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Cauchy Hypersurfaces and Global Lorentzian Geometry

Miguel Sánchez [1]

[1] *Depto. Geometría y Topología, Universidad de Granada*

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Abstract

Our purpose is to give a taste on some global problems in General Relativity, to an audience with a basic knowledge on intrinsic Differential Geometry. We focus on the following problems related to the fundamental concept of *Cauchy hypersurface*: (1) smoothability and structure of globally hyperbolic spacetimes, (2) initial value problem, (3) singularity theorems, (4) cosmic censorship and Penrose inequality. Finally, some open questions are commented.

Keywords: Global Lorentzian Geometry, Cauchy hypersurface, globally hyperbolic, smoothability, Einstein equation, initial value problem, singularity theorems, ADM mass, cosmic censorship hypotheses, Penrose inequality

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1. Introduction

General Relativity is an impressive physical theory, but nowadays it can be regarded essentially as a branch of Geometry (Lorentzian Geometry), in a similar sense of Rational Mechanics. Nevertheless, the physical intuitions which lead to geometric results are subtler and less evident in General Relativity than in Mechanics and, thus, many mathematicians are reluctant to study it. But once this difficulty is overcome, a new geometric world is open, including unsuspected new problems in (positive definite) Riemannian Geometry. Our purpose is to provide to the mathematically oriented reader a brief overview of this wonderful world, which maybe closer to his formation than he may expect. We focus on global problems, which are usually the most interesting for mathematicians (see M. Santander's contribution to these proceedings for local ones). We hope to be also of some interest for physicists. Frequently, they are very familiar with local Differential Geometry, but global problems are neglected as mathematical speculations, with scarce interest for more experimental purposes. Nevertheless, global questions orientate the full theory and have implications in more practical ones.

From the mathematical viewpoint, one can distinguish different approaches in the research on Lorentzian Geometry as:

1. *Looking at Riemannian Geometry.* That is, trying to adapt the preexisting Riemannian tools and results to the Lorentzian case, as far as possible. This was the case at the beginning of General Relativity, and also the typical starting point of a standard mathematician –who has studied Riemannian Geometry but not Lorentzian one. This is not as trivial as it sounds, because Lorentzian and Riemannian geometries, in spite of sharing common roots, diverge fast.

2. *Developing specific Lorentzian tools.* Say, concepts as causality, boundaries, conformal extensions (Penrose diagrams), asymptotic behaviors (spatial and null infinities) or black holes, appear in a highly particular way in Lorentzian Geometry, with no analog in the Riemannian case. Here, physical intuitions are a very important guide, but we emphasize that these concepts have a completely tidy mathematical definition.

3. *Feed back to Riemannian Geometry.* Sometimes, the problem in Lorentzian Geometry admits a full reduction to a purely Riemannian problem. This problem may be unexpected from a purely Riemannian approach, but now it becomes natural. The initial value problem for Einstein equation, the positive mass theorems (which yield the last step in the solution to Yamabe problem!) or Penrose inequality, provide remarkable examples of this situation.

In what follows, four global problems in Lorentzian Geometry which come from General Relativity are briefly explained. In our choice of problems, Cauchy hypersurfaces will play an important role. The reason is twofold: on one hand, they play a central role in global problems, on the other, they are very intuitive and easy to be understood.

2. An emergency kit for Lorentzian Geometry.

A *Lorentzian manifold* is a $n(\geq 2)$-manifold M, endowed with a non-degenerate metric g of index 1, $(-,+,\ldots,+)$ (see [3, 30, 41, 54] for background). Following [41], a tangent vector $v \in T_pM$ is *causal* if it is either *timelike*, i.e. $g(v,v) < 0$, or *lightlike*, i.e. $g(v,v) = 0, v \neq 0$, otherwise, v *is spacelike* (in particular, this convention means that $v = 0$ is spacelike). These definitions for vectors are extendible naturally to curves and hypersurfaces.

A spacetime (M,g) is a connected Lorentzian manifold which is implicitly assumed to be "time-oriented", i.e. a causal cone at each $T_pM, p \in M$ (the "*future*" causal cone, in opposition to the non-chosen one, or "past" cone) has been continuously chosen. In particular, in a spacetime, one can speak on a *timelike curve* $\gamma : I \subseteq \mathbb{R} \to M$, which will be called "(the trajectory of) an observer", if it is future-directed and normalized to speed one $g(\gamma', \gamma') \equiv -1$; the observer is called *freely falling* if γ is a geodesic. Future-directed lightlike geodesics are regarded as "(trajectories of) light rays". No good interpretation holds for spacelike curves, even if they are geodesics, except for very special classes of spacetimes.

