

# Regular black holes and energy conditions

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We establish the relationship between the space-time structure of regular spherically-symmetrical black holes and the character of violation of the strong energy condition (SEC). It is shown that it is violated in any static region under the event horizon in such a way that the Tolman mass is negative there. In non-static regions there is constraint of another kind which, for a perfect fluid, entails violation of the dominant energy condition.

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The famous theorem [1] establishes the connection between the validity of strong energy condition (SEC) and the appearance of singularities inside black holes. Correspondingly, if a black hole is regular, SEC is necessarily violated somewhere inside the horizon. Examples of such black holes were constructed [2], [3]. Researches on regular black holes are continuing to develop (see works [4], [5], review [6] and references therein) that makes it important to formulate more precisely criteria, how to evaluate the degree of such violation. In the present work we suggest such a very simple and clear criterion. It is formulated in terms of the well-known quantity - Tolman mass [7] and, thus, has an integral character. Additionally, our derivation can be considered as a simplified version of the proof of the theorem [1] for the particular situation of the spherically-symmetrical black holes.

We restrict ourselves by spherically-symmetrical black hole metrics (though generalization to the distorted case is straightforward). For definiteness, we choose the gauge in which the metric can be written as

$$ds^2 = -dt^2 f + \frac{du^2}{f} + r^2(u) d\omega^2. \quad (1)$$

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We assume that, in general, there are zeros of the function  $f$  at  $u = u_1, u = u_2, \dots, u = u_N \equiv u_h$  where roots are enumerated from the left to the right. The outmost root viewed from the outside corresponds to the black hole event horizon. We also assume that there is a regular center in the system. This means that (i) there exist the point  $r = 0$  in the system such that (ii) all curvature invariants are finite there. To achieve this goal, the metric functions  $f$  and  $r$  should obey the conditions which in the coordinates (1) read  $f \left( \frac{dr}{du} \right)^2 = 1 + O(r^2)$ ,  $f = f(0) + O(r^2)$  as  $r \rightarrow 0$  (see e.g. [5], [8] or references therein). The typical example is the (anti)de Sitter metric for which  $r = u$  and  $f = 1 + const * r^2$ .

We always can achieve  $u = 0$  by shifting to the appropriate constant, so that the region  $0 \leq u < u_1$  is static,  $f > 0$  there. As the outer region, by the definition, should be also static, the number  $N$  of zeros (horizon) is even. Then, it is clear that for non-degenerate roots

$$f'(u_{2k-1}) < 0, f'(u_{2k}) > 0. \quad (2)$$

The adjacent roots can merge giving there  $f'(u_{2k-1}) = f'(u_{2k}) = 0$ . However, as this does not affect our consideration, for simplicity of presentation we assume that roots are nondegenerate (unless otherwise specified).

Now let us write down the expression for the  $tt$  component of the Ricci tensor for the metric which does not depend on  $t$ :

$$R_t^t = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{-g} g^{tt} \Gamma_{tt}^i) \quad (3)$$

where we took into account from the very beginning that components  $g_{ti} = 0$  ( $i = r, \theta, \phi$ ). For the metric (1) it gives us

$$-R_t^t r^2 = \frac{1}{2} \frac{d}{du} (r^2 f') \quad (4)$$

where  $' \equiv \frac{d}{du}$ . Now we integrate this equality and exploit the  $tt$  components of the Einstein equations written in the form  $R_\mu^\nu = 8\pi(T_\mu^\nu - \frac{1}{2}T_\alpha^\alpha \delta_\mu^\nu)$ . Here, we use the system of units with  $G = c = 1$ . The stress-energy tensor  $T_\mu^\nu$  supporting the metric (1) has the form

$$T_\mu^\nu = \text{diag}(-\rho, p_r, p_\perp, p_\perp). \quad (5)$$

Upon integration, we obtain immediately that

$$4m_T(a, b) = (r^2 f') \big|_a^b \quad (6)$$

where, by definition,

$$m_T(a, b) = 4\pi \int_a^b dr r^2 (T_k^k - T_0^0), \quad (7)$$

is the Tolman mass calculated within the corresponding interval.

Let us consider the equality (6) in different intervals.

