}ESSENTIAL{

Introductory Physics

for PHY 1A10 and 1A20

Zeroth Edition

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AHBETE's physics notes series
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Preface

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Part I

Chapter 1

Basics

1.1 Physical Quantities and Units

A physical quantity is a property of a material or system that can be quantified by some measurements. When measuring a quantity, we always compare it with some reference standard. Such a standard is called a unit of quantity. The most commonly used unit system is the International System (SI). Table. 1.1 lists base quantities and their units.

Physical Quantity	Unit
Time	Second (s)
Length	Meter (m)
Mass	Kilogram (kg)
Electric current	Ampere (A)
Temperature	Kelvin (K)
Amount of substance	Mole (mol)
Luminous internists	Candela (cd)

Table 1.1: Base quantities and units

Sometimes it would be more convenient to introduce a larger or smaller unit. A prefix can indicate the scaling factor of the unit. Table. 1.2 lists some prefixes.

Other important topics are skipped here: **unit consistency and conversions**; **uncertainty and significant figures**; and **estimates and orders of magnitude**.

Power of ten	Prefix
10^{-24}	yocto- (y)
10^{-21}	zepto- (z)
10^{-18}	atto- (a)
10^{-15}	femto- (f)
10^{-12}	pico- (p)
10^{-9}	nano- (n)
10^{-6}	micro- (μ)
10^{-3}	milli- (m)
10^{-2}	centi- (c)
10^{3}	kilo- (k)
10^{6}	mega- (M)
10^{9}	giga- (G)
10^{12}	tera- (T)
10^{15}	peta- (P)
10^{18}	exa- (E)
10^{21}	zetta- (Z)
10^{24}	yotta- (Y)

Table 1.2: Unit prefixes.

1.2 Vectors

When a physical quantity is described by a single number, we call it a scalar quantity; while a vector quantity has both magnitude and direction.

1.2.1 Vector Addition

Vector addition is both commutative and associative.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}, \tag{1.1}$$

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}). \tag{1.2}$$

Pictorially, above properties are very intuitive.

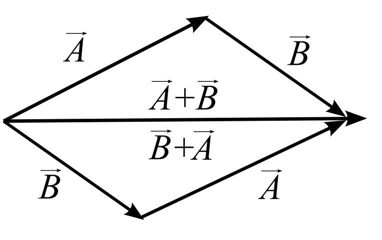

Figure 1.1: Vector addition is commutative.

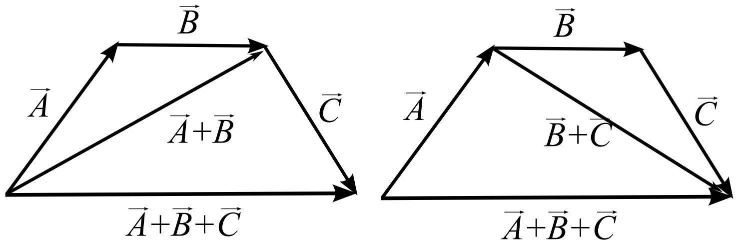

Figure 1.2: Vector addition is associative.

For the vector "subtraction", we can consider it as adding the inverse (vector with same magnitude but opposite direction).

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}). \tag{1.3}$$

1.2.2 Components of Vectors

Sometimes it is easier to decompose a vector and to work componentwise. Taking a vector $\vec{A}$ in xy-plane as an example,

$$\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{\imath} + A_y\hat{\jmath}. \tag{1.4}$$

Here $\vec{A}_x$ is the projection on x direction and $\vec{A}_y$ is the projection on y direction. Expressing using unit vectors, $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors in x and y direction respectively and A_x and A_y are the length of the projection. Using components of a vector, we can also workout its direction and magnitude.

$$\cos\theta = \frac{A_x}{|\vec{A}|}, \sin\theta = \frac{A_y}{|\vec{A}|}, \tan\theta = \frac{A_y}{A_x}, \tag{1.5}$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}. \tag{1.6}$$

The vector sum is also more explicit expressing in component form.

$$\vec{C} = \vec{A} + \vec{B} \rightarrow \begin{cases} C_x = A_x + B_x \\ C_y = A_y + B_y \end{cases}. \tag{1.7}$$

1.2.3 Vector Multiplication

Other than addition, we can also preform multiplication to vectors. In the context of this lecture note, there are three type of multiplication: scalar multiplication, scalar product (or dot product), and cross product (or vector product).

Scalar Multiplication

We may multiply a scalar to a vector.

$$\vec{D} = a\vec{A} \to \begin{cases} D_x = aA_x \\ D_y = aA_y \end{cases} . \tag{1.8}$$

Scalar Product

Scalar product takes two vector and returns a scalar.

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta = A_x B_x + A_y B_y + A_z B_z. \tag{1.9}$$

Here θ is the angle between two vector.

Cross Product

The cross product is only meaningful in three dimensions. It takes two vectors, and returns a vector that is perpendicular to both original vectors.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.10}$$

Suppose $\vec{C} = \vec{A} \times \vec{B}$, componentwise we have

$$C_x = A_y B_z - A_z B_y,\ C_y = A_z B_x - A_x B_z,\ C_z = A_x B_y - A_y B_x. \tag{1.11}$$

Their magnitudes follow the relation

$$|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta. \tag{1.12}$$

The direction of $\vec{C}$ can be found using the right-hand rule as show in Fig.1.3.

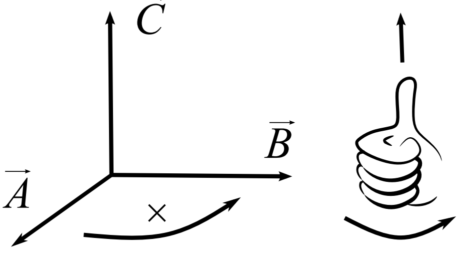

Figure 1.3: The right-hand rule. If you curl the fingers of your right hand so that they follow a rotation from vector A to vector B, then the thumb will point in the direction of the vector product.

Chapter 2

Linear Kinematics and Dynamics

2.1 Motion in One Dimension

Average speed is the change in distance over change in time.

