New Regular Black Hole Solution from Nonlinear Electrodynamics

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Using a nonlinear electrodynamics coupled to General Relativity a new regular *exact* black hole solution is found. The nonlinear theory reduces to the Maxwell one in the weak limit, and the solution corresponds to a charged black hole for $|q| \leq 2s_{\rm c}m \approx 1.05\,m$, with metric, curvature invariants, and electric field regular everywhere.

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In the last years an increasing revival of nonlinear electrodynamic theories is observed [1]. A nonlinear electrodynamics was first proposed by Born and Infeld [2] at the 30's in order to obtain a finite–energy electron model; they succeeded in determining an electron of finite radius. After these first achievements very few investigations were carried out, as Plebański mentioned in 1970 at the introduction of his monograph [3]: *If in recent times the interest in NLE cannot be said to be very popular, it is not due to the fact that one could rise some serious objections against this theory. It is simply rather difficult in its mathematical formulation, what causes that it is very unlikely to derive some concrete results in closed form.* The mayor responsible of the actual revival, more than 25 years later of the moment of formulation of quoted statement, is the fact that these nonlinear theories appear as effective theories at different levels of string/M–theory. For instance, the Born–Infeld action arises as part of the low energy effective action of the open superstring theory [4,5]. Moreover, generalizations of the quoted finite–energy solutions (BIons) are very important in the description of Dp–branes —solitons of string theory described by Dirac–Born–Infeld like actions [6–8]. This kind of nonlinear theories has also been used as inspiration to build up supersymmetric extensions [9–11], and non–Abelian generalizations [12,13] (these last ones are used to generalize the colored solutions of Einstein–Yang–Mills theory [14]). More recently, in the light of the AdS/CFT correspondence (cf. [15] for a review), the nonlinear electrodynamics string approach has been used to obtain solutions describing baryon configurations which are consistent with confinement [16].

In other frameworks, various important results have been obtained. For instance, nonlinear electrodynamics ideas are source of inspiration to formulate other models, as it is the case of Born–Infeld Skyrmions, where nonlinear terms are essential in order to obtain stationary solutions [17,18]. It has been established that Maxwell and Born–Infeld theories are singled out among all electromagnetic theories since they bear both dual invariance [19,20] and "good propagations" (in the sense that excitations propagate without shocks) [3,21]. It is also remarkable that all the nonlinear electrodynamics satisfy the zeroth and first laws of black hole mechanics [22].

In this paper we are interested in another less explored application of nonlinear electrodynamics: namely its coupling to Einstein theory to obtain regular black hole solutions. Previous attempts on this direction with nonlinear electrodynamics either have been totally unsuccessful or only partially solve the singularity problem; in certain approaches the singular region is artificially cut off from the whole space–time, and in others cases the singularity is only weakened [23–25]. We would like to point out that the well–known Born–Infeld Lagrangian is useless in this topic, since it gives rise to a singular black hole solution, at least in the spherically symmetric case [26].

The study on global regularity of black hole solutions is quite important in order to understand the final state of gravitational collapse of initially regular configurations, since the Penrose cosmic censorship conjecture claims that if singularities predicted by General Relativity [27,28] occur, they must be dressed by event horizons (cf. [29,30] for recent reviews). Previous regular black hole models has been proposed [31–36]; all of them have been referred to as "Bardeen black holes" [37], since Bardeen was the first author producing a surprising regular black hole model [31]. None of these models is an exact solution to Einstein equations; there is no known physical sources associated with any of them. Regular black hole solutions to Einstein equations with physically reasonable sources were reported in [38,39]. Other commonly used approaches to avoid the existence of singularities are based on the search of more general gravity theories. The best candidate today to produce singularity–free models, even at the classical level, is string/M–theory, due to its intrinsic non–locality [40].

We show in this contribution that it is possible to find regular solutions within General Relativity simply by coupling to the Einstein equations an appropriate nonlinear electrodynamics, which in the weak field approximation becomes the usual Maxwell theory. In what follows we shall exhibit a singularity–free black hole solution to the quoted about system, which behaves asymptotically as the Reissner–Nordström solution.