1)  $a = 0, b = u_1$ . Thus, we take the interval between the center and the first inner horizon. As  $u = 0$  corresponds to the center,  $r(0) = 0$ . Further, using eq. (2) we immediately obtain that

$$m_T(0, u_1) < 0. \quad (8)$$

2)  $a = u_{2k}, b = u_{2k+1}$ . This interval corresponds just to the static region between two successive horizons. Then,  $f'(u_{2k}) > 0, f'(u_{2k+1}) < 0$ . Again, we obtain

$$m_T(u_{2k}, u_{2k+1}) < 0. \quad (9)$$

3)  $a = u_{2k-1}, b = u_{2k}$ . This interval corresponds to the non-static region. In such a case  $m_T$  does not have a direct physical meaning of the mass but, for simplicity, we will still call it Tolman mass. Now,

$$m_T(u_{2k-1}, u_{2k}) > 0. \quad (10)$$

As a result of properties (1) - (3), in the interval  $u_k < u < u_{k+n}$  between corresponding horizons, the Tolman mass as the function of  $u$  has  $n - 1$  zeros.

4)  $a = r_N, b = \infty$ . Assuming also that at infinity the metric is asymptotically flat, we can write

$$f = 1 - \frac{2m}{r} + o\left(\frac{1}{r}\right), \quad r \approx u \quad (11)$$

where  $m$  is the Schwarzschild mass.

Then, we obtain the standard mass formula [9]

$$m = m_T + \frac{\kappa A}{4\pi} \quad (12)$$

where  $m_T$  is the total Tolman mass in the outer region,  $A = 4\pi r_h^2$  is the area of the event horizon,  $\kappa = \frac{f'(u_h)}{2}$  is the surface gravity.

The strong energy condition (SEC) requires that  $Y \equiv \rho + p_r + 2p_\perp \geq 0$ . Meanwhile, in the static region  $T_k^k - T_t^t = Y$  and we see from (8), (9) that in each of static regions in our system  $\int dr^2 Y < 0$ , except from the outer one. Thus, SEC is necessarily violated somewhere in each such a region and, moreover, this violation is so strong that it amounts to the negativity of the corresponding Tolman mass. If, by contrary,  $Y > 0$  everywhere in the static region, the solution with a regular centre and a horizon is impossible in accordance with the general theorem [1] and its manifestation in the context of spherically-symmetrical space-times [8]. Actually, for such space-times, we generalized the aforementioned observation of [8] and related the violation of SEC and its degree to the space-time structure having, in general, an arbitrary number of horizons.

We also formulated the energy condition to be satisfied in the non-static region that does not have meaning of SEC. Indeed, in the non-static regions the above formulas written in terms of  $T_\mu^\nu$  are still valid but the meaning of separate components of  $T_\mu^\nu$  changes. Now, in the region where  $f < 0$  the coordinate  $u$  is time-like,  $u \equiv T$ . In a similar way, the coordinate  $t$  is space-like,  $t \equiv x$ . Correspondingly,  $T_u^u = T_T^T = -\rho$ ,  $T_t^t = T_X^X = p_x$ . Then, the condition (10) takes the form

$$m_T(u_{2k-1}, u_{2k}) = 4\pi \int_{u_{2k-1}}^{u_{2k}} dr^2 (2p_\perp - p_x - \rho) > 0. \quad (13)$$

It cannot be in general related to the standard energy conditions directly. However, in the isotropic case  $p_x = p_\perp \equiv p$  that corresponds to the perfect fluid we have

$$m_T(u_{2k-1}, u_{2k}) = 4\pi \int_{u_{2k-1}}^{u_{2k}} dr^2 (p - \rho) > 0 \quad (14)$$

and we obtain the violation of the dominant energy condition (DEC)  $p \leq \rho$ . It follows that the perfect fluid obeying DEC everywhere cannot be a source for black holes with a regular centre.

In the highly anisotropic case, when  $|p_\perp| \ll |p_x|$ , it is seen from (13) that the condition  $p_x + \rho \geq 0$  cannot be satisfied everywhere in a given non-static region, so WEC (weak energy condition), DEC and SEC are violated. In the opposite case,  $|p_x| \ll |p_\perp|$ , there is no definite constraint of this kind. For example, if  $\frac{\rho}{2} < p_\perp < \rho$ , DEC is satisfied.

