习题 10.2　偏导数与全微分

1. 求下列函数的一阶偏导数.

$(1)\, z=\dfrac{x^2+y^2}{xy}$；

【解析】① $z=\dfrac{x^2+y^2}{xy}=\dfrac{x}{y}+\dfrac{y}{x}$；

② $\dfrac{\partial z}{\partial x}=\dfrac{1}{y}-\dfrac{y}{x^2}=\dfrac{x^2-y^2}{x^2y}$，$\dfrac{\partial z}{\partial y}=-\dfrac{x}{y^2}+\dfrac{1}{x}=\dfrac{y^2-x^2}{xy^2}$.

$(2)\, z=\ln(x+\sqrt{x^2+y^2})$；

【解析】$\dfrac{\partial z}{\partial x}=\dfrac{1}{x+\sqrt{x^2+y^2}}\cdot\left[1+\dfrac{1}{2\sqrt{x^2+y^2}}\cdot 2x\right]=\dfrac{1}{\sqrt{x^2+y^2}}$，

$$\frac{\partial z}{\partial y}=\frac{1}{x+\sqrt{x^2+y^2}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y=\frac{y}{\left(x+\sqrt{x^2+y^2}\right)\cdot\sqrt{x^2+y^2}}.$$

$(3)\, u=x^{\frac{y}{z}}$；

【解析】$\dfrac{\partial u}{\partial x}=\dfrac{y}{z}x^{\frac{y}{z}-1}$，$\dfrac{\partial u}{\partial y}=\left(x^{\frac{y}{z}}\cdot\ln x\right)\cdot\dfrac{1}{z}=\dfrac{\ln x}{z}\cdot x^{\frac{y}{z}}$，$\dfrac{\partial u}{\partial z}=\left(x^{\frac{y}{z}}\cdot\ln x\right)\cdot\left(-\dfrac{y}{z^2}\right)=-\dfrac{y\ln x}{z^2}\cdot x^{\frac{y}{z}}$.

$(4)\, u=\arctan(x-y)^2$.

【解析】$\dfrac{\partial u}{\partial x}=\dfrac{1}{1+(x-y)^4}\cdot 2(x-y)\cdot 1=\dfrac{2(x-y)}{1+(x-y)^4}$，$\dfrac{\partial u}{\partial y}=\dfrac{1}{1+(x-y)^4}\cdot 2(x-y)\cdot(-1)=-\dfrac{2(x-y)}{1+(x-y)^4}$.

2. 设 $f(x,y,z)=(z-a^{xy})\sin\ln x$，求 $f(x,y,z)$ 在点 $(1,0,2)$ 处的 3 个一阶偏导数.

【解析】

方法一：先分别对 x,y,z 求出偏导函数，然后在代值；

方法二：亦可以用偏导数定义的方法求之；

方法三：$f'_x(1,0,2)=\left[\dfrac{d}{dx}f(x,0,2)\right]_{x=1}=\left[\dfrac{d}{dx}\sin(\ln x)\right]_{x=1}=\left[\dfrac{1}{x}\cos(\ln x)\right]_{x=1}=1$；

$$f'_y(1,0,2)=\left[\frac{d}{dy}f(1,y,2)\right]_{y=1}=\left[\frac{d}{dy}(0)\right]_{y=1}=0；$$

$$f'_z(1,0,2)=\left[\frac{d}{dz}f(1,0,z)\right]_{z=1}=\left[\frac{d}{dz}(0)\right]_{z=1}=0.$$

3. 设 $u=e^{\frac{x}{y^2}}$，证明 $2x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

【解析】① $\dfrac{\partial u}{\partial x}=e^{\frac{x}{y^2}}\cdot\dfrac{1}{y^2}$，$\dfrac{\partial u}{\partial y}=e^{\frac{x}{y^2}}\cdot x\cdot(-2)\cdot\dfrac{1}{y^3}=-\dfrac{2x}{y^3}e^{\frac{x}{y^2}}$；

② $2x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2x\cdot e^{\frac{x}{y^2}}\cdot\dfrac{1}{y^2}-y\cdot\dfrac{2x}{y^3}e^{\frac{x}{y^2}}=0$.

