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No-hair conjecture for Einstein-Plebański nonlinear electrodynamics static black holes

Alberto Garcia,[1] Eva Hackmann,[2] Claus Lämmerzahl,[2,3] and Alfredo Macías[4,*]

[1]*Departamento de Física, CINVESTAV-IPN, Apartado Postal 14-740, C. P. 07000, México, D. F., Mexico*

[2]*ZARM, University of Bremen, Am Fallturm, 28359 Bremen, Germany*

[3]*Institute for Physics, University Oldenburg, 26111 Oldenburg, Germany*

[4]*Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, A. P. 55-534, Mexico D. F. 09340, Mexico*

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The *no-hair conjecture* statement is enhanced to include static regular and singular black-hole solutions, based on the class of nonlinear electrodynamics proposed by Plebański coupled to Einstein theory. In particular, we focus, as example, on regular black-hole solutions, i.e., black holes where the space-time metric everywhere is nonsingular, in the framework of General Relativity theory. Time-reversal invariance implies the existence of two separate non-overlapping cases: a purely gravito-electric case, and a purely gravito-magnetic one. We prove the enhanced *no-hair conjecture* for both cases. Moreover, the generalization of the conjecture including also the couplings to non-Abelian matter is also discussed.

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I. INTRODUCTION

A black hole is a region of space-time which is so incredibly dense that not even light can escape from its surface. The boundary of this region is known as event horizon, and it is defined as the set of points where the escape velocity from its gravitational field is equal to the speed of light [1].

A black hole is formed when a star of sufficient mass undergoes gravitational collapse, with most or all of its mass compressed into a sufficiently small area of space, producing infinite space-time curvature at a *place* called a *singularity* (For more rigorous definitions of a singularity see for instance Hawking *et al.* [2], and Wald [3] chapter 9).

Therefore, a black hole is an object whose entire mass M lies within a hypersurface of radius r_H. Event horizon is the name given to r_H, because from that radius the escape velocity from the black-hole's gravity is the speed of light. Black holes draw mass in through gravitational forces, but none of the masses can ever escape.

According to the Penrose *cosmic censorship conjecture* these singularities must be dressed by event horizons. No causal connection could exist between the interior of a black hole with the exterior fields, thus all events occurring at the singular region would have no influence on the exterior region, i.e., no information can reach us from this region. The black hole inside is hidden, it encloses a mysterious unknown. The understanding of the inside of a black hole is one of the outstanding problems in gravitational theory.

Black holes have never been directly observed, though predictions of their effects have matched observations [4,5]. There are different types of black holes. The most common type of black holes is known as stellar mass black holes as they are roughly up to a few times the mass of our Sun. This type of black holes is formed when large main sequence stars (10–15 times the mass of our Sun) run out of nuclear fuel in their cores. The result is a massive supernova explosion, leaving a black hole core behind, where the star once existed. The other type of black holes is called supermassive black holes. These are black holes with masses millions or billions times the mass of the Sun. It is possible that they form from the collapse of a single, extremely high massive star.

We will not consider cosmological black holes, since in such a case one should face two different kinds of event horizons, i.e., the black hole event horizon and the cosmological event horizon of an observer, c.f. Gibbons *et al.* [6].

Almost 40 years ago Wheeler enunciated the Israel-Carter conjecture, today colloquially known as *black holes have no hair* [7]. He was inspired by Israel's uniqueness theorems for the Schwarzschild and Reissner-Nordström black holes [8,9], and by Carter's [10] and Wald's [11] uniqueness theorems for the Kerr black hole.

Wheeler anticipated that collapse leads to a black hole endowed with mass, charge, and angular momentum, but, so far as we can now judge, no other free parameters exist by which he meant that the collapse ends with a Kerr-Newman black hole. Wheeler stressed that other quantum numbers such as baryon [12] number or strangeness can have no place in the external observer's description of a black hole.

What is so special about mass, electric charge and angular momentum? According to Bekenstein [12], they are all conserved quantities subject to a Gauss-type law. One can thus determine these properties of a black hole by measurements from afar. Obviously this reasoning has to be theoretically complemented by including magnetic (monopole) charge as a fourth parameter because it is also conserved in Einstein-Maxwell theory, it also submits to a Gauss type law, and duality of the theory permits

*amac@xanum.uam.mx

Kerr-Newman-like solutions with magnetic charge alongside or instead of electric charge.

In the updated version of Wheeler's conjecture, the forbidden *hair* is any field not of gravitational or electromagnetic (Yang-Mills) nature associated with a black hole. The *no-hair conjecture*, the statement that is used as a realization of Wheeler's graphical statement *black holes have no hair*, has often been interpreted in many different ways by different authors, their interpretation depends of the particular definition of a black hole in the different theories of gravity, in four or in higher dimensions, sometimes leading to confusions and misunderstandings for the reader. Therefore, in this work we shall try to be more precise, self-contained, and restrict ourselves only to the case of general static, regular and singular, black holes in Einstein's General Relativity.

On the other hand, recently considerable interest in regular black-hole solutions, i.e., black holes where the space-time metric everywhere is nonsingular, based on the coupling of Einstein theory to the class of nonlinear electrodynamics proposed by Plebański [13,14], has been renewed, since this system admits regular solutions. The study on the global regularity of black-hole solutions is quite important in order to try to understand the final state of gravitational collapse of initially regular configurations.

Bardeen, in 1968 [15], advanced a pioneering model for a regular black hole. This model, as Ayón-Beato *et al.* [16] showed, *is indeed a solution of Einstein field equations* in which there is a black hole with a horizon but without a singularity, i.e., the first regular black hole. The matter field content resulted in a kind of magnetic matter field, monopole charge of a self-gravitating magnetic field described by a nonlinear electrodynamics of the Plebański class, yielding a modification of the Reissner-Nordström metric [16].

The Bardeen model fulfills the weak energy condition and is regular everywhere, although the invariant of the associated electromagnetic field exhibits the usual singular behavior ($\sim 1/r^4$) of magnetic monopoles.

Pellicer *et al.* [17] in 1969, were the first to consider Einstein-Plebański nonlinear electrodynamics systems, they found a class of nonsingular, static, spherically symmetric solutions, corresponding to a point-charge source. The metrics of these solutions are shown to approach the Reissner-Nordström metric at large distances from the source, if one makes the proper identification of mass. In some sense, the Ayón-García regular black hole can be considered to some extent as a particular case of the Pellicer *et al.* solution.

In 1979 Bronnikov *et al.* [18] also found point particle solutions, however see [19] for a more recent review, in particular, the type I nonlinear electrodynamics solutions [20].

Almost all the subsequent regular black hole spherically symmetric solutions are based on Bardeen's proposal. There has been a tremendous development on the implementation and on the analysis of the properties of regular solutions by Lake *et al.* [21,22].

Other black-hole models with regular centers exist, like the ones by Bronnikov *et al.* [20], and Dymnikova [23]. Nevertheless, as Dymnikova [23] mentions, such models, although they satisfy the condition ${T^0}_0 = {T^1}_1$ of spherical symmetry, cannot be derived from a Lagrangian in general relativity. Another kind of the them involves unusual exotic phantom anomalous scalar matter, i.e., scalar matter whose kinetic term is negative, and other models involving higher dimensions [20]. Since a phantom violates the strong and weak energy conditions and does not lead to the correct sign for the mass in the corresponding Klein-Gordon equation, it represents an *ad hoc* speculation trick nowadays used in cosmology in order to produce negative pressure [24–27]. Nevertheless, such models will not be considered in this paper.

