Notes on Yang–Mills–Higgs monopoles and dyons on $\mathbb{R}^D$, and Chern–Simons–Higgs solitons on $\mathbb{R}^{D-2}$: dimensional reduction of Chern–Pontryagin densities

TOPICAL REVIEW

Notes on Yang–Mills–Higgs monopoles and dyons on $\mathbb{R}^D$, and Chern–Simons–Higgs solitons on $\mathbb{R}^{D-2}$: dimensional reduction of Chern–Pontryagin densities

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Abstract

We review work on the construction of Monopoles in higher dimensions. These are solutions to a particular class of models descending from Yang–Mills systems on even-dimensional bulk, with spheres as codimensions. The topological lower bounds on the Yang–Mills action translate into Bogomol'nyi lower bounds on the residual Yang–Mills–Higgs systems. Mostly, consideration is restricted to eight-dimensional bulk systems, but extension to the arbitrary case follows systematically. After presenting the monopoles, the corresponding dyons are also constructed. Finally, new Chern–Simons densities expressed in terms of Yang–Mills and Higgs fields are presented. These are defined in all dimensions, including in even-dimensional spacetimes. They are constructed by subjecting the dimensionally reduced Chern–Pontryagin densities to a further descent by two steps.

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Contents

1. Introduction and summary

The central task of these notes is to explain how to subject nth Chern–Pontryagin (CP) density $\mathcal{C}^{(n)}$:

$$\mathcal{C}^{(n)} = \frac{1}{\omega(\pi)} \varepsilon_{M_1 M_2 M_3 M_4 \ldots M_{2n-1} M_{2n}} \operatorname{Tr} F_{M_1 M_2} F_{M_3 M_4} \ldots F_{M_{2n-1} M_{2n}} \qquad (1.1)$$

defined on the $2n$-dimensional space to a dimensional descent to $\mathbb{R}^D$ by considering (1.1) on the direct product space $\mathbb{R}^D \times S^{2n-D}$. The resulting residual density on $\mathbb{R}^D$ will be denoted as $\Omega_D^{(n)}$.

The density $\mathcal{C}^{(n)}$ is, by construction, a total divergence

$$\mathcal{C}^{(n)} = \nabla \cdot \Omega^{(n)}, \qquad (1.2)$$

and it turns out that under certain restrictions, the dimensional descendant of (1.2) is also a total divergence. After demonstrating this result, it will be applied to the construction of monopoles, dyons and Chern–Simons (CS) solitons in all dimensions.

Some special choices, or restrictions, are made for practical reasons. Firstly, we have restricted to the codimension S^{2n-D}, the $(2n - D)$-sphere, as this is the most symmetric compact coset space that is defined both in even and in odd dimensions. It can of course be replaced by any other symmetric and compact coset space.

Secondly, the gauge field of the bulk gauge theory is chosen to be a $2^{n-1} \times 2^{n-1}$ array with complex valued entries. Given our choice of spheres for the codimension, this leads to residual gauge fields on which we take their values in the Dirac matrix representation of the residual gauge group $SO(D)$.

As a result of the above two choices, it is possible to make the symmetry imposition (namely the dimensional reduction) such that the residual Higgs field is described by a D-component isovector multiplet. This choice is made specifically, with the requirement that the resulting Higgs models support topologically stable, finite energy, solitons ('monopoles') whose asymptotic gauge fields describe Dirac–Yang (DY) [1–3] monopoles. These are $SO(D)$ monopoles defined on all $\mathbb{R}^D$ ($D \geqslant 3$), generalizing the usual $SO(3)$ 't Hooft–Polyakov [4, 5] monopole on $\mathbb{R}^3$.

The third and last restriction is to limit our concrete calculations to the case of codimensions S^{2n-D} for $n = 2, 3$ and 4 only, i.e. to the descents of the second, third and fourth CP densities only. The resulting residual CP densities capture all qualitative features of the generic nth case, the calculus for $n \geqslant 3$ being inordinately more complicated, without yielding a new qualitative insight.

Inspite of the title, alluding to applications to the construction of various solitons, the central result of these notes is the demonstration that the CP density $\mathcal{C}_D^{(n)}$ on $\mathbb{R}^D$ descended from the $2n$-dimensional bulk CP density $\mathcal{C}^{(n)}$ is a *total divergence*

$$\mathcal{C}_D^{(n)} = \nabla \cdot \mathbf{\Omega}^{(n,D)} \tag{1.3}$$

like $\mathcal{C}^{(n)}$ formally is on the bulk.

The various applications ensuing from this central result will be covered subsequently, rather briefly, since these present an open ended list of exercises. These fall under the following broad headings.

- When $D \geqslant 3$, the reduced density $\Omega_D^{(n)}$ can be interpreted as the *monopole* charge density of a static Yang–Mills–Higgs (YMH) theory on $\mathbb{R}^D$, and when $D = 2$ it can be interpreted as the *vortex* charge of the YMH system on $\mathbb{R}^2$. In the $D \geqslant 3$ case, these are the D-dimensional generalizations of the 't Hooft–Polyakov monopole on $\mathbb{R}^3$. In the $D = 2$ case, these are the generlizations of the Abrikosov–Nielsen–Olesen (ANO) vortices [6, 7] of Abelian Higgs models featuring higher order Higgs-dependent functions.

  In this application of the main result, a hierarchy of YMH models supporting *monopoles*[1] in all dimensions can be constructed systematically by subjecting the topological inequality of Yang–Mills (YM) systems in higher (even) dimensions, to dimensional reduction. The higher (even)-dimensional YM hierarchy [8] in question will be described in section 2, and this will be followed by a brief description of the ensuing Higgs models after dimensional reduction.

  It must be stressed that this procedure is not unique, and one can alternatively follow a scheme where the topological charge is defined directly as the *winding number* [9] of the Higgs field, which is suitably covariantized. We eschew this alternative because in that case the ensuing Bogomol'nyi lower bounds result in energy densities that are described by kinetic terms which are not exclusively quadratic in the velocity fields.

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[1] We are not very concerned with the construction of generalized *vortices* on $\mathbb{R}^2$, since such Abelian models cannot support dyons, that being the second main application to be discussed.

- The construction of monopoles is followed in a natural way by the construction of dyons and dyon-like solutions in $(D + 1)$-dimensional Minkowski spacetime. This involves the formal extension of the models supporting monopoles on $\mathbb{R}^D$ to Minkowskian theories on $(D + 1)$-dimensional flat spacetime, supporting static electric YM fields in addition to monopoles. In this respect, these solutions are the D-dimensional generalizations of the Julia–Zee [10] dyon[2] on $\mathbb{R}^3$. However, only those defined on $\mathbb{R}^3$ describe an electric flux like the latter, and they are referred to as dyons. The electric field carrying static solutions on $\mathbb{R}^D$ ($D \geqslant 4$) are referred to as pseudo-dyons. Also unlike the Julia–Zee dyon which does have a BPS limit, solutions in dimensions other than $\mathbb{R}^{4p-1}$ do not have a BPS limit. These last dyonic configurations are presented as a byproduct of monopolic configurations, presented in [11]. Apart from these, none of the dyonic configurations presented below in section 8 have been studied quantitatively (numerically) to date.
- From the reduced density $\mathbf{\Omega}^{(n,D)} \equiv \Omega_i^{(n,D)}$, where i is the index of the spacelike coordinate x_i with $i = 1, 2, \ldots, D$, one can formally identify a CS density as the Dth component of $\mathbf{\Omega}^{(n,D)}$. This quantity can then be interpreted as a CS term on $(D - 1)$- dimensional Minkowski space, i.e. on the spacetime $(t, \mathbb{R}^{D-2})$. The solitons of the corresponding CS–Higgs (CSH) theory can be constructed systematically. Note that this is not the usual CS term defined in terms of a pure YM field on odd-dimensional spacetime, but rather these new CS terms are defined by both the YM and the Higgs fields. Most importantly, the definition of these new CS terms is not restricted to odd-dimensional spacetimes, but also covers even-dimensional spacetimes. To date, such CSH solutions have not been studied concretely.

2. The YM hierarchy

We seek finite energy solutions of YMH systems in arbitrary spacelike dimensions $\mathbb{R}^D$. The construction of YM instantons and YMH and monopoles in higher dimensions was first suggested in [13]. YMH systems can be derived from the dimensional descent of a suitable member of the YM hierarchy on the Euclidean space $\mathbb{R}^D \times K^N$ such that K^N is a compact coset space. Here, $D+N$ is even since the YM hierarchy introduced in [8] is defined in even dimensions as CP densities are defined in even dimensions only.

The YM hierarchy of $SO(4p)$ gauge fields in the chiral (Dirac matrix) representations consisting only of the p-YM term in (2.2), the $p = 1$ member of which is the usual YM system supporting the celebrated BPST instanton [12], was introduced in [8] to construct self-dual instantons in $4p$ dimensions. (The self-duality equation for the $p = 2$ case was solved independently in [14], whose authors subsequently stated in their *erratum* that this solution was the instanton of the $p = 2$ member of the hierarchy introduced earlier in [8].) The instantons of the generic system consisting of the sum of many terms (2.2) with different p, while stable, are not self-dual and cannot be evaluated in the closed form and are constructed numerically [15]. Restricting ourselves here to finite action (instanton) solutions only, it is worth mentioning an alternative hierarchy which supports self-dual instantons in $4p + 2$ dimensions [16, 17]. While it is straightforward to construct spherically symmetric solutions with the gauge group $SO(4p + 2)$ in the chiral Dirac representations, these self-duality equations are even more overdetermined than those of the $4p$-dimensional hierarchy. The action densities of these systems are not positive definite so that while the self-duality equations do solve the second order field equations, they do not saturate a Bogomol'nyi bound and hence are not necessarily

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[2] For $D = 2$ the resulting solitons are not *monopoles*, but *vortices*. In those cases, the subsequent construction of a dyon is obstructed by the Julia–Zee theorem, *unless* if the model is augmented with a suitable CS term.

stable. The self-duality equations employed in the works so far mentioned are nonlinear in the YM curvature. There have been other formulations of higher-dimensional self-duality [18–20] which differ from the former in that the self-duality equations proposed there are linear in the YM curvature. These suffer from the same lack of stability of the solutions of [16, 17], and some of them do not even have finite energy. There is also a large mathematical literature on higher-dimensional finite action instantons satisfying linear (anti-)self-duality equations, on suitable Kähler manifolds, e.g. on quaternionic Kähler manifolds in [21]. There have been more recent developments in this direction in the literature (see for example [22], where a comprehensive list of references is also given).

We start from the definition of the YM hierarchy. Since we will be concerned with the dimensional reduction over the N-dimensional codimension K^N, we will denote the connection and the curvature in the bulk with $(\mathcal{A}, \mathcal{F})$ to distinguish these from the corresponding quantities (A, F) on the residual space $\mathbb{R}^D$. Using the notation $\mathcal{F}(2) = \mathcal{F}_{\mu\nu}$ for the 2-form YM curvature, the $2p$-form YM tensor

$$\mathcal{F}(2p) = \mathcal{F}(2) \wedge \mathcal{F}(2) \wedge \ldots \wedge \mathcal{F}(2), \qquad p\text{-times} \tag{2.1}$$

is a p fold totally antisymmetrized product of the 2-form curvature.

The p-YM system of the YM hierarchy is defined, on $\mathbb{R}^{4p}$, by the Hamiltonian[3] density[4]

$$\mathcal{H}_{\text{YM}_p} = \text{Tr}\,\mathcal{F}(2p)^2. \tag{2.2}$$

In $2n$ dimensions, partitioning n as $n = p+q$, the Hodge dual of the $2q$-form field $F(2q)$, namely $(^\star\mathcal{F}(2q))(2p)$, is a $2p$-form.

Starting from the inequality

$$\text{Tr}[\mathcal{F}(2p) - \kappa^{(p-q)}\,{}^\star\mathcal{F}(2q)]^2 \geqslant 0, \tag{2.3}$$

it follows that

$$\text{Tr}[\mathcal{F}(2p)^2 + \kappa^{2(p-q)}\,\mathcal{F}(2q)^2] \geqslant 2\kappa^{(p-q)}\,\mathcal{C}^{(n)}, \tag{2.4}$$

where $\mathcal{C}^{(n)} \equiv \mathcal{C}^{(n=p+q)}$ is the nth CP density. In (2.3) and (2.4), the constant κ has the dimension of length if $p > q$ and the inverse if $p < q$.

The element of the YM systems labelled by (p, q) in (even) $2(p+q)$ dimensions is defined by Lagrangians defined by the densities on the *left-hand side* of (2.4). When in particular $p = q$, then these systems are conformally invariant and we refer to them as the p-YM members of the YM hierarchy.

The inequality (2.4) presents a topological lower bound which guarantees that finite action solutions to the Euler–Lagrange equations exist. Of particular interest are solutions to first order self-duality equations which solve the second order Euler–Lagrange equations, when (2.4) can be saturated.

For $\mathbf{M}^{2n} = \mathbb{R}^{2n}$, the self-duality equations support nontrivial solutions only if $q = p$:

$$\mathcal{F}(2p) = {}^\star\mathcal{F}(2p). \tag{2.5}$$

For $p = 1$, i.e. in four Euclidean dimensions, (2.5) is the usual YM self-duality equation supporting instanton solutions. Of these, the spherically symmetric [8, 12] and axially symmetric [23–25] instantons on $\mathbb{R}^{4p}$ are known. For $p \geqslant 2$, i.e. in dimensions eight and higher, only spherically symmetric [8] and axially symmetric [24, 25] solutions can be constructed, because in these dimensions (2.5) are overdetermined [27].

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[3] We call this a Hamiltonian rather than a Lagrangian because by definition it is positive definite. The corresponding Lagrangian with a given Minkowskian signature can then be defined systematically.

[4] The canonical definition is $\mathcal{H}_p = \frac{1}{(2p)!^2}\,\text{Tr}\,\mathcal{F}(2p)^2$, but here we use a simpler, unconventional, normalization for convenience.

In the large r asymptotic region, all these 'instanton' fields on $\mathbf{R}^{2n}$, whether self-dual or not, asymptotically behave as pure-gauge

$$\mathcal{A} \rightarrow g^{-1}\, \mathrm{d}g.$$

For $\mathbf{M}^{2n} = G/H$, namely on compact coset spaces, the self-duality equations support nontrivial solutions for all p and q:

$$\mathcal{F}(2p) = \kappa \,{}^{\star}\mathcal{F}(2q), \tag{2.6}$$

where the constant κ is some power of the 'radius' of the (compact) space. The simplest examples are $\mathbf{M}^{2n} = S^{2n}$, the $2n$-spheres [28–31], and $\mathbf{M}^{2n} = \mathbf{CP}^{n}$, the complex projective spaces [32, 33].

Gravitating members of the YM hierarchy [34] were studied and applicated to dynamical compactification in [35, 36]. (In [36] in particular, the dimensional descent from ten-dimensional spacetime yielding a YMH system was considered, but unlike in the present notes this was not done with a view to the construction of solitons.)

The above definitions of the YM systems can be formally extended to all dimensions, including all odd dimensions. The only difference this makes is that in odd dimensions, topological lower bounds enabling the construction of instantons are lost since CP charges are defined only on even dimensions.

Incidentally, on the subject of solutions to higher-dimensional YM systems consisting of higher order curvature terms, one might mention in passing that Meron [37] solutions in all even dimensions [38] can also be constructed.

3. Higgs models on $\mathbf{R}^{D}$

Higgs fields have the same dimensions as gauge connections and appear as the extra components of the latter under dimensional reduction, when the extra dimension is a compact symmetric space. The dimensional reduction of gauge fields over a compact codimension is implemented by the imposition of the symmetry of the compact coset space on the coordinates of the codimensions. In this respect, the calculus of dimensional reduction does not differ from that of imposition of symmetries generally.

The calculus of imposition of symmetry on gauge fields that has been used in the works being reviewed here is that of Schwarz [39–41]. This formalism was adapted to the dimensional reduction over arbitrary codimensions in [44–46]. An alternative formalism for the dimensional reduction of gauge fields [42] is familiar in the literature, but the calculus of [39] was found to be more convenient for extension to codimensions of arbitrary dimensions.

In the following, we will denote the codimension-N by the index $I = 1, 2, \ldots, N$ and the residual dimension-D by the index $i = 1, 2, \ldots, D$. The bulk dimension-$(D + N) = 2n$, which we will take to be even, permits the definition of the bulk nth CP density $\mathcal{C}^{(n)}$. The dimensional descent of the densities $\mathcal{C}^{(n)}$ is the first task of these notes. This amounts to the imposition of symmetry appropriate to the codimension, followed by the integration over this codimension. Integrating inequality (2.4) over this compact volume

$$\int_{\mathbb{R}^{D} \times K^{2(p+q)-D}} \mathrm{Tr}[\mathcal{F}(2p)^{2} + \kappa^{2}\,\mathcal{F}(2q)^{2}] \geqslant 2\kappa \int_{\mathbb{R}^{D} \times K^{2(p+q)-D}} \mathcal{C}^{(n)} \tag{3.1}$$

results in the reduced YMH energy density functional on the left-hand side, and on the right-hand side the required residual CP density. In (3.1), $\mathcal{F}(2p)$ is the $2p$-form curvature of the 1-form connection $\mathcal{A}$ on the higher-dimensional space $\mathbf{R}^{D} \times K^{2(p+q)-D}$.

Imposing the symmetry appropriate to $K^{2(p+q)-D}$ on the gauge fields results in the breaking of the original gauge group to the residual gauge group g for the fields on $\mathbf{R}^{D}$. This residual

gauge group g depends on the precise mode of dimensional reduction, namely on the choice of the codimension $K^{2(p+q)-D}$. The details will be specified below in section 4. Performing then the integration over the compact space K^{4p-D} leads to the (static) Hamiltonian $\mathcal{H}[A, \phi]$ of the residual YMH model on $\mathbf{R}^D$. In (3.1), the residual connection and its curvature are denoted by (A, F) taking their values in the algebra of the residual gauge group g, and ϕ is the Higgs multiplet whose structure under g depends on the detailed choice of $K^{2(p+q)-D}$, implying the following gauge transformations:

$$A \rightarrow g^{-1}Ag + g^{-1}\,\mathrm{d}g$$

and depending on the choice of K^{4p-D},

$$\phi \rightarrow g\phi\,g^{-1} \qquad \text{or} \qquad \phi \rightarrow g\phi, \qquad \text{etc.}$$

The inequality (3.1) leads to

$$\int_{\mathbb{R}^D} \mathcal{H}[A, \phi] \geqslant \int_{\mathbb{R}^D} \mathcal{C}_D^{(n)} = \int_{\mathbb{R}^D} \nabla \cdot \mathbf{\Omega}^{(n,D)}[A, \phi] = \int_{\Sigma^{D-1}} \mathbf{\Omega}^{(n,D)}[A, \phi], \quad (3.2)$$

where $\mathcal{H}[A, \phi] = \mathcal{H}[F, D\phi, |\phi|^2, \eta^2]$ is the residual Hamiltonian in terms of the residual gauge connection A and its curvature F, the Higgs fields ϕ and its covariant derivative $D\phi$ and the inverse of the compactification 'radius' η. The latter is simply the VEV of the Higgs field, seen clearly from the typical form of the components of the curvature F on the extra (compact) space K^{4p-D}:

$$F|_{K^{2(p+q)-D}} \sim (\eta^2 - |\phi|^2) \otimes \Lambda^{(N)} \quad \Rightarrow \quad \lim_{r\rightarrow\infty} |\phi|^2 = \eta^2, \qquad (3.3)$$

where $\Lambda^{(N)}$ are, symbolically, spin-matrices/Clebsch–Gordan coefficients. Specifically, $\Lambda^{(N)}$ are representation matrices of the stability group of the symmetry group of K^N. The precise definition of these matrices will be given below in section 4 for the special cases considered here, namely for $K^{2(p+q)p-D} = S^{2(p+q)p-D}$.

While the precise overall numerical factor in the density $\mathbf{\Omega}^{(n,D)}[A, \phi]$ in (3.2) can be evaluated, in practice these quantities will be evaluated without regard to such factors, since in each case we will normalize the monopole charge by requiring that in the spherically symmetric case this be the *unit* charge.

Of course, the simplest choice for these (topological) inequalities is when $p = q$, i.e. when $D + N = 4p$, in which cases these inequalities lead to Bogomol'nyi equations that do not feature a dimensional constant κ. This is the case with the familiar examples of $(D = 3, N = 1)$ for the BPS monopole and the $(D = 2, N = 2)$ of the critical Abelian Higgs vortices, which are *self-dual* solutions. It should be noted at this stage that subjecting the self-duality equations (2.5) for $p = q$ to this dimensional descent results in Bogomol'nyi equations on $\mathbb{R}^D$, which for $p \geqslant 2$ in $\mathbb{R}^D \times K^{4p-D}$ (cf (3.1)) turn out to be overdetermined [27] with only two exceptions, these being when $D = 4p - 1$ and $D = 2$.

The residual gauge connection here is defined on the Euclidean space $\mathbb{R}^D$, so we will refer to it as the *magnetic* component. Subsequently, we will also define the *electric* component, with reference to the Higgs multiplet ϕ.

3.1. Choice of codimension and residual gauge group

In the particular plan of dimensional descent of gauge fields pursued here, one is guided by the choice of the residual gauge group. Here, we are guided by the requirement that our residual gauge connections be described by DY [1–3] monopoles asymptotically. This leads immediately to the choice of $SO(D)$ for the residual gauge field on $\mathbb{R}^D$.

As stated at the outset, our explicit considerations are restricted to the codimensions $K^N = S^N$, i.e. to N-spheres. The symmetry group of S^N is $SO(N+1)$ with the stability group $SO(N)$, and it is the representation matrices of these stability groups which feature in Schwarz's calculus of dimensional reduction, as the latter exploits the symmetry imposition equations at a fixed point (say the north pole) of S^N. In particular, we have chosen to employ the gamma (Dirac) matrix representations of $SO(N)$, so there arises a distinction between *odd* and *even* N due to the existence of a chirality operator of the Clifford algebras in *even* dimensions. Thus, the concrete examples of $\Lambda^{(N)}$ in (3.3) now are

$$\Gamma_{IJ} = -\tfrac{1}{4}[\Gamma_I, \Gamma_J] \tag{3.4}$$

in terms of the N-dimensional (Dirac) gamma matrices Γ_I.

In what follows, we will use a uniform index notation to label the coordinate $x_M = (x_i, x_I)$, with x_i (lowercase Latin $i = 1, 2, \ldots, D$) the coordinate on the residual space $\mathbb{R}^D$ and with x_I (uppercase Latin $I = 1, 2, \ldots, N$) the coordinate on the codimension S^N. (We reserve the Greek letters $\mu, \nu, \ldots$ for the Minkowskian index $\mu = (0, i)$ for later use in describing dyons.)

Since the dimensionality of the bulk is $2n = D + N$, this distinction between *odd* and *even* N will be reflected in distinct features of the residual D-dimensional fields for *odd* and *even* D, respectively.

The gamma matrices, Γ_I, used to represent the $SO(N)$ algebra are

$$\text{for even } N, \qquad 2^{\frac{N}{2}} \times 2^{\frac{N}{2}} \text{ complex-valued arrays and}$$

$$\text{for odd } N, \qquad 2^{\frac{N-1}{2}} \times 2^{\frac{N-1}{2}} \text{ complex-valued arrays.}$$

But in the fixed-point calculus of Schwarz, the bulk gauge field $\mathcal{A}$ is a direct product of the residual gauge field A times an element of the N-dimensional Clifford algebra. Thus, if we choose the bulk gauge connections $\mathcal{A}$, irrespective of whether N is odd or even, to be $2^{(n-1)} \times 2^{(n-1)}$ anti-Hermitian arrays, then the residual gauge connections A will be

$$\text{for even } N, \qquad 2^{\frac{D}{2}-1} \times 2^{\frac{D}{2}-1} \text{ complex-valued arrays and}$$

$$\text{for odd } N, \qquad 2^{\frac{D-1}{2}} \times 2^{\frac{D-1}{2}} \text{ complex-valued arrays.}$$

Let us consider first the case of odd N (and hence odd D). Here, the residual connection A is a $2^{\frac{D-1}{2}} \times 2^{\frac{D-1}{2}}$ anti-Hermitian matrix (not necessarily traceless). This allows for the option of choosing the residual gauge group to be $SO(D)$,[5] such that A takes its values in one or the other of the two chiral (Dirac) matrix representations of the algebra of $SO(D)$. With this choice of residual gauge group, the asymptotic configurations of the residual YM connections are described by DY [1–3] monopoles, as required.

