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Four Parametric Regular Black Hole Solution

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We present a regular class of *exact* black hole solutions of Einstein equations coupled with a nonlinear electrodynamics source. For weak fields the nonlinear electrodynamics becomes the Maxwell theory, and asymptotically the solutions behave as the Reissner–Nordström one. The class is endowed with four parameters, which can be thought of as the mass m, charge q, and a sort of dipole and quadrupole moments α and β, respectively. For $\alpha \geq 3$, $\beta \geq 4$, and $|q| \leq 2s_{\rm c}m$ the corresponding solutions are regular charged black holes. For $\alpha = 3$, they also satisfy the weak energy condition. For $\alpha = \beta = 0$ we recover the Reissner–Nordström singular solution and for $\alpha = 3$, $\beta = 4$ the family includes a previous regular black hole reported by the authors.

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One of the active research topics on black hole Physics in recent years has been related with its interior behavior. It is clear now that black holes are not necessarily singular and several examples of regular *exact* black hole solutions has been reported in the literature [1, 2, 3, 4]. The research on this subject started with the pioneering work of Bardeen [5] who proposed the first regular black hole model (see also Refs. [4, 6, 7]). The Bardeen model is a regular black hole satisfying the weak energy condition and it was a crucial guidance on the ulterior investigations related with spacetime singularities (see Ref. [8, pag. 265]). Subsequently, other regular black hole models were proposed [6, 7, 9, 10, 11, 12, 13, 14]. These models independently of being interesting by themselves are not exact solutions of the Einstein equations, and consequently, no recognizable physical sources can be associated with them. This situation was changed with the discovery of the results of Refs. [1, 2, 3, 4] using nonlinear electrodynamics, where the first regular *exact* black hole solutions in the literature were presented.

Some of the further investigations are directly concerned with these configurations. They include, for example, the study of the propagation of the photons governed by the related nonlinear electrodynamics [15]. The phenomenon of vacuum polarization on these backgrounds has been exhibited for conformally coupled quantum fields [16] and in the case of more general nonminimal couplings [17]. Different prescriptions for calculate the energy distributions have been explored on these spacetimes in order to shed some light on the energy localization problem [18]. Another interesting results concerning the role that such spacetimes can play as explicit realizations of the confinement mechanism has been obtained by the authors of Ref. [19, 20]. Additionally, it has recently proved that such black holes are stable under external perturbations [21].

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In this paper we present a class of four parametric solutions which, under physically reasonable assumptions, are regular exact black hole solutions of the Einstein equations coupled to a nonlinear electrodynamics acting as a source. In the weak field approximation this class of solutions reduces to the Reissner–Nordström one of the usual linear Maxwell theory. Moreover, the black hole solutions satisfy the weak energy condition for a particular range of the parameters. Within this class, for a particular choice of the parameters, one arrives at our previous reported solution [1].

The gravitational field of the derived solution is given by the metric

$$\begin{aligned} \boldsymbol{g} \;=\; & -\left(1-\frac{2mr^{\alpha-1}}{(r^2+q^2)^{\alpha/2}}+\frac{q^2r^{\beta-2}}{(r^2+q^2)^{\beta/2}}\right)\boldsymbol{dt}^2 \\ & +\left(1-\frac{2mr^{\alpha-1}}{(r^2+q^2)^{\alpha/2}}+\frac{q^2r^{\beta-2}}{(r^2+q^2)^{\beta/2}}\right)^{-1}\boldsymbol{dr}^2+r^2\boldsymbol{d\Omega}^2, \end{aligned} \tag{1}$$

and the related source is the electric field

$$E=q\left(\frac{\alpha m\left[5r^2-(\alpha-3)q^2\right]r^{\alpha-1}}{2(r^2+q^2)^{\alpha/2+2}}+\frac{\left[4r^4-(7\beta-8)q^2r^2+(\beta-1)(\beta-4)q^4\right]r^{\beta-2}}{4(r^2+q^2)^{\beta/2+2}}\right), \tag{2}$$

where m, q, α and β are parameters. In order to interpret them we exhibit the asymptotic behavior of the solution up to $O(1/r^5)$, which is given by

$$\begin{aligned} -g_{tt} \;&=\; 1-\frac{2m}{r}+\frac{q^2}{r^2}+\alpha\frac{mq^2}{r^3}-\beta\frac{q^4}{2r^4}+O\left(\frac{1}{r^5}\right), \\ E \;&=\; \frac{q}{r^2}+\alpha\frac{5qm}{2r^3}-\beta\frac{9q^3}{4r^4}+O\left(\frac{1}{r^5}\right). \end{aligned}$$

Up to $O(1/r^3)$ we recover the Reissner-Nordström behavior and, consequently, the parameters m and q are related with the mass and the electric charge, respectively. Moreover, from the asymptotic behavior of the electric field, the parameters α and β could be associated with a sort of dipole and quadrupole moments of the nonlinear source, respectively. In correspondence with this interpretation, one notices that our solution reduces, for $\alpha=\beta=0$, to the Reissner–Nordström black hole. For $\alpha=3$, $\beta=4$ one arrives at our previous reported solution [1], hence, the present spacetimes can be thought as a generalization of the regular black holes of [1].

