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Black Holes with Short Hair

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Abstract

We present spherically symmetric black hole solutions for Einstein gravity coupled to anisotropic matter. We show that these black holes have arbitrarily short hair, and argue for stability by showing that they can arise from dynamical collapse. We also show that a recent 'no short hair' theorem does not apply to these solutions.

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I. INTRODUCTION

It is widely believed that black holes retain only very limited information about the matter that collapsed to form them. This information is reflected in the number of parameters characterizing the black hole solution. For stationary axis–symmetric black holes in general relativity with no matter coupling, the only such parameters are the mass M and angular momentum J. With coupling to electromagnetism, the parameters include the electric charge Q. These results are known collectively as the 'no hair' theorems [1–6].

The three parameters M, J, and Q are special in that they can be detected by performing experiments in the asymptotically flat region of the black hole spacetime. This is because they are captured by conserved surface integrals in the asymptotically flat region. They are not what is referred to as 'hair'. Any parameters in a black hole spacetime that are not captured by asymptotic surface integrals may be called 'hair'. Thus, no hair theorems imply that all the parameters characterizing black hole spacetimes are those that are 'visible' at spatial infinity.

For Einstein gravity coupled to matter fields other than electromagetism, black holes might be parametrized by quantities other than M, J, and Q. In these cases, the parameters might all be captured by surface integrals at infinity, leading to an associated 'no hair theorem'. Alternatively, such theories might yield black hole solutions with hair.

In this paper we describe spherically symmetric and static black hole solutions that arise from coupling Einstein gravity to an 'anisotropic fluid', which is defined below. For certain special cases this matter gives the de–Sitter, and Reissner–Nordstrom black holes. For other cases, the resulting black holes have hair in the sense described above. In the next section we describe the anisotropic medium and construct a class of static, spherically summetric solutions to the coupled Einstein–matter equations of motion. In section III we specialize to a class of black hole solutions and show that these black holes have short hair. We also give arguments indicating that these black holes are stable. Section IV discusses the relationship of our black hole solutions with the so called 'no short hair' conjecture of Ref. [7]. We conclude in section V with a short discussion.

II. SOLUTIONS WITH ANISOTROPIC MATTER

Consider a static, spherically symmetric metric of the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2.1}$$

The nonvanishing components of the Einstein tensor for this metric are

$$\begin{aligned} G_t^t = G_r^r &= \frac{rf' - 1 + f}{r^2} \\ G_\theta^\theta = G_\phi^\phi &= \frac{rf'' + 2f'}{2r} \ , \end{aligned} \tag{2.2}$$

where the prime denotes a derivative with respect to r. It follows that any matter source that gives rise to a metric of the form Eq. (2.1) must satisfy $T_t^t = T_r^r$ and $T_\theta^\theta = T_\phi^\phi$.

In particular, the "radial pressure" T_r^r need not equal the "angular pressure" $T_\theta^\theta = T_\phi^\phi$. Therefore, generically, such a matter source is not isotropic in the static time slices. What we seek, then, is a medium that is capable of supporting anisotropic stresses.

A material with the desired properties is the elastic medium discussed by DeWitt [8]. We begin with DeWitt's action, as written in Ref. [9],

$$S[g_{ab}, Z^i] = -\int_{\mathcal{M}} d^4x\sqrt{-g}\,\rho(Z^i, h_{jk}) \ . \tag{2.3}$$

