习题 10.3　多元复合函数微分法

1. 求下列复合函数的偏导数.

(1) $z=\sin(2u+3v), u=xy, v=x^2+y^2$，求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

【解析】

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\cos(2u+3v)\cdot 2\cdot y+\cos(2u+3v)\cdot 3\cdot 2x=\cos(2xy+3x^2+3y^2)\cdot(2y+6x),$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=\cos(2u+3v)\cdot 2\cdot x+\cos(2u+3v)\cdot 3\cdot 2y=\cos(2xy+3x^2+3y^2)\cdot(2x+6y).$$

(2) $z=u^2\ln v$，其中 $u=\dfrac{x}{y}, v=3x-2y$，求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

【解析】

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=2u\ln v\cdot\frac{1}{y}+\frac{u^3}{v}\cdot 3=\frac{2x}{y^2}\ln(3x-2y)+\frac{3x}{(3x-2y)y^2},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=2u\ln v\cdot\left(-\frac{x}{y}\right)+\frac{u^2}{v}\cdot(-2)=-\frac{2x}{y^2}\ln(3x-2y)-\frac{2x^2}{(3x-2y)y^2}.$$

2. 设 $z=f(xy,\dfrac{x}{y})+g(\dfrac{y}{x})$，其中 f 具有 二阶连续偏导数，g 具有二阶连续导数，求 $\dfrac{\partial^2 z}{\partial x\partial y}$.

【解析】① $\dfrac{\partial z}{\partial x}=f_1'\cdot y+f_2'\cdot\dfrac{1}{y}+g'\cdot\left(-\dfrac{y}{x^2}\right),$

② $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial z}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial z}{\partial y}\left[f_1'\cdot y+f_2'\cdot\dfrac{1}{y}+g'\cdot\left(-\dfrac{y}{x^2}\right)\right]$

$$=y\left[f_{11}''\cdot x+f_{12}''\cdot\left(-\frac{x}{y^2}\right)\right]+f_1'+\left[f_{21}''\cdot x+f_{22}''\cdot\left(-\frac{x}{y^2}\right)\right]\cdot\frac{1}{y}+f_2'\cdot\left(-\frac{1}{y^2}\right)+g''\cdot\frac{1}{x}\left(-\frac{y}{x^2}\right)+g'\cdot\left(-\frac{1}{x^2}\right)$$

$$=xyf_{11}''-\frac{x}{y^3}f_{22}''+f_1'-\frac{1}{y^2}f_2'-\frac{y}{x^3}g''-\frac{1}{x^2}g'.$$

3. 设 $z=\dfrac{y}{f(u)}$，其中 $u=x^2-y^2$，$f(u)$ 为可导函数，求 $\dfrac{1}{x}\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\dfrac{\partial z}{\partial y}$.

【解析】① $\dfrac{\partial z}{\partial x}=-\dfrac{yf'(u)}{f^2(u)}\cdot\dfrac{\partial u}{\partial x}=-\dfrac{2xyf'(u)}{f^2(u)},\quad \dfrac{\partial z}{\partial y}=\dfrac{1}{f(u)}-\dfrac{yf'(u)}{f^2(u)}\cdot\dfrac{\partial u}{\partial y}=\dfrac{1}{f(u)}+\dfrac{2y^2f'(u)}{f^2(u)},$

② $\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=-\frac{2yf'(u)}{f^2(u)}+\frac{1}{yf(u)}+\frac{2yf'(u)}{f^2(u)}=\frac{1}{yf(u)}=\frac{z}{y^2}.$

4. 设 $f(u,v,w)$ 具有二阶连续偏导数，求函数 $z=f(\sin x,\cos y,e^{x+y})$ 的二阶连续偏导数 $\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial y^2}$ 及 dz .

