3. Elastic Wave Equation

M. Ravasi

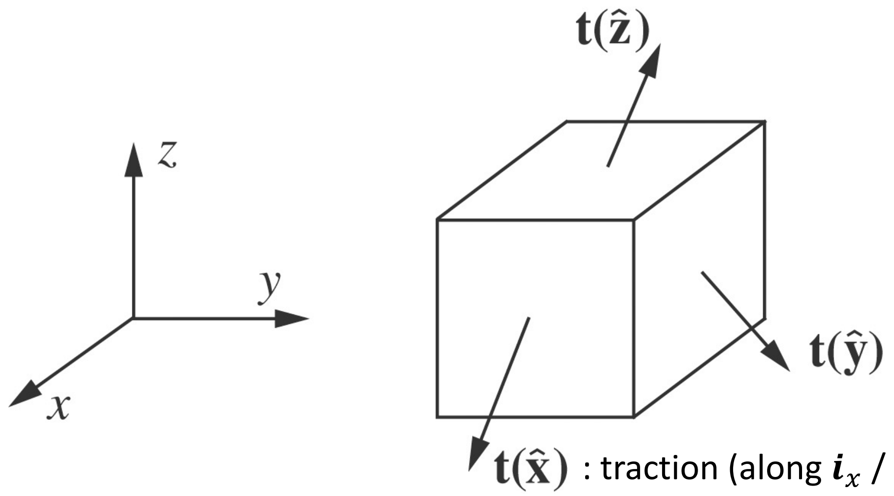
ERSE 210 Seismology

Elastic wave propagation

- **P-waves or compressional:** displacement longitudinal to propagation (like acoustic waves)

- **S-waves or shear:** displacement transverse to propagation (due to shearing of medium, not possible in acoustic media)

