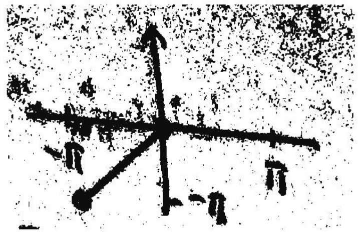
习题 13.4　Fourier 级数

1. 设 $f(x)$ 是以 2π 为周期的周期函数，且

$$f(x)=\begin{cases} x & -\pi < x < 0; \\ 0 & 0 \le x \le \pi. \end{cases}$$

将 $f(x)$ 展开成以 2π 为周期的 Fourier 级数.



【解析】（1）间断点 $x = k\pi(k=\pm1,\pm3,\pm5,\cdots)$ 处收敛于 $-\dfrac{\pi}{2}$；

（2）$x \ne k\pi(k=\pm1,\pm3,\pm5,\cdots)$ 收敛于 $f(x)$；

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}x\,dx=-\frac{\pi}{2},$$

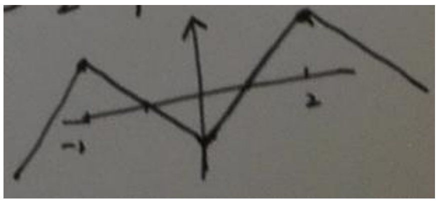
$$a_n=\frac{1}{\pi}\int_{-\pi}^{0}x\cos nx\,dx=\frac{1}{\pi n}\int_{-\pi}^{0}x\,d(\sin nx)=\frac{1-\cos n\pi}{\pi n^2}=\frac{1-(-1)^n}{\pi n^2}=\begin{cases}\dfrac{2}{\pi n^2}, & n=1,3,5,\cdots\\ 0, & n=2,4,6,\cdots\end{cases}$$

,

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}x\sin nx\,dx=-\frac{1}{\pi n}\int_{-\pi}^{0}x\,d(\cos nx)=-\frac{1}{n}\cos n\pi=\frac{(-1)^{n+1}}{n},$$

$$f(x)=-\frac{\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{\pi n^2}\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\right].$$

2. 将函数 $f(x)=x-1(0\le x\le 2)$ 展开成以 4 为周期的余弦级数.



【解析】所给函数定义在半周期上，因此作偶延拓及周期延拓展成余弦函数.

$$a_0=\frac{2}{l}\int_0^l f(x)dx=\int_0^2(x-1)dx=0;$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx=\int_0^2(x-1)\cos\frac{n\pi x}{2}dx=\frac{4}{n^2\pi^2}[(-1)^n-1]=\begin{cases}-\dfrac{8}{\pi^2 n^2}, & n=1,3,5,\cdots\\ 0, & n=2,4,6,\cdots\end{cases}$$

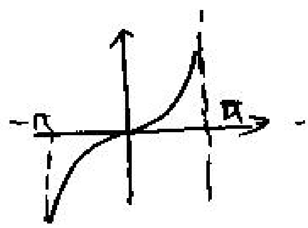
;

则 $f(x)=-\dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}\qquad x\in[0,2]$

3. 将函数 $f(x)=x^2(0\le x\le\pi)$ 分别展开成正弦级数和余弦级数.

【解析】（1）正弦级数

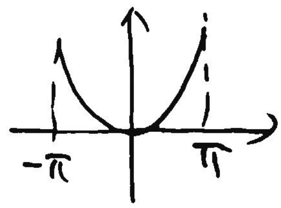
$a_n=0\quad(n=0,1,2,\cdots)$；



$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx = \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}[(-1)^n - 1]\ ;$$

所以 $f(x) = \sum_{n=1}^{\infty}\left\{\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}[(-1)^n - 1]\right\}\cdot \sin nx \qquad x \in (0,\pi)$；

（2）余弦级数



$b_n = 0 \quad (n = 0,1,2,\cdots)$；

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx = (-1)^n \frac{4}{n^2}\ ;$$

所以 $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cdot \cos nx \qquad x \in [0,\pi]$