Regular black-hole solutions to Einstein equations with physically reasonable sources have been reported by Ayón-Beato and García [28–30]. In order to avoid the problem of multiple Lagrangian branches arising in the standard formulation in terms of the electromagnetic fields for gravito-electric solutions, they worked with the natural variables for the exact analytic gravito-electric regular black-hole solutions which are the ones in terms of the field excitations $P^{\mu\nu}$ and $\mathcal{H}(P)$, while the standard ones in terms of the fields $F^{\mu\nu}$ and $\mathcal{L}(F)$ are the natural variables for the gravito-magnetic exact analytic solutions [31], to the systems Einstein-Plebański nonlinear electrodynamics. Both sets of variables are related by a Legendre transformation, i.e., $\mathcal{L} = P_{\mu\nu}F^{\mu\nu} - \mathcal{H}$ [13].

The objective of this contribution is to prove the *enhanced no-hair conjecture* for the gravito-electric and gravito-magnetic regular black holes [29,31]. Moreover, we analyze its coupling with non-Abelian matter. It is worthwhile to stress the fact that our results apply to all static Einstein-Maxwell, Einstein-Plebański nonlinear electrodynamics, and to Einstein-Yang-Mills configurations, even those with singularities.

The outline of the paper is as follows: In Sec. II the no-hair conjecture is revisited and the *enhanced no-hair conjecture* is formulated. In Sec. III the coupling with non-Abelian matter is reviewed and the *long-hair conjecture* is presented. In Sec. IV a classification of the spherically symmetric solutions in General Relativity is presented. In Sec. V the Ayón-García regular black hole is briefly revisited. In Sec. VI the proof of the *enhanced no-hair conjecture* for the gravito-electric and for the gravito-magnetic regular black holes is presented. Section VII is devoted to the conclusions. Finally, since non-Abelian hair is generic, in the Appendix the proof of the *long-hair conjecture* for different non-Abelian matter couplings to regular black holes is summarized.

II. ENHANCED NO-HAIR CONJECTURE STATEMENT

The *no-hair conjecture* statement postulates that all black-hole solutions of the Einstein-Maxwell equations of gravitation coupled to electromagnetism in general relativity can be completely characterized by only three externally observable classical parameters: mass, electric charge, and angular momentum.

All other information (for which *hair* is a metaphor) about the matter which formed a black hole or is falling into it, "disappears" behind the black hole event horizon and is therefore permanently inaccessible to external observers, i.e., all matter fields present in a black hole space-time would eventually be either radiated to infinity or "sucked" into the black hole, except when those fields are associated with conserved charges defined at asymptotic infinity [32].

There is black-hole hair when the space-time metric and the configuration of the other fields of a stationary black-hole solution are not completely specified by the conserved charges defined at asymptotic infinity.

The *no-hair conjecture* was originally formulated for black holes within the context of a four-dimensional space-time, obeying the Einstein field equations of general relativity with zero cosmological constant, in the presence of electromagnetic fields, or optionally other Abelian fields such as scalar fields and massive vector fields (Proca fields, spinor fields, etc.) [33]. The *no-hair conjecture* is concerned only with properties which are independent of the frame of reference (point of view of the observer). The conjecture therefore says nothing about a black-hole's position or velocity. In other words it is a need to specify conserved Noether charges defined at asymptotic infinity, in order to characterize completely a stationary/static black-hole solution [34].

Since Schwarzschild black hole is a unique static solution, it is specified completely by the value of its ADM mass, defined by a two-surface integral at infinity, and it is associated with an asymptotic conservation law. Additionally, since the same happens with the Reissner-Nordström black-hole solution, it is specified completely by the values of its electric charge and of its ADM mass, respectively. These numbers represent the conserved attributes of an object which can be determined from a distance by examining its gravitational and electromagnetic fields. All other variations in the black hole will either escape to infinity or be swallowed up by the black hole.

The event horizon of a stationary black hole must be a Killing horizon, i.e., a null hypersurface $\mathcal{K}$ to which a Killing field is normal, and has complete orbits on $\mathcal{K}$. This Killing field ξ^μ normal to $\mathcal{K}$ needs not coincide with the Killing field which is time-like near infinity [35]. We can classify the Killing horizons according to the value of the surface gravity κ as extremal ($\kappa = 0$) or nonextremal ($\kappa \neq 0$), and, moreover, that a Killing horizon is of the bifurcating type if it is the union of two intersecting null hypersurfaces.

An event horizon is regular, if it is contained, together with an open neighborhood of itself, in a smooth nonsingular pseudo-Riemannian manifold.

The regularity of the geometry at the horizon ensures that tidal gravitational forces are bounded there. Test particles following geodesics feel nothing peculiar as they cross the horizon.

Now, we are in position to formulate the statement of the *enhanced no-hair conjecture* to include black holes of the system Einstein-Plebański class of nonlinear electrodynamics.

The enhanced no-hair conjecture *statement postulates that* all black-hole *solutions, regular or singular*, of the Einstein-Maxwell *and of the* Einstein-Plebański class of *nonlinear electrodynamics systems of equations of gravitation coupled to electromagnetism, can be completely characterized by only three externally observable classical parameters: mass, electric/magnetic charge, and angular momentum.*

In order to prove the *enhanced no-hair conjecture* one first writes the black-hole metric

$$ds^2 = -Ndt^2 + \frac{1}{F(r)}dr^2 + r^2 d\Omega, \tag{1}$$

in advanced Eddington-Finkelstein coordinates [36]. Such coordinates are advantageous since they can be extended through the horizon, in fact a portion of it. One then has

$$ds^2 = -V^2 du^2 + 2du dr + r^2 d\Omega, \tag{2}$$

where $V^2 = -\xi^\mu \xi_\mu$ is the square of the Killing field ($\xi^\mu = (\partial/\partial u)^\mu$) that is time-like at infinity (ξ_μ is related with g_{00}, and its norm $\xi^\mu \xi_\mu = -1$, at infinity), and u is a null coordinate varying in the range $(0, \infty)$, and $r > 0$.

In view of the fact that the metric (2) is regular at the horizon ($r = r_H$), we shall explicitly check whether or not Einstein equations hold at the horizon.

The normal vector to the hypersurfaces $r = \text{const}$ is given by $l_\mu = \nabla_\mu r = \delta^r_\mu$, $l^\mu = g^{\mu r}$ with norm $l_\mu l^\mu = g^{rr} = F(r)$. Therefore, at the horizon, i.e., $r = r_H$, the metric function $F(r_H) = 0$ the l_μ vector becomes a null vector.

The fact that $l^\mu = (\partial/\partial r)^\mu$ is a smooth vector field (space-like everywhere but null only at the horizon), even across the horizon ($r = r_H$) implies that the quantity $R_{\mu\nu} l^\mu l^\nu$ is finite everywhere and, in particular, at the horizon. The next step would be to compute $T_{\mu\nu} l^\mu l^\nu|_{r_H}$ and compare its value with that of $R_{\mu\nu} l^\mu l^\nu|_{r_H}$. Therefore, at the horizon the vectors l^μ and ξ^μ coincide.

Additionally, one chooses n^μ as a future-directed null vector ($n_\mu l^\mu = -1$), orthogonal to the space-like cross sections of the horizon, the quantities $R_{\mu\nu} n^\mu n^\nu$ and $R_{\mu\nu} l^\mu n^\nu$ must vanish at the horizon [37]. Therefore, there exist no trails of the hair in the whole domain of outer communication of the black-hole horizon.

In full, the enhanced form of the *no-hair conjecture* states that only the observer at spatial infinity can measure the conserved charges characterizing a static black hole.

III. LONG-HAIR CONJECTURE: NON-ABELIAN MATTER HAIR

There is black-hole hair when the spacetime metric and the configuration of the other fields of a stationary black-hole solution are not completely specified by the conserved charges defined at asymptotic infinity.

Recently, there has been a surge of interest in studying static black-hole solutions for gravity coupled to nonlinear matter fields. This is due in part to the discovery that, in spite of the widely held belief to the contrary, nontrivial static black-hole solutions exist in the Einstein-Yang-Mills theory [38], Einstein-Skyrme theory [39], Einstein-Yang-Mills-dilaton theory [40], and the Einstein-Yang-Mills-Higgs theory [41].