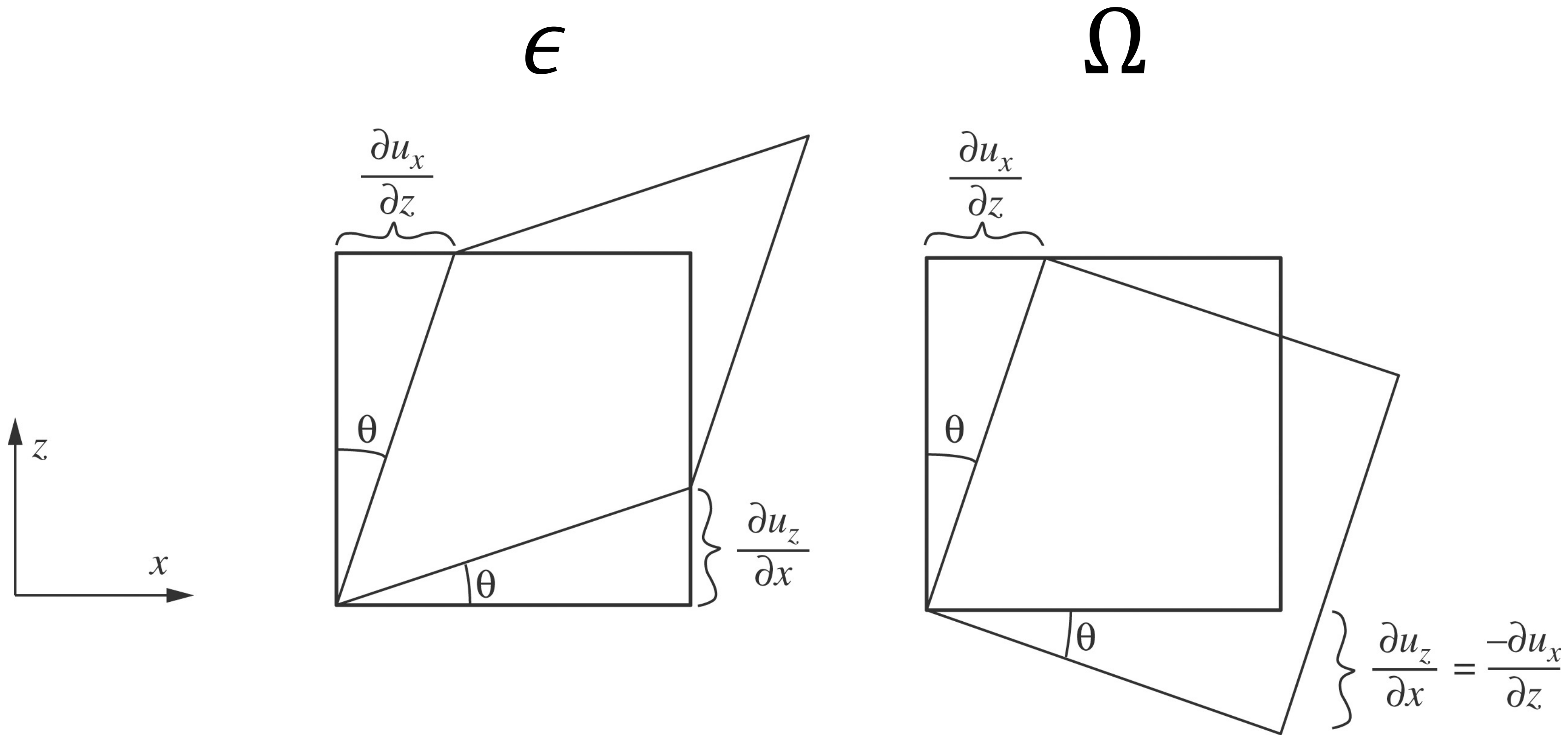
Stresses



: traction (along $\boldsymbol{i}_x$ / across yz plane)

$$\tau = \begin{bmatrix} \boldsymbol{t}(\boldsymbol{i}_x) \\ \boldsymbol{t}(\boldsymbol{i}_y) \\ \boldsymbol{t}(\boldsymbol{i}_z) \end{bmatrix}$$

Strains



ϵ
Ω
$\frac{\partial u_x}{\partial z}$
θ
$\frac{\partial u_z}{\partial x}$
$\frac{\partial u_x}{\partial z}$
θ
$\frac{\partial u_z}{\partial x} = \frac{-\partial u_x}{\partial z}$
z
x

Hooke's law

$$\tau_{ij} = C_{ijkl}\,\varepsilon_{kl} = \frac{1}{2} C_{ijkl}\,(\partial_l u_k + \partial_k u_l)$$

Compliance or Elasticity tensor

Isotropic Hooke's law

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\, \varepsilon_{ij}$$

Lamè parameters

Elastic velocities

P-wave velocity $$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

S-wave velocity $$\beta = \sqrt{\frac{\mu}{\rho}}$$

Elastic wave equation constituents

- **Principle of Inertia** $$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \tau_{ij} + f_i$$

- **Hooke's law** $$\tau_{ij} = C_{ijkl}\, \varepsilon_{kl}$$

Isotropic Elastic wave equation

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \nabla\lambda(\nabla \cdot \boldsymbol{u}) + \nabla\mu \cdot [\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\boldsymbol{T}}] + (\lambda + 2\mu)\nabla\nabla \cdot \boldsymbol{u} - \mu\nabla\times\nabla\times\boldsymbol{u}$$

