Refining trapped surfaces

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1. Introduction

Trapped surfaces [Penrose 1965] play an important role in gravitational physics,
both for black holes and in cosmology, e.g. in the singularity theorems
[Penrose 1965, Hawking 1967, Penrose & Hawking 1970].

Marginal surfaces, a limit of trapped surfaces, admit a local, dynamical theory of black holes
[Hayward 1994–2009, Ashtekar et al. 1999–2004, Booth & Fairhurst 2004–2007, Gourgoulhon & Jaramillo 2006...].

One might expect the boundary of the region of trapped surfaces to consist of
marginal surfaces, i.e. to be a trapping horizon.

However, this is not so: trapped surfaces can poke through the trapping horizon
[Schnetter & Krishnan 2006, Ben-Dov 2007, Bengtsson & Senovilla 2009].

On the other hand, this boundary does not have the special physical properties
that trapping horizons have, such as a first law involving surface gravity [Hayward 1998]
and a local Hawking temperature [Hayward et al. 2009, di Criscienzo et al. 2009].

There is thus a conflict between the mathematics and physics.

The physics seems to be clear, so the mathematics must yield.
That is, trapped surfaces as simply defined need to be refined in some way.

Conjecture: the boundary of a region of suitably refined trapped surfaces
is a trapping horizon.

The following gives eight refinements, which are related:
they form an interwoven hierarchy, with some direct relations and some relations
which assume the Einstein equation, or more exactly the null energy condition (NEC).

2. Minimal trapped surfaces

Consider spatial surfaces S embedded in a given space-time,

and normal vectors η, with L_η denoting the Lie derivative along η,

and $L(\eta) = L_\eta$.

The expansion 1-form θ is defined by $\star\theta = L(\star 1)$

where $\star$ is the Hodge operator induced on S by the space-time metric g,

i.e. $\star 1$ is the area form and θ its logarithmic normal derivative.

The expansion vector (a.k.a. mean-curvature vector) is $H = g^{-1}(\theta)$.

There is also a Hodge operator $*$ in the normal space,

e.g. $*\theta$ is the dual expansion 1-form.

This induces a duality operation on normal vectors by $g(\eta_*) = *g(\eta)$,

or equivalently $g(\eta_*, \eta) = 0$, $g(\eta_*, \eta_*) = -g(\eta, \eta)$.

In particular, there is the dual expansion vector H_*.

A surface is trapped if H is temporal, or equivalently if H_* is spatial.

Assuming a time-orientable space-time, the surface is future (respectively past) trapped

if H is future (respectively past) temporal.

Any surface is extremal in the H_* direction: $H_* \cdot \theta = 0$.

Then it is natural to ask whether the surface is not merely extremal but minimal.

Definition 1. A (strictly) minimal trapped surface is a trapped surface for which, for

some variation, $Q = H_*^a H_*^b \nabla_b \theta_a > 0$ where ∇ is the covariant derivative operator of g.

The strict sign will turn out to be convenient.

Rewriting as $Q = -H^a H_*^b \nabla_b (*\theta)_a$ is convenient for calculations using differential forms.

3. Outer trapped surfaces

In spherical symmetry, outer trapped spheres can be defined by $\kappa > 0$,

where κ is surface gravity [Hayward 1998].

The relevant object in general turns out to be the curvature $K = {*d*}\theta + \frac{1}{2}g^{-1}(\theta, \theta)$

where $*d*$ is the normal codifferential or divergence.

A previous definition of quasi-local surface gravity was $\kappa = \dfrac{1}{16\pi R}\displaystyle\oint_S \star K$ [Mukohyama & Hayward 2000]

where $R = \sqrt{A/4\pi}$ is the area radius, i.e. area is $A = \displaystyle\oint_S \star 1 = 4\pi R^2$.

This κ enters a quasi-local first law for trapping horizons involving the Hawking mass.

Definition 2. An outer trapped surface is a trapped surface for which,

for some variation, $K > 0$.

Lemma 1. Assuming the Einstein equation with units $G = 1$, $2Q = -g^{-1}(\theta, \theta)K - 16\pi H \cdot \Psi$,

where $\Psi = (T + \Theta) \cdot H + w\theta$, T is the energy tensor,

$w = -\frac{1}{2}\mathrm{tr}\,(T + \Theta)$ is an energy density, where the trace is in the normal space,

and Θ is an effective energy tensor for gravitational radiation [Hayward 2000–2009].

All proofs are a calculations using the null-null components of the Einstein equation.

Proposition 1. NEC and minimal trapped implies outer trapped.

Proof. NEC $\Rightarrow$ $H \cdot \Psi \geq 0$. For a trapped surface, $g^{-1}(\theta, \theta) < 0$, then inspect signs.

4. Increasingly trapped surfaces

Noting that $-g^{-1}(\theta,\theta)$ vanishes for marginal surfaces and is positive for trapped surfaces, it can be taken as a measure of how trapped a surface is.

The idea then is to ask whether it is increasing to the future (respectively past) for a future (respectively past) trapped surface.

Definition 3. An increasingly trapped surface is a trapped surface for which, for some variation, $H\cdot dg^{-1}(\theta,\theta)<0$.

Lemma 2. Assuming the Einstein equation,
$-H\cdot dg^{-1}(\theta,\theta)=16\pi H\cdot\Psi-g^{-1}(\theta,\theta)(K-g^{-1}(\theta,\theta))$.

