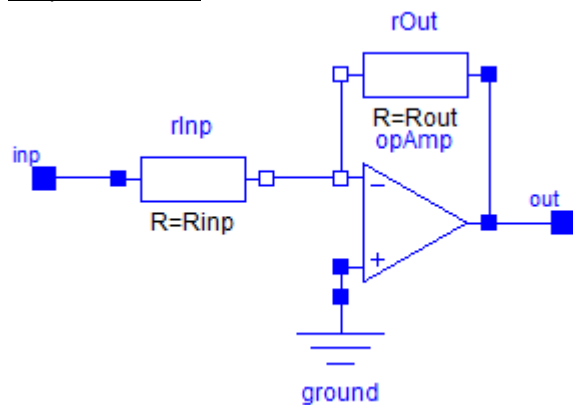
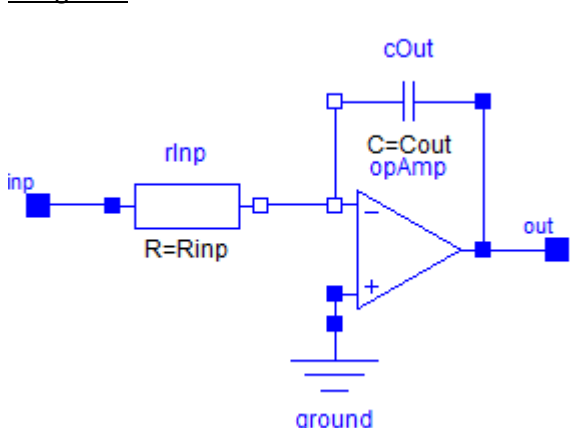
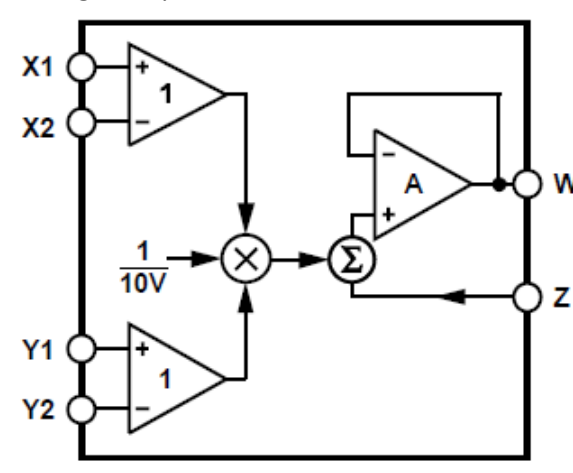


**Comparison of chaotic circuits**

Lotka-Volterra	$\frac{dx_i}{dt} = r_i \cdot x_i \cdot \left( 1 - \sum_{j=1}^n a_{ij} \cdot x_j \right)$
van der Pol Circuit	$\begin{aligned} \frac{dx}{d\tau} &= y \\ y &= \mu \cdot \left[ 1 - \frac{1}{3} \cdot x^2 \right] \cdot x - z + \left\{ A \cdot \cos(w \cdot \tau - \pi) \right\}^0 \\ \frac{dz}{d\tau} &= x \end{aligned}$
Lorenz System	$\begin{aligned} \frac{dx}{dt} &= \sigma \cdot (y - x) \\ \frac{dy}{dt} &= x \cdot (\rho - z) - y \\ \frac{dz}{dt} &= x \cdot y - \beta \cdot z \end{aligned}$
Roessler System	$\begin{aligned} \frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + a \cdot y \\ \frac{dz}{dt} &= b + (x - c) \cdot z \end{aligned}$
Chua's Circuit	nonlinear conductor (two NICs) with partly negative slope
Chaotic Diode Circuit	$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= -z \\ \frac{dz}{dt} &= -x + a \cdot (e^y - 1) - b \cdot z \end{aligned}$
Chaotic Oscillator A simple chaotic oscillator for educational purposes	$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= a \cdot y - x - z \\ \frac{dz}{dt} &= b + y - c \cdot (e^z - 1) \end{aligned}$
Colpitts Oscillator	LC oscillator with transistor (orig. vacuum tube)
Shinriki Oscillator	two antiparallel Z-diodes and nonlinear conductor (NIC) with partly negative slope
Jerk Circuit	$\ddot{x} + G(\ddot{x}, \dot{x}, x) = 0$
Rikitake System	Two coupled disc dynamos.
Simplest Chaotic Circuit	A series resonance circuit with a special memristor

## OpAmp-Circuits

Algebraic-Differential Equation Systems can be simulated with analog computers using operational amplifiers and analog multipliers.