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t}. \tag{2.1}$$

Instantaneous speed can be found by taking the limit $\Delta t \to 0$ to the above equation.

$$v = \frac{dx}{dt}. \tag{2.2}$$

Similar to the distance-speed relation, the average and instantaneous acceleration can be found using the speed via following equations

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t}, \tag{2.3}$$

and

$$a = \frac{dv}{dt}. \tag{2.4}$$

On the other hand, we can also find the speed if we have an expression of acceleration as a function of time by doing the integration.

$$v_f = v_i + \int_0^t a dt. \tag{2.5}$$

Here the subscription i and f are for initial and final. Similarly,

$$x_f = x_i + \int_0^t v dt. \tag{2.6}$$

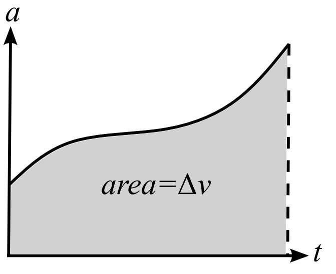

Figure 2.1: Acceleration vs. time plot. The area under the curve gives the change in velocity by Eq. (2.5).

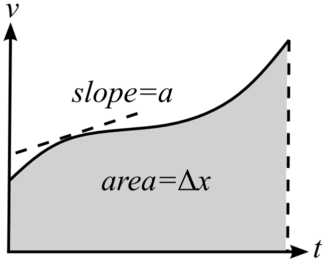

Figure 2.2: Velocity vs. time plot. The area under the curve gives the change in position by Eq. (2.6), and the gradient of the curve at a specific time gives the instantaneous acceleration by Eq. (2.4).

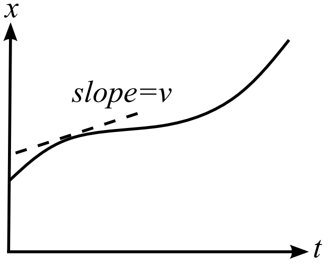

Figure 2.3: Displacement vs. time plot. The gradient of the curve at a specific time gives the instantaneous velocity by Eq. (2.2).

Under constant acceleration, the following expressions can be useful:

$$v_f = v_i + at, \tag{2.7}$$

$$x_f = x_i + v_i t + \frac{1}{2}at^2, \tag{2.8}$$

$$v_f^2 = v_i^2 + 2a(x_f - x_i), \tag{2.9}$$

$$x_f - x_i = \left(\frac{v_i + v_f}{2}\right)t. \tag{2.10}$$

2.2 Motion in Three Dimensions

In three dimensions, the position can be represented using a vector

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}. \tag{2.11}$$

The velocity vector is then

$$\vec{v} = \frac{d\vec{r}}{dt}, \tag{2.12}$$

and the acceleration vector is

$$\vec{a} = \frac{d\vec{v}}{dt}. \tag{2.13}$$

2.2.1 Projectile Motion

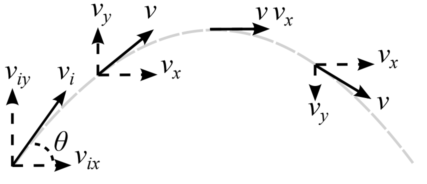

Figure 2.4: Projectile motion. Horizontal component of velocity is constant assuming no air resistance, and vertical component has acceleration due to gravity.

The horizontal component of velocity is

$$v_x = v_i \cos\theta, \tag{2.14}$$

where θ is the shooting angle and the vertical component is

$$v_y = v_i \sin\theta - gt. \tag{2.15}$$

Therefore the horizontal and vertical displacements are

$$x = (v_i \cos\theta)t, \tag{2.16}$$

$$y = (v_i \sin\theta)t - \frac{1}{2}gt^2. \tag{2.17}$$

2.2.2 Uniform Circular Motion

The centripetal acceleration is

$$a_c = \frac{v^2}{r} = \frac{4\pi r^2}{T^2}, \tag{2.18}$$

where r is the radius of the path, and T is the period to complete a revolution.

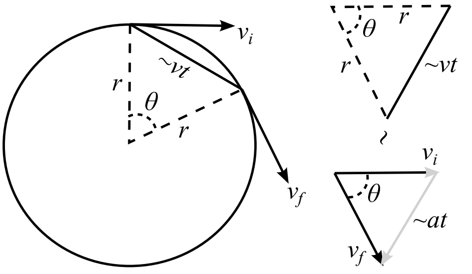

Figure 2.5: Circular motion. Over a time interval, the triangle constricted by **radius** r and **displacement** $\theta r \approx vt$, and the triangle constricted by **velocities** at the beginning v_i and at the end v_f and the **change in velocity vector** $\Delta v \approx at$ are similar triangles.

2.3 Newton's Laws of Motion

Newton's First Law

Body acted on by no net force moves with constant velocity and zero acceleration, i.e., for a body in equilibrium

$$\sum \vec{F} = 0. \tag{2.19}$$

Newton's Second Law

If a net external force acts on a body, the body accelerates. The direction of acceleration is the same as the direction of the net force. The mass of the body times the acceleration of the body equals the net force vector.

$$\sum \vec{F} = m\vec{a}. \tag{2.20}$$

Newton's Third Law

If body A exerts a force on body B, then body B exerts a force on body A. These two forces have the same magnitude but are opposite in direction. These two forces act on different bodies.

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}. \tag{2.21}$$

2.4 Work, Energy and Power

Work done by a force is

$$W = \vec{F} \cdot \vec{x} = Fx\cos(\theta), \tag{2.22}$$

where θ is the angle between force and displacement, or more generally if the force is not constant,

$$W = \int_{x_i}^{x_f} F dx. \tag{2.23}$$

The kinetic energy of a moving body is

$$E_K = \frac{1}{2}mv^2. \tag{2.24}$$

Power is the time rate of doing work. The average power is then

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t}, \tag{2.25}$$

and the instantaneous power is

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}. \tag{2.26}$$

2.5 Potential Energy

Change in gravitational potential energy near the surface is

$$\Delta U_{\text{grav}} = mg\Delta h, \tag{2.27}$$

where Δh is change in height.

For a conservative force, there is a potential, and they are related through

$$F = -\frac{dU}{dx}. \tag{2.28}$$

A more general form would be,

$$F = -\nabla U, \tag{2.29}$$

and here ∇ is the gradient operator.

The elastic potential energy is

$$\Delta U_{\text{elas}} = \frac{1}{2}k\Delta x^2, \tag{2.30}$$

where k is the spring constant. Recall: for an ideal spring we have $\Delta F = k\Delta x$.

2.6 Momentum

Momentum is defined as

$$\vec{P} = m\vec{v}. \tag{2.31}$$

In terms of momentum, the Newton's second law can be expressed as

$$\vec{F} = \frac{d\vec{P}}{dt}. \tag{2.32}$$

When a force is applied over a time interval, it produce a impulse,

$$\vec{J} = \int_0^t \vec{F} dt, \tag{2.33}$$

which in many cases is the change of the momentum.

