless, note that this solution is not likely to be physical. The inequality states that the total energy of the system is less than the energy of the electromagnetic field, so that the mass of the body would be negative.

$\boldsymbol{(m^2 - q^2) = 0}$: This case is known as an extremal black hole, since there is a single degenerate event horizon at $r_{H\pm} = r_H = m$. Here the mass is exactly balanced by the charge; however, such balance is unstable because adding a little bit of mass or charge would take it to one of the other 2 cases.

$\boldsymbol{(m^2 - q^2) > 0}$: This situation would apply in a real gravitational collapse, for the energy of the electromagnetic field is less than the total energy of the system. We find two event horizons located at $r_{H\pm} = m \pm \sqrt{m^2 - q^2}$. More precisely, at $r_{H+}$ there is an event horizon totally analogous to $r = 2M$ in the Schwarzschild metric; whereas $r_{H-}$ corresponds to a Cauchy horizon.

A good analysis of the global structure for the Reissner-Nordström metric, in the three aforementioned cases, can be found in [16]. Now we turn to the more interesting case of regular black holes.

# 4 Regular black holes

A black hole is said to be regular if the invariants $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $R^{\mu\nu}R_{\mu\nu}$ and $R$ are all finite everywhere in spacetime. The first model describing a black hole with such characteristics was proposed by James M. Bardeen [1] in 1968 [6]. The Bardeen regular spacetime is described by the metric

$$\mathbf{g_B} = ds^2 = -\left(1 - \frac{2mr^2}{(r^2+g^2)^{3/2}}\right)dt^2 + \left(1 - \frac{2mr^2}{(r^2+g^2)^{3/2}}\right)^{-1} dr^2 + r^2 d\Omega^2\,, \tag{74}$$

[1] Not to be confused with John Bardeen, the one who was awarded the Nobel Price in Physics twice, and also James's father.

which is a conveniently modified Schwarzchild metric, where $g$ is used as a regularizing parameter for such spacetime. An asymptotic expansion, i.e.

$$f(r) = 1 - \frac{2m}{r} + \frac{3mg^2}{r^3} + O\left(\frac{1}{r^5}\right) , \tag{75}$$

shows that the constant $m$ can be associated with the mass of the system, but it gives no physical interpretation for the parameter $g$. For this reason, the Bardeen metric is considered just as a model and not as an exact solution of the Einstein's equations. It was only up to the year 2000, when Ayón-Beato and García provided a physical interpretation for such parameter [22]; they showed that $g$ is the monopole charge of a self-gravitating magnetic field described by a particular nonlinear electrodynamics Lagrangian $L(F)$. Furthermore, it can be shown that this regular spacetime satisfies the weak energy condition.

Quite a few years later, in a 1998 Letter [7], Eloy Ayón-Beato and Alberto García presented another metric describing a regular black hole. The latter is a solution of a particular nonlinear electrodynamics theory coupled to the Einstein's equations. Such electrodynamics considers only a radial electric field as the source, and it is formulated in the $H(P)$ framework. Also, the authors prove that their solution satisfies the weak energy condition (WEC). In the rest of this work we are going to analyse the most important features of this system.

Such spacetime is described by the metric

$$\begin{aligned} \mathbf{g} = ds^2 = &-\left(1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2}\right) dt^2 \\ &+ \left(1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2}\right)^{-1} dr^2 + r^2 d\Omega^2 , \end{aligned} \tag{76}$$

which is a suitable manipulation of the Reissner-Nordström metric, where the dimensions are fixed after introducing the regularizing parameter $q$ in the denominator. Asymptotic expansions of $f(r)$ around $\infty$ and 0 yield

$$\begin{aligned} f(r) &\xrightarrow[r\to\infty]{} 1 - \frac{2m}{r} + \frac{q^2}{r^2} + O\left(\frac{1}{r^4}\right) \\ f(r) &\xrightarrow[r\to 0]{} 1 + \left(\frac{1}{q^2} - \frac{2m|q|}{q^4}\right) r^2 + O\left(r^4\right) . \end{aligned} \tag{77}$$

From the asymptotic behaviour at infinity one recovers the Reissner-Nordström metric, hence the constants $m$ and $q$ are interpreted as the mass and the electric charge of the system, respectively. Notice that the first two terms in $\mathbf{g}_{00}$ are exactly the same as $\mathbf{g}_{\mathbf{B}00}$, so that the remaining term in $\mathbf{g}_{00}$ fixes the asymptotic limit (75) of Bardeen's metric and makes it Reissner-Nordström-like.