The dynamics of the theory we are using is governed by the action

$$\mathcal{S}=\int dv\left(\frac{1}{16\pi}R-\frac{1}{4\pi}\mathcal{L}(F)\right), \tag{1}$$

where R is scalar curvature, and $\mathcal{L}$ is a function of $F\equiv\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ is the electromagnetic strength. We would like to recall that there are more general Lagrangians depending also on the second invariant, $F^{*}_{\mu\nu}F^{\mu\nu}$, but for the objectives of this work it is enough to consider only an action as the given in (1). Alternatively, one can describe the considered system using another function, obtained by means of a Legendre transformation [41]

$$\mathcal{H}\equiv 2F\mathcal{L}_F-\mathcal{L}, \tag{2}$$

where, $\mathcal{L}_F\equiv\partial\mathcal{L}/\partial F$. Defining $P_{\mu\nu}\equiv\mathcal{L}_F F_{\mu\nu}$, it can be shown that $\mathcal{H}$ is a function of $P\equiv\frac{1}{4}P_{\mu\nu}P^{\mu\nu}=(\mathcal{L}_F)^2F$, i.e., $d\mathcal{H}=(\mathcal{L}_F)^{-1}d((\mathcal{L}_F)^2F)=\mathcal{H}_P dP$, where $\mathcal{H}_P\equiv\partial\mathcal{H}/\partial P$. Using $\mathcal{H}$, the nonlinear electromagnetic Lagrangian is expressed as $\mathcal{L}=2P\mathcal{H}_P-\mathcal{H}$, depending now on the anti–symmetric tensor $P_{\mu\nu}$. The Einstein–nonlinear–electrodynamics field equations resulting from action (1) are

$$G_{\mu}{}^{\nu}=2(\mathcal{H}_P P_{\mu\lambda}P^{\nu\lambda}-\delta_{\mu}{}^{\nu}(2P\mathcal{H}_P-\mathcal{H})), \tag{3}$$

$$\nabla_{\mu}P^{\alpha\mu}=0. \tag{4}$$

The $\mathcal{H}$ function has to satisfy the correspondence to Maxwell theory, i.e., $\mathcal{H}\approx P$ for weak fields ($P\ll 1$). We would like to point out that in this description the usual electromagnetic strength tensor is given by

$$F_{\mu\nu}\equiv\mathcal{H}_P P_{\mu\nu}, \tag{5}$$

this tensor is the physically relevant quantity, although by using the auxiliary tensor $P_{\mu\nu}$ many results can be achieved more easily.

The particular non–linear electrodynamics source used to derive our regular black hole solution is determined by the following function $\mathcal{H}$:

$$\mathcal{H}(P)=P\left(1-\tanh^2\left(s\sqrt[4]{-2q^2P}\right)\right), \tag{6}$$

where s stands for $s\equiv|q|/2m$; q and m are free parameters which we anticipate to be associated with charge and mass respectively. Notice that this last function fulfills the correspondence condition to Maxwell theory. In order to obtain the desired solution, we consider a static and spherically symmetric configuration

$$\boldsymbol{g}=-\left(1-\frac{2M(r)}{r}\right)\boldsymbol{dt}^2+\left(1-\frac{2M(r)}{r}\right)^{-1}\boldsymbol{dr}^2+r^2\boldsymbol{d\Omega}^2, \tag{7}$$

and assume the following ansatz for the anti–symmetric field

$$P_{\mu\nu}=2\delta^{t}_{[\mu}\delta^{r}_{\nu]}D(r). \tag{8}$$

With these choices, equations (4) are easily integrated,

$$P_{\mu\nu}=2\delta^{t}_{[\mu}\delta^{r}_{\nu]}\frac{q}{r^2}\quad\longrightarrow\quad P=-\frac{D^2}{2}=-\frac{q^2}{2r^4}, \tag{9}$$

where the integration constant was chosen as q, since it plays the role of the electric charge, which becomes apparent from the evaluation of the electric field $E=F_{tr}=\mathcal{H}_P D$. Using expression (6) for $\mathcal{H}$, and (9), the electromagnetic field strength results in

$$F_{\mu\nu}=2\delta^{t}_{[\mu}\delta^{r}_{\nu]}E(r),\qquad E(r)=\frac{q}{4mr^3}\left(1-\tanh^2(q^2/2mr)\right)\left(4mr-q^2\tanh(q^2/2mr)\right). \tag{10}$$

From these expressions two conclusions follow: the electric field is regular everywhere, and asymptotically behaves as $E=q/r^2+O(1/r^4)$, i.e., a Coulomb field with electric charge q.