<p><u>Amplifier/Adder</u></p> 	$\frac{inp.v}{R_{inp}} + \frac{out.v}{R_{out}} = 0$ $-out.v = k \cdot inp.v$ $k = \frac{R_{out}}{R_{inp}}$ <p>Input resistance = <math>R_{inp}</math>  Output resistance <math>\rightarrow 0</math>  It is possible to add several inputs.</p>
<p><u>Integrator</u></p> 	$\frac{inp.v}{R_{inp}} + C_{out} \cdot \frac{d out.v}{dt} = 0$ $-out.v = v_0 + \frac{1}{\tau} \cdot \int_0^t inp.v \cdot dt$ $\tau = R_{inp} \cdot C_{out}$ <p>Input resistance = <math>R_{inp}</math>  Output resistance <math>\rightarrow 0</math>  It is possible to integrate the sum of several inputs.</p>
<p><u>Analog Multiplier</u></p> 	<p>Functional Block Diagram of AD633  Division by 10 V (scaling) inhibits overflow.  Additional summing input Z is omitted.  Negative inputs of X- and Y-amplifiers are connected to ground.</p> <p>Possible implementations:</p> <ul style="list-style-type: none"> <li>• Gilbert cell</li> <li>• <math>y = e^{\ln(x_1) + \ln(x_2)}</math></li> <li>• <math>y = \frac{(x_1 + x_2)^2 - (x_1 - x_2)^2}{4}</math></li> </ul>

## **Lotka-Volterra**

[Vano2006]

As a 2-dimensional predator-prey-model no chaos is reported:

$$\begin{aligned}\frac{dx}{dt} &= r_x \cdot x - d_x \cdot x \cdot y \\ \frac{dy}{dt} &= d_x \cdot x \cdot e_y \cdot y - d_y \cdot y\end{aligned}$$

We might interpret  $x$  as number of hares (prey) and  $y$  as foxes (predator).

$r_x$  is the reproduction rate of hares,  $d_x$  the deathrate of hares due to foxes.

$e_y$  is the efficiency in growing foxes from hares,  $d_y$  the (natural) deathrate of foxes.

A n-dimensional Lotka-Volterra model is defined for  $2 \leq i \leq n$  ( $n$  designates the number of species):

$$\frac{dx_i}{dt} = r_i \cdot x_i \cdot \left( 1 - \sum_{j=1}^n a_{ij} \cdot x_j \right)$$

The vector  $[r]$  describes the reproduction rates of the species whereas the quadratic matrix  $[a]$  describes the competition between species.

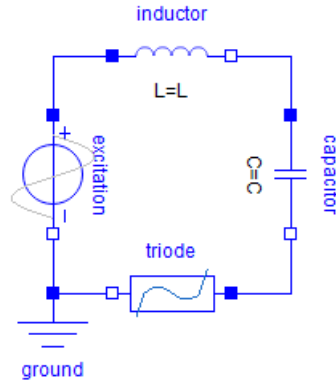
The above mentioned predator-prey case with 2 species can be expressed as:

$$\begin{aligned}[r] &= \begin{bmatrix} r_x \\ -d_y \end{bmatrix} \\ [a] &= \begin{bmatrix} 0 & \frac{d_x}{r_x} \\ \frac{d_x \cdot e_y}{d_y} & 0 \end{bmatrix}\end{aligned}$$

# van der Pol Circuit

[Buscarino2014], [VanDerPol1927]

Balthasar van der Pol reported 1927 strange phenomena about oscillations in a series resonance circuit containing a vacuum electron triode. Due to the nonlinear characteristic of the triode the autonomous circuit is able to maintain periodic oscillations, and with harmonic excitation it is able to produce chaos.



$$i = C \cdot \frac{dv_C}{dt} \rightarrow v_C = v_{C0} + \frac{1}{C} \cdot \int_{t_0}^t i \cdot dt'$$

$$L \cdot \frac{di}{dt} + \tilde{R}(i) \cdot i + v_C = \begin{cases} 0 \\ \hat{V} \cdot \cos(\omega \cdot t - \pi) \end{cases}$$

$$\tilde{R}(i) = -R_0 \cdot \left[ 1 - \frac{1}{3} \cdot \left( \frac{i}{I_0} \right)^2 \right]$$

These are the equations of the physical model.

Note the phase shift of the excitation!

$$L \cdot \frac{d^2 i}{dt^2} - R_0 \cdot \left[ 1 - \left( \frac{i}{I_0} \right)^2 \right] \cdot \frac{di}{dt} + \frac{1}{C} \cdot i = \begin{cases} 0 \\ \hat{V} \cdot \omega \cdot \sin(\omega \cdot t) \end{cases}$$

$$\omega_0 = \frac{1}{\sqrt{L \cdot C}}$$

$$\mu = R_0 \cdot \sqrt{\frac{C}{L}}$$

$$A = \frac{\omega_0 \cdot C \cdot \hat{V}}{I_0}$$

$$w = \frac{\omega}{\omega_0}$$

$$\tau = \omega_0 \cdot t$$

$$x = \frac{i}{I_0}$$

$$y = \frac{dx}{d\tau} = \frac{1}{\omega_0 \cdot I_0} \cdot \frac{di}{dt}$$

$$\frac{I_0}{\omega_0 \cdot C} \cdot z = v_C = v_{C0} + \frac{1}{C} \cdot \int_{t_0}^t i \cdot dt'$$

