Category Theory

Projektarbejde i Datalogi 10ECTS (E24.520202U002.A)

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This report is written for the course on Category Theory undertaken as a project module: “Projektarbejde i Datalogi 10ECTS (E24.520202U002.A)”, under the supervision of Lars Birkedal and Alejandro Aguirre in Aarhus University as part of a MSc Computer Science program.

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§1 Categories

1.1 Notes

Definition 1.1 (*Category*) [1, definition 1.1] A category is a collection of objects and morphisms between them satisfying some properties.

$$C = (\mathbf{Ob}_C, \mathbf{Hom}_C)$$

$$A, B, C, \ldots \in \mathbf{Ob}_C \qquad \begin{matrix} {\scriptstyle f:A\to B} \\ {\scriptstyle g:B\to C} \\ \vdots \end{matrix} \in \mathbf{Hom}_C$$

PROPERTIES	DEFINITION
composition	$\forall f : A \to B, g : B \to C. \exists g \circ f : A \to C$
identity	$\exists! 1_A : A \to A$
associativity	$h \circ (g \circ f) = (h \circ g) \circ f$
unital	$f \circ 1_A = f = 1_B \circ f$

Definition 1.2 (*Functor*) [1, definition 1.2] A functor is a structure preserving map from one category to another.

$$F : C \to D$$

STRUCTURE	DEFINITION
domains	$F(f) : F(A) \to F(B)$
identity	$F(1_A) = 1_{F(A)}$
composition	$F(g \circ f) = F(g) \circ F(f)$

Definition 1.3 (*Isomorphism*) [1, definition 1.3] An isomorphism is a morphism that has an inverse. The objects on them are isomorphic.

$$\text{isIso}(f) = \exists! g. g \circ f = 1_A \wedge f \circ g = 1_B$$
$$A \cong B = \exists f : A \to B.\ \text{isIso}(f)$$

Definition 1.4 (*Constructions on Categories*) [1, definition 1.6] Here are categories that are constructed out of prior categories.

CATEGORY	OBJECTS	MORPHISMS
product $A \times B$	$\mathbf{Ob}_C \times \mathbf{Ob}_B$	$\mathbf{Hom}_C \times \mathbf{Hom}_B$
dual C^{op}	$\mathbf{Ob}_C$	$A, B \mapsto \mathbf{Hom}_C(B, A)$
arrow $C^{\to}$	$\mathbf{Hom}_C$	$f, f' \mapsto \{(g, g') \mid g' \circ f = f' \circ g\}$
slice C/A	$\{X \mid \mathbf{Hom}_C(X, A) \neq \emptyset\}$	$\mathbf{Hom}_C$
co-slice $C \setminus A$	$\{X \mid \mathbf{Hom}_C(A, X) \neq \emptyset\}$	$\mathbf{Hom}_C$

Definition 1.5 (*Universal Mapping Property: Free Monoids and Categories*) [1, definition 1.7,1.9,1.10]

- Alphabet: $A = \{a_0, a_1, a_2, \ldots\}$

- Words: concatenations of a_i form words. along with $-$; an empty word, forms a monoid
- Free Monoid: elements of A freely generate the monoid $M(A)$
- Forgetful Functor: $|-|$: Monoids $\rightarrow$ Sets, gives an underlying alphabet of a monoid

UMP	DEFINITION
existence	$\forall i, f. \exists! \overline{f}.\ |\overline{f}| \circ i = f$
uniqueness	$|\overline{f_a}| \circ i = f \wedge$ $|\overline{f_b}| \circ i = f \rightarrow$ $\overline{f_a} = \overline{f_b}$

$$
\begin{tikzcd}
M(A) \arrow[r, dashed, "\overline{f}"] \arrow[d, Rightarrow] & N \arrow[d, Rightarrow] & \mathbf{Mon} \\
|M(A)| \arrow[r, "|\overline{f}|"] & |N| & \\
A \arrow[u, "i"] \arrow[ur, "f"] & & \mathbf{Sets}
\end{tikzcd}
$$

- No Junk: all words are concatenations of A; implied by uniqueness
- No Noise: equality of words by equality of alphabets; implied by existence
- Free Category: if A are the set of edges of a directed graph, the words are morphisms of a category excluding identity morphisms

Definition 1.6 (*Category Size*) [1, definition 1.11,1.12] A size of a category depends of its consituents are sets

SIZE	DEFINITION
small	$\mathbf{Ob}_C, \mathbf{Hom}_C$ are sets
locally small	$\mathbf{Hom}_C(A, B)$ are sets
large	otherwise

1.2 Mandatory Exercises

<u>Exercise 1:</u> Let T_0 send a set X to its powerset $\mathcal{P}(X)$...

We show that $T = (T_0, T_1)$ is a functor by Definition 1.2

STRUCTURE	DEFINITION
domains	$T_1(f) = [A \mapsto \{f(x) \mid x \in A\}]$
identity	$T_1(1_A) = T_1(\text{id}) = \text{id} = 1_{T_0(A)}$
composition	$T_1(g \circ f) = T_1(g) \circ T_1(f)$

Proof *(composition)*

$$
\begin{aligned}
T_1(g \circ f) &= [A \mapsto \{g(f(x)) \mid x \in A\}] \\
&= [A \mapsto \{g(y) \mid y \in \{f(x) \mid x \in A\}\}] \\
&= [A \mapsto \{g(y) \mid y \in T_1(f)(A)\}] \\
&= [A \mapsto [B \mapsto \{g(y) \mid y \in B\}](T_1(f)(A))] \\
&= [A \mapsto T_1(g)(T_1(f)(A))] \\
&= [A \mapsto (T_1(g) \circ T_1(f))(A)] \\
&= T_1(g) \circ T_1(f)
\end{aligned}
$$

$\therefore T$ is a functor from **Sets** to **Sets** □

EXERCISE 2: Define the category $\mathbb{K}$ as follows ...

$\mathbb{K}$ morphisms are left total relations for objects in **Sets**. The relations are encoded as functions mapping an element of X to a subset of Y. Thus we can show $\mathbb{K}$ is a category by Definition 1.1.

PROPERTIES	DEFINITION
composition	$g \circ f = \cup_{y \in f(x)} g(y)$
identity	$1_X = [x \mapsto \{x\}]$
associativity	$h \circ (g \circ f) = (h \circ g) \circ f$
unital	$1_{T(Y)} \circ f \circ 1_X = f$

Proof *(associativity)*

$$
\begin{aligned}
h \circ (g \circ f) &= \left[x \mapsto \cup_{z \in (g \circ f)(x)} h(z)\right] \\
&= \left[x \mapsto \cup_{z \in \cup_{y \in f(x)} g(y)} h(z)\right] \\
&= \left[x \mapsto \cup_{y \in f(x)} \cup_{z \in g(y)} h(z)\right] \\
&= \left[x \mapsto \cup_{y \in f(x)} (h \circ g)(y)\right] \\
&= [x \mapsto ((h \circ g) \circ f)(x)] \\
&= (h \circ g) \circ f
\end{aligned}
$$

□

Proof *(unital)*

$$
\begin{aligned}
1_{T(Y)} \circ f \circ 1_X &= \left[x \mapsto \cup_{x' \in 1_{X(x)}} \cup_{y \in f(x')} 1_{T(Y)}(y)\right] \\
&= \left[x \mapsto \cup_{x' \in \{x\}} \cup_{y \in f(x')} [y \mapsto \{y\}](y)\right] \\
&= \left[x \mapsto \cup_{x' \in \{x\}} \cup_{y \in f(x')} \{y\}\right] \\
&= \left[x \mapsto \cup_{x' \in 1_{X(x)}} f(x')\right] \\
&= \left[x \mapsto \cup_{x' \in \{x\}} f(x')\right] \\
&= [x \mapsto f(x)] = f
\end{aligned}
$$

□

Let F, G be an isomorphism by Definition 1.3

$$F : \mathbb{K} \to \mathbf{Rel}$$

$$F(X) = X$$

$$F(f : X \to T(Y)) = \{(x, y) \mid y \in f(x)\}$$

$$G : \mathbf{Rel} \to \mathbb{K}$$

$$G(X) = X$$

$$G(R) = [x \mapsto \{y \mid (x, y) \in R\}]$$

Proof *($G \circ F = 1_{\mathbb{K}}$)*

Proof *($F \circ G = 1_{\mathbf{Rel}}$)*

$$\begin{aligned}&(G \circ F)(f)\\&= G(\{(x, y) \mid y \in f(x)\})\\&= [x \mapsto \{y \mid (x, y) \in \{(x, y) \mid y \in f(x)\}\}]\\&= [x \mapsto \{y \mid y \in f(x)\}]\\&= [x \mapsto f(x)] = f\end{aligned}$$

□

$$\begin{aligned}&(F \circ G)(R)\\&= F([x \mapsto \{y \mid (x, y) \in R\}])\\&= \{(x, y) \mid y \in [x \mapsto \{y \mid (x, y) \in R\}](x)\}\\&= \{(x, y) \mid y \in \{y \mid (x, y) \in R\}\}\\&= \{(x, y) \mid (x, y) \in R)\} = R\end{aligned}$$

□

$\therefore$ $\mathbb{K}$ is a category that is isomorphic to $\mathbf{Rel}$ via F, G.

Exercise 3: Let $\mathbb{C}$ be a category with binary products ...

Remark: epi, mono, showing uniqueness of morphism is not explored in SA chapter 1, we will revisit this exercise in the future.

1.3 Relevant Exercises

Exercise 1: The objects of $\mathbf{Rel}$ are sets ...

We show $\mathbf{Rel}$ is a category by Definition 1.1

PROPERTIES	DEFINITION
composition	$(S \circ R) = \{(a, c) \mid \exists b.(a, b) \in R \wedge (b, c) \in S\}$
identity	$1_A = \{(a, a) \mid a \in A\}$
associativity	$(S \circ R) \circ Q = S \circ (R \circ Q)$
unital	$1_B \circ S \circ 1_A = S$

Proof *(associativity)*

$$\begin{aligned}(S \circ R) \circ Q &= \{(b, d) \mid \exists c.(b, c) \in R \wedge (c, d) \in S\} \circ Q\\&= \{(a, d) \mid \exists b.(a, b) \in Q \wedge (b, d) \in \{(b, c) \mid \exists c.(b, c) \in R \wedge (c, d) \in S\}\}\\&= \{(a, d) \mid \exists b.(a, b) \in Q \wedge \exists c.(b, c) \in R \wedge (c, d) \in S\}\\&= \{(a, d) \mid \exists c.(a, c) \in \{(a, c) \mid \exists b.(a, b) \in Q \wedge (b, c) \in R\} \wedge (c, d) \in S\}\\&= \{(a, d) \mid \exists c.(a, c) \in (R \circ Q) \wedge (c, d) \in S\}\\&= S \circ (R \circ Q)\end{aligned}$$

□

Proof *(unital)*

$$\begin{aligned}1_B \circ S \circ 1_A &= 1_B \circ \{(a, b) \mid \exists a.(a, a) \in 1_A \wedge (a, b) \in S\}\\&= 1_B \circ \{(a, b) \mid \exists a.(a, b) \in S\}\\&= 1_B \circ S\\&= \{(a, b) \mid \exists b.(a, b) \in S \wedge (b, b) \in 1_B\}\\&= \{(a, b) \mid \exists b.(a, b) \in S\}\\&= S\end{aligned}$$

□

$\therefore$ **Rel** is a category

Let G be a functor such that Definition 1.2 holds.

STRUCTURE	DEFINITION
domains	$G(f) = \{(a, f(a))\|a \in A\}$
identity	$G(1_A) = \{(a, 1_A(a))\|a \in A\} = 1_{G(A)}$
composition	$G(g \circ f) = G(g) \circ G(f)$

Proof *(composition)*

$$\begin{aligned}
G(g \circ f) &= \{(a, (g \circ f)(a)) \mid a \in A\} \\
&= \{(a, g(b)) \mid f(a) = b \wedge a \in A\} \\
&= \{(a, c) | f(a) = b \wedge g(b) = c \wedge a \in A\} \\
&= \{(a, c) | (a, b) \in \{(a, f(a)) | a \in A\} \wedge (b, c) \in \{(b, g(b)) | b \in B\}\} \\
&= \{(a, c) | \ (a, b) \in G(f) \wedge (b, c) \in G(g)\} \\
&= G(g) \circ G(f)
\end{aligned}$$

$\therefore G : \mathbf{Sets} \to \mathbf{Rel}$ □

Let C be a functor as Definition 1.2 holds.