Proposition 2. NEC and outer trapped implies increasingly trapped.

Proof. For a trapped surface, $g^{-1}(\theta,\theta)<0$, NEC $\Rightarrow H\cdot\Psi\geq 0$ as before, then inspect signs.

Definition 4. An anyhow increasingly trapped surface is a future (respectively past) trapped surface for which, for all variations along a future (respectively past) causal normal vector ζ, $\zeta\cdot dg^{-1}(\theta,\theta)<0$.

Definition 5. A somehow increasingly trapped surface is a future (respectively past) trapped surface for which, for some variation along a future (respectively past) causal normal vector ζ, $\zeta\cdot dg^{-1}(\theta,\theta)<0$.

Clearly anyhow increasingly trapped implies increasingly trapped, which implies somehow increasingly trapped.

5. Doubly outer trapped surfaces

In spherical symmetry, outer trapped implies anyhow increasingly trapped,
but this does not hold in general.

Instead, a stricter version of outer trapped has this property, as follows.

Introduce two more curvatures: $K_{(\pm)} = *d*\theta \mp *d\theta + \frac{1}{2}g^{-1}(\theta,\theta)$
where $(\pm)$ indicates a label rather than an index. Then $2K = K_{(+)} + K_{(-)}$.

Definition 6. A doubly outer trapped surface is a trapped surface for which,
for all variations, $K_{(+)} > 0$, $K_{(-)} > 0$.

Clearly doubly outer trapped implies outer trapped.

Lemma 3. Assuming the Einstein equation,
$$-\zeta \cdot dg^{-1}(\theta,\theta) = 16\pi\zeta \cdot \Psi + g^{-1}(\theta,\theta)\zeta \cdot \theta - \tfrac{1}{2}\zeta \cdot (\theta - *\theta)K_{(-)} - \tfrac{1}{2}\zeta \cdot (\theta + *\theta)K_{(+)}.$$

Proposition 3. NEC and doubly outer trapped implies anyhow increasingly trapped.

Proof. For a trapped surface, $g^{-1}(\theta,\theta) < 0$, while for ζ in the appropriate causal quadrant,
$\zeta \cdot \theta < 0$ and $\zeta \cdot (\theta \pm *\theta) \leq 0$, then NEC $\Rightarrow \zeta \cdot \Psi \geq 0$, then inspect signs.

The proof also makes clear that outer trapped generally does not imply
anyhow increasingly trapped.

6. Involute trapped surfaces

Minimal trapped generally does not imply doubly outer trapped.

But if it is refined further, such a result can be obtained.

Definition 7. An involute trapped surface is a future (respectively past) trapped surface for which, for all variations along a future (respectively past) causal normal vector ζ, $Q = -\zeta^a H^b_* \nabla_b (*\theta)_a > 0$.

Clearly involute trapped implies minimal trapped.

Involute means curved or curled inwards, as of a leaf.

Lemma 4. Assuming the Einstein equation,

$-2Q = 16\pi\zeta \cdot \Psi + \frac{1}{2}\zeta \cdot (\theta - *\theta)K_{(+)} + \frac{1}{2}\zeta \cdot (\theta + *\theta)K_{(-)}$.

Proposition 4. NEC and involute trapped implies doubly outer trapped.

Proof. As before, $\zeta \cdot (\theta \pm *\theta) \leq 0$ and NEC $\Rightarrow \zeta \cdot \Psi \geq 0$.

Considering the null normal vectors $\zeta = l_\pm$, both $K_{(\pm)}$ must be positive.

The proof also makes clear that minimal trapped generally does not imply doubly outer trapped.

Definition 8. An *somehow involute* trapped surface is a future (respectively past) trapped surface for which, for some variation along a future (respectively past) causal normal vector ζ, $Q = -\zeta^a H^b_* \nabla_b (*\theta)_a > 0$.

Clearly minimal trapped implies somehow involute trapped.

One might ask whether somehow involute trapped implies somehow increasingly trapped.

The answer is negative, except for a special case.

Proposition 5. NEC and somehow involute trapped implies somehow increasingly trapped if $K_{(+)} = K_{(-)}$.

Proof. The involute condition gives $\zeta \cdot (\theta - *\theta)K_{(+)} + \zeta \cdot (\theta + *\theta)K_{(-)} < 0$.

However, to get increasingly trapped generally requires $\zeta \cdot (\theta - *\theta)K_{(-)} + \zeta \cdot (\theta + *\theta)K_{(+)} < 0$.

Thus it works if $K_{(+)} = K_{(-)}$.

7. Summary

The hierarchy of trapped surfaces is illustrated as follows:

$$\begin{array}{lllll}
\text{involute} & \Rightarrow & \text{doubly outer} & \Rightarrow & \text{anyhow increasingly} \\
\Downarrow & \text{NEC} & \Downarrow & \text{NEC} & \Downarrow \\
\text{minimal} & \Rightarrow & \text{outer} & \Rightarrow & \text{increasingly} \\
\Downarrow & \text{NEC} & & \text{NEC} & \Downarrow \\
\text{somehow involute} & & \Rightarrow & & \text{somehow increasingly} \\
 & & \text{NEC, } K_{(+)} = K_{(-)} & &
\end{array}$$

where the vertical implications are straightforward,

while the horizontal implications require NEC,

and in the last case, the symmetry where the curvatures $K_{(\pm)}$ are equal.

Otherwise, the threads are respectively geometrical warp and physical weft.