Some concepts work formally equal in Lorentzian and Riemannian Geometry, as the Levi-Civita connection and the curvature tensor, or with small modifications (sectional curvature is defined only for "non-lightlike" planes). But the following ones are *very different*:

1. Conformal structures are "visualizable": two Lorentzian metrics g, g' are conformal ($g' = \Omega g$, $\Omega > 0$) if and only if they have equal causal cones.

2. Inequalities of curvature are naturally imposed in separate causal types. For example, the inequality for the Ricci curvature $(\mathrm{Ric}(v,v)/g(v,v)) \leq 0)$ for any timelike vector v, is a natural one –in fact, this is called the *timelike convergence equation*. Nevertheless, the same inequality for all $v \in M$ with $g(v,v) \neq 0$ implies constant Ricci curvature.

3. Geodesics, even though defined formally equal than in the Riemannian case, may behave in a very different way: (i) M compact does not imply geodesic completeness or connectedness, (ii) completeness for the three causal types of geodesics (timelike, lightlike, spacelike) are logically independent, (iii) conjugate points along a spacelike geodesic may accumulate (and contain an open interval)...

   Good analogies only happen for timelike geodesics: locally they "maximize length", in a similar way as Riemannian geodesics minimize it. Lightlike geodesics also "maximize locally among connecting causal curves", but they present some important differences (see, for example, [40, Section 2]).

3. Causality and Cauchy hypersurfaces

Associated to the conformal structure of a spacetime one can define two binary relations: (a) Chronological relation: $p \ll q$ iff there exists a future-directed timelike curve from p to q, (b) Causal relation: $p < q$ iff there exists a future-directed causal curve from p to q. Then, the future and past of points (or analogously subsets) can be defined in a natural way, say: chronological future $I^+(p) = \{q \in M : p \ll q\}$, causal future $J^+(p) = \{q \in M : p \leq q\}$ ($p \leq q$ means either $p < q$ or $p = q$, of course), and analogously with the pasts $I^-(p), J^-(p)$. Locally, either of these relations characterizes the conformal structure, and traditionally Causality is identified to conformal geometry in Lorentzian signature (even though a convenient modification has been recently introduced by García-Parrado and Senovilla [26], see also [25]). These concepts suggest natural conditions to impose to a spacetime:

- To avoid paradoxes travelling to the past ("grandfather's paradox").

  From less to more restrictive: (1) chronology: no closed timelike curves exist, (2) causality: no closed causal curves, (3) strong causality: no "almost closed" causal curve exists (for each $p \in M$ and any neighbourhood $V \ni p$ there exists a neighborhood $U \subset V$ such that any causal curve with endpoints in U is totally contained in V), (4) stable causality: close metrics to the original one (in the C^0-topology of metrics) are causal.

- To avoid that "information from p to $q(\geq p)$ can escape to (or suddenly appear from) infinite" or the absence of "naked singularities": compactness of the diamonds $J^+(p) \cap J^-(q), \forall p, q \in M$. The colorful physical names are suggested because, when the property does not hold, there exists a sequence of causal curves γ_n from p to q which admits as a (lower) limit an inextendible causal curve γ which starts at p but does not approach q –i.e., before reaching q, γ disappears from spacetime losing possible information. Nevertheless, γ lies in the past of q and, thus, this "singular behaviour" (which can be called more properly singular in the context of gravitational collapse, see below) is visible from q.

Notice that these conditions become natural even from a phylosophical viewpoint and, in fact, a spacetime is called *globally hyperbolic* when it comprises both, that is, the diamonds are compact and the spacetime is strongly causal (and, then, stably causal too).

Global hyperbolicity is the strongest "commonly accepted" assumption for physically reasonable spacetimes –it lies at the top of the standard "causal hierarchy" of spacetimes. Nevertheless, it is not excessively restrictive: it plays a role similar to completeness for Riemannian manifolds. Even more, it implies that the space of causal curves joining each two $p \neq q$ is compact and, thus, if $p < q$ there exists a future-directed length-maximizing causal geodesic from p to q.

An important theorem due to Geroch [27] characterizes globally hyperbolic spacetimes as those spacetimes which admit a a *Cauchy hypersurface*, i.e., a subset $S \subset M$ which is intersected exactly once by any inextendible timelike curve ("the whole 'space' at an instant of time"). Necessarily, any Cauchy hypersurface is a (embedded) topological hypersurface, but perhaps it is not a smooth one. This leads us to our first problem.

4. Smoothability and structure of glob. hyp. s.-t.

More precisely, Geroch proved that if a (topological) Cauchy hypersurface exists then the spacetime is globally hyperbolic and, conversely:

Theorem 4.1 *If M is globally hyperbolic, there exists a continuous function $t : M \to \mathbb{R}$ such that:*

(1) t is strictly increasing on any future-directed causal curve (i.e., t is a t*ime function).*

(2) Each level $S_a := t^{-1}(a)$ is a Cauchy hypersurface $\forall a \in \mathbb{R}$.