Conservation of momentum: if the vector sum of the external forces on a system is zero, the total momentum of the system is constant.

Chapter 3

Rotational Kinematics and Dynamics

3.1 Kinematics

In analogy to the linear motion, angular velocity and angular acceleration can be found using

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}, \tag{3.1}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}, \tag{3.2}$$

where θ is the angular displacement.

Again, under constant angular acceleration, the following relations can be handy.

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2, \tag{3.3}$$

$$\theta_f - \theta_i = \frac{1}{2}(\omega_i + \omega_f)t, \tag{3.4}$$

$$\omega_f = \omega_i + \alpha t, \tag{3.5}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i). \tag{3.6}$$

The linear variables and angular variables are related through the radius of the motion r.

$$v = r\omega, \tag{3.7}$$

$$a_{\text{tan}} = r\alpha, \tag{3.8}$$

$$a_{\text{rad}} = \omega^2 r. \tag{3.9}$$

3.2 Energy

In linear motion, there is“linear inertia” (AKA the mass), which is used in calculating kinetic energy, momentum and so on. The rotational version of it is the moment of inertia. When a body is rotating about its center of mass, the moment of inertia is

$$I = \sum_i m_i r_i^2, \tag{3.10}$$

or for a continuous case

$$I = \int \rho(r) r^2 dr. \tag{3.11}$$

When the rotation is not about the center of mass, there will be an additional rotation of inertia

$$I = I_{\mathbf{cm}} + Mr^2. \tag{3.12}$$

Kinetic energy of a rotating body is

$$E_K = \frac{1}{2} I\omega^2, \tag{3.13}$$

which resembles the kinetic energy in the linear case.

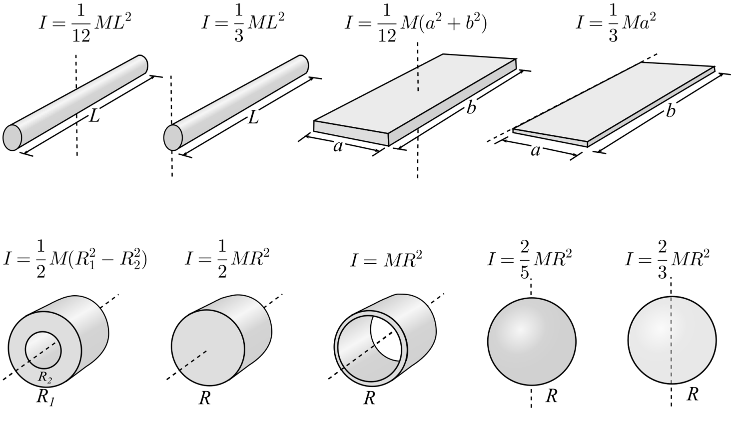

Figure 3.1: Rotation inertia.

3.3 Dynamics

The torque is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}, \tag{3.14}$$

where F is the force acting on a rigid body and r is the vector from the pivot to the point where the force is acting on.

The rotational analogue of Newton's second law is

$$\tau = I\alpha. \tag{3.15}$$

Similar to the linear case, we can also express the work done by a torque by

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta, \tag{3.16}$$

which can be useful when finding the change in kinetic energy.

3.4 Angular Momentum

The angular momentum for a rigid body rotating around a symmetry axis

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = I\omega. \tag{3.17}$$

Formally, the angular momentum is more commonly expressed in terms of position vector $\vec{r}$ and linear momentum vector $\vec{p}$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}. \tag{3.18}$$

The rotational analogue of Newton's second law in term of angular momentum is

$$\vec{\tau} = \frac{d\vec{L}}{dt}. \tag{3.19}$$

Conservation of angular momentum: when the net external torque acting on a system is zero, the total angular momentum of the system is conserved.

3.4.1 Gyroscopic Precession

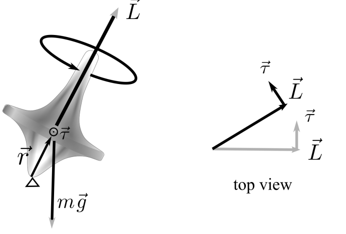

Figure 3.2: Gyroscopic precession.

From Eq.(3.19),

$$d\vec{L} = \tau dt. \tag{3.20}$$

The procession angular speed is given by

$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}}{dt} = \frac{\tau}{L} = \frac{mgr}{I\omega}. \tag{3.21}$$

Chapter 4

Mechanics

The following figures show some typical scenarios.

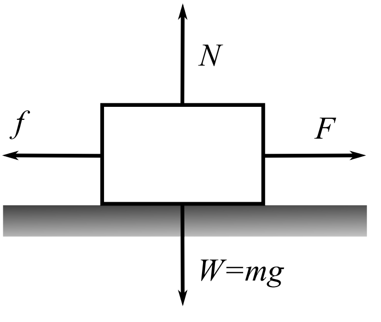

Figure 4.1: Force acting on a block on a horizontal surface.

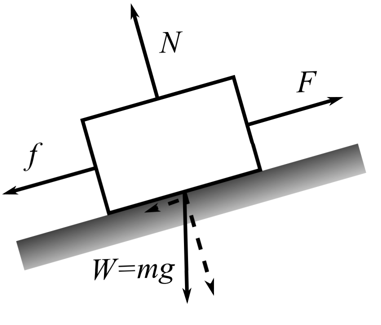

Figure 4.2: Force acting on a block on a inclined surface.

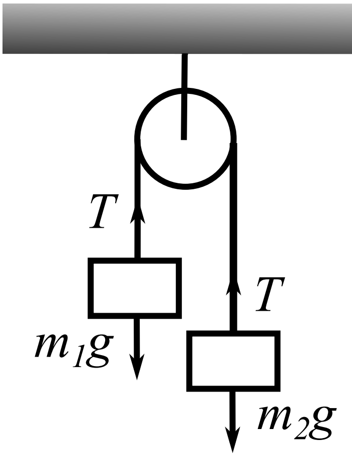

Figure 4.3: Unequal masses attached through a pulley.

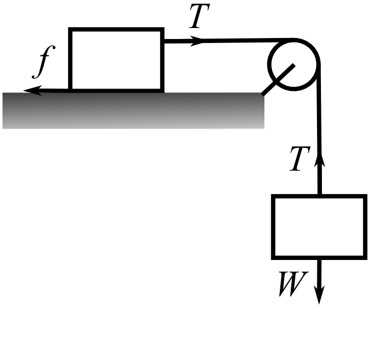

Figure 4.4: Unequal masses attached through a pulley, with one mass on a horizontal surface.