STRUCTURE	DEFINITION
domains	$C(S) = \{(b, a)\|(a, b) \in S\}$
identity	$C(1_A) = \{(a, a)\|(a, a) \in 1_A\} = 1_{C(A)}$
composition	$C(S \circ R) = C(S) \circ C(R)$

Proof *(composition)*

$$\begin{aligned}
C(S \circ R) &= \{(c, a) | (a, c) \in S \circ R\} \\
&= \{(c, a) | \exists b. (a, b) \in R \wedge (b, c) \in S\} \\
&= \{(c, a) | \exists b. (c, b) \in C \wedge (b, a) \in C(R)\} \\
&= C(S) \circ C(R)
\end{aligned}$$

$\therefore C : \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Rel}$ □

Exercise 2: Consider the following isomorphisms of categories and determine which hold.

ISOMORPHISM	HOLDS
$\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$	yes by $C \circ C = 1_{\mathbf{Rel}^{\mathrm{op}}} = 1_{\mathbf{Rel}}$
$\mathbf{Sets} \cong \mathbf{Sets}^{\mathrm{op}}$	no since not all functions are bijective; has inverses
$P(X)^{\mathrm{op}} \cong P(X)$	yes as $P(X)$ is a subcategory of **Rel** thus C is proof

Exercise 3: Show that...

Proof *(isomorphisms in* $\mathbf{Sets}$ *are bijections)*

$$\begin{aligned} A \cong B &\leftrightarrow \exists f, g. g \circ f = 1_A \wedge f \circ g = 1_B \\ &\leftrightarrow \exists f, g. \forall x, y. g(f(x)) = x \wedge f(g(y)) = y \\ &\leftrightarrow \text{isBijective}(f) \end{aligned}$$

□

Proof *(isomorphisms in* $\mathbf{Mon}$ *are bijective homomorphisms)*

$$\begin{aligned} A \cong B &\leftrightarrow \exists f, g. g \circ f = 1_A \wedge f \circ g = 1_B \\ &\leftrightarrow \exists f, g. \forall a_0, a_1, b_0, b_1. \\ &g(f(a_0 \times_A a_1)) = g(f(a_0) \times_B f(a_1)) = g(f(a_0)) \times_A g(f(a_1)) = a_0 \times_A a_1 \\ &\wedge f(g(b_0 \times_B b_1)) = f(g(b_0) \times_A g(b_1)) = f(g(b_0)) \times_B f(g(b_1)) = b_0 \times_B b_1 \\ &\leftrightarrow \text{isBijectiveHomomorphism}(f) \end{aligned}$$

□

Proof *(isomorphisms in* $\mathbf{Posets}$ *are not bijective homomorphisms)*

Let $a_0, a_1 \in A$ be a poset where $a_0 \leq a_1$.

Let $b_0, b_1 \in B$ be a poset where there is no ordering between b_0 and b_1.

Even though we can define a homomorphism / bijective function from $[a_i \mapsto b_i]$,

there are no monotone functions / bijective homomorphisms f such that $f(a_0) \leq f(a_1)$ □

<u>Exercise 5:</u> For any category $\boldsymbol{C}$, define a functor U...

Let U be a functor such that Definition 1.2 holds.

STRUCTURE	DEFINITION
domains	$U(f : X \to C) = X$ $U\Big(f \in \mathbf{Hom}_{C/C}(X, Y)\Big) = f$
identity	$U(1_X) = 1_X = 1_{U(X)}$
composition	$U(g \circ f) = g \circ f = U(f) \circ U(g)$

We then define the functor F as follows:

STRUCTURE	DEFINITION
domains	$F(f : X \to C) = f$ $F\Big(g \in \mathbf{Hom}_{C/C}(f, f')\Big) = (g, 1_C)$
identity	$F\big(1_{f : X \to C}\big) = (1_X, 1_C)$
composition	$F(g_2 \circ g_1) = F(g_2) \circ F(g_1)$

Proof *(composition)*

$$
\begin{aligned}
F(g_2 \circ g_1) &= (g_2 \circ g_1, 1_C) \\
&= (g_2 \circ g_1, 1_C \circ 1_C) \\
&= (g_2, 1_C) \circ (g_1, 1_C) = F(g_2) \circ F(g_1)
\end{aligned}
$$

$$
\begin{tikzcd}
X \arrow[r, "g_1"] \arrow[d, "f"'] & Y \arrow[r, "g_2"] \arrow[d, "f'"'] & Z \arrow[d, "f''"'] \\
C \arrow[r, "1_C"] & C \arrow[r, "1_C"] & C
\end{tikzcd}
$$

□

Proof *($\mathbf{dom} \circ F = U$)*

$$
\begin{aligned}
(\mathbf{dom} \circ F)(f) = \mathbf{dom}(f) = X = U(f) \\
(\mathbf{dom} \circ F)(g) = \mathbf{dom}(g, 1_C) = g = U(g)
\end{aligned}
$$

□

<u>Exercise 6:</u> Construct the ‘coslice category’...

We define the co-slice category as before in Definition 1.4.

$$
\begin{aligned}
\mathbf{Ob}_{C \backslash C} &= \{X | \mathbf{Hom}_{\boldsymbol{C}}(C, X) \neq \emptyset\} \\
\mathbf{Hom}_{C \backslash C}(f : C \to X, g : C \to Y) &= \mathbf{Hom}_{\boldsymbol{C}}(X, Y)
\end{aligned}
$$

with the dual operator on slice categories as

$$
\begin{aligned}
&\left(\mathbf{Ob}_{C/C}, \mathbf{Hom}_{C/C}\right)^{\mathrm{op}} \\
&= \left(\left\{f^{-1} : C \to X | f : X \to C \in \mathbf{Ob}_{C/C}\right\}, \mathbf{Hom}_{C/C}\right) \\
&= \left(\mathbf{Ob}_{C \backslash C}, \mathbf{Hom}_{C \backslash C}\right)
\end{aligned}
$$

<u>Exercise 7:</u> Let $2 = \{a, b\}$ be any set with exactly two elements...

Given

$$F(f : X \to 2) = (f^{-1}(a), f^{-1}(b))$$

we can define its inverse as

$$F^{-1}(x_0, x_1) = [x_0 \mapsto a, x_1 \mapsto b]$$

such that

Proof *(inverse)*

$$
\begin{aligned}
(F^{-1} \circ F)(f) &= F^{-1}(f^{-1}(a), f^{-1}(b)) \\
&= [f^{-1}(a) \mapsto a, f^{-1}(b) \mapsto b] \\
&= f
\end{aligned}
$$

$$
\begin{aligned}
(F \circ F^{-1})(x_0, x_1) &= F([x_0 \mapsto a, x_1 \mapsto b]) \\
&= (x_0, x_1)
\end{aligned}
$$

$\therefore$ $\mathbf{Sets}\ /2 \cong \mathbf{Sets} \times \mathbf{Sets}$ □

moreover if we have $1 = \{\star\}$

$$
\begin{aligned}
F(x) &= [x \mapsto \star] \\
F(f : X \to 1) &= f^{-1}(\star)
\end{aligned}
$$

similarly we have

Proof *(inverse)*

$$
\begin{aligned}
(F^{-1} \circ F)(f) &= F^{-1}(f^{-1}(\star)) \\
&= f
\end{aligned}
\qquad\qquad
\begin{aligned}
(F \circ F^{-1})([x \mapsto \star]) &= F(x) \\
&= [x \mapsto \star]
\end{aligned}
$$

$\therefore$ **Sets** $/1 \cong$ **Sets** □

EXERCISE 11: Show that the free monoid functor...

Proof *(from effect)* the functor M holds Definition 1.2 as follows

STRUCTURE	DEFINITION
domains	$M(f) = [x_0 \ldots \mapsto f(x_0) \ldots]$
identity	$M(1_A) = M(\mathrm{id}_{\mathbf{Sets}}) = \mathrm{id}_{\mathbf{Mon}} = 1_{M(A)}$
composition	$M(g \circ f) = M(g) \circ M(f)$

$$
\begin{aligned}
M(g \circ f) &= [x_0 \ldots \mapsto g(f(x_0)) \ldots] \\
&= [y_0 \ldots \mapsto g(y_0) \ldots] \circ [x_0 \ldots \mapsto f(x_0) \ldots] \\
&= M(g) \circ M(f)
\end{aligned}
$$

□

Proof *(from UMP)* Let $\mathrm{UMP} : (X \to |M(X)|) \to (X \to |M(Y)|) \to (M(X) \to M(Y))$, then

$$
M(f : X \to Y) = \mathrm{UMP}(i_X, i_Y \circ f) = \overline{f}
$$

$$
\begin{tikzcd}
M(X) \arrow[r, dashed, "\overline{f}"] \arrow[rr, dashed, bend left, "\overline{g \circ f}"] \arrow[d, Rightarrow] & M(Y) \arrow[r, dashed, "\overline{g}"] \arrow[d, Rightarrow] & M(Z) \arrow[d, Rightarrow] & \mathbf{Mon} \\
{|M(X)|} \arrow[r, "|\overline{f}|"] & {|M(Y)|} \arrow[r, "|\overline{g}|"] & {|M(Z)|} & \\
X \arrow[u, hook, "i_X"'] \arrow[r, "f"] \arrow[ur] & Y \arrow[u, hook, "i_Y"] \arrow[r, "g"] \arrow[ur] & Z \arrow[u, hook, "i_Z"'] & \mathbf{Sets}
\end{tikzcd}
$$

$$
\begin{aligned}
M(g \circ f) &= \mathrm{UMP}(i_X, i_Z \circ g \circ f) \\
&= \overline{g \circ f} \\
\text{(by uniqueness)} &= \overline{g} \circ \overline{f} \\
&= M(g) \circ M(f)
\end{aligned}
$$

□

§2 Abstract Structure

2.1 Notes

Definition 2.1 (*Monomorphisms and Epimorphisms*) [1, definition 2.1]

monomorphisms have unique pre-composites from the same object.

$$\text{isMono}(f : A \rightarrowtail B) = f \circ g = f \circ h \Rightarrow g = h$$

epimorphisms have unique post-composites to the same object

$$\text{isEpi}(f : A \twoheadrightarrow B) = g \circ f = h \circ f \Rightarrow g = h$$

Definition 2.2 (*split mono and epi*) [1, definition 2.7]

Let $e : X \twoheadrightarrow A$, then if

$$e \circ s = 1_A$$

- s is a **section** / **splitting** of e, and is monic
- e is a **retraction** of s, and is epic
- A is a **retract** of X

Let $m : A \rightarrowtail X$, then if

$$s \circ m = 1_A$$

- s is a **section** / **splitting** of m, and is epic
- m is a **retraction** of s, and is monic
- A is a **retract** of X

- if f is a right inverse of g / g is a left inverse of f, then $g \circ f = 1$, then f is a split mono of epic g
- if isomorphic then f, g are both split epi and mono of each other

Theorem 2.3 (*Interpretation of split epis (optional)*) [1, example 2.8]

- an object P is *projective* if for any e, f there is an $\overline{f}$ as follows. $\text{Proj}(e, f) = \overline{f}$

$$\text{isProjective}(P) = \forall e : E \twoheadrightarrow X, f : P \to X . \exists \overline{f} : P \to E . e \circ \overline{f} = f$$

$$\begin{tikzcd}
 & E \arrow[d, two heads, "e"] \\
P \arrow[ur, "\overline{f}"] \arrow[r, "f"] & X
\end{tikzcd}$$