Helmoltz decomposition theorem

Solenoidal or rotational potential

$$\boldsymbol{u} = \nabla\phi + \nabla\times\boldsymbol{\Psi}$$

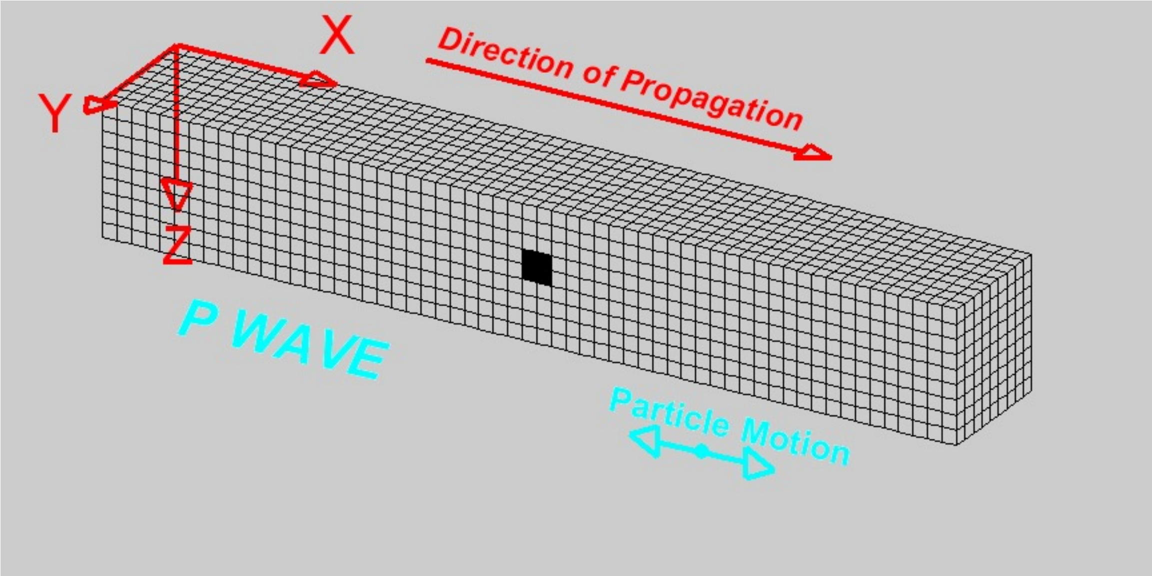
Irrotational or central potential

Potential elastic wave equations

- **P-wave:** $\nabla \cdot (WE) \quad \rightarrow \quad \nabla^2\phi - \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}$

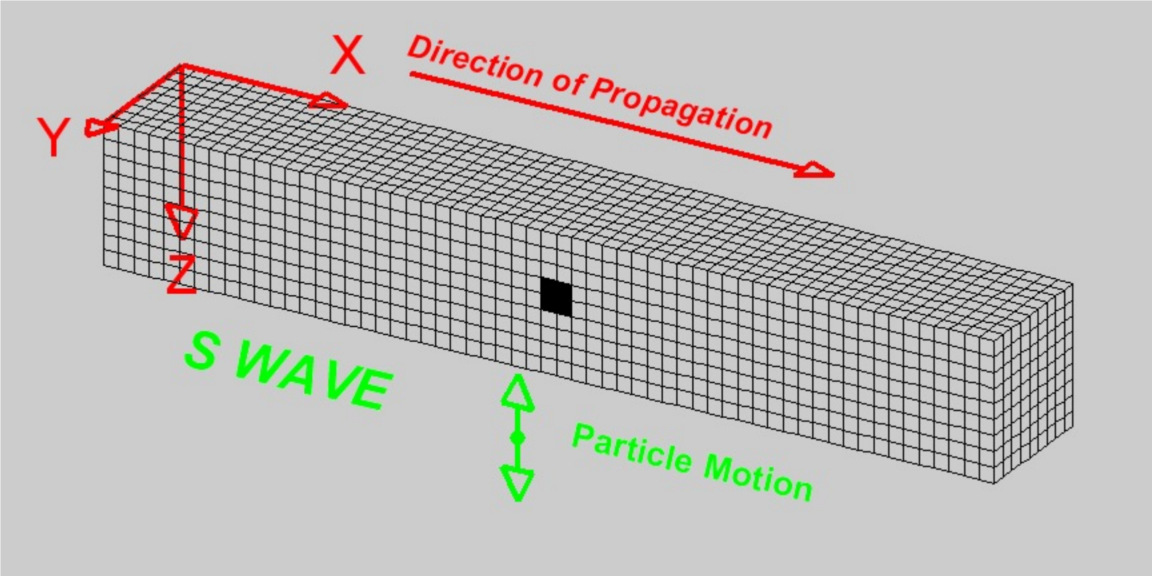
- **S-wave:** $\nabla \times (WE) \quad \rightarrow \quad \nabla^2\mathbf{\Psi} - \frac{1}{\beta^2}\frac{\partial^2\mathbf{\Psi}}{\partial t^2}$

Compressional waves



Source: https://web.ics.purdue.edu/~braile/edumod/waves/Pwave.htm

Shear waves (transverse SV)



Source: https://web.ics.purdue.edu/~braile/edumod/waves/Swave.htm