$$z = z_0 + \int_{\tau_0}^{\tau} x \cdot d\tau'$$

$\frac{d^2 x}{d\tau^2} - \mu \cdot [1 - x^2] \cdot \frac{dx}{d\tau} + x = \begin{cases} 0 \\ A \cdot w \cdot \sin(w \cdot \tau) \end{cases}$ $\frac{dx}{d\tau} = y$ $\frac{dy}{d\tau} = \mu \cdot [1 - x^2] \cdot y - x + \begin{cases} 0 \\ A \cdot w \cdot \sin(w \cdot \tau) \end{cases}$ $\frac{dz}{d\tau} = x$	<p>Alternative Formulation with 2 states:</p> $\frac{dx}{d\tau} = y$ $y = \mu \cdot \left[ 1 - \frac{1}{3} \cdot x^2 \right] \cdot x - z + \begin{cases} 0 \\ A \cdot \cos(w \cdot \tau - \pi) \end{cases}$ $\frac{dz}{d\tau} = x$
---	--

Note:

Instead of using a series resonance circuit and deriving a scaled differential equation for the current, we could use an equivalent parallel resonance circuit and derive a scaled differential equation for the voltage.

$$v = L \cdot \frac{di_L}{dt} \rightarrow i_L = i_{L0} + \frac{1}{L} \cdot \int_{t_0}^t v \cdot dt'$$

$$i_L + \tilde{G}(v) \cdot v + C \cdot \frac{dv}{dt} = i_e$$

Initialization:

The physical model has 2 states:  $i$  and  $v_C$ . Current  $i$  acts as an initial value for the nonlinear resistor.

The analytic equations have 3 states:  $x$ ,  $y$  and  $z$ .

The third state has been introduced artificially by first differentiating the voltage equation, generating an equation with second derivative of  $i$ . Splitting this equation into two first order differential equations, we generate  $i$  and  $\frac{di}{dt}$  as states. Calculating capacitor voltage  $v$ , we get the third state.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. We chose natural eigen frequency  $\omega_0$  as time scale:

$$x' = \frac{x}{k_x}$$

$$y' = \frac{y}{k_y}$$

$$z' = \frac{z}{k_z}$$

$$t' = \frac{\tau}{\omega_0}$$

We also have to take into account that the analog multiplier divides by  $V_S$  to avoid overflow of the output.

After that, none of the computing block should encounter an overflow.

This leads to the following set of equations:

$$\frac{1}{\omega_0} \cdot \frac{dx'}{dt'} = \frac{k_y}{k_x} \cdot y'$$

$$y' = \mu \cdot \frac{k_x \cdot V_S}{k_y} \cdot \left[ 1 - \frac{k_x^2 \cdot V_S}{3} \cdot \frac{x'^2}{V_S} \right] \cdot \frac{x'}{V_S} - \frac{k_z}{k_y} \cdot z' + \left\{ \frac{A}{k_y} \cdot \cos(w \cdot \omega_0 \cdot t' - \pi) \right\}$$

$$\frac{1}{\omega_0} \cdot \frac{dz'}{dt'} = \frac{k_x}{k_z} \cdot x'$$

These equations can easily get implemented as blocks or as an electronic circuit.

Calculating back from per-unit-parameters:

$$\mu = 0.2, w = 1.15, A = [0..1]$$

and some assumptions:

$$C = \frac{100}{2\pi} \mu F, \omega_0 = 2\pi \cdot 1000 \frac{rad}{s}, I_0 = 0.5 A$$

we obtain physical parameters:

$$L = \frac{1}{\omega_0^2 \cdot C} = \frac{10}{2\pi} mH$$

$$R_0 = \mu \cdot \sqrt{\frac{L}{C}} = \frac{\mu}{\omega_0 \cdot C} = 2 \Omega$$

$$\hat{V} = A \cdot \frac{I_0}{\omega_0 \cdot C} = [0 \dots 5] V$$

Investigating the nonlinear resistance of the triode:

$$v_R = -R_0 \cdot I_0 \cdot \left[ \left( \frac{i}{I_0} \right) - \frac{1}{3} \cdot \left( \frac{i}{I_0} \right)^3 \right] = -R_0 \cdot i \cdot \left[ 1 - \frac{1}{3} \cdot \left( \frac{i}{I_0} \right)^2 \right]$$

$$\frac{v_R}{i} = -R_0 \cdot \left[ 1 - \frac{1}{3} \cdot \left( \frac{i}{I_0} \right)^2 \right]$$

$$\frac{dv_R}{di} = -R_0 \cdot \left[ 1 - \left( \frac{i}{I_0} \right)^2 \right]$$