This action is a functional of the spacetime metric g_{ab} and the Lagrangian coordinates $Z^i(x)$, $i = 1, 2, 3$. The Lagrangian coordinates are a set of labels $\zeta^i = Z^i(x)$ that tell which particle passes through the spacetime point x. Thus, $Z^i(x)$ are three scalar fields whose gradients are orthogonal to the matter world lines. The action $S[g_{ab}, Z^i]$ is the proper volume integral of the Lagrangian $-\rho$, where ρ is the proper energy density in the rest frame of the matter. The energy density ρ depends on Z^i explicitly, and also implicitly through the matter space metric h_{ij}. The matter space metric is defined by

$$h^{ij} = (\partial_a Z^i) g^{ab} (\partial_b Z^j) \ , \tag{2.4}$$

and is interpreted as the metric in the rest frame of the matter. That is, $h_{ij} d\zeta^i d\zeta^j$ is the square of the proper orthogonal distance separating particle world lines with Lagrangian coordinates ζ^i and $\zeta^i + d\zeta^i$. Using h_{ij}, the spacetime metric g_{ab} may be written in the form

$$g_{ab} = -U_a U_b + h_{ij}(\partial_a Z^i)(\partial_b Z^j) \ , \tag{2.5}$$

where U^a is the fluid velocity, defined as the future pointing unit vector orthogonal to the gradients $\partial_a Z^i$.

The stress–energy–momentum tensor for the elastic medium is

$$\begin{aligned} T_{ab} &\equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial h^{ij}}(\partial_a Z^i)(\partial_b Z^j) \\ &= \rho U_a U_b + t_{ij}(\partial_a Z^i)(\partial_b Z^j) \ , \end{aligned} \tag{2.6}$$

where we have used (2.5) and defined the matter stress tensor

$$t_{ij} = \left(2\frac{\partial \rho}{\partial h^{ij}} - \rho h_{ij}\right) = \frac{2}{\sqrt{h}}\frac{\partial(\sqrt{h}\rho)}{\partial h^{ij}} \ . \tag{2.7}$$

The matter equations of motion, obtained by varying the action with respect to Z^i, are equivalent to the conservation equations $\nabla_a T^{ab} = 0$. When the Einstein equations hold, the contracted Bianchi identity implies that the matter equations are also satisfied.

An *isotropic* elastic medium is one whose principal pressures are all equal. Its stress tensor is that of a perfect fluid, and is obtained by letting ρ depend only on the proper volume of the Lagrangian coordinate cell $d^3\zeta$ and the number of particles in that cell. This is realized [9] by setting

$$\rho = \rho(N) \ , \tag{2.8}$$

with the scalar N defined by the ratio of densities

$$N = \underline{N}(Z^i)/\sqrt{h} \ , \tag{2.9}$$

where $\underline{N}(\zeta^i)$ is the number of particles per unit coordinate cell $d^3\zeta$. Thus, N is the proper number density, i.e., the number of particles per unit proper volume in the material rest frame. With these definitions, the stress–energy–momentum tensor (2.6) assumes the familiar form: Using the identity

$$\frac{\partial\rho(N)}{\partial h^{ij}} = \frac{N}{2}\frac{\partial\rho}{\partial N}h_{ij} \ , \tag{2.10}$$

and (2.5), the tensor (2.6) becomes

$$T_{ab} = \rho U_a U_b + \left(N\frac{\partial\rho}{\partial N} - \rho \right)(g_{ab} + U_a U_b) \ . \tag{2.11}$$

Thus, the matter action (2.3) with $\rho = \rho(N)$ is the action of a perfect fluid with pressure $P = N(\partial\rho/\partial N) - \rho$. This definition of pressure can be written as $d(\rho/N) = -Pd(1/N)$, which is the first law of thermodynamics applied to a cell with a fixed number of particles.