Thus, in general, the presence of matter with non-Abelian structure does not allow us to apply the usual arguments used to prove the *enhanced no-hair conjecture*. In all these cases black-hole hair exists. It seems that the nonlinear character of the non-Abelian matter content plays an essential role and it is responsible for the failure of the argument and, thus, for the existence of black-hole hair.

Núñez *et al.* [42] proposed the so-called *no-short-hair conjecture* as an alternative to the *no-hair conjecture*, for the cases of couplings with non-Abelian matter.

They assume that the nonlinear behavior of the matter fields must be present both in a region very close to the horizon (a region from which presumably the fields would tend to be sucked in) and in a region relatively distant from the horizon (a region from which presumably the fields would tend to be radiated away), with the self-interaction being responsible for binding together the fields in these two regions. They show that in all the above mentioned cases black-hole hair exists and it must extend beyond a surface situated at 3/2 the horizon radius r_H [42,43].

We briefly extend the *no-short-hair conjecture* of Núñez *et al.*, and formulate the *long-hair conjecture* to include regular and singular black holes arising from the systems Einstein-Plebański nonlinear electrodynamics.

Let us consider asymptotically flat static spherically symmetric black hole space-times and write the line element as follows:

$$ds^2 = -e^{-2\delta} f dt^2 + f^{-1} dr^2 + r^2 d\Omega, \tag{3}$$

where $\delta = \delta(r)$, $f = 1–2m(r)/r$, and there is a regular event horizon at r_H, where $m(r_H) = r_H/2$, and $\delta(r_H)$ is finite. The asymptotic flatness requires that $f \to 1$ and $\delta \to 0$ at infinity. It is interesting to note that Eq. (3) represents a special parametrization with $N = e^{-2\delta} f$ and $F = f^{-1}$ of the general spherically symmetric metric, Eq. (1).

We assume that the matter fields will also respect the symmetries of the space-time, as it happens in each of the cases where black-hole hair has been found.

The Einstein equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, together with the equations of motion for the field under consideration, form a dependent set as they are related by the conservation equation $T^{\mu}{}_{\nu;\mu} = 0$. The corresponding Einstein equations thus read:

$$f' = 8\pi r T^t{}_t + (1 - f)/r, \tag{4}$$

$$\delta' = (4\pi r/f)[T^t{}_t - T^r{}_r]. \tag{5}$$

where the prime stands for differentiation with respect to r. The conservation equation has only one nontrivial component $T^{\mu}{}_{r;\mu} = 0$, the r component, which by means of Eqs. (4) and (5) can be written as

$$e^{\delta}(e^{-\delta} r^4 T^r{}_r)' = \frac{r^3}{2f}[(3f - 1)(T^r{}_r - T^t{}_t) + 2fT], \tag{6}$$

with T the trace of the energy-momentum tensor.

Let us assume that the line element Eq. (3), for asymptotically flat static spherically symmetric black hole space-times, satisfies Einstein equations with matter fields satisfying the week energy conditions, i.e., the energy density, $\rho \equiv -T^t{}_t$, is positive semidefinite and that it bounds the pressures, in particular, $|T^r{}_r| \leq T^t{}_t$.

Now, the statement of the *long-hair conjecture* reads:

Theorem: Let Eq. (3) represent the line element of an asymptotically flat static spherically symmetric black hole space-time, regular or singular, satisfying Einstein's equations with matter fields satisfying the Weak Energy Conditions and such that the trace of the energy-momentum tensor is nonpositive, and such that the energy density ρ goes to zero faster than r^{-4}, then the function $\Xi = e^{-\delta} r^4 T^r{}_r$ is negative semidefinite at the horizon and it is decreasing between r_H and r_0, where $r_0 > \frac{3}{2} r_H$ and for some $r > r_0$ the function Ξ begins to increase towards its asymptotic value $\Xi = 0$.

In order to prove it, we begin by noting that the proper radial distance is given by $du = f^{-1/2} dr$, so Eq. (6) can be written as

$$\begin{aligned} \frac{d}{du}(e^{-\delta} r^4 T^r{}_r) = \frac{e^{-\delta} r^3}{2}(&f^{1/2}[3(T^r{}_r - T^t{}_t) + 2T] \\ &- f^{-1/2}[(T^r{}_r - T^t{}_t]), \end{aligned} \tag{7}$$

Since the components $T^t{}_t$, $T^r{}_r$, and $T^{\theta}{}_{\theta}$ must be regular at the horizon, i.e., the scalar $T^{\mu\nu}T_{\mu\nu}$ is regular at the horizon, and since u is a good coordinate at the horizon, the left-hand side of Eq. (7) must be finite in the limit $r \to r_H$, and since $f(r_H) = 0$ one finds that

$$T^r{}_r(r_H) = T^t{}_t(r_H) = -\rho(r_H) \leq 0, \tag{8}$$

therefore $\Xi(r_H) \leq 0$. The right-hand side Eq. (6) is negative definite unless $3f - 1 \geq 0$. This follows from the weak energy condition, which requires that $T^r{}_r - T^t{}_t > 0$ and the assumption the trace of the energy-momentum tensor T is nonpositive. Hence $\Xi = e^{-\delta} r^4 T^r{}_r$ is decreasing at least until the point where $3f - 1$ becomes positive, this happens at $r_1 = 3m(r_1)$, therefore, $r_0 > r_1$. Since $m(r)$ is an increasing function, one then has $r_0 > 3m(r_1) > 3m(r_H) = \frac{3}{2} r_H$.

Under these conditions, the asymptotic behavior of the fields cannot start before r is sufficiently large, since this behavior is characterized by the fact that $T^r{}_r$ approaches zero, at least as r^{-4}. And, in particular, in the asymptotic regime the function Ξ is not simultaneously negative and decreasing.

From the inequality $\Xi(\frac{3}{2} r_H) < \Xi(r_H) \leq 0$, and from the negativity of the radial pressure $T^r{}_r$, it follows that

$$\left| T^r{}_r\left(\frac{3}{2} r_H\right) \right| > \left(\frac{2}{3}\right)^4 |T^r{}_r(r_H)| e^{[\delta((3/2) r_H) - \delta(r_H)]}. \quad (9)$$

From the fact that $T^r{}_r(r_H) = T^t{}_t(r_H) = -\rho(r_H)$, $\Xi \leq 0$ and using the relation $e^{-\delta(r_H)} = 2 r_H \kappa / [1 + 8\pi r_H^2 T^t{}_t(r_H)]$, with κ the surface gravity of the black hole, one obtains that

$$\left| T^r{}_r\left(\frac{3}{2} r_H\right) \right| > \left(\frac{2}{3}\right)^4 \frac{2 r_H \kappa}{1 + 8\pi r_H^2 T^t{}_t(r_H)} |T^t{}_t(r_H)| e^{\delta((3/2) r_H)}. \quad (10)$$

From Eq. (5) and the weak energy condition, it follows that $\delta(r) > 0$, since $\delta(\infty) = 0$, then one obtains that

$$\left| T^r{}_r\left(\frac{3}{2} r_H\right) \right| > \left(\frac{2}{3}\right)^4 \frac{2 r_H \kappa}{1 + 8\pi r_H^2 T^t{}_t(r_H)} |T^t{}_t(r_H)|. \quad (11)$$

The bound Eq. (11) on the value of the radial pressure at $r = \frac{3}{2} r_H$ is expressed completely in terms of physical quantities evaluated at r_H, it means that the matter fields start their asymptotic behavior at some $r = \frac{3}{2} r_H$. Although Eq. (11) has a clear physical interpretation, it is probably not the best bound that can be put on the value of $T^r{}_r(\frac{3}{2} r_H)$, since there is no general lower bound on the surface gravity.