- intuitively e has *lifts* of f; permitting *"more"* arrows; a *"free"* structure
- split epis $e : E \twoheadrightarrow X$ are categorical choice functions by using fibers $e^{-1}(x)$
- axiom of choice implies all sets are projective

Definition 2.4 (*Initial and Terminal Objects and Duality*) [1, definition 2.9, 2.10] a category is initial / terminal if it has initial / terminal objects 0 / 1

$$\text{UMP}_{\text{terminal}}(0) = \forall X . \exists ! 0 . 0_X \in \mathbf{Hom}(0, X)$$
$$\text{UMP}_{\text{initial}}(1) = \forall X . \exists ! 1 . 1_X \in \mathbf{Hom}(X, 1)$$

- $0^{\text{op}} = 1$, initial and terminal objects are duals
- $f : A \rightarrowtail B^{\text{op}} = f' : B \twoheadrightarrow A$, monomorphisms and epimorphisms are duals

Theorem 2.5 (*Generalized elements from initial and terminal objects*) [1, section 2.3]

In **Sets**

- $1 = \{\star\}$, thus morphisms $\overline{x} : 1 \to X$ represent $x \in X$; meaning $X \cong \mathbf{Hom}_{\mathbf{Sets}}(1, X)$
- $\overline{x}$ are *generalized elements* / *global elements* / *points* / *constants* of X
- if $\overline{x}$ is an epimorphism, then X has a unique inhabitant since $f \circ \overline{x} = g \circ \overline{x} \Rightarrow f = g$
- instead of 1 we can also use T to *"probe"* the internal structure of X

Definition 2.6 (*Products*) [1, definition 2.15] P is a product of A and B under p_1, p_2 if there exists a unique u; P, A, B, p_1, p_2 is called a diagram of the limit. $\mathrm{UMP}(x_1, x_2) = u$

$$\mathrm{UMP}_{\text{product}}(x_1, x_2, u, p_1, p_2) = \forall x_1, x_2. \exists! u. x_1 = p_1 \circ u \wedge x_2 = p_2 \circ u$$

$$\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"'] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B
\end{tikzcd}$$

note that there can be more than one product of A, B i.e. $p_{1'}, p_{2'}$ can be distinct from p_1, p_2

Definition 2.7 (*Categories with Products for every pair of objects*) [1, section 2.6]

Let $\times$ be a functor

$$\times : C \times C \to C$$
$$\times\,(A, A') = A \times A'$$
$$\times\,(f : A \to B, f' : A' \to B') = f \times f' : A \times A' \to B \times B'$$

- this implies $A \times (B \times C) \cong (A \times B) \times C$, by UMP uniqueness they are isomorphic; associative
- terminal objects behave as nullary products; an identity for products
- single objects behave as unary products

$$\begin{tikzcd}
X \arrow[d, dashed, "!"'] \\
1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & X \arrow[dl, dashed, "\overline{p}"'] \arrow[d, dashed, "\overline{p}"'] \arrow[dr, dashed, "\overline{p}"] & \\
P & P \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & P
\end{tikzcd}$$

- thus a category C has all finite products if it has $\times$ and 1
- a category C has all small products if every set of objects has a product object

Definition 2.8 (*Products as Hom-sets*) [1, section 2.7]

$\mathbf{Hom}_C(X, -) : C \to \mathbf{Sets}$ is called the covariant representable functor of X, where notationally:

$$\mathbf{Hom}_C(X, g) = g_* = [f \mapsto g \circ f]$$

iff for every object X, the canonical function is an isomorphism

$$\vartheta_X : \mathbf{Hom}(X, P) \cong \mathbf{Hom}(X, A) \times \mathbf{Hom}(X, B)$$

such that

$$\vartheta_X(u) = (p_{1*}(u), p_{2*}(u))$$
$$\vartheta_X(u) = (x_1, x_2)$$

then the C has all finite products

moreover a functor F preserves products iff $F(A \times B) \cong F(A) \times F(B)$

thus, the covariant representable functor of X preserves products

2.2 Mandatory Exercises

EXERCISE 3 FROM WK1: Let $\mathbb{C}$ be a category with binary products

Proof *(is the projection $\pi_X : X \times Y \to X$ an epimorphism in general? Is it a monomorphism?)* It is a monomorphism since there is a unique u into $X \times Y$ by UMP of product. It is not necessarily an epimorphism as there can be more than one distinct morphism out of X □

Proof *(show $(f, g) \circ h = (f \circ h, g \circ h)$)*

$$\begin{tikzcd}
 & W \arrow[dl, "f \circ h"'] \arrow[d, "h"] \arrow[d, dashed, bend left, "(f \circ h, g \circ h)" near end] \arrow[dr, "g \circ h"] & \\
 & Z \arrow[dl, "f"] \arrow[d, dashed, "(f,g)"'] \arrow[dr, "g"'] & \\
X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y
\end{tikzcd}$$

$$\mathrm{UMP}(f \circ h, g \circ h) = (f, g) \circ h$$

□

Proof *(Let $f : Z \to X$ and $g : W \to Y$ be ...)*

$$\begin{tikzcd}
 & A \arrow[dl, "h_Z"'] \arrow[d, dashed, "u'"'] \arrow[dr, "h_W"] & \\
Z \arrow[d, "f"'] & Z \times W \arrow[l, "\pi_Z"'] \arrow[r, "\pi_W"] \arrow[d, dashed, "u"'] & W \arrow[d, "g"] \\
X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y
\end{tikzcd}$$

$$\mathrm{UMP}(h_Z, h_W) = u'$$
$$\mathrm{UMP}(f \circ h_Z, g \circ h_W) = u \circ u'$$

by uniqueness of u' and $u \circ u'$, u too is unique □

Proof *(Show $(f \times g) \circ (h \times k) = (f \circ h) \times (g \circ k)$)*

$$\begin{tikzcd}
 & T \arrow[dl, "u_A"'] \arrow[d, dashed, "u"'] \arrow[dr, "u_B"] & \\
A \arrow[d, "h"'] & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] \arrow[d, dashed, "(h,k)"'] & B \arrow[d, "k"] \\
Z \arrow[d, "f"'] & Z \times W \arrow[l, "\pi_Z"'] \arrow[r, "\pi_W"] \arrow[d, dashed, "(f,g)"'] & W \arrow[d, "g"] \\
X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y
\end{tikzcd}$$

$$\mathrm{UMP}(u_A, u_B) = u$$
$$\mathrm{UMP}(h \circ u_A, k \circ u_B) = (h, k) \circ u$$
$$\mathrm{UMP}(f \circ h \circ u_A, g \circ k \circ u_B) = (f, g) \circ (h, k) \circ u$$
$$(f \circ h \circ u_A) \times (g \circ k \circ u_B) = (f, g) \circ (h, k) \circ u$$
$$((f \circ h) \times (g \circ k)) \circ u = (f, g) \circ (h, k) \circ u$$
$$(f \circ h) \times (g \circ k) = (f \times g) \circ (h \times k)$$

□

<ins>Exercise 1:</ins> Show that every poset considered as a catgory has equalizers and coequalizers of all pairs of morphisms

Remark: SA Chapter 2 did not cover equalizers and coequalizers, we will revisit the question in the future.

<u>Exercise 2:</u> Let the functor $F : \mathbb{C} \to \mathbb{D}$ be an isomorphism of categories. Show the following.

Proof *(if $\mathbb{C}$ has binary products so does $\mathbb{D}$ and F preserves them)* A product is a $\mathrm{UMP}(f : X \to A, g : X \to B) = u : X \to A \times B$

Product UMP	definition
existence	$F(f) : F(X) \to F(A), F(g) : F(X) \to F(B), \exists F(u) : F(X) \to F(A \times B)$
uniqueness	$\exists u', F(u).F^{-1}(u') = F^{-1}(F(u)) \Rightarrow u' = F(u)$

since u is unique by definition of product, and F is an isomorphism, if there exists another $u' : F(X) \to F(A \times B)$ it too maps to unique u bijectively, making $u' = F(u)$ □

Proof *(if $\mathbb{C}$ has binary coproducts so does $\mathbb{D}$ and F preserves them)* same argument as above □

Remark: skipping the equalizers and coequalizers for now

<u>Exercise 3:</u> Let $\mathbb{C}$ be a category and X an object of $\mathbb{C}$. Show the following

Proof *(The slice category $\mathbb{C}/X$ always has a terminal object)* recall the objects are morphisms $f : \_ \to X$ and morphisms are morphisms between the domain objects. All objects; $f : \_ \to X$ thus have morphisms to object $\mathrm{id}_X : X \to X$, making it a terminal object

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "f"'] & X \arrow[dl, "\mathrm{id}_X"'] \\
X &
\end{tikzcd}$$

□

Proof *(If $\mathbb{C}$ has an initial object then so does $\mathbb{C}/X$)* if 0 is the initial object of $\mathbb{C}$ then $0_X : 0 \to X$ is the initial object. If $f : Y \to X$ is an object of $\mathbb{C}/X$ then $0_Y : 0 \to Y$ is the morphism from the initial object of $\mathbb{C}/X$

$$\begin{tikzcd}
0 \arrow[r, "0_Y"] \arrow[d, "0_X"'] & Y \arrow[dl, "f"'] \\
X &
\end{tikzcd}$$

□

Remark: skipping the equalizers question for now

2.3 Relevant Exercises

<u>Exercise 1</u>: Show that a function between sets is an epimorphism if and only if it is surjective. Conclude that isos in **Sets** are exactly the epi-monos.

Proof *(surjective implies epimorphism)*

We can define surjective functions as follows and derive the property of epimorphisms

$$\begin{aligned}
&\forall y \in Y . \exists x . f(x) = y \\
&= \forall x . f(x) = y \wedge (g(y) \text{ is well defined}) \\
&= \forall x . f(x) = y \wedge (g(y) = h(y) \Rightarrow g(y) = h(y)) \\
&= \forall x . g(f(x)) = h(f(x)) \Rightarrow g = h \\
&= g \circ f = h \circ f \Rightarrow g = h \\
&= \mathrm{isEpi}(f)
\end{aligned}$$

□

Proof *(epimorphism implies surjective)*

if f is not surjective then $\mathrm{im}(f)$ is a subset of Y

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[loop above, "\mathrm{id}_Y"] \arrow[r, bend left, "\pi"] & \mathrm{im}(f) \arrow[l, hook, bend left, "i"]
\end{tikzcd}$$

if we let $g = \mathrm{id}_Y$ and $h = i \circ \pi$ we can see

$$\begin{aligned}
\mathrm{isEpi}(f) = \mathrm{id}_Y \circ f = i \circ \pi \circ f &\Rightarrow \mathrm{id}_Y = i \circ \pi \\
&= Y = \mathrm{im}(f) \\
&= \mathrm{surjective}(f)
\end{aligned}$$

□

Proof *(isos are epi-monos)* Isomorphisms are bijective functions, and thus are both surjective and injective. Thus We just have to show injective functions are exactly monomorphisms.

we can define injectivity as follows and derive the property of monomorphisms

$$\begin{aligned}
&\forall w_1, w_2 . w_1 \neq w_2 \Rightarrow f(w_1) \neq f(w_2) \\
&= \forall w_1, w_2 . f(w_1) = f(w_2) \Rightarrow w_1 = w_2 \\
&= \forall w . f(g(w)) = f(h(w)) \Rightarrow g(w) = h(w) \\
&= f \circ g = f \circ h \Rightarrow g = h \\
&= \mathrm{isMono}(f)
\end{aligned}$$

if f is not injective then $\mathrm{preim}(f)$ is a strict subset of X as there can be more than two pre-images for a single element in the codomain.

$$\begin{tikzcd}
\mathrm{preim}(f) \arrow[r, hook, bend left, "i"] & X \arrow[l, bend left, "\pi"] \arrow[loop above, "\mathrm{id}_X"] \arrow[r, "f"] & Y
\end{tikzcd}$$

if we let $g = i \circ \pi$ and $h = \mathrm{id}_X$ we can see

$$\begin{aligned}
\mathrm{isMono}(f) = f \circ \mathrm{id}_X = i \circ \pi \circ f &\Rightarrow \mathrm{id}_X = i \circ \pi \\
&= X = \mathrm{preim}(f) \\
&= \mathrm{injective}(f)
\end{aligned}$$

□

<u>Exercise 2</u>: Show that in a poset category, all arrows are both monic and epic.