The result is obtained at a topological level and, in fact, one may obtain from the constructive proof a non-smooth t (by *smooth* we mean as differentiable as permitted by the order of differentiability of the spacetime). Each level $t =$ constant becomes a topological Cauchy hypersurface, which is also crossed exactly once by any inextendible causal curve. It is not difficult to check that M is then homeomorphic to $\mathbb{R} \times S$.

Thus, from the mathematical viewpoint, there are obvious problems of smoothability. The simplest one is to wonder if there exists a smooth and spacelike Cauchy hypersurface [45]. More accurately, we can wonder if there exists a *Cauchy temporal function* $\mathcal{T}$, where "temporal" means that $\mathcal{T}$ is a smooth function with past-directed timelike gradient (and, thus $\mathcal{T}$ is a time function) and, "Cauchy" means that its levels are Cauchy hypersurfaces (now necessarily smooth and spacelike). Remarkably, the existence of such a $\mathcal{T}$ implies not only a smooth splitting $M \equiv \mathbb{R} \times S$ but also an *orthogonal one*:

$$g = -\beta d\mathcal{T}^2 + g_{\mathcal{T}}, \tag{1}$$

where $\beta : M \equiv \mathbb{R} \times S \to \mathbb{R}^+$ is a positive function, and $g_{\mathcal{T}}$ is a Riemannian metric on each level $S_{\mathcal{T}}$, smoothly varying with $\mathcal{T}$.

The interest of these questions are obvious. On one hand, (smooth) spacelike Cauchy hypersurfaces are the natural ones for many properties (Einstein equation, Penrose inequality...; see the next sections); on the other, the orthogonal splitting is useful for many properties: Morse Theory, quantization, to find global coordinates, etc. These problems must not be overlooked by physicists as minor questions of mathematical rigour. Recall that the (almost phylosophical) requirements in the definition of global hyperbolicity are satisfactory from the physical viewpoint. But the *a priori* assumption of a splitting of the spacetime as (1) (the type of expression truly useful for many physicists) is a completely unjustified one.

These questions have been answered affirmatively very recently. Briefly (see [6, Section 2] for an expanded summary), a smoothing procedure claimed by Seifert [47] (cited in [30] and then in many references) presented some gaps. Thus, Sachs and Wu [45, p. 1155] posed the existence of a *smooth* Cauchy hypersurface in any globally hyperbolic spacetime as an open "folk" problem. Dieckmann's attempt [21] was not sufficient, and the problem persisted [3]. The full solution has been obtained by proving first the existence of a *spacelike* Cauchy hypersurface [5] and, then, the existence of a Cauchy temporal function [7] (including the splitting (1)). Other related questions have been solved also recently (see [8, 46]).

5. Initial value problem

Einstein field equation can be written (in suitable units) as

$$\mathrm{Ric} - \frac{1}{2} S g = 8\pi T. \tag{2}$$

Here, the geometric terms at the left hand side (Ricci tensor Ric, scalar curvature S) are related to a symmetric 2-tensor at the right hand, the "stress-energy" T, which describes the distribution of matter/energy.

More properly, we must emphasize that the unknown quantity is not only the metric g (with Ric and S): equations for T must be added to get a coupled system with (2). Nevertheless, we will consider for simplicity (in addition to dimension(M)=4, when necessary):

- Along this section, $T = 0$ (vacuum), i.e. (2) becomes Ric$\equiv 0$.
- Along the next sections, solutions with T non-determined but satisfying only any of the (mild) "energy conditions" as: (1) Weak: $T(v, v) \geq$

0 for any timelike v (density energy is nonnegative), (2) Dominant: $-T(v,\cdot)^\flat \equiv -g^{ij}T_{jk}v^k$ is either future-directed causal or 0 for any future timelike v (energy flow is causal), (3) Strong: equivalent via Einstein equation to the timelike convergence condition, $\mathrm{Ric}(v,v) \geq 0$ for timelike v (gravity, on average, attracts).

The well-posedness of Einstein equation requires an input of initial data on a 3–manifold Σ which permits to obtain a ("unique, maximal") spacetime (and eventually a T) such that Σ is embedded in M consistently with the initial data. The problem is complicated: of course, a classical theorem such as Cauchy-Kovalewski's is not applicable, and, even more, in principle the system of equations is not hyperbolic. Nevertheless, there exist a highly non-trivial procedure –based on the existence of *harmonic coordinates*– which allows one to find an equivalent (quasi-linear, diagonal, second order) hyperbolic system. The standard global result was obtained by Choquet-Bruhat and Geroch [15]:

Theorem 5.1 *Let (Σ, h) be a (connected) Riemannian 3-manifold, and σ a symmetric two covariant tensor which satisfies the compatibility conditions of a second fundamental form (Gauss and Codazzi eqns.) Then there exist a unique spacetime (M, g) satisfying the following conditions:*

(i) $\Sigma \hookrightarrow M$, consistently with h, σ (i.e., $h = g|_\Sigma$ etc.)