On the other hand, the asymptotic behaviour around $r = 0$ is identified with a de Sitter spacetime, where the cosmological constant is given by

$$\Lambda = 3\left(\frac{2m|q|}{q^4} - \frac{1}{q^2}\right) . \tag{78}$$

Note that the topology of de Sitter spacetime is $\mathbb{R} \times \mathbb{S}^3$, whereas the topology for Reissner-Nordström is $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^2$; in both cases the first $\mathbb{R}$ corresponds to the time coordinate and the rest to the spatial part. Therefore, at some point in space there is a topology change in the spatial slices from $\mathbb{R} \times \mathbb{S}^2$ at infinity to $\mathbb{S}^3$ approaching the origin of coordinates. As it is pointed out by Borde [23] $\mathbb{S}^3$ is a compact or "closed" surface; such compactness allows the avoidance of singularities, since it violates one of the hypotheses of the Penrose singularity theorem [24]. A very interesting question is to find in what point of space occurs this topology change and what is the reason of such change, since it is a manifestation of a discontinuity in some function describing the system.

With the aim of studying locally the existence of singularities in the system, one must obtain all the curvature invariants for the metric $\mathbf{g}$. A straightforward calculation from (37) yields the Kretschmann scalar

$$\begin{aligned} R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} =& \frac{16m^2(5q^4 - 2q^2r^2 + 2r^4)}{(q^2+r^2)^5} - \frac{2m(2q^4 - 11q^2r^2 + 2r^4)}{(q^2+r^2)^{7/2}} \\ & - \frac{16mq^2(5q^4 - 4q^2r^2 + 3r^4)}{(q^2+r^2)^{11/2}} + \frac{4q^4(5q^4 - 6q^2r^2 + 5r^4]}{(q^2+r^2)^6} \\ & + q^2(2q^4 + 16q^2r^2 + 6r^4) , \end{aligned} \tag{79}$$

the Ricci tensor squared

$$\begin{aligned} R^{\mu\nu}R_{\mu\nu} =& \frac{18m^2q^4(8q^4 - 4q^2r^2 + 13r^4)}{(q^2+r^2)^7} - \frac{6mq^4(24q^6 + 56q^2r^4 - 30q^4r^2 - 10r^6)}{(q^2+r^2)^{15/2}} \\ & + \frac{q^4(72q^8 - 156q^6r^2 + 272q^4r^4 - 80q^2r^6 + 8r^8)}{2(q^2+r^2)^8} , \end{aligned} \tag{80}$$

and the scalar curvature

$$R = -\frac{6q^2m(r^2 - 4q^2)}{(q^2+r^2)^{7/2}} + \frac{12q^4(r^2 - q^2)}{(q^2+r^2)^4} . \tag{81}$$

Despite that the three expressions above are complicated, by inspection one easily realizes that this solution is indeed a regular one. In order to find out if this spacetime describes also a black hole, one must study the points where $g^{11} = -g_{00} = f(r)$ vanishes. It follows that for the relative value

$$|q| \leq 0.652m \tag{82}$$

we are in the presence of a black hole. The main lines of this calculation can be found in the cited paper. In particular, under the strict equality there are inner and event horizons as in the third case of the Reissner-Nordström solution. Also, the equality corresponds to an extreme black hole, where both horizons shrink into a single one. We see that this system has a similar global structure as the Reissner-Norsdström black hole except for the singularity at $r = 0$, which has been removed thanks to nonlinear electrodynamics.

Now, in order to find out which is the electric field acting as the source of the above described geometry, we simply use equation (43) and compute it. The result is

$$E(r) = qr^4 \left( \frac{r^2 - 5q^2}{(r^2+q^2)^4} + \frac{15}{2} \frac{m}{(r^2+q^2)^{7/2}} \right) . \tag{83}$$