Recall a poset category has elements of the poset as objects and morphisms as witness the $\leq$ relation holds for the two objects. Since all arrows between any two objects are unique, $g = h$ regardless if they are post-composites of an epi or pre-composites of a mono. Thus all arrows are monic and epic.

<u>Exercise 3:</u> (Inverses are unique.) If an arrow $f : A \to B$ has inverses $g, g' : B \to A$ (i.e. $g \circ f = 1_A$ and $f \circ g = 1_B$ and similarly for g'), then $g = g'$.

$g \circ f = 1_A$ makes f a split mono of epic g and g' by Definition 2.2. Since g and g' are inverses of f, thus $f \circ g = f \circ g' = 1_B$. Moreover since f is a monomorphism,

$$\text{isMono}(f) = f \circ g = f \circ g' \Rightarrow g = g'$$

<u>Exercise 4:</u> With regard to a commutative triangle... in any category $\mathbf{C}$, show

Proof *(if f and g are isos (resp. monos, resp. epis), so is h)* we know by the commutative triangle $h = g \circ f$

since f, g are isos we also know there exists unique f^{-1}, g^{-1} such that

$$\begin{aligned} 1_A &= f^{-1} \circ g^{-1} \circ g \circ f \\ &= f^{-1} \circ g^{-1} \circ h \end{aligned} \qquad\qquad \begin{aligned} 1_B &= g \circ f \circ f^{-1} \circ g^{-1} \\ &= h \circ f^{-1} \circ g^{-1} \end{aligned}$$

$$\therefore h^{-1} = f^{-1} \circ g^{-1} \Rightarrow \text{isIso}(h)$$

□

Proof *(if h is monic, so is f)* if f is not monic, there exists $i \neq j$ where $f \circ i = f \circ j \Rightarrow i \neq j$

now that we know i and j are distinct, we can see h is also not monic as $h \circ i = h \circ j \not\Rightarrow i = j$

thus proven by contrapositive □

Proof *(if h is epic, so is g)* similar argument as monic □

Proof *(if h is monic, g need not be.)*

$$\begin{tikzcd} E \arrow[r, bend left, "i_c"] \arrow[r, bend right, "j_c"'] & A \arrow[r, "f"] \arrow[dr, "h"] & B \arrow[d, "g"'] & D \arrow[l, bend right, "j_b"'] \arrow[l, bend left, "i_b"] \\ & & C & \end{tikzcd}$$

it is very well possible $i_c = j_c$ and $i_b \neq j_b$ □

<u>Exercise 5:</u> Show that the following are equivalent for an arrow $f : A \to B$ in any category

- f is an isomorphism
- f is both a mono and a split epi
- f is both a split mono and an epi
- f is both a split mono and a split epi

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "1_A"] \arrow[dr, bend right, "g \circ f"'] & B \arrow[d, "g"] \arrow[dr, "1_B"] \arrow[dr, bend left, "f \circ g"] & \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

$$\begin{aligned}
\mathrm{isIso}(f) &= g \circ f = 1_A \land f \circ g = 1_B \\
&= (f \circ g \circ f = f \circ 1_A) \land (f \circ g = 1_B) \\
&= \mathrm{isMono}(f) \land \mathrm{isSplitEpi}(f) \\
&= g \circ f = 1_A \land f \circ g = 1_B \\
&= (g \circ f = 1_A) \land (f \circ g \circ f = 1_B \circ f) \\
&= \mathrm{isSplitMono}(f) \land \mathrm{isEpi}(f) \\
&= (g \circ f = 1_A) \land (f \circ g = 1_B) \\
&= \mathrm{isSplitMono}(f) \land \mathrm{isSplitEpi}(f)
\end{aligned}$$

<u>Exercise 13:</u> In any category with binary products, show directly that $A \times (B \times C) \cong (A \times B) \times C$

$$\begin{tikzcd}
B & A \times B \arrow[l, "\pi_B"'] \arrow[d, "\pi_A"'] & (A \times B) \times C \arrow[l, "\pi_{A,B}"'] \arrow[r, "\pi_C"] \arrow[dl] \arrow[dr] \arrow[d, dashed, leftrightarrow, "\mathrm{assoc}"] & C & X \arrow[l, "h"'] \arrow[d, "g"] \arrow[dl, dashed] \\
X \arrow[u, "g"] \arrow[r, "f"'] \arrow[ur, dashed] & A & A \times (B \times C) \arrow[l, "\pi_A"'] \arrow[r, "\pi_{B,C}"] \arrow[ul] \arrow[ur] & B \times C \arrow[u, "\pi_C"'] \arrow[r, "\pi_B"] & B
\end{tikzcd}$$

$$\mathrm{UMP}(\pi_{A,B}, \pi_C) = \mathrm{assoc}$$

$$\mathrm{UMP}(\pi_A, \pi_{B,C}) = \mathrm{assoc}^{-1}$$

$$(A \times B) \times C \cong A \times (B \times C) = \mathrm{isIso}(\mathrm{assoc})$$

Note: there is some abuse of notation where different projection morphisms are notated with the same name. But the UMPs holds nonetheless.

<u>Exercise 14:</u> For any index set I, define the ...

Proof *(For any index set I, define the prouct $\Pi_{i \in I} X_i$ of an I-indexed family of objects $(X_i)_{i \in I}$ in a category, by giving a UMP generalizing that for binary products (the case $I = 2$))*

$$\mathrm{UMP}_I(\vec{f}) = \begin{cases} \mathrm{UMP}_{I-1}\left(\vec{f} / \{f_1, f_2\} \cup \{\mathrm{UMP}(f_1, f_2)\}\right) & I > 1 \\ f_1 \end{cases}$$

this is left associative e.g.

$$\begin{aligned}
\mathrm{UMP}_3(\{f, g, h\}) &= \mathrm{UMP}_2(\{f \times g, h\}) \\
&= \mathrm{UMPI}_1(\{(f \times g) \times h)\}) \\
&= (f \times g) \times h
\end{aligned}$$

we can define UMP_I via right associativity by popping the last two elements instead. $\square$

Proof *(Show that in* **Sets***, for any set X the set X^I of all functions $f : I \to X$ has this UMP, with respect to the "constant family" where $X_i = X$ for all $i \in I$, and thus $X^I \cong \Pi_{i \in I} X$)*

$$
\begin{aligned}
\mathrm{UMP}_I(X^I) &= \mathrm{UMP}_I(\{f_1, f_2, f_3 ...\}) \\
&= \mathrm{UMP}_{I-1}(\{f_1 \times f_2, f_3, ...\}) \\
&= \mathrm{UMP}_1(\{f_1 \times f_2 \times f_3 ...\}) \\
&= f_1 \times f_2 \times f_3 ...
\end{aligned}
$$

□

<u>Exercise 15:</u> Given a category $\mathbf{C}$ with objects A and B, define the category $\mathbf{C}_{A,B}$...

$$
\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"'] \arrow[dr, "x_2"] & \\
A & 1 \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B
\end{tikzcd}
$$

$$
\begin{aligned}
\forall X.\ \mathrm{UMP}(x_1, x_2) &= u \\
1_X : (X, x_1, x_2) &\to (1, \pi_A, \pi_B)
\end{aligned}
$$

<u>Exercise 17:</u> In any category $\mathbf{C}$ with products, define the graph ...

we can deduce the following if f is monic

$$
\begin{aligned}
\mathrm{isMonic}(1_A \times f) &= (1_A \times f) \circ g = (1_A \times f) \circ h \Rightarrow g = h \\
&= (1_A \circ g \times f \circ g) = (1_A \circ h \times f \circ h) \Rightarrow g = h \\
&= 1_A \circ g = 1_A \circ h \land f \circ g = f \circ h \Rightarrow g = h \\
&= g = h \land f \circ g = f \circ h \Rightarrow g = h \\
&= f \circ g = f \circ h \Rightarrow g = h
\end{aligned}
$$

thus f must be monic by the contrapositive

$$
g \neq h \Rightarrow f \circ g \neq f \circ h
$$

then we have the functor

$$
\begin{aligned}
\Gamma(A) &= A \\
\Gamma(f) &= \mathrm{im}(1_A \times f)
\end{aligned}
$$

that preserves the structure

STRUCTURE	DEFINITION
domain	$\Gamma(f) : \Gamma(A) \to \Gamma(B)$
identity	$\Gamma(1_A) = \mathrm{im}(1_A \times 1_A) = 1_{\Gamma(A)}$
composition	$\Gamma(g \circ f) = \Gamma(g) \circ \Gamma(f)$

Proof *(composition)*

$$
\begin{aligned}
\Gamma(g \circ f) &= \mathrm{im}(1_A \times g \circ f) \\
&= \mathrm{im}((1_A \times g) \circ (1_A \times f)) \\
&= \mathrm{im}(1_A \times g) \circ \mathrm{im}(1_A \times f) \\
&= \Gamma(g) \circ \Gamma(f)
\end{aligned}
$$

$\square$

<u>Exercise 18:</u> Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Sets}$ from ...

If U is a forgetful functor then $U(N)$ is the underlying set of the monoid. The elements of the underlying set can be defined by the representable objects in the category of monoids i.e. morphisms from the terminal object.

$$\mathbf{Hom}_{\mathbf{Mon}}(1, N) \cong U(N)$$

Then, $U(f)$ is the underlying function of the monoid homomorphism $f : N \to M$. We can construct the functions as follows.

$$\{\mathbf{Hom}_{\mathbf{Mon}}(1, N) \to \mathbf{Hom}_{\mathbf{Mon}}(1, M)\} \cong U(f)$$

In the category of monoids we know that:

$$p = (\pi_N \circ p, \pi_M \circ p)$$

$$
\begin{tikzcd}
N & N \times M \arrow[l, "\pi_N"'] \arrow[r, "\pi_M"] & M \\
& X \arrow[ul] \arrow[u, dashed, "u"'] \arrow[ur] &
\end{tikzcd}
$$

Let $x : 1 \to X$, thus we can construct

$$
\begin{aligned}
\{u \circ x\} &\cong \{\pi_N \circ u \circ x\} \times \{\pi_M \circ u \circ x\} \\
\mathbf{Hom}_{\mathbf{Mon}}(1, N \times M) &\cong \mathbf{Hom}_{\mathbf{Mon}}(1, N) \times \mathbf{Hom}_{\mathbf{Mon}}(1, M) \\
U(N \times M) &\cong U(N) \times U(M)
\end{aligned}
$$

Thus U preserves products

§3 Duality

3.1 Notes

Definition 3.1 (*Duality*) [1, Proposition 3.1, 3.2]

Formal Duality if Σ follows from the axioms of categories, then so does its dual Σ^*

$$(\mathrm{CT} \Rightarrow \Sigma) \Rightarrow (\mathrm{CT} \Rightarrow \Sigma^*)$$

specifically, if for a category $\boldsymbol{C}$ that Σ holds, then in its dual $\boldsymbol{C}^{\mathrm{op}}$ the interpretation Σ^* holds.

$$(\boldsymbol{C} \Rightarrow \Sigma) \Rightarrow (\boldsymbol{C}^{\mathrm{op}} \Rightarrow \Sigma^*)$$

Conceptual Duality moreover if Σ holds for all categories $\boldsymbol{C}$, then it also holds in all $\boldsymbol{C}^{\mathrm{op}}$. Thus

$$\forall \boldsymbol{C}.(\boldsymbol{C} \Rightarrow \Sigma) \Rightarrow (\boldsymbol{C}^{\mathrm{op}} \Rightarrow \Sigma)$$

with both we can conclude that

$$\Rightarrow ((\boldsymbol{C}^{\mathrm{op}})^{\mathrm{op}} \Rightarrow \Sigma^*) \Rightarrow (\boldsymbol{C} \Rightarrow \Sigma^*)$$