The case of even N (and hence even D) is more subtle and restrictive. Here, the residual connection A consists of two $2^{\frac{D}{2}-1} \times 2^{\frac{D}{2}-1}$ complex-valued arrays, each being anti-Hermitian. However, as we will see below in the explicit-dimensional reduction equations, these two matrices are 'doubled up' *via* the chiral operator Γ_{D+1}. The resulting residual connection A is the $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$ direct sum of the two (*left* and *right*) chiral components. This allows one to ascribe to A the residual gauge group $SO(D)$, as in the case of odd D, but now with A in the chirally symmetric (Dirac) matrix representation of $SO(D)$. Again, in the case of even D, the (restricted) choice of $SO(D)$ is the natural one, describing asymptotically a DY monopole.

Since the options exercised above are specific to odd, resp. even, dimensions, these two cases will be treated separately when the concrete-dimensional descent is presented. However, the scheme just described presents a unified framework for both odd and even D, with $SO(D)$ being the residual gauge group for both.

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[5] Had we chosen a larger rank gauge connection in the bulk, we would have ended up with a larger residual gauge group, which however would include the *convenient* gauge group $SO(D)$.

The above-described prescription for dimensional descent pertains to the $\mathcal{A}_i(x_i, x_I)$ components of the bulk gauge fields, resulting in $SO(D)$ residual gauge fields A on $\mathbb{R}^D$. The descent of the components $\mathcal{A}_I(x_i, x_I)$ in turn result in Higgs multiplets in which, as is well known [9, 47], the topology of the monopole is encoded. Again the guiding feature is that of having the asymptotic connection on $\mathbb{R}^D$ described by a DY [1–3] monopole, which for the asymptotic Higgs field means gauging it away to the constant (trivial) configuration oriented along the x_D-axis, resulting in the vanishing of its covariant derivative. There will be a Dirac line singularity along the positive or negative x_D-axis, which is a gauge artefact. As will be seen in the next section below, there is a striking difference in the odd and even D cases. In both cases, the Higgs field Φ is *not* restricted to take its values in the *algebra* of $SO(D)$ (except in the special case of $D = 3$). In odd, (resp.) even D, Φ consists of a $2^{\frac{D-1}{2}} \times 2^{\frac{D-1}{2}}$, (resp.) $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$, complex-valued array. As such it can be described by elements of the *left* or *right* chiral (resp.), chirally symmetric *left* plus *right* (resp.), representations of the $SO(D+1)$ algebra for odd and even D. A unified expression for the asymptotic Higgs field, tailored to present a DY field, is

$$\Phi|_{r\to\infty} \simeq \hat{x}_i \, \Sigma_{i,D+1} \tag{3.5}$$

$$\Phi|_{r\to\infty} \simeq \hat{x}_i \, \Gamma_{i,D+1}, \tag{3.6}$$

where $\hat{x}_i$ is the unit radius vector in $\mathbb{R}^D$ and Σ_{ab} are the *left* or *right* $SO(D)$ chiral representation matrices

$$\Sigma_{ab}^{(\pm)} = -\frac{1}{4}\left(\frac{1 \pm \Gamma_{D+1}}{2}\right)\Gamma_{ab}, \qquad a, b = 1, 2, \ldots, D+1. \tag{3.7}$$

In (3.5)–(3.6), the spin matrices $\Sigma_{i,D+1}$ and $\Gamma_{i,D+1}$ are the orthogonal complements of the (Dirac) representations of $SO(D)$ in $SO(D+1)$. As such, they are tantamount to the description of the Higgs multiplet as a D-component isovector in both cases.

It should be stressed here that although the Higgs field is not restricted to take its values in the algebra of $SO(D)$, the gauge field does take its values in the algebra of $SO(D)$. Thus, the residual gauge group indeed remains $SO(D)$.

The above-described prescription of dimensional reduction is fairly general, within the context of the restrictions opted for. The exception, which we have eschewed here, is when the descent is over the codimension S^2, i.e. when $N = 2$. In this case, the symmetry group of S^2, namely $SO(3)$, has the stability group $SO(2)$. The latter, being Abelian, affords a very much richer family of solutions to the symmetry equations imposed for the descent [41], allowing for Higgs multiplets not *necessarily* restricted as above. The more general Higgs multiplets that result are superfluous for our purposes and are not considered here.

The YM field in the YMH models on R^D discussed thus far is purely magnetic supporting a 'magnetic' monopole. But when it comes to YMH models, as stated earlier, the presence of the Higgs field enables the support of the electric component of the YM connection A_0 and hence enables the description of a Julia–Zee-type dyon in $d = (D+1)$-dimensional spacetime. In the usual [10] sense as the dyon in $3 + 1$ dimensions, the Higgs field partners the newly introduced 'electric' YM potential A_0. Thus, we can describe $SO(d)$ dyons[6] in d-dimensional spacetime.

When the dimension of the spacetime $d = D + 1$ is even, then the gauge field $A_\mu = (A_i, A_0)$ and Higgs field Φ in the residual space are expanded in the basis of the chiral

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[6] While both the Higgs field and A_0 take their values in the Dirac matrix basis $\Gamma_{i,D+1}$, in the familiar (3+1)-dimensional case the 'enlarged' algebra $SO(4)$ splits into the two $SU(2)$ subalgebras, whence the *magnetic* component of the connection A_i and the *electric* component of the connection A_0 are both described by $SU(2)$ matrices. In all higher dimensions, this is not the case and the full algebra employed is $SO(D+1)$, where $d = D + 1$ is the dimension of the spacetime.

$SO(d)$ matrices $\left(\Sigma_{ij}^{(\pm)}, \Sigma_{i,d}^{(\pm)}\right)$, e.g. in $d = 3 + 1$ spacetime. By contrast, when the dimension of the spacetime d is odd, then the chirally symmetric $SO(d)$ matrices $\Gamma_{\mu\nu} = (\Gamma_{ij}, \Gamma_{i,d})$ are used in the same situation, e.g. in $d = 4 + 1$ spacetime [51] when $D = 4$.

In this enlarged context, namely that allowing the construction of dyons in addition to monopoles, the case of $D = 2$ is excluded, due to the well-known obstruction of the Julia–Zee theorem [10] to the existence of dyons in $d = 2 + 1$ spacetime.

4. Dimensional reduction of gauge fields

In this section, we present the formalism of the dimensional reduction of YM fields employed in the sections following it. We list the results of the calculus of dimensional reduction for the classes of descents considered in a unified notation. This calculus is based on the formalism of Schwarz [39], which is especially transparent due to the choice of displaying the results only at a fixed point of the compact symmetric codimensional space K^N (the north or south pole for S^N). Our formalism is a straightforward extension of [39–41].

The criterion of constructing monopoles that are asymptotically DY restricts our calculus to the framework described above. This does not include the restriction to the codimensions $K^N = S^N$, but we do this anyway for the sake of simplicity. (It is also the case that employing $K^N = CP^N$ [43], for example, does not lead to any qualitatively new results.)

4.1. Descent over S^N: N odd

For the descent from the bulk dimension $2n = D + N$ down to *odd D* (over odd N), the components of the residual connection evaluated at the north pole of S^N are given by

$$\mathcal{A}_i = A_i(\vec{x}) \otimes \mathbb{1} \tag{4.1}$$

$$\mathcal{A}_I = \Phi(\vec{x}) \otimes \tfrac{1}{2}\Gamma_I. \tag{4.2}$$

The unit matrix in (4.1), like the N-dimensional gamma matrix in (4.2), is $2^{\frac{1}{2}(N-1)} \times 2^{\frac{1}{2}(N-1)}$ arrays. Choosing the $2^{n-1} \times 2^{n-1}$ bulk gauge group to be, say, $SU(n-1)$ allows the choice of $SO(D)$ as the gauge group of the residual connection $A_i(x)$. This choice is made such that the asymptotic connections describe a DY monopole.

For the same reason, the choice for the multiplet structure of the Higgs field is made to be less restrictive. The (anti-Hermitian) field Φ, which is not necessarily traceless[7], can be and *is* taken to be in the algebra of $SO(D + 1)$, in particular, in one or the other of the chiral representations of $SO(D + 1)$, $D + 1$ here being even:

$$\Phi = \phi^{ab}\Sigma_{ab}, \qquad a = i, D+1, \qquad i = 1, 2, \ldots, D. \tag{4.3}$$

(Only in the $D = 3$ case does the Higgs field take its values in the algebra of $SO(3)$, since the representations $SO(3)$ coincide with those of chiral $SO(4)$.)

In anticipation of the corresponding situation of even D in the next subsection, one can specialize (4.3) to a D-component *isovector* expression of the Higgs field

$$\Phi = \phi^i \Sigma_{i,D+1}, \tag{4.4}$$

with the purpose of having a unified notation for both even and odd D, where the Higgs field takes its values in the components $\Sigma_{i,D+1}$ orthogonal to elements Σ_{ij} of the algebra of $SO(D + 1)$. This specialization is not necessary and is in fact inappropriate should one consider, e.g., axially symmetric fields. It is, however, adequate for the presentation here,

---

[7] In practice, when constructing soliton solutions, Φ is taken to be traceless without loss of generality.

being consistent with the asymptotic expressions (3.5), and is sufficiently general to describe spherically symmetric monopoles[8].

In (4.1) and (4.2), and everywhere henceforth, we have denoted the components of the residual coordinates as $x_i = \vec{x}$. The dependence on the codimension coordinate x_I is suppressed since all fields are evaluated at a fixed point (north or south pole) of the codimension space.

The resulting components of the curvature are

$$\mathcal{F}_{ij} = F_{ij}(\vec{x}) \otimes \mathbb{1} \tag{4.6}$$

$$\mathcal{F}_{iI} = D_i \Phi(\vec{x}) \otimes \tfrac{1}{2}\Gamma_I \tag{4.7}$$

$$\mathcal{F}_{IJ} = S(\vec{x}) \otimes \Gamma_{IJ}, \tag{4.8}$$

where $\Gamma_{IJ} = -\tfrac{1}{4}[\Gamma_I, \Gamma_J]$ are the Dirac representation matrices of $SO(N)$, the stability group of the symmetry group of the N-sphere. In (4.7), $D_i\Phi$ is the covariant derivative of the Higgs field Φ

$$D_i\Phi = \partial_i\Phi + [A_i, \Phi] \tag{4.9}$$

and S is the quantity

$$S = -(\eta^2\mathbb{1} + \Phi^2), \tag{4.10}$$

where η is the inverse of the radius of the N-sphere.

4.2. Descent over S^N: N even

The formulae corresponding to (4.1)–(4.8) for the case of *even D* are somewhat more complex. The reason is the existence of a chiral matrix Γ_{N+1}, in addition to the Dirac matrices Γ_I, $I = 1, 2, \ldots, N$. Instead of (4.1)–(4.2), we now have

$$\mathcal{A}_i = A_i(\vec{x}) \otimes \mathbb{1} + B_i(\vec{x}) \otimes \Gamma_{N+1}$$
$$\mathcal{A}_I = \phi(\vec{x}) \otimes \tfrac{1}{2}\Gamma_I + \psi(\vec{x}) \otimes \tfrac{1}{2}\Gamma_{N+1}\Gamma_I,$$

where A_i, B_i, ϕ and ψ are again anti-Hermitian matrices, but with only A_i being traceless. The fact that B_i is not traceless here results in an Abelian gauge field in the reduced system.

Anticipating what follows, it is much more transparent to re-express these formulae in the form

$$\mathcal{A}_i = A_i^{(+)}(\vec{x}) \otimes P_+ + A_i^{(-)}(\vec{x}) \otimes P_- + \frac{i}{2}a_i(\vec{x})\Gamma_{N+1} \tag{4.11}$$

$$\mathcal{A}_I = \varphi(\vec{x}) \otimes \frac{1}{2}P_+\Gamma_I - \varphi(\vec{x})^\dagger \otimes \frac{1}{2}P_-\Gamma_I, \tag{4.12}$$

where now $P_\pm$ are the $2^{\frac{N}{2}} \times 2^{\frac{N}{2}}$ projection operators

$$P_\pm = \tfrac{1}{2}(\mathbb{1} \pm \Gamma_{N+1}). \tag{4.13}$$

In (4.11), the residual gauge connections $A_i^{(\pm)}$ are anti-Hermitian and traceless $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$ arrays, and the Abelian connection a_i results directly from the trace of the field B_i. The $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$

---

[8] While all concrete considerations in the following are restricted to spherically symmetric fields, it should be emphasized that relaxing spherical symmetry results in the Higgs multiplet getting out of the orthogonal complement $\Sigma_{i,D+1}$ to $\Sigma_{i,j}$. Indeed, subject to axial symmetry one has

$$\Phi = f_1(\rho, z)\Sigma_{\alpha\beta}\hat{x}_\beta + f_2(\rho, z)\Sigma_{\beta,D+1}\hat{x}_\beta + f_3(\rho, z)\Sigma_{D,D+1}, \tag{4.5}$$

where $x_i = (x_\alpha, z)$, $|x_\alpha|^2 = \rho^2$ and with $\hat{x}_\alpha = x_\alpha/\rho$. Clearly, the term in (4.5) multiplying the basis $\Sigma_{\alpha\beta}$ does not occur in (4.4).

'Higgs' field φ in (4.12) is neither Hermitian nor anti-Hermitian. Again, to achieve the desired breaking of the gauge group, to lead eventually to the requisite Higgs *isomultiplet*, we choose the gauge group in the bulk to be $SU(n-1)$, where $2n = D + N$.

The components of the curvature are readily calculated to give

$$\mathcal{F}_{ij} = F_{ij}^{(+)}(\vec{x}) \otimes P_+ + F_{ij}^{(-)}(\vec{x}) \otimes P_- + \frac{\mathrm{i}}{2} f_{ij}(\vec{x}) \Gamma_{N+1} \tag{4.14}$$

$$\mathcal{F}_{iI} = D_i \varphi(\vec{x}) \otimes \frac{1}{2} P_+ \Gamma_I - D_i \varphi^\dagger(\vec{x}) \otimes \frac{1}{2} P_- \Gamma_I \tag{4.15}$$

$$\mathcal{F}_{IJ} = S^{(+)}(\vec{x}) \otimes P_+ \Gamma_{IJ} + S^{(-)}(\vec{x}) \otimes P_- \Gamma_{IJ}, \tag{4.16}$$

the curvatures in (4.14) being defined by

$$F_{ij}^{(\pm)} = \partial_i A_j^{(\pm)} - \partial_j A_i^{(\pm)} + \left[A_i^{(\pm)}, A_j^{(\pm)} \right] \tag{4.17}$$

$$f_{ij} = \partial_i a_j - \partial_j a_i. \tag{4.18}$$

The covariant derivative in (4.15) is now defined as

$$D_i \varphi = \partial_i \varphi + A_i^{(+)} \varphi - \varphi A_i^{(-)} + \mathrm{i}\, a_i\, \varphi \tag{4.19}$$

$$D_i \varphi^\dagger = \partial_i \varphi^\dagger + A_i^{(-)} \varphi^\dagger - \varphi^\dagger A_i^{(+)} - \mathrm{i}\, a_i\, \varphi^\dagger, \tag{4.20}$$

and the quantities $S^{(\pm)}$ in (4.16) are

$$S^{(+)} = \varphi \varphi^\dagger - \eta^2, \qquad S^{(-)} = \varphi^\dagger \varphi - \eta^2. \tag{4.21}$$

In what follows, we will suppress the Abelian field a_i, since only when less stringent symmetry than spherical is imposed is it that it would contribute. In any case, using the formal replacement

$$A_i^{(\pm)} \leftrightarrow A_i^{(\pm)} \pm \frac{\mathrm{i}}{2} a_i \mathbb{1}$$

yields the algebraic results to be derived below, in the general case.

We now refine our calculus of descent over even codimensions further. We see from (4.11) that $A_i^{(\pm)}$ being $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$ arrays, they can take their values in the two chiral representations, respectively, of the algebra of $SO(D)$. It is therefore natural to introduce the full $SO(D)$ connection

$$A_i = \begin{bmatrix} A_i^{(+)} & 0 \\ 0 & A_\mu^{(-)} \end{bmatrix}. \tag{4.22}$$

Next, we define the D-component *isovector* Higgs field

$$\Phi = \begin{bmatrix} 0 & \varphi \\ -\varphi^\dagger & 0 \end{bmatrix} = \phi^i \Gamma_{i,D+1} \tag{4.23}$$

in terms of the Dirac matrix representation of the algebra of $SO(D+1)$ with $\Gamma_{i,D+1} = -\frac{1}{2} \Gamma_{D+1} \Gamma_i$.

Note here the formal equivalence between the Higgs multiplet (4.23) in even D to the corresponding one (4.4) in odd D. This formal equivalence turns out to be very useful in the calculus employed in the following sections. In contrast with the former case of odd D, however, the form (4.23) for even D is much more restrictive. This is because in this case the Higgs multiplet is restricted to take its values in the components $\Gamma_{i,D+1}$ orthogonal to the elements Γ_{ij} of $SO(D)$ by definition, irrespective of what symmetry is imposed. Referring

to footnote 8, it is clear that relaxing the spherical symmetry here does not result in Φ getting out of the orthogonal complement of Γ_{ij}, when D is even.

From (4.22), follows the $SO(D)$ curvature

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] = \begin{bmatrix} F_{ij}^{(+)} & 0 \\ 0 & F_{ij}^{(-)} \end{bmatrix} \qquad (4.24)$$

and from (4.22) and (4.23) follows the covariant derivative

$$D_\mu \Phi = \partial_i \Phi + [A_i, \Phi] = \begin{bmatrix} 0 & D_i\varphi \\ -D_i\varphi^\dagger & 0 \end{bmatrix}. \qquad (4.25)$$

From (4.23) there simply follows the definition of S for even D:

$$S = -(\eta^2 \mathbb{1} + \Phi^2) = \begin{bmatrix} S^{(+)} & 0 \\ 0 & S^{(-)} \end{bmatrix}. \qquad (4.26)$$

Note 1. Given that $D + N$ is even, an Abelian gauge field Higgs system with $SO(D) = SO(2)$ results only from descents over even-dimensional codimensions. The finite energy solutions in these cases are Abelian *vortices*, which are qualitatively different from all other cases describing *monopoles*. Unlike monpoles, which asymptotically are (singular) DY monopoles, vortices do not support an asymptotic curvature. Likewise, models supporting vortices cannot be adapted to support Julia–Zee-type dyons. Another such difference between monopoles and vortices is that in the former case the boundary conditions (on the large sphere—not circle!) can be adjusted to support monopole–antimonopole pairs and chains, while this option is not open for the latter case with an asymptotic circle.

The dimensional reduction formulae for $D = 2$ can in principle be read from the even-descent formulae (4.11)–(4.26), but it is convenient to display them distinctly in a more transparent notation, which is given in section 4.2.1 below.

Note 2. Descent by one dimension only is special and the calculus involved is rather trivial. In that case, the symmetry group of the YM field in the bulk does not break. In the (odd) $N = 1$ case, the matrices in (4.1)–(4.2) contract to real numbers; hence, (A_i, A_I) take their values in the algebra of the bulk gauge group like $(\mathcal{A}_i, \mathcal{A}_I)$.

In addition, the components of the bulk gauge curvature on the codimension space, (4.8), vanishes and the residual YMH model does not feature a symmetry-breaking Higgs potential.

Note 3. In the extreme case where $D = 1$, the residual connection $\mathcal{A}_{\rangle}$ in (4.1) has only one component and hence its curvature vanishes. It follows that this connection is gauge equivalent to zero; hence, one ends up with a residual system described by a scalar field Φ in (4.2). The covariant derivative in (4.7) then becomes a partial derivative, and the components (4.8) on the codimension lead to symmetry-breaking potentials. The resulting systems are more nonlinear versions of the ϕ^4 model on $\mathbb{R}^1$. These will henceforth be ignored.

4.2.1. Descent over S^N: N even and $D = 2$. Dimensional descendants of the connection $(\mathcal{A}_i, \mathcal{A}_I)$ on $\mathbb{R}^2 \times S^N$, expressed at the north pole of S^N are given by

$$\mathcal{A}_i = \frac{\mathrm{i}}{2} A_i(\vec{x}) \Gamma_{N+1} \qquad (4.27)$$

$$\mathcal{A}_I = \frac{\mathrm{i}}{2} [\varphi(\vec{x}) \, P_+ \Gamma_I + \varphi^\star(\vec{x}) \, P_- \Gamma_I], \qquad (4.28)$$

where A_i is now the residual Abelian gauge field and φ is a complex-valued scalar field.

The components of the curvature are readily calculated to give

$$\mathcal{F}_{ij} = \frac{\mathrm{i}}{2} F_{ij}(\vec{x}) \Gamma_{N+1} \tag{4.29}$$

$$\mathcal{F}_{iI} = \frac{\mathrm{i}}{2}[D_i\varphi(\vec{x})\ P_+\Gamma_I + D_i\varphi^*(\vec{x})\ P_-\Gamma_I] \tag{4.30}$$

$$\mathcal{F}_{IJ} = S(\vec{x})\,\Gamma_{IJ} \tag{4.31}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ is the residual Abelian curvature, the covariant derivative $D_I\varphi$ is

$$D_i\varphi = \partial_i\varphi + \mathrm{i}A_i\varphi \tag{4.32}$$

and S now is

$$S = -\eta^2 + |\varphi|^2. \tag{4.33}$$

5. Dimensional reduction of CP densities

The crucial step in producing a YMH model on $\mathbb{R}^D$ that can support finite energy topologically stable monopoles, i.e. that one can establish a topological lower bound on the energy, is to find the monopole charge density which is a dimensional descendent of the relevant CP density. In other words, it is crucial to show that the density on the right-hand side of (3.2) is a total divergence. Denoting the CP density on $\mathbb{R}^D$ by $\mathcal{C}_D^{(n)}$ descended from $\mathcal{C}^{(n)}$ on $\mathbb{R}^D \times S^{2n-D}$, the result to be demonstrated is

$$\mathcal{C}_D^{(n)} = \nabla \cdot \mathbf{\Omega}^{(n,D)}. \tag{5.1}$$

The actual proof that the residual CP density is a total divergence proceeds directly by taking arbitrary variations δA_i and $\delta\Phi$ of it, and to show that these lead to *trivial* variational equations. On the other hand, should one wish to exploit the densities $\mathbf{\Omega}^{(n,D)}$ in the role of CS densities in a $(D-1, 1)$ Minkowskian theory, then one would need the explicit expressions of $\Omega[A_\mu, \Phi]$. This is the task carried out in the present section.

For practical reasons we restrict ourselves to the second, third and fourth CP densities. These examples illustrate how to systematically extend the analysis to higher order CP densities relevant to higher dimensions. In each case, we will consider all possible descents, yielding the topological charges of the YMH models on the residual $\mathbb{R}^D$. Thus, from the second CP density one arrives at the vortex number of the Abelian Higgs model [6, 7] on $\mathbb{R}^2$ and the ('t Hooft–Polyakov) monopole charge of the Georgi–Glashow model on $\mathbb{R}^3$. From the third CP density one arrives at the topological charges of (one of the) generalized Abelian Higgs models [57, 58] on $\mathbb{R}^2$, and the monopole charges of the generalized YMH models on $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^5$ descended from the bulk YM model

$$\mathcal{H} = \mathrm{Tr}\,\mathcal{F}(2)^2 + \kappa^2\,\mathrm{Tr}\,\mathcal{F}(4)^2. \tag{5.2}$$

From the fourth CP density one arrives at the topological charges of (one of the other) generalized Abelian Higgs models [57, 58] on $\mathbb{R}^2$ and, the monopole charges of generalized YMH models on $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$, $\mathbb{R}^6$ and $\mathbb{R}^7$ descended from the two bulk YM models

$$\mathcal{H}_1 = \mathrm{Tr}\,\mathcal{F}(4)^2, \tag{5.3}$$

$$\mathcal{H}_2 = \mathrm{Tr}\,\mathcal{F}(2)^2 + \kappa^2\,\mathrm{Tr}\,\mathcal{F}(6)^2. \tag{5.4}$$

There is an important distinction between the 'vortex numbers' and 'monopole charges' stated above. The former are the topological charges of Abelian gauge field systems on R^2; hence, the curvature on the large circle has no curvature so that there exists no DY [3] fields asymptotically. By contrast, all the non-Abelian gauge field systems on R^D, $D \geqslant 3$, do have asymptotic DY fields, with the important consequence that the asymptotic gauge connection is *half-pure gauge*, i.e. decays as r^{-1}, typical of monopoles rather than instantons.