The asymptotic behaviour of such field is given by

$$E(r) = \frac{q}{r^2} + O\left(\frac{1}{r^3}\right) , \tag{84}$$

which is the Coulomb field derived from Maxwell's equations. This asymptotic limit is consistent with the one for the geometry; both the electric field and the metric behave like the Reissner-Nordström spacetime as one approaches infinity. Also note that the electric field is regular everywhere in space, it follows by inspection of (83). This finiteness feature is what Born and Infeld were looking for with their Lagrangian; except that they proposed a maximum electric field at the origin, and the Ayón-Beato-García electric field vanishes at $r = 0$ but achieves a maximum at some $r^* > 0$. The last statement can be verified with the derivative of $E(r)$ with respect to $r$, or qualitatively from the following plot

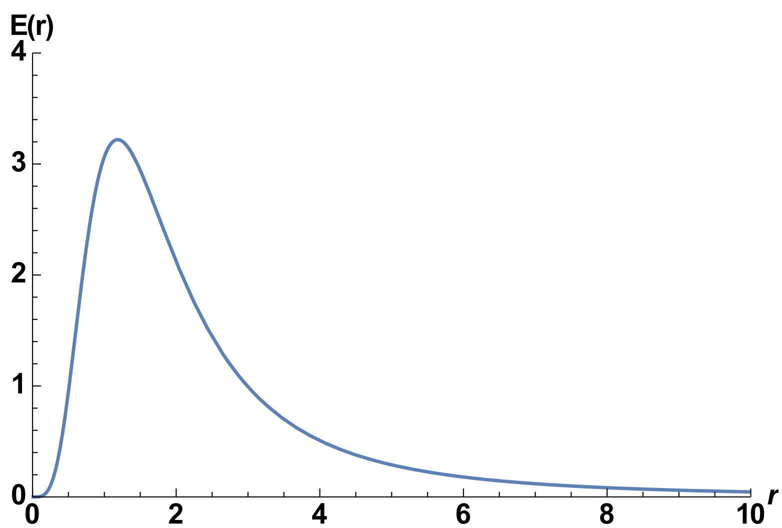


FIG Ayón-Beato-García electric field for $q = 1$ and $m = 5$, these values satisfy the inequality of the black hole regime.

Finally, we are led to find the structural function for the system. Using (53) we can calculate it in terms of $r$ and $q$, that is

$$H(r,q) = -\frac{3mq^2}{\left(q^2+r^2\right)^{5/2}} - \frac{q^2\left(r^2-3q^2\right)}{\left(q^2+r^2\right)^3}\,; \tag{85}$$

replacing $\frac{q^2}{2r^4} \to P$ one gets the canonical description of the system presented by the authors [2]

$$H(P) = -P\frac{1-3\sqrt{2q^2P}}{(1+\sqrt{2q^2P})^3} - \frac{3}{2q^2s}\left(\frac{\sqrt{2q^2P}}{1+\sqrt{2q^2P}}\right)^{5/2}\,. \tag{86}$$

A series expansion around infinity shows that this function behaves as

$$H(P) = -P - \frac{3\left(2q^2\right)^{1/4}P^{5/4}}{s} + 6\sqrt{2q^2}P^{3/2} + \frac{15\left(2q^2\right)^{3/4}P^{7/4}}{2s} - 30q^2P^2 + O\left(P^{9/4}\right)\,; \tag{87}$$

where the first term corresponds to the Maxwell theory, so the proposed nonlinear electrodynamics fulfills the plausibility condition. Nevertheless, the following terms indicate that $H(P)$ cannot be regarded as some weak field limit of the Euler-Heisenberg Lagrangian, like the Born-Infeld theory does. This could have been predicted from the differences in the behaviour of both electric fields pointed out above. Therefore, in the scope of a weak field limit of QED it is difficult to give a physical interpretation to this nonlinear electrodynamics theory.

At this point, we have already described all the relevant features of the system. A simple analysis of the curvature invariants and the vanishing of $g^{11}$ shows that the proposed metric (76) describes a regular black hole for certain ranges of $m$ and $q$. The source of such desirable geometry is the radial electric field (83), which can be derived from the structural function (86). Although it is complicated to find a physically realisable system that behaves as this nonlinear electrodynamics theory indicates; in principle there is nothing that forbids it from happening in more exotic contexts, for instance in string theory.

Nevertheless, the above solution seems to contradict a no-go theorem proved by K. A. Bronnikov long time before [25, 26]. In short, Bronnikov asserts that all the characteristics of the regular black hole proposed by Ayón-Beato and García lead to a contradiction [11]. Let us enunciate such theorem and make an outline of the proof.