Within the null tetrad formalism, with $\boldsymbol{g}=2\boldsymbol{e}^1\boldsymbol{e}^2+2\boldsymbol{e}^3\boldsymbol{e}^4$, we define the null tetrad

$$\left.\begin{matrix}\boldsymbol{e}^1\\ \boldsymbol{e}^2\end{matrix}\right\}=\frac{r}{\sqrt{2}}\left(\boldsymbol{d\theta}\pm i\sin\theta\boldsymbol{d\varphi}\right), \qquad \left.\begin{matrix}\boldsymbol{e}^3\\ \boldsymbol{e}^4\end{matrix}\right\}=\frac{1}{\sqrt{2}}\left(\frac{\boldsymbol{dr}}{\sqrt{-g_{tt}}}\pm\sqrt{-g_{tt}}\boldsymbol{dt}\right).$$

With respect to this tetrad our solution is algebraically characterized by three nonvanishing quantities [22]: the nonzero traceless Ricci tensor components

$$\begin{aligned} C_{12}=-C_{34} \;=\; & -\frac{\alpha mq^2\left[5r^2-(\alpha-3)q^2\right]r^{\alpha-3}}{2(q^2+r^2)^{\alpha/2+2}} \\ & -\frac{q^2\left[4r^4-(7\beta-8)q^2r^2+(\beta-1)(\beta-4)q^4\right]r^{\beta-4}}{4(q^2+r^2)^{\beta/2+2}}, \end{aligned} \tag{3}$$

the nonvanishing Weyl complex coefficient

$$C^{(3)} = -\frac{m\left[6r^4-(7\alpha-12)q^2r^2+(\alpha-2)(\alpha-3)q^4\right]r^{\alpha-3}}{3(q^2+r^2)^{\alpha/2+2}} + \frac{q^2\left[12r^4-3(3\beta-8)q^2r^2+(\beta-3)(\beta-4)q^4\right]r^{\beta-4}}{6(q^2+r^2)^{\beta/2+2}}, \tag{4}$$

and the scalar curvature

$$R = -\frac{2\alpha q^2 m\left[r^2-(\alpha+1)q^2\right]r^{\alpha-3}}{(q^2+r^2)^{\alpha/2+2}} + \frac{\beta q^4\left[3r^2-(\beta-1)q^2\right]r^{\beta-4}}{(q^2+r^2)^{\beta/2+2}}. \tag{5}$$

Since the curvature invariants $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ can be evaluated using these quantities, we conclude from the behavior of the above expressions at $r=0$, that the solution is regular for the two following cases: $\alpha\geq 3$, $\beta\geq 4$, and $|q|=2m$, $\alpha\geq 1$, with $\beta=\alpha+1$.

In this paper we study the case $\alpha\geq 3$, $\beta\geq 4$. For these values of the parameters metric (1) is a black hole for a certain range of values of the charge. To establish this last statement we accomplish the substitutions $x=r/|q|$ and $s=|q|/2m$, and rewrite g_{tt} as

$$-g_{tt} = A(x,s,\alpha,\beta) \equiv 1-\frac{x^{\alpha-1}}{s(x^2+1)^{\alpha/2}}+\frac{x^{\beta-2}}{(x^2+1)^{\beta/2}}, \tag{6}$$

which, for any nonvanishing value of s, with α and β fixed, has a single positive minimum. Thus, for any α and β there exists a single real critical value of x, $x_{\rm c}$, and one of s, $s_{\rm c}$, to be determined from $A(x_{\rm c},s_{\rm c},\alpha,\beta)=0$ and $\partial_x A(x_{\rm c},s_{\rm c},\alpha,\beta)=0$, namely

$$u^\beta - u^{\beta-\alpha}\sqrt{u^2-1}/s+u^2-1=0, \qquad \left[(\alpha-1)u^2-\alpha\right]u^{\beta-\alpha}-\sqrt{u^2-1}\left[(\beta-2)u^2-\beta\right]s=0,$$