5.1. Topological densities on $\mathbb{R}^D$ from the second CP density $\mathcal{C}^{(2)}$

In this case $D + N = 4$ and hence there are only two possible descents, with $N = 1$ and 2 (to $D = 3$ and 2, respectively), since we exclude the descent to $D = 1$.

5.1.1. $N = 1$, $D = 3$. Using (4.6) and (4.7), one has

$$
\begin{aligned}
\mathcal{C}_3^{(2)} &= \varepsilon_{ijk4} \operatorname{Tr} \mathcal{F}_{ij} \mathcal{F}_{k4} \\
&\equiv \varepsilon_{ijk} \operatorname{Tr} F_{ij} D_k \Phi \\
&\overset{\text{def.}}{=} \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}^{(2,3)}
\end{aligned}
$$

which is manifestly a total derivative defining the CS density

$$
\Omega_k^{(2,3)} = \varepsilon_{ijk} \operatorname{Tr} F_{ij} \Phi. \tag{5.5}
$$

5.1.2. $N = 2$, $D = 2$. The CP term

$$
\tfrac{1}{2} \mathcal{C}_2^{(2)} = \varepsilon_{ij} \varepsilon_{IJ} \operatorname{Tr}(\mathcal{F}_{ij} \mathcal{F}_{IJ} - 2 \mathcal{F}_{iI} \mathcal{F}_{jJ}) \tag{5.6}
$$

is subjected to dimensional reduction using the symmetry constraints (4.29), (4.30) and (4.31), and performing the traces over the codimension indices I, J.

The result is

$$
\begin{aligned}
\tfrac{1}{2} \mathcal{C}_2^{(2)} &= \varepsilon_{ij} (S F_{ij} - 2i D_i \varphi^* D_j \varphi) \\
&\overset{\text{def.}}{=} \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}^{(2,2)}
\end{aligned} \tag{5.7}
$$

which is manifestly a total divergence defining the CS density

$$
\Omega_i^{(2,2)} \simeq \varepsilon_{ij} (\eta^2 A_j + i \varphi^* D_j \varphi). \tag{5.8}
$$

5.2. Topological densities on $\mathbb{R}^D$ from the third CP density $\mathcal{C}^{(3)}$

In this case, $D + N = 6$ and hence there are four possible descents, with $N = 1, 2, 3$ and 4 (to $D = 5, 4, 3$ and 2, respectively).

5.2.1. $N = 1$, $D = 5$. Using (4.6) and (4.7), one has

$$
\begin{aligned}
\mathcal{C}_5^{(3)} &= \varepsilon_{ijklm6} \operatorname{Tr} \mathcal{F}_{ijkl} \mathcal{F}_{m6} \\
&\equiv \varepsilon_{ijklm} \operatorname{Tr} F_{ij} F_{kl} D_m \Phi \\
&\overset{\text{def.}}{=} 3! \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}^{(3,5)},
\end{aligned} \tag{5.9}
$$

which is manifestly a total derivative defining the CS density

$$
\Omega_m^{(3,5)} = \varepsilon_{ijklm} \operatorname{Tr} F_{ij} F_{kl} \Phi. \tag{5.10}
$$

5.2.2. $N = 2$, $D = 4$. The CP term

$$
\begin{aligned}
\mathcal{C}_4^{(3)} &= \varepsilon_{ijkl} \varepsilon_{IJ} \operatorname{Tr}(\mathcal{F}_{ijkl} \mathcal{F}_{IJ} - 8 \mathcal{F}_{ijkI} \mathcal{F}_{lJ} + 6 \mathcal{F}_{ijIJ} \mathcal{F}_{kl}) \\
&= 18 \, \varepsilon_{ijkl} \, \varepsilon_{IJ} \operatorname{Tr}(\mathcal{F}_{ij} \mathcal{F}_{kl} \mathcal{F}_{IJ} - 4 \mathcal{F}_{iI} \mathcal{F}_{jJ} \mathcal{F}_{kl})
\end{aligned} \tag{5.11}
$$

is subjected to dimensional reduction using the symmetry constraints (4.14), (4.15) and (4.16), and performing the traces over the codimension indices I, J.

The result can be recast in transparent form by further using (4.22)–(4.23) and (4.24), (4.25) and (4.26). This leads to the compact[9] expression

$$\mathcal{C}_4^{(3)} = 18\varepsilon_{ijkl}\operatorname{Tr}\Gamma_5(SF_{ij}F_{kl} + 2D_i\Phi D_j\Phi F_{kl})$$
$$\stackrel{\text{def.}}{=} \boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(3,4)}. \tag{5.12}$$

From definition (4.26) of S, it is clear that the leading Higgs independent term

$$\eta^2\varepsilon_{ijkl}F_{ij}F_{kl} = \eta^2\boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(4)} \tag{5.13}$$

is manifestly a total divergence in terms of the usual CS density of the pure YM field on $\mathbb{R}^4$

$$\Omega_i^{(4)} = \varepsilon_{ijkl}\operatorname{Tr}\Gamma_5 A_j\left[F_{kl} - \tfrac{2}{3}A_k A_l\right]. \tag{5.14}$$

The rest of the (Higgs-dependent) terms in (5.12) can also be shown to be a total divergence, such that

$$\Omega_i^{(3,4)} = 3!^2\varepsilon_{ijkl}\operatorname{Tr}\Gamma_5\left[-2\eta^2 A_j\left(F_{kl} - \tfrac{2}{3}A_k A_l\right) + (\Phi D_j\Phi - D_j\Phi\Phi)F_{kl}\right]. \tag{5.15}$$

5.2.3. $N = 3$, $D = 3$. The CP density

$$\begin{aligned}\tfrac{1}{4}\mathcal{C}_3^{(3)} &= \varepsilon_{ijk}\varepsilon_{IJK}\operatorname{Tr}(\mathcal{F}_{iIjk}\mathcal{F}_{JK} + \mathcal{F}_{iIJK}\mathcal{F}_{jk} + 3\mathcal{F}_{jkJK}\mathcal{F}_{iI}) \\ &= 3\varepsilon_{ijk}\varepsilon_{IJK}\operatorname{Tr}(3\mathcal{F}_{jk}\mathcal{F}_{iI}\mathcal{F}_{JK} + 3\mathcal{F}_{jk}\mathcal{F}_{JK}\mathcal{F}_{iI} - 4\mathcal{F}_{iI}\mathcal{F}_{jJ}\mathcal{F}_{kK})\end{aligned} \tag{5.16}$$

is subjected to the dimensional reduction using the symmetry constraints (4.6), (4.7) and (4.8).

This results directly in

$$\mathcal{C}_3^{(3)} = 3!^2\varepsilon_{ijk}\operatorname{Tr}[3F_{jk}(D_i\Phi S + SD_i\Phi) + 2D_i\Phi D_j\Phi D_k\Phi]$$
$$\stackrel{\text{def.}}{=} \boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(4,3)} \tag{5.17}$$

which is manifestly a total divergence defining the CS density

$$\Omega_k^{(3,3)} = (18)^2\varepsilon_{ijk}\operatorname{Tr}[-(3\eta^2 - \Phi^2)\Phi F_{ij} + \Phi D_i\Phi D_j\Phi]. \tag{5.18}$$

The calculus described in this subsection was first developed in [43].

5.2.4. $N = 4$, $D = 2$. This example was first considered in [44]. The CP term

$$\begin{aligned}\tfrac{1}{8}\mathcal{C}_2^{(3)} &= \varepsilon_{ij}\varepsilon_{IJKL}\operatorname{Tr}(6\mathcal{F}_{ijIJ}\mathcal{F}_{KL} - 8\mathcal{F}_{iIKL}\mathcal{F}_{jJ} + \mathcal{F}_{IJKL}\mathcal{F}_{ij}) \\ &= 18\varepsilon_{ij}\varepsilon_{IJKL}\operatorname{Tr}(\mathcal{F}_{ij}\mathcal{F}_{IJ}\mathcal{F}_{KL} - 4\mathcal{F}_{iI}\mathcal{F}_{jJ}\mathcal{F}_{KL})\end{aligned} \tag{5.19}$$

is subjected to the dimensional reduction using the symmetry constraints (4.29), (4.30) and (4.31), and performing the traces over the codimension indices I, J, K, L.

The result is

$$\tfrac{1}{8}\mathcal{C}_2^{(3)} = 9\,\mathrm{i}\varepsilon_{ij}(S^2 F_{ij} - 4\,\mathrm{i}SD_i\varphi^* D_j\varphi)$$
$$\stackrel{\text{def.}}{=} \boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(3,2)} \tag{5.20}$$

which is manifestly a total divergence defining the CS density

$$\Omega_i^{(3,2)} \simeq \varepsilon_{ij}[\eta^4 A_j - \mathrm{i}(2\eta^2 - |\varphi|^2)\,\varphi^* D_j\varphi]. \tag{5.21}$$

---

[9] In the ensuing manipulations, note that A_i and F_{ij} commute with Γ_5, while Φ and $D_i\Phi$ anticommute with it.

5.3. Topological densities on $\mathbb{R}^D$ from the fourth CP density $\mathcal{C}^{(4)}$

In this case $D + N = 8$ and hence there are six possible descents, with $N = 1, 2, 3, 4, 5$ and 6 (to $D = 7, 6, 5, 4, 3$ and 2, respectively). For future convenience, namely when studying the Bogomol'nyi lower bounds on descendants of the energy density (5.3), we express the fourth CP density in the form

$$\mathcal{C}_D^{(4)} = \text{Tr } \mathcal{F}(4) \wedge \mathcal{F}(4)$$

rather than as $\text{Tr } \mathcal{F}(2) \wedge \mathcal{F}(2) \wedge \mathcal{F}(2) \wedge \mathcal{F}(2)$. This is just a practical option which enables the treatment of all the above-listed cases in a uniform manner.

5.3.1. $N = 1$, $D = 7$.   Using (4.6) and (4.7), one has

$$
\begin{aligned}
\mathcal{C}_7^{(4)} &= \varepsilon_{ijklmnp} \text{Tr } \mathcal{F}_{ij} \mathcal{F}_{kl} \mathcal{F}_{mn} \mathcal{F}_{p6} \\
&= \varepsilon_{ijklmnp} \text{Tr } F_{ij} F_{kl} F_{mn} D_p \Phi \\
&\overset{\text{def.}}{=} 3!^2 \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}^{(4,7)}
\end{aligned}
$$

which is manifestly a total derivative defining the CS density

$$\Omega_p^{(4,7)} = \varepsilon_{ijklmnp} \text{Tr } F_{ij} F_{kl} F_{mn} \Phi. \tag{5.22}$$

5.3.2. $N = 2$, $D = 6$.   The CP term

$$
\begin{aligned}
\tfrac{1}{4}\mathcal{C}_6^{(4)} &= \varepsilon_{ijklmn} \varepsilon_{IJ} \text{Tr}(3\mathcal{F}_{ijkl}\mathcal{F}_{mnIJ} - 4\mathcal{F}_{ijkI}\mathcal{F}_{lmnJ}) \\
&= 36\varepsilon_{ijklmn}\varepsilon_{IJ} \text{Tr}(\mathcal{F}_{ij}\mathcal{F}_{kl}\mathcal{F}_{mn}\mathcal{F}_{IJ} - 4\mathcal{F}_{ij}\mathcal{F}_{kl}\mathcal{F}_{mI}\mathcal{F}_{nJ} - 2\mathcal{F}_{ij}\mathcal{F}_{mI}\mathcal{F}_{kl}\mathcal{F}_{nJ})
\end{aligned} \tag{5.23}
$$

is subjected to the dimensional reduction using the symmetry constraints (4.14), (4.15) and (4.16), and performing the traces over the codimension indices I, J.

   The result can be recast in transparent form by further using (4.22)–(4.23) and (4.24), (4.25), (4.26). This leads to the compact expression (see footnote 9)

$$
\begin{aligned}
\frac{1}{144}\mathcal{C}_6^{(4)} &= \varepsilon_{ijklmn} \text{Tr}\Gamma_7[SF_{ij}F_{kl}F_{mn} + 2F_{ij}F_{kl}D_m\Phi D_n\Phi + F_{ij}D_m\Phi F_{kl}D_n\Phi] \\
&\overset{\text{def.}}{=} \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}^{(4,6)}.
\end{aligned} \tag{5.24}
$$

From definition (4.26) of S, it is clear that the leading Higgs-independent term

$$\eta^2 \varepsilon_{ijklmn} F_{ij} F_{kl} F_{mn} = \eta^2 \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}^{(6)} \tag{5.25}$$

is manifestly a total divergence in terms of the usual CS density of the pure YM field on $\mathbb{R}^6$:

$$\Omega_i^{(6)} = 2\varepsilon_{ijklmn} \text{Tr } \Gamma_7 A_j \left[F_{kl}F_{mn} - F_{kl}A_m A_n + \tfrac{2}{5} A_k A_l A_m A_n \right]. \tag{5.26}$$

The rest of the (Higgs-dependent) terms in (5.24) can also be shown to be a total divergence, such that

$$-\tfrac{1}{144}\Omega_i^{(4,6)} = \Omega_i^{(6)} + \varepsilon_{ijklmn} \text{Tr } \Gamma_7 D_j\Phi(\Phi F_{kl}F_{mn} + F_{kl}\Phi F_{mn} + F_{kl}F_{mn}\Phi). \tag{5.27}$$

5.3.3. $N = 3$, $D = 5$.   The CP density

$$
\begin{aligned}
\tfrac{1}{8}\mathcal{C}_5^{(4)} &= \varepsilon_{ijklm}\varepsilon_{IJK} \text{Tr}\left(\mathcal{F}_{ijkl}\mathcal{F}_{mIJK} + 6\mathcal{F}_{ijIJ}\mathcal{F}_{klmK} \right) \\
&= 36\varepsilon_{ijklm}\varepsilon_{IJK} \text{Tr}(\mathcal{F}_{ij}\mathcal{F}_{kl}\mathcal{F}_{IJ}\mathcal{F}_{mK} + \mathcal{F}_{ij}\mathcal{F}_{kl}\mathcal{F}_{mK}\mathcal{F}_{IJ} \\
&\quad + \mathcal{F}_{ij}\mathcal{F}_{mK}\mathcal{F}_{kl}\mathcal{F}_{IJ} - 4\mathcal{F}_{iI}\mathcal{F}_{jJ}\mathcal{F}_{kK}\mathcal{F}_{mn})
\end{aligned} \tag{5.28}
$$

is subjected to the dimensional reduction using the symmetry constraints (4.6), (4.7) and (4.8).

This results directly in

$$\tfrac{1}{8}\mathcal{C}_5^{(4)} = 3\varepsilon_{ijklm}\,\mathrm{Tr}[D_m\Phi(3\eta^2 F_{ij}F_{kl} + F_{ij}F_{kl}\Phi^2 + \Phi^2 F_{ij}F_{kl} + F_{ij}\Phi^2 F_{kl})$$
$$- 2F_{ij}D_k\Phi D_l\Phi D_m\Phi]$$
$$\overset{\text{def.}}{=} \boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(4,5)} \tag{5.29}$$

which is manifestly a total divergence defining the CS density

$$\tfrac{1}{8}\Omega_m^{(4,5)} = (18)^2\varepsilon_{ijklm}\,\mathrm{Tr}\big[\Phi\big(\eta^2 F_{ij}F_{kl} + \tfrac{2}{9}\Phi^2 F_{ij}F_{kl} + \tfrac{1}{9}F_{ij}\Phi^2 F_{kl}\big)$$
$$- \tfrac{2}{9}(\Phi D_i\Phi D_j\Phi - D_i\Phi\Phi D_j\Phi + D_i\Phi D_j\Phi\Phi)F_{kl}\big]. \tag{5.30}$$

5.3.4. $N=4$, $D=4$. The CP term

$$\tfrac{1}{2}\mathcal{C}_4^{(4)} = \varepsilon_{ijkl}\varepsilon_{IJKL}\,\mathrm{Tr}(\mathcal{F}_{ijkl}\mathcal{F}_{IJKL} - 16\mathcal{F}_{ijkK}\mathcal{F}_{lLIJ} + 18\mathcal{F}_{ijIJ}\mathcal{F}_{klKL})$$
$$= 36\varepsilon_{ijkl}\varepsilon_{IJKL}\,\mathrm{Tr}[2\mathcal{F}_{ij}\mathcal{F}_{kl}\mathcal{F}_{IJ}\mathcal{F}_{KL} + \mathcal{F}_{ij}\mathcal{F}_{IJ}\mathcal{F}_{kl}\mathcal{F}_{KL}$$
$$- 8(\mathcal{F}_{iI}\mathcal{F}_{jJ}\mathcal{F}_{kl}\mathcal{F}_{KL} + \mathcal{F}_{iI}\mathcal{F}_{jJ}\mathcal{F}_{KL}\mathcal{F}_{kl} + \mathcal{F}_{iI}\mathcal{F}_{kl}\mathcal{F}_{jJ}\mathcal{F}_{KL})$$
$$+ 8\mathcal{F}_{iI}\mathcal{F}_{jJ}\mathcal{F}_{kK}\mathcal{F}_{lL}] \tag{5.31}$$

is subjected to the dimensional reduction using the symmetry constraints (4.14), (4.15) and (4.16), and performing the traces over the codimension indices I, J, K, L.

The result can be recast in the transparent form by further using (4.22)–(4.23) and (4.24), (4.25), (4.26). This leads to the compact expression (see footnote 9)

$$\tfrac{1}{2}\mathcal{C}_4^{(4)} = 18\varepsilon_{ijkl}\,\mathrm{Tr}\,\Gamma_5[2S^2 F_{ij}F_{kl} + F_{ij}SF_{kl}S + 4(D_i\Phi D_j\Phi\{S, F_{kl}\} + D_i\Phi F_{kl}D_j\Phi S)$$
$$+ 2D_i\Phi D_j\Phi D_k\Phi D_l\Phi]$$
$$\overset{\text{def.}}{=} \boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(4,4)}. \tag{5.32}$$

From definition (4.26) of S, it is clear that the leading Higgs independent term

$$\eta^4\varepsilon_{ijkl}F_{ij}F_{kl} = \eta^4\boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(4)} \tag{5.33}$$

is manifestly a total divergence in terms of the usual CS density of the pure YM field on $\mathbb{R}^4$

$$\Omega_i^{(4)} = \varepsilon_{ijkl}\,\mathrm{Tr}\,\Gamma_5 A_j\big[F_{kl} - \tfrac{2}{3}A_k A_l\big]. \tag{5.34}$$

The rest of the (Higgs-dependent) terms in (5.32) can also be shown to be a total divergence, such that

$$\Omega_i^{(4,4)} = \varepsilon_{ijkl}\,\mathrm{Tr}\,\Gamma_5\big\{6\eta^4 A_j\big(F_{kl} - \tfrac{2}{3}A_k A_l\big) - 6\eta^2(\Phi D_j\Phi - D_j\Phi\Phi)F_{kl}$$
$$+ [(\Phi^2 D_j\Phi\Phi - \Phi D_j\Phi\Phi^2) - 2(\Phi^3 D_j\Phi - D_j\Phi\Phi^3)]F_{kl}\big\}. \tag{5.35}$$

5.3.5. $N=5$, $D=3$. The CP density

$$\tfrac{1}{4}\mathcal{C}_3^{(4)} = 2\varepsilon_{ijk}\varepsilon_{MIJKL}\,\mathrm{Tr}(\mathcal{F}_{ijkM}\mathcal{F}_{IJKL} + 6\mathcal{F}_{ijIJ}\mathcal{F}_{kMKL})$$
$$= 4!^2\,\varepsilon_{ijk}\,\varepsilon_{MIJKL}\,\mathrm{Tr}(\mathcal{F}_{ij}\,\mathcal{F}_{kM}\,\mathcal{F}_{IJ}\,\mathcal{F}_{KL} + \mathcal{F}_{ij}\,\mathcal{F}_{IJ}\,\mathcal{F}_{kM}\,\mathcal{F}_{KL}$$
$$+ \mathcal{F}_{ij}\,\mathcal{F}_{IJ}\,\mathcal{F}_{kM}\,\mathcal{F}_{KL} - 4\,\mathcal{F}_{iI}\,\mathcal{F}_{jJ}\,\mathcal{F}_{kM}\,\mathcal{F}_{KL}) \tag{5.36}$$

is subjected to the dimensional reduction using the symmetry constraints (4.6), (4.7) and (4.8).

This results directly in

$$\mathcal{C}_3^{(4)} = 4\,4!^2\,\varepsilon_{ijk}\,\mathrm{Tr}(3\eta^4 F_{ij}D_k\Phi + \eta^2[3F_{ij}(\Phi^2 D_k\Phi + D_k\Phi\Phi^2) - 2D_i\Phi D_j\Phi D_k\Phi]$$
$$+ [F_{ij}(\Phi^4 D_k\Phi + D_k\Phi\Phi^4 + \Phi^2 D_k\Phi\Phi^2) - 2\Phi^2 D_i\Phi D_j\Phi D_k\Phi])$$
$$\overset{\text{def.}}{=} \boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(4,3)} \tag{5.37}$$

which is manifestly a total divergence defining the CS density

$$\Omega_k^{(4,3)} = (18)^2 \, \varepsilon_{ijk} \, \mathrm{Tr}\big[3\eta^4 \Phi F_{ij} + 2\eta^2(\Phi^3 F_{ij} - \Phi D_i \Phi D_j \Phi)$$
$$+ \tfrac{1}{5}[3\Phi^5 F_{ij} - 2(2\Phi^3 D_i \Phi D_j \Phi - \Phi^2 D_i \Phi \Phi D_j \Phi)]\big]. \tag{5.38}$$

This example was considered in [51].

5.3.6. $N = 6$, $D = 2$. The CP term

$$\tfrac{1}{4}\mathcal{C}_2^{(4)} = \varepsilon_{ij}\varepsilon_{IJKLMN} \, \mathrm{Tr}(3\mathcal{F}_{ijIJ}\mathcal{F}_{KLMN} - 4\mathcal{F}_{iIKL}\mathcal{F}_{jJMN})$$
$$= 3!^2 \varepsilon_{ij}\varepsilon_{IJKLMN} \, \mathrm{Tr}(\mathcal{F}_{ij}\mathcal{F}_{IJ}\mathcal{F}_{KL}\mathcal{F}_{MN} - 4\mathcal{F}_{iI}\mathcal{F}_{jJ}\mathcal{F}_{KL}\mathcal{F}_{MN} - 2\mathcal{F}_{iI}\mathcal{F}_{KL}\mathcal{F}_{jJ}\mathcal{F}_{MN}) \tag{5.39}$$

is subjected to the dimensional reduction using the symmetry constraints (4.29), (4.30) and (4.31), and performing the traces over the codimension indices I, J, K, L.

The result is

$$\tfrac{1}{4}\mathcal{C}_2^{(4)} = 3!^2\varepsilon_{ij}\{-\eta^6 F_{ij} + 3\eta^4(|\varphi|^2 F_{ij} - 2\mathrm{i}\, D_i\varphi^* D_j\varphi) - 3\eta^2|\varphi|^2(|\varphi|^2 F_{ij} - 4\mathrm{i}\, D_i\varphi^* D_j\varphi)$$
$$+ (|\varphi|^2)^2(|\varphi|^2 F_{ij} - 6\mathrm{i}\, D_i\varphi^* D_j\varphi)\} \tag{5.40}$$
$$\overset{\text{def.}}{=} \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}^{(4,2)}$$

which is manifestly a total divergence defining the CS density[10]

$$\Omega_i^{(4,2)} \simeq -\varepsilon_{ij}(\eta^6 A_j + \mathrm{i}[3\eta^4 - 3\eta^2|\varphi|^2 + (|\varphi|^2)^2]\varphi^* D_j\varphi). \tag{5.41}$$

5.4. Gauge transformation properties of $\boldsymbol{\Omega}^{(n,D)}$

The descended CP densities $\mathcal{C}_D^{(n)}$ on $\mathbb{R}^D$ presented above are total divergences

$$\mathcal{C}_D^{(n)} = \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}^{(n,D)},$$

where the densities $\boldsymbol{\Omega}^{(n,D)}$ are the descended CS densities on $\mathbb{R}^D$. The most remarkable feature of the descended CS densities $\boldsymbol{\Omega}^{(n,D)}$ on $\mathbb{R}^D$ is that for *odd D* they are *gauge invariant* functions of the YMH fields, while those on *even D* are *gauge variant* densities. The most familiar examples featuring this property are (a) the magnetic field $\mathbf{B} = \mathrm{Tr}\Phi\, \vec{B}$ of the 't Hooft–Polyakov monopole on $\mathbb{R}^3$, cf (5.5), which is *gauge invariant*, and (b) the magnetic Maxwell potential A_j in (5.8) yielding the vortex number of the Abelian Higgs model on $\mathbb{R}^2$, which is *gauge variant*.

