Category Theory

Projektarbejde i Datalogi 10ECTS (E24.520202U002.A)

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This report is written for the course on Category Theory undertaken as a project module: “Projektarbejde i Datalogi 10ECTS (E24.520202U002.A)”, under the supervision of Lars Birkedal and Alejandro Aguirre in Aarhus University as part of a MSc Computer Science program.

Contents

§1 Version2

This section onwards departs from the previous due to new understanding of proof writing in category theory, particularly with UMPs.

<u>Exercise 3, chapter 4:</u> regular mono stable under pullback

Given h', e' is a pullback of h, e

$$\begin{aligned} \text{Pullback}_{h,e}(e', h') \Leftrightarrow he' = eh' \wedge & \quad (1.1)\\ \forall f, g. hf = eg \Rightarrow & \quad (1.2)\\ \exists! u. h'u = g \wedge e'u = f & \quad (1.3) \end{aligned}$$

Given e is a monomorphism

$$\begin{aligned} \text{isMono}(e) \Leftrightarrow \forall g, h. eg = eh \Rightarrow & \quad (2.1)\\ g = h & \quad (2.2) \end{aligned}$$

We shown e' is a monomorphism

$$\begin{aligned} \text{isMono}(e') \Leftrightarrow \forall u_1, u_2. e'u_1 = e'u_2 \Rightarrow & \quad (3.1)\\ u_1 = u_2 & \quad (3.2) \end{aligned}$$

By Equation 1.1 and Equation 3.1 above, we have

$$(eh')u_1 = h(e'u_1) = h(e'u_2) = eh'u_2$$

This satisfies Equation 2.1 by $g = h'u_1, h = h'u_2$ thus quotienting them

$$h'u_1 = h'u_2$$

Both u_1, u_2 now satisfies Equation 1.3 and by uniqueness we get the consequent of Equation 3.2

$$u_1 = u_2$$

$\therefore e'$ is a mono. □

Given e is an equalizer; we assume for some a, a'

$$\begin{aligned} \text{Equalizer}_{a,a'}(e) \Leftrightarrow ae = a'e \wedge & \quad (4.1)\\ \forall z. az = a'z \Rightarrow & \quad (4.2)\\ \exists! u. eu = z & \quad (4.3) \end{aligned}$$

We show equalizers are stable under pullback; e' is an equalizer on ah and $a'h$.

$$\begin{aligned} \text{Equalizer}_{ah,a'h}(e') \Leftrightarrow ahe' = a'he' \wedge & \quad (5.1)\\ \forall z. ahz = a'hz \Rightarrow & \quad (5.2)\\ \exists! u. e'u = z & \quad (5.3) \end{aligned}$$

Equation 5.1 derived by Equation 1.1 and Equation 4.1

$$a(he') = (ae)h' = a'(eh') = a'he'$$

Supposing Equation 5.2 for $z = f$

$$\forall f. ahf = a'hf$$

this satisfies Equation 4.3 giving us some unique g by Equation 4.3

$$\exists! g . eg = hf$$

this satisfies Equation 1.3 giving us some unique u_1 by Equation 1.3

$$\exists! u_1 . h' u_1 = g \wedge e' u_1 = f$$

this satisfies 3.3 but stronger, thus not necessarily unique, assuming another morphism u_2 exists

$$\forall u_2 . e' u_2 = f$$

both u_1, u_2 satisfies Equation 3.2 thus quotienting them, specifically:

$$e' u_1 = e' u_2 \Rightarrow u_1 = u_2$$

Thus the consequent of Equation 5.3 holds

$$\exists! u . e' u = f$$

$\therefore e'$ is an equalizer on ah and $a'h$. ■

$$\dfrac{\dfrac{\dfrac{\text{Pullback}_{h,e}(e', h')}{1.1} \quad \dfrac{}{3.1_{u_1,u_2}} \quad \dfrac{\text{isMono}(e)}{2.1_{h'u_1,h'u_2}} \quad \dfrac{\text{Pullback}_{h,e}(e', h')}{1.3}}{3.1_{u_1,u_2} \Rightarrow 3.2}}{\text{isMono}(e')}$$

$$\dfrac{\dfrac{\dfrac{\text{Pullback}_{h,e}(e', h')}{1.1} \quad \dfrac{\text{Equalizer}_{a,a'}(e)}{2.1}}{5.1} \quad \dfrac{\dfrac{\dfrac{\text{Equalizer}_{a,a'}(e) \quad \dfrac{}{5.2_f}}{4.2_{hf} \Rightarrow 4.3_g} \quad \text{Pullback}_{h,e}(e', h')}{1.2_{f,g} \Rightarrow 1.3_{u_1}} \quad \dfrac{}{u_2} \quad \dfrac{\text{isMono}(e')}{3.1_{u_1,u_2} \Rightarrow 3.2}}{5.2_f \Rightarrow 5.3_u}}{\text{Equalizer}_{ah,a'h}(e')}$$