4. 设 $z = x\ln(xy)$，求 $\dfrac{\partial^3 z}{\partial x^2 \partial y}$ 与 $\dfrac{\partial^3 z}{\partial x \partial y^2}$.

【解析】① $\dfrac{\partial z}{\partial x} = \ln(xy) + x \cdot \dfrac{1}{xy} \cdot y = 1 + \ln(xy)$， $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{xy} \cdot y = \dfrac{1}{x}$， $\dfrac{\partial^3 z}{\partial x^2 \partial y} = 0$；

② $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{y}$， $\dfrac{\partial^3 z}{\partial x \partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 z}{\partial x \partial y}\right) = -\dfrac{1}{y^2}$.

5. 设 $z = \arctan\dfrac{x+y}{1-xy}$， 求 $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial x \partial y}, \dfrac{\partial^2 z}{\partial y \partial x}, \dfrac{\partial^2 z}{\partial y^2}$.

【解析】

① $\dfrac{\partial z}{\partial x} = \dfrac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2} \cdot \dfrac{1 \cdot (1-xy) - (x+y)(-y)}{(1-xy)^2} = \dfrac{1}{1+x^2}$， $\dfrac{\partial z}{\partial y} = \dfrac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2} \cdot \dfrac{(1-xy) - (x+y)(-x)}{(1-xy)^2} = \dfrac{1}{1+y^2}$；

② $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{2x}{(1+x^2)^2}$， $\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{2y}{(1+y^2)^2}$；

③ $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = 0.$

6. 求下列函数的全微分.

$(1)\, z = \arctan\dfrac{x+y}{x-y}$

【解析】① $\dfrac{\partial z}{\partial x} = \dfrac{1}{1+\left(\dfrac{x+y}{x-y}\right)^2} \cdot \dfrac{(x-y)-(x+y)}{(x-y)^2} = \dfrac{-y}{x^2+y^2}$， $\dfrac{\partial z}{\partial y} = \dfrac{x}{x^2+y^2}$；

② $dz = \dfrac{-ydx + xdy}{x^2+y^2}$.

$(2)\, u = \ln\left(x^2 - y^2 + e^z\right)$

【解析】① $\dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2 - y^2 + e^z}$， $\dfrac{\partial z}{\partial y} = \dfrac{-2y}{x^2 - y^2 + e^z}$， $\dfrac{\partial z}{\partial z} = \dfrac{e^z}{x^2 - y^2 + e^z}$；

② $dz = \dfrac{2xdx - 2ydy + e^z dz}{x^2 - y^2 + e^z}$.

7. 设 $u = \left(\dfrac{y}{z}\right)^{\frac{1}{x}}$，求 $du(1,1,1)$.

【解析】① $\dfrac{\partial u}{\partial x} = \left(\dfrac{y}{z}\right)^{\frac{1}{x}} \cdot \ln\dfrac{y}{x} \cdot \left(-\dfrac{1}{x^2}\right)$， $\left.\dfrac{\partial u}{\partial x}\right|_{(1,1,1)} = 0$；

$\dfrac{\partial u}{\partial y} = \dfrac{1}{x} \cdot \left(\dfrac{y}{z}\right)^{\frac{1}{x}-1} \cdot \dfrac{1}{z}$， $\left.\dfrac{\partial u}{\partial y}\right|_{(1,1,1)} = 1$；

$\dfrac{\partial u}{\partial z} = \dfrac{1}{x} \cdot \left(\dfrac{y}{z}\right)^{\frac{1}{x}-1} \cdot \left(-\dfrac{y}{z^2}\right)$， $\left.\dfrac{\partial u}{\partial z}\right|_{(1,1,1)} = -1$；

② $du(1,1,1) = dy - dz$.