It is remarkable that the CS densities in even-dimensional residual space $\mathbb{R}^D$ are cast into two distinct parts: one *gauge variant* and the other *gauge invariant*. The leading term is the gauge variant part which does not feature the Higgs field and contributes to the surface integral over S^{D-1}, yielding the monopole (or vortex) charge. The rest, which is the Higgs dependent part, is gauge invariant and does not contribute to the topological charge. The reason for this in the case of monopoles, which is the subject of interest here, is that both the YM curvature and the covariant derivative of the Higgs field decay as r^{-2}. This property is explained by the fact that monopoles are DY fields as exposed in section 6 below, or otherwise stated the monopole solutions must obey the *finite energy conditions*.

What is perhaps more remarkable, if not unexpected, is the fact that these gauge variant leading terms are formally identical to the generic CS densities defined in terms of the YM connection and curvature in $(D + 1)$-dimensional spacetimes. The only difference between

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[10] The dimensional reduction of CP densities on $\mathbb{R}^2 \times S^{4p-2}$ down to $\mathbb{R}^2$ can be carried out systematically.

these types of densities is that for monopoles on $\mathbb{R}^D$ the gauge connection takes its values in the Dirac representation of $SO(D)$, while the connection of the generic CS densities takes its values in the *chiral* representation of $SO(D)$.

The gauge invariant densities $\mathbf{\Omega}^{(n,D)}$ for odd D also split up into leading terms which contribute to the topological charge surface integral, and another part which decays too fast and has a vanishing contribution. In this case, both parts feature the Higgs field.

5.5. Normalization of the topological charge

At this stage, it is in order to state the normalization constants of the higher-dimensional monopole charges, even though the simplest presentation involves the spherically symmetric field configurations and hence anticipates results in section 6 below. Normalization involves setting the charge of the *spherically symmetric* monopole equal to *unity*. The topological charge (volume) integral over $\mathbb{R}^D$ then reduces to a one-dimensional integral whose integrand now is a total derivative and is integrated trivially. This will not be carried out explicitly here since it is easily carried out in each case.

What is important to consider here is the normalization of the monopole charge on $\mathbb{R}^D$ when D is even, relative to the normalization of the corresponding instanton charge. These instantons are those alluded to in section 2. Since the decay properties of the (asymptotically pure gauge) instantons are quite different from the milder decay of the monopoles in the same dimensions, it might be expected that the respective normalizations will be different. In fact, they are identical.

For pedagogical simplicity, it is sufficient to consider the $D = 4$ case, and in particular the respective spherically symmetric field configurations. The spherically symmetric ansatz for the connection of the $D = 4$ monopole is given by (6.2) below, and the finite energy asymptotics of the function $w(r)$ there is that given by (6.4). The corresponding ansatz for the $D = 4$ instanton is formally identical to that stated in (6.1), with no Higgs field, and where now $\Sigma_{ab}^{\pm}$ is that given by (3.7) and not by (6.3). The asymptotics for finite energy/action in this case are

$$\lim_{r \to 0} w = \pm 1 \qquad \lim_{r \to \infty} w = \mp 1, \tag{5.42}$$

which are quite different from (6.4), since unlike the latter the instanton is pure gauge at infinity.

Now in both these cases the reduced Pontryagin density is proportional to the same total derivative and its integral is

$$\int_{w(0)}^{w(\infty)} \frac{\mathrm{d}}{\mathrm{d}r} \left(w - \frac{1}{3} w^3 \right) \mathrm{d}r.$$

Thus, in the case of the instanton, the limits in (5.42) result in double the integral of the monopole with limits (6.4). However, the trace in the second CP density in the case of the monopole is twice as large as that for the instanton, since in the former case the matrices involved are the 4×4 Dirac matrices while in the latter case they are the 2×2 chiral matrices. This counting repeats in every $4p$ dimension. Hence, the normalization of the $D = 4$ monopole and the $D = 4$ instanton is equal. This holds for all even D.

6. Spherical symmetry and DY monopoles

It is natural to introduce the DY monopoles at this stage, after having stated the spherically symmetric ansätze for the $SO(D)$ gauge fields on $\mathbb{R}^D$, because the DY [1–3] fields are the

asymptotic gauge fields of the spherically symmetric monopole solutions themselves. In that context, the asymptotics result from the *finite energy boundary conditions*. These boundary conditions also result from requiring that the monopole charge be a *topological charge*, i.e. that for the monopole on $\mathbb{R}^D$, the surface integral on the large spheres S^{D-1} be convergent and normalized to an integer. Imposing spherical symmetry on the monopole charge densities presented in the previous section, which are all manifestly total divergences, yields total derivative expressions.

The Dirac [1] monopole can be constructed by gauge transforming the asymptotic 't Hooft–Polyakov monopole [4, 5] in $D = 3$, which can be taken to be spherically symmetric[11], such that the $SO(3)$ isovector Higgs field is gauged to a (trivial) constant, and the $SU(2) \sim SO(3)$ gauge group of the YM connection breaks down to $U(1) \sim SO(2)$, the resulting Abelian connection developing a line singularity on the positive or negative $(x_3 =)z$-axis.

In exactly the same way, the Yang [2] monopole can be constructed by transforming the $(D = 5)$-dimensional monopole such that the $SO(5)$ isovector Higgs field is gauged to a (trivial) constant, and the $SO(5)$ gauge group of the YM connection breaks down to $SO(4)$, the resulting non-Abelian connection developing a line singularity on the positive or negative x_5-axis. In fact, the residual non-Abelian connection can take its values in one or other chiral representations of $SU(2)$, as formulated by Yang [2], but this is a low-dimensional accident which does not apply to the higher-dimensional analogues to be defined below, all of which are $SO(D-1)$ connections. Just like the 't Hooft–Polyakov monopole is the regular counterpart of the Dirac monopole, so is the $(D = 5)$-dimensional 'monopole' the regular counterpart of the Yang monopole.

The above two definitions of the Dirac and of the Yang monopoles will be the template for our definition of what we will refer to as the hierarchy of DY monopoles in all dimensions. The two examples just given are both in odd ($D = 3$ and $D = 5$) dimensions, but the DY hierarchy is in fact defined in all, including even, dimensions.

We start by stating the spherically symmetric ansatz for the $SO(D)$ gauge connection A_i and the *iso-D-vector* Higgs field Φ, for odd and even D, respectively, as

$$A_i^{(\pm)} = \frac{1}{r}(1 - w(r))\Sigma_{ij}^{(\pm)}\hat{x}_j, \qquad \Phi = 2\eta h(r)\hat{x}_i\Sigma_{i,D+1}^{(\pm)} \qquad \text{for odd } D \qquad (6.1)$$

$$A_i = \frac{1}{r}(1 - w(r))\Gamma_{ij}\hat{x}_j, \qquad \Phi = 2\eta h(r)\hat{x}_i\Gamma_{i,D+1} \qquad \text{for even } D. \qquad (6.2)$$

In relations (6.1)–(6.2), $\hat{x}_i = \frac{x_i}{r}$, $i = 1, 2, \ldots, D$, is the unit radius vector. Γ_i are the Dirac gamma matrices in D dimensions with the chiral matrix Γ_{D+1} for even D, so that

$$\Gamma_{ij} = -\tfrac{1}{4}[\Gamma_i, \Gamma_j]$$

are the Dirac representations of $SO(D)$. The matrices Σ_{ij}, employed only in the odd D case, are

$$\Sigma_{ij}^{(\pm)} = -\frac{1}{4}\left(\frac{\mathbb{1} \pm \Gamma_{D+2}}{2}\right)[\Gamma_i, \Gamma_j], \qquad (6.3)$$

Γ_{D+1} being the chiral matrix in $D + 1$ dimensions, and $\Sigma_{ij}^{(\pm)}$ being one or the other of the two possible chiral representations of the $SO(D)$ subgroup of $SO(D + 1)$.

Just as the Dirac monopole can be defined as a gauge transform of the asymptotic spherically symmetric 't Hooft–Polyakov monopole, our definition for the DY fields in arbitrary

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[11] Strictly speaking, it is not necessary to restrict to spherically symmetric fields only. By choosing to start with the asymptotic axially symmetric fields characterized with vorticity n, the gauge transformed connection is just n times the usual Dirac monopole field.

D dimensions starts from the asymptotic (non-Abelian) $SO(D)$ YM field A_i and the D-tuplet Higgs field Φ, with

$$\lim_{r \to \infty} w(r) = 0, \qquad \lim_{r \to \infty} w(r) = 1 \tag{6.4}$$

$$\lim_{r \to \infty} h(r) = 1, \qquad \lim_{r \to \infty} h(r) = 0 \tag{6.5}$$

leading to

$$A_i^{(\pm)} = \frac{1}{r} \Sigma_{ij}^{(\pm)} \hat{x}_j, \qquad \Phi = 2\eta\, \hat{x}_i\, \Sigma_{i,D+1}^{(\pm)} \qquad \text{for odd } D \tag{6.6}$$

$$A_i = \frac{1}{r} \Gamma_{ij} \hat{x}_j, \qquad \Phi = 2\eta\, \hat{x}_i\, \Gamma_{i,D+1} \qquad \text{for even } D. \tag{6.7}$$

The DY monopoles result from the action of the following $SO(D)$ gauge group element:

$$g_\pm = \frac{(1 \pm \cos\theta_1)\mathbb{1} \pm \Gamma_D \Gamma_\alpha \hat{x}_\alpha \sin\theta_1}{\sqrt{2(1 \pm \cos\theta_1)}}, \tag{6.8}$$

having parametrized the $\mathbb{R}^D$ coordinate $x_i = (x_\alpha, x_D)$ in terms of the radial variable r and the polar angles

$$(\theta_1, \theta_2, \ldots, \theta_{D-2}, \varphi) \tag{6.9}$$

with the index alpha running over $\alpha = 1, 2, \ldots, D - 1$. The meaning of the $\pm$ sign in (6.8) is as follows [52]. Choosing these signs the Dirac line singularity will be along the negative or positive x_D-axis, respectively. (In the case of odd D if we choose the opposite sign on Σ in (6.6) the situation will be reversed.) In other words, the DY field will be the $SO(D - 1)$ connection on the upper or lower half $D - 1$ sphere, S^{D-1}, respectively, the transition gauge transformation being given by $g_+^{-1} g_-$. Note that the dimensionality of the matrices g, (6.8), and those of both (6.6) and (6.7), match in each case.

The result of the action of (6.8) on (6.6) or (6.7),

$$A_i \to g^{-1} A_i g + g^{-1} \partial_i g$$
$$\Phi \to g^{-1} \Phi g,$$

yields the required DY fields $\hat{A}_i^{(\pm)} = \left(\hat{A}_\alpha^{(\pm)}, \hat{A}_D^{(\pm)} \right)$:

$$\hat{A}_\alpha^{(\pm)} = \frac{1}{r(1 \pm \cos\theta_1)} \Sigma_{\alpha\beta} \hat{x}_\beta \quad , \qquad \hat{A}_D^{(\pm)} = 0 \qquad \text{for odd } D \tag{6.10}$$

$$\hat{A}_\alpha^{(\pm)} = \frac{1}{r(1 \pm \cos\theta_1)} \Gamma_{\alpha\beta} \hat{x}_\beta \quad , \qquad \hat{A}_D^{(\pm)} = 0 \qquad \text{for even } D, \tag{6.11}$$

and the Higgs field is gauged to a constant, i.e. it is trivialized.

The components of the DY curvature $\hat{F}_{ij}^{(\pm)} = \left(\hat{F}_{\alpha\beta}^{(\pm)}, \hat{F}_{\alpha D}^{(\pm)} \right)$ follow from (6.10)–(6.11) straightforwardly. To save space we give only the curvature corresponding to (6.10):

$$\hat{F}_{\alpha\beta}^{(\pm)} = -\frac{1}{r^2} \left[\Gamma_{\alpha\beta} + \frac{1}{(1 \pm \cos\theta_1)} \hat{x}_{[\alpha} \Gamma_{\beta]\gamma} \hat{x}_{[\gamma} \right] \tag{6.12}$$

$$\hat{F}_{\alpha D}^{(\pm)} = \pm \frac{1}{r^2} \Gamma_{\alpha\gamma} \hat{x}_\gamma, \tag{6.13}$$

where the notation $[\alpha\beta]$ implies the antisymmetrization of the indices, and the components of the curvature for even D corresponding to (6.11) follow by replacing Γ in (6.12)–(6.13) with $\Sigma^{(\pm)}$. The parametrization (6.10)–(6.11) and (6.12)–(6.13) for the DY field appeared in [53] and [52].

That the DY field (6.10)–(6.11) in D dimensions, constructed by gauge-transforming the asymptotic fields (6.6)–(6.7) of a $SO(D)$ EYM system, is a $SO(D-1)$ YM field is obvious. For $D = 3$ and $D = 5$, these are the Dirac [1] and Yang [2] monopoles, respectively.

In retrospect, we point out that to construct DY monopoles, it is not even necessary to start from a YMH system, but ignoring the Higgs field and simply applying the gauge transformation (6.8) to the YM members of (6.6)–(6.7) results in the DY monopoles (6.10)–(6.11). In other words, the only function of the Higgs fields in (6.6)–(6.7) is the definition of the gauge group element (6.8) designed to gauge it away.

It is perhaps reasonable to emphasize there that DY monopoles exist in all dimensions D, whether D is odd or even, as presented above. The most prominent difference between these cases is that for odd D, one has a *gauge covariant*, $D-1$ form generalization [52] for the definition of a 't Hooft electromagnetic tensor.

7. Monopoles on $\mathbb{R}^D$: $D \geqslant 3$

These are topologically stable static finite energy solutions generalizing the 't Hooft–Polyakov monopole on $\mathbb{R}^3$. The fundamental inequality yielding the appropriate Bogomol'nyi lower bound is (3.2), which is descended from inequality (2.4). The simplest option here is to take $p = q$ in (2.4), in which case the density on the right-hand side will consist of a single term (2.2) scale invariant in $4p$ dimensions. For the purposes of the present notes, attention will be restricted to bulk systems in six and eight dimensions, since these two suffice to expose all qualitative features of monopoles in higher dimensions. In the first case, $p + q = 3$, the only choice we have is $p = 1$ and $q = 2$, i.e. $p \neq q$, while in the second case $p + q = 8$ allows one to opt for the simpler choice of $p = q = 4$, which is what will be done here.

The Bogomol'nyi bounds (3.1), descending from (2.4), are in general not saturated. When $p \neq q$, the presence of the dimensionful constant κ obstructs the construction of self-dual solutions exactly as is the case in Skyrme [54] theory. When $p = q$ on the other hand, the inequality can be saturated in the bulk but its descendants on the residual space result in Bogomol'nyi equations that are in general overdetermined [27] and do not support any nontrivial *self-dual* monopole solutions. Such monopoles are solutions of second order Euler–Lagrange equations. The exceptions[12] are the monopoles supported by YMH models on $\mathbb{R}^{4p-1}$ descended from the p-YM systems on $\mathbb{R}^{4p}$, given in [11], which might be referred to as p-BPS monopoles. An analytic proof of existence for the p-BPS monopoles is given in [48–50]. The $p = 1$ case is the BPS monopole which is known in closed form, all other p-BPS monopoles being constructed only numerically.

The presentation here is restricted to YMH systems on $\mathbb{R}^D$, $D \geqslant 3$, supporting monopoles. The vortices on $\mathbb{R}^2$ supported by models descending from higher-dimensional YM systems [57, 58] are not included here since consideration is restricted to monopoles only.

In any dimension $\mathbb{R}^D$, there is an infinite tower of YMH models, each descending from the p-YM member (2.2) of the YM hierarchy on the bulk of dimension $D+N$ for all N, i.e. such that $4p \geqslant D + N$. In these notes, only the first nontrivial elements of these towers are presented explicitly, namely that consideration is restricted to monopoles of YMH models descending from the $p = 1$ and $p = 2$ members of the YM hierarchy defined on the bulk dimensions $D + N = 4$, $D + N = 6$ and $D + N = 8$.

---

[12] The only other exceptions are the vortices supported by the Abelian Higgs models on $\mathbb{R}^2$ descended from the p-YM systems on $\mathbb{R}^{4p}$ given in [57, 58]. These do have a BPS limit, whose solutions might be referred to as p-BPS vortices. An analytic proof of existence for the $p = 2$ BPS vortices can be found in [26], which can readily be adapted to the arbitrary p case.

In the $D + N = 4$ case, the monopole charge density is the descendant of the second CP density, and the YMH models in question are the descendants of the usual ($p = 1$) YM system. There are the two possibilities, ($D = 3, N = 1$) and ($D = 2, N = 2$), the first of which is the usual 't Hooft–Polyakov monopole in the BPS limit and the second is the usual Abelian Higgs model supporting the ANO vortex, consideration of which is excluded.

In the $D + N = 6$ case, the monopole charge density is the descendant of the third CP density, and the YMH models in question are the descendants of the sum of the $p = 1$ and $p = 2$ members of the YM hierarchy. There are the four possible monopoles ($D = 5, N = 1$), ($D = 4, N = 2$) and ($D = 3, N = 3$), the vortex case ($D = 2, N = 4$) again being excluded. In fact, ($D = 5, N = 1$) and ($D = 4, N = 2$) are also excluded. The reason is that the energy of the usual ($p = 1$) YM term diverges in both $D = 5$ and $D = 4$ dimensions, since the asymptotic connection is a DY and decays as r^{-1}.

In the $D + N = 8$ case, the monopole charge density is the descendant of the fourth CP density, and the YMH models are those descending from the $p = 2$ members of the YM hierarchy. In this case one does not have the option of employing the descendants of the sum of the $p = 1$ and $p = 2$ members of the YM hierarchy, since the only possibility is the (generalized [57, 58]) ANO vortex [6, 7] on $\mathbb{R}^2$, outside of interest here. This results in the possibilities ($D = 7, N = 1$), ($D = 6, N = 2$), ($D = 5, N = 3$), ($D = 4, N = 4$) and ($D = 3, N = 5$).

To date, only two of the above-mentioned models resulting from a descent over S^N with $N \geqslant 2$ have been studied quantitatively. These are the ($D = 3, N = 5$) [51] and the ($D = 4, N = 4$) [46] monopoles, respectively. In addition, the monopoles resulting from a descent over S^1 with $N \geqslant 1$, i.e. those on R^3, R^5 and $\mathbb{R}^7$, are readily constructed as special cases of the monopoles on arbitrary dimensions $\mathbb{R}^{2n+1}$ given in [11, 13, 30, 59].

Since all descents are performed from compact bulk dimensions to Euclidean residual dimensions, the resulting residual systems are described as static Hamiltonians[13]. In what follows, the (candidate static) Hamiltonian densities with (D, N) are denoted as $\mathcal{H}_D^{(N)}$.

7.1. Monopole on $\mathbb{R}^3$ with descended second CP charge

This is the usual 't Hooft–Polyakov monopole on $\mathbb{R}^3$ in the BPS limit. It is also the first one in the hierarchy of monopoles [11, 13, 30, 59] on $\mathbb{R}^{2n+1}$.

7.2. Vortex on $\mathbb{R}^2$ with descended second CP charge

Since the density whose surface integral yielding winding number (5.8) pertaining to this case was given in section 5.1.2, it is sufficient here to state that the model supporting these vortices is the usual Abelian Higgs model [6, 7].

7.3. Monopole on $\mathbb{R}^3$ with descended third CP charge

This is the monopole whose topological lower bound is given by the volume integral of the topological charge density (5.16) or the surface integral of (5.17). It is the *only* monopole in this class, since the energy integrals of those on $\mathbb{R}^5$ and $\mathbb{R}^4$ are divergent due to the presence of the usual ($p = 1$) YM term. Another special feature of this model is that it features a dimensionful constant κ, exactly like the usual Skyrme model. In this sense, the question of saturating the topological lower bound does not arise. It is of course likewise possible

---

[13] The description of 'static Hamiltonian' for the descended system is used here, in anticipation of employing the name 'static Lagrangian' in the next section, where the corresponding towers of Julia–Zee dyons will be described.

to estimate the amount by which the lowest energy monopole exceeds this lower bound numerically.

What distinguishes the model employed here from the rest of the examples given later[14] is that the energy density functional in six dimensions that is bounded from below by the third CP density must feature both $\mathcal{F}_{ij}$ and $\mathcal{F}_{ijkl}$ curvature terms. This means that there will appear a dimensionful constant κ, whose function is to compensate the difference in the dimensions of these two curvature terms. In this respect, the resulting model is akin to the Skyrme model.

The static Hamiltonian can be labelled by $(D, N) = (3, 3)$, which is the dimensional descendant of the $p = 1$ and the $p = 2$ members of the YM hierarchy on $\mathbb{R}^3 \times S^3$, appearing in (5.2).

The static Hamiltonian is readily calculated using (4.6)–(4.8)

$$
\begin{aligned}
\mathcal{H}_3^{(3)} &= \text{Tr}\left(\mathcal{F}_{ij}^2 + 2\mathcal{F}_{iI}^2 + \mathcal{F}_{IJ}^2\right) - \frac{\kappa^4}{12}\text{Tr}\left(4\mathcal{F}_{ijkI}^2 + 6\mathcal{F}_{ijII}^2 + 4\mathcal{F}_{iIJK}^2\right) \\
&= \text{Tr}\left(2F_{ij}^2 + \frac{3}{2}D_i\Phi^2 - 3S^2\right) \\
&\quad - \frac{\kappa^4}{4}\text{Tr}\left(\{F_{[ij}, D_{k]}\Phi\}^2 - 3(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi])^2 - \frac{9}{2}\{S, D_i\Phi\}^2\right),
\end{aligned}
\tag{7.1}
$$

where the constant κ has the dimension of *length* (7.1), which is the positive/negative definite (employing anti-Hermitian fields) is bounded from below by the topological charge density

$$
\varrho_3^{(3)} = \kappa^2\varepsilon_{ijk}\,\text{Tr}(3F_{ij}\{S, D_k\Phi\} + 2D_i\Phi D_j\Phi D_k\Phi),
\tag{7.2}
$$

read from (5.17).

7.3.1. Spherical symmetry. Subject to spherical symmetry (6.1), the density (7.1), viewed as a static Hamiltonian, reduces to the one-dimensional subsystem

$$
\begin{aligned}
H_3^{(3)} &\simeq ([2w'^2 + r^{-2}(1 - w^2)^2] + \lambda_1\eta^2[r^2h'^2 + 2w^2h^2] + \lambda_2\eta^4r^2(1 - h^2)^2) \\
&\quad + \kappa^4(\lambda_3\eta^2([(1 - w^2)h]')^2 + \lambda_4\eta^4(2([(1 - h^2)w]')^2 \\
&\quad + r^{-2}[(1 - h^2)(1 - w^2) + 2w^2h^2]^2) + \lambda_5\eta^6(1 - h^2)^2(r^2h'^2 + 2w^2h^2)),
\end{aligned}
\tag{7.3}
$$

whose final normalization will be fixed after the monopole charge of the hedgehog solution is fixed to *unity*. The constants $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ must be positive for the topological lower bound to remain valid.

Subject to the same symmetry, the monopole charge density (7.2) bounding $H_3^{(3)}$ from below reduces to

$$
\rho_3^{(3)} = \kappa^2\eta^3\frac{\mathrm{d}}{\mathrm{d}r}\left\{\left(h - \frac{1}{3}h^3\right) - (1 - h^2)w^2h\right\}
\tag{7.4}
$$

which, as expected, is a total derivative. It is clear that the only contribution to the integral of (7.4) comes from the first term, since the second term yields *nil*, according to the boundary values (6.4)–(6.5) resulting from the finite energy conditions. This was of course known in advance since the *monopole* gauge fields are asymptotically DY fields.