A much better bound on $T^r{}_r(\frac{3}{2} r_H)$ can be obtained by considering Eq. (9). Integrating Eq. (5) results

$$\delta\left(\frac{3}{2} r_H\right) - \delta(r_H) = 4\pi \int_{r_H}^{(3/2 r_H)} \frac{r[T^t{}_t - T^r{}_r]}{f} dr. \quad (12)$$

Then, one can substitute the particular form of $T^t{}_t - T^r{}_r$ corresponding to the specific coupling of general relativity with non-Abelian matter one is considering and obtain the corresponding bound.

This set of results has been obtained under the assumption of spherical symmetry and asymptotic flatness, since all the cases in which hair has been discovered also involved this simplifying assumption.

In full, the *long-hair conjecture* states that only an observer situated at a distance $r > r_0$ outside the horizon can measure the global charges characterizing a static black hole coupled to non-Abelian matter.

IV. CLASSIFICATION OF THE SPHERICALLY SYMMETRIC SOLUTIONS IN GR

Assuming spherical symmetry, the Einstein tensor associated with the metric:

$$\boldsymbol{g} = -N(r)\boldsymbol{dt}^2 + \frac{\boldsymbol{dr}^2}{F(r)} + r^2(\boldsymbol{d\theta}^2 + \sin^2\theta \boldsymbol{d\phi}^2), \quad (13)$$

can be written in a very symmetric form:

$$G_\mu{}^\nu = \mathrm{diag}(A + a, A, B, B), \quad (14)$$

where

$$A = \frac{1}{r}\frac{F}{N}\frac{d}{dr}N + \frac{F}{r^2} - \frac{1}{r^2};\ a = \frac{1}{r}\frac{d}{dr}F - \frac{1}{r}\frac{F}{N}\frac{d}{dr}N,$$
$$B = \frac{1}{2}\frac{F}{N}\frac{d^2}{dr^2}N - \frac{1}{4}\frac{F}{N^2}\left(\frac{d}{dr}N\right)^2 + \left(\frac{1}{2r}\frac{F}{N} + \frac{1}{4N}\frac{d}{dr}F\right)\frac{d}{dr}N + \frac{1}{2r}\frac{d}{dr}F. \quad (15)$$

We can classify the different types of matter by means of their energy-momentum tensor. One carries out the classification of the Einstein tensor, because of the Einstein equations, this classification yields to a classification of the energy-momentum tensor, in the global sense, independently of the particular mixture of matter fields, which could be present. This leads to the Plebański-Pirani algebraic types.

Therefore, for spherically symmetry, one has the following cases:

(1) If $A \neq B$ and $a \neq 0$, $\rightarrow G^\mu{}_\nu$ corresponds to anisotropic fluid.
(2) If $A = B$ and $a \neq 0$, $\rightarrow G^\mu{}_\nu$ corresponds to perfect fluid.
(3) $A \neq B$ and $a = 0$, $\rightarrow G^\mu{}_\nu$ corresponds to linear/nonlinear electromagnetic Abelian and nonlinear Yang-Mills non-Abelian matter. Moreover, it also allows the interpretation of an anisotropic fluid for some specific eigenvectors.
(4) If $A = B$ and $a = 0$, $\rightarrow G^\mu{}_\nu$ corresponds to vacuum (cosmological constant vacuum).

These are the only kinds of matter consistent with spherically symmetric gravitational fields in Einstein theory. The vacuum (cosmological constant vacuum), and the linear/nonlinear electromagnetic Abelian matter, are compatible with black-hole solutions and no-one of them can lead to black-hole hair. Hence, they fulfill the *enhanced no-hair conjecture*.

On the other hand, since the energy-momentum tensor $T_{\mu\nu}$ for Yang-Mills non-Abelian matter shares similar Lorentz structure with the one of Maxwell electromagnetic field, it is consistent with spherical symmetric solutions, like black holes. Nevertheless, it does not fulfill the *enhanced no-hair conjecture* and leads to the existence of *long-hair.*

Therefore, the statement of the generalized version of the *enhanced no-hair conjecture* states that: *All spherically symmetric, regular or singular, black-hole solutions to the Einstein field equations coupled to any type of self-gravitating matter field are completely and uniquely determined by their mass, electric/magnetic charge, angular momentum, (as it is the case for the coupling with Abelian matter, satisfying the* enhanced no-hair conjecture), *and a set of global charges (as it is the case for the coupling with non-Abelian matter, satisfying the* long-hair conjecture).

V. THE AYÓN-GARCÍA REGULAR BLACK HOLE

Ayón and García [28] presented a static, spherically symmetric solution of general relativity coupled to a nonlinear electrodynamics, belonging to the Plebański class. The metric is globally regular and, for certain values of the ratio between electric charge and mass, describes black holes with a global structure similar to Reissner-Nordström as a proper spatial asymptotic structure but with a regular center instead of a singularity. Two other solutions having similar properties can be found in [29,30].

The Ayón-García regular black hole [28] can be obtained from the following action of the Einstein-Plebański nonlinear electrodynamic theory[13,14]:

$$\mathcal{S} = \int d\tau\left[\frac{1}{16\pi}R - \frac{1}{4\pi}\mathcal{L}(F)\right], \tag{16}$$

where R is scalar curvature and $\mathcal{L}(F)$ is a function of the electromagnetic invariant $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

One can also describe the system under consideration by means of the function $\mathcal{H}(P)$ obtained from the following Legendre transformation:

$$\mathcal{H} = F_{\mu\nu}P^{\mu\nu} - \mathcal{L} = 2F\mathcal{L}_F - \mathcal{L} = 2P\mathcal{H}_P - \mathcal{L}, \tag{17}$$

where $\mathcal{L}_F = \frac{\partial\mathcal{L}}{\partial F}$ and $\mathcal{H}_P = \frac{\partial\mathcal{H}}{\partial P}$. The components of $P_{\mu\nu} = \mathcal{L}_F F_{\mu\nu}$ are just the electromagnetic field excitations **D** and **H**, and $\mathcal{H}(P)$ is a function of its invariant $P = \frac{1}{4}P_{\mu\nu}P^{\mu\nu} = (\mathcal{L}_F)^2F$, i.e.,

$$d\mathcal{H} = \mathcal{H}_P dP = \mathcal{H}_P\frac{dP}{dF}dF = \mathcal{H}_P d[F(\mathcal{L}_F)^2] \tag{18}$$

$$= \mathcal{H}_P\mathcal{L}_F[\mathcal{L}_F + 2F\mathcal{L}_{FF}]dF \tag{19}$$

$$= [\mathcal{L}_F + 2F\mathcal{L}_{FF}]dF, \tag{20}$$

since $\mathcal{H}_P\mathcal{L}_F = 1$. In terms of $\mathcal{H}$ the nonlinear electromagnetic Lagrangian $\mathcal{L}(F)$ reads:

$$\mathcal{L}(F) = 2P\mathcal{H}_P - \mathcal{H}(P). \tag{21}$$

The function $\mathcal{H}$ determining the nonlinear electromagnetic source reads:

$$\mathcal{H}(P) = P\frac{[1 - 3\sqrt{-2q^2P}]}{[1 + \sqrt{-2q^2P}]^3} - \frac{3}{q^2 s}\left[\frac{\sqrt{-2q^2P}}{1 + \sqrt{-2q^2P}}\right]^{3/2}, \tag{22}$$

where $s = |q|/2m$ and the invariant $P < 0$. The fulfillment of the weak energy condition requires that $\mathcal{H} < 0$, $\mathcal{H}_P > 0$. The linear Maxwell case is recovered when $\mathcal{H} \approx P$ for weak fields, i.e., $P \ll 1$.

The Einstein and nonlinear electromagnetic field equations arising from the action (16) read:

$$G_\mu{}^\nu = 2[\mathcal{H}_P P_{\mu\lambda}P^{\nu\lambda} - \delta_\mu^\nu(2P\mathcal{H}_P - \mathcal{H})] \tag{23}$$

$$\nabla_\mu P^{\alpha\mu} = 0. \tag{24}$$

The Ayón-García ansatz reads: $-g_{tt} = g_{rr}^{-1} = 1–2m/r + Q(r)/r^2$, and $P_{\mu\nu} = \delta^t_{[\mu}\delta^r_{\nu]}D(r) = \delta^t_{[\mu}\delta^r_{\nu]}q/r^2$. Therefore $P = -D^2/2 = -q^2/2r^4$.