<u>EXERCISE DUAL:</u> regular epi stable under pushout

§2 Appendix

$$
\begin{aligned}
\text{isCategory}(\mathcal{C}) = & \quad \forall A, B, C \in \mathbf{Ob}_{\mathcal{C}}, f \in \mathbf{Hom}_{\mathcal{C}}(A, B), g \in \mathbf{Hom}_{\mathcal{C}}(B, C). g \circ f \in \mathbf{Hom}_{\mathcal{C}}(A, C) \\
& \wedge \forall X \in \mathbf{Ob}_{\mathcal{C}}. \exists! 1_X \in \mathbf{Hom}(X, X) \\
& \wedge \forall h, g, f. h \circ (g \circ f) = (h \circ g) \circ f \\
& \wedge \forall A, B \in \mathbf{Ob}_{\mathcal{C}}, f \in \mathbf{Hom}_{\mathcal{C}}(A, B). f \circ 1_A = f = 1_B \circ f \\
\text{isFunctor}(F : C \to D) = & \quad \forall f \in \mathbf{Hom}_C(A, B). F(f) \in \mathbf{Hom}_D(F(A), F(B)) \\
& \wedge \forall 1_X \in \mathbf{Hom}_C(X, X). F(1_X) = 1_{F(X)} \\
& \wedge \forall g, f \in \mathbf{Hom}_C. F(g \circ f) = F(g) \circ F(f) \\
\text{isIso}(f : A \to B) = & \quad \exists! g. g \circ f = 1_A \wedge f \circ g = 1_B \\
A \cong B = & \quad \exists (f : A \to B).\ \text{isIso}(f) \\
C \times D = & \quad (\mathbf{Ob}_C \times \mathbf{Ob}_D, [(C_1, D_1), (C_2, D_2) \mapsto \mathbf{Hom}_C(C_1, C_2) \times \mathbf{Hom}_D(D_1, D_2)]) \\
C^{\text{op}} = & \quad (\mathbf{Ob}_C, [A, B \mapsto \{f^{\text{op}} : A \to B \mid f \in \mathbf{Hom}_C(B, A)\}]) \\
C^{\to} = & \quad (\mathbf{Hom}_C, [f, f' \mapsto \{(g, g') \mid g' \circ f = f' \circ g\}]) \\
C / A = & \quad (\{f \mid f \in \mathbf{Hom}_C(X, A)\}, [f, f' \mapsto \mathbf{Hom}_C(\mathbf{dom}(f), \mathbf{dom}(f'))]) \\
C \setminus A = & \quad (\{f \mid f \in \mathbf{Hom}_C(A, X)\}, [f, f' \mapsto \mathbf{Hom}_C(\mathbf{cod}(f), \mathbf{cod}(f'))]) \\
\text{isMono}(f) = & \quad \forall g, h. f \circ g = f \circ h \Rightarrow g = h \\
\text{isEpi}(f) = & \quad \forall g, h. g \circ f = h \circ f \Rightarrow g = h \\
\text{isSplitMono}(m, s) = & \quad s \circ m = 1_{\mathbf{dom}(m)} \\
\text{isSplitEpi}(e, s) = & \quad e \circ s = 1_{\mathbf{cod}(e)} \\
\text{areIso}(f, g) = & \quad \text{isSplitEpi}(f, g) \wedge \text{isSplitMono}(f, g) \\
\text{isProjective}(P) = & \quad \forall e : E \twoheadrightarrow X, f : P \to X. \exists \overline{f} : P \to E. e \circ \overline{f} = f \\
\\
\text{UMP}_{\text{freemonoid}}\left(|\overline{f}|\right) = & \quad \forall i, f. \exists! \overline{f}.\ |\overline{f}| \circ i = f \\
\text{UMP}_{\text{terminal}}(0_X) = & \quad \forall X. \exists! 0. 0_X \in \mathbf{Hom}(0, X) \\
\text{UMP}_{\text{initial}}(1_X) = & \quad \forall X. \exists! 1. 1_X \in \mathbf{Hom}(X, 1) \\
\text{UMP}_{\text{product}}(p_1, p_2) = & \quad \forall x_1, x_2. \exists! u. x_1 = p_1 \circ u \wedge x_2 = p_2 \circ u \\
\text{UMP}_{\text{coproduct}}(q_1, q_2) = & \quad \forall x_1, x_2. \exists! u. x_1 = u \circ q_1 \wedge x_2 = u \circ q_2 \\
\text{UMP}_{\text{equalizer}}(e, f, g) = & \quad \forall z. (f \circ z = g \circ z) \wedge \exists! u. e \circ u = z \\
\text{UMP}_{\text{coequalizer}}(q, f, g) = & \quad \forall z. (z \circ f = z \circ g) \wedge \exists! u. u \circ q = z \\
\text{UMP}_{\text{pullback}}(p_1, p_2, f, g) = & \quad (f \circ p_1 = g \circ p_2) \\
& \wedge \forall z_1, z_2. \exists! u. z_1 = p_1 \circ u \wedge z_2 = p_2 \circ u \\
\text{UMP}_{\text{limit}}(\{p_i\}, D) = & \quad \forall \{c_j\}. \exists! u. \forall j. p_j \circ u = c_j \\
\text{isExponential}(C^B, \varepsilon) = & \quad \forall A, (f : A \times B \to C). \exists! \left(\tilde{f} : A \to C^B\right). \varepsilon \circ \left(\tilde{f} \times 1_B\right) = \bar{\tilde{f}} = f \\
\text{isCCC}(\mathbf{Ob}, \mathbf{Hom}) = & \quad \text{isCategory}(\mathbf{Ob}, \mathbf{Hom}) \\
& \wedge \forall A, B. \exists! A \times B. \text{UMP}_{\text{product}}(p_1 : A \times B \to A, p_2 : A \times B \to B) \\
& \wedge \forall B, C.\ \text{isExponential}(C^B, \varepsilon)
\end{aligned}
$$