The integral of (7.4) is equal to $\frac{2}{3}$, which means that both it and the static Hamiltonian (7.3) must each be multiplied by $\frac{3}{2}$ for the monopole charge of the Hedgehog to be *unity*.

---

[14] The models whose monopole charge density is the descendant of the fourth CP density are descended from eight-dimensional bulk space and hence one has the option of employing a bulk action density with $p = q = 2$ in (3.1), featuring only $\mathcal{F}(4)$, or, one with $p = 2$ and $q = 3$, featuring both $\mathcal{F}(2)$ and $\mathcal{F}(6)$. The latter choice is eschewed since the added technical complexity does not bring any new qualitative features of the monopole whose Hamiltonian includes the 6-form term $\text{Tr}\mathcal{F}_{ijkIJK}^2 \to \text{Tr}\,S^2\{F_{[ij}, D_{k]}\Phi\}^2$.

In this and all subsequent monopoles considered, the number of Bogomol'nyi equations if greater than the number of functions parametrizing the fields, the systems are overdetermined. But in the case in hand, the overdetermination is even more pronounced. Here, the situation is more similar to the usual Skyrme [54] model which likewise features a dimensionful constant, in addition to the Bogomol'nyi equations numbering 2, for *one* Hedgehog function $f(r)$. One would expect therefore that the Bogomol'nyi bound is, like in the Skyrme model, violated more severely than in the case of the corresponding first order equations in the following, which pertain to systems descended from a single member of the YM hierarchy, and which consequently do not feature any additional dimensional constant other than η, the inverse radius of the codimensional compact space.

7.4. Vortex on $\mathbb{R}^2$ with descended third CP charge

Since the density whose surface integral yielding the winding number (5.21) pertaining to this case was given in section 5.2.4, we state the generalized [57, 58] Abelian Higgs model in this case. This consists of the usual Abelian Higgs model, *plus* the density descended from the $p = 2$ YM density displayed in (7.34) in section 7.10.

7.5. Monopole on $\mathbb{R}^7$ with descended fourth CP charge

This is the monopole whose topological lower bound is given by the volume integral of the topological charge density (5.22), or the surface integral of (5.22). The static Hamiltonian can be labelled by $(D, N) = (7, 1)$, which is the dimensional descendant of the $p = 2$ member of the YM hierarchy on $\mathbb{R}^7 \times S^1$, appearing in (5.2). The static Hamiltonian is readily calculated using (4.6)–(4.8):

$$
\begin{aligned}
\mathcal{H}_7^{(1)} &= \text{Tr}\big(\mathcal{F}_{ijkl}^2 + 4\mathcal{F}_{ijk8}^2\big) \\
&= \text{Tr}\big(F_{ijkl}^2 + 4\{F_{[ij}, D_{k]}\Phi\}^2\big),
\end{aligned} \tag{7.5}
$$

which is bounded from below by the topological charge density

$$
\varrho_7^{(1)} = 8\varepsilon_{ijklmnp}\text{Tr}\, F_{ij}F_{kl}F_{np}D_m\Phi. \tag{7.6}
$$

The topological lower bound in the class of models on $\mathbb{R}^{4p-1}$ descended from the p-YM system on $\mathbb{R}^{4p-1} \times S^1$ is very special in that the Bogomol'nyi equations

$$
F_{ijkl} = \tfrac{1}{3!}\varepsilon_{ijklmnp}\{F_{[mn}, D_{p]}\Phi\} = \tfrac{1}{2}\varepsilon_{ijklmnp}\{F_{mn}, D_p\Phi\}
$$

which saturate the topological lower bound are not overdetermined. The general case on $\mathbb{R}^{4p-1}$ is given in [11], and here we present the monopole in this class on $\mathbb{R}^7$.

7.5.1. Spherical symmetry. Subject to spherical symmetry (6.1), density (7.5) reduces to the one-dimensional subsystem[15]

$$H_7^{(1)} = \frac{1}{2}(1-w^2)^2\left[4w'^2 + 3\frac{(1-w^2)^2}{r^2}\right]$$
$$+ \frac{1}{2}\eta^2\left[\frac{r^2}{3}([(1-w^2)h]')^2 + 4(1-w^2)^2w^2h^2\right], \tag{7.7}$$

which is bounded from below by the density

$$\rho_7^{(1)} = \eta\frac{d}{dr}[(1-w^2)^{2p-1}h], \tag{7.8}$$

which as expected is a total derivative. Both (7.5) and (7.8) are normalized such that the hedgehog has *unit* monopole charge.

The Bogomol'nyi equations for this hedgehog configuration are

$$w' \mp \eta wh = 0,$$
$$\eta\, r[(1-w^2)\,h]' \pm \frac{3}{r}(1-w^2)^2 = 0, \tag{7.9}$$

which are not overdetermined and can be solved numerically [11]. These are obviously the direct generalizations of the BPS monopoles on $\mathbb{R}^3$, so they will be referred to as the tower of *p*-BPS monopoles on $\mathbb{R}^{4p-1}$ in these notes.

An analytic proof of existence to the BPS equations (7.9) on $\mathbb{R}^7$ was given in [48, 49]. Clearly, that proof can be readily adapted to the solutions of the BPS equations on $\mathbb{R}^{4p-1}$ of [11].

7.6. Monopole on $\mathbb{R}^6$ with descended fourth CP charge

This is the monopole whose topological lower bound is given by the volume integral of the topological charge density (5.23) or the surface integral of (5.24).

The static Hamiltonian can be labelled by $(D, N) = (6, 2)$, which is the dimensional descendant of the $p = 2$ member of the YM hierarchy on $\mathbb{R}^6 \times S^2$, appearing in (5.3).

The static Hamiltonian is readily calculated from (4.14)–(4.16), in the compact notation of (4.24), (4.25) and (4.26):

$$\mathcal{H}_6^{(2)} = \text{Tr}\big(\mathcal{F}_{ijkl}^2 + 4\mathcal{F}_{ijkI}^2 + 6\mathcal{F}_{ijIJ}^2\big)$$
$$\simeq \text{Tr}\big(F_{ijkl}^2 + 4\lambda_1\{F_{[ij}, D_{k]}\Phi\}^2 - 3\lambda_2(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi])^2\big), \tag{7.10}$$

---

[15] In the case of arbitrary p, the corresponding expression for the static energy density of the monopole on $\mathbb{R}^{4p-1}$ is [11]

$$H_{4p-1}^{(1)} = \frac{1}{2}(1-w^2)^{2(p-1)}\left[2pw'^2 + (2p-1)\frac{(1-w^2)^2}{r^2}\right]$$

$$+ \frac{1}{2}\eta^2\left[\frac{r^2}{2p-1}([(1-w^2)^{p-1}h]')^2 + 2p(1-w^2)^{2(p-1)}w^2h^2\right],$$

which is bounded from below by the density

$$\rho_7^{(1)} = \eta\frac{d}{dr}[(1-w^2)^{2p-1}h],$$

the lower bound being saturated by the Bogomol'nyi equations

$$w' \mp \eta wh = 0, \qquad \eta r[(1-w^2)^{p-1}h]' \pm \frac{(2p-1)}{r}(1-w^2)^p = 0.$$

the second line of which is expressed up to an overall numerical factor, since the final normalization will be made by requiring the Hedgehog to have *unit* monopole charge. Also in the second line of (7.22), the fictitious dimensionless and *positive* constants (λ_1, λ_2) are inserted since the Bogomol'nyi inequality remains valid as long as these constants are all positive. The Bogomol'nyi inequalities in question can be saturated *only* when each of these constants is equal to 1. (7.22) is a positive definite Hamiltonian density, in which negative signs appear under the trace since we have used an anti-Hermitian connection.

The Hamiltonian density (7.10) is bounded from below by the topological charge density

$$\varrho_6^{(2)} = \varepsilon_{ijklmn} \text{Tr}\gamma_7 [SF_{ij}F_{kl}F_{mn} + 2F_{ij}F_{kl}D_m\Phi D_n\Phi + F_{ij}D_m\Phi F_{kl}D_n\Phi] \tag{7.11}$$

read from (5.24).

7.6.1. Spherical symmetry. Subject to spherical symmetry, (6.2), density (7.10) reduces to the one-dimensional subsystem

$$\begin{aligned}
H_6^{(2)} \simeq\ & r^{-1}(2[(1-w^2)w']^2 + r^{-2}(1-w^2)^4) \\
& + \tfrac{2}{9}\lambda_1\eta^2 r[((1-w^2)h]')^2 + 3^2 r^{-2}(1-w^2)^2 w^2 h^2] \\
& + \tfrac{1}{36}\lambda_2\eta^4 r^3[([(1-h^2)w]')^2 + 2r^{-2}[(1-w^2)(1-h^2) + 2w^2 h^2]^2]
\end{aligned} \tag{7.12}$$

whose overall normalization will be chosen such that the monopole charge of the hedgehog solution is fixed to *unity*, and for convenience each of the λ's is rescaled.

We now rewrite (7.12) with a given choice of the constants $\lambda_1 = \lambda_2 = 1$:

$$\begin{aligned}
H_6^{(2)} \simeq\ & \big(2r^{-1}\{(1-w^2)w' - \tfrac{1}{6}\eta^2 r[(1-h^2)(1-w^2) + 2w^2 h^2]\}^2 \\
& + \tfrac{1}{3}\eta^2[(1-h^2)(1-w^2) + 2w^2 h^2](1-w^2)w') \\
& + \big(\tfrac{2}{9}\eta^2 r\{[(1-w^2)h]' + 3r^{-1}(1-w^2)wh\}^2 - \tfrac{4}{3}\eta^2(1-w^2)wh[(1-w^2)h]'\big) \\
& + \tfrac{1}{6}\big(r^3\{\eta^2[(1-h^2)w]' - r^{-3}(1-w^2)^2\}^2 + \tfrac{1}{3}\eta^2(1-w^2)^2[(1-h^2)w]'\big),
\end{aligned} \tag{7.13}$$

such that the Bogomol'nyi lower bound is exposed. This density is bounded from below by

$$\rho_6^{(2)} = \eta^2 \frac{\text{d}}{\text{d}r}\left\{\left(w - \frac{2}{3}w^3 + \frac{1}{5}w^5\right) - [(1-w^2)wh^2]\right\} \tag{7.14}$$

which is a total derivative descending from (5.24) or (5.27). It is clear that the only contribution to the integral of (7.14) comes from the first term only, since the second term yields *nil*, according to the boundary values (6.4)–(6.5) resulting from the finite energy conditions. This was of course known in advance since the *monopole* gauge fields are asymptotically DY fields.

The integral of (7.14) is equal to $\frac{8}{15}$, which means that both it and the static Hamiltonian (7.12) must each be multiplied by $\frac{15}{8}$ for the monopole charge of the Hedgehog to be *unity*.

Finally we state the Bogomol'nyi equations following from (7.13):

$$\begin{aligned}
(1-w^2)w' &= \pm\tfrac{1}{6}\eta^2 r[(1-h^2)(1-w^2) + 2w^2 h^2] \\
r[(1-w^2)h]' &= \mp 3(1-w^2)wh \\
\eta^2[(1-h^2)w]' &= \pm r^{-3}(1-w^2)^2
\end{aligned} \tag{7.15}$$

which are overdetermined [27].

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7.7. Monopole on $\mathbb{R}^5$ with descended fourth CP charge

This is the monopole whose topological lower bound is given by the volume integral of the topological charge density (5.28), or the surface integral of (5.29).

The static Hamiltonian can be labelled by $(D, N) = (5, 3)$, which is the dimensional descendant of the $p = 2$ member of the YM hierarchy on $\mathbb{R}^5 \times S^3$, appearing in (5.3).

The static Hamiltonian is readily calculated from (4.6)–(4.8),

$$\mathcal{H}_5^{(3)} = \mathrm{Tr}\big(\mathcal{F}_{ijkl}^2 + 4\mathcal{F}_{ijkI}^2 + 6\mathcal{F}_{ijIJ}^2 + 4\mathcal{F}_{iIJK}^2\big)$$
$$\simeq \mathrm{Tr}\big(F_{ijkl}^2 + 4\lambda_1\{F_{[ij}, D_{k]}\Phi\}^2 - 18\lambda_2(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi])^2 - 54\lambda_3\{S, D_i\Phi\}^2\big), \tag{7.16}$$

the second line of which is expressed up to an overall numerical factor, since the final normalization will be made by requiring the Hedgehog to have *unit* monopole charge. Also in the second line of (7.22), the fictitious dimensionless and *positive* constants $(\lambda_1, \lambda_2, \lambda_3)$ are inserted since the Bogomol'nyi inequality remains valid as long as these constants are all positive. The Bogomol'nyi inequalities in question can be saturated *only* when each of these constants is equal to 1. (7.16) is a positive definite Hamiltonian density, in which negative signs appear under the trace since we have used an anti-Hermitian connection.

The Hamiltonian density (7.22) is bounded from below by the topological charge density

$$\varrho_5^{(3)} = \varepsilon_{ijklm}\,\mathrm{Tr}[(SF_{ij}F_{kl} + F_{ij}SF_{kl} + F_{ij}F_{kl}S)D_m\Phi + 2F_{ij}D_k\Phi D_l\Phi D_m\Phi] \tag{7.17}$$

read from (5.29).

7.7.1. Spherical symmetry.
Subject to spherical symmetry, (6.1), the density (7.16) reduces to the one-dimensional subsystem

$$H_5^{(3)} \simeq r^{-2}(1 - w^2)^2\left[4w'^2 + r^{-2}(1 - w^2)^2\right]$$
$$+ \lambda_1\eta^2\left[([(1 - w^2)h]')^2 + 6r^{-2}(1 - w^2)^2w^2h^2\right]$$
$$+ \tfrac{1}{3}\lambda_2\eta^4 r^2\left[2([(1 - h^2)w]')^2 + 3r^{-2}[(1 - w^2)(1 - h^2) + 2w^2h^2]^2\right]$$
$$+ \tfrac{1}{4}\lambda_3\eta^6 r^4(1 - h^2)^2\left[h'^2 + 4r^{-2}w^2h^2\right] \tag{7.18}$$

whose normalization is chosen such that the monopole charge of the hedgehog solution is fixed to *unity*, and for convenience each of the λ's is rescaled.

We now rewrite (7.18), with $\lambda_1 = \lambda_2 = \lambda_3 = 1$, in the following way:

$$H_5^{(3)} \simeq \big(4\{r^{-1}(1 - w^2)w' + \tfrac{1}{2}\eta^3 r(1 - h^2)wh\}^2 - 4\eta^3(1 - h^2)(1 - w^2)hww'\big)$$
$$+ \big(\{\eta[(1 - w^2)h]' - \eta^2[(1 - h^2)(1 - w^2) + 2w^2h^2]\}^2$$
$$+ 2\eta^3[(1 - h^2)(1 - w^2) + 2w^2h^2][(1 - w^2)h]'\big)$$
$$+ \big(\tfrac{2}{3}\{\eta^2 r[(1 - h^2)w]' + 3\eta r^{-1}(1 - w^2)wh\}^2 - 4\eta^3(1 - w^2)wh[(1 - h^2)w]'\big)$$
$$+ \big(\tfrac{1}{4}\{\eta^3 r^2(1 - h^2)h' - 2r^{-2}(1 - w^2)^2\}^2 + \eta^3(1 - h^2)(1 - w^2)^2h'\big), \tag{7.19}$$

such that the Bogomol'nyi bound is exposed. This density is bounded from below by

$$\rho_5^{(3)} = \eta^3\frac{\mathrm{d}}{\mathrm{d}r}\{(3h - h^3) - [3(2 - w^2) - (6 - 5w^2)h^2]w^2h\} \tag{7.20}$$

which is a total derivative descending from (5.29) or (5.30). It is clear that the only contribution to the integral of (7.20) comes from the first term only, since the second term yields *nil*, according to the boundary values (6.4)–(6.5) resulting from the finite energy conditions. This was of course known in advance since the *monopole* gauge fields are asymptotically DY fields.

The integral of (7.20) is equal to 2, which means that both it and the static Hamiltonian (7.12) must each be multiplied by $\frac{1}{2}$ for the monopole charge of the Hedgehog to be *unity*.

Finally, we state the Bogomol'nyi equations following from (7.19):

$$
\begin{aligned}
(1 - w^2)w' &= \mp \tfrac{1}{2}\eta^3 r^2 (1 - h^2)wh \\
[(1 - w^2)h]' &= \pm \eta[(1 - h^2)(1 - w^2) + w^2 h^2] \\
\eta r^2 [(1 - h^2)w]' &= \mp 3(1 - w^2)wh \\
\eta^3 r^4 (1 - h^2)h' &= \pm 2(1 - w^2)^2
\end{aligned}
\tag{7.21}
$$

which are overdetermined [27].

7.8. Monopole on $\mathbb{R}^4$ with descended fourth CP charge

This is the monopole whose topological lower bound is given by the volume integral of the topological charge density (5.31) or the surface integral of (5.32).

The static Hamiltonian can be labelled by $(D, N) = (4, 4)$, which is the dimensional descendant of the $p = 2$ member of the YM hierarchy on $\mathbb{R}^4 \times S^4$, appearing in (5.3).

The static Hamiltonian is readily calculated from (4.14) to (4.16), in the compact notation of (4.24), (4.25) and (4.26):

$$
\begin{aligned}
\mathcal{H}_4^{(4)} &= \mathrm{Tr}\big(\mathcal{F}_{ijkl}^2 + 4\mathcal{F}_{ijkI}^2 + 6\mathcal{F}_{ijIJ}^2 + 4\mathcal{F}_{iIJK}^2 + \mathcal{F}_{IJKL}^2\big) \\
&\simeq \mathrm{Tr}\big(F_{ijkl}^2 + 4\lambda_1 \{F_{[ij}, D_{k]}\Phi\}^2 - 18\lambda_2(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi])^2 \\
&\quad - 54\lambda_3 \{S, D_i\Phi\}^2 + 54\lambda_4 S^4\big),
\end{aligned}
\tag{7.22}
$$

the second line of which is expressed up to an overall numerical factor, since the final normalization will be made by requiring the Hedgehog to have *unit* monopole charge. Also in the second line of (7.22), the fictitious dimensionless and *positive* constants $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are inserted since the Bogomol'nyi inequality remains valid as long as these constants are all positive. The Bogomol'nyi inequalities in question can be saturated *only* when each of these constants is equal to 1. (7.22) is a positive definite Hamiltonian density, in which negative signs appear under the Trace since we have used an anti-Hermitian connection.

The Hamiltonian density (7.22) is bounded from below by the topological charge density

$$
\begin{aligned}
\varrho_4^{(4)} &= \varepsilon_{ijkl}\,\mathrm{Tr}\,\gamma_5[2S^2 F_{ij}F_{kl} + F_{ij}SF_{kl}S + 4(D_i\Phi D_j\Phi SF_{kl} + D_i\Phi D_j\Phi F_{kl}S + D_i\Phi F_{kl}D_j\Phi S) \\
&\quad + 2D_i\Phi D_j\Phi D_k\Phi D_l\Phi],
\end{aligned}
\tag{7.23}
$$

read from (5.32).

7.8.1. Spherical symmetry.
Subject to spherical symmetry, (6.2), the density (7.22) reduces to the one-dimensional subsystem

$$
\begin{aligned}
H_4^{(4)} &\simeq r^{-3}[(1 - w^2)w']^2 + \tfrac{1}{3}\lambda_1 \eta^2 r^{-1}[([(1 - w^2)h]')^2 + 3r^{-2}(1 - w^2)^2 w^2 h^2] \\
&\quad + \tfrac{1}{2}\lambda_2 \eta^4 r^{-2}[([(1 - h^2)w]')^2 + r^{-2}[(1 - w^2)(1 - h^2) + 2w^2 h^2]^2] \\
&\quad + \lambda_3 \eta^6 r^3[[(1 - h^2)h']^2 + 3r^{-2}(1 - h^2)^2 w^2 h^2] + \lambda_4 \eta^8 r^3 (1 - h^2)^4,
\end{aligned}
\tag{7.24}
$$

whose normalization is chosen such that the monopole charge of the hedgehog solution is fixed to *unity*, and for convenience each of the λ's is rescaled.

We now rewrite (7.24) with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$ in the following way:

$$
\begin{aligned}
H_4^{(4)} \simeq \; & (r^{-3}\{(1-w^2)w' - \eta^4 r^3(1-h^2)^2\}^2 + 4\eta^4(1-h^2)^2(1-w^2)w') \\
& + \tfrac{1}{3}(\eta^2 r^{-1}\{[(1-w^2)h]' + 3\eta^2 r(1-h^2)wh\}^2 - 6\eta^4(1-h^2)wh[(1-w^2)h]') \\
& + \tfrac{1}{2}(\eta^4 r^{-2}\{[(1-h^2)w]' - r^{-1}[(1-h^2)(1-w^2)+2w^2h^2]\}^2 \\
& + 2\eta^4[(1-h^2)(1-w^2)+2w^2h^2][(1-h^2)w]') \\
& + (r^3\{\eta^2(1-h^2)h' + \eta r^{-3}(1-w^2)wh\}^2 - 2\eta^4(1-h^2)(1-w^2)whh'),
\end{aligned}
\tag{7.25}
$$

such that the Bogomol'nyi bound is exposed. This density is bounded from below by

$$
\rho_4^{(4)} = \frac{\mathrm{d}}{\mathrm{d}r}\{(3w - w^3) - [(2-h^2)(3-w^2) - 4(1-h^2)w^2]h^2 w\}
\tag{7.26}
$$

which is a total derivative descending from (5.32) or (5.35). It is clear that the only contribution to the integral of (7.26) comes from the first term only, since the second term yields *nil*, according to the boundary values (6.4)–(6.5) resulting from the finite energy conditions. This was of course known in advance since the *monopole* gauge fields are asymptotically DY fields.

The integral of (7.26) is equal to 2, which means that both it and the static Hamiltonian (7.24) must each be multiplied by $\frac{1}{2}$ for the monopole charge of the Hedgehog to be *unity*.

Finally we state the Bogomol'nyi equations following from (7.25):

$$
\begin{aligned}
(1-w^2)w' &= \pm\tfrac{1}{2}\eta^4 r^3(1-h^2)^2 \\
[(1-w^2)h]' &= \mp 3\eta^2 r(1-h^2)wh \\
[(1-h^2)w]' &= \pm r^{-1}[(1-h^2)(1-w^2)+w^2h^2] \\
(1-h^2)h' &= \mp\eta^{-2}r^{-3}(1-w^2)wh
\end{aligned}
\tag{7.27}
$$

which are overdetermined [27].

7.9. Monopole on $\mathbb{R}^3$ with descended fourth CP charge

This is the monopole whose topological lower bound is given by the volume integral of the topological charge density (5.36) or the surface integral of (5.37).

The static Hamiltonian can be labelled by $(D, N) = (3, 5)$, which is the dimensional descendant of the $p = 2$ member of the YM hierarchy on $\mathbb{R}^5 \times S^3$, appearing in (5.3).

The static Hamiltonian is readily calculated using again (4.6)–(4.8):

$$
\begin{aligned}
\mathcal{H}_3^{(5)} &= \left(4\mathcal{F}_{ijkI}^2 + 6\mathcal{F}_{ijIJ}^2 + 4\mathcal{F}_{iIJK}^2 + \mathcal{F}_{IJKL}^2\right) \\
&\simeq \mathrm{Tr}(\{F_{[ij}, D_{k]}\Phi\}^2 - 6\lambda_1(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi])^2 - 27\lambda_1\{S, D_i\Phi\}^2 + 54\lambda_3 S^4),
\end{aligned}
\tag{7.28}
$$

again inserting dimensionless non-negative coefficients $(\lambda_1, \lambda_2, \lambda_3)$. (7.28) is bounded from below by the topological charge density

$$
\varrho_3^{(5)} = \varepsilon_{ijk}\,\mathrm{Tr}(F_{ij}S^2 D_k\Phi + F_{ij}D_k\Phi S^2 + F_{ij}SD_k\Phi S + 2SD_i\Phi D_j\Phi D_k\Phi),
\tag{7.29}
$$

read from (5.37).