where $u^2\equiv 1+1/x^2$. To solve these equations, one substitutes s from the first equation into the second one arriving at $(\alpha-1)u^{\beta+2}-\alpha u^\beta-(\beta-\alpha-1)u^4+[2(\beta-\alpha)-1]u^2-(\beta-\alpha)=0$, which has only one positive real root for u. For instance, the corresponding critical values for $\alpha=3$, $\beta=4$ are $s_{\rm c}\approx 0.317$, $x_{\rm c}\approx 1.58$, for $\alpha=3$, $\beta=5$ are $s_{\rm c}\approx 0.326$, $x_{\rm c}\approx 1.48$, for $\alpha=3$, $\beta=6$ are $s_{\rm c}\approx 0.335$, $x_{\rm c}\approx 1.41$, etc. For $s<s_{\rm c}$ the quoted minimum is negative, for $s=s_{\rm c}$ the minimum vanishes and for $s>s_{\rm c}$ the minimum is positive. Since the solution is regular everywhere the singularities appearing in (1), for $s\leq s_{\rm c}$, due to the vanishing of A are only coordinate singularities describing the existence of horizons. Consequently, we are in the presence of black hole solutions for $|q|\leq 2s_{\rm c}m$. For the corresponding values of the charge we have, under the strict inequality $|q|<2s_{\rm c}m$, inner and event horizons $r_\pm$ for the Killing field $\boldsymbol{k}=\boldsymbol{\partial/\partial t}$, defined by the real solutions of the equation $-k_\mu k^\mu=A=0$. For $|q|=2s_{\rm c}m$, the horizons shrink into a single one, corresponding to an extreme black hole ($\nabla_\nu(k_\mu k^\mu)=0$). The extension of the metric beyond the horizons $r_\pm$ becomes apparent by passing to the standard advanced and retarded Eddington–Finkelstein coordinates, in terms of which the metric is smooth everywhere for $\alpha\geq 3$, $\beta\geq 4$, even in the extreme case. Summarizing, our spacetime possesses, for $\alpha\geq 3$, $\beta\geq 4$, a similar global structure as the Reissner–Nordström black hole except that the singularity, at $r=0$, of this last solution has been smoothed out and $r=0$ is now simply the origin of the spherical coordinates. As it has been previously pointed out this fact originates a change in the spatial topology of spacetime from open to closed [7].

For $|q| > 2s_c m$, there are no horizons and the corresponding exact solution for $\alpha \geq 3$, $\beta \geq 4$ represents a globally regular spacetime in this coordinates.

The fields (1) and (2) are a solution of the Einstein-nonlinear electrodynamics field equations which follows form the action [23, 24]

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{1}{16\pi} R - \frac{1}{4\pi} \left(\frac{1}{2} P^{\mu\nu} F_{\mu\nu} - \mathcal{H}(P) \right) \right], \tag{7}$$

where R is the scalar curvature and $\mathcal{H}(P)$ is the so called structural function depending on the invariant $P \equiv \frac{1}{4} P_{\mu\nu} P^{\mu\nu}$ of the antisymmetric tensor $P_{\mu\nu}$. In general the structural function also depends on the other invariant $Q = \frac{1}{4} {}^*P_{\mu\nu} P^{\mu\nu}$ where * stands for the Hodge dual, but for the static configurations we analyze here this invariant is zero. In our case the structural function determining the nonlinear electrodynamics giving the regular black hole family (1) is

$$\mathcal{H}(P) = P \frac{\left(1 - (\beta - 1)\sqrt{-2q^2 P}\right)}{\left(1 + \sqrt{-2q^2 P}\right)^{\beta/2+1}} - \frac{\alpha}{2q^2 s} \frac{\left(\sqrt{-2q^2 P}\right)^{5/2}}{\left(1 + \sqrt{-2q^2 P}\right)^{\alpha/2+1}}, \tag{8}$$

where $s = |q|/2m$ and the invariant P being a negative quantity. This structural function satisfies the plausible condition of correspondence to Maxwell theory, i.e., $\mathcal{H} \approx P$ for weak fields ($P \ll 1$).

Variation of action (7) with respect to the antisymmetric field $P_{\mu\nu}$ gives the material or constitutive equations relating this field to the electromagnetic one

$$F_{\mu\nu} = \mathcal{H}_P P_{\mu\nu}. \tag{9}$$

The Einstein and nonlinear electrodynamics equations arise from action (7) under variation with respect to the metric $g_{\mu\nu}$ and the electromagnetic potential A_μ, respectively

$$G_\mu{}^\nu = 2(\mathcal{H}_P P_{\mu\lambda} P^{\nu\lambda} - \delta_\mu{}^\nu (2P\mathcal{H}_P - \mathcal{H})), \tag{10}$$

$$\nabla_\mu P^{\alpha\mu} = 0. \tag{11}$$

In order to obtain the solution (1), (2), we consider the following static and spherically symmetric ansatz for the metric

