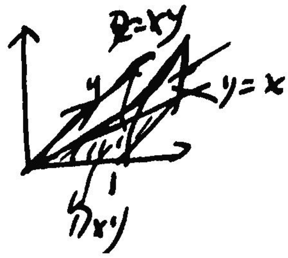
习题 11.3　三重积分

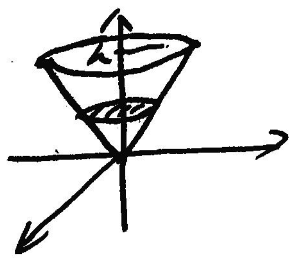
1. 计算下列三重积分:



$(1)\iiint\limits_V xy^2z^3dxdydz$，其中$V$由$z=xy, y=x, x=1$和$z=0$所围成.

【解析】V投影到xoy面的投影区域$D_{xy}:\begin{cases}0\le x\le 1\\0\le y\le x\end{cases}$，$0\le z\le xy$

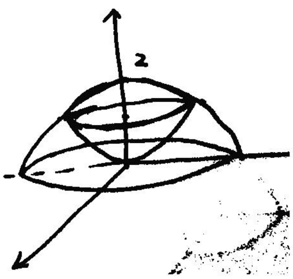
$$原式=\iint\limits_{D_{xy}}dxdy\int_0^{xy}xy^2z^3dz=\frac{1}{4}\iint\limits_{D_{xy}}xy^2\cdot(xy)^4dxdy=\frac{1}{4}\int_0^1dx\int_0^x x^5y^6dy=\frac{1}{364}.$$



$(2)\iiint\limits_V zdxdydz$，其中$V$由$z=\frac{h}{R}\sqrt{x^2+y^2}$与平面$z=h\ (R>0,h>0)$所围成.

【解析】$D_z: x^2+y^2\le\left(\frac{R}{h}z\right)^2$，$0\le z\le h$

$$原式=\int_0^h zdz\iint\limits_{D_z}dxdy=\int_0^h z\cdot\pi\left(\frac{R}{h}z\right)^2dz=\pi\cdot\frac{R^2}{h^2}\int_0^h z^3dz=\frac{\pi}{4}R^2h^2$$

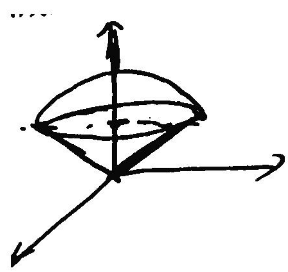


$(3)\iiint\limits_V zdxdydz$，其中$V$由球面$z=\sqrt{4-x^2-y^2}$与抛物面$z=\frac{1}{3}(x^2+y^2)$所围成.

【解析】$D_{xy}: x^2+y^2\le 3$，利用柱坐标方程计算；$x=r\cos\theta, y=r\sin\theta, z=z$

则$V': 0\le\theta\le 2\pi, 0\le r\le\sqrt{3}, \frac{1}{3}r^2\le z\le\sqrt{4-r^2}$；

$$原式=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}rdr\int_{\frac{1}{3}r^2}^{\sqrt{4-r^2}}zdz=2\pi\int_0^{\sqrt{3}}r\cdot\frac{1}{2}\left[4-r^2-\frac{4}{9}r^4\right]dr=\frac{13}{4}\pi$$



$(4)\iiint\limits_V zdxdydz$，其中$V$由$x^2+y^2+(z-a)^2\le a^2$及$x^2+y^2\le z^2$所围成.

【解析】$\begin{cases}x^2+y^2+(z-a)^2=a^2\\x^2+y^2=z^2\end{cases}\Rightarrow z=0$或$z=a$，$D_{xy}: x^2+y^2\le a^2$；

选择球坐标方程：$x=r\sin\varphi\cos\theta, y=r\sin\varphi\sin\theta, z=r\cos\varphi$，

得$0\le\theta\le 2\pi, 0\le\varphi\le\frac{\pi}{4}, 0\le r\le 2a\cos\varphi$，

$$原式=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^{2a\cos\theta}(r\cos\varphi)(r^2\sin\varphi)dr$$

$$=2\pi\int_0^{\frac{\pi}{4}}\cos\varphi\sin\varphi d\varphi\int_0^{2a\cos\theta}r^3dr=2\pi\int_0^{\frac{\pi}{4}}4a^4\cos^5\varphi\sin\varphi d\varphi=\frac{7}{6}\pi a^4$$

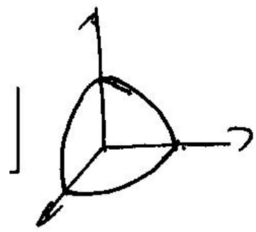
(5) $\iiint\limits_{V}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dxdydz$，其中 V 为椭球体 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$.

【解析】利用广义球坐标方程：$x=ra\sin\varphi\cos\theta, y=rb\sin\varphi\sin\theta, z=rc\cos\varphi$，得

$$0\le\theta\le 2\pi, 0\le\varphi\le\pi, 0\le r\le 1, dxdydz=abcr^2\sin\varphi ,$$

原式 $=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^1\sqrt{1-r^2}\cdot abr^2\sin\varphi dr=\frac{\pi^2}{4}abc$

2. 设物体占有的空间区域为球面 $x^2+y^2+z^2=1$ 及三个坐标面在第一卦限内的部分，点 (x,y,z) 处的体密度为 $\rho(x,y,z)=xyz$，求物体的质量.



【解析】$M=\iiint\limits_{V}xyzdV$，其中 $V: x^2+y^2+z^2\le 1, x\ge 0, y\ge 0, z\ge 0$；

选择球坐标方程：$x=r\sin\varphi\cos\theta, y=r\sin\varphi\sin\theta, z=r\cos\varphi$，

得 $0\le\theta\le\frac{\pi}{2}, 0\le\varphi\le\frac{\pi}{2}, 0\le r\le 1$，即

原式 $=\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^1 r\sin\varphi\cos\theta\cdot r\sin\varphi\sin\theta\cdot r^2\sin\varphi dr$

$=\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta\int_0^{\frac{\pi}{2}}\sin^3\varphi d\varphi\int_0^1 r^5dr=\frac{1}{2}\times\frac{1}{4}\times\frac{1}{6}=\frac{1}{48}$