习题 10.4　隐函数求导法则

1. 设 $\frac{x}{z}=\ln\frac{z}{y}$，求 $\frac{\partial z}{\partial x}$ 及 $\frac{\partial z}{\partial y}$.

【解析】① $F(x,y,z)=\frac{x}{z}-\ln\frac{z}{y}=\frac{x}{z}-\ln z+\ln y$，

② $F_x'=\frac{1}{z}$，$F_y'=\frac{1}{y}$，$F_z'=-\frac{x}{z^2}-\frac{1}{z}$，

③ $\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=-\frac{\frac{1}{z}}{-\frac{x}{z^2}-\frac{1}{z}}=\frac{x}{x+z}$，$\frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}=-\frac{\frac{1}{y}}{-\frac{x}{z^2}-\frac{1}{z}}=\frac{z^2}{y(x+z)}$.

2. 设 $e^z-xyz=0$，求 $\frac{\partial^2 z}{\partial x^2}$.

【解析】① $F(x,y,z)=e^z-xyz$，

② $F_x'=-yz$，$F_z'=e^z-xy$，$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=\frac{yz}{e^z-xy}$，

③

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{y\cdot\frac{\partial z}{\partial x}\cdot(e^z-xy)-yz\left(e^z\cdot\frac{\partial z}{\partial x}-y\right)}{\left(e^z-xy\right)^2}=\frac{y\cdot\frac{yz}{e^z-xy}\cdot(e^z-xy)-yz\left(e^z\cdot\frac{yz}{e^z-xy}-y\right)}{\left(e^z-xy\right)^2}$$

$$=\frac{2y^2z(e^z-xy)-y^2z^2e^z}{\left(e^z-xy\right)^3}.$$

3. 设 $u=u(x,y),v=v(x,y)$ 是由方程组 $\begin{cases}u^2-v+x=0\\u+v^2-y=0\end{cases}$ 确定的 x,y 的隐函数，求 $\frac{\partial u}{\partial x},\frac{\partial v}{\partial y}$.

【解析】方程组两边对 x,y 分别求微分，得

$$\begin{cases}2udu-dv+dx=0\\du+2vdv-dy=0\end{cases}$$

消去 du：$(4uv+1)dv=-dx+2udy\Rightarrow dv=\frac{-dx+2udy}{4uv+1}\Rightarrow\frac{\partial v}{\partial y}=\frac{2u}{4uv+1}$；

消去 dv：$(4uv+1)du=-2vdx+dy\Rightarrow du=\frac{-2vdx+dy}{4uv+1}\Rightarrow\frac{\partial u}{\partial x}=\frac{-2v}{4uv+1}$.

4. 设 $\begin{cases}x=e^u+u\sin v\\y=e^u-u\cos v\end{cases}$，求 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$.

【解析】方程组两边对 x,y 分别求偏导，得

$$1=e^{u}\cdot\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\cdot\sin v+u\cdot\cos v\cdot\frac{\partial v}{\partial x} \qquad (1)$$

$$0=e^{u}\cdot\frac{\partial u}{\partial y}+\frac{\partial u}{\partial y}\cdot\sin v+u\cdot\cos v\cdot\frac{\partial v}{\partial y} \qquad (2)$$

$$0=e^{u}\cdot\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\cdot\cos v+u\cdot\sin v\cdot\frac{\partial v}{\partial x} \qquad (3)$$

$$1=e^{u}\cdot\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}\cdot\cos v+u\cdot\sin v\cdot\frac{\partial v}{\partial y} \qquad (4)$$

（1）（3）联立，解得

$$\frac{\partial u}{\partial x}=\frac{\sin v}{e^{u}(\sin v-\cos v)+1},\quad \frac{\partial v}{\partial x}=\frac{\cos v-e^{u}}{u\left[e^{u}(\sin v-\cos v)+1\right]};$$

（2）（4）联立，解得

$$\frac{\partial u}{\partial y}=\frac{-\cos v}{e^{u}(\sin v+\cos v)+1},\quad \frac{\partial v}{\partial y}=\frac{\sin v+e^{u}}{u\left[e^{u}(\sin v-\cos v)+1\right]}.$$

5. 设 $u=f(x,y,z)$ 有连续偏导数，$y=y(x),z=z(x)$ 分别由方程 $e^{xy}-y=0$ 和 $e^{z}-xz=0$ 所确定，求 $\dfrac{du}{dx}$.

【解析】① $\dfrac{du}{dx}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dx}+\dfrac{\partial f}{\partial z}\cdot\dfrac{dz}{dx}$，

② $e^{xy}-y=0\Rightarrow e^{xy}(y+xy')-y'=0\Rightarrow y'=\dfrac{ye^{xy}}{1-xe^{xy}}$，

③ $e^{z}-xz=0\Rightarrow e^{z}\cdot z'-z-xz'=0\Rightarrow z'=\dfrac{z}{e^{z}-x}$，

④ $\dfrac{du}{dx}=f_x'+f_y'\cdot\dfrac{ye^{xy}}{1-xe^{xy}}+f_z'\cdot\dfrac{z}{e^{z}-x}$.

6. 设 $y=g(x,z)$，而 z 是由方程 $f(x-z,xy)=0$ 所确定的 x,y 的函数，其中 g,f 具有一阶偏导连续，$f_1'-xf_2'g_2'\neq 0$，求 $\dfrac{dz}{dx}$.

【解析】$f(x-z,xy)=0$ 两边对 x 求导，即

$$f_1'\cdot\left(1-\frac{dz}{dx}\right)+f_2'\cdot\left[y+x\left(g_1'+g_2'\cdot\frac{dz}{dx}\right)\right]=0,$$

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$$\frac{dz}{dx}=\frac{f_1'+yf_2'+xf_2'\cdot g_2'}{f_1'-xf_2'\cdot g_2'}.$$