(ii) Vacuum: Ric$\equiv 0$ (this can be extended to more general T).

(iii) Σ is a Cauchy hypersurface of (M, g).

(iv) Maximality: if (M', g') satisfies (i)—(iii) then it is isometric to an open subset of (M, g).

We emphasize that, for the well posedness of the problem, property (iii) becomes essential. In fact, the existence of a solution spacetime can be proven because no timelike curve crosses Σ twice, and the uniqueness because all timelike curves cross Σ at least once.

Remark 5.2 *(SCCC).* Even though the solution (M, g) provided by Theorem 5.1 is maximal, it may be extendible as a spacetime, that is, (M, g) may be isometric to an open proper subset of another spacetime $(\bar{M}, \bar{g})$ –even a vacuum one. In this case, Σ cannot hold as a Cauchy hypersurface of the extension, and two possibilities arise: (a) $(\bar{M}, \bar{g})$ is not globally hyperbolic or (b) the initial Σ was not "chosen adequately", as an input hypersurface for a whole physically meaningful spacetime. Thus, one can wonder: can the inextendability of (M, g) be characterized?

This question becomes extremely important in General Relativity because, of course, one thinks that "our" physical spacetime is inextendible. And one can wonder if it must be "predictable" from initial data and, thus, globally hyperbolic.

The *Strong Cosmic Censorship Conjecture (SCCC)* asserts that, for generic physically reasonable data (including a "good choice" of Σ), (M, g) is inextendible. Of course, a non-trivial problem of the conjecture, is to explain carefully what "generic physically reasonable data" means.

A systematically studied problem is to characterize/classify the solutions of (vacuum) Einstein equation. By using Theorem 5.1, this is rather a purely Riemannian problem (roughly: given data as, say, (Σ, h), classify the σ's which satisfy Gauss and Codazzi equations). There are two specially important methods of solution (see [2] for a detailed exposition):

- *Conformal.* Initial data are divided into two subsets: a subset of *freely specified* conformal data (the conformal class of h, a scalar field τ, and a symmetric divergence free 2-tensor $\tilde{\sigma}$), and a subset of *determined* data (a function $\phi > 0$, a vector field $W \in \chi(\Sigma)$), which are derived from the free data by means of differential equations. The interpretation and equations for these data vary with two types of conformal method (the method (A) or semi-decoupling, whose origin goes back to Lichnerowicz [36], and the method (B) or conformally covariant). The problem is then to show if there exists solutions for the equations of the determined data, and classify them.

- *Gluing solutions.* As a difference with the conformal method, this is not a general one, but it is very fruitful in relevant particular cases. Corvino and Schoen [18, 19] glue any bounded region of an asymptotically flat spacetime with the exterior region of a slice of Kerr's –this case becomes specially interesting as the "no hair theorems" highlight Kerr spacetime at the final state of the evolution of a black hole. The useful gluing by Isenberg et al. ([33], see also the initial data engineering [17]) constructs consistent initial data for Einstein equation from the connected sum of previously obtained data (for example, construction of wormholes).

For the general conformal method, the results depend on different criteria —topology of Σ, asymptotic behaviour, regularity (analytic, smooth, Hölder class...), metric conformal class (Yamabe type)... The most important one is the mean curvature H. Essentially, when H is constant almost all is known (at least if $T = 0$); in fact, if Σ is either compact without boundary, or asymptotically flat or hyperbolic, it is completely determined which solutions exist (and they exist for all but certain special cases). When H is nearly constant there are many results, but also many open questions; otherwise, there are very few results.

6. Singularity theorems

In some concrete spacetimes, singularities might be defined "by hand" but a general definition is difficult [28], for example:

1. The singularity will not be a point of the spacetime.
2. It should be no placed "at infinity" -but no natural notion of infinity exists in general.
3. Curvature tensor R is expected to diverge, but all its scalar invariants $(\sum R_{ijkl}R^{ijkl}, \sum \nabla_s R_{ijkl}\nabla^s R^{ijkl}, S...)$ may vanish when $R \neq 0$.

At any case, some sort of "strange disappearance" happens if the spacetime is *inextendible, but an incomplete causal geodesic exists*, and these two conditions will be regarded as *sufficient* for the existence of a singularity. Then, the aim of the so-called *singularity theorems* is to prove that causal incompleteness occurs under general natural conditions on T (an energy condition) and on the causality of the manifold, as global hyperbolicity. Nevertheless, recall that, rather than "singularity" results, they may be "incompleteness" ones: the physical conclusion of these theorems could be that a physically realistic spacetime cannot be globally hyperbolic, rather than being singular. So, they become "true singularity" results when an assumption as global hyperbolicity is removed... or if SCCC (Remark 5.2) is true!