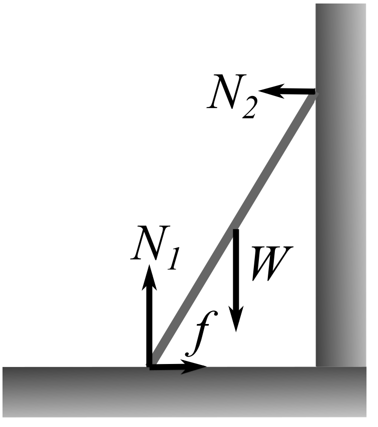

Figure 4.5: A rod leans on a wall.

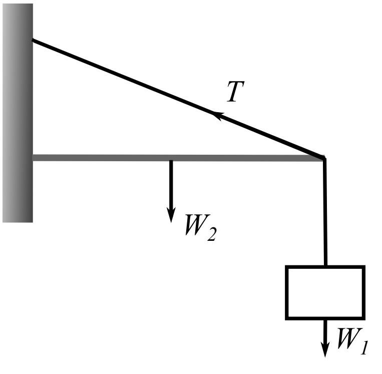

Figure 4.6: A rod with a mass attached on it supported by a cable.

Chapter 5

Material*

Stress characterizes the strength of the forces causing the deformation, i.e., force per unit area. **Strain** describes the resulting deformation. When they are small enough, their relationship are pretty linear, and the proportionality is called **elastic modulus**. Therefore, generally speaking, we have

$$\text{Elastic modulus} = \frac{\text{Stress}}{\text{Strain}}. \tag{5.1}$$

This can also be considered as the generalized version of Hooke's law. The following are some specific examples.

Tensile (or Compressive) Stress and Strain

The tensile (or compressive) stress at the cross section is the ratio of the force acting perpendicular to the cross section to the cross-sectional area,

$$\text{Tensile (Compressive) stress} = \frac{F_\perp}{A}. \tag{5.2}$$

The tensile (or compressive) strain is the ratio of the change in length to the original length

$$\text{Tensile (Compressive) strain} = \frac{\Delta l}{l}. \tag{5.3}$$

The elastic modulus in this case is called **Young's modulus**,

$$Y = \frac{\text{Tensile (Compressive) stress}}{\text{Tensile (Compressive) strain}} = \frac{F_\perp l}{A \Delta l}. \tag{5.4}$$

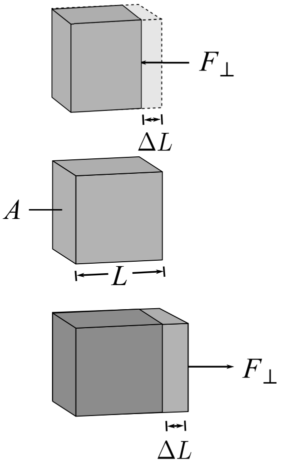

Figure 5.1: Tensile or compressive stress.

Bulk Stress and Strain

When the stress is a uniform pressure on all sides, we use **bulk stress** and **bulk strain**. Bulk strain is the ratio of the volume change to the original volume,

$$\text{Bulk strain} = \frac{\Delta V}{V}. \tag{5.5}$$

Under small change in pressure, the pressure change should be proportional to bulk strain and the proportionality in this case is **bulk modulus**,

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = \frac{\Delta p}{\Delta V/V}. \tag{5.6}$$

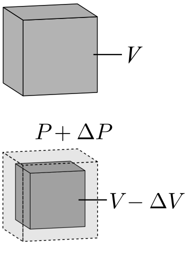

Figure 5.2: Bulk stress.

Shear Stress and Strain

When forces are applied tangent to opposite surfaces, the **shear stress** is defined as the force divided by the area

$$\text{Shear stress} = \frac{F_{\parallel}}{A}. \tag{5.7}$$

The **shear stain** is the ratio of the displacement x to the transverse dimension h,

$$\text{Shear strain} = \frac{x}{h}. \tag{5.8}$$

The elastic modulus is the **shear modulus**, denoted by S,

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F \parallel /A}{x/h}. \tag{5.9}$$

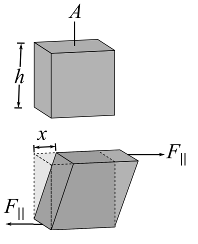

Figure 5.3: Shear stress.

Chapter 6

Gravitation

6.1 Newton's Law of Gravitation

The gravity between two objects is proportional to their masses and inversely proportional to the distance square. Expressing this relation as an equation, we have

$$F = \frac{Gm_1m_2}{r^2}, \tag{6.1}$$

where the proportionality constant G is the gravitational constant and it has numerical value $G = 6.67430 \times 10^{-11} N \cdot m^2/kg^2$.

Gravity is a central force and a conservative force. (Note: Not all central force fields are conservative or spherically symmetric. However, a central force is conservative if and only if it is spherically symmetric or rotationally invariant.) As a conservative force, there is a potential associated

$$U = \frac{-Gm_1m_2}{r}. \tag{6.2}$$

6.1.1 Circular Orbits

A satellite at circular orbits has speed

$$v = \sqrt{\frac{GM}{r}}, \tag{6.3}$$

and period

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}} = \frac{2\pi r^{3/2}}{\sqrt{GM}}. \tag{6.4}$$

6.1.2 Kepler's Laws

- Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

- A line from the sun to a given planet sweeps out equal areas in equal times. In fact, this is due to conservation of angular momentum.

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2} r^2 \frac{d\theta}{dt} \\ &= \frac{1}{2} rv \sin\phi \\ &= \frac{1}{2m} \left| \vec{r} \times m\vec{v} \right| = \frac{L}{2m}.\end{aligned}$$

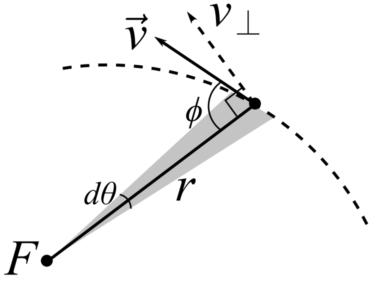

Figure 6.1: Kepler's second law. The area swept per time is proportional to the cross product of velocity and distance towards focus, which is the angular momentum.