$$x = \frac{i}{I_0}$$

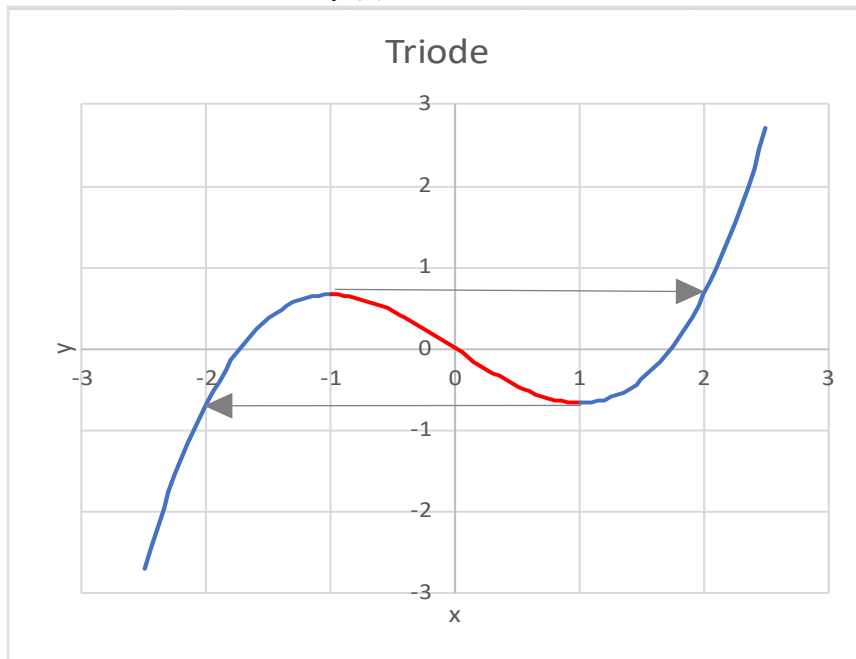
$$y = \frac{v_R}{R_0 \cdot I_0}$$

$$y = - \left( x - \frac{x^3}{3} \right) = -x \cdot \left( 1 - \frac{x^2}{3} \right)$$

$$\frac{y}{x} = - \left( 1 - \frac{x^2}{3} \right)$$

$$\frac{dy}{dx} = -(1 - x^2)$$

Characteristic of the triode  $y(x)$ :



Zero crossings  $y = 0$ :  $x = \{-\sqrt{3}, 0, +\sqrt{3}\}$  with slopes  $\frac{dy}{dx} = \{+2, -1, +2\}$

Extrema:  $x = \{-1, +1\}$  with  $x = \left\{ +\frac{2}{3}, -\frac{2}{3} \right\}$

Inflection point:  $[x, y] = [0, 0]$

If current  $x$  is prescribed, voltage  $y$  can be unambiguously determined.

If  $y$  is prescribed, in the range  $-2 \leq x \leq +2$  i.e.  $-\frac{2}{3} \leq y \leq +\frac{2}{3}$   $x$  has 2 or 3 possible solutions.

For this application, this restriction has no influence.

Shifting the characteristic up and to the right, it looks like the  $i(v)$  characteristic of a tunnel (Esaki) diode.

Inversion of the triode characteristic shows hysteretic behavior (split into 2 branches):

$$x \geq +1: y + \frac{2}{3} = \frac{(x-1)^3}{3} + (x-1)^2$$

$$x \leq -1: y - \frac{2}{3} = \frac{(x+1)^3}{3} - (x+1)^2$$

## Lorenz System

[Buscarino2014], [Lorenz1963]

Developed 1963 by Edward Lorenz to model atmospheric convection.

$x$  is proportional to the rate of convection,  $y$  to the horizontal temperature variation and  $z$  to the vertical temperature variation.  $\sigma$  depicts the Prandtl number,  $\rho$  the Rayleigh number and  $\beta$  the physical dimensions.

The original parameters were:  $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$

$$\begin{aligned}\frac{dx}{dt} &= \sigma \cdot (y - x) \\ \frac{dy}{dt} &= x \cdot (\rho - z) - y \\ \frac{dz}{dt} &= x \cdot y - \beta \cdot z\end{aligned}$$

$\beta = \frac{1}{3}$  leads to a periodic solution.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. This can be compared with calculating per-unit values by dividing by reference values:

$$\begin{aligned}x' &= \frac{x}{k_x} \\ y' &= \frac{y}{k_y} \\ z' &= \frac{z}{k_z} \\ t' &= \frac{t}{\tau}\end{aligned}$$

We also have to take into account that the analog multiplier divides by  $V_S$  to avoid overflow of the output.

After that, none of the computing block should encounter an overflow.

This leads to the following set of equations:

$$\begin{aligned}\frac{1}{\tau} \cdot \frac{dx'}{dt'} &= -\sigma \cdot x' + \sigma \cdot \frac{k_y}{k_x} \cdot y' \\ \frac{1}{\tau} \cdot \frac{dy'}{dt'} &= \rho \cdot \frac{k_x}{k_y} \cdot x' - y' - \frac{k_x \cdot k_z \cdot V_S}{k_y} \cdot \frac{x' \cdot z'}{V_S} \\ \frac{1}{\tau} \cdot \frac{dz'}{dt'} &= \frac{k_x \cdot k_y \cdot V_S}{k_z} \cdot \frac{x' \cdot y'}{V_S} - \beta \cdot z'\end{aligned}$$

These equations can easily get implemented as blocks or as an electronic circuit.