7.9.1. Spherical symmetry. Subject to spherical symmetry (6.1), the static Hamiltonian density (7.28) reduces to the one-dimensional subsystem

$$
\begin{aligned}
H_3^{(5)} \simeq \; & \eta^2 r^{-2}([(1-w^2)h]')^2 + \lambda_1\eta^4[2([(1-h^2)w]')^2 + r^2[(1-w^2)(1-h^2)+w^2h^2]^2] \\
& + \tfrac{8}{9}\lambda_2\eta^6 r^2[(1-h^2)^2 h'^2 + 2r^{-2}(1-h^2)^2 w^2 h^2] \\
& + \tfrac{1}{9}\lambda_3\eta^8 r^2(1-h^2)^4,
\end{aligned}
\tag{7.30}
$$

whose final normalization will be fixed after the monopole charge of the hedgehog solution is fixed to *unity*, and for convenience each of the λ's is rescaled.

Rewriting (7.30) with $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$$
\begin{aligned}
H_3^{(5)} \simeq \Big(r^{-2} \Big\{ &\eta[(1-w^2)h]' - \tfrac{1}{3}\eta^4 r^2 (1-h^2)^2 \Big\}^2 + \tfrac{2}{3}\eta^5 (1-h^2)^2 [(1-w^2)h]' \Big) \\
&+ 2 \Big(\big\{ \eta^2[(1-h^2)w]' + \tfrac{2}{3}\eta^3(1-h^2)wh \big\}^2 - \tfrac{4}{3}\eta^5 (1-h^2)wh[(1-h^2)w]' \Big) \\
&+ 2 \Big(\big\{ \tfrac{2}{3}\eta^3(1-h^2)h' - \eta^2 r^{-2}[(1-h^2)(1-w^2) + 2w^2h^2] \big\}^2 \\
&+ \tfrac{4}{3}\eta^5[(1-h^2)(1-w^2) + 2w^2h^2](1-h^2)h' \Big)
\end{aligned}
\tag{7.31}
$$

the Bogomol'nyi bound can be conveniently exposed.

Subject to the same symmetry, the monopole charge density (7.29) bounding $H_3^{(5)}$ from below reduces to

$$
\rho_3^{(5)} = \eta^5 \frac{\mathrm{d}}{\mathrm{d}r} \left\{ \left(h - \frac{2}{3}h^3 + \frac{1}{5}h^5 \right) - (1-h^2)^2 w^2 h \right\}
\tag{7.32}
$$

which is a total derivative. It is clear that the only contribution to the integral of (7.32) comes from the first term, since the second term yields *nil*, according to the boundary values (6.4)–(6.5) resulting from the finite energy conditions. This was of course known in advance since the *monopole* gauge fields are asymptotically DY fields.

The integral of (7.32) is equal to $\frac{8}{15}$, which means that both it and the static Hamiltonian (7.30) must each be multiplied by $\frac{15}{8}$ for the monopole charge of the Hedgehog to be *unity*.

Finally, we state the Bogomol'nyi equations following from (7.31)

$$
\begin{aligned}
[(1-w^2)h]' &= \pm \frac{1}{3}\eta^3 r^2 (1-h^2)^2 \\
[(1-h^2)w]' &= \mp \frac{2}{3}\eta^2 (1-h^2)wh \\
\eta\, r^2 (1-h^2)h' &= \pm \frac{3}{2}[(1-h^2)(1-w^2) + 2w^2h^2]
\end{aligned}
\tag{7.33}
$$

which are overdetermined [27].

7.10. Vortex on $\mathbb{R}^2$ with descended fourth CP charge

The density whose surface integral yields the winding number (5.41) pertaining to this case was given in section 5.3.6, so we state the generalized [57, 58] Abelian Higgs model in this case. This consists only of the density descended from the $p = 2$ YM density given by

$$
\mathcal{H}_2^{(6)} = [(\eta^2 - |\varphi|^2)F_{ij} + i\, D_{[i}\varphi\, D_{j]}\varphi^*]^2 + 24(\eta^2 - |\varphi|^2)^2 |D_i\varphi|^2 + 6(\eta^2 - |\varphi|^2)^4.
\tag{7.34}
$$

Note here that there remain only three distinct terms in the residual density. This leads to a peculiar feature of this Abelian Higgs model derived from the dimensional descent of $p = 2$ YM as the usual one pertaining to the $p = 1$ case: after rescaling, there remains only one single dimensionless parameter λ. This situation persists[16] for all p.

---

[16] For the same reason, the residual density on $\mathbb{R}^2$ can be expressed compactly for arbitrary p as

$$
\mathcal{H}_2^{(4p-1)} = (\eta^2 - |\varphi|^2)^{2(p-2)}([(\eta^2 - |\varphi|^2)F_{ij} + i\, D_{[i}\varphi D_{j]}\varphi^*]^2
$$

$$
+ 4p(2p-1)(\eta^2 - |\varphi|^2)^2 |D_i\varphi|^2 + 2(2p-1)(\eta^2 - |\varphi|^2)^4).
$$

7.11. Bogomol'nyi bounds and bound states

A noteworthy feature of monopoles on $\mathbb{R}^D$ discussed above is that with the exception of those pertaining to YMH models on $\mathbb{R}^{4p-1}$ descended from the pth member of the YM hierarchy, their energies do not saturate the topological lower bound. The first member, that with $p = 1$, of this exceptional class is the 't Hooft–Polyakov [4, 5] monopole itself. (In addition to these monopoles, all the vortices [6, 7, 57, 58] on $\mathbb{R}^2$ in models which are descended from a *single* member of the YM hierarchy do saturate the Bogomol'nyi lower bound.)

The first order Bogomol'nyi equations of the monopoles that do not saturate the topological lower bound, in common with those of the skyrmion [54], are overdetermined [27]. There is however a marked quantitative difference between the excess of the energy of the skyrmion above the topological lower bound, and the corresponding excess in the case of monopoles in YMH models descended from *one single* member of the YM hierarchy. In the latter case, this excess is several orders of magnitude smaller than the excess in the former, as pointed out in [51]. In this sense, the higher-dimensional monopoles are quantitatively closer to the energy bound saturating vortices [6, 7, 57, 58] than the skyrmion [54]. Also the YMH models supporting higher-dimensional monopoles *always* feature at least one dimensionful constant that cannot be scaled away, *like* the Abelian Higgs models, and *unlike* the usual Skyrme model (without pion mass potential and sextic kinetic terms). In the latter case the (non-self-dual) skyrmions have a nonzero interaction energy independently of the value of a dimensionful parameter, while in the case of Abelian Higgs vortices it is well known from the work of [55] that the interaction energy depends on the value of a parameter, which for a critical value results in BPS noninteracting configurations. This feature is repeated also for all generalized vortices [58].

The corresponding situation for higher-dimensional monopoles is very close to that of vortices, whose YMH models always feature at least one dimensionful coupling constant that cannot be scaled away. While there is (are) no value(s) of this constant(s) for which the topological energy bound is saturated, in certain 'almost self-dual' configurations, this bound is approached [51] quantitatively very closely. It is therefore not unreasonable to expect that for various values of this parameter the interaction energy of the monopoles may exhibit bound states. It was verified in [56], in the physically most relevant case on $\mathbb{R}^3$ describing the monopole presented in section 7.9 above that some configurations do feature positive binding energy. It is also not surprising that monopoles of the model presented in section 7.3 should also feature positive binding energy, even though the lowest energy in that case is not quantitatively close to the Bogomol'nyi lower bound.

7.12. Gauge decoupling limits: global monopoles

The two best-known solitons of gauged Higgs models are the vortices of the Abelian Higgs model [6, 7] on $\mathbb{R}^2$ and the monopoles of the 't Hooft–Polyakov [4, 5] model on $\mathbb{R}^3$. In both these cases the energy is divergent in the gauge decoupling limit as in that limit the Derrick scaling requirement is not satisfied. As seen in sections 7 and 7.1, respectively, these models descend from the usual $p = 1$ YM system on $\mathbb{R}^2 \times S^2$ and $\mathbb{R}^3 \times S^1$. This situation changes drastically when considering the gauged Higgs models descending from $p \geqslant 2$ YM systems, simply because there these models feature high enough nonlinear terms in the covariant derivative of the Higgs field that after gauge decoupling the model still contains the requisite terms to satisfy the Derrick scaling requirement yielding finite energy solutions.

The gauge decoupled versions of all the non-Abelian and Abelian models described in this section are immediately found by suppressing the gauge connection and curvature. In the spherically or radially symmetric cases, this is achieved by setting the $w(r) = 1$. In

particular, the topological charges in that case are the winding numbers resulting from the substitution $w(r) = 1$ in the monopole, and $w(r) = n$ in the vortex charge densities. The resulting symmetry-breaking models, featuring a scalar isovector field and no gauge fields, support solitons. In the two-dimensional case, such models were first employed in [60], and in the three-dimensional case in [61]. These were variously described as Goldstone models [62, 63]. In the three dimensions both spherically and axially symmetric solitons, as well as soliton–antisolitons, were constructed in [62]. The solitons in arbitrary dimensions [63] can also be described. In the Abelian analogue on $\mathbb{R}^2$, such vortices were studied in detail in [64].

The $D = 3$ Goldstone model is apparently similar to the Skyrme model [54], but unlike the latter, none of its higher topological charge solitons have a positive binding energy [62], and is unable to describe bound states of nucleons. The Goldstone model has some qualitative similarity to the Skyrme model [54]. Unlike the latter, however, none of its higher charge solitons have positive binding energy and hence cannot describe bound states of nucleons. The Goldstone model is not an alternative to the Skyrme Model.

Solitons of these Goldstone models can be considered to be global monopoles [65–67] in the non-Abelian case and global strings [68, 69] in the Abelian case. The gravitating versions of such solitons may find application as topological defects in the context of phase transitions in the early Universe. The difference from those employed previously [65–69] is that the ones proposed here are both topologically stable and have finite energy.

8. Dyon and pseudo-dyon solutions on $\mathbb{R}^D$: $D \geqslant 3$

The solutions described in this section are static finite energy solutions of the Euler–Lagrange equations in $(D+1)$-dimensional Minkowski space. They are close analogues of the Julia–Zee dyon [10] in the sense that the 'magnetic' components of the gauge field describe monopoles, namely that they are asymptotically DY fields. In this respect, they are completely different from the *dyonic instantons* [70] in $4 + 1$ dimensions. The 'magnetic' components of the gauge field of the latter describe instantons[17], which by contrast are asymptotically pure gauge.

The Lagrangians from which the Euler–Lagrange equations are derived are those pertaining to the (static) Hamiltonians presented in section 7, which support topologically stable monopoles on $\mathbb{R}^D$. The (static) solutions described here support both the magnetic components F_{ij} and the electric components F_{i0} of the YM curvature $F_{\mu\nu} = (F_{ij}, F_{i0})$. In this respect, these solutions are similar to the Julia–Zee (JZ) dyon [10], the electric component A_0 of the YM connection being introduced as a partner of the Higgs field Φ. This is the crux of the construction of the solutions presented here. However, there are some clear departures between the electric-YM field carrying (static) solutions on $\mathbb{R}^D$ for $D \geqslant 4$, and the JZ dyon on $\mathbb{R}^3$. Before proceeding to describe *pseudo*-dyons in higher dimensions and the *excited*-dyons in three (space) dimensions, it is in order to comment on the JZ dyon itself since the analyticity and stability features of this are not at all on the same firm footing as that of the 't Hooft–Polyakov monopole and the higher-dimensional monopoles described above in section 7.

Concerning existence, the Euler–Lagrange equations of the Julia–Zee dyon arise from the variation of a Lagrangian given on a Minkowskian space, i.e. an action density that is not positive definite. Thus, the proofs of existence of monopoles do not carry over. The existence proof [71] of the Julia–Zee dyon is a much more involved problem. To date, the main description of the JZ dyon is through numerical construction, except in the BPS limit where it is known in the closed form.

---

[17] One might think that in higher dimensions it may be possible to exploit the $4p$-dimensional self-dual instantons described in section 2 but no such solution is identified to date.

The JZ dyon presented in [10] pertains to the full Georgi–Glashow model, with nonvanishing Higgs self-interaction potential. (For an in-depth numerical analysis of this system see [72].) However, it was soon realised that in the absence of the Higgs potential the dyon solutions satisfy the first order Bogomol'nyi equations [73]. In this limit, the monopole satisfies first order Bogomol'nyi equations saturating the topological lower bound. More importantly in this limit, there is a complete symmetry between the Higgs field Φ and the (non-Abelian) electric potential A_0 in the Lagrangian, and hence also in the field equations. As a result, the electric potential obeys the very same first order Bogomol'nyi equations as the Higgs field does. Hence, the existence of the JZ dyon in the BPS limit follows directly from the existence of the BPS monopole.

The new dyon-like configurations described in the present section, in addition to sharing some properties with the JZ dyon (in and out of the BPS limit), also differ from the latter. A qualitative discussion of these features is listed here, before presenting the specific examples. The new dyon-like configurations fall into two main categories:

- *excited*-dyons in 3 + 1 Minkowskian dimensions and
- *pseudo*-dyons in $D + 1$ Minkowskian dimensions, $D \geqslant 4$.

Qualitative properties of these two types of dyon-like configurations are listed here.

- *Excited*-dyons in 3 + 1 dimensions. These partner the monopoles descended from higher-dimensional YM, down to $\mathbb{R}^3$. Examples of these are the monopole in section 7.2 descended from six-dimensional YM, and the monopole in section 7.7 descended from eight-dimensional YM. They have the following qualitative features.

  * The gauge group of the residual (excited) monopole on $\mathbb{R}^3$ being $SO(3)$, the Higgs field Φ is an iso-triplet and hence is also its partner A_0. Thus, both A_0 and the magnetic A_i are iso-triplets. This feature is in common with the Julia–Zee dyon [10].
  * In common with the Julia–Zee dyon, these excited dyons are ascribed an electric flux. This is the surface integral of the electric component, $E_i = \mathcal{F}_{i0}$, of the 't Hooft electromagnetic tensor $\mathcal{F}_{\mu\nu} = (\mathcal{F}_{ij}, \mathcal{F}_{i0})$:

$$
\begin{aligned}
Q &= \frac{1}{4\pi} \int \mathbf{E} \cdot \mathrm{d}\mathbf{S} \\
&= \frac{1}{4\pi} \int \mathrm{Tr}(\Phi F_{i0}) \, \mathrm{d}S_i .
\end{aligned}
\tag{8.1}
$$

  * In contrast with the Julia–Zee dyon, *excited* dyons do not have a BPS limit. This is because their partner monopoles *do not saturate their respective Bogomol'nyi bounds*. In the first case, the YMH model supporting the monopole in section 7.1 features the dimensionful constant κ. The presence of κ obstructs the possibility of finding *self-dual* solutions in exactly the same way as in the Skyrme model. In the second case, namely the monopole in section 7.7 where the YMH model displays only dimensionless constants $\lambda_{(i)}$, the *self-duality* equations are *overdetermined* and again there are no solutions saturating the topological lower bound. The nonexistence of *self-dual* solutions is tantamount to the absence of complete symmetry between the functions parametrizing the Higgs field Φ and those parametrizing the electric YM connection A_0. This will be seen below when the examples in question are presented concretely, and is always the case when the dimensional descent resulting in the monopole is over a codimension greater than 1, when a Higgs potential is present.

    The result is that the excited dyons in both these (categories of) cases are solutions of second-order Euler–Lagrange equations, and not first order self-duality

equations symmetric between electric and magnetic functions. Consequently proving the existence of such dyons is even more involved than that of the JZ dyon.

- *Pseudo*-dyons in $D + 1$ dimensions, $D \geqslant 4$. These partner the monopoles descended from higher-dimensional YM, down to $\mathbb{R}^D$, $D \geqslant 4$. Examples of these are the monopoles in sections 7.3–7.6 descended from eight dimensions. The salient feature in which they differ from the dyons and excited dyons on $\mathbb{R}^3$ is that here there is no simple or natural definition of a (scalar) electric flux. In the context of models describing monopoles, definition (8.1) cannot be extended[18] to $\mathbb{R}^D$ for $D \geqslant 4$. This is because the electric field $F_{i0} = D_i A_0$ decays like $D_i \Phi$, namely as r^{-2}, causing the integral (8.1) to diverge.

  Another special feature of pseudo-dyons (on $\mathbb{R}^d$, $D \geqslant 4$) contrasting with dyonsand excited dyons (on $\mathbb{R}^3$) is the fact that the electric and magnetic components for the YM connection have different multiplet structure. Both Φ and A_0 take their values in the orthogonal complement $L_{i,D+1}$ of L_{ij} ($i = 1, 2, \ldots, D$) of the $SO(D + 1)$ algebra $L_{ab} = (L_{i,D+1}, L_{ij})$ ($a = i, D + 1$), while A_i takes values in L_{ij}, i.e. in $SO(D)$ algebra. Except when $D = 3$, when the algebra of $SO(4)$ splits up into two *chiral* $SU(2)$ pieces, the electric connection A_0 and the magnetic connection A_i do not belong to the same isotopic multiplet, in contrast to $D = 3$, when A_i, A_0 and Φ are all iso-triplets.

  Pseudo-dyons fall in two very different categories: those in $4p$-dimensional Minkowskian spacetimes, and those in the rest.

  * *Pseudo-dyons in $4p$-dimensional spacetimes.* In common with the JZ dyon in the BPS limit, the pseudo-dyon in $7 + 1$ Minkowski space partnering the monopole on $\mathbb{R}^7$ in section 7.3 satisfies first-order self-duality equations that are symmetric between the electric and magnetic functions. In fact, such pseudo-dyons exist in all $4p$-dimensional Minkowskian spacetimes. These are constructed numerically [11], and the analytic proof of their existence is given in [50], the first ($p = 1$) member of that hierarchy being the Julia–Zee dyon itself, in the BPS limit. However, only the latter is a genuine dyon with a scalar electric flux, while all the higher-dimensional members are pseudo-dyons with no natural definition for an electric flux in flat Minkowski space.

  * *Pseudo-dyons in all spacetime dimensions different from $4p$.* The monopoles partnering these dyon-like configurations *do not* saturate their respective Bogomol'nyi lower bounds, even in the case of descents from higher-dimensional YM models not featuring a dimensional constant, i.e. those descending from a *single p-*YM system. Examples of this are the Bogomol'nyi equations (7.15), (7.21) and (7.27) for $D = 6, 5, 4$, respectively. These are all overdetermined and are satisfied only by the trivial solution. Their static Hamiltonians, as well as Lagrangians, feature a Higgs potential which destroys the symmetry between the electric and magnetic potentials in the latter. These are solutions to the second order Euler–Lagrange equations and can be constructed numerically.

---

[18] The magnetic and electric charge densities in $3 + 1$ are the 2-form $\mathcal{F}_{ij}$ and the 1-form $\mathcal{F}_{i0}$ of the 't Hooft electromagnetic tensor. The first of these, $\mathcal{F}_{ij}$, can be seen as the 2-form defined from $\Omega_k^{(2,3)}$, $\Omega_k^{(3,3)}$ and $\Omega_k^{(4,3)}$, displayed in (5.5), (5.18) and (5.38), respectively. Then, the electric charge density can be obtained, formally, by replacing the 2-form indices $[ij]$ with the 1-form $[i0]$ index. In $D + 1$ dimensions, the magnetic part [52] of the 't Hooft electromagnetic tensor $\mathcal{F}_{\mu_1 \mu_2 \ldots \mu_{D-1}}$ is the $(D-1)$-form $\mathcal{F}_{i_1 i_2 \ldots i_{D-1}}$, defined from $\Omega_k^{(p,D)}$ displayed in section 5. It is then natural to define the corresponding electric charge density as the antisymmetric $D - 2$ form $\mathcal{F}_{i_1 i_2 \ldots i_{D-2}, 0}$. On the flat space with spherical boundary restricted to here, there is no natural definition of a flux for this electric tensor. It is possible though that in the presence of gravity spaces with suitable boundaries can be found to accommodate the definition of a scalar electric flux.

There remains to present the examples arising from the dimensional descent of YM in six and eight dimensions. Since one is dealing with static fields only, the component of the YM curvature $F_{i0} = D_i A_0$ is the covariant derivative of the electric YM connection A_0, and it plays a similar role to the covariant derivative $D_i \Phi$ of the Higgs field. Like the Higgs field, the asymptotic behaviour of A_0, consistent with finiteness of energy, will be such that its magnitude tends to a constant at infinity. As long as this magnitude is not larger than that of the Higgs VEV η, the solutions can support a soliton. If the asymptotic value of the magnitude of A_0 is larger than η, the solutions become oscillatory.

The construction of the Lagrangian (for the static fields) on $(D + 1)$-dimensional Minkowski space follows systematically from the static Hamiltonians constructed in section 7 above, as in [10]. In a symbolic way, this construction can be achieved by the following replacements in (7.1), (7.5), (7.10), (7.16), (7.22) and (7.28):

$$
\begin{aligned}
F_{ij} &\to F_{\mu\nu} \\
D_i \Phi &\to D_\mu \Phi
\end{aligned}
\tag{8.2}
$$

taking account always to insert the correct sign in the Lagrangian according to the Minkowskian signature chosen.

While the existence of these dyons is not dependent on the degree of symmetry the full YMH systems are subjected to, it is nevertheless convenient to demonstrate this in the spherically symmetric case. Especially so, since all discussion in the present notes is restricted to spherically symmetric YMH monopoles (and dyons).

To this end, we extend the (static) spherically symmetric ansätze (6.1) and (6.2) for the A_i and Φ fields on $\mathbb{R}^D$ to also include the ansatz for the static spherically electric component A_0 of the YM connection $A_\mu = (A_i, A_0)$ in $d = (D + 1)$-dimensional Minkowski spacetime

$$
A_i^{(\pm)} = \frac{1}{r}(1 - w(r))\Sigma_{ij}^{(\pm)}\hat{x}_j, \qquad A_0^{(\pm)} = u(r)\Sigma_{j,D+1}^{(\pm)}\hat{x}_j, \qquad \Phi = 2\eta h(r)\hat{x}_i \Sigma_{i,D+1}^{(\pm)} \text{ for odd } D
\tag{8.3}
$$

$$
A_i = \frac{1}{r}(1 - w(r))\Gamma_{ij}\hat{x}_j, \qquad A_0 = u(r)\Gamma_{j,D+1}\hat{x}_j, \qquad \Phi = 2\eta h(r)\hat{x}_i \Gamma_{i,D+1} \text{ for even } D.
\tag{8.4}
$$

The (static) Lagrangians on $d = (D + 1)$-dimensional Minkowski spacetime, subject to spherical symmetry on $\mathbb{R}^D$, pertaining to the monopoles discussed in the previous section will be presented here. These Lagrangian support dyons for the models on $\mathbb{R}^3$ and pseudo-dyons for the models on $\mathbb{R}^D$, $D \geqslant 4$.

8.1. Dyon in $d = 3 + 1$ Minkowski space with second CP magnetic charge

This is the JZ dyon in the BPS limit. Its stability follows immediately from the fact that the equations of motion of the electric potential are identical to the equations of motion of the Higgs field, the latter being the Bogomol'nyi equations of the absolutely stable BPS monopole.

Adding the symmetry breaking Higgs self-interaction potential to this system spoils the symmetry between the electric potential and the Higgs field, such that the ensuing equations of motion are no longer solved by the first order Bogomol'nyi equations.

8.2. Excited-dyon in $d = 3 + 1$ Minkowski space with the third CP magnetic charge

This solution is a genuine dyon in that it supports a nonvanishing electric flux, but it is not *the* JZ dyon, the latter being the dyon in $d = 3 + 1$ Minkowski space with *second* CP magnetic

charge. The magnetic charge of the present example is that descended from the *third* CP charge, and this dyon is referred to as an *excited* dyon[19].

In some convenient normalization, the (static) Lagrangian corresponding to the Hamiltonian (7.1) is

$$
\mathcal{L}_3^{(3)} = \mathrm{Tr}\left(2F_{\mu\nu}^2 - \frac{3}{2}D_\mu\Phi^2 - 3S^2 \right) - \frac{\kappa^4}{4}\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu\rho\sigma}^2 - \{F_{[\mu\nu}, D_{\rho]}\Phi\}^2 \right.
$$
$$
\left. - 3(\{S, F_{\mu\nu}\} + [D_\mu\Phi, D_\nu\Phi])^2 + \frac{9}{2}\{S, D_\mu\Phi\}^2 \right). \tag{8.5}
$$

It should be noted here that (8.5) does not result simply from the replacement (8.2), but in addition the first term $\frac{1}{4}F_{\mu\nu\rho\sigma}^2$ in the second line is inserted by hand. Introducing this term in the Lagrangian leads to enhanced symmetry between the doublet of functions (u, h), already present in the *fourth* line of (8.6), also in its *second* line. This is not necessary for the existence of the dyon[20].