【解析】① $\frac{\partial z}{\partial x}=f_1'\cdot\cos x+f_3'\cdot e^{x+y}$， $\frac{\partial z}{\partial y}=f_1'\cdot(-\sin y)+f_3'\cdot e^{x+y}$,

② $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(f_1'\cdot\cos x+f_3'\cdot e^{x+y}\right)$

$$=-\sin x\cdot f_1'+\cos x\cdot\left(f_{11}''\cdot\cos x+f_{13}''\cdot e^{x+y}\right)+e^{x+y}\cdot f_3'+e^{x+y}\cdot\left(f_{31}''\cdot\cos x+f_{33}''\cdot e^{x+y}\right)$$

$$=-\sin x\cdot f_1'+e^{x+y}\cdot f_3'++\cos^2 x\cdot f_{11}''+2\cos xf_{13}''\cdot e^{x+y}+e^{2(x+y)}\cdot f_{33}''.$$

③ $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(f_1'\cdot\cos x+f_3'\cdot e^{x+y}\right)$

$$=\cos x\cdot\left[f_{12}''\cdot\left(-\sin y\right)+f_{13}''\cdot e^{x+y}\right]+e^{x+y}\cdot f_3'+e^{x+y}\cdot\left[f_{32}''\cdot(-\sin y)+f_{33}''\cdot e^{x+y}\right]$$

$$=e^{x+y}\cdot f_3'-\cos x\cdot\sin y\cdot f_{12}''+\cos x\cdot e^{x+y}\cdot f_{13}''-\sin y\cdot e^{x+y}\cdot f_{23}''+f_{33}''\cdot e^{2(x+y)}$$

④ $\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left[f_1'\cdot(-\sin y)+f_3'\cdot e^{x+y}\right]$

$$=-\cos y\cdot f_2'-\sin y\left[f_{22}''\cdot\left(-\sin y\right)+f_{23}''\cdot e^{x+y}\right]+e^{x+y}\cdot f_3'+e^{x+y}\cdot\left[f_{32}''\cdot(-\sin y)+f_{33}''\cdot e^{x+y}\right]$$

$$=-\cos y\cdot f_2'+e^{x+y}\cdot f_3'+\sin^2 y\cdot f_{22}''-2e^{x+y}\cdot\sin y\cdot f_{23}''+f_{33}''\cdot e^{2(x+y)}$$

⑤ $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\left(f_1'\cdot\cos x+f_3'\cdot e^{x+y}\right)dx+\left(f_1'\cdot(-\sin y)+f_3'\cdot e^{x+y}\right)dy$

5. 设 $u=f(r),r=\sqrt{x^2+y^2+z^2}$ ，若 u 满足调和方程 $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$，试求函数 u .

【解析】① $\frac{\partial u}{\partial x}=f'(r)\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{r}f'(r)$, $\frac{\partial u}{\partial y}=\frac{y}{r}f'(r)$, $\frac{\partial u}{\partial z}=\frac{z}{r}f'(r)$,

② $\frac{\partial^2 u}{\partial x^2}=\left[\frac{x}{r}f'(r)\right]_x'=\frac{1}{r}f'(r)+x\left(-\frac{1}{r^2}\right)\cdot\frac{x}{r}f'(r)+\frac{x}{r}f''(r)\cdot\frac{x}{r}=\frac{1}{r}f'(r)-\frac{x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)$ ；

$\frac{\partial^2 u}{\partial y^2}=\frac{1}{r}f'(r)-\frac{y^2}{r^3}f'(r)+\frac{y^2}{r^2}f''(r)$ ；　　$\frac{\partial^2 u}{\partial z^2}=\frac{1}{r}f'(r)-\frac{z^2}{r^3}f'(r)+\frac{z^2}{r^2}f''(r)$ ；

③ $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{r}f'(r)+f''(r)=0$ ，

解微分方程得：$\frac{f''(r)}{f'(r)}=-\frac{2}{r}\Rightarrow \ln f'(r)=-2\ln r+\ln C_1\Rightarrow f'(r)=\frac{C_1}{r^2}\Rightarrow f(r)=-\frac{C_1}{r}+C_2$；

即 $u=-\frac{C_1}{\sqrt{x^2+y^2+z^2}}+C_2$， C_1,C_2 为任意实数.