Recall the following Hawking's singularity theorem (see [30] or [41] for a detailed exposition):

Theorem 6.1 *Let (M, g) be a spacetime such that:*

1.- It is globally hyperbolic.

2.- Some spacelike Cauchy hypersurface S strictly expanding $H \geq C > 0$ (H: futur. mean curvat. -expansion means "on average")

3.- Strong energy holds: $Ric(v, v) \geq 0$ for timelike v.

Then, any past-directed timelike geodesic γ is incomplete.

Sketch of proof. The last two hypotheses imply that any past-directed geodesic ρ normal to S contains a focal point if it has length $L' \geq \frac{1}{C}$. Thus, once S is crossed, no γ can have a point p at length $L > \frac{1}{C}$ (otherwise, a length-maximizing timelike geodesic from p to S with length $L' \geq L$ would exist by global hyperbolicity, a contradiction). $\square$

This result is very appealling from a physical viewpoint, because the expansion assumption seems completely justified by astronomical observations. From a mathematical viewpoint, the reader can appreciate the similarities with the following result, which can be proved now as an exercise:

Theorem 6.2 *Let (M, g) be a Riemannian manifold such that:*

1.- It is complete.

2.- Some closed (as a subset) hypersurface S separates M as the disjoint union $M = M_- \cup S \cup M_+$, and S is strictly expanding towards M_+: $H \leq -C < 0$ (with appropiate sign convention).

3.- $Ric(v, v) \geq 0$ for all v.

Then $dist(p, S) \leq 1/C$, $\forall p \in M_-$

Singularity theorems combine previous ideas with (highly non-trivial) elements of Causality. Essentially, there are two types:

1. Proving the existence of an incomplete timelike geodesics in a cosmological setting.

   This is the case of Theorem 6.1, and some hypotheses there (specially glob. hyp.) are weakened or replaced by others. For example, Hawking himself proved that, if S is compact, global hyperbolicity can be replaced by assuming that S is achronal (i.e., non-crossed twice by a timelike curve). In this case, the timelike incompleteness conclusion holds, but in a less strong sense: at least one timelike incomplete geodesic exist.

2. Proving the existence of an incomplete lightlike geodesic in the context of gravitational collapse and black holes.

For the latter, the notion of (closed, future) *trapped* surface K (or $n-2$ submanifold) becomes fundamental. Its mathematically simplest definition says that K is a compact embedded spacelike surface without boundary, such that its mean curvature vector field $\vec{H}$ is future-directed and timelike on all K [49] –essentially, this means that the the area of any portion of K is initially decreasing along *any* future evolution; when it is only non-increasing, K will be said *weakly trapped*. Trapped surfaces are implied by spherical gravitational collapse. One would expect that, at least in asymptotically flat spacetimes (see next section), they must appear if enough matter is condensed in a small region and, under suitable conditions, must imply the existence of a *black hole* (see [20] and references therein). That is, the physical claim is that “gravitational collapse implies incompleteness”, and a support for this claim is provided by the following Penrose's theorem (the first modern singularity theorem [43] -after the works by Raychaudhuri and Komar):

Theorem 6.3 *Let (M, g) be a spacetime such that:*

1. *Admits a non-compact Cauchy hypersurface.*

2. *Contains a trapped surface.*

3. *Ric*$(k,k) \geq 0$ *for lightlike* k.

Then there exist an incomplete future-directed lightlike geodesic.

Singularity theorems are very accurate, even though it would be desirable to obtain general results on the nature of the incompleteness, or ensuring divergences of R in some natural sense.

7. Mass, Penrose inequality and CCC

Asymptotically flat 4-spacetimes are useful to model the spacetime around an isolated body. They can be defined in terms of Penrose conformal embeddings, even though the definition is somewhat involved (see for example [54, 24]). Nevertheless, in what follows it is enough to bear in mind that, in an asymptotically flat (4-)spacetime there exists spacelike Cauchy hypersurfaces Σ which admits an *asymptotically flat* chart $(\Sigma \backslash K, (x_1, x_2, x_3))$ as follows. For some compact $K \subset \Sigma$ and some closed ball $\overline{B_0(R)}$ of $\mathbb{R}^3$, $\Sigma \backslash K$ is isometric to $\mathbb{R}^3 \backslash \overline{B_0(R)}$ endowed with the metric:

$$h_{ij} = \delta_{ij} + O(1/r), \quad \partial_k h_{ij} = O(1/r^2), \quad \partial_k \partial_l h_{ij} = O(1/r^3),$$

in Cartesian coordinates (this means that Σ is intrinsically asymptotically flat, as a Riemannian 3-manifold; in particular, Ric and S, decay as $O(1/r^3)$ for large r), and, even more, its second fundamental form σ satisfies: $\sigma_{ij} = O(1/r^2)$. (This definition can be extended to include more than one end, each one isometric to $(\Sigma \backslash K, (x_1, x_2, x_3))$ as above.)