Definition 3.2 (*Coproducts*) [1, Definition 3.3]

$$\mathrm{UMP}_{\mathrm{coproduct}}(x_1, x_2, u, q_1, q_2) = \forall x_1, x_2.\exists! u. x_1 = u \circ q_1 \land x_2 = u \circ q_2$$

$$\begin{tikzcd}
 & X & \\
A \arrow[ur, "x_1"] \arrow[r, "q_1"] & P \arrow[u, dashed, "u"'] & B \arrow[ul, "x_2"'] \arrow[l, "q_2"']
\end{tikzcd}$$

notice the UMP is dual to products; flipped arrows and composition

Examples of Coproducts:

CATEGORY	EQUATION	DESCRIPTION	EXAMPLE OF
Sets	$A + B = A \times 1 \cup B \times 2$	disjoint union of sets	from definition
Monoids	$M(A + B) \cong M(A) + M(B)$	monoid of $A + B$	preserves coproducts
Top/Powerset	$\mathcal{P}(X + Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y)$	$2^{X+Y} \cong 2^X \times 2^Y$	dual in underlying
Posets	$p + q = p \lor q$	least upper bound	from definition
Proofs	$\varphi + \psi = \varphi \lor \psi$	disjunction	from definition
Free Monoids	$A + B \cong M(\lvert A\rvert + \lvert B\rvert)/\sim$	quotient unit & mul	underlying quotient
Abelian Groups	$A + B \cong A \times B$	commutative	self dual

Definition 3.3 (*Equalizers*) [1, Definition 3.13, 3.16] $\mathrm{UMP}(f, g, z) = u$, notice e is monic by uniqueness of u

Definition 3.4 (*Coequalizers*) [1, Definition 3.18, 3.19] $\mathrm{UMP}(f, g, z) = u$, notice q is epic by unqiueness of u

$$\begin{tikzcd} E \arrow[r, tail, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dashed, "u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

equalizers as subsets as two outgoing morphisms f, g are equalized to one e

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\ & & Z \end{tikzcd}$$

coequalizers as quotients as two incoming morphisms f, g are coequalized to one q

$$\mathrm{UMP}_{\text{equalizer}}(e, f, g) = \forall z.\exists! u. f \circ z = g \circ z \Rightarrow e \circ u = z$$
$$\mathrm{UMP}_{\text{coequalizer}}(q, f, g) = \forall z.\exists! u. z \circ f = z \circ g \Rightarrow u \circ q = z$$

UMP	EXISTENCE	UNIQUENESS
equalizer	$\forall f, g.\exists e. f \circ e = g \circ e$	$\forall z.\exists! u. e \circ u = z$
coequalizer	$\forall f, g.\exists q. q \circ f = q \circ g$	$\forall z.\exists! u. u \circ q = z$

Definition 3.5 (*Presentation of Algebras*) [1, Definition 3.24]
- Given a finite set, let it be the **generators** of the **free algebra**
  - $F(3) = F(x, y, z)$ where x, y, z are generators.
- the relations it must hold are defined by coequalizers
  - $xy = z$ is quotiented by q if $\mathrm{UMP}(xy, z, q) = \mathrm{id}_q$
  - more generally the morphisms can be the relations themselves; $r_1 \mapsto xy = z, r_2 \mapsto y^2 = 1$
- this is called a **presentation** of the algebra; they are not unique

3.2 Mandatory Exercises

<u>EXERCISE 1 FROM WK2:</u> Show that every poset considered as a category has equalizers and coequalizers of all pairs of morphisms

A poset as a category, has objects as elements of the poset and morphisms as witness of the ordering relation. Thus for any two morphisms $f, g : A \to B$ in the poset

$$\begin{tikzcd} E \arrow[r, tail, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dashed, "u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

e is an equzlier when $E = A \wedge B$ greatest lower bound since for any other Z it will then unqiuely be $Z \leq E$ such that $u : Z \to E$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\ & & Z \end{tikzcd}$$

q is a coequalizer when $Q = A \vee B$ least upper bound since for any other Z it will then unqiuely be $Z \geq Q$ such that $u : Q \to Z$

Moreover from the definition of the morphisms in the category we know that if there are parallel morphisms they have to be equal.

<u>EXERCISE 2 FROM WK2:</u> Let the functor $\mathcal{F} : \mathbb{C} \to \mathbb{D}$ be an isomorphism of categories. Show the following.

Proof *(if $\mathbb{C}$ has equalizers so does $\mathbb{D}$ and $\mathcal{F}$ preserves them.)*

an equalizer has the following UMP

$$\text{UMP}(f, g, z) = u$$

thus, we know $F(u)$ is unique by the composition preserving property of functors

$$\text{UMP}(F(f), F(g), F(z)) = F(u)$$

□

Proof *(if $\mathbb{C}$ has coequalizers so does $\mathbb{D}$ and $\mathcal{F}$ preserves them.)* by duality of the above argument, so does coequalizers. □

> <u>Exercise 3 from wk2:</u> Let $\mathbb{C}$ be a category and X and object of $\mathbb{C}$. Show the following. If $\mathbb{C}$ has equalizers so does $\mathbb{C}/X$

$$\begin{tikzcd}
& X \arrow[dl, "x_Z"'] \arrow[d, "x_A"] \arrow[dr, "x_B"] & \\
E \arrow[r, tail, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "z"'] & &
\end{tikzcd}$$

$$\text{UMP}(f, g, z) = u$$

since the morphisms of the slice categories are also the morphisms in the underlying category, the equalizer in the slice category is the same as the equalizer in the underlying category. This is only true if X has a morphism to all objects of the equalizer.

Note: we will move mandatory assignment for week 3 to week 4 since most of them are on pullbacks which is the next chapter.

3.3 Relevant Exercises

> <u>Exercise 1:</u> In any category $\boldsymbol{C}$, show that ... is a coproduct diagram just if for...

For products we have the following as given

$$\begin{aligned}
\text{UMP}(c_1^{-1} \circ f^{-1}, c_2^{-1} \circ f^{-1}) &= f^{-1} \\
\mathbf{Hom}_{\boldsymbol{C}^{\text{op}}}(Z, C) &\cong \mathbf{Hom}_{\boldsymbol{C}^{\text{op}}}(C, A) \times \mathbf{Hom}_{\boldsymbol{C}^{\text{op}}}(C, B) \\
f^{-1} &\cong (c_1^{-1} \circ f^{-1}, c_2^{-1} \circ f^{-1})
\end{aligned}$$

taking its dual we derive

$$\begin{aligned}
f &\cong (f \circ c_1, f \circ c_2) \\
\mathbf{Hom}_{\boldsymbol{C}}(C, Z) &\cong \mathbf{Hom}_{\boldsymbol{C}}(A, C) \times \mathbf{Hom}_{\boldsymbol{C}}(B, C)
\end{aligned}$$

<u>Exercise 2:</u> Show in detail that the free monoid functor M preserves coproducts for any sets A, B ...

$$\begin{tikzcd}
 & M(X) & \\
M(A) \arrow[ur, "M(a)"] \arrow[r, "M(q_a)"] & M(A+B) \arrow[u, dashed, "M(u)"] & M(B) \arrow[ul, "M(b)"'] \arrow[l, "M(q_b)"'] \arrow[d, "1_{M(B)}"] \\
M(A) \arrow[u, "1_{M(A)}"] \arrow[ur, "M(q_a)"] \arrow[r] & M(A)+M(B) \arrow[u, dashed, "u'"] & M(B) \arrow[ul, "M(q_b)"'] \arrow[l]
\end{tikzcd}$$

$$\mathrm{UMP}(a, b) = u$$
$$\mathrm{UMP}(M(a), M(b)) = M(u)$$
$$\mathrm{UMP}(M(a), M(b)) = M(u) \circ u'$$

by uniqueness property of UMP

$$M(u) = u' \circ M(u)$$
$$1_{M(A+B)} = u'$$
$$M(A+B) = M(A) + M(B)$$

<u>Exercise 3:</u> Verify that the construction given in the text of the coproduct of monoids $A + B$ as a quotient of the free monoid $M(|A| + |B|)$ really is a coproduct in the category of monoids.

$$\begin{tikzcd}
C \arrow[r, shift left, "c_1"] \arrow[r, shift right, "c_2"'] & M(|A|+|B|) \arrow[rr, bend left, "q"] \arrow[drr, bend right, "{[f,g]'}"'] & A \arrow[r, "i_A"] \arrow[dr, "f"] & M(|A|+|B|)/\sim \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "i_B"'] \arrow[dl, "g"'] \\
 & & & M &
\end{tikzcd}$$

1. $[f, g] : M(|A| + |B|)/\sim \to M$ is defined as follows

$$[f, g]([a] \cdot \mathrm{rest}) = f(a) \cdot_M [f, g](\mathrm{rest})$$
$$[f, g]([b] \cdot \mathrm{rest}) = g(b) \cdot_M [f, g](\mathrm{rest})$$

2. q is defined as a quotient of $M(|A| + |B|)$ such that

$$[f, g]' \circ c_1 = [f, g]' \circ c_2 \Rightarrow [f, g] \circ q = [f, g]'$$

3. thus a coequalizer by $\mathrm{UMP}(c_1, c_2, q) = [f, g]$
4. thus $[f, g]$ is unique by q being a coequalizer
5. thus the coproduct $A + B$ holds for $\mathrm{UMP}(f, g) = [f, g]$
6. therefore, we have $M(|A| + |B|)/\sim = A + B$

Exercise 4: Show that the product of two powerset boolean algebras $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is also a powerset, ...

$$\begin{tikzcd}
 & X \arrow[dl, "a"'] \arrow[d, dashed, "u"'] \arrow[dr, "b"] & \\
\mathcal{P}(A) & \mathcal{P}(A+B) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathcal{P}(B)
\end{tikzcd}$$

$$\begin{aligned}
\pi_1 &= [S \mapsto S \cap A\}] \\
\pi_2 &= [S \mapsto S \cap B] \\
a &= [x \mapsto u(x) \cap A\}] \\
&= [S \mapsto S \cap A] \circ [x \mapsto u(x)] \\
&= \pi_1 \circ u \\
b &= \pi_2 \circ u \\
\mathrm{UMP}(a, b) &= u
\end{aligned}$$

Exercise 6: Verify that the category of monoids has all equalizers and finite products.

$$\begin{tikzcd}
A_2 & A_0 \arrow[l, hook', "i'"'] \arrow[r, hook, "i"] & A_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
 & Z \arrow[ul, "z'"] \arrow[u, dashed, "h"'] \arrow[ur, "z"'] & &
\end{tikzcd}$$

1. let $A_0 = \{a \mid a \in A_1 \wedge f(a) = g(a)\}$ making $A_0 \subseteq A_1$ and i its inclusion
2. moreover $\forall x.h(x) = z(x)$ making h uniquely determined from z
3. thus equalizer UMP $f \circ z = g \circ z \Rightarrow i \circ h = z$ holds
4. let $A_0 \subseteq A_2$ with i' its inclusion
5. thus $\forall x.h(x) = z'(x)$ as well satisfying the UMP for product
6. therefore $A_0 = A_1 \times A_2 \subseteq A_1 \cap A_2$

Exercise 10: In the proof of proposition 3.24 in the text it is shown that any monoid M has a specific presentation $T^2M \rightrightarrows TM \to M$ as a coequalizer of free monoids. Show that coequalizers of this particular form are preserved by the forgetful functor $\mathbf{Mon} \to \mathbf{Sets}$

$$\begin{tikzcd}
|T^2M| \arrow[r, shift left, "|\varepsilon|"] \arrow[r, shift right, "|\mu|"'] & |TM| \arrow[r, "|\pi|"] \arrow[dr, "z"'] & |M| \arrow[d, dashed, "u"] \\
 & & Z
\end{tikzcd}$$