The ${}_{t}{}^{t}$ component of Einstein equations (3) yields

$$M'(r)=-r^2\mathcal{H}(P). \tag{11}$$

Substituting $\mathcal{H}$ from (6), with $P = -q^2/2r^4$, the first integral of (11) is

$$M(r) = -m \tanh(q^2/2mr) + K, \tag{12}$$

where K is an integration constant which can be evaluated using the mass definition, $m \equiv \lim_{r\to\infty} M(r)$, thus $K = m$ and consequently $M(r) = m\left(1 - \tanh(q^2/2mr)\right)$. Entering (12) into (7), one finally obtains the following metric

$$\boldsymbol{g} = -\left(1 - \frac{2m\left(1-\tanh(q^2/2mr)\right)}{r}\right)\boldsymbol{dt}^2 + \left(1 - \frac{2m\left(1-\tanh(q^2/2mr)\right)}{r}\right)^{-1}\boldsymbol{dr}^2 + r^2\boldsymbol{d\Omega}^2. \tag{13}$$

It can be noted that the metric asymptotically behaves as the Reissner–Nordström solution, i.e., $g_{tt} = 1 - 2m/r + q^2/r^2 + O(1/r^4)$, thus the parameters m and q can be correctly associated with mass and charge respectively.

We shall show that for a certain range of the charge parameter our metric solution (13) describes a black hole, which in addition is regular everywhere. Making the substitutions $y = 2mr/q^2$, $s = |q|/2m$ we rewrite g_{tt} as

$$-g_{tt} = A(y,s) \equiv 1 - \frac{1-\tanh(1/y)}{s^2 y}. \tag{14}$$

Independently of the nonvanishing value of s, the last function has a single minimum for $y_{\rm m} \approx 1.56$, which is the only positive solution for y of the equation $\partial_y A(y,s) = 0$, which explicitly amounts to

$$\left(1-\tanh(1/y)\right)\left(1-y+\tanh(1/y)\right) = 0.$$

Next, the solution of the equation $A(y_{\rm m}, s) = 0$ gives the single positive root $s_{\rm c} \approx 0.53$. At $y_{\rm m}$, for $s < s_{\rm c}$ the quoted minimum is negative, for $s = s_{\rm c}$ the minimum vanishes and for $s > s_{\rm c}$ the minimum is positive. From the analytical expressions of the curvature invariants for metric (13)

$$R = \frac{2\tanh(1/y)\left(1-\tanh^2(1/y)\right)}{q^2 s^4 y^5}, \tag{15}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{2\left(1-\tanh^2(1/y)\right)^2}{q^4 s^8 y^{10}}\left(2y^2 - 2y\tanh(1/y) + \tanh^2(1/y)\right), \tag{16}$$

$$\begin{aligned} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{4\left(1-\tanh(1/y)\right)^2}{q^4 s^8 y^{10}}\Big(&3y^4 + y^2\left(\tanh(1/y)+1\right)\left(7\tanh(1/y)+5-6y\right) + \\ &\tanh(1/y)\left(\tanh(1/y)+1\right)^2\left(\tanh(1/y)-4y\right)\Big), \end{aligned} \tag{17}$$