An *anisotropic* elastic medium is one whose principal pressures are not all equal. If two of the three principal pressures are equal, the fluid has planar symmetry. The basic idea for introducing anisotropy is to perform a '$2+1$' decomposition of the matter space metric h_{ij}, and treat the plane differently from the line. This may be done as follows. Let us view the Lagrangian coordinates ζ^i as points in a manifold $\mathcal{S}$, which is the space of particle flow lines, and introduce a foliation of $\mathcal{S}$ by a function $\varphi(\zeta^i)$ with $\partial_i\varphi \neq 0$. Now set $\rho = \rho(n, m)$, where n is the particle number per unit proper area in the $\varphi = \text{const}$ slices and m is the particle number per unit proper length in the direction orthogonal to these slices. This introduces the anisotropy. For definiteness and simplicity, let us choose $\varphi(\zeta^i) = \zeta^1$. Then $n = \underline{n}/\sqrt{\sigma}$ where $\underline{n}(\zeta^i)$ is the number of particles per unit coordinate area $d^2\zeta = d\zeta^2 \wedge d\zeta^3$, and $\sigma = h_{22}h_{33} - (h_{23})^2$ is the determinant of the metric in the $\zeta^1 = \text{const}$ slices. Similarly, $m = \underline{m}/\alpha$ where $\underline{m}$ is the number of particles per unit coordinate length $d\zeta^1$ and $\alpha = \sqrt{h/\sigma}$ is the 'lapse function' that measures proper length in the direction orthogonal to $\zeta^1 = \text{const}$.

We thus have the following '$2+1$' form for h_{ij}:

$$h_{ij} = \begin{pmatrix} \alpha^2 + \beta^\mu\beta_\mu & \beta_\nu \\ \beta_\mu & \sigma_{\mu\nu} \end{pmatrix} \ , \tag{2.12}$$

where β^μ is the 'shift vector' and the indices μ, ν take the values 1, 2. Equivalently, we have $h_{ij} = u_i u_j + \sigma_{ij}$, where $u_i = \alpha\delta_i^1$ is the unit vector orthogonal to the $\zeta^1 = \text{const.}$ surfaces. The corresponding spacetime tensors are

$$u_a = u_i\partial_a Z^i = \alpha\partial_a Z^1 \ , \qquad \sigma_{ab} = (h_{ij} - u_i u_j)(\partial_a Z^i)(\partial_b Z^j) \ , \tag{2.13}$$

which represent, respectively, the unit normal and metric of the isotropic 2-surfaces in the rest frame of the matter.

The stress tensor for the anisotropic medium, with $\rho(n, m)$, is obtained by computing $(\partial\rho/\partial h^{ij})$ using the above definitions, and substituting into (2.7). We find

$$\frac{\partial\rho(n,m)}{\partial h^{ij}} = \frac{n}{2}\frac{\partial\rho}{\partial n}\sigma_{ij} + \frac{m}{2}\frac{\partial\rho}{\partial m}u_i u_j \tag{2.14}$$

which gives

$$t_{ij} = \left(n\frac{\partial\rho}{\partial n} - \rho\right)\sigma_{ij} + \left(m\frac{\partial\rho}{\partial m} - \rho\right)u_i u_j \ . \tag{2.15}$$

Finally, the full stress–energy–momentum tensor for the anisotropic matter is

$$T_{ab} = \rho U_a U_b + \left(n\frac{\partial\rho}{\partial n} - \rho\right)\sigma_{ab} + \left(m\frac{\partial\rho}{\partial m} - \rho\right)u_a u_b \ . \tag{2.16}$$

We note, as required, that the pressure in the 2-surfaces is $n(\partial\rho/\partial n) - \rho$, which differs from the pressure $m(\partial\rho/\partial m) - \rho$ in the direction orthogonal to 2-surfaces.

Our goal now is to find an equation of state, $\rho = \rho(n,m)$, that will yield the metric (2.1) as a solution of the Einstein and matter equations of motion, *for arbitrary* $f(r)$. The Einstein tensor (2.2) shows that the spheres $t =$ const, $r =$ const are isotropic. Thus, we seek a solution where u_a is radial. In particular, let $Z^1(x) = r$, $Z^2(x) = \theta$, and $Z^3(x) = \phi$ so that $u_a = \alpha\delta^r_a$. From (2.16) the condition $T^t_t = T^r_r$ now shows that the equation of state must be independent of m, so $\rho = \rho(n)$. In this case we have

$$T^t_t = T^r_r = -\rho \tag{2.17}$$

$$T^\theta_\theta = T^\phi_\phi = n\frac{\partial\rho}{\partial n} - \rho \ , \tag{2.18}$$

and the Einstein equations yield (with $G = 1$)

$$\frac{rf' - 1 + f}{r^2} = -8\pi\rho \tag{2.19}$$

$$\frac{rf'' + 2f'}{2r} = 8\pi\left(n\frac{\partial\rho}{\partial n} - \rho\right) \ . \tag{2.20}$$