1. recall $|q| = [a \mapsto q(a)]$ from q with arguments as single alphabets
2. since $\pi(x_1, ..., x_n) = x_1 \cdot ... \cdot x_n$, then $|\pi|(x) = x$, thus $|\pi| = 1_{|TM|}$ and $u = z$
5. therefore $z \circ |\varepsilon| = z \circ |\mu| \Rightarrow u \circ |\pi| = z$, preserving the coequalizer

Exercise 11: Prove that **Sets** has all coequalizers by constructing the coequalizer of a parallel pair of functions $A \underset{g}{\overset{f}{\rightrightarrows}} B \to Q = B/(f = g)$ by quotienting B by a suitable equivalence relation R on B, generated by the pairs $(f(x), g(x))$ for all $x \in A$. Define R to be the intersection of all equivalence relations on B containing all such pairs.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "\mu"] & Q \arrow[d, dashed, "\mu"] \\
& & B \times B
\end{tikzcd}$$

1. we know $R(b_1, b_2) = R(f(x), g(x))$
2. let $q(b) = \{b \mid R(b, b)\}$ and $\mu(b) = (b, b)$
2. thus $\mu \circ f = \mu \circ g \Rightarrow \mu \circ q = \mu$

$$\begin{aligned}
(\mu \circ f)(x) = (\mu \circ g)(x) &\Rightarrow (\mu \circ q)(b) = \mu(b) \\
\mu(f(x)) = \mu(g(x)) &\Rightarrow \{\mu(b) \mid R(b, b)\} = \mu(b) \\
(f(x), f(x)) = (g(x), g(x)) &\Rightarrow \{\mu(b) \mid R(b, b)\} = \mu(b) \\
R(f(x), g(x)) &\Rightarrow \{\mu(b) \mid R(b, b)\} = \mu(b) \\
R(b, b) &\Rightarrow \{\mu(b) \mid R(b, b)\} = \mu(b) \\
R(b, b) &\Rightarrow \mu(b) = \mu(b) \\
R(b, b) &\Rightarrow \top
\end{aligned}$$

4. therefore the UMP holds for coequalizer

Exercise 12: Verify the coproduct-coequalizer construction mentioned in the text for coproduct of rooted posets, that is, posets with a least element 0 and monotone maps preserving 0. Specifically, show that the coproduct $P +_0 Q$ of two such posets can be constructed as a coequalizer in posets, $1 \underset{0_Q}{\overset{0_p}{\rightrightarrows}} P + Q \to P +_0 Q$. You may assume as given the fact that the category of posets has all coequalizers.

$$\begin{tikzcd}
1 \arrow[r, shift left, "0_P"] \arrow[r, shift right, "0_Q"'] & P + Q \arrow[r, "q"] \arrow[dr, "f"] & P +_0 Q \arrow[d, dashed, "f"] \\
& & R
\end{tikzcd}$$

1. $0_P, 0_Q$ is the representation of the least element in P, Q respectively
2. q quotients the least elements of P, Q i.e. $0_P = 0_Q$
3. thus $f \circ 0_P = f \circ 0_Q \Rightarrow f \circ q = f$ making q the coequalizer

EXERCISE 13: Show that the category of monoids has all coequalizers as follows

EXERCISE 1: Given any pair of monoid homomorphisms $f, g : M \to N$, show that the following equivalence relation on N agree:

EXERCISE A: $n{\sim}n' \Leftrightarrow$ for all monoids X and homomorphisms $h : N \to X$, one has $hf = hg$ implies $hn = hn'$

1. let m be such that $f(m) = n, g(m) = n'$
2. thus

$$\begin{gathered} n{\sim}n' \\ f(m){\sim}g(m) \\ (h \circ f)(m){\sim}(h \circ g)(m) \\ h(n){\sim}h(n') \end{gathered}$$

EXERCISE B: the intersection of all equivalence relations $\sim$ on N satisfying $fm{\sim}gm$ for all $m \in M$ as well as $n{\sim}n' \wedge m{\sim}m' \Rightarrow n \cdot m{\sim}n' \cdot m'$

1. ??

EXERCISE 2: Taking $\sim$ to be the equivalence relation defined in (1), show that the quotient set $N/{\sim}$ is a monoid under $[n] \cdot [m] = [n \cdot m]$, and the projection $N \to N/{\sim}$ is the coequalizer of f and g.

?? whats $[n]$ again?

EXERCISE 14: Consider the following category of sets:

EXERCISE A: Given a function $f : A \to B$ describe the equalizer of the function $f \circ p_1, f \circ p_2 : A \times A \to B$ as a binary relation on A and show that it is an equivalence relation called the kernel of f

?? what is a kernel?

EXERCISE B: Show that the kernel of the quotient $A \to A/R$ by an equivalence relation R is R itself

?? i need to learn more algebra about kernels

EXERCISE C: Given any binary relation $R \subseteq A \times A$, let $[R]$ be the equivalence relation on A generated by R. Show that the quotient $A \to A/[R]$ is the coequalizer of the two projections

EXERCISE D: Using the foregoing, show that for any binary relation R on a set A one can characterize the equivalence relation $[R]$ generated by R as the kernel of the coequalizer of the two projections of R.

notes in commutative diagram below eqn 3.8 in 3.24, it should be T^2M and not just T^2 right?

§4 Limits and Colimits

4.1 Notes

Definition 4.1 (*Subobjects*) [1, Definition 5.1, Remark 5.2, Example 5.3]

- A subobject of an object X is a monomorphism $m : M \rightarrowtail X$ in $\boldsymbol{C}$
- $\mathrm{Sub}_{\boldsymbol{C}}(X)$ is a category of subobjects of X with morphisms arrows in $\boldsymbol{C}/X$
  - the morphisms between any two objects in $\mathrm{Sub}_{\boldsymbol{C}}(X)$ are unique (by mono m)
  - thus $\mathrm{Sub}_{\boldsymbol{C}}(X)$ is a **preorder category**
  - thus if theres a morphism both ways, it is an isomorphism modelling $\equiv$
- if we quotient the isomorphic objects into equivalence classes, we get a **poset category**
  - in this interpretation $\mathrm{Sub}_{\mathbf{Sets}(X)} \cong P(X)$; powerset; objects are subsets
  - thus morphisms $f : M \rightarrowtail M'$ is also monic since composites of monos are also monos
  - thus we have a functor $\mathrm{Sub}(M') \to \mathrm{Sub}(X) = [f \mapsto m' \circ f]$
  - *local set membership* $m \underset{X}{\in} M' \Leftrightarrow \exists f . m = m' \circ f$

PREORDER	POSET

$$
\begin{tikzcd}
M \arrow[r, dashed, "f"] \arrow[dr, tail, "m"'] & M' \arrow[d, tail, "m'"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
M' \arrow[r, tail, "m'"] & X \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Y \\
M \arrow[u, dashed, "f"] \arrow[ur, tail, "m"'] & &
\end{tikzcd}
$$

- since equalizers are monos they are subobjects too $m \underset{X}{\in} M' \Leftrightarrow g(m) = g(h)$
  - we can then regard M' as the subobject of generalized elements m such that $g(m) = h(m)$

Definition 4.2 (*Pullbacks*) [1, Definition 5.4, Proposition 5.5, Corollary 5.6]

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"] & C
\end{tikzcd}
$$

- a pullback is a $\mathrm{UMP}(f, g) = (\pi_1, \pi_2)$
  - $P = A \times_C B = \{\langle p_1, p_2 \rangle \in A \times B \mid f \circ z_1 = g \circ z_2\}$
  - making u an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$
  - P is a subobject of Z that makes it commute in the square
  - if π_1 is monic, g is monic too

UMP	DEFINITION
existence	$\forall p_1, p_2 . \exists! u . f \circ \pi_1 \circ u = g \circ \pi_2 \circ u$
uniqueness	$\pi_1 \circ u = p_1 \wedge \pi_1 \circ u' = z_1 \Rightarrow u = u'$

- in $\mathbf{Sets}$ the square made by $f : A \to B$ and $\overline{f} : f^{-1}(V) \to V$; its inverse images to images with inverses, form a pullback, generalizing inverses
- horizontally composed squares ↓⇉↓⇉↓
  - if the two squares are pullbacks, so is the outer rectangle
  - if the right square and outer rectangle are pullbacks, so is the left square
- the pullback of a commutative triangle is a commutative triangle; prism diagram
- a pullback is a functor, fix $g : B \to C$, the functor is $g^* : \boldsymbol{C}/C \to \boldsymbol{C}/C' = [f \mapsto \pi_2]$
- for $f : A \to B$ the $f^{-1} : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ and $f^* : \boldsymbol{C}/B \to \boldsymbol{C}/A$ form a pullback too

Definition 4.3 (*Diagram*) [1, Definition 5.15]

- a diagram is a functor $D : \boldsymbol{J} \to \boldsymbol{C}$ where $\boldsymbol{J}$ is an index category
- a cone is an object C and family of arrows $c_j : C \to D_j$
  - any arrows $\alpha : i \to j$ makes the triangle $D_\alpha : D_i \to D_j, c_i, c_j$ commute
- a cone morphisms $\vartheta : (C, c_j) \to (C', c_{j'})$, maps the object and family of arrows to another
  - the triagle $\vartheta, c_j, c_{j'}$ also commutes
  - this forms a category $\mathbf{Cone}(D)$

Definition 4.4 (*Limits*) [1, 5.16-5.25]
- a limit of a diagram D is a terminal object in $\mathbf{Cone}(D)$ with $\mathrm{UMP}(C, c_j, D) = u$
  - the limit is notated as $p_i : \varprojlim_{j} D_j \to D_i$; the arrrows from the limit cone object to the diagram
  - a limit exists if for all cones, there exists a unique morphism to the limit cone

UMP	DEFINITION
existence	$\forall (C, c_j), D.\exists! \left(u : C \to \varprojlim_{j} D_j\right).\forall j.p_j \circ u = c_j$
uniqueness	if there are two there will be an iso

$$\begin{tikzcd} & C \arrow[ddl, bend right, "c_i"'] \arrow[d, dashed, "u"'] \arrow[ddr, bend left, "c_j"] & \\ & \varprojlim_{j} D_j \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ D_i \arrow[rr, "D_\alpha"] & & D_j \end{tikzcd}$$

DIAGRAM; $\boldsymbol{J}$	LIMIT	DIAGRAM
$\bullet\ \bullet$	product	$D_1 \xleftarrow{p_1} \varprojlim_{j} D_j \xrightarrow{p_2} D_2$
$\bullet \rightrightarrows \bullet$	equalizer	$\begin{tikzcd} \varprojlim_{j} D_j \arrow[r, "c_1"] \arrow[rr, bend right, "c_2"'] & D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2 \end{tikzcd}$
$\emptyset$	terminal object	$\varprojlim_{j} D_j$
$\bullet \to \bullet \leftarrow \bullet$	pullback	$\begin{tikzcd} \varprojlim_{j} D_j \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D_2 \arrow[d, "g"] \\ D_1 \arrow[r, "f"] & C \end{tikzcd}$

LHS	RHS
binary products, equalizers	pullbacks
finite products, equalizers	pullbacks, terminal object
finite products, equalizers	finite limits
finite products and equalizers of a cardinality	finite limits of the cardinality

Table 25: lhs iff rhs

Definition 4.5 (*Contravariant Functors*) [1, Definition 5.26] a functor of the form $F : \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{D}$

Definition 4.6 (*Functor preserving Limits*) [1, Definition 5.24-5.25]

- a functor $F : \boldsymbol{C} \to \boldsymbol{D}$ preserves limits of type $\boldsymbol{J}$ if $p_j : L \to D_j$ is a limit, then the cone $Fp_j : FL \to FD_j$ is a limit for the diagram $FD : \boldsymbol{J} \to \boldsymbol{D}$
- a functor preserving limits is said to be **continuous**

$$F\left(\underset{\longleftarrow}{\mathbf{lim}} D_j\right) \cong \underset{\longleftarrow}{\mathbf{lim}} F(D_j)$$

- representable functor $\mathbf{Hom}(C, -)$ preserves all limits
- contravariant representable functors i.e. $\mathbf{Hom}(-, C)$, map colimits to limits