one concludes they are all regular everywhere. Hence, for $s \le s_{\rm c}$ the singularities appearing in (13) due to the vanishing of A are only coordinate–singularities describing the existence of event horizons, consequently, we are in the presence of black hole solutions for $|q| \le 2s_{\rm c}m \approx 1.05\,m$. It should be notice that metrics possessing regular standard invariants, as (15)–(17), could still present non–regular behavior of the differential invariants, $R_{;\alpha_1\cdots\alpha_n}R^{;\alpha_1\cdots\alpha_n}$, $R_{\mu\nu;\alpha_1\cdots\alpha_n}R^{\mu\nu;\alpha_1\cdots\alpha_n}$, $R_{\mu\nu\alpha\beta;\alpha_1\cdots\alpha_n}R^{\mu\nu\alpha\beta;\alpha_1\cdots\alpha_n}$ [42]. Because of the structure of the metric components of our solution (13) and their derivatives, the components of the Riemman tensor behave generically at $r = 0$ as $\sim g(r)/r^a$, where $g(r)$ is a function vanishing exponentially as $r \to 0$. This fact explain the bounded behavior of the invariants (15)–(17). Moreover, the quoted behavior at $r = 0$ extends to the covariant differential of any order of the Riemman tensor (the derivative of an exponentially vanishing function is again an exponentially vanishing function, and the contributions due to the Christoffel symbols entering in the covariant derivatives do not alter the quoted pattern). In its turn, this again yields a bounded behavior on the differential invariants of our solution.

For the corresponding values of mass and charge we have, under the strict inequality $|q| < 2s_{\rm c}m$, inner and event horizons for the Killing field $\boldsymbol{k} = \boldsymbol{\partial/\partial t}$, defined by $k_\mu k^\mu = g_{tt} = 0$. For the equality $|q| = 2s_{\rm c}m$, the horizons shrink into a single one, where also $\nabla_\nu(k_\mu k^\mu) = 0$, i.e., this case corresponds to an extreme black hole as in the Reissner–Nordström solution. The extension of the metric beyond the horizons, up to $r = 0$, becomes apparent by passing to the standard advanced and retarded Eddington–Finkelstein coordinates, in terms of which the metric is well–behaved everywhere, even in the extreme case. The maximal extension of this metric can be achieved by following the main lines presented in [43] for the Reissner–Nordström solution, taking care of course, of the more involved integration in our case of the tortoise coordinate $r^* \equiv \int A^{-1} dr$. The global structure of our space–time is similar to the structure of the Reissner–Nordström black hole, except that the usual singularity of this solution, at $r = 0$, has been smoothed out and now it simply corresponds to the origin of the spherical coordinates.

It must be noted that for $q = 0$ our solution (13) becomes the singular Schwarzschild one. For $q \neq 0$, in the derived solution, the charge appears by means of a nonlinear electromagnetic field $F_{\mu\nu}$, which is singularity–free and regularizes the associated gravitational field. In nonlinear electrodynamics, one can think of the space–time coupled to the nonlinear field as produced by a charge distribution through the whole space; in each point of it, the nonlinear interaction of gravitation and electromagnetism is responsible for the regular character of the fields involved. One arrives at this effective charge distribution from the non–zero divergence of the electric field, as can be established from the Maxwell equations (4). Nonlinear electrodynamics can be entirely constructed in terms of functions of $F_{\mu\nu}$ and its invariants. Historically the introduction of the auxiliary field $P_{\mu\nu}$ obeyed the necessity of establishing a relationship between the new electromagnetic theory with theories currently at hand, as the Maxwell theory of continuous media, through the so–called "material equations" [2,3]. In turn this $P_{\mu\nu}$ field resulted to be useful in the derivation of exact solutions in General Relativity [41]; Maxwell equations (4) are linear in terms of $P_{\mu\nu}$. Since $P_{\mu\nu}$ is a secondary field —function of $F_{\mu\nu}$— the presence of a Coulomb–like singularity in it is irrelevant, the physics has to be extracted from $F_{\mu\nu}$. Since $F_{\mu\nu}$ is regular everywhere the conclusion we arrive at is that the nonlinearity of the electrodynamic source together with the well–known nonlinearity of the gravitational field are the only responsible for the regular behavior of the metric and the matter content. This last conclusion is very important, since indicates that singularities occurring in Physics can be due to extrapolations of the linear interactions to strong regimes, where a nonlinear description would be more appropriate.

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[1] Los Alamos Database, more than 180 papers match in the period 96–99 with the search pattern: *Born AND Infeld* (`http://xxx.lanl.gov/find/hep-ph,hep-th,quant-ph/1/Born%3BInfeld/0/1/0/96,97,98,99/0/0`).
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