## **Roessler System**

[Roessler1976], [Roessler1979]

A simple system of 3 ordinary nonlinear differential equations to study chaos without physical background.

$$\begin{aligned}\frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + a \cdot y \\ \frac{dz}{dt} &= b + (x - c) \cdot z\end{aligned}$$

$a = 0.2$ ,  $b = 0.2$  and  $c = 1$  give periodic results. Changing  $c = 5.7$  reveals chaotic results.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. This can be compared with calculating per-unit values by dividing by reference values:

$$\begin{aligned}x' &= \frac{x}{k_x} \\ y' &= \frac{y}{k_y} \\ z' &= \frac{z}{k_z} \\ t' &= \frac{t}{\tau}\end{aligned}$$

We also have to take into account that the analog multiplier divides by  $V_S$  to avoid overflow of the output. After that, none of the computing block should encounter an overflow.

This leads to the following set of equations:

$$\begin{aligned}\frac{1}{\tau} \cdot \frac{dx'}{dt} &= -\frac{k_y}{k_x} \cdot y' - \frac{k_z}{k_x} \cdot z' \\ \frac{1}{\tau} \cdot \frac{dy'}{dt} &= \frac{k_x}{k_y} \cdot x' + a \cdot y' \\ \frac{1}{\tau} \cdot \frac{dz'}{dt} &= \frac{b}{k_z} + k_x \cdot V_S \cdot \frac{x' \cdot z'}{V_S} - c \cdot z'\end{aligned}$$

These equations can easily get implemented as blocks or as an electronic circuit.

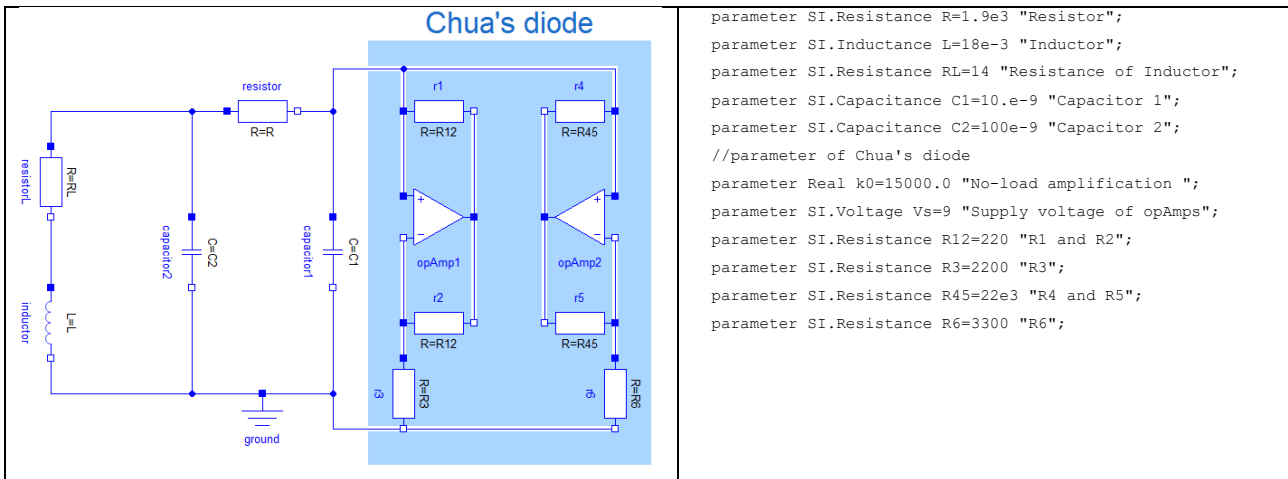


## Chua's Circuit

[Berkeley], [Buscarino2014], [Chua1983], [Kennedy1993a], [Muthuswamy2009], [Zhong1985]