In order to evaluate the electric field $E = F_{tr} = \mathcal{H}_P D$ Ayón-García used Eq. (22) and the ${}_t^t$-component of the Einstein equations, i.e., $2\mathcal{H}(p) = (rQ' - Q)/r^4$, where $Q' = dQ/dr$, to evaluate the function $Q(r)$ of their ansatz. Hence, the function Q is obtained to be

$$Q = 2mr - \frac{2mr^4}{(r^2 + q^2)^{3/2}} + \frac{q^2r^4}{(r^2 + q^2)^2}. \tag{25}$$

Therefore, the gravitational field of the regular black hole of Ayón-García is described by the line element

$$ds^2 = -\left[1 - \frac{2mr^2}{(r^2 + q^2)^{3/2}} + \frac{q^2r^2}{(r^2 + q^2)^2}\right]dt^2 + \left[1 - \frac{2mr^2}{(r^2 + q^2)^{3/2}} + \frac{q^2r^2}{(r^2 + q^2)^2}\right]^{-1}dr^2 + r^2d\Omega, \tag{26}$$

and the associated electric field E reads:

$$E = qr^4\left[\frac{r^2 - q^2}{(r^2 + q^2)^4} + \frac{15}{2}\frac{m}{(r^2 + q^2)^{7/2}}\right]. \tag{27}$$

It is interesting to note that asymptotically this solution behaves as the Reissner-Nordström one, i.e., $-g_{tt} = 1–2m/r + q^2/r^2 + \cdots$, and $E = q/r^2 + \cdots$.

It is important to note that, as it was pointed out in [31], there exists a duality between spherically symmetric solutions written in the F and P frameworks, i.e., any solution for a given Lagrangian $\mathcal{L}(F)$ characterized by a certain metric structural function $A(r)$ and the electromagnetic field components F_{01} and F_{23} has a counterpart with the same $A(r)$ but with F replaced by $-P$, $\mathcal{L}$ by $\mathcal{H}$, F_{01} by P_{23} and F_{23} by P_{01}, and conversely. The functional dependence $-\mathcal{H}(-P)$ in the dual solution is the same as $\mathcal{L}(F)$ in the original solution. In particular, any regular magnetic solution obtained for a given $\mathcal{L}(F)$ has a purely electric counterpart with similar dependence $\mathcal{H}(P)$ and viceversa. Therefore, there exists a magnetically charged solution dual to the electrically charged Ayón-García regular black-hole solution [44].

It is worthwhile to point out the fact that in order to avoid the necessity of multiple Lagrangian branches the natural variables for the exact analytic gravito-electric regular black-hole solutions are the ones in terms of the field excitations $\mathbf{D}$ and $\mathbf{H}$, while the standard ones in terms of the fields $\mathbf{E}$ and $\mathbf{B}$ are the natural variables for the gravito-magnetic exact analytic solutions [31], in the systems Einstein-Plebański class of nonlinear electrodynamics.

VI. ENHANCED NO-HAIR CONJECTURE PROOF FOR STATIC REGULAR BLACK HOLES

In a static black hole, the Killing field $\boldsymbol{\xi}$ coincides with the null generator of the event horizon $\mathcal{K}$ and is time-like and hypersurface orthogonal in all the domain of outer communications $\langle\langle\mathcal{J}\rangle\rangle$. This allows us to choose, by simply connectedness of $\langle\langle\mathcal{J}\rangle\rangle$, a global coordinates system (t, x^i), $i = 1, 2, 3$, in all $\langle\langle\mathcal{J}\rangle\rangle$, such that $\boldsymbol{\xi} = \boldsymbol{\partial}/\boldsymbol{\partial t}$ and the metric reads:

$$\boldsymbol{g} = -V\boldsymbol{dt}^2 + \gamma_{ij}\boldsymbol{dx}^i\boldsymbol{dx}^j, \tag{28}$$

where V and $\boldsymbol{\gamma}$ are t-independent, $\boldsymbol{\gamma}$ is positive definite in all $\langle\langle\mathcal{J}\rangle\rangle$, and V is positive in all $\langle\langle\mathcal{J}\rangle\rangle$, and vanishes in $\mathcal{K}$. From (28) it can be noted that staticity is equivalent to the existence of a time-reversal isometry $t \mapsto -t$.

We will assume that the nonlinear electromagnetic field shares the same symmetries as the metric one, namely, it is stationary, $\mathcal{L}_{\boldsymbol{\kappa}}\boldsymbol{A} = 0$. The staticity of the metric is extended to the nonlinear electromagnetic field A^α and the nonlinear electromagnetic equations (24), i.e., they are invariant under time-reversal transformations.

The time-reversal invariance of nonlinear electromagnetic equations (24) requires that, in the coordinates (28), A^t and P^{ti} remain unchanged while A^i and P^{ij} change their sign, or the opposite scheme, i.e., A^t and P^{ti} change sign as long as A^i and P^{ij} remain unchanged under time-reversal.

Therefore A^i and P^{ij} must vanish in the first case, and A^t and P^{ti} vanish in the second one. Hence, time-reversal invariance implies the existence of two separated nonoverlapping cases: a purely gravito-electric case (i.e., Ayón–García regular black hole.) and a purely gravito-magnetic case (i.e., Bronnikov regular black hole.).

First let us prove the *enhanced no-hair conjecture* for purely gravito-electric configurations, we will follow the procedure proposed by Ayón-Beato [45].

Now let $\mathcal{V} \subset \langle\langle\mathcal{J}\rangle\rangle$ be the open region bounded by the space-like hypersurface Σ, the space-like hypersurface Σ' and the pertinent portions of the horizon $\mathcal{K}$, and the spatial infinity i^o. The space-like hypersurface Σ' is obtained by shifting each point of Σ a unit parametric value along the integral curves of the Killing field $\boldsymbol{\xi}$. Multiplying the nonlinear electromagnetic equations (24) by A_μ and integrating by parts over $\mathcal{V}$ using the Gauss theorem, one obtains

$$\left[\int_{\Sigma'} - \int_{\Sigma} + \int_{\mathcal{K}\cap\overline{\mathcal{V}}} + \int_{i^o\cap\overline{\mathcal{V}}}\right] A_\alpha P^{\beta\alpha} d\Sigma_\beta = \int_{\mathcal{V}} (2F\mathcal{L}_F) d\tau = 2\int_{\mathcal{V}} \frac{\mathcal{P}}{\mathcal{L}_F} d\tau = 2\int_{\mathcal{V}} \mathcal{H}_P \mathcal{P} d\tau. \tag{29}$$

The boundary integral over Σ' cancels out the corresponding one over Σ, since Σ' and Σ are isometric hypersurfaces. The boundary integral over $i^o \cap \overline{\mathcal{V}}$ vanishes by the usual falloff of the fields at infinity. We will show that the integrand of the boundary integral at the portion of the horizon $\mathcal{K} \cap \overline{\mathcal{V}}$ vanishes. To achieve this goal we use the standard measure at the horizon $d\Sigma_\beta = 2n_{[\beta}l_{\mu]}l^\mu d\sigma$, where $\boldsymbol{l}$ is the null generator of the horizon, $\boldsymbol{n}$ is the other future-directed null vector ($n_\mu l^\mu = -1$), orthogonal to the space-like cross sections of the horizon, and $d\sigma$ is the surface element. The standard measure follows from choosing the natural volume 3-form at the horizon, i.e., $\boldsymbol{\eta}_3 = {}^*(\boldsymbol{n} \wedge \boldsymbol{l}) \wedge \boldsymbol{l}$. By using the quoted measure the integrand over the horizon can be written as

$$A_\alpha P^{\beta\alpha} d\Sigma_\beta = (A_\alpha P^{\beta\alpha} l_\beta + A_\alpha P^{\beta\alpha} n_\beta l_\mu l^\mu) d\sigma. \tag{30}$$