Definition 4.7 (*Pushouts*) [1, Definition 5.6] dual of a pullback; where the pushout quotients $\forall a. f(a) \sim g(a)$

Definition 4.8 (*Colimits*) [1, Definition 5.6] dual of a limit

4.2 Mandatory Exercises

Exercise 1: Suppose the square below is a pullback. Show the following

Exercise 1.1: If g is a split epimorphism then k is

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, two heads, "g"'] \\ C \arrow[r, "f"] \arrow[u, bend left, tail, "s'"] & D \arrow[u, bend right, tail, "s"'] \end{tikzcd}$$

1. By definition of a pullback we know $A = C \times_D B = \{(c, b) \mid f(c) = g(b)\}$
2. since g is a split epi $\exists s. g \circ s = 1_D$ where $s' = [c \mapsto (c, (s \circ f)(c)]$

$$\begin{aligned} k \circ s' &= [c \mapsto k(c, (s \circ f)(c))] \\ &= [c \mapsto c] \\ &= 1_C \end{aligned}$$

3. thus s' is a section of k
5. k is also an epimorphism

$$\begin{aligned} m \circ k = n \circ k &\Leftrightarrow m \circ k \circ s' = n \circ k \circ s' \\ &\Leftrightarrow m \circ 1_C = n \circ 1_C \\ &\Leftrightarrow m = n \end{aligned}$$

6. therefore k is a split epi

Exercise 1.2: If g is an isomorphism then k is

$$\begin{aligned} s' \circ k &= [(c, b) \mapsto (s' \circ k)(c, b)] \\ &= [(c, b) \mapsto s'(c)] \\ &= [(c, b) \mapsto (c, s \circ f(c))] \\ &= [(c, b) \mapsto (c, s \circ g(b))] && \text{by UMP of pullback} \\ &= [(c, b) \mapsto (c, b)] && \text{by isomorphism of} g, s \\ &= 1_A \end{aligned}$$

therefore the section s' is also a left inverse making k an isomorphism

Exercise 1.3: If g is a split monomorphism then k is not necessarily a split monomorphism

1. a counter example is if $A = \{(1, 1), (1, 2)\}$
2. then $\forall p. k(p) = 1$
3. therefore $\nexists s'. s' \circ k = 1_A$

Exercise 2: Recall that the two-pullback lemma, or the pullback pasting lemma, show that if the left square and the outer rectangle are pullbacks then the right square can fail to be a pullback

$$\begin{tikzcd}
\{\star\} \arrow[r, "{[\star \mapsto \star]}"] \arrow[d, "1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{\star, \star'\} \arrow[r, "f"] \arrow[d, "{[x \mapsto \star]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{\star, \star'\} \arrow[d, "{[x \mapsto \star]}"] \\
\{\star\} \arrow[r, "1"] & \{\star\} \arrow[r, "1"] & \{\star\}
\end{tikzcd}$$

1. the left square is a pullback since $\exists! u . 1 \circ 1 \circ u = [x \mapsto \star] \circ [\star \mapsto \star] \circ u$
   - where $u = [x \mapsto \star]$
2. the outer rectangle is a pullback since $\exists! u . 1 \circ 1 \circ 1 \circ u = [x \mapsto \star] \circ f \circ [\star \mapsto \star] \circ u$
   - where $u = [x \mapsto \star]$
3. the right rectangle is not a pullback since u is not unique in $1 \circ [x \mapsto \star] \circ u = [x \mapsto \star] \circ f \circ u$
   - where $u = [x \mapsto \star]$ or $u = [x \mapsto \star']$ both satisfies

> Exercise 3: A regular monomorphism is an arrow $e : E \rightarrowtail A$ which is an equalizer of some pair of arrows $f, g : A \to B$. Recall that by Proposition 3.16 of SA e is in particular a monomorphism. Show that in a pullback square below, if e is a regular monomorphism then so is e' for any object C and any arrow h. This property is often called "regular monos are stable under pullback".

$$\begin{tikzcd}
E' \arrow[r, "h'"] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "e"] \\
C \arrow[r, "h"] & A
\end{tikzcd}$$

Given
1. $C \overset{e'}{\leftarrow} E' \overset{h'}{\to} E$ are a pullback of $C \overset{h}{\to} A \overset{e}{\leftarrow} E$
2. $e : E \to A$ is a regular monomorphism

We prove that e' is a regular monomorphism by showing
- e' is a monomorphism by (4)
- e' is an equalizer for any C and h by (5) and (6)

$$\begin{tikzcd}
X \arrow[r, dashed, "u"] \arrow[rr, bend left, "f"] \arrow[dr, bend right, "g"'] & E' \arrow[r, "h'"] \arrow[d, "e'"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, tail, "e"'] \\
& C \arrow[r, "h"] & A
\end{tikzcd}$$

3. Given by (1) and Definition 4.2, for any X
   1. $(h \circ e' = e \circ h') \wedge$
   2. $\forall X, (f : X \to E), (g : X \to C), \exists!(u : X \to E').$
   3. $\quad (e' \circ u = g) \wedge$
   4. $\quad (h' \circ u = f)$

Explanation: Given the pullback, for any X and morphisms $f : X \to E, g : X \to C$, there exists a unique u such that the square commutes even with u precomposed and that the triangles X, E', C and X, E', E commutes as well.

$$\begin{tikzcd} X \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] & E' \arrow[r, "h'"] \arrow[d, tail, "e'"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, tail, "e"'] \\ & C \arrow[r, "h"] & A \end{tikzcd}$$

Let $\forall(u_1, u_2 : X \to E')$. be the scope of the following formulas

4. By (3.1) where $h \circ e' = e \circ h'$ we know the following

   $$((e \circ h') \circ u_1 = (h \circ e') \circ u_1) \wedge ((e \circ h') \circ u_2 = (h \circ e') \circ u_2)$$

5. Supposing $e' \circ u_1 = e' \circ u_2$ we derive from (4) that

   $e' \circ u_1 = e' \circ u_2 \Rightarrow$

   $e \circ h' \circ u_1 = h \circ e' \circ u_1 = h \circ e' \circ u_2 = e \circ h' \circ u_2$

6. Given (2) where e is a monomorphism we know

   $$e \circ h' \circ u_1 = e \circ h' \circ u_2 \Rightarrow h' \circ u_1 = h' \circ u_2$$

7. The consequent of (5) satisfies the antecedent of (6), thus by transitivity we have

   $$e' \circ u_1 = e' \circ u_2 \Rightarrow h' \circ u_1 = h' \circ u_2$$

8. Given (1) and (3), there must exist a unique $v : X \to E'$, since both u_1, u_2 satisfies v, they must be equal

   $h' \circ u_1 = h' \circ v \wedge e' \circ u_1 = e' \circ v \wedge$

   $h' \circ u_2 = h' \circ v \wedge e' \circ u_2 = e' \circ v \Rightarrow u_1 = u_2$

9. (7) satisfies the antecedent of (8), thus by transitivity we have that e' is a monomorphism

   $$e' \circ u_1 = e' \circ u_2 \Rightarrow u_1 = u_2$$

Explanation: The square commutes (4), additionally with $e' \circ u_1 = e' \circ u_2$ as the premise for e' to be a monomorphism, we have that $e \circ h' \circ u_1 = e \circ h' \circ u_2$ in (5). This satisfies the premise for e to be a monomorphism in (6). Thus we get the consequent $h' \circ u_1 = h' \circ u_2$ in (7). Moreover, since morphisms $X \to E'$ have to be unique by UMP of the pullback, they must be equal (8). Therefore e' is a monomorphism (9).

$$\begin{tikzcd} X \arrow[r, dashed, "u"] \arrow[dr, "x_C"] & E' \arrow[r, "h'"] \arrow[d, tail, "e'"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, tail, "e"'] \\ & C \arrow[r, "h"] & A \arrow[d, shift right, "a'"'] \arrow[d, shift left, "a"] \\ & & B \end{tikzcd}$$

Let $\forall(a, a' : A \to B)$. be the scope of the following formulas

10. By (2) where e is an equalizer

    $$a \circ e = a' \circ e$$

11. By (3.1); $h \circ e' = e \circ h'$, and (10), e' equalizes $a \circ h$ and $a' \circ h$

    $(a \circ e) \circ h' = (a' \circ e) \circ h' =$

    $(a \circ h) \circ e' = (a' \circ h) \circ e'$

12. Given (1) and (3), there must exist a unique $u : X \to E'$ for any X and $x_C : X \to C$ such that $e' \circ u = x_C$ in (3.3).

    $a \circ h \circ e' \circ u = a \circ h \circ x_C \wedge$

    $a' \circ h \circ e' \circ u = a' \circ h \circ x_C$

13. Since e' is universal (12), equalizes on h (11) and is a monomorphism (9), e' must be a regular monomorphism.

Explanation: It is given e is an equalizer for a, a' in (10) if we precompose it by h' in (11) it will still hold. Since the square commutes the identity holds for a, a' composed with $h \circ e'$ as well. Moreover e' is universal by the the uniqueness of u induced by the pullback. Therefore e' is an equalizer for $h \circ a$ and $h \circ a'$. Making it both a monomorphism and an equalizer and thus a regular monomorphism ■

second attempt:

$$
\begin{aligned}
&\mathrm{UMP}_{\mathrm{pullback}}(e', h', h, e) \wedge \mathrm{UMP}_{\mathrm{equalizer}}(e, a, a') \wedge \mathrm{isMono}(e) \\
=\ &((he' = eh') \wedge (\forall f, g. \exists! u. f = e'u \wedge g = h'u)) \wedge \\
&(\forall v_1 . av_1 = a'v_1 \wedge \exists! u_1 . eu_1 = v_1) \wedge \\
&(\forall v_2, v_3 . ev_2 = ev_3 \Rightarrow v_2 = v_3) \\
&\text{let } v_1 = hf, u_1 = h'u, v_2 = h'u_1, v_3 = h'u_2 \\
=\ &(he' = eh') \wedge (\forall f, g. \exists! u. (f = e'u \wedge g = h'u) \wedge \\
&(a(hf) = a'(hf) \wedge e(h'u) = (hf)) \wedge \\
&(\forall u_1, u_2 . eh'u_1 = eh'u_2 \Rightarrow h'u_1 = h'u_2)) \\
=\ &... \\
=\ &(\forall f. haf = ha'f \wedge \exists! u. e'u = f) \wedge \\
&(\forall u_1, u_2 . e'u_1 = e'u_2 \Rightarrow u_1 = u_2) \wedge ... \\
=\ &\mathrm{UMP}_{\mathrm{equalizer}}(e', h \circ a, h \circ a') \wedge \mathrm{isMono}(e') \wedge ...
\end{aligned}
$$

Exercise 4: Let $e : A \rightarrowtail B$ be a regular monomorphism. Show that if the square is a pushout then e is the equalizer of x and y

$$
\begin{tikzcd}
A \arrow[r, tail, "e"] \arrow[d, tail, "e"'] & B \arrow[d, tail, "y"] \\
B \arrow[r, "x"] & Q \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

1. we have the following pushout with e as a regular monomorphism

$$
\begin{tikzcd}
A \arrow[r, tail, "e"] \arrow[d, tail, "e"'] & B \arrow[d, tail, "y"] \arrow[dr, "x_B"] & \\
B \arrow[r, "x"] \arrow[rr, bend right, "x_B"'] & Q \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "u"] & X
\end{tikzcd}
$$

2. let $u : Q \to X$ be a pushout for any X
3. thus $e \circ x \circ u = e \circ y \circ u$
4. thus $e \circ x = e \circ y \Rightarrow x = y$, satisfies the existence equalizer UMP

$$
\begin{tikzcd}
Y \arrow[r, dashed, "v"] \arrow[rr, bend left, "y_B"] \arrow[dr, "y_B"'] & A \arrow[r, tail, "e"] \arrow[d, tail, "e"'] & B \arrow[d, tail, "y"] \arrow[dr, "x_B"] & \\
& B \arrow[r, "x"] \arrow[rr, bend right, "x_B"'] & Q \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "u"] & X
\end{tikzcd}
$$

5. for any Y, $v : Y \to A$ must be unique by e being a mono
6. thus $v \circ e = y_B$, satisfies the uniqueness equalizer UMP
7. therefore by (4) and (6) e is the equalizer of x and y

Exercise 5: Let $F : \mathbb{C} \to \mathbb{D}$ be a finite limit preserving functor. Show that for any monomorphism $m : A \rightarrowtail B$ the morphism $F(m) : F(A) \to F(B)$ is also a monomorphism, i.e. F preserves monomorphisms. Dualizing, show that if $F : \mathbb{C} \to \mathbb{D}$ preserves finite colimits then it preserves epimorphisms.