$$\boldsymbol{g} = -\left(1 - \frac{2M(r)}{r}\right) \boldsymbol{dt}^2 + \left(1 - \frac{2M(r)}{r}\right)^{-1} \boldsymbol{dr}^2 + r^2 \boldsymbol{d\Omega}^2, \tag{12}$$

and the antisymmetric field $P_{\mu\nu} = 2\delta^t_{[\mu}\delta^r_{\nu]} D(r)$. With these choices the equation (11) integrates as

$$P_{\mu\nu} = 2\delta^t_{[\mu}\delta^r_{\nu]} \frac{q}{r^2} \quad \longrightarrow \quad P = -\frac{D^2}{2} = -\frac{q^2}{2r^4}, \tag{13}$$

where we have chosen the integration constant as q since, as it was previously anticipated, it plays the role of the electric charge. The evaluation of the electric field $E = F_{tr} = \mathcal{H}_P D$ from the constitutive equations (9) gives just the formula (2). The Einstein equations (10) corresponding to the $G_t{}^t$ component yields

$$M'(r) = -r^2 \mathcal{H}(P). \tag{14}$$

Substituting $\mathcal{H}$ from (8) with $P = -q^2/2r^4$, and using that $m = \lim_{r\to\infty} M(r)$ one can write the integral of (14) as

$$M(r) = m - q^2 \int_r^\infty \mathrm{d}y \left(\frac{\alpha m y^{\alpha-1}}{(y^2+q^2)^{\alpha/2+1}} + \frac{[y^2-(\beta-1)q^2]\, y^{\beta-2}}{2(y^2+q^2)^{\beta/2+1}} \right). \tag{15}$$

Using the auxiliary variable $u = 1 + q^2/y^2$, the α–dependent and β–dependent integrals above can be expressed as

$$I_\alpha = \frac{\alpha m}{2} \int \frac{\mathrm{d}u}{u^{\alpha/2+1}}, \quad \text{and} \quad I_\beta = \frac{q^2}{2} \int \left[\frac{1}{u^{\beta/2}} \mathrm{d}\left(\frac{1}{y}\right) - \beta/2 \frac{\mathrm{d}u}{u^{\beta/2+1}} \frac{1}{y} \right], \tag{16}$$

respectively, from which the primitive of both integrals follows straightforwardly. Thus, one arrives finally at the expression

$$M(r) = \frac{m r^\alpha}{(r^2+q^2)^{\alpha/2}} - \frac{q^2 r^{\beta-1}}{2(r^2+q^2)^{\beta/2}}, \tag{17}$$

which allows to conclude the regular dependence (1) for the metric.

A physically plausible requirement is the fulfillment of the weak energy condition, i.e., the positivity of the local energy density along any timelike field. If $\boldsymbol{X}$ is a timelike field ($X_\mu X^\mu = -1$) the local energy density along $\boldsymbol{X}$ for the matter content at the right hand side of Einstein equations (10), can be expressed as

$$4\pi T_{\mu\nu} X^\mu X^\nu = H_\alpha H^\alpha \mathcal{H}_P - \mathcal{H}, \tag{18}$$

where $H_\alpha \equiv {}^*P_{\alpha\mu} X^\mu$ is the magnetic field intensity associated to $\boldsymbol{X}$, and it is also a spacelike field ($H_\alpha H^\alpha > 0$) since by definition it is orthogonal to $\boldsymbol{X}$. The requirement of positivity of the above expression allows to conclude that the weak energy condition is satisfied if $\mathcal{H} < 0$ and $\mathcal{H}_P > 0$. For the range of parameters in the structural function (8) allowing the existence of regular black hole solutions the last conditions imply: $\alpha = 3$, $\beta \geq 4$. Summarizing, we are in the presence of regular black hole solutions satisfying the weak energy condition for $\alpha = 3$. We would like to point out that there is no contradiction with the Penrose singularity theorem [25] for these cases, since, additionally to the requirement of the null energy condition (positivity of the local energy density along signals) and the existence of a closed trapped surface (black hole), such theorem also demands the existence of a noncompact Cauchy surface, which is absent for our spacetimes (see the discussion of Ref. [7]).

Particular attention deserves the previously quoted regular case $|q| = 2m$, $\alpha \geq 1$, $\beta = \alpha + 1$. This solution is not a black hole and it satisfies the weak energy condition only for $\alpha = 1$ ($\beta = 2$).

It is worthwhile to mention that for $\alpha = \beta = 0$ the considered nonlinear electrodynamics (8) contains the Maxwell theory, and the corresponding solution (1), (2) reduces to the Reissner–Nordström black hole.

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