- The periods of the planets are proportional to the 3/2 powers of the major axis lengths of their orbits, i.e.,

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}.$$

6.2 Spherical Mass Distributions

As an example, let us consider the gravitational potential energy of a spherical mass shell. The mass of the spherical shell is M and the radius is R. The first case would be when we observe outside the shell at a distance r from the center of the shell. We can start with a 'belt' (the dark region in Fig. 6.2) with mass

$$dM = \frac{M}{A} dA = \frac{M}{4\pi R^2} (2\pi R \sin\phi)(R d\phi). \tag{6.5}$$

The distance from the belt surface to the observing point s can be expressed as

$$s = \sqrt{r^2 - 2rR\cos\phi + R^2}. \tag{6.6}$$

From Eq. (6.2),

$$dU = \frac{GmdM}{s}. \tag{6.7}$$

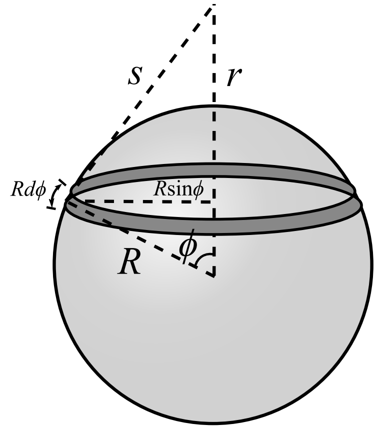

Figure 6.2: Setup of calculating the gravitational potential energy of a spherical mass shell.

Plugging Eq. (6.5) and (6.6) into Eq. (6.7) and integrating give

$$\begin{aligned} U &= \int dU \\ &= -\int \frac{Gm}{\sqrt{r^2 - 2rR\cos\phi + R^2}} \frac{M}{4\pi R^2} (2\pi R \sin\phi)(Rd\phi) \\ &= -\frac{GMm}{r}. \end{aligned} \tag{6.8}$$

The second case would be observing inside the shell, and with a similar setup the result becomes

$$U = -\frac{GMm}{R}. \tag{6.9}$$

6.3 Black holes

Escape speed is the minimum speed that an object needs to escape from the gravitational influence of a second body, thus reaching an infinite distance from the second body, instead of forming a circular or elliptical orbit. When this object with escape speed initially reaches an infinite distance from the second body, it has zero mechanical energy. Therefore, to find the escape speed at the surface of the second body, we have the following relation

$$\begin{aligned} E_{\text{total}} =& T + U \\ =& \frac{1}{2}mv^2 - \frac{GMm}{R} = 0. \end{aligned}$$

Here, T stands for the kinetic energy and U stands for the potential energy. The left-hand side of the second line is the energy evaluated at the surface, and the right-hand side is the energy evaluated at infinity. This relation gives the escape velocity

$$v = \sqrt{\frac{2GM}{R}}. \tag{6.10}$$

Now imagine an object that is so massive (and dense), so that the escape speed is the light speed. Since "all light emitted from such a body would be made to return towards it", it is called a black hole. The radius inside which even light cannot escape is called the Schwarzschild radius, and it is

$$R_S = \frac{2Gm}{c^2}. \tag{6.11}$$

Chapter 7

Periodic Motion and Waves

A repetitive motion is called periodic motion or oscillation. There are several parameters describing periodic motions. The amplitude A is the maximum magnitude of the displacement from equilibrium. The period T, is the time for one cycle. The frequency f is the number of cycles per unit time. The frequency and period are related through

$$f = \frac{1}{T}. \tag{7.1}$$

The angular frequency ω is 2π times of the frequency.

$$\omega = 2\pi f = \frac{2\pi}{T}. \tag{7.2}$$

7.1 Simple Harmonic Motion

When the restoring force is directly proportional to the displacement from the equilibrium, the oscillation is called simple harmonic motion (SHM). A body under such a motion is called a harmonic oscillator. In general, a harmonic oscillator has the displacement, velocity and acceleration

$$x = A\cos(\omega t + \phi), \tag{7.3}$$

$$v = A\omega\sin(\omega t + \phi), \tag{7.4}$$

$$a = -A\omega^2\cos(\omega t + \phi). \tag{7.5}$$

Comparing the equation for displacement and acceleration, we can have

$$a = -\omega^2 x. \tag{7.6}$$

7.1.1 Case 1: Circular Motion

Components of circular motion can be treated similar to simple harmonic motion. Let us consider an object rotating around the origin in a circle with radius A. The components of displacement are

$$x = A\cos\theta, \tag{7.7}$$
$$y = A\sin\theta. \tag{7.8}$$

Since it is a circular motion, so the acceleration follows

$$|a| = \omega^2 A, \tag{7.9}$$

and its x-component is

$$\begin{aligned} a_x =& |a|\cos\theta \\ =& \omega^2 A\cos\theta \\ =& \omega^2 x, \end{aligned} \tag{7.10}$$

and similarly, its y-component is

$$a_y = \omega^2 y. \tag{7.11}$$

7.1.2 Case 2: Spring-Mass System

For a spring-mass system, we have

$$F = -kx, \tag{7.12}$$

which can be expressed as

$$a = -\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{7.13}$$

Comparing with Eq. (7.6), we may conclude that

$$\omega = \sqrt{\frac{k}{m}}. \tag{7.14}$$

For an ideal case with no energy dissipation, the total energy of the system is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2. \tag{7.15}$$

7.1.3 Case 3: Pendulum

Imagine an ideal pendulum (oscillating under small angle) with length L and mass m as shown in Fig. 7.1. The component of weight contribute to the acceleration is

$$F_\theta = mg\sin\theta = m\frac{d^2x}{dx^2}. \tag{7.16}$$

From the geometry, we can have

$$\sin\theta = \frac{x}{L}. \tag{7.17}$$

Therefore,

$$\frac{d^2x}{dt^2} = \frac{g}{L}x, \tag{7.18}$$

and then in this case we have

$$\omega = \sqrt{\frac{g}{L}}. \tag{7.19}$$

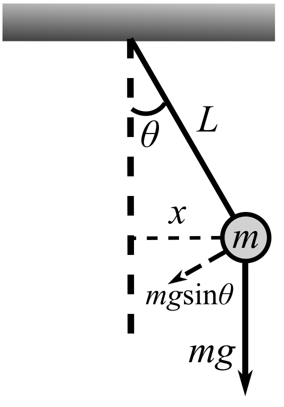

Figure 7.1: A idealized simple pendulum.