The total ADM (Arnowit, Deser, Misner) mass of an asymptotically flat Riemannian 3-manifold can be defined as the limit in any asymptotic chart:

$$m = \frac{1}{16\pi} \lim_{r\to\infty} \sum_{i,j=1}^{3} \int_{S_r} \left(\partial_i h_{ij} - \partial_j h_{ii})\right) n^j \, dA,$$

where n is the outward unit vector to S_r, the sphere of radius r. Notice that m depends only on the Riemannian 3-manifold (in fact, if this manifold is seen as a hypersurface of an asymptotically flat spacetime, the appropiate name for m would be *ADM energy*, and the definition of mass would depend on σ.) This definition of mass is not mathematically elegant, but recall:

1. ADM mass appears naturally in a Hamiltonian approach, as an asymptotic boundary term for the variations of $\int S$. The definition is not

trivial because only pointwise energy density and total mass (even pointwise momentum and total momentum) are well-defined: no intermediate step makes sense —nevertheless, it is worth pointing out the attempts to define a quasilocal mass [52].

2. There exists a classical Newtonian analog when the spacetime is Ricci-flat outside $\mathbb{R}\times K$, K compact, and there exists a timelike Killing vector field ξ with $\lim_{r\to\infty}|\xi| = 1$, such that $\Sigma \perp \xi$. In this case, the divergence theorem yields:
   $m = \frac{1}{4\pi}\int_K |\xi|^{-1}\mathrm{Ric}(\xi,\xi)dV = \int_K |\xi|\rho dV$
   i.e., the "integral of the poissonian density ρ measured at ∞".

3. The expression in coordinates for m is manageable:
   - If $h_{ij} = u^4\delta_{ij}$ with $u(x) = a + \frac{b}{|x|} + O(\frac{1}{|x|^2})$ then $m = 2ab$.
     In particular, this is the case if u is "harmonically flat" i.e. harmonic with finite limit at ∞.
   - Otherwise, when $S \geq 0$ then h is perturbable to the harmonically flat case with arbitrarily small error for m and preserving $S \geq 0$ (Schoen and Yau [51]; Corvino [18] extended the result for $m > 0$ without error in the mass).

4. Classical outer Schwarzschild metric can be written as:
   $M = \mathbb{R}\times\Sigma$, where $\Sigma = \mathbb{R}^3\backslash\overline{B_0(|m|/2)}$;
   $g = -\left((1-\frac{m}{2|x|})/u\right)^2 dt^2 + h$, $h_{ij} = u^4\delta_{ij}$ with $u = 1 + \frac{m}{2|x|}$
   (in particular $\sigma \equiv 0$). Of course, classical Schwarzschild mass m agrees ADM mass.

One expects from the physical background that, when the dominant property holds, the ADM mass will be positive for any asymptotically flat Cauchy Σ. Two technical points are relevant here: (a) When Σ is totally geodesic ($\sigma \equiv 0$) the dominant property yields $S \geq 0$. (b) Under our definition of asymptotic flatness, Σ is necessarily complete, but the Riemannian part of exterior Schwarzschild spacetime $(\mathbb{R}^3\backslash\overline{B_0(|m|/2)}, h)$ is incomplete for any $m \neq 0$. Of course, this is not a problem for the computation of the limit in the expression of the ADM mass, and one can also extend and modify $(\mathbb{R}^3\backslash\overline{B_0(|m|/2)}, h)$ in a bounded region to obtain a complete Riemannian manifold Σ^c with the same asymptotic behaviour. Moreover, in the globally hyperbolic case $m > 0$, one can obtain such a Σ^c (say, corresponding to the spacetime created by a star of the same mass) with: (i) the same asymptotic behaviour, (ii) $S \geq 0$. Clearly, this property is not expected in the non-globally hyperbolic case $m < 0$. And, in fact, it is forbidden by the *Riemann positive mass theorem*:

Theorem 7.1 *Let* (Σ, h) *be any asymptotically flat (complete) Riemannian manifold with* $S \geq 0$. *Then,* $m \geq 0$ *and equality holds iff* (Σ, h) *is Euclidean space* $\mathbb{E}^3 = (\mathbb{R}^3, \delta)$.