Setting $A_0 = 0$ results in the Hamiltonian (8.5) supporting the partner monopole, but of course in the presence of this extra term the energy of the resulting (excited-)dyon is affected.

Subject to spherical symmetry (6.1), and further redefining the numerical coefficients, the density (8.5) reduces to the one-dimensional subsystem

$$
L_3^{(3)} \simeq ([2w'^2 + r^{-2}(1-w^2)^2] - [r^2u'^2 + 2w^2u^2] + \eta^2[r^2h'^2 + 2w^2h^2] + \eta^4r^2(1-h^2)^2)
$$
$$
+ \kappa^4(-([(1-w^2)u]')^2 + \eta^2([(1-w^2)h]')^2
$$
$$
+ \eta^4(2([(1-h^2)w]')^2 + r^{-2}[(1-h^2)(1-w^2) + 2w^2h^2]^2)
$$
$$
- \eta^4(1-h^2)^2(r^2u'^2 + 2w^2u^2) + \eta^6(1-h^2)^2(r^2h'^2 + 2w^2h^2)). \tag{8.6}
$$

The symmetry between the functions u and h in (8.6) is absent. These solutions do not satisfy first-order Bogomol'nyi equations as is the case for the BPS limit of the JZ dyon in (8.1) above, and for the (pseudo) dyon on $\mathbb{R}^7$ in (8.3). It means that the existence of these excited dyons, like the Julia–Zee dyon (not in the BPS limit), do not follow from the existence of the partner monopole.

The static energy density of the excited dyon is given by expression (8.6), with all minus signs replaced by plus signs.

8.3. Pseudo-dyon in $d = 7 + 1$ Minkowski space with fourth CP magnetic charge

This is the dyon living on the 2-BPS monopole on $\mathbb{R}^7$ explicitly discussed in section 7.3. It is in fact a *pseudo*-dyon in the nomenclature used above, in the sense that it describes a nonvanishing electric YM potential A_0, but no scalar electric flux in addition to the magnetic charge. It is a solution to the Euler–Lagrange equations of the corresponding Lagrangian in $d = (7 + 1)$-dimensional Minkowski spacetime.

The Lagrangian corresponding to the Hamiltonian (7.5) is

$$
\mathcal{L}_7^{(1)} = \mathrm{Tr}\left(F_{\mu\nu\rho\sigma}^2 - 4\{F_{[\mu\nu}, D_{\rho]}\Phi\}^2 \right), \qquad \mu = i, 0, \qquad i = 1, 2, \dots, 7
$$
$$
= \mathrm{Tr}\left((F_{ijkl}^2 - 4F_{ijk0}^2) - 4(-\{F_{[ij}, D_{k]}\Phi\}^2 + 3\{F_{[ij}, D_{0]}\Phi\}^2) \right). \tag{8.7}
$$

---

[19] Clearly, there is an infinite tower of such excited dyons, each pertaining to the nth CP magnetic monopole ($n \geqslant 4$, $n = 4$ being the highest order considered here). Even within this remit, there is in addition the excited monopole and dyon descending from eight-dimensional bulk, which we ignore here for simplicity. See footnote 14.

[20] If the symmetry between the doublet of functions (u, h) were complete, then the system would be solved by first-order (Bogomol'nyi) equations identical for u and h. This is not the case here since the symmetry between (u, h) is in any case violated in the first and last lines of (8.6). However, it may be reasonable to maximize this incomplete symmetry, at least on aesthetic grounds.

Subject to the static spherically symmetric ansatz (8.3), the reduced one-dimensional (static) Lagrangian density (8.7) is

$$L_7^{(1)} = \frac{1}{2}(1-w^2)^2\left[4w'^2 + 3\frac{(1-w^2)^2}{r^2}\right] + \frac{1}{2}\eta^2\left[\frac{r^2}{3}([(1-w^2)h]')^2 + 4(1-w^2)^2w^2h^2\right]$$
$$-\frac{1}{2}\left[\frac{r^2}{3}([(1-w^2)u]')^2 + 4(1-w^2)^2\,w^2u^2\right]. \tag{8.8}$$

In contrast with the previous example, the symmetry between the functions (u, h) in (8.8) is complete, and as a result the system is solved by the Bogomol'nyi equations (7.9) of the partner monopole, and identical first-order equations where the electric function u replaces the Higgs function h, *via* the following substitution:

$$h(r) = f(r)\cosh\gamma, \qquad u(r) = \eta\,f(r)\sinh\gamma, \tag{8.9}$$

with a constant parameter γ. This hyperbolic rotation renders the action functional (8.7) identical to the energy functional (7.7), with $h(r)$ replaced by $f(r)$. Thus, the solution [11] of the Bogomol'nyi equations (7.9), augmented by the same first-order equations with u replacing h, yield the dyon field.

This situation can occur in every $4p$-dimensional spacetime, provided that the partner monopole pertains to a model that is descended from the bulk YM system *via* the descent by *one codimension only!* (In those cases, the residual system does *not* feature a Higgs potential.) In the following four examples, the dyons are constructed as solutions to the second order Euler–Lagrange equations arising from the respective Lagrangian, and their existence cannot be inferred *via* a rotation like (8.9) from the existence of the corresponding (partner) monopole.

8.4. Pseudo-dyon in $d = 6 + 1$ Minkowski space with fourth CP magnetic charge

This is a pseudo-dyon with no electric charge defined in Minkowski space. The Lagrangian corresponding to the Hamiltonian (7.10), with convenient normalizations, is

$$\mathcal{L}_6^{(2)} \simeq \mathrm{Tr}\big(F_{\mu\nu\rho\sigma}^2 - \{F_{[\mu\nu}, D_{\rho]}\Phi\}^2 - (\{S, F_{\mu\nu}\} + [D_\mu\Phi, D_\nu\Phi])^2\big). \tag{8.10}$$

Subject to spherical symmetry, (6.2), the density (8.10) reduces to the one-dimensional subsystem

$$L_6^{(2)} \simeq r^{-1}(1-w^2)^2[2w']^2 + r^{-2}(1-w^2)^2) - r[([(1-w^2)u]')^2 + 9r^{-2}(1-w^2)^2w^2u^2]$$
$$+ \eta^2 r[([(1-w^2)h]')^2 + 9r^{-2}(1-w^2)^2w^2h^2]$$
$$+ \eta^4 r^3[([(1-h^2)w]')^2 + 2r^{-2}[(1-w^2)(1-h^2) + 2w^2h^2]^2]. \tag{8.11}$$

This model has no BPS limit and the consequent absence of symmetry between the functions u and h in (8.11) means that the existence of this (pseudo-)dyon does not follow from the existence of its partner monopole.

8.5. Pseudo-dyon in $d = 5 + 1$ Minkowski space with fourth CP magnetic charge

The Lagrangian of this pseudo-dyon corresponding to the Hamiltonian (7.16) is

$$\mathcal{L}_5^{(3)} \simeq \mathrm{Tr}\big(F_{\mu\nu\rho\sigma}^2 - \{F_{[\mu\nu}, D_{\rho]}\Phi\}^2 - (\{S, F_{\mu\nu}\} + [D_\mu\Phi, D_\nu\Phi])^2 + \{S, D_\mu\Phi\}^2\big). \tag{8.12}$$

Subject to spherical symmetry, (6.2), the density (8.12) reduces to the one-dimensional subsystem

$$L_5^{(3)} \simeq r^{-2}(1-w^2)^2[4w'^2 + r^{-2}(1-w^2)^2] - [([(1-w^2)u]')^2 + 6r^{-2}(1-w^2)^2w^2u^2]$$
$$+ \eta^2[([(1-w^2)h]')^2 + 6r^{-2}(1-w^2)^2w^2h^2]$$
$$+ \eta^4 r^2[2([(1-h^2)w]')^2 + 3r^{-2}[(1-w^2)(1-h^2) + 2w^2h^2]^2]$$
$$- \eta^4 r^4(1-h^2)^2[u'^2 + 4r^{-2}w^2u^2] + \eta^6 r^4(1-h^2)^2[h'^2 + 4r^{-2}w^2h^2]. \tag{8.13}$$

Again, this model has no BPS limit and hence the existence of this A_0 carrying solution does not follow from the existence of its partner monopole.

8.6. Pseudo-dyon in $d = 4 + 1$ Minkowski space with fourth CP magnetic charge

The Lagrangian of this pseudo-dyon corresponding to the Hamiltonian (7.22) is

$$\mathcal{L}_4^{(4)} \simeq \mathrm{Tr}\big(F_{\mu\nu\rho\sigma}^2 + \{F_{[\mu\nu}, D_{\rho]}\Phi\}^2 - (\{S, F_{\mu\nu}\} + [D_\mu\Phi, D_\nu\Phi])^2 - \{S, D_\mu\Phi\}^2\big). \qquad (8.14)$$

Subject to spherical symmetry, (6.2), the density (8.14) reduces to the one-dimensional subsystem

$$
\begin{aligned}
L_4^{(4)} \simeq\ & r^{-2}(1-w^2)^2[4w'^2 + r^{-2}(1-w^2)^2] - [([(1-w^2)u]')^2 + 6r^{-2}(1-w^2)^2w^2u^2] \\
& + \eta^2[([(1-w^2)h]')^2 + 6r^{-2}(1-w^2)^2w^2h^2] \\
& + \eta^4 r^2[2([(1-h^2)w]')^2 + 3r^{-2}[(1-w^2)(1-h^2) + 2w^2h^2]^2] \\
& - \eta^4 r^4 (1-h^2)^2[u'^2 + 4r^{-2}w^2u^2] + \eta^6 r^4 (1-h^2)^2[h'^2 + 4r^{-2}w^2h^2] \quad (8.15)
\end{aligned}
$$

with no BPS limit, such that the existence of this pseuodo-dyon does not follow from the existence of its partner monopole.

8.7. Excited–Dyon in $d = 3 + 1$ Minkowski space with fourth CP magnetic charge

Like the excited-dyon in section 8.2, this solution is also a genuine dyon as it supports a nonvanishing electric flux. It differs from the latter, however, since its Lagrangian does not sit on top of that of the Georgi–Glashow model, as a consequence of the distinct-dimensional descent giving rise to its partner monopole which in this case is stabilized by the *fourth* CP magnetic charge. It can nonetheless be described as an(other) *excited* dyon.

In some convenient normalization, the (static) Lagrangian corresponding to the Hamiltonian (7.28) is

$$\mathcal{L}_3^{(5)} \simeq \mathrm{Tr}\big(\tfrac{1}{4} F_{\mu\nu\rho\sigma}^2 - \{F_{[\mu\nu}, D_{\rho]}\Phi\}^2 - (\{S, F_{\mu\nu}\} + [D_\mu\Phi, D_\nu\Phi])^2 + \{S, D_\mu\Phi\}^2 + S^4\big). \quad (8.16)$$

Again as in section 8.2, the term $\tfrac{1}{4} F_{\mu\nu\rho\sigma}^2$ is inserted by hand in (8.16) for the purpose of maximizing the symmetry between the functions u and h in (8.17) below.

Subject to spherical symmetry (6.1) and with some further redefinitions of the numerical coefficients, the density (8.16) reduces to the one-dimensional subsystem

$$
\begin{aligned}
H_3^{(5)} \simeq\ & -r^{-2}([(1-w^2)u]')^2 + \eta^2 r^{-2}([(1-w^2)h]')^2 \\
& + \eta^4[2([(1-h^2)w]')^2 + r^2[(1-w^2)(1-h^2) + w^2h^2]^2] \\
& - r^2(1-h^2)^2[u'^2 + 2r^{-2}w^2u^2] + \eta^6 r^2(1-h^2)^2[h'^2 + 2r^{-2}w^2h^2] \\
& + \eta^8 r^2(1-h^2)^4. \qquad (8.17)
\end{aligned}
$$

The static energy density of the excited dyon is given by expression (8.17), with all minus signs replaced by plus signs.

9. New CS terms

CS densities can be defined in all odd-dimensional spaces irrespective of the signature, i.e. on Minkowskian or Euclidean spaces. They are defined in terms of the gauge connection and curvature, both Abelian or non-Abelian. In the Abelian case, they are very simple quantities and their properties can be easily analysed. Here, we are concerned exclusively with non-Abelian CS densities. We are not concerned at all with CS densities of (Abelian) antisymmetric potentials $A_{\mu_1\mu_2\ldots\mu_n}$ and their field strengths $F_{\mu_1\mu_2\ldots\mu_n\mu_{n+1}}$.

The usual CS densities in odd dimensions defined in terms of the non-Abelian gauge connection are first recalled. This is followed by the subsection in which the new CS terms are introduced in both odd and even dimensions. These are defined in terms of the non-Abelian connection and its partner Higgs field which appears in the dimensionally descended CP terms presented above in section 5. As will be seen below, the new CS densities are exclusively non-Abelian.

9.1. Usual non-Abelian CS terms in odd dimensions

Topologically massive gauge field theories in $(2 + 1)$-dimensional spacetimes were first introduced in [74, 75]. The salient feature of these theories is the presence of a CS dynamical term. To define a CS density, one needs to have a gauge connection, and hence also a curvature. Thus, CS densities can be defined both for Abelian (Maxwell) and non-Abelian (YM) fields. They can also be defined for the gravitational [76] field since in that system too one has a (Lévi-Cività or otherwise) connection, akin to the YM connection in that it carries frame indices analogous to the isotopic indices of the YM connection. Here we are interested exclusively in the (non-Abelian) YM case, in the presence of an *isovector*-valued Higgs field.

The definition of a CS density follows from the definition of the corresponding CP density (1.1). As stated by (1.2), this quantity is a total divergence and the density $\boldsymbol{\Omega}^{(n)} = \Omega_M^{(n)}$ $(M = 1, 2, \ldots, 2n)$ in that case has $(2n)$-components. The CS density is then defined as one fixed component of $\boldsymbol{\Omega}^{(n)}$, say the $2n$th component

$$\Omega_{\mathrm{CS}}^{(n)} = \Omega_{2n}^{(n)} \tag{9.1}$$

which is now given in one dimension less, where $M = \mu, 2n$ and $\mu = 1, 2, \ldots (2n - 1)$.

This definition of a (dynamical) CS term holds in all odd-dimensional spacetimes $(t, \mathbb{R}^D)$, with $x_\mu = (x_0, x_i)$, $i = 1, 2, \ldots, D$, with D being an even integer. That D must be even is clear since $D + 2 = 2n$, the $2n$ dimensions in which the CP density (1.1) is defined, is itself even.

The properties of CS densities are reviewed in [77]. Most remarkably, CS densities are defined in odd (space or spacetime) dimensions and are *gauge variant*. The context here is that of a $(2n - 1)$-dimensional Minkowskian space. It is important to realize that dynamical CS theories are defined on spacetimes with Minkowskian signature. The reason is that the usual CS densities appearing in the Lagrangian are by construction *gauge variant*, but in the definition of the energy densities the CS term itself does not feature, resulting in a Hamiltonian (and hence energy) being *gauge invariant* as it should be[21].

Of course, the CP densities and the resulting CS densities can be defined in terms of both Abelian and non-Abelian gauge connections and curvatures. The context of the present notes is the construction of soliton solutions[22], unlike in [74, 75]. Thus, in any given dimension, our choice of gauge group must be made with due regard to regularity, and the models chosen must be consistent with the Derrick scaling requirement for the finiteness of energy. Accordingly, in all but $2 + 1$ dimensions, our considerations are restricted to non-Abelian gauge fields.

Clearly, such constructions can be extended to all odd-dimensional spacetimes systematically. We list Ω_{CS}, defined by (9), for $D = 2, 4, 6$:

$$\Omega_{\mathrm{CS}}^{(2)} = \varepsilon_{\lambda\mu\nu} \operatorname{Tr} A_\lambda \left[F_{\mu\nu} - \tfrac{2}{3} A_\mu A_\nu \right] \tag{9.2}$$

---

[21] Should one employ a CS density on a space with Euclidean signature, with the CS density appearing in the static Hamiltonian itself, then the energy would not be *gauge invariant*. Hamiltonians of this type have been considered in the literature, e.g. in [78]. In yet another context, CS densities on Euclidean spaces, defined in terms of the composite connection of a sigma model, find application as the topological charge densities of Hopf solitons [79].

[22] The term soliton solutions is used here rather loosely, implying only the construction of regular and finite energy solutions, without insisting on topological stability in general.

$$\Omega_{\text{CS}}^{(3)} = \varepsilon_{\lambda\mu\nu\rho\sigma} \operatorname{Tr} A_\lambda \left[F_{\mu\nu} F_{\rho\sigma} - F_{\mu\nu} A_\rho A_\sigma + \tfrac{2}{5} A_\mu A_\nu A_\rho A_\sigma \right] \tag{9.3}$$

$$\begin{aligned}
\Omega_{\text{CS}}^{(4)} = \varepsilon_{\lambda\mu\nu\rho\sigma\tau\kappa} \operatorname{Tr} A_\lambda \Big[& F_{\mu\nu} F_{\rho\sigma} F_{\tau\kappa} - \tfrac{4}{5} F_{\mu\nu} F_{\rho\sigma} A_\tau A_\kappa - \tfrac{2}{5} F_{\mu\nu} A_\rho F_{\sigma\tau} A_\kappa \\
& + \tfrac{4}{5} F_{\mu\nu} A_\rho A_\sigma A_\tau A_\kappa - \tfrac{8}{35} A_\mu A_\nu A_\rho A_\sigma A_\tau A_\kappa \Big].
\end{aligned} \tag{9.4}$$

Note that (9.2) and (9.3) coincide with the leading terms in (5.35) and (5.27), respectively, except for the chiral matrices Γ_5 and Γ_7 in the latter.

Concerning the choice of gauge groups, one notes that the CS term in $D + 1$ dimensions features the product of D powers of the (algebra-valued) gauge field/connection in front of the trace, which would vanish if the gauge group *is not larger than* $SO(D)$. In that case, the YM connection would describe only a 'magnetic' component, with the 'electric' component necessary for the nonvanishing of the CS density absent. As in [80], the most convenient choice is $SO(D + 2)$. Since D is always even, the representation of $SO(D + 2)$ is the *chiral* representation in terms of (Dirac) spin matrices. This completes the definition of the usual non-Abelian CS densities in $D + 1$ spacetimes.

From (9.2)–(9.4), it is clear that the CS density is *gauge variant*. The Euler–Lagrange equations of the CS density is nonetheless *gauge invariant*, such that for examples (9.2)–(9.4), the corresponding arbitrary variations are

$$\delta_{A_\lambda} \Omega_{\text{CS}}^{(2)} = \varepsilon_{\lambda\mu\nu} F_{\mu\nu} \tag{9.5}$$

$$\delta_{A_\lambda} \Omega_{\text{CS}}^{(3)} = \varepsilon_{\lambda\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{9.6}$$

$$\delta_{A_\lambda} \Omega_{\text{CS}}^{(4)} = \varepsilon_{\lambda\mu\nu\rho\sigma\kappa\eta} F_{\mu\nu} F_{\rho\sigma} F_{\kappa\eta}. \tag{9.7}$$

This and other interesting properties of CS densities are given in [77]. A remarkable property of a CS density is its transformation under the action of an element, g, of the (non-Abelian) gauge group. We list these for the two examples (9.2)–(9.3):

$$\Omega_{\text{CS}}^{(2)} \to \tilde{\Omega}_{\text{CS}}^{(2)} = \Omega_{\text{CS}}^{(2)} - \tfrac{2}{3} \varepsilon_{\lambda\mu\nu} \operatorname{Tr} \alpha_\lambda \alpha_\mu \alpha_\nu - 2\varepsilon_{\lambda\mu\nu} \partial_\lambda \operatorname{Tr} \alpha_\mu A_\nu \tag{9.8}$$

$$\begin{aligned}
\Omega_{\text{CS}}^{(3)} \to \tilde{\Omega}_{\text{CS}}^{(3)} = \Omega_{\text{CS}}^{(3)} & - \tfrac{2}{5} \varepsilon_{\lambda\mu\nu\rho\sigma} \operatorname{Tr} \alpha_\lambda \alpha_\mu \alpha_\nu \alpha_\rho \alpha_\sigma \\
& + 2\,\varepsilon_{\lambda\mu\nu\rho\sigma}\, \partial_\lambda \operatorname{Tr} \alpha_\mu \big[A_\nu \big(F_{\rho\sigma} - \tfrac{1}{2} A_\rho A_\sigma \big) + \big(F_{\rho\sigma} - \tfrac{1}{2} A_\rho A_\sigma \big) A_\nu \\
& - \tfrac{1}{2} A_\nu \alpha_\rho A_\sigma - \alpha_\nu \alpha_\rho A_\sigma \big],
\end{aligned} \tag{9.9}$$

where $\alpha_\mu = \partial_\mu g\, g^{-1}$, as distinct from the algebra-valued quantity $\beta_\mu = g^{-1} \partial_\mu g$ that appears as the inhomogeneous term in the gauge transformation of the non-Abelian curvature (in our convention).

As seen from (9.8)–(9.9), the gauge variation of Ω_{CS} consists of a term which is explicitly a total divergence and another term

$$\omega^{(n)} \simeq \varepsilon_{\mu_1\mu_2\ldots\mu_{2n-1}} \operatorname{Tr} \alpha_{\mu_1} \alpha_{\mu_2} \ldots \alpha_{\mu_{2n-1}}, \tag{9.10}$$

which is *effectively total divergence*, and in a concrete group representation parametrization becomes *explicitly total divergence*. This can be seen by subjecting (9.10) to variations with respect to the function g and taking into account the Lagrange multiplier term resulting from the (unitarity) constraint $g^\dagger g = g\, g^\dagger = \mathbb{1}$.

The volume integral of the CS density then transforms under a gauge transformation as follows. Given the appropriate asymptotic decay of the connection (and hence also the curvature), the surface integrals in (9.8)–(9.9) vanish. The only contribution to the gauge variation of the CS action/energy then comes from the integral of the density (9.10), which (in

the case of Euclidean signature) for the appropriate choice of gauge group yields an integer, up to the angular volume as a multiplicative factor.

All above-stated properties of the CS density hold irrespective of the signature of the space. Here, the signature is taken to be Minkowskian, such that the CS density in the Lagrangian does not contribute to the energy density directly. As a consequence the energy of the soliton is gauge invariant and does not suffer the gauge transformation (9.8)–(9.9). Should a CS density be part of a static Hamiltonian (on a space of Euclidean signature), then the energy of the soliton would change by a multiple of an integer.

9.2. New CS terms in all dimensions

The plan to introduce a completely new type of CS term. The usual CS densities $\Omega_{\mathrm{CS}}^{(n)}$, (9), are defined with reference to the total divergence expression (1.2) of the nth CP density (1.1), as the $2n$th component $\Omega_{2n}^{(n)}$ of the density $\Omega^{(n)}$. Likewise, the new CS terms are defined with reference to the total divergence expression (5.1) of the dimensionally reduced nth CP density, with the dimension D of the residual space replaced formally by $\bar{D}$

$$\mathcal{C}_{\bar{D}}^{(n)} = \nabla \cdot \Omega^{(n,\bar{D})}. \tag{9.11}$$

The densities $\Omega^{(n,\bar{D})}$ can be read off from $\Omega^{(n,D)}$ given in section 5, with the formal replacement $D \to \bar{D}$. The new CS term is now identified as the $\bar{D}$th component of $\Omega^{(n,\bar{D})}$. The final step in this identification is to assign the value $\bar{D} = D + 2$, where D is the spacelike dimension of the $(D + 1)$-dimensional Minkowski space, with the new CS term defined as

$$\tilde{\Omega}_{\mathrm{CS}}^{(n,D+1)} \stackrel{\mathrm{def}}{=} \Omega_{D+2}^{(n,D+2)}. \tag{9.12}$$

The departure of the new CS densities from the usual CS densities is stark, and these differ in several essential respects from the usual ones described in the previous subsection. The most important new features in question are as follows.