For a real pendulum (oscillating under a small angle), unlike a simple pendulum which can be considered as consisting of a massless string and a point mass, it has an extended mass distribution. Therefore, we need to use the knowledge from torque and moment of inertia. Consider the case as shown in Fig. 7.2. The distance from the pivot to the center of mass is d, the object has mass m, and the line connecting the pivot to the center is at angle θ with respect to vertical, thus the torque due to weight is

$$\tau = mgd\sin\theta. \tag{7.20}$$

On the other hand, in analogy to Newton's second law

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}. \tag{7.21}$$

Therefore,

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta, \tag{7.22}$$

and

$$\omega = \sqrt{\frac{mgd}{I}}. \tag{7.23}$$

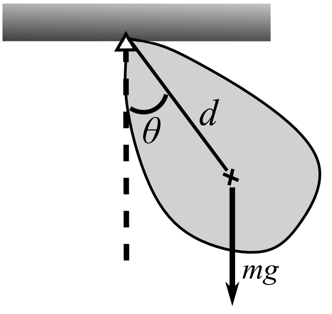

Figure 7.2: A real pendulum.

7.2 Damped Oscillations

In real life, the amplitude of an oscillator tends to decrease. The decrease in amplitude caused by dissipative forces is called damping, and such motion is called damped oscillation. Normally, the damping force is proportional to the velocity, i.e., $F \propto -bv$. Thus the dynamic of the system is

$$\begin{aligned} \sum F &= -kx - bv \\ ma &= -kx - bv \\ m\frac{d^2x}{dt^2} &= -kx - b\frac{dx}{dt}. \end{aligned} \tag{7.24}$$

The solution could be

$$x = A\mathrm{e}^{-(b/2m)t}\cos(\omega' t + \phi), \tag{7.25}$$

with

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \tag{7.26}$$

Here A is the initial amplitude, ϕ is the initial phase. The exponential part indicates the decay of the amplitude, while the cosine part signals the system is still oscillating.

Additional Notes - more general solutions

The equation of motion can be expressed as

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Using the characteristic equation to help us find the solution:

$$\begin{aligned} mr^2 + br + k &= 0, \\ r_1, r_2 &= \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -p \pm \sqrt{p^2 - w_0^2}, \end{aligned}$$

with $p = b/2m$ and $\omega_0 = \sqrt{k/m}$. There are three situations

$$\begin{cases} b > 2\sqrt{km} \\ b = 2\sqrt{km} \\ b < 2\sqrt{km} \end{cases} \Rightarrow \begin{cases} A\mathrm{e}^{r_1} + B\mathrm{e}^{r_2} \text{ (underdamping)} \\ C\mathrm{e}^{pt} + Dt\mathrm{e}^{pt} \text{ (critical damping)} \\ \mathrm{e}^{-pt}(E\cos(\omega' t) + F\sin(\omega' t)) \text{ (overdamping)} \end{cases}$$

The energy dissipation for an underdamping case is

$$\begin{aligned} \frac{dE}{dt} =& \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) \\ =& mv\frac{dv}{dt} + kx\frac{dx}{dt} \\ =& mva + kxv \\ =& v(ma + kx) \\ =& -bv^2. \end{aligned} \tag{7.27}$$

7.3 Forced Oscillation

When a periodic force is applied to a damped oscillator, the motion is called a forced oscillation. The dynamics can be described as

$$m\ddot{x} + b\dot{x} + kx = F(t), F(t) = F_0\cos(\omega_d t), \tag{7.28}$$

where $F(t)$ is the applied force, and its frequency is ω_d. When the system reaches steady state, it has the amplitude

$$A = \frac{F_0}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}}. \tag{7.29}$$

Additional Notes - more details

With the knowledge of the ordinary differential equation, the solution to the above differential equation is a linear combination of homogeneous solution (which solves the case when $F(t) = 0$) and particular solution (which deals with $F(t)$ part). We already have the homogeneous solution from damped oscillation. An educated guess for the particular solution is

$$x_p = A\cos(\omega_d t) + B\sin(\omega_d t). \tag{7.30}$$

Then we just need to plug the particular solution into the differential equation and solve for A and B by matching cosine and sine terms on both sides.

Skipping some mathematical details,

$$A = \frac{(w_0^2 - w_d^2)F_0}{m\left((2\omega_d p)^2\right) + m(\omega_0^2 - \omega_d^2)^2}, B = \frac{2\omega_d p F_0}{m\left((2\omega_d p)^2\right) + m(\omega_0^2 - \omega_d^2)^2}, \tag{7.31}$$

where ω_0 and p are the same as defined in the damped case. We can combine the sine and cosine to a single term, so that

$$x_p = \frac{F_0}{m\sqrt{(2\omega_d p)^2 + (\omega_0^2 - \omega_d^2)^2}} \cos(\omega_d t - \phi), \tan\phi = \frac{B}{A}. \tag{7.32}$$

7.4 Wave

For a propagation wave, its speed in terms of its wavelength and frequency is

$$v = \lambda f. \tag{7.33}$$

Considering a wave propagating in the positive direction of x-axis and oscillating in the direction of y-axis, its displacement as a function of both time and position is

$$y(x,t) = A\cos(kx - \omega t), \tag{7.34}$$

where $k = 2\pi/\lambda$ is the wave number (wave vector in 3D case) and $\omega = 2\pi f$ is the angular frequency. In terms of wave number and angular frequency, the propagation speed is

$$v = \frac{\omega}{k} \tag{7.35}$$

7.4.1 Wave Equation

Consider the second derivative of displacement with respect to position and time respectively,

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y, \tag{7.36}$$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 y. \tag{7.37}$$

Comparing the above two equations, we can then have

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2} \\ \frac{\partial^2 y}{\partial x^2} &= \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \end{aligned} \tag{7.38}$$

This is the wave equation.

7.5 Waves on a String

Let us consider a small segment of string as shown in Fig. 7.3. As the tension on the string aligns with the string, we have

$$\frac{F_{V1}}{F_H} = -\left.\frac{\partial y}{\partial x}\right|_{x_1}, \quad \frac{F_{V2}}{F_H} = -\left.\frac{\partial y}{\partial x}\right|_{x_2}. \tag{7.39}$$

The net force should be in the vertical direction only. (Particles on the string only move up and down. If they are moving in the horizontal direction, we would observe the whole string moving horizontally as well. It is the phase moving in the horizontal direction.)

$$\begin{aligned} F_{\text{net}} =& F_{V1} + F_{V2} \\ =& F_H\left(\left.\frac{\partial y}{\partial x}\right|_{x_2} - \left.\frac{\partial y}{\partial x}\right|_{x_1}\right). \end{aligned} \tag{7.40}$$

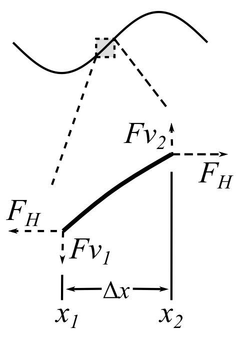

Figure 7.3: Waves on a string.