Using the relationship $(rf' - 1 + f)' = rf'' + 2f'$ between the numerators of the left–hand sides of these equations, we find

$$\frac{(-r^2\rho)'}{2r} = n\frac{\partial\rho}{\partial n} - \rho \ . \tag{2.21}$$

This reduces to $n'/n = -2/r$, and implies

$$n = \frac{K}{r^2} \ , \tag{2.22}$$

where K is a positive dimensionless constant. Now define the function $F(n)$ by

$$F(n) \equiv F(K/r^2) \equiv f(r) \ . \tag{2.23}$$

Then the Einstein equation (2.19) gives

$$\rho(n) = \frac{n}{8\pi K}\left(1 - F + 2n\frac{\partial F}{\partial n}\right) \ . \tag{2.24}$$

This determines the equation of state $\rho(n)$ in terms of the *arbitrary function* $f(r)$ that appears in the metric (2.1).

Note that the correspondence between $f(r)$ and $\rho(n)$ is not one–to–one. One can see this by solving Eq. (2.24) for $F(n)$:

$$F(n) = 1 + 4\pi K\sqrt{n}\int dn\, n^{-5/2}\rho(n) \ . \tag{2.25}$$

The integral over n yields a term $4\pi KK'\sqrt{n}$ which is proportional to the integration constant K'. This term corresponds to $-2M/r$ in the metric function $f(r)$, where $M = -2\pi K^{3/2}K'$.

Let us reexpress and summarize these results. Consider an elastic medium (2.3–2.4) with equation of state $\rho = \rho(n)$, where $n = \underline{n}/\sqrt{\sigma}$, $\underline{n} = K\sin Z^2$, and $\sigma = h_{22}h_{33} - (h_{23})^2$. A family of solutions of the Einstein and matter equations of motion is $Z^1(x) = r$, $Z^2(x) = \theta$, $Z^3(x) = \phi$, and $g_{ab}(x)$, where $g_{ab}(x)$ is given by (2.1). The function $f(r)$ that appears in the metric is defined by $f(r) = F(K/r^2)$, where $F(n)$ is determined by the equation of state through Eq. (2.25). This family of solutions is parametrized by an arbitrary constant M that appears in a term $-2M/r$ in the metric function $f(r)$. Equation (2.24) shows that, for a suitable equation of state $\rho = \rho(n)$, any metric of the form (2.1) can be obtained as a solution of the Einstein and matter equations of motion.

The anisotropic elastic medium described here satisfies the (weak, dominant, or strong) energy conditions only if restrictions are placed on the equation of state. For example, for the weak energy condition the proper energy density must be nonnegative, $\rho \geq 0$, and the principle stresses must be greater than or equal to $-\rho$. This later requirement implies $\partial\rho/\partial n \geq 0$. The inequalities $\rho \geq 0$ and $\partial\rho/\partial n \geq 0$ can be translated into restrictions on the function $F(n)$, which in turn restrict the form of the metric function $f(r)$.

III. BLACK HOLES WITH SHORT HAIR

In the previous section we showed that the general spherically symmetric metric (2.1) arises from an anisotropic matter source. However all such metrics are not physical because, as mentioned above, the matter does not in general satisfy physical energy conditions. In this section we consider a class of black hole solutions of the Einstein–anisotropic fluid theory that are physical. We show that these black holes have arbitrarily short hair, and argue that they are stable under scalar perturbations. The stability argument relies on the observation that these black holes can arise from spherically symmetric gravitational collapse.