As is mentioned in introductory paragraphs, if we want to have a regular black hole, SEC should be violated somewhere. More precisely, in our context only the sign of  $Y$  is important whereas two other manifestations of SEC  $p_r + \rho \geq 0$ ,  $p_\perp + \rho \geq 0$  are irrelevant in accordance with the observation made in [8]. One may ask a question - is it possible to have SEC violated everywhere due to  $Y < 0$ ? In the intermediate non-static region the condition (13) should be satisfied. In combination with  $Y < 0$ , this would give us in some subregion of such a region the conditions  $p_x + \rho < 2p_\perp < -p_x - \rho$ , so that WEC is to be also violated,  $p_x + \rho < 0$  and the presence of the region with phantom matter is unavoidable. The sign of  $p_\perp + \rho$  is not now fixed in the situation under discussion.

The fact that the explicit form of the energy condition in non-static regions changed as compared to the static ones is quite natural. The SEC is obtained from the requirement  $R_{\mu\nu}u^\mu u^\nu \geq 0$  where  $u^\mu$  is a time-like vector. In the static region it is convenient to take  $u^\mu$  to be the four-velocity of an observer at rest, whence the explicit expression for  $Y$  is obtained. But in the non-static region, we, as a matter of fact, integrated the same inequality but with a space-like four-vector  $u^\mu$ .

It is instructive to compare our approach with that in the recent paper [10] where the idea of replacing the central singularity by the de Sitter core [3] was again discussed. The model comprising the vacuum Schwarzschild and de Sitter region separated by smooth distribution of matter was considered there. It was noticed in [10] that on the border between vacuum and matter under the Schwarzschild horizon DEC is violated. Actually, this observation can be understood as follows. If some configuration is smooth (no shells are present), in the static region this entails, as is known, the continuity of the radial pressure. In the non-static region for the metric of the type (1) the radial pressure and energy density mutually exchange their roles as is explained above. Correspondingly, this requires the continuity of  $\rho$ . As in the vacuum region  $\rho = 0$ , the same holds on the boundary from the inner side and, as result, DEC is necessarily violated for any  $p_x \neq 0$ . For the metric of the type (1) with  $f < 0$ , this circumstance is independent of whether or not the horizon is present. Meanwhile, the condition (13) relies on the presence of the horizon. In doing so, even if DEC is violated, WEC and SEC can be satisfied.

The special case arises if two successive roots are degenerate. Then, both in the static

or non-static situation the corresponding quantity  $m_T = 0$ . If, in addition, the vacuum-like equation of state  $p_x + \rho = 0$  holds, DEC can be satisfied on the border but SEC is violated somewhere since  $p_\perp$  should alter its sign to ensure the zero Tolman mass, so  $Y$  should change sign also.

We see that in regular black holes the violation of the strong energy condition (unavoidable due to the singularity theorem) received simple formulation in terms of the Tolman mass. This violation occurs just in static regions and the natural measure of its degree is the Tolman mass calculated between two adjacent horizons or between the centre and the first horizon. In the non-static regions, the standard energy conditions (not only SEC) can be either violated or satisfied. For a perfect fluid DEC is violated.

It is worth noting that, instead of integrating eq. (4) between horizons, we could do it in the intervals between successive minima and maxima of  $f(u)$ . Then, as  $f' = 0$  at limits of the integral, each of such integrals is equal to zero, so contributions of regions in which SEC is violated and those where (13) is satisfied, exactly compensate each other, thus giving another measure of violation of SEC.

Up to now, we discussed the regular black hole having a centre. Meanwhile, it turns out that the class of such space-time is more diverse and can include, in particular, models with  $r \rightarrow r_0$  or  $r \rightarrow \infty$  inside the horizon [4], [5]. In such a case the condition (8) loses its meaning but other conditions retain their validity.

To summarize, we found the direct and simple relations between the space-time structure of regular black holes and energy conditions which must be violated or satisfied and having different meaning in static and non-static regions. In particular, the integral measure of violation of SEC is established in terms of the Tolman mass. We would like to stress that it is this kind of mass which turned out to be relevant for this purpose, whereas, say, the Hernandez-Misner-Sharp mass [11] defined according to  $\frac{2m}{r} = 1 - (\nabla r)^2 = 1 - f r'^2$  would not give such information reducing simply to  $m = \frac{r}{2}$  at each root of  $f$ .

It is of interest to extend the results of the present paper to the case of rotating regular black holes. The separate issue is application of the approach developed in the present paper to wormhole physics where alternation of regions where the standard energy conditions are satisfied or violated can also occur [12].

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