On the other hand, applying Newton's second law to this segment gives

$$F = \Delta m a = \mu \Delta x \frac{\partial^2 y}{\partial t^2}, \tag{7.41}$$

where μ is the linear density of the string. Therefore,

$$\frac{\left(\left.\frac{\partial y}{\partial x}\right|_{x_2} - \left.\frac{\partial y}{\partial x}\right|_{x_1}\right)}{\Delta x} = \frac{\mu}{F_H}\frac{\partial^2 y}{\partial t^2} \tag{7.42}$$

Taking the limit $\Delta x \to 0$ makes left-hand side the second partial derivative of y with respect to x,

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F_H}\frac{\partial^2 y}{\partial t^2}. \tag{7.43}$$

On the large scale, the horizontal force is just the tension T on the string. Comparing the above result with the wave equation, we conclude that the speed of wave on a string is

$$v = \sqrt{\frac{T}{\mu}}. \tag{7.44}$$

In general, the speed of a propagation wave is

$$v = \sqrt{\frac{\text{Restoring Force}}{\text{Inertia}}}. \tag{7.45}$$

7.6 Energy of Waves

Suppose we still have a wave that can be expressed as

$$y(x,t) = A\cos(kx - \omega t). \tag{7.46}$$

The energy transmitted by the wave per time is

$$\begin{aligned} P_y =& F_y v_y \\ =& \left(F\frac{\partial y}{\partial x}\right)\frac{\partial y}{\partial t} \\ =& Fk\omega A^2 \sin^2(kx - \omega t). \end{aligned} \tag{7.47}$$

For the wave on a string, we have

$$P_y = \sqrt{\mu F}\omega^2 A^2 \sin^2(kx - \omega t). \tag{7.48}$$

The time-average power delivered is

$$P_{\text{avg}} = \frac{1}{2}\sqrt{\mu F}\omega^2 A^2. \tag{7.49}$$

7.7 Superposition and Standing Waves

When two or more waves overlap, the resultant displacement is the sum of the displacements of the individual waves.

$$y(x,t) = y_1(x,t) + y_2(x,t) + \dots . \tag{7.50}$$

This is known as the principle of superposition.

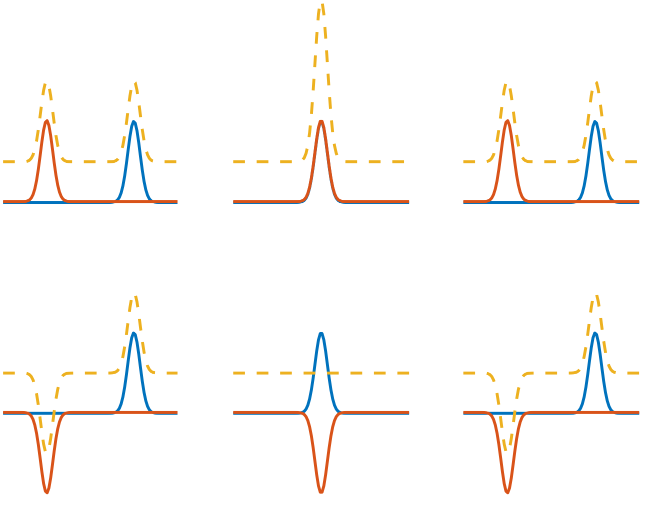

Figure 7.4: (Color online) The principle of superposition. The red wave is traveling towards the right, and the blue wave is traveling towards the left. The yellow dashed line shows the resultant displacement. In the top row, two peaks add to form an even higher peak, which is known as the constructive interference. In the bottom row, a peak and a trough meet and cancel with each other, which is known as the destructive interference.

When a sinusoidal wave of a particular frequency is reflected from a fixed end of a string, the incident and reflected waves superposed and form a standing wave. Points with zero amplitude are called nodes and points with the largest amplitude are called antinodes.

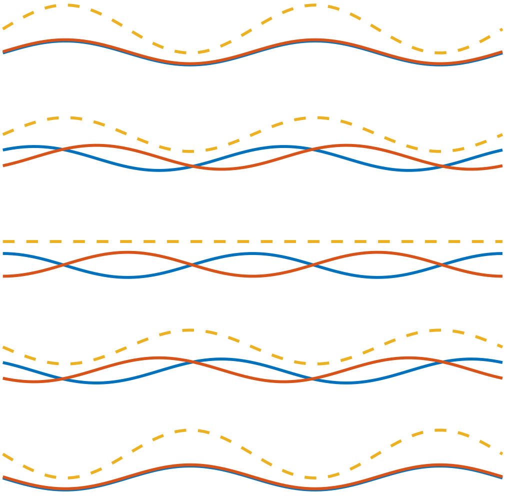

Figure 7.5: (Color online) A standing wave on a string. The red wave travels toward the right and the blue wave is traveling towards the left. The yellow dashed line shows the resultant displacement. Notice that the yellow resultant wave is only oscillating but not traveling at all.

7.7.1 Standing Waves with Strings and Pipes

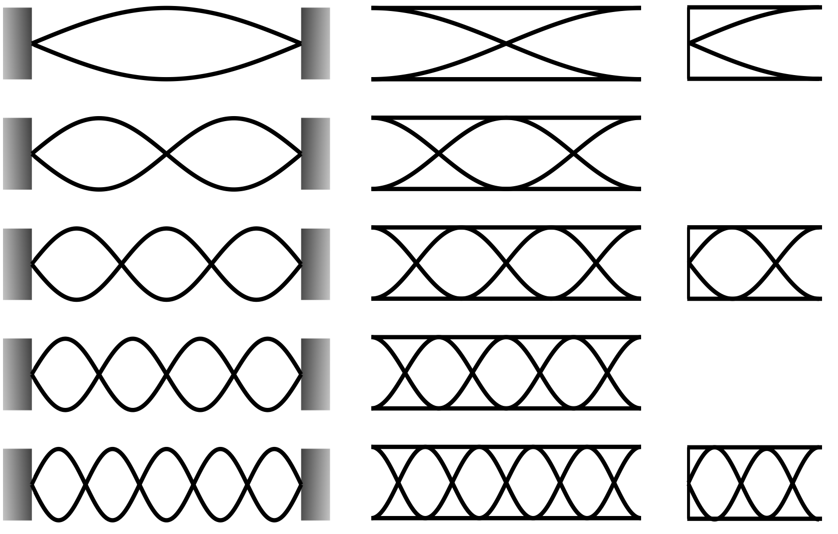

Figure 7.6: Standing waves on a string, inside a pipe with open ends, and inside a pipe with only one open end. From top to bottom are the first harmonic (fundamental), the second harmonic, the third harmonic, the fourth harmonic, and the fifth harmonic. For standing waves on a string, the wavelength of the n^{th} harmonic is $\lambda_n = 2L/n$, where L is the length of the string. For a pipe with two open ends, we have $\lambda_n = 2L/n$, where L is the length of the pipe. For a pipe with only one open end, we have $\lambda_n = 4L/n$ and n can be odd integers only.