Consider the equation of state

$$\rho(n) = Cn^{k+1} \tag{3.1}$$

for the anisotropic fluid, where C and k are constants. From Eq. (2.16), the stress–energy–momentum tensor is

$$T_{ab} = \rho U_a U_b - \rho u_a u_b + P\sigma_{ab} \ , \tag{3.2}$$

where

$$P \equiv n\frac{\partial\rho}{\partial n} - \rho = k\rho \tag{3.3}$$

defines the principal pressures in the $\theta - \phi$ surfaces.

T_{ab} is a type I stress tensor, so the weak energy condition requires $\rho \geq 0$ and $P \geq -\rho$. This clearly holds for all $C \geq 0$, $k \geq -1$. The strong energy condition requires, in addition to the weak energy condition, that the sum of the energy density ρ and the principal pressures should be non–negative. In our case this implies $P \geq 0$, so the strong energy condition holds for all $C \geq 0$, $k \geq 0$. The dominant energy condition requires, in addition to the weak energy condition, that energy fluxes should never be spacelike. In our case this leads to $\rho \geq P$. Therefore the dominant energy condition holds for $C \geq 0$, $-1 \leq k \leq 1$.

From Eq. (2.25) we see that the metric resulting from the equation of state $\rho(n) = Cn^{k+1}$ is (2.1) with

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^{2k}}{r^{2k}}\right) , \qquad (k \neq 1/2) . \tag{3.4}$$

Here, we have chosen $C > 0$ and defined $Q^{2k} = 8\pi C K^{k+1}/(2k-1)$. The energy density and pressures are

$$\rho = \frac{Q^{2k}(2k-1)}{8\pi r^{2k+2}} , \qquad P = k \, \frac{Q^{2k}(2k-1)}{8\pi r^{2k+2}} . \tag{3.5}$$

Notice that for $k = 1$ this solution is identical to the Reissner–Nordstrom metric. For our purposes, the more interesting situation occurs for $k > 1$. In that case the parameter Q is *not* captured by a surface integral at spatial infinity due to the rapid fall off r^{-2k} of the corresponding term in $f(r)$. Therefore, when $k > 1$, Q qualifies as hair. Since we can take k to be arbitrarily large, *the hair Q can be arbitrarily short.* This means that the dependence of the metric on Q may be confined to an arbitrarily small neighbourhood of the curvature singularity at $r = 0$. As an example, consider the limit $Q \ll M$. In this case the outer event horizon is at $r_+ \approx 2M$ and the inner (Cauchy) horizon is at $r_- \approx Q(Q/2M)^{1/(2k-1)} + 2M(Q/2M)^{4k/(2k-1)}/(2k-1)$. The ratio of the energy densities at the event and inner horizons is $\rho(r_+)/\rho(r_-) = (r_-/r_+)^{2k+2} \approx (Q/2M)^{2k(2k+2)/(2k-1)}$. This ratio can be made arbitrarily small by decreasing $Q/2M$ or by increasing k. Thus the hair sprouting from the event horizon can be arbitrarily short. We emphasize that the anisotropic fluid that gives rise to these short–hair solutions satisfies both the weak and strong energy conditions.

We now consider the issue of stability for the black holes with short hair. In particular, we present evidence that the black holes (3.4) are stable under the spherically symmetric perturbations

$$f(r) \to f(r) + \delta f(r,t) , \quad \rho(r) \to \rho(r) + \delta\rho(r,t) , \quad P(r) \to P(r) + \delta P(r,t) \tag{3.6}$$

about the static solutions. As discussed below, these are not the most general possible spherically symmetric perturbations.

Stability of a static or stationary black hole is studied by performing a standard but tedious analysis to obtain the equations governing metric and fluid perturbations to linearized order. The black hole is stable if initial metric perturbations do not grow without bound. Alternatively, one can attempt to show that the black hole arises from the long time limit of gravitational collapse. Indeed, if a static or stationary black hole is known to arise as the endpoint of a dynamical evolution, its stability is not seriously in question. We will consider an argument for stability based on the second approach; however, it should be emphasized that such an approach is physical, and is not a substitute for a complete mathematical analysis of the first type.