Remark 7.2 This celebrated result by Schoen and Yau [50] (shortly after re-proved spectacularly by Witten [55], see also [31] and references therein for mathematical subtleties) is a purely Riemannian one. From this case, more general "positive mass" results follow, which include the case $\sigma \not\equiv 0$ [51]. By the way, recall that the solution of Yamabe problem was completed by using above result (see the nice survey [35]).

Next, let us consider a recent -and no less spectacular- further step (for a more detailed exposition, see [10]). But, first two notions will be briefly explained:

1.- *WCCC*. A question related with SCCC (see Remark 5.2) is the so-called *weak cosmic censorship conjecture* (WCCC), which is stated in the framework of asymptotically flat spacetimes. In such spacetimes, a natural notion of asymptotic future null infinity $\mathcal{J}^+$ can be defined ($\mathcal{J}^+$ is a subset of the image of M for a suitable conformal embedding in a bigger spacetime $\bar{M}$) and, then, also a rigorous notion of the *black hole* region B of M ($B = M \backslash J^-(\mathcal{J}^+)$) –this region corresponds to the intuitive idea of a "spatially bounded region from where nothing can scape". WCCC asserts that, generically, for any asymptotically flat spacetime M obtained as the evolution of physically reasonable initial data[1], the region outside the black hole B will be globally hyperbolic[2]. The physical interpretation of this assertion is that no singularity (except at most an "initial" one) can be observed from $M \backslash B$, that is, singularities must lie inside a black hole and cannot be seen from outside (singularities are not "naked").

2.- *Outermost trapped surfaces.* Given a totally geodesic asymptotically flat slice Σ, those trapped surfaces (more precisely, compact spacelike surfaces whose expansion respect to the outer future lightlike direction is at no point

[1]Tipically, this data must satisfy: (i) (Σ, h, σ) is asymptotically flat, (ii) T satisfies the dominant property, and the equations for T constitute a quasilinear, diagonal, second order hyperbolic system, (iii) the fall-off of the initial value of T on Σ is fast enough for the h-distance, and this distance is also assumed to be complete.

[2]WCCC also asserts additional properties on the asymptotic structure. In particular, the asymptotic flatness of the evolved spacetime M should be deduced from the evolution of the data, that is, no a priori imposition of asymptotic flatness would be necessary for M. Thus, WCCC cannot be regarded as a particular case of SCCC.

positive) contained in Σ which are boundaries of a 3-manifold, are known to satisfy:

1. Such trapped surfaces correspond to compact minimal surfaces of Σ.

2. The outermost boundary compact minimal surfaces (necessarily topological spheres, each one the "apparent horizon in Σ of a black hole") are well-defined.

3. Let $\mathcal{H}$ be the union of the outermost minimal surfaces. Under WCCC, if $\mathcal{H}$ is connected and A_0 denotes its area, physical considerations ensure that the "contribution to the mass" m_0 of the corresponding black hole would satisfy: $m_0 \geq \sqrt{\frac{A_0}{16\pi}}$.

Therefore, choosing any asymptotic Σ one expects for its mass m_Σ:

$$m_\Sigma \geq \sqrt{\frac{A_0}{16\pi}} \tag{3}$$

(at least if $\sigma = 0$). But (3) is an inequality in pure Riemannian Geometry. Thus, the following precise result must hold:

Theorem 7.3 *Let (Σ, h) be a complete Riemannian 3-manifold with $S \geq 0$, and let $\mathcal{H}_0$ be the largest outermost (connected) minimal surface, with area A_0. Then inequality (3) holds, and the equality holds if and only if (Σ, h) is Schwarzschild Riemannian metric outside $\mathcal{H}_0$.*

This is the celebrated "Riemann-Penrose inequality", proved by Huisken and Ilmanen [32] (who re-prove then the Riemann positive mass theorem), and shortly after extended by Bray to the full area of the (maybe non-connected) $\mathcal{H}$, with a different proof [9] based on positive mass theorem.

Penrose inequality is a more general conjecture, which includes the case $\sigma \neq 0$. It is still open, and it becomes a major problem in Differential Geometry.

8. Some open questions

In order to list open questions in Global Lorentzian Geometry, it is convenient to distinguish between questions in Mathematical Relativity and questions which are mathematically natural, independent of physical motivations. Along this talk we have emphasized in the first ones. But recall that the works on Lorentzian manifolds inspired only in reasons of mathematical naturality and beauty, have also an overall support from General Relativity (in fact, vague comments on possible applications to Physics are frequently invoked in the introductions) and many times true applications to General Relativity appear.