**Theorem.** *The field system of equation (6) having a Maxwell asymptotic ($L \to 0$, $dL/dF \to -1$ as $F \to 0$), does not admit a static spherically symmetric solution with a regular center and a nonzero electric charge.*

[2] Recall that their convention is $P = \frac{1}{4}P_{\mu\nu}P^{\mu\nu}$ but here the convention is $P = -\frac{1}{4}P_{\mu\nu}P^{\mu\nu}$.

*Proof.* At a regular spacetime point the three invariants $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $R^{\mu\nu}R_{\nu\mu}$ and $R$ are finite. In particular, for a static spherically symmetric metric (36) the Ricci tensor squared reads $R^{\mu\nu}R_{\mu\nu} = (R^{\mu}_{\mu})^2$, thereby each component $R^{\circ}_{\circ}$ must be finite. Hence, each component of the tensors $G^{\mu}_{\nu}$ and $T^{\mu}_{\nu}$ must be finite too. Then, from (30) one gets the inequality

$$4\pi\left(T^0_0 - T^2_2\right) = \frac{dL}{dF}E(r)^2 = 2F\frac{dL}{dF} < \infty\,. \tag{88}$$

On the other hand, one has

$$2F\left(\frac{dL}{dF}\right)^2 = \frac{q^2}{r^4}\left(\frac{dL}{dF}\right)^{-2}\left(\frac{dL}{dF}\right)^2 = \frac{q^2}{r^4} \xrightarrow{r\to 0} \infty\,. \tag{89}$$

Thereby, at a regular center we obtain the conditions $F\frac{dL}{dF} < \infty$ and $F\left(\frac{dL}{dF}\right)^2 \xrightarrow{r\to 0} \infty$, and combining these two we are led to $F \to 0$ and $\frac{dL}{dF} \to \infty$. The latter is a strongly non-Maxwell behaviour at small $F$, so the plausibility condition cannot be satisfied. □

In his paper Bronnikov asserts that the theorem can be extended to systems where both electric and magnetic charges are present, the proof being similar. He also highlights that there was not an asymptotic flatness requirement.

So what is wrong with the Ayón-Beato and García solution? Bronnikov says that it contradicts his theorem because of the description of the system in the P framework. His arguments are the following:

$P$ is just an auxiliary field and the dynamics are really specified in the F framework, but the two frameworks are only equivalent if the function $F(P)$ is monotonic. Since $E(r)$ vanishes both at $r = 0$ and $r \to \infty$ and $F = \frac{1}{2}E(r)^2$, then $F(P)$ must have a maximum at some $P^*$. Hence, the derivative $\frac{d^2L}{dF^2}$ tends to infinities of opposite signs as one approaches $F^* = F(P^*)$ from the right and from the left. This leads to a branching of the function $L(F)$, whose graph forms a cusp at $F^*$. Additional branching is related to extrema of the structural function $H(P)$. Therefore, $L(F)$ corresponds to different Lagrangians at different parts of space, and the Ayón-Beato-García regular solution is not really derived from a consistent theory.

This explanation seems totally right, and one is led to think that Ayón-Beato and García just proposed an *ad hoc* $H(P)$ obtained from the desired regular geometry. Bronnikov's arguments are repeated in different papers that deal with regular black holes, for example in [27].

However, Irina Dymnikova pointed out [28] that the Bronnikov's theorem requires the fulfillment of the Maxwell asymptotic *at the center*, so the proof reads that "a regular electrically charged structure is not compatible with the Maxwell weak field limit in the center". This statement is obvious, since the Coulomb's electric field diverges at $r = 0$ and the old and well-known Reissner-Nordstrom solution is singular there. It is nonsense to ask for a Maxwell asymptotic in the center if one is looking for a regular black hole solution; the physically reasonable requirement is a Maxwell asymptotic at infinity, where the fields are weak enough and one should recover the linear theory.

Dymnikova also showed [28] that for electrically charged solutions of general relativity coupled to nonlinear electrodynamics with the symmetry $T_0^0 = T_1^1$ the WEC always implies a de Sitter asymptotic at approaching a regular center. An explicit example of the latter is the Ayón-Beato and García solution! She explains that the branching in $L(F)$ is inevitable as Bronnikov says, but the fulfillment of the WEC avoids the branching in $H(P)$. When the structural function is monotonic there is no problem in going back to the $F$ framework from the $H$ one; hence, Bronnikov's branching arguments do not apply for the Ayón-Beato and García solution.