To begin, let us observe that the stress–energy–momentum tensor (3.2) can be rewritten as

$$T_{ab} = \rho(r)(v_a w_b + v_b w_a) + P(r)(g_{ab} + v_a w_b + v_b w_a) \ , \tag{3.7}$$

where the null vectors v_a and w_a are defined by

$$U_a = \frac{1}{\sqrt{2}}(v_a + w_a) \ , \qquad u_a = \frac{1}{\sqrt{2}}(v_a - w_a) \ . \tag{3.8}$$

For our solutions $U_a = (\sqrt{f}, -1/\sqrt{f}, 0, 0)$ and $u_a = (0, 1/\sqrt{f}, 0, 0)$ in the coordinates (v, r, θ, ϕ), where v is an advanced time coordinate defined by $dv = dt + (1/f)dr$.

In [10] a stress tensor similar to (3.7) was used to obtain dynamical spherically symmetric collapse solutions. That tensor is

$$T_{ab} = \frac{\dot{m}(r,v)}{4\pi r^2} v_a v_b + \rho(r,v)(v_a w_b + v_b w_a) + P(r,v)(g_{ab} + v_a w_b + v_b w_a) \ . \tag{3.9}$$

The general solution of the Einstein equations for this source, with the equation of state $P = k\rho$, is

$$ds^2 = -\left(1 - \frac{2\bar{f}(v)}{r} + \frac{2\bar{g}(v)}{(2k-1)r^{2k}}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{3.10}$$

where

$$m(r,v) = \bar{f}(v) - \frac{\bar{g}(v)}{(2k-1)r^{2k-1}} \ , \quad (k \neq 1/2), \tag{3.11}$$

$$P(r,v) = k \ \frac{\bar{g}(v)}{4\pi r^{2k+2}} = k\rho(r,v) \ , \tag{3.12}$$

and $\bar{f}(v)$ and $\bar{g}(v)$ are arbitrary functions. Now we can choose the functions $\bar{f}$ and $\bar{g}$ such that $\lim_{v\to\infty} \bar{f}(v) = M$ and $\lim_{v\to\infty} \bar{g}(v) = Q^{2k}(2k-1)/2$. Then the metric (3.10) at large values of advanced time coincides with the short hair black hole (2.1), (3.4). In this way we see that the static short hair solutions result as the end point of gravitational collapse. The metric (3.10) is to the short hair black hole what the Vaidya metric is to the Schwarzschild black hole.

We now elaborate on the connection between the stress–energy–momentum tensors (3.7) and (3.9) induced by the perturbations (3.6). Notice first that if the functions f, ρ and P are all taken to have general (r, v) dependence, the most general stress–energy–momentum tensor *must be of the form* (3.9) [10]. Thus the perturbations (3.6), when substituted into the tensor (3.7), must lead to a tensor of the general form (3.9); the extra $v_a v_b$ term is effectively induced by the time dependence of f. Now since we have the general solution

(3.10) with the source (3.9), we have implicitly the solution of the equations governing the perturbations (3.6), to all orders in the perturbations. This is because the general solution (3.10) can always be separated into a static part plus a time dependent 'perturbation'. Thus, this argument shows stability under perturbations of the type (3.6).

The most general spherical scalar perturbations are however not given by (3.6). To obtain the most general perturbations of this type we would need to add to the stress–energy–momentum tensor (3.9) an 'outgoing' (r, v) dependent term proportional to $w_a w_b$. Such a term comes effectively from including a metric perturbation $\delta h(r, v)\ dvdr$. Although an exact collapse solution with some matter outgoing to infinity is not known, one can argue, by analogy with the numerical spherically symmetric scalar field collapse [11], that an outgoing component with positive energy density will not affect stability when black hole formation occurs. In that system it is observed, in the so called 'supercritical' region of parameter space (where a black hole forms), that an initially ingoing pulse oscillates radially as it collapses, with some scalar waves escaping to infinity with each oscillation as the collapse to a black hole proceeds. Similar behaviour has been observed numerically in the collapse of perfect fluids [12], and may occur for the type of anisotropic fluid discussed here. While this analogy is suggestive, we only have an argument for stability for the spherical 'ingoing' perturbations (3.6).