In order to show the vanishing of

$$\int_{\mathcal{K}\cap\overline{\mathcal{V}}} A_\alpha P^{\beta\alpha} d\Sigma_\beta = \int_{\mathcal{K}\cap\overline{\mathcal{V}}} (A_\alpha P^{\beta\alpha} l_\beta + A_\alpha P^{\beta\alpha} n_\beta l_\mu l^\mu) d\sigma \tag{31}$$

it is enough to establish the vanishing of $A_\alpha P^{\beta\alpha} l_\beta$ and the bounded of $A_\alpha P^{\beta\alpha} n_\beta$ at the horizon. The behavior of this quantities at the horizon can be established by studying some invariants constructed from the curvature. By using the Einstein equations (23), we obtain the following relations:

$$\frac{4\pi}{\kappa} R = 8[\mathcal{H}_P P - \mathcal{H}], \tag{32}$$

$$\frac{16\pi^2}{\kappa^2} R_{\mu\nu}R^{\mu\nu} = 3P^2 + 4G^2 - 32\mathcal{H}_P P\mathcal{H} + 16\mathcal{H}^2, \tag{33}$$

where $P \equiv P_{\alpha\beta}P^{\alpha\beta}/4$ and $G \equiv {}^*P_{\alpha\beta}P^{\alpha\beta}/4$. The smoothness of the horizon leads to a well-behaved scalar curvature on it, hence the invariant $\mathcal{H}_P P - \mathcal{H}$ has to remain also bounded on the horizon.

The last two terms in (33) are non-negative in both gravito-electric and gravito-magnetic cases, the remaining terms are also non-negative, and consequently each one is bounded at the horizon, in particular, the invariants P, G, and $\mathcal{H}$. Other invariants can be built from the Ricci curvature (23) by means of $\boldsymbol{l}$ and $\boldsymbol{n}$, which are well-defined smooth vector fields on the horizon. The first invariant reads:

$$\frac{4\pi}{\kappa}R_{\mu\nu}n^\mu n^\nu = 2\mathcal{H}_P B_\mu B^\mu - 2n_\mu n^\mu \mathcal{H}, \qquad (34)$$

where $B^\mu \equiv P^{\mu\nu}n_\nu$ is the gravito-magnetic field at the horizon. The last term above vanishes because the bounded behavior of the invariant $\mathcal{H}$. Since $\boldsymbol{B}$ is orthogonal to the null vector $\boldsymbol{n}$, it must be space-like or null ($B_\mu B^\mu \geq 0$), therefore each one of the remaining terms on the right-hand side of (34) must be bounded. The next invariant to be considered, which vanishes at the horizon by applying the Raychaudhuri equation to the null generator reads:

$$0 = \frac{4\pi}{\kappa}R_{\mu\nu}l^\mu l^\nu = 2\mathcal{H}_P D_\mu D^\mu - 2l_\mu l^\mu \mathcal{H}, \qquad (35)$$

where $D^\mu \equiv P^{\mu\nu}l_\nu$ is the gravito-electric field at the horizon. Once again the bounded behavior of the invariant $\mathcal{H}$ can be used to achieve the vanishing of the last term of (35). Since $\boldsymbol{D}$ is orthogonal to the null generator $\boldsymbol{l}$, it must be space-like or null ($D_\mu D^\mu \geq 0$), consequently each term on the right-hand side of (35) vanishes independently, which implies that $A_\mu l^\mu = 0$ and that $\boldsymbol{D}$ is proportional to the null generator $\boldsymbol{l}$ at the horizon, i.e., $\boldsymbol{D} = -(D_\alpha n^\alpha)\boldsymbol{l}$. The last studied invariant gives the following relation,

$$\frac{4\pi}{\kappa}R_{\mu\nu}l^\mu n^\nu - 2\mathcal{H} = 2\mathcal{H}_P(D_\mu n^\mu)^2, \qquad (36)$$

where it has been used that $\boldsymbol{D} = -(D_\alpha n^\alpha)\boldsymbol{l}$. Since $\mathcal{H}$ and $\mathcal{H}_P$ are bounded at the horizon, it follows that the right-hand side of (36) vanishes, thus $D_\mu n^\mu$ is bounded at the horizon as consequence of the bounded behavior of the left-hand side of (36).

Summarizing, the study of the quoted invariants at the horizon leads to the following conclusions: $D_\mu n^\mu$, $A_\mu n^\mu$, $A_\mu A^\mu$, and $B_\mu B^\mu$ are bounded at the horizon, $A_\mu l^\mu = 0$ and $\boldsymbol{D} = -(D_\alpha n^\alpha)\boldsymbol{l}$ in the same region.

Now, in order to show the fulfillment of the sufficient conditions for the vanishing of the integrand (30) over the horizon, i.e., $A_\alpha P^{\beta\alpha}l_\beta$ vanishes and $A_\alpha P^{\beta\alpha}n_\beta$ remains bounded at the horizon. Using the definition $D^\mu \equiv P^{\mu\nu}l_\nu$ and that $\boldsymbol{D} = -(D_\alpha n^\alpha)\boldsymbol{l}$, we obtain for the first quantity at the horizon

$$A_\alpha P^{\beta\alpha}l_\beta = (D_\mu n^\mu)(A_\nu l^\nu) = 0, \qquad (37)$$

where the vanishing follows from the fact that $D_\mu n^\mu$ is bounded and $A_\nu l^\nu$ vanishes at the horizon.

For the second quantity we note that $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal to the null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, respectively. Therefore, $\boldsymbol{A}$ must be space-like or proportional to $\boldsymbol{l}$, and $\boldsymbol{B}$ must be space-like or proportional to $\boldsymbol{n}$. Using a null tetrad basis, constructed with $\boldsymbol{l}$, $\boldsymbol{n}$, and a pair of linearly independent space-like vectors, these last ones being tangent to the space-like cross sections of the horizon, the $\boldsymbol{A}$ and $\boldsymbol{B}$ vectors can be written as

$$\boldsymbol{A} = -(A_\alpha n^\alpha)\boldsymbol{l} + \boldsymbol{A}^\perp, \qquad (38)$$

$$\boldsymbol{B} = -(B_\alpha l^\alpha)\boldsymbol{n} + \boldsymbol{B}^\perp, \qquad (39)$$

where $\boldsymbol{A}^\perp$ and $\boldsymbol{B}^\perp$ are the projections, orthogonal to $\boldsymbol{l}$ and $\boldsymbol{n}$, on the space-like cross sections of the horizon. Using (38) and (39) it is clear that $A_\mu A^\mu = A^\perp_\mu A^{\perp\mu}$, and $B_\mu B^\mu = B^\perp_\mu B^{\perp\mu}$, i.e., the contribution to these bounded magnitudes comes only from the space-like sector orthogonal to $\boldsymbol{l}$ and $\boldsymbol{n}$. With the help of (38) and (39) the other quantity appearing in the integrand (30) can be written as

$$A_\alpha P^{\beta\alpha}n_\beta = -A_\alpha B^\alpha = -(A_\alpha n^\alpha)(D_\beta n^\beta) - A^\perp_\alpha B^{\perp\alpha}, \qquad (40)$$

where the identity $B_\alpha l^\alpha = -D_\alpha n^\alpha$ has been used. The first term in (40) is bounded because $A_\alpha n^\alpha$ and $D_\beta n^\beta$ are bounded. To the second term the Schwarz inequality applies, since $\boldsymbol{A}^\perp$ and $\boldsymbol{B}^\perp$ belong to a space-like subspace. Thus, $(A^\perp_\alpha B^{\perp\alpha})^2 \leq (A^\perp_\mu A^{\perp\mu})(B^\perp_\nu B^{\perp\nu}) = (A_\mu A^\mu)(B_\nu B^\nu)$ and since $A_\mu A^\mu$ and $B_\nu B^\nu$ are bounded at the horizon, the second term of (40) is also bounded.