The open questions in Mathematical Relativity commented in the previous three sections (notice that all the most relevant questions regarding smoothability seems to have been solved definitively [8]) can be summarized as:

1. Cosmic Censorship Conjecture (weak and strong), including full Penrose's inequality.

2. Cauchy problem (blow up criteria, global regularity for large data...)

3. Definitively satisfactory definition of singularities.

Of course, they are not by any means the unique relevant questions in Mathematical Relativity. For example, the following ones have attracted the interest of many researchers [and also were discussed in the special session of the meeting]:

4. Christodoulou and Klainerman [16] have proved the non-linear stability of Lorentz-Minkowski spacetime $\mathbb{L}^4$ as a solution of Einstein equation. This means that a small perturbation of the initial conditions for $\mathbb{L}^4$ yields a spacetime with properties close to $\mathbb{L}^4$ (and, for example, not to a spacetime with singularities). In spite of its apparent simplicity, the proof is extremely difficult -recall that [16] is a 500 pages book. The result is a landmark in Mathematical Relativity, and opens the study of the stability under weaker falloff hypotheses of the initial data or the stability of other spacetimes, as those with constant curvature.

5. There are different ways to attach a boundary to a spacetime. When one focuses on causal properties, a specially successful one uses conformal embeddings (in fact, these embeddings allow the definition of asymptotic flatness). Nevertheless, this procedure is extrinsic, and an intrinsic way seems to be desirable. In 1972, Geroch, Kronheimer and Penrose gave a first natural definition of *causal boundary* [29]. Nevertheless, their definition was truly consistent only for stably causal spacetimes, and many modifications have been proposed since then [11, 44, 53]. The last one is a new recipe proposed by Marolf and Ross [37] to overcome the problems of previous ones in the framework of plane waves –and their applications to string theory. Thus, to find a definitively satisfactory definition of causal boundary is a remarkable challenge.

6. We cannot forget that General Relativity is one of the two fundamental physical theories, being Quantum Theory the other one. The unification of both theories is a physical challenge of first order and, since decades ago, it yields a permanent inspiration for some of the mathematics of highest level.

Among the open problems which are interesting as purely mathematical ones, independent of relativistic interpretations, the following ones also attracts the interest of many researchers:

1. Classification of submanifolds with natural geometric properties in spaceforms and other physical or mathematically relevant spacetimes. The result by Cheng and Yau [14] which characterizes spacelike hyperplanes as the unique closed (as a subset) maximal spacelike hypersurfaces of $\mathbb{L}^4$, becomes classical. Many other results have been obtained, including submanifolds of constant or prescribed mean curvature, non-spacelike submanifolds, totally umbilical ones, stability, etc. The ambient spacetimes have also a variety of types, as constant curvature, globally hyperbolic, spacetimes with priviledged symmetries or others (see, among many other examples, [22, 38, 1] and references therein).

2. Classification of complete flat (or constant curvature) Lorentzian manifolds. This problem has its roots in Hilbert's 18th problem of classifying crystallographic groups and the celebrated Bieberbach theorem. The so-called Auslander conjecture (the fundamental group of a compact flat affine manifold is virtually policyclic) and the discovery by Margulis –following suggestions by Milnor– of spacetimes which yield counterexamples to the conjecture if compactness is replaced by completeness (proper quotients of $\mathbb{L}^n$ by free groups of isometries), are highlights which have motivated a deep research; see, for example, the review [13]. A related problem deals with (geodesic) completeness. In fact, the question whether a compact Lorentzian manifold M of constant curvature is geodesically complete (answered affirmatively in [12, 34]), becomes essential because it ensures that M can be obtained as a quotient of a spaceform by a suitable group.

3. Critical curves for indefinite functionals on Lorentzian manifolds. Starting from seminal works by Benci, Fortunato and Giannoni [4], geodesics and other curves on spacetimes are being systematically studied by means of (infinite-dimensional) methods of Global Analysis on manifolds, for example, see the book [39]. Applications to both, the (positive-definite) Riemannian case and classical problems in General Relativity, have been also found, see [23].

4. We must emphasize that there are many natural and apparently simple questions on Lorentzian manifolds which remain unknown. For example (see [42]), let (M, g) be a compact Lorentzian manifold. It is easy to prove that the lightlike completeness of g is invariant in its conformal class but this is not known for non-lighlike geodesics. That is, if g is complete, must a conformal metric be complete? Recall that, if there exists a counterexample, this would show the independence of two types of causal completeness in the compact case. The answer to such questions need fresh ideas which, surely, will be useful for many fields.

Finally, I would like to express my personal feeling on this subject. I do not know if I have been able to transmit some of the beauty global geometric ideas with roots in Einstein's General Relativity –if not, I apologize: this would be my fault totally. At the break of 20th century, Hilbert claimed the existence of a (highly untangibly) Cantor's paradise: "No one shall expel us from the paradise that Cantor has created for us". Now, at the beginning of 21th, we can also enjoy a more palpable Einstein's paradise –and, of course, nobody shall expel us from it.

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