IV. COMMENT ON A 'NO SHORT HAIR' THEOREM

As we have seen, some of the static spherically symmetric black hole solutions (those for $k > 1$) arising from gravitational coupling to anisotropic media have hair—that is, parameters in the metric which are not captured by surface integrals at spatial infinity. The energy density falls to zero with radius r faster than $1/r^4$. The physical picture is that some matter protrudes from the black hole event horizon without being pulled in, but does not 'extend to infinity' as in the electric field case. Matter can hover in a strong gravitational field without collapsing completely only if its internal pressures are sufficiently large. This appears to be the case for the anisotropic fluid black holes with $k > 1$.

More generally one can ask: How close to a black hole event horizon can matter hover if the stress–energy–momentum tensor satisfies some reasonable physical conditions? Or in other words, how 'short' can hair be? This question has been posed, and there is a theorem based on certain assumptions [7]. Specifically, the theorem states that if matter is such that

(i) the weak energy condition holds,

(ii) the energy density ρ falls to zero faster than r^{-4},

(iii) $T := T_{ab}g^{ab} \leq 0$,

then it follows that Pr^4 ($P =$ radial pressure) *is negative and decreasing until at least 3/2 the radius of the event horizon, beyond which this quantity increases to zero.* This behaviour is interpreted as 'long hair', and it is suggested [7] that it will occur for any non–linear theory coupled to gravity. The three conditions above effectively imply that radial pressure is negative, at least near the horizon.

We have presented black hole solutions that have arbitrarily short hair for the parameter $k > 1$. For our 'short hair' black holes, conditions (i) and (ii) hold, but condition (iii) does not: $T = 2\rho(k-1)$, which is positive for $k > 1$. Thus, our solutions do not contradict the no short hair theorem because the condition $T \leq 0$ for it to hold is not satisfied. Thus, while this theorem does apply to several cases as discussed in [7], it does not apply to the anisotropic fluid.

V. DISCUSSION

We have presented a class of black hole solutions arising from Einstein gravity coupled to a type of anisotropic matter, and described some of their properties. Perhaps their most unusual feature is the possibility of arbitrarily short hair.

It is interesting to observe that these black holes can also be seen as arising from a generalized electromagnetism. Consider the stress–energy–momentum tensor

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{\alpha}{4}\, g_{ab}F_{cd}F^{cd}\right) , \tag{5.1}$$

where F_{ab} is the field strength and α is a constant.[1] The field equation for F_{ab} is that obtained from $\nabla_a T^{ab} = 0$. A solution of the Einstein equations coupled to this matter is the metric (2.1) and (3.4), with

$$F_{ab} = \sqrt{\frac{(2k-1)(k+1)}{2}}\,\frac{Q^k}{r^{k+1}}\,(dt \wedge dr)_{ab} , \tag{5.2}$$

and $\alpha = 2k/(k+1)$. We note the black hole solutions viewed as arising from this matter source can be shown to be stable via an analysis very similar to that given in [13] for the Reissner-Nordstrom solution. Indeed, the only difference in the details are appropriate replacements of Q^2/r^2 by Q^{2k}/r^{2k}.

It would be of interest to study the collapse of anisotropic fluids numerically, especially in light of the black hole solutions we have found. In particular, as has been done for the perfect fluid [12], one could seek a 'critical' solution with a self–similarity ansatz. If found, such a solution could be used as the starting point of perturbation theory [12,14] for finding critical exponents associated with the parameters M and Q.

Note added: After completing this work we became aware of related results by Magli and Kijowski [15]. These authors consider equilibriuim, spherically symmetric stars composed of anisotropic elastic matter. We thank G. Magli for bringing this to our attention.

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[1] We do not know an action which leads to the stress–energy–momentum tensor (5.1).

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