TODO LIST:

list of functors: $|-|, (-)^A, (-)^{\mathrm{op}}, \mathbf{Hom}(A,-), \mathbf{Hom}(-,A)$

todo in chapter4 notes
- $(f)^A = \widetilde{f \circ \varepsilon}$
- define Sub and Cone category

$$\begin{tikzcd}
X \arrow[r, dashed, "u"] \arrow[rr, bend left] \arrow[dr] & E' \arrow[r, "h'"] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, tail, "e"'] & \\
& C \arrow[r, "h"] & A \arrow[r, shift left, "a'"] \arrow[r, shift right, "a"'] & B
\end{tikzcd}$$

Lemma 1: e' is mono

$$\dfrac{\forall f g. e' f = e' g \qquad \dfrac{}{u_1}\,\forall X \to E \qquad \dfrac{}{u_2}\,\forall X \to E}{e' u_1 = e' u_2}\,\text{mono-antecedent}$$

might be an error here, need to sh ow h' is a mono, then elim $\mathrm{Mono}(eh')$*, quantifiers can't just be substituted by arbitrary composites*

$$\dfrac{\dfrac{\mathrm{Mono}(e) \qquad h' u_1 \qquad h' u_2}{e h' u_1 = e h' u_2 \Rightarrow h' u_1 = h' u_2}\,\text{mono-elim} \qquad \dfrac{\text{Pullback }(e', h')}{h e' = e h'}\,\text{pullback-elim}^{\text{diagram}}}{h e' u_1 = h e' u_2 \Rightarrow h' u_1 = h' u_2}\,=$$

$$\dfrac{\dfrac{e' u_1 = e' u_2 \qquad h e' u_1 = h e' u_2 \Rightarrow h' u_1 = h' u_2}{h e' u_1 = h e' u_1 \Rightarrow h' u_1 = h' u_2}\,=}{h' u_1 = h' u_2}\,\Rightarrow\text{-elim}$$

$$\dfrac{\dfrac{\text{Pullback }(e', h') \qquad u_1 \qquad u_2}{e' u_1 = e' u_2 \wedge h' u_1 = h' u_2 \Rightarrow u_1 = u_2}\,\text{pullback-elim}^{\text{unique}} \qquad e' u_1 = e' u_2 \qquad h' u_1 = h' u_2}{u_1 = u_2}\,\Rightarrow\text{-elim}$$

$$\dfrac{\dfrac{e' u_1 = e' u_2 \qquad u_1 = u_2}{e' u_1 = e' u_2 \Rightarrow u_1 = u_2}\,\Rightarrow\text{-intro}}{\mathrm{Mono}(e')}\,\text{mono-intro}$$

Lemma 2: e' is regular / equalizer

$$\dfrac{\dfrac{\dfrac{\mathrm{Equalizer}(e) \qquad \dfrac{}{a}\,\forall A \to B \qquad \dfrac{}{a'}\,\forall A \to B}{a e = a' e}\,\text{equalizer-elim}^{\text{diagram}}}{a e h' = a' e h'}\,= \qquad \dfrac{\text{Pullback }(e', h')}{h e' = e h'}\,\text{pullback-elim}^{\text{diagram}}}{a h e' = a' h e'}\,=$$

$$\dfrac{\dfrac{\text{Mono}(e') \qquad u_1 \qquad u_2}{e'u_1 = e'u_2 \Rightarrow u_1 = u_2}\text{mono-elim}}{he'u_1 = he'u_2 \Rightarrow u_1 = u_2} =$$

$$\dfrac{he'u_1 = he'u_2 \Rightarrow u_1 = u_2 \qquad ahe' = a'he'}{\text{Equalizer}(he')}\text{equalizer-intro}$$

Proof: he' is a regular mono

$$\dfrac{\dfrac{he'u_1 = he'u_2 \Rightarrow u_1 = u_2}{\text{Mono}(he')}\text{mono-intro} \qquad \text{Equalizer}(he')}{\text{RegularMono}(he')}\text{regular-mono-intro}$$

■

assumptions

1. composing a morphism on both sides of an equality remains equal
   - i.e. $f = g, h \vdash hf = hg$ and $f = g, h \vdash fh = gh$
2. substituting a quantifier argument with another quantifier argument but composed is the same
   - i.e. $\text{Mono}(e), ab, c \vdash eab = ec \Rightarrow ab = c$ where a is a quantified argument or fixed morphism and b otherwise
   - which should be true by definition of existence of composition morphism in a category
   - this might be wrong

Bibliography