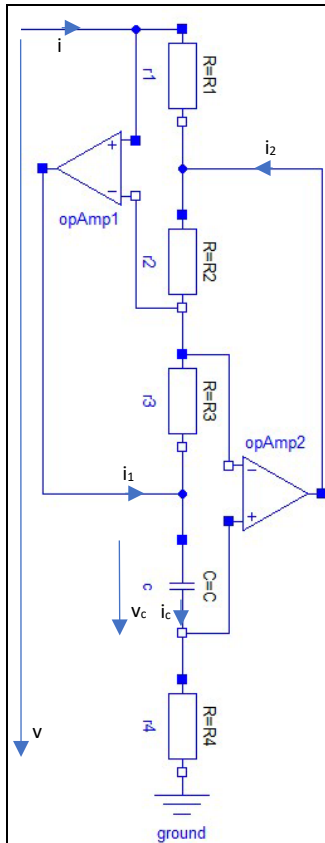
$$\begin{aligned}
 L \cdot \frac{di_L}{dt} &= v_2 - R_L \cdot i_L \\
 C_2 \cdot \frac{dv_2}{dt} &= -i_L - \frac{v_2 - v_1}{R} \\
 C_1 \cdot \frac{dv_1}{dt} &= -i_{NL} + \frac{v_2 - v_1}{R} \\
 -i_{NL}(v_1) &= \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b \cdot (v_1 + V_e) - G_a \cdot V_e \\ -V_e < v_1 < +V_e \rightarrow G_a \cdot v_1 \\ +V_e < v_1 < +\infty \rightarrow G_b \cdot (v_1 - V_e) + G_a \cdot V_e \end{cases} \\
 -\frac{i_{NL}}{v_1} &= \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b - (G_a - G_b) \cdot \frac{V_e}{v_1} \\ -V_e < v_1 < +V_e \rightarrow G_a \\ +V_e < v_1 < +\infty \rightarrow G_b + (G_a - G_b) \cdot \frac{V_e}{v_1} \end{cases} \\
 -\frac{di_{NL}}{dv_1} &= \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b \\ -V_e < v_1 < +V_e \rightarrow G_a \\ +V_e < v_1 < +\infty \rightarrow G_b \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \tau_L \cdot \dot{v}_{RL} &= v_2 - v_{RL} \\
 \tau_2 \cdot \dot{v}_2 &= +v_1 - v_2 - \frac{R}{R_L} \cdot v_{RL} \\
 \tau_1 \cdot \dot{v}_1 &= -v_1 + v_2 + R \cdot g \cdot v_1 \\
 g(v_1) &= \begin{cases} |v_1| > V_e \rightarrow G_b + (G_a - G_b) \cdot \frac{V_e}{|v_1|} \\ |v_1| < V_e \rightarrow G_a \end{cases}
 \end{aligned}$$



This implementation of Chua's Diode with opAmps combines two NICs – see **Investigation of the negative impedance converter (NIC)**.

### Chua's Circuit: Inductor Replacement



OpAmp input currents neglectible

OpAmp differential input voltage neglectible

$$R_1 \cdot i + R_2 \cdot (i + i_2) = 0$$

$$R_3 \cdot (i + i_2) + v_c = 0$$

$$i + i_2 + i_1 = i_c = C \cdot \frac{dv_c}{dt}$$

$$i + i_2 = -\frac{R_1}{R_2} \cdot i$$

$$v_c = \frac{R_1 \cdot R_3}{R_2} \cdot i$$

$$i_c = C \cdot \frac{R_1 \cdot R_3}{R_2} \cdot \frac{di}{dt}$$

$$v = R_4 \cdot i_c = C \cdot \frac{R_1 \cdot R_3 \cdot R_4}{R_2} \cdot \frac{di}{dt}$$

This TwoPin is *not* a OnePort  $i_c \neq i$  and  $i_1 + i_2 \neq 0$ !

The ground at the bottom is necessary.

$$R_1 = 100 \, \Omega$$

$$R_2 = 1 \, k\Omega$$

$$R_3 = 1 \, k\Omega$$

$$R_4 = 1,8 \, k\Omega$$

$$C = 100 \, nF$$

$$L = C \cdot \frac{R_1 \cdot R_3 \cdot R_4}{R_2} = 18 \, mH$$

## Chaotic Diode Circuit

[Pham2016]

$$\begin{aligned} C \cdot \frac{dv_1}{dt} &= -\frac{v_2}{R} \\ C \cdot \frac{dv_2}{dt} &= -\frac{v_3}{R} \\ C \cdot \frac{dv_3}{dt} &= -\frac{v_1}{R} - \frac{v_3}{R_b} - \frac{v_4}{R} \\ \frac{v_4}{R_a} &= -I_{ds} \cdot \left( e^{\frac{v_2}{nV_t}} - 1 \right) \end{aligned}$$

$$\begin{aligned} \tau &= R \cdot C \\ \tau \cdot \frac{dv_1}{dt} &= -v_2 \\ \tau \cdot \frac{dv_2}{dt} &= -v_3 \\ \tau \cdot \frac{dv_3}{dt} &= -v_1 - \frac{R}{R_b} \cdot v_3 + R_a \cdot I_{ds} \cdot \left( e^{\frac{v_2}{nV_t}} - 1 \right) \end{aligned}$$