- The field content of the new CS systems includes Higgs fields in addition to the YM fields, as a consequence of the dimensional reduction of gauge fields described in section 4. It should be emphasized that the appearance of the Higgs field here is due to the imposition of symmetries in the descent mechanism, in contrast with its presence in the models [81–83] supporting $(2 + 1)$-dimensional CS vortices, where the Higgs field was introduced by hand with the expedient of satisfying the Derrick scaling requirement.

- The usual dynamical CS densities defined with reference to the nth CP density live in $(2n - 1)$-dimensional Minkowski space, i.e. only in odd-dimensional spacetime. By contrast, the new CS densities defined with reference to the nth CP densities live in $(D + 1)$-dimensional Minkowski space for all D subject to

$$2n - 2 \geqslant D \geqslant 2, \tag{9.13}$$

  i.e. in both odd and even dimensions. Indeed, in any given D there is an infinite tower of new CS densities characterized by the integer n subject to (9.13). This is perhaps the most important feature of the new CS densities.

- The smallest simple group consistent with the nonvanishing of the *usual* CS density in $(2n - 1)$-dimensional spacetime is $SO(2n)$, with the gauge connection taking its values in the *chiral* Dirac representation. By contrast, the gauge groups of the new CS densities in $(D+1)$-dimensional spacetime are fixed by the prescription of the dimensional descent from which they result. As *per* the prescription of descent described in section 4, the gauge group now will be $SO(D+2)$, independently of the integer n, while the Higgs field takes its values in the orthogonal complement of $SO(D+2)$ in $SO(D+3)$. As such, it forms an iso-$(D+2)$-vector multiplet.

- Certain properties of the new CS densities are remarkably different for D even and D odd.

  * Odd D. Unlike in the usual case (9.2)–(9.3), the new CS terms are *gauge invariant*. The gauge fields are $SO(D+2)$ and the Higgs are in $SO(D+3)$. D being odd, $D+3$ is even and hence the fields can be parametrized with respect to the *chiral* (Dirac) representations of $SO(D+3)$. An important consequence of this is the fact that now, both (electric) A_0 and (magnetic) A_i fields lie in the same isotopic multiplets, in contrast to the *pseudo*-dyons described in the previous section.

  * Even D. The new CS terms now consist of a *gauge variant* part expressed only in terms of the gauge field, and a *gauge invariant* part expressed in terms of both gauge and Higgs fields. The leading, *gauge variant*, term differs from the corresponding usual CS terms (9.2)–(9.3) only due to the presence of a (chiral) Γ_{D+3} matrix in front of the trace. The gauge and Higgs fields are again in $SO(D+2)$ and $SO(D+3)$, respectively, but now, D being even $D+3$ is odd and hence the fields are parametrized with respect to the (chirally doubled up) full Dirac representations of $SO(D+3)$. Hence, the chiral matrix appears in front of the trace.

As in the usual CS models, the regular finite energy solutions of the new CS models are not topologically stable. These solutions can be constructed numerically.

Before proceeding to display some typical examples in the following subsection, it is in order to make a small diversion at this point to make a clarification. The new CS densities proposed are functionals of both the YM and the 'isovector' Higgs field. Thus, the systems to be described below are Chern–Simons–Yang–Mills–Higgs models in a very specific sense, namely that the Higgs field is an intrinsic part of the new CS density. This is in contrast with Yang–Mills–Higgs–Chern–Simons or Maxwell–Higgs–Chern–Simons models in $(2+1)$-dimensional spacetimes that have appeared ubiquitously in the literature. It is important to emphasize that the latter are entirely different from the systems introduced here, simply because the CS densities they employ are the *usual* ones, namely (9.2) or more often its Abelian[23] version

$$\Omega_{U(1)}^{(2)} = \varepsilon_{\lambda\mu\nu} A_\lambda F_{\mu\nu},$$

while the CS densities employed here are *not* simply functionals of the gauge field, but also of the (specific) Higgs field. To put this in perspective, let us comment on the well-known *Abelian* CSH solitons in $2+1$ dimensions constructed in [81, 82] supporting self-dual vortices, which happen to be unique inasfar as they are also topologically stable. (Their non-Abelian counterparts [83] are not endowed with topological stability.) The presence of the Higgs field in [81–83] enables the Derrick scaling requirement to be satisfied by virtue of the presence of the Higgs self-interaction potential. In the Abelian case, in addition, it results in the topological stability of the vortices. If it were not for the topological stability, it would not be necessary to have a Higgs field merely to satisfy the Derrick scaling requirement. That can be achieved instead, e.g., by introducing a negative cosmological constant and/or gravity, as was done in the $(4+1)$-dimensional case studied in [80]. Thus, the involvement of the Higgs field in conventional (*usual*) CS theories is not the only option. The reason for emphasizing the optional status of the Higgs field in the usual $(2+1)$-dimensional CSH models is that in the new models proposed here the Higgs field is intrinsic to the definition of the (new) CS density itself.

[23] There are, of course, Abelian CS densities in all odd spacetime dimensions but these do not concern us here since in all $D+1$ dimensions with $D = 2n \geqslant 4$, no regular solitons can be constructed.

9.3. Examples

As discussed above, the new dynamical CS densities

$$\tilde{\Omega}_{\text{CS}}^{(n,D+1)}[A_\mu, \Phi]$$

are characterized by the dimensionality of the space D and the integer n specifying the dimension $2n$ of the bulk space from which the relevant residual system is arrived at. As in section 4 above, we restrict attention here to $n = 2, 3$ and 4.

The case $n = 2$ is empty, since according to (9.13) the largest spacetime in which a new CS density can be constructed is $2n - 2$, i.e. in $(1 + 1)$-dimensional Minkowski space which we ignore.

The case $n = 3$ is not empty, and affords us two nontrivial examples. The largest spacetime $2n - 2$, in which a new CS density can be constructed in this case is $3 + 1$ and the next in $2 + 1$ Minkowski space. These two densities can be read off (5.10) and (5.15), respectively:

$$\tilde{\Omega}_{\text{CS}}^{(3,3+1)} = \varepsilon_{\mu\nu\rho\sigma} \text{Tr}\, F_{\mu\nu} F_{\rho\sigma} \Phi \tag{9.14}$$

$$\tilde{\Omega}_{\text{CS}}^{(3,2+1)} = \varepsilon_{\mu\nu\lambda} \text{Tr}\, \gamma_5 \left[-2\eta^2 A_\lambda \left(F_{\mu\nu} - \tfrac{2}{3} A_\mu A_\nu \right) + (\Phi\, D_\lambda \Phi - D_\lambda \Phi\, \Phi)\, F_{\mu\nu} \right]. \tag{9.15}$$

The case $n = 4$ affords four nontrivial examples, those in $5+1, 4+1, 3+1$ and $2+1$ Minkowski space. These densities can be read off (5.22), (5.27), (5.30) and (5.35), respectively:

$$\tilde{\Omega}_{\text{CS}}^{(4,5+1)} = \varepsilon_{\mu\nu\rho\sigma\tau\lambda} \text{Tr}\, F_{\mu\nu} F_{\rho\sigma} F_{\tau\lambda} \Phi \tag{9.16}$$

$$\begin{aligned}
\tilde{\Omega}_{\text{CS}}^{(4,4+1)} = \varepsilon_{\mu\nu\rho\sigma\lambda} \text{Tr}\, \Gamma_7 \big[& A_\lambda \left(F_{\mu\nu} F_{\rho\sigma} - F_{\mu\nu} A_\rho A_\sigma + \tfrac{2}{5} A_\mu A_\nu A_\rho A_\sigma \right) \\
& + D_\lambda \Phi (\Phi F_{\mu\nu} F_{\rho\sigma} + F_{\mu\nu} \Phi F_{mn} + F_{\mu\nu} F_{mn} \Phi) \big]
\end{aligned} \tag{9.17}$$

$$\begin{aligned}
\tilde{\Omega}_{\text{CS}}^{(4,3+1)} = \varepsilon_{\mu\nu\rho\sigma} \text{Tr}\big[& \Phi \left(\eta^2 F_{\mu\nu} F_{\rho\sigma} + \tfrac{2}{9} \Phi^2 F_{\mu\nu} F_{\rho\sigma} + \tfrac{1}{9} F_{\mu\nu} \Phi^2 F_{\rho\sigma} \right) \\
& - \tfrac{2}{9} (\Phi D_\mu \Phi D_\nu \Phi - D_\mu \Phi \Phi D_\nu \Phi + D_\mu \Phi D_\nu \Phi \Phi) F_{\rho\sigma} \big]
\end{aligned} \tag{9.18}$$

$$\begin{aligned}
\tilde{\Omega}_{\text{CS}}^{(4,2+1)} = \varepsilon_{\mu\nu\lambda} \text{Tr}\, \Gamma_5 \big\{ & 6\eta^4 A_\lambda \left(F_{\mu\nu} - \tfrac{2}{3} A_\mu A_\nu \right) \\
& - 6\eta^2 (\Phi\, D_\lambda \Phi - D_\lambda \Phi\, \Phi) F_{\mu\nu} \\
& + [(\Phi^2 D_\lambda \Phi \Phi - \Phi D_\lambda \Phi \Phi^2) - 2(\Phi^3 D_\lambda \Phi - D_\lambda \Phi \Phi^3)] F_{\mu\nu} \big\}.
\end{aligned} \tag{9.19}$$

It is clear that in any $(D + 1)$-dimensional spacetime an infinite tower of CS densities $\tilde{\Omega}_{\text{CS}}^{(n,D+1)}$ can be defined for all positive integers n. Of these, those in even-dimensional spacetimes are gauge invariant, e.g. (9.14), (9.16) and (9.18), while those in odd-dimensional spacetimes are gauge variant, e.g. (9.15), (9.17) and (9.19), the gauge variations in these cases being given formally by (9.8) and (9.9), with g replaced by the appropriate gauge group here.

Static soliton solutions to models whose Lagrangians consist of the above-introduced types of CS terms together with YMH terms are currently under construction [84]. The only constraint in the choice of the detailed models employed is the requirement that the Derrick scaling requirement be satisfied. Such solutions are constructed numerically. In contrast to the monopole solutions, they are not endowed with topological stability because the gauge group must be larger than $SO(D)$ for which the solutions to the constituent YMH model is a stable monopole. Otherwise the CS term would vanish.

9.4. Imposition of spherical symmetry

Since the context of the present notes is one of constructing solitons, it may be helpful to state the ansätze for the fields subjected to spherical symmetry. The monopole and dyon-like solitons presented in sections 7 and 8 were also discussed in the context of spherical symmetry. One reason is that in that framework the DY nature of the monopole fields becomes transparent and exposition of the overdetermination of some of the Bogomol'nyi equations is natural. Also dyonic properties are easy to analyse subject to that symmetry. Less symmetric monopole and dyonic fields can also be studied for the $SO(D)$ gauge fields and iso-D-vector Higgs fields of sections 7 and 8 with a manageable additional effort.

In the context of the new CS density, however, where the gauge fields are $SO(D+2)$ and the Higgs multiplets are iso-$(D+2)$-vectors, imposition of less stringent symmetry than spherical is impractical. Besides, the spherically symmetric expressions in this case help to illustrate the gauge group and multiplet structure presented in the previous subsection concretely.

In $(D+1)$-dimensional spacetime, the gauge connection $A_\mu = (A_i, A_0)$, $i = 1, 2, \ldots D$, takes its values in $SO(D+2)$ and subject to spherical symmetry is parametrized by the pair of triplets $\vec{\xi}(r)$, $\vec{\chi}(r)$ and the triplet[24] $\vec{A}_r(r)$. The Higgs field Φ takes its values in the orthogonal complement of $SO(D+2)$ in $SO(D+3)$ and is parametrized by the triplet $\vec{\phi}(r)$. The explicit expression of this ansatz for the Higgs field and the gauge connection are

$$\Phi = \phi^M \gamma_{M,D+3} + \phi^{D+3} \hat{x}_j \gamma_{j,D+3} \tag{9.20}$$

$$A_0 = -(\varepsilon\chi)^M \hat{x}_j \, \gamma_{jM} - \chi^{D+3} \gamma_{D+1,D+2} \tag{9.21}$$

$$A_i = \left(\frac{\xi^{D+3}+1}{r}\right) \gamma_{ij}\hat{x}_j + \left[\left(\frac{\xi^M}{r}\right)(\delta_{ij} - \hat{x}_i\hat{x}_j) + (\varepsilon A_r)^M \hat{x}_i\hat{x}_j\right] \gamma_{jM}$$
$$+ A_r^{D+3} \hat{x}_i \, \gamma_{D+1,D+2}, \tag{9.22}$$

where ε_{MN} is the two-dimensional Lévi-Cività symbol. In (9.20)–(9.22), the index M runs over two values $M = D+1$, $D+2$, such that $\xi \equiv (\xi^M, \xi^{D+3})$, $\vec{\chi} \equiv (\chi^M, \chi^{D+3})$, $\vec{A}_r \equiv (A_r^M, A_r^{D+3})$ and $\vec{\phi} \equiv (\phi^M, \phi^{D+3})$. The spin matrices $\gamma_{ab} \equiv (\gamma_{ij}, \gamma_{iM}, \gamma_{i,D+3}, \gamma_{MN}, \gamma_{M,D+3})$ are the generators of $SO(D+3)$ in a unified notation to cover both the Dirac, Γ_{ab}, and the chiral, Σ_{ab}, as the case may be. It can now clearly be seen from (9.20)–(9.22) that all components of the gauge connection $A_\mu = (A_i, A_0)$ take their values in the algebra of $SO(D+2)$, with the Higgs field in the orthogonal complement of $SO(D+2)$ in $SO(D+3)$. This is in essential contrast with the Julia–Zee-type dyons for which the electric component of the gauge connection A_0 does not have the same multiplet structure as the magnetic component A_i. It has instead precisely the same multiplet structure as the Higgs field as seen in (8.3)–(8.4).

The above choice of the multiplet structure of A_0 is only one of two possibilities, namely the one that differs from the Julia–Zee case. In the latter case, to support solutions regular at the origin in $\mathbb{R}^D$ with $D \geqslant 4$, A_0 has to take its values outside the algebra of $SO(D)$, and like the Higgs field Φ, in the orthogonal complement in $SO(D+1)$. Here, however, the gauge group is not $SO(D)$, but rather $SO(D+2)$ permitting nonvanishing (new-)CS density. As a consequence it was possible to let A_0 take its values in the algebra of $SO(D+2)$ like the magnetic component of the connection A_i and unlike Φ. It is, however, just as legitimate

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[24] This triplet, $\vec{A}_r(r)$, plays the role of a connection in the residual one-dimensional system after the imposition of symmetry and encodes the $SO(3)$ arbitrariness of this ansatz. In one dimension, there is no curvature; hence, it can be gauged [80] away in practice.

to choose instead that A_0 take its values in the orthogonal complement of $SO(D+2)$ in $SO(D+3)$. In that case, the spherically symmetric ansatz (9.21) for A_0 would be replaced by

$$A_0 = \chi^M \gamma_{M,D+3} + \chi^{D+3} \hat{x}_j \gamma_{j,D+3}. \tag{9.23}$$

Solitons of such CSH models in $3+1$ and $2+1$ dimensions are now under active consideration [85].

10. Outlook

Solitons in higher dimensions, both in flat space and their gravitating versions, have the potential to be employed in the construction of string theory solitons. (The presentation in these notes is restricted to considerations in flat space. The gravitating versions follow systematically, and are deferred.) Solitons of gauge fields play a very special role in this context, because both Abelian and non-Abelian matter feature prominently in heterotic string theory and in supergravities.

The present notes are restricted to non-Abelian gauge fields, and more specifically to YMH systems. Solitons of the YM systems, the instantons, are topologically stable in even spacelike dimensions, stabilized by CP charges. The gauge connections of these solutions are asymptotically pure gauge, resulting in vanishing curvature on the boundary. Solitons of the YMH systems of the type considered in these notes (namely with the Higgs field taking values of an isotopic D-vector), on the other hand, are topologically stable monopoles which can be defined in both even and odd spacelike dimensions. These are asymptotically DY monopoles. The major difference of monopoles and instantons is that the gauge connection of a monopole is asymptotically one-half pure-gauge and not pure-gauge, resulting in nonvanishing curvature on the boundary. This difference can be important in certain applications, in particular to some examples in the AdS–CFT correspondence.

Another important feature of YMH systems generally, versus YM systems, is that the solutions to the former exhibit symmetry breaking, and the presence of the dimensionful vacuum expectation value of the Higgs field breaks the scale. In the presence of a Higgs self-interaction potential, the resulting soliton is exponentially localized.

The main limitation of monopoles is that they are asymptotically DY fields. As such the YM curvature decays as r^{-2} asymptotically, and hence in all spacelike dimensions higher than *three* the *usual* YM energy/action is divergent. Only high enough order YM curvature terms decay appropriately to yield finite energy in any given (higher) dimension. Thus, the requirement of finite energy in higher dimensions restricts the models to be employed to consist only of higher order YM curvature terms, in the absence of the usual quadratic YM term[25] as the case may be.

This limitation is somewhat mitigated in the case of YMH models derived from the dimensional descent over codimensions N, with $N \geqslant 2$. In all those cases, the quadratic YM density can appear in a specific guise, namely *via* the term

$$\mathrm{Tr}\left(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi]\right)^2, \qquad S = (\eta^2 + \Phi^2)$$

in the Hamiltonian density, e.g., in (7.28), (7.22), (7.16), (7.1) and (7.10). Clearly, the term $\eta^4 \mathrm{Tr}\, F_{ij}^2$ is present here, albeit in a rather couched manner. In the cases arising from the descent over unit codimension $N = 1$, such terms are absent since the presence of a Higgs potential in the residual system is predicated in the presence of nontrivial

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[25] It may be worth to digress here to mention that pure YM systems in the absence of a Higgs field are not subject to this limitation. In the absence of a Higgs field, the solitons of pure YM systems are instantons [8, 12], which are *pure-gauge* at infinity. This faster decay enables the inclusion of the usual (quadratic) YM term in higher dimensions.

components of the curvature on the codimension. Unfortunately, these models on $\mathbb{R}^{4p-1}$ are interesting as they are the only ones for which the monopole saturates the Bogomol'nyi lower bound.

This limitation need not necessarily be a disadvantage since such applications have been usefully made in the literature: in strings from five branes [86], closed strings from instantons [88] and cosmic strings from open heterotic string [89], where the only YM term appearing is F^4. The actual field configurations employed in [86, 88, 89], as in the earliest prototype [87] for the heterotic string, are those of *instantons* on $\mathbb{R}^8$ and $\mathbb{R}^4$, respectively, and not of *monopoles*. But the usual BPS *monopole* on $\mathbb{R}^3$ is also used [90] in this context. Obviously, the BPS *monopole* on $\mathbb{R}^7$ of section 7.3 above can likewise be used analogously to the use of the eight-dimensional instanton in [86]. This limitation applies to the construction of monopoles, dyons and the solitons of the new CS terms (which are functionals of both gauge and Higgs fields, defined in all dimensions) presented in sections 7, 8 and 9, respectively.

Another direction for applications is the construction of supersymmetric self-gravitating solitons [91], where both *instantonic* (in the absence of Higgs fields) and *monopolic* field configurations are employed. In the case where monopole field configurations are employed in [91], the relation of the latter with the mechanism applied in [90] has been explored in [92]. In the case where instanton field configurations are employed in [91], this has been extended to cover the case where instead of the instanton, the *dyonic instanton* [70] is employed [93]. Unfortunately this latter scheme cannot be extended to higher dimensions, due to the absence of higher-dimensional dyonic instantons. The higher-dimensional extensions of such models, with the exception of the last, [93], can proceed systematically, either using the higher-dimensional *instantons* alluded to in section 2 or the *monopoles* in section 7. Like in the generalizations [86, 89] of [87], here too higher order YM and YMH terms will appear in the absence of the usual *quadratic* YM and YMH densities.

Concerning the new CS densities introduced here, these can be applied to problems of gravitating gauge fields. Indeed CS densities play a central role in [91] and [93], the latter being related to the new CS terms. (This analysis is pending.) One interesting outcome of the application of CS densities is their effect on the thermodynamic stability of black holes, as discovered in [94].

In all possible further generalizations of [87, 90, 86, 88, 89], e.g., [86, 88, 89], and of [91], one *caveat* is that the field configurations exploited must satisfy self-duality or Bogomol'nyi equations. In the absence of a Higgs field, e.g. for [86, 88, 89] and some of the systems considered in [91], this criterion is satisfied if one employs the instantons on $\mathbb{R}^{4p}$. Alternatively, this can be achieved by using instead, instantons on S^{2n}, CP^n and other symmetric compact coset spaces. In the presence of a Higgs field, these configurations are the monopoles in section 7, and most of these do not saturate the Bogomol'nyi bound. The only exceptions, whose Bogomol'nyi equations are not overdetermined, are the monopoles on $\mathbb{R}^{4p-1}$ and the (generalized) vortices on $\mathbb{R}^2$.

Looking forward, it may be worthwhile to consider question of the limitation of YMH solitons in dimensions higher than $3+1$, where the usual (quadratic) YM and YMH terms are absent. This is a feature of Higgs theories. The presence of a Higgs field, along with a symmetry-breaking Higgs self-interaction potential, is necessary only when exponential localization is required, and the scale is broken by the vacuum expectation value of the Higgs field. Should the presence of (usual quadratic) YM term in dimensions higher than $3+1$ be required, there are two alternatives to using a Higgs model. With both these choices, the solitons employed are instantonic and asymptotically there is power decay. These two options are as follows.

- In the first variant, the Higgs field is simply absent. The resulting configurations are instantons, which are asymptotically *pure-gauge* and decay fast enough. As a consequence finite energy solitons are consistent with the presence of (usual) quadratic YM terms. In all dimensions higher than $4 + 1$, these theories are necessarily scale breaking and the instantons display power decay. (See section 2.) In the appropriate (necessarily even spacelike) dimensions, these solitons are stabilized by a Pontryagin charge, e.g. the example in [15] on $\mathbb{R}^6$.

- The second variant involves gauged, S^n-valued sigma models[26], which also exhibit instantonic configurations. These are the solitons of the $SO(N)$ gauged $O(D + 1)$ sigma models on $\mathbb{R}^D$ with $2 \leqslant N \leqslant D$. The $SO(2)$ gauged solitons (vortices) of the $O(3)$ sigma model on $\mathbb{R}^2$ were given in [96], the $SO(3)$ gauged solitons of the $O(4)$ sigma model on $\mathbb{R}^3$ in [97, 98], and the $SO(4)$ gauged solitons of the $O(5)$ sigma model on $\mathbb{R}^4$ in [98]. The generic case of $SO(D)$ gauged $O(D + 1)$ sigma models on $\mathbb{R}^D$ is discussed in [99]. But what is peculiar to gauged $O(D + 1)$ sigma models on $\mathbb{R}^D$ is that they can be $SO(N)$ gauged with $2 \leqslant N \leqslant D$. Such an example is given in [100, 101] for the $SO(2)$ gauged $O(4)$ sigma models on $\mathbb{R}^3$. This is in contrast to (gauged) Higgs models on $\mathbb{R}^D$ of the type described in section 7, with iso-D-vector Higgs which support monopoles. In the latter case, the gauge group must be $SO(D)$ since the covariant derivative of the hedgehog field at infinity must vanish.

Using either of these variants for the purpose of constructing solitons falls outside the remit of this review. What is perhaps more interesting and hence may deserve a comment is the corresponding definition of new CS terms in the context of the second variant mentioned above. These would be the analogues of the new CS densities defined in terms of the gauge and Higgs field, proposed in section 9. In this case they would be defined instead in terms of the gauge and the sphere-valued sigma model fields. These putative Chern–Simons forms would be extracted from the topological charge densities of *gauged S^n*-valued sigma models. The latter are not expressed explicitly as *total divergences*, but are essentially total divergences in the sense that when subjected to the variational calculus, the resulting Euler–Lagrange equations turn out to be trivial. Not having a local expression for this topological charge as a total divergence, the extraction of a CS form from it is a challenge which we have eschewed so far. The advantage of such CS forms over the Higgs analogue (of section 9) is that the instantonic decay of the gauge field now would enable the retention of the usual YM term. The advantage over the first variant on the other hand, namely in the absence of a scalar field, is that the topological charge density of gauged $O(D + 1)$ sigma models on $\mathbb{R}^D$ is defined for all (even and odd) D, in contrast to the purely gauge field case where D must be even, only.

Acknowledgments

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[26] The proposal of such a construction is one of the results in [95].

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