Finally, the vanishing of (37) and the bounded behavior of (40), and the null character of $\boldsymbol{l}$ at the horizon lead to the vanishing of the integrand (30) over the event horizon.

With no contribution from boundary integrals in (29) we shall write the volume integral, using the coordinates from (28), for each one of the different cases discussed at the beginning of this section.

For the purely gravito-electric case we have

$$\int_{\mathcal{V}} -V\left(\frac{1}{2}g_{ij}P^{ti}P^{tj}\right)d\tau = 0. \qquad (41)$$

The non-positiveness of the above integrand, which is the sum of squared terms, implies that the integral is vanishing only if P^{ti} and A^t vanish everywhere in $\mathcal{V}$, and hence in all $\langle\!\langle \mathcal{J} \rangle\!\rangle$.

For the purely gravito-magnetic case the volume integral reads:

$$\int_{\mathcal{V}}\left(\frac{1}{2}g_{ik}g_{jl}P^{kl}P^{ij}\right)d\tau = 0, \qquad (42)$$

in this case the non-negativeness of the above integrand is responsible for the vanishing of P^{ij} and A^i in all $\langle\!\langle \mathcal{J} \rangle\!\rangle$.

VII. CONCLUSIONS

We enhanced the standard no-hair conjecture statement to include regular and singular black holes arising from the Einstein–Plebański nonlinear electrodynamic systems. According to the statement of the *enhanced no-hair conjecture*, regular or singular black holes are therefore extraordinarily simple objects, whose geometry, exterior to their event horizon, is a member of the Kerr-Newman family and is completely determined by just three quantities: mass, angular momentum and electric/magnetic charge. Furthermore, these quantities are global conserved Noether charges which can, at least in principle, be measured at spatial infinity, far away from the black-hole event horizon.

We also formulated the *long-hair conjecture* to include the mentioned Einstein-Plebański black holes, it states that only an observer situated at a distance $r > r_0 > r_H$ can measure the global charges characterizing a static black hole coupled to non-Abelian matter, i.e., the hair is always longer than certain radius $r_0 = \frac{3}{2} r_H$.

These two statements can be gathered in the so-called generalized version of the *enhanced no-hair conjecture* [46], which states that *All spherically symmetric, regular or singular, black-hole solutions to the Einstein field equations coupled to any type of self-gravitating matter field are completely and uniquely determined by their mass, electric/magnetic charge, angular momentum, (as it is the case for the coupling with Abelian matter, satisfying the* enhanced no-hair conjecture), *and a set of global charges, (as it is the case for the coupling with non-Abelian matter, satisfying the* long-hair conjecture).

Moreover, since time-reversal invariance implies the existence of two different non-overlapping cases: a purely gravito-electric case, and a purely gravito-magnetic one. We proved the generalized enhanced *no-hair conjecture* statement for both cases.

Additionally, since the natural variables for the exact analytic gravito-electric regular black-hole solutions are the field excitations **D** and **H**, while the standard ones in terms of the fields **E** and **B** are the natural variables for the gravito-magnetic exact analytic solutions, they represent two separate non-overlapping cases. Both sets of variables are related by a Legendre transformation, i.e., $\mathcal{L} = P_{\mu\nu} F^{\mu\nu} - \mathcal{H}$.

Consequently, the possibility of having exact dyonic solutions remains as an open problem. What we have until now are solutions with a gravito-electric exact piece and a magnetic piece expressed in quadratures and vice versa, depending which variables one considers [47].

Advances in astrometry [48] and future gravitational wave detectors [49] may even be able to probe the validity of the *no-hair conjecture* for astrophysical black holes by verifying that the mass, angular momentum and quadrupole moment Q^2 of the black hole satisfy the relation $Q^2 = J^2/M$ which holds for Kerr black holes.

There are many results which rule out scalar field hair in quite general models, particularly in asymptotically flat space-times [50–52]. In recent years many other examples of black-hole solutions with nontrivial scalar field hair have been found. Nevertheless, all the black-hole solutions involving scalar hair are of numerical nature and all of them are unstable against linear perturbations [53,54].

A detailed analysis and a classification of all possible trajectories in the space-time of the Ayón-García regular black hole is presently in progress and will be reported elsewhere [55–58].

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APPENDIX: COUPLING OF REGULAR BLACK HOLES WITH NON-ABELIAN MATTER

The strategy for showing the validity of the *long-hair conjecture* is the following. From the matter field equations it is possible to obtain a generic function [51] of the form $\Xi \propto e^{-\delta}(K - U)$ which enters the conservation Eq. (6). If we demand the existence of a black-hole solution in each case, we will show that Ξ must be negative on the horizon and positive semidefinite at infinity. Therefore, Ξ must be positive in some region (between the horizon and infinity) which is determined by the condition $3f > 1$. Finally, we show that this region corresponds to values of the radial coordinate r outside the hairosphere. This proves the validity of the conjecture.

We now investigate all the particular cases in which black-hole hair has been found. For the sake of simplicity, in each case we will quote for α, β, K and U in $\Xi \propto e^{-\delta}(K - U)$ only the contribution corresponding to the additional matter field, dropping the term coming from the effective regular black hole which does not allow the presence of hair. Because of the additivity of the stress-energy tensor discussed above, this simplification does not affect the behavior of the generic function Ξ.

(1) *The SU(2) Yang-Mills field* [59] for which the matter Lagrangian has the form

$$\mathcal{L}_{\rm YM} = -\frac{\sqrt{-g}}{16\pi f^2} F_{\mu\nu}{}^a F^{\mu\nu}{}_a, \tag{A1}$$

where $F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + \epsilon^a{}_{bc} A_\mu{}^b A_\nu{}^c$ is the field strength for the gauge field $A_\mu{}^a$, and f represents the gauge coupling constant. We use the static spherically symmetric ansatz for the potential

$$\begin{aligned} A &= \sigma_a A_\mu{}^a dx^\mu \\ &= \sigma_1 w d\theta + (\sigma_3 \cot\theta + \sigma_2 w) \sin\theta d\varphi, \end{aligned} \tag{A2}$$

where σ_i $(i = 1, 2, 3)$ are the Pauli matrices and w is a function of r only. The field equations for this case may be written as in Eqs. (4) and (5), i.e.,

$$f' = 8\pi r T^t{}_t + (1 - f)/r, \tag{A3}$$

$$\delta' = (4\pi r/f)[T^t{}_t - T^r{}_r]. \tag{A4}$$

where $T^r{}_r \propto K - U$ with $K = f w'^2$, $U = (1 - w^2)^2/(2r^2)$. From the matter field equations we obtain

$$\Xi' \equiv [r^2 e^{-\delta}(K - U)]' = r e^{-\delta}(3f - 1) w'^2. \tag{A5}$$

From the expressions for K and U we see that Ξ is negative at the horizon because $K(r_H) = 0$, (since $f(r_H) = 0$), and $U(r_H) > 0$. On the other hand, the asymptotic flatness condition implies that $\Xi \to 0$ as $r \to \infty$. Accordingly, Ξ must be an increasing function of r in some intermediate region. It follows then that the right-hand side of Eq. (A5) must become positive at some point, i.e. we must have $3f > 1$.