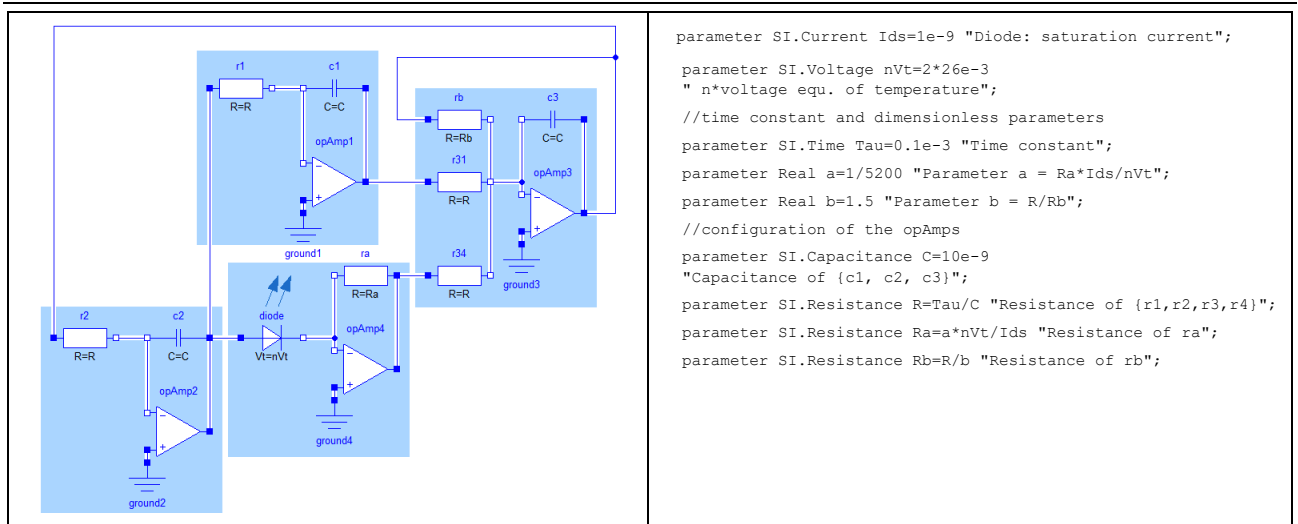
$$a = \frac{R_a \cdot I_{ds}}{nV_t}$$

$$b = \frac{R}{R_b}$$

$$\tau \cdot \dot{x}_1 = -x_2$$

$$\tau \cdot \dot{x}_2 = -x_3$$

$$\tau \cdot \dot{x}_3 = -x_1 + a \cdot (e^{x_2} - 1) - b \cdot x_3$$



## Chaotic Oscillator

[Tamasevicius2005], [Tamasevicius2007]

$$i_L = C \cdot \frac{dv_C}{dt}$$

$$L \cdot \frac{di_L}{dt} = \left(k - 1 - \frac{R_L}{R}\right) \cdot R \cdot i_L - v_C - v_{C^*}$$

$$k = 1 + \frac{R_2}{R_1}$$

$$C^* \cdot \frac{dv_{C^*}}{dt} = I_0 + i_L - I_{DS} \cdot \left(e^{\frac{v_{C^*}}{nV_t}} - 1\right)$$

$$I_0 \approx \frac{V_b}{R_0}$$

$$\tau = \sqrt{L \cdot C}$$

$$Z = \sqrt{\frac{L}{C}}$$

$$a = \left(k - 1 - \frac{R_L}{R}\right) \cdot \frac{R}{Z}$$

$$b = \frac{Z \cdot I_0}{nV_t}$$

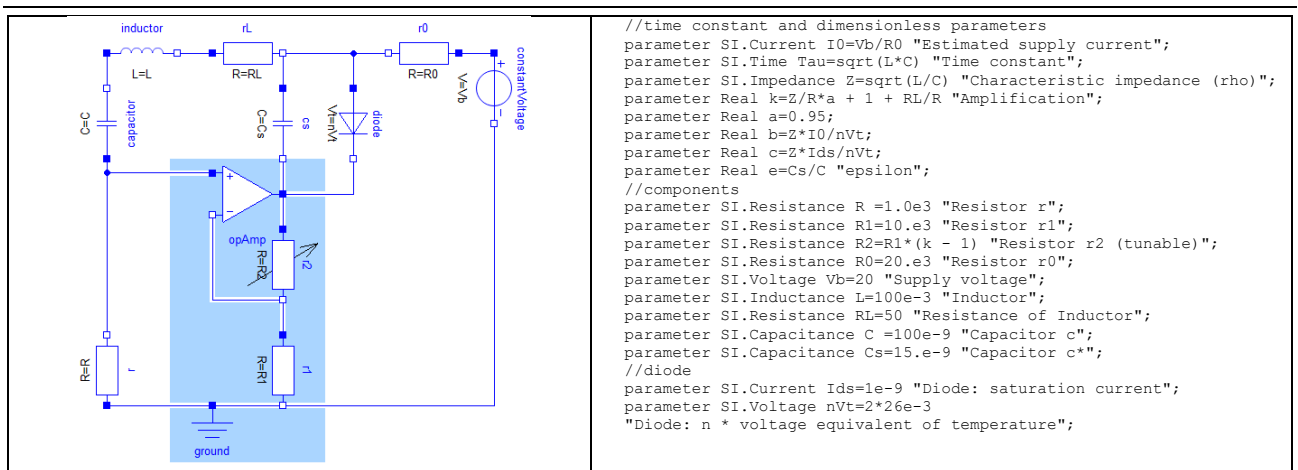
$$c = \frac{Z \cdot I_{DS}}{nV_t}$$

$$e = \frac{C^*}{C}$$

$$\tau \cdot \frac{\dot{v}_C}{nV_t} = \frac{Z \cdot i_L}{nV_t}$$

$$\tau \cdot \frac{Z \cdot i_L}{nV_t} = \left(k - 1 - \frac{R_L}{R}\right) \cdot \frac{R}{Z} \cdot \frac{Z \cdot i_L}{nV_t} - \frac{v_C}{nV_t} - \frac{v_{C^*}}{nV_t}$$

