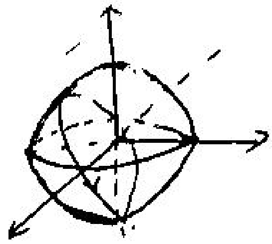
习题 12.7　Stokes 公式



1. 利用 Stokes 公式计算 $\oint_L ydx + zdy + xdz$，其中 L 为圆周

$L: x^2 + y^2 + z^2 = a^2, x + y + z = 0$ 从 z 轴正向看

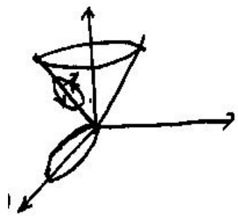
去沿逆时针方向.

【解析】(1) $\sum: z = -x - y$，方向向上；$\vec{n} = (1,1,1),\ \vec{n}^0 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$；

(2) $\oint_L ydx + zdy + xdz = \iint\limits_{\Sigma} \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} dS = \frac{1}{\sqrt{3}} \iint\limits_{\Sigma} [(0-1) - (1-0) + (0-1)]\, dS$

$$= \frac{1}{\sqrt{3}} \iint\limits_{\Sigma} 1dS = -\sqrt{3}\pi a^2$$

2. 计算　$I = \oint_L xydx + z^2dy + zxdz$，其中 L 为锥面 $z = \sqrt{x^2 + y^2}$ 与柱面 $x^2 + y^2 = 2ax\,(a > 0)$ 的交线，从 z 轴正向看去沿逆时针方向.



【解析】$\sum: z = \sqrt{x^2 + y^2}$，方向向上；

$$I = \iint\limits_{\Sigma} \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & z^2 & zx \end{vmatrix} = \iint\limits_{\Sigma} (-2z)dydz - zdzdx - xdxdy$$

$$= -\iint\limits_{\Sigma} 2zdydz + zdzdx + xdxdy \qquad \vec{n} = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right),\ \vec{n}^0 = \frac{1}{|\vec{n}|}\vec{n}$$

$$= -\iint\limits_{\Sigma} (2z\cos\alpha + z\cos\beta + x\cos\gamma)dS = -\iint\limits_{\Sigma} \left(2z \cdot \frac{\cos\alpha}{\cos\gamma} + z \cdot \frac{\cos\beta}{\cos\gamma} + x\right)\cos\gamma dS$$

$$= -\iint\limits_{\Sigma} \left(2z \cdot \frac{-x}{\sqrt{x^2+y^2}} + z \cdot \frac{-y}{\sqrt{x^2+y^2}} + x\right)dxdy = \iint\limits_{\Sigma} (2x + y - x)dxdy = \iint\limits_{\Sigma} (x + y)dxdy$$

$$= \iint\limits_{D_{xy}} (x + y)dxdy \qquad D_{xy}: x^2 + y^2 \le a^2\ (a > 0)$$

$$\overset{\text{对称性}}{=} \iint\limits_{D_{xy}} xdxdy \overset{\text{极坐标}}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} r\cos\theta \cdot rdr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \cdot \left(\frac{1}{3}r^3\Big|_0^{2a\cos\theta}\right)d\theta = \frac{8a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta$$

$$= \frac{16a^3}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = \frac{16a^3}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi a^3$$