(2) *The Skyrme field* with the matter Lagrangian [39,60–62]

$$\mathcal{L}_{\rm SK} = \sqrt{-g}\frac{h^2}{4}\,\mathrm{Tr}(\nabla_\mu W \nabla^\mu W^{-1}) + \frac{\sqrt{-g}}{32e^2}\,\mathrm{Tr}[(\nabla_\mu W)W^{-1}, (\nabla_\nu)W^{-1}]^2, \quad \text{(A6)}$$

where ∇_μ is the covariant derivative, W is the $SU(2)$ chiral field, and h^2 and e^2 are the coupling constants. For the $SU(2)$ chiral field we use the *hedgehog* ansatz $W(r) = \exp(\sigma \cdot \mathbf{r} F(r))$ where σ are the Pauli matrices and $\mathbf{r}$ is a unit radial vector.

To write down the field equations we follow [39,60,61] and use the variables $\tilde{r} = ehr$, and $\tilde{m}(\tilde{r}) = ehm(r)$ so that the function f defined above [in Eq. (4)] remains invariant. Dropping the tilde, the resulting equations are equivalent to Eqs. (A3) and (A4) with $K = f[r^2/2 + \sin^2 F(r)]F'^2$, $U = \sin^2 F[1 + \sin^2 F/(2r^2)]$, $\alpha = -8\pi h^2$, and $\beta = -8\pi h^2/(fr)$. From the matter field equations we find

$$\Xi' = -e^{-\delta}\left[rfF'^2 + \frac{1-f}{rf}K - 2r\left(1 + \frac{U}{r^2} - \sqrt{1 + 2\frac{U}{r^2}}\right)\right]. \quad \text{(A7)}$$

From the explicit expressions for K and U it follows that $\Xi(r_H) < 0$, and since the asymptotic behavior of the field equations implies $F(r) \approx 1/r^2$ at infinity, we have that $\Xi \to +0$ at infinity. Therefore, the right-hand side of Eq. (A7) must be positive in some region. Moreover, there must be a point where $\Xi = 0$, i.e. $K = U$, and $\Xi' > 0$. At this point, the right-hand side of Eq. (A7) becomes

$$-r\mu F'^2 - 2r\left(\sqrt{1 + 2\frac{K}{r^2}} - 1\right) + \frac{K}{rf}(3f - 1) > 0. \quad \text{(A8)}$$

Since the first and second terms of the last equation are negative, we conclude that $3f > 1$ at this point. This is the same condition as in case 1.

(3) In the case of $SU(2)$ *Yang-Mills-dilaton* field with an arbitrary (positive semidefinite) potential term $V(\phi)$ (which is expected to arise in superstring-inspired models [63]), the corresponding matter Lagrangian is given by [64]

$$\mathcal{L}_{\rm YMD} = \frac{\sqrt{-g}}{4\pi}\left(\frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{4h^2}e^{2\gamma\phi}F_{\mu\nu}{}^a F^{\mu\nu}{}_a - V(\phi)\right), \quad \text{(A9)}$$

where h is the gauge coupling constant, γ is the dimensionless dilatonic coupling constant, and $F_{\mu\nu}{}^a$ is the SU(2) Yang-Mills field strength. The ansatz for the gauge field configuration is the same as that given in case 1, and $\phi = \phi(r)$.

The corresponding field equations can be written in the generic form Eqs. (A3) and (A4) with $K = K_1 + K_2$, where $K_1 = f\exp(2\gamma\phi)w'^2/h^2$, $K_2 = fr^2\phi'^2/2$ and $U = r^2 V(\phi) + \exp(2\gamma\phi)\times (1 - w^2)^2/(2h^2 r^2)$. Following the same procedure, from the matter field equations we find

$$\Xi' \equiv [r^2 e^{-\delta}(K - U)]' = re^{-\delta}\left[-2K_2 - 4r^2 V(\phi) + (3f - 1)\frac{K}{f}\right]. \quad \text{(A10)}$$

The behavior of the generic function Ξ is as in the previous cases, and since the first and second terms of the right-hand side of Eq. (A10) are negative, we again find the condition $3f > 1$ in order to obtain asymptotically flat solutions.

(4) For a $SU(2)$ *Yang-Mills-Higgs* field the matter Lagrangian is given by [41]

$$\mathcal{L}_{\rm YMH} = -\frac{\sqrt{-g}}{4\pi}\left[\frac{1}{4h^2}F_{\mu\nu}{}^a F^{\mu\nu}{}_a + (D_\mu \Phi)^\dagger (D^\mu \Phi) + V(\Phi)\right], \quad \text{(A11)}$$

where D_μ is the usual gauge-covariant derivative, Φ is a complex doublet Higgs field, and $F^{\mu\nu}{}_a$ is the $SU(2)$ Yang-Mills field given above. The arbitrary potential $V(\Phi)$ must be positive semidefinite. In this case, the ansatz for the Yang-Mills field is the same as before, and for the Higgs field we have

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \varphi(r) \end{pmatrix} \quad \text{(A12)}$$

The field equations are equivalent to Eqs. (A3) and (A4) with $K = K_1 + K_2$, where $K_1 = fr^2\varphi'^2/2$, and $K_2 = fw'^2/h^2$, $U = r^2 V(\varphi) + (1 - w^2)^2/(2^h 2r^2) + (1 + w)^2\varphi^2/4$. Finally, the matter field equations lead to

$$\Xi' \equiv [r^2 e^{-\delta}(K-U)]' = re^{-\delta}\left[-2K_2 - 4r^2 V(\varphi) - \frac{1}{2}(1+w)^2\varphi^2 + (3f-1)\frac{K}{f}\right]. \quad \text{(A13)}$$

As in the previous cases, the required behavior of the function Ξ and the fact that the first three terms of the right-hand side of Eq. (A11) are negative lead to the condition $3f > 1$ for the region of interest.

(5) In the case of a *non-Abelian Proca* field the matter Lagrangian is [41]

$$\mathcal{L}_{\rm NAP} = -\frac{\sqrt{-g}}{16\pi h^2} F_{\mu\nu}{}^{a} F^{\mu\nu}{}_{a} - \frac{\sqrt{-g}m^2}{32\pi} A^a_\mu A^\mu_a, \quad \text{(A14)}$$

where m is the mass parameter and the ansatz for the potential is as in the Yang-Mills case (A2). Again, the field equations are given by Eqs. (A3) and (A4) with $K = fw'^2$, $U = (1-w^2)^2/(2r^2) + m^2(1+w)^2$. On the other hand, from the matter field equations we obtain

$$\Xi' \equiv [r^2 e^{-\delta}(K-U)]' = re^{-\delta}\left[(3f-1)w'^2 - \frac{f^2 m^2}{2}(1+w)^2\right]. \quad \text{(A15)}$$

As in the previous cases, the required behavior of the generic function Ξ leads to the condition $3f > 1$.

In all cases presented here, there is a change in the behavior of the generic function Ξ: It always starts at the horizon as a negative and decreasing function and needs to increase towards its asymptotic value. We have shown that this change always occurs beyond the point characterized by $3f > 1$. Since $f = 1–2M(r)/r$, the change occurs at the point $r_0 > 3M(r_0)$. On the other hand, $M(r)$ is an increasing function because from the general field equation (4) we have that $M' = -(\tilde{\kappa}/2)r^2 T^t{}_t = (\tilde{\kappa}/2)r^2\rho_\Xi > 0$, where ρ_Ξ is the energy density of the matter field which we suppose to be positive semidefinite in accordance with the weak energy condition. Being an increasing function, $M(r)$ reaches its minimum value on the horizon, where $f(r_{\rm H}) = 0$ and $M(r_{\rm H}) = r_{\rm H}/2$. Consequently, the turning point r_0 satisfies the inequality $r_0 > 3M(r_0) \geq 3M(r_{\rm H}) = 3r_{\rm H}/2$.

This result shows that the asymptotic behavior of the matter fields present in the gravitational field of a static spherically symmetric black hole, regular or singular, can start only after the value of the radial coordinate r is sufficiently large, and the lowest value determines the radius $r_{\rm hair}$ of the so-called hairosphere.

This proves the validity of the *long-hair conjecture* for regular black-hole solutions in the presence of non-Abelian matter fields, in which black-hole hair has been found.

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