$$\tau \cdot e \cdot \frac{\dot{v}_{C^*}}{nV_t} = \frac{Z \cdot I_0}{nV_t} + \frac{Z \cdot i_L}{nV_t} - \frac{Z \cdot I_{DS}}{nV_t} \cdot \left(e^{\frac{v_{C^*}}{nV_t}} - 1\right)$$



## Colpitts Oscillator

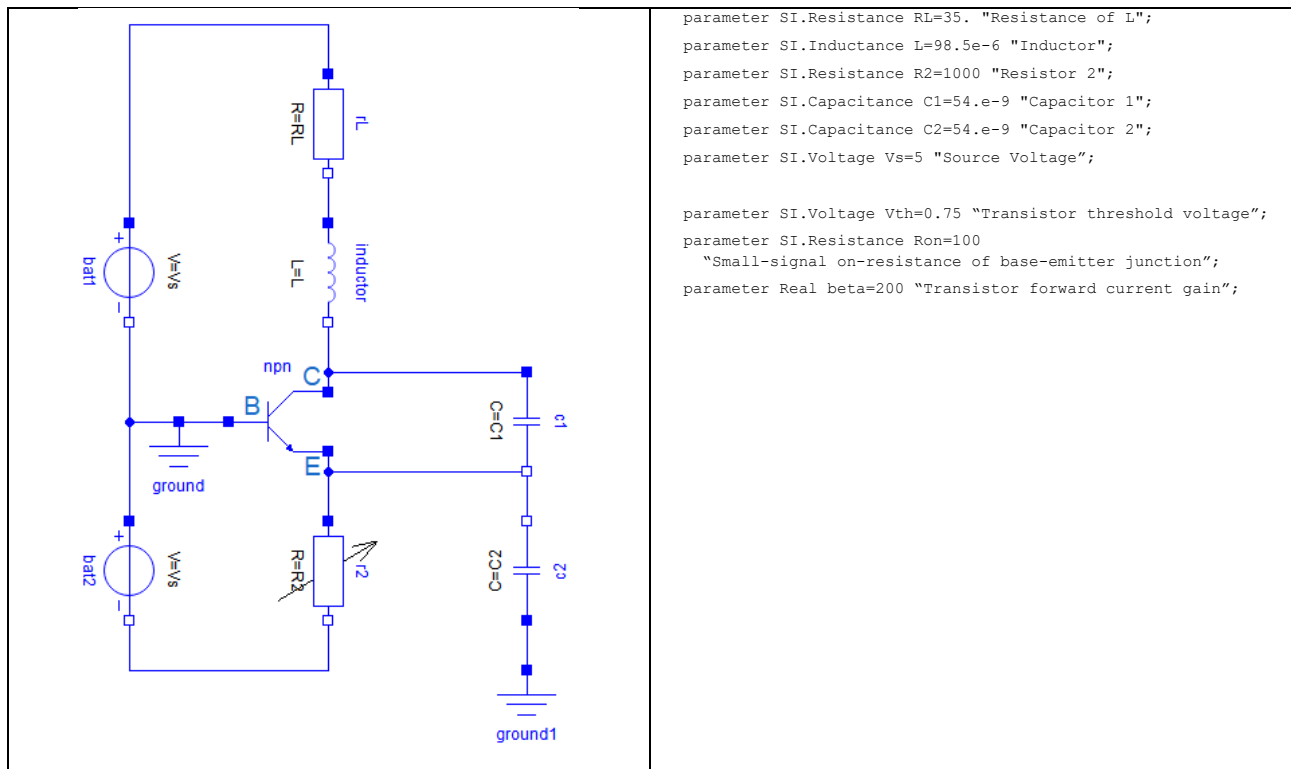
[Buscarino2014], [Kennedy1994]

$$C_1 \cdot \frac{dv_1}{dt} = i_L - \beta \cdot i_B$$

$$C_2 \cdot \frac{dv_2}{dt} = -\frac{V_{s-} + v_2}{R_2} - i_L - i_B$$

$$L \cdot \frac{di_L}{dt} = V_{s+} - v_1 + v_2 - R_L \cdot i_L$$

$$i_B = \begin{cases} v_2 = v_{BE} \leq V_{th} \rightarrow 0 \\ v_2 = v_{BE} > V_{th} \rightarrow \frac{v_2 - V_{th}}{R_{on}} \end{cases}$$



## Shinriki Oscillator

[Lueck1995], [Shinriki1981]