7.8 Sound*

7.8.1 The Decibel Scale

Ear sensitivity is logarithmic scaled, so a logarithmic intensity scale is usually used. The sound intensity level β is defined as

$$\beta = 10(\text{dB}) \log \frac{I}{I_0}. \tag{7.51}$$

Here I is the intensity measured, I_0 is a reference intensity, which is 10^{-12} W/m^2, at approximately 1000 Hz.

7.8.2 Beats

When two sounds are slightly off-tune, that is, they have slightly different frequency, the resultant sound would have varying loudness - amplitude. This variation in loudness is called the beats and the frequency of changing loudness is called the beat frequency. The process of calculating the beat frequency is very similar to a "catch-up" problem. Between two beats, the sound wave with higher frequency oscillates n times and the other sound wave oscillates $n-1$ times. Therefore, we have

$$T_{\text{beat}} = nT_1,\ T_{\text{beat}} = (n-1)T_2. \tag{7.52}$$

Eliminating n gives

$$T_{\text{beat}} = \frac{T_1T_2}{T_2 - T_1}. \tag{7.53}$$

The reciprocal of the above form is

$$\begin{aligned}
f_{\text{beat}} =& \frac{1}{T_{\text{beat}}} = \frac{T_2 - T_1}{T_1T_2} \\
=& \frac{T_2 - T_1}{T_1T_2} \\
=& \frac{1}{T_1} - \frac{1}{T_2} \\
=& f_1 - f_2.
\end{aligned} \tag{7.54}$$

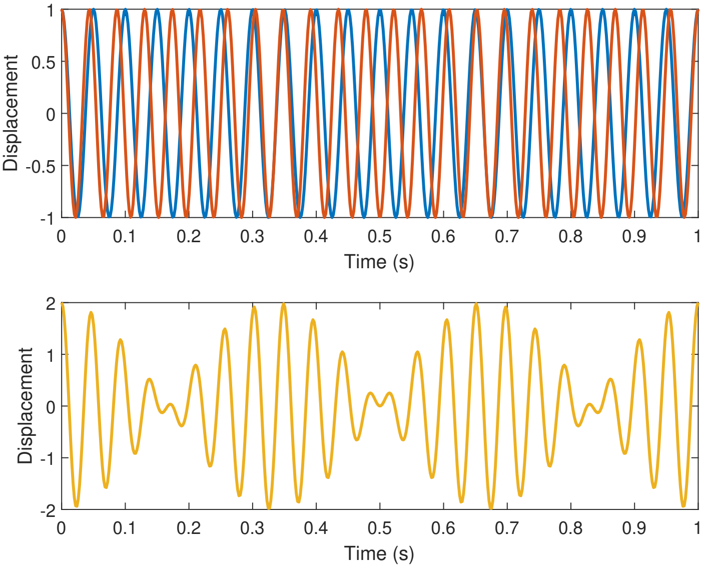

Figure 7.7: Beat frequency. The above figure shows waves of frequencies 50 Hz and 53 Hz. The bottom figure shows the resultant wave, the frequecy is about (53-50)=3 Hz.

7.9 The Doppler Effect

When an object is approaching or receding, the pitch of the emitted sound by the moving object heard by a stationary listener changes. This is called the Doppler effect. We are going to express this phenomenon mathematically. First, let us consider moving receiver and stationary source as shown in the top row of Fig. 7.8. Start with the definition of the frequency,

$$f = \frac{v}{\lambda}. \tag{7.55}$$

When the receiver is moving, from the view of the receiver, the wave speed changes, so we have

$$f = \frac{v \pm v_R}{v/f_0}. \tag{7.56}$$

Now, let us make the source move as well. Effectively, the wavelength listened by the receiver changes as shown in the bottom two cases of Fig. 7.8, which can be expressed as

$$\lambda = (v \pm v_s)/f_0. \tag{7.57}$$

Therefore, when both receiver and source are moving, the frequency heard by receiver is

$$f = \frac{v \pm v_R}{v \pm v_S} f_0. \tag{7.58}$$

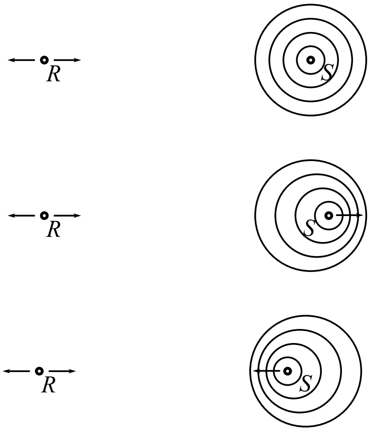

Figure 7.8: The Doppler effect.

7.9.1 Shock Waves*

When an object moves faster than the speed of sound, a cone of precipitated droplet can be observed. Such a cone is actually the wavefront of the shock wave. By observing the open angle of the cone, we are able to tell how fast the object is. As shown in Fig. 7.9, the hypotenuse of the triangle is the distance traveled by the object over some time t, and the radius of the circle (wavefront of sound emitted at a specific time) is the distance traveled by the sound over the same time. Therefore, the angle α can be expressed as

$$\sin\alpha = \frac{vt}{v_s t} = \frac{v}{v_s}. \tag{7.59}$$

The ration of the speed to the sound speed is also known as Mach number.

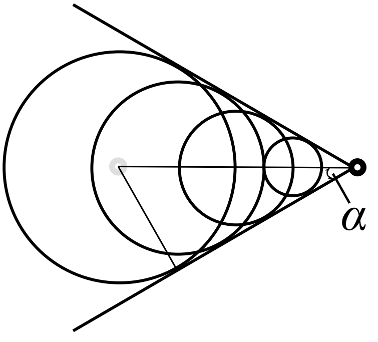

Figure 7.9: A shock wave.

Part II

Chapter 8

Fluid Mechanics

Chapter 9

Thermodynamics

Chapter 10

Electromagnetism

Chapter 11

Optics

Chapter 12

Modern Physics