Exercise 6: Give an example of each of a functor $\mathbf{Sets} \to \mathbf{Sets}$ that:

Exercise 6.1: Both preserves and creates terminal objects;

Exercise 6.2: Preserves, but does not create, terminal objects

Exercise 6.3: Neither preserves nor creates terminal objects

Exercise 6.4: Finally show that any functor $\mathbf{Sets} \to \mathbf{Sets}$ which creates terminal objects also preserves them.

4.3 Relevant Exercises

§5 Exponentials

5.1 Notes

Definition 5.1 (*Exponentials*) [1, Definition 6.1]

$$\text{isExponential}(C^B, \varepsilon) = \forall A, (f : A \times B \to C).\exists!\left(\tilde{f} : A \to C^B\right).\varepsilon \circ \left(\tilde{f} \times 1_B\right) =\!\!-\left(\tilde{f}\right) = f$$

Definition 5.2 (*Cartesian Closed Category (CCC)*) [1, Definition 6.2]

$$\begin{aligned}\text{isCCC}(\mathbf{Ob}, \mathbf{Hom}) = {} & \text{isCategory}(\mathbf{Ob}, \mathbf{Hom}) \\ & \wedge \forall A, B.\exists! A \times B.\text{UMP}_{\text{product}}(p_1 : A \times B \to A, p_2 : A \times B \to B) \\ & \wedge \forall B, C.\ \text{isExponential}(C^B, \varepsilon)\end{aligned}$$

todo: categorical logic

- Heyting Algebra ~ Intuitionistic Propositional Calculus
- CCC ~ λ-calculus
- Kripke models of logic; variable sets
- theory: a set of basic types and terms and equations between them (generators; recall the section on coequalizer)

5.2 Mandatory Exercises

5.3 Relevant Exercises

§6 Appendix

$$
\begin{aligned}
&\text{isCategory}(\mathcal{C}) = && \forall A, B, C \in \mathbf{Ob}_{\mathcal{C}}, f \in \mathbf{Hom}_{\mathcal{C}}(A, B), g \in \mathbf{Hom}_{\mathcal{C}}(B, C).g \circ f \in \mathbf{Hom}_{\mathcal{C}}(A, C) \\
& && \land \forall X \in \mathbf{Ob}_{\mathcal{C}}.\exists! 1_X \in \mathbf{Hom}(X, X) \\
& && \land \forall h, g, f.h \circ (g \circ f) = (h \circ g) \circ f \\
& && \land \forall A, B \in \mathbf{Ob}_{\mathcal{C}}, f \in \mathbf{Hom}_{\mathcal{C}}(A, B).f \circ 1_A = f = 1_B \circ f \\
&\text{isFunctor}(F : C \to D) = && \forall f \in \mathbf{Hom}_C(A, B).F(f) \in \mathbf{Hom}_D(F(A), F(B)) \\
& && \land \forall 1_X \in \mathbf{Hom}_C(X, X).F(1_X) = 1_{F(X)} \\
& && \land \forall g, f \in \mathbf{Hom}_C.F(g \circ f) = F(g) \circ F(f) \\
&\text{isIso}(f : A \to B) = && \exists! g.g \circ f = 1_A \land f \circ g = 1_B \\
&A \cong B = && \exists (f : A \to B).\ \text{isIso}(f) \\
&C \times D = && (\mathbf{Ob}_C \times \mathbf{Ob}_D, [(C_1, D_1), (C_2, D_2) \mapsto \mathbf{Hom}_C(C_1, C_2) \times \mathbf{Hom}_D(D_1, D_2)]) \\
&C^{\text{op}} = && (\mathbf{Ob}_C, [A, B \mapsto \{f^{\text{op}} : A \to B \mid f \in \mathbf{Hom}_C(B, A)\}]) \\
&C^{\to} = && (\mathbf{Hom}_C, [f, f' \mapsto \{(g, g') \mid g' \circ f = f' \circ g\}]) \\
&C/A = && (\{f \mid f \in \mathbf{Hom}_C(X, A)\}, [f, f' \mapsto \mathbf{Hom}_C(\mathbf{dom}(f), \mathbf{dom}(f')))) \\
&C \setminus A = && (\{f \mid f \in \mathbf{Hom}_C(A, X)\}, [f, f' \mapsto \mathbf{Hom}_C(\mathbf{cod}(f), \mathbf{cod}(f'))]) \\
&\text{isMono}(f) = && \forall g, h.f \circ g = f \circ h \Rightarrow g = h \\
&\text{isEpi}(f) = && \forall g, h.g \circ f = h \circ f \Rightarrow g = h \\
&\text{isSplitMono}(m, s) = && s \circ m = 1_{\mathbf{dom}(m)} \\
&\text{isSplitEpi}(e, s) = && e \circ s = 1_{\mathbf{cod}(e)} \\
&\text{areIso}(f, g) = && \text{isSplitEpi}(f, g) \land \text{isSplitMono}(f, g) \\
&\text{isProjective}(P) = && \forall e : E \twoheadrightarrow X, f : P \to X.\exists \overline{f} : P \to E.e \circ \overline{f} = f \\
&\text{UMP}_{\text{freemonoid}}\left(|\overline{f}|\right) = && \forall i, f.\exists! \overline{f}.\ |\overline{f}| \circ i = f \\
&\text{UMP}_{\text{terminal}}(0_X) = && \forall X.\exists! 0.0_X \in \mathbf{Hom}(0, X) \\
&\text{UMP}_{\text{initial}}(1_X) = && \forall X.\exists! 1.1_X \in \mathbf{Hom}(X, 1) \\
&\text{UMP}_{\text{product}}(p_1, p_2) = && \forall x_1, x_2.\exists! u.x_1 = p_1 \circ u \land x_2 = p_2 \circ u \\
&\text{UMP}_{\text{coproduct}}(q_1, q_2) = && \forall x_1, x_2.\exists! u.x_1 = u \circ q_1 \land x_2 = u \circ q_2 \\
&\text{UMP}_{\text{equalizer}}(e, f, g) = && \forall z.(f \circ z = g \circ z) \land \exists! u.e \circ u = z \\
&\text{UMP}_{\text{coequalizer}}(q, f, g) = && \forall z.(z \circ f = z \circ g) \land \exists! u.u \circ q = z \\
&\text{UMP}_{\text{pullback}}(p_1, p_2, f, g) = && (f \circ p_1 = g \circ p_2) \\
& && \land \forall z_1, z_2.\exists! u.z_1 = p_1 \circ u \land z_2 = p_2 \circ u \\
&\text{UMP}_{\text{limit}}(\{p_i\}, D) = && \forall \{c_j\}.\exists! u.\forall j.p_j \circ u = c_j \\
&\text{isExponential}(C^B, \varepsilon) = && \forall A, (f : A \times B \to C).\exists!\left(\tilde{f} : A \to C^B\right).\varepsilon \circ \left(\tilde{f} \times 1_B\right) =- \left(\tilde{f}\right) = f \\
&\text{isCCC}(\mathbf{Ob}, \mathbf{Hom}) = && \text{isCategory}(\mathbf{Ob}, \mathbf{Hom}) \\
& && \land \forall A, B.\exists! A \times B.\text{UMP}_{\text{product}}(p_1 : A \times B \to A, p_2 : A \times B \to B) \\
& && \land \forall B, C.\ \text{isExponential}(C^B, \varepsilon)
\end{aligned}
$$

TODO LIST:

list of functors: $|-|, (-)^A, (-)^{\mathrm{op}}, \mathbf{Hom}(A,-), \mathbf{Hom}(-,A)$

todo in chapter4 notes
- $(f)^A = \widetilde{f \circ \varepsilon}$
- define Sub and Cone category

$$\begin{tikzcd}
X \arrow[r, dashed, "u"] \arrow[rr, bend left, "f"] \arrow[dr, bend right, "g"'] & E' \arrow[r, "h'"] \arrow[d, "e'"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, tail, "e"'] \\
& C \arrow[r, "h"] & A \arrow[r, shift left, "a'"] \arrow[r, shift right, "a"'] & B
\end{tikzcd}$$

Lemma 1: e' is mono

$$\dfrac{\forall fg.e'f = e'g \quad u_1 \quad u_2}{e'u_1 = e'u_2}\,\text{mono-antecedent}$$

$$\dfrac{\dfrac{\text{Mono}(e) \quad h'u_1 \quad h'u_2}{eh'u_1 = eh'u_2 \Rightarrow h'u_1 = h'u_2}\,\text{mono-elim} \qquad \dfrac{\text{Pullback }(e', h')}{he' = eh'}\,\text{pullback-elim}^{\text{diagram}}}{he'u_1 = he'u_2 \Rightarrow h'u_1 = h'u_2}\,=$$

$$\dfrac{\dfrac{e'u_1 = e'u_2 \quad he'u_1 = he'u_2 \Rightarrow h'u_1 = h'u_2}{he'u_1 = he'u_1 \Rightarrow h'u_1 = h'u_2}\,=}{h'u_1 = h'u_2}\,\Rightarrow\text{-elim}$$

$$\dfrac{\dfrac{\text{Pullback }(e', h') \quad u_1 \quad u_2}{e'u_1 = e'u_2 \wedge h'u_1 = h'u_2 \Rightarrow u_1 = u_2}\,\text{pullback-elim}^{\text{unique}} \qquad e'u_1 = e'u_2 \quad h'u_1 = h'u_2}{u_1 = u_2}\,\Rightarrow\text{-elim}$$

$$\dfrac{\dfrac{e'u_1 = e'u_2 \quad u_1 = u_2}{e'u_1 = e'u_2 \Rightarrow u_1 = u_2}\,\Rightarrow\text{-intro}}{\text{Mono}(e')}\,\text{mono-intro}$$

Lemma 2: e' is regular / equalizer

$$\dfrac{\dfrac{\dfrac{\text{Equalizer}(e) \quad a \quad a'}{ae = a'e}\,\text{equalizer-elim}^{\text{diagram}}}{aeh' = a'eh'}\,= \qquad \dfrac{\text{Pullback }(e', h')}{he' = eh'}\,\text{pullback-elim}^{\text{diagram}}}{ahe' = a'he'}\,=$$

$$\dfrac{\dfrac{\text{Mono}(e') \quad u_1 \quad u_2}{e'u_1 = e'u_2 \Rightarrow u_1 = u_2}\,\text{mono-elim}}{he'u_1 = he'u_2 \Rightarrow u_1 = u_2}\,=$$

$$\frac{he'u_1 = he'u_2 \Rightarrow u_1 = u_2 \qquad ahe' = a'he'}{\text{Equalizer}(e')}\text{equalizer-intro}$$

Proof: e' is a regular mono

$$\frac{\text{Mono}(e') \qquad \text{Equalizer}(e')}{\text{RegularMono}(e')}\text{regular-mono-intro}$$

■

assumptions

1. composing a morphism on both sides of an equality remains equal
   - i.e. $f = g, h \vdash hf = hg$ and $f = g, h \vdash fh = gh$
2. substituting a quantifier argument with another quantifier argument but composed is the same
   - i.e. $\text{Mono}(e), ab, c \vdash eab = ec \Rightarrow ab = c$ where a is a quantified argument or fixed morphism and b otherwise
   - which should be true by definition of existence of composition morphism in a category

Bibliography

[1] S. Awodey, *Category Theory*. Ebsco Publishing, 2006.