In this section we have discovered many interesting features of the regular black hole solution presented by Ayón-Beato and García back in 1998. The solution is obtained as a manipulation of the Reissner-Nordström metric, where the electric charge $q$ is conveniently used to achieve regularity. Thereby, the structural function $H(P)$ is indeed an *ad hoc* theory for electrodynamics; the associated Lagrangian $L(F)$ can be obtained without problems thanks to the fulfillment of the WEC. Moreover, the metric has a de Sitter asymptotic behaviour near the origin, which is consistent with the increasing interval of the electric field from $r = 0$ to $r = r^*$. This suggests the presence of a repulsive interaction there which shifts the maximum value of the electric field. Note that although the first correction to the Maxwell structural function is not Born-Infeld-like, the existence of a cut-off in the electric field follows the spirit of Born and Infeld.

The above mentioned repulsive interval is related to the violation of the strong energy condition, which reads

$$T_{\mu\nu}t^\mu t^\nu + \frac{1}{2}T \geq 0. \tag{90}$$

The authors show that the weak energy condition is always fulfilled for the studied solution. Therefore, the previous inequality can be violated if

$$T = \frac{1}{4\pi}\left(2\frac{dL}{dF}E(r)^2 - 4L(F)\right). \tag{91}$$

is negative enough. After a series expansion of (90) around $r = 0$, one realises that near the origin the violation of the SEC is expressed as

$$\frac{|q|}{m} \leq 6\,, \tag{92}$$

which is crearly fulfilled in the black hole regime defined by $|q| \leq 0.652m$.

For completeness we show here other regular black hole solutions proposed by Ayón Beato and García. In a very similar way, the following year (1999) they presented two more regular spacetimes [8, 9]. In these solutions the Schwarzchild metric is conveniently regularized with more sophisticated functions of $q$, $m$ and $r$; leading to some electric field and structural function, in each case. The two systems are explicitly described by

$$
\left.\begin{aligned}
\mathbf{g_1} = ds^2 = &-\left(1-\frac{2m[1-\tanh(q^2/2mr)]}{r}\right)dt^2 \\
&\qquad\qquad\qquad + \left(1-\frac{2m[1-\tanh(q^2/2mr)]}{r}\right)dr^2 + r^2 d\Omega^2 \\
E_1(r) = &\frac{q}{4mr^3}\left(1-\tanh(q^2/2mr)\right)\left(4mr - q^2\tanh(q^2/2mr)\right) \\
H_1(P) = &P\left(\tanh\left(s\sqrt[4]{2q^2P}\right)-1\right)
\end{aligned}\right\} \tag{93}
$$

and

$$
\left.\begin{aligned}
\mathbf{g_2} = ds^2 = &-\left(1-\frac{2mr^2e^{-q^2/2mr}}{(r^2+q^2)^{3/2}}\right)dt^2 + \left(1-\frac{2mr^2e^{-q^2/2mr}}{(r^2+q^2)^{3/2}}\right)^{-1}dr^2 + r^2 d\Omega^2 \\
E_2(r) = &\frac{qe^{-q^2/2mr}}{(r^2+q^2)^{7/2}}\left(r^5 + \frac{(60m^2-q^2)r^4}{8m} + \frac{q^2r^3}{2} - \frac{q^4r^2}{4m} - \frac{q^4r}{2} - \frac{q^6}{8m}\right) \\
H_2(P) = &\frac{-Pe^{-s\sqrt[4]{2q^2P}}}{\left(1+\sqrt{2q^2P}\right)^{5/2}}\left(1+\sqrt{2q^2P}+\frac{3}{s}\sqrt[4]{2q^2P}\right).
\end{aligned}\right\} \tag{94}
$$

An additional was solution was presented in 2005 [10]. This spacetime is a generalization of their first solution and is described by

$$
\left.\begin{aligned}
\mathbf{g_3} = ds^2 = &-\left(1-\frac{2mr^{\alpha-1}}{(r^2+q^2)^{\alpha/2}}+\frac{q^2r^{\beta-2}}{(r^2+q^2)^{\beta/2}}\right)dt^2 \\
&\qquad + \left(1-\frac{2mr^{\alpha-1}}{(r^2+q^2)^{\alpha/2}}+\frac{q^2r^{\beta-2}}{(r^2+q^2)^{\beta/2}}\right)^{-1}dr^2 + r^2 d\Omega^2 \\
E_3(r) = &q\left(\frac{\alpha m[5r^2-(\alpha-3)q^2]r^{\alpha-1}}{2(r^2+q^2)^{\alpha/2+2}}\right) \\
&\qquad + q\left(\frac{[4r^4-(7\beta-8)q^2r^2+(\beta-1)(\beta-4)q^4]r^{\beta-2}}{4(r^2+q^2)^{\beta/2+2}}\right) \\
H_3(P) = &-P\frac{1-(\beta-1)\sqrt{2q^2P}}{(1+\sqrt{2q^2P})^{\beta/2+1}} - \frac{\alpha}{2q^2s}\frac{(\sqrt{2q^2P})^{5/2}}{1+\sqrt{2q^2P})^{\alpha/2+1}}.
\end{aligned}\right\} \tag{95}
$$

where $s = |q|/2m$. Once again, an asymptotic expansion shows the three systems behave as the Reissner-Nordström spacetime, so -as expected- $m$ and $q$ are regarded as the mass and the electric charge of the system, respectively.

The three aforementioned solutions always describe regular spacetimes, since all their curvature invariants are finite everywhere. This can be verified by a straightforward calculation using the formulas in (37). To find out if such spacetimes describe also black holes, one must study the points where $g^{11} = -g_{00} = f(r)$ vanishes. In each particular case, it follows that for certain relative values between $q$ and $m$ we are in the presence of a black hole. All the calculations can be checked in the cited papers, here we present the results

$$\begin{aligned} \mathbf{g_1} &: \ |q| \leq 1.05m \\ \mathbf{g_2} &: \ |q| \leq 0.6m \\ \mathbf{g_3} &: \ |q| \leq s_c m \, , \end{aligned} \tag{96}$$

where $s_c$ is a critical value that depends on $\alpha$ and $\beta$. Just as in the first solution, under the strict equality there are inner and event horizons as in the third case of the Reissner-Nordström solution. Also, the equality corresponds to an extreme black hole, where both horizons shrink into a single one. Thus, we see that the three systems described above have the same global structure as the Reissner-Norsdström black hole except for the singularity at $r = 0$, which -in each case- has been removed thanks to the considered nonlinear electrodynamics model.

# 5 Conclusions

This work starts by introducing the action of general relativity coupled to nonlinear electrodynamics. Then, applying the least action principle, a derivation of the Einstein's equations, the energy-momentum tensor for nonlinear electrodynamics, the generalized Maxwell's equations and electric field is done. Also, the FP dual formalism is briefly presented. All this in the scope of a static, spherically symmetric geometry. Next, the derived equations are applied to study of the well-known Reissner-Nordström black hole and the Ayón-Beato-García regular black hole.

Regarding the Reissner-Nordström spacetime, the metric is derived in a different way than the usual textbooks on general relativity do. From the Newtonian weak field limit and Gauss's law one concludes that the constants of integration are related to the mass and the electric charge of the system. Then, the existence of a singularity in this spacetime is understood in terms of the strong energy condition, which is explicitly calculated for the energy-momentum tensor of classical electrodynamics. At last, a short analysis about the structure of the event horizons is done.

The heart of this work lies in the study of the regular black hole solution presented by Ayón-Beato and García in 1998. It is pointed out that the metric is obtained by imposing regularity conditions on the Reissner-Nordström metric via the electric charge $q$; so the corresponding structural function is an *ad hoc* theory of nonlinear electrodynamics that can be obtained from the metric. A calculation of the curvature invariants, the electric field and the structural function is done directly using the equations derived in section 2. It is also shown that the metric has a de Sitter behaviour near the origin of coordinates, which is consistent with the increasing interval of the electric field from

$r = 0$ to a certain $r^*$. Therefore, a physical interpretation of the structural function can be given in terms of a cut-off in the electric field that is Born-Infeld-like.

Although the solution presented by the authors is an *ad hoc* manipulation of the Reissner-Nordström spacetime, it sheds light on the characteristics that some models of gravity might have in order to avoid spacetime singularities. The latter could be useful regarding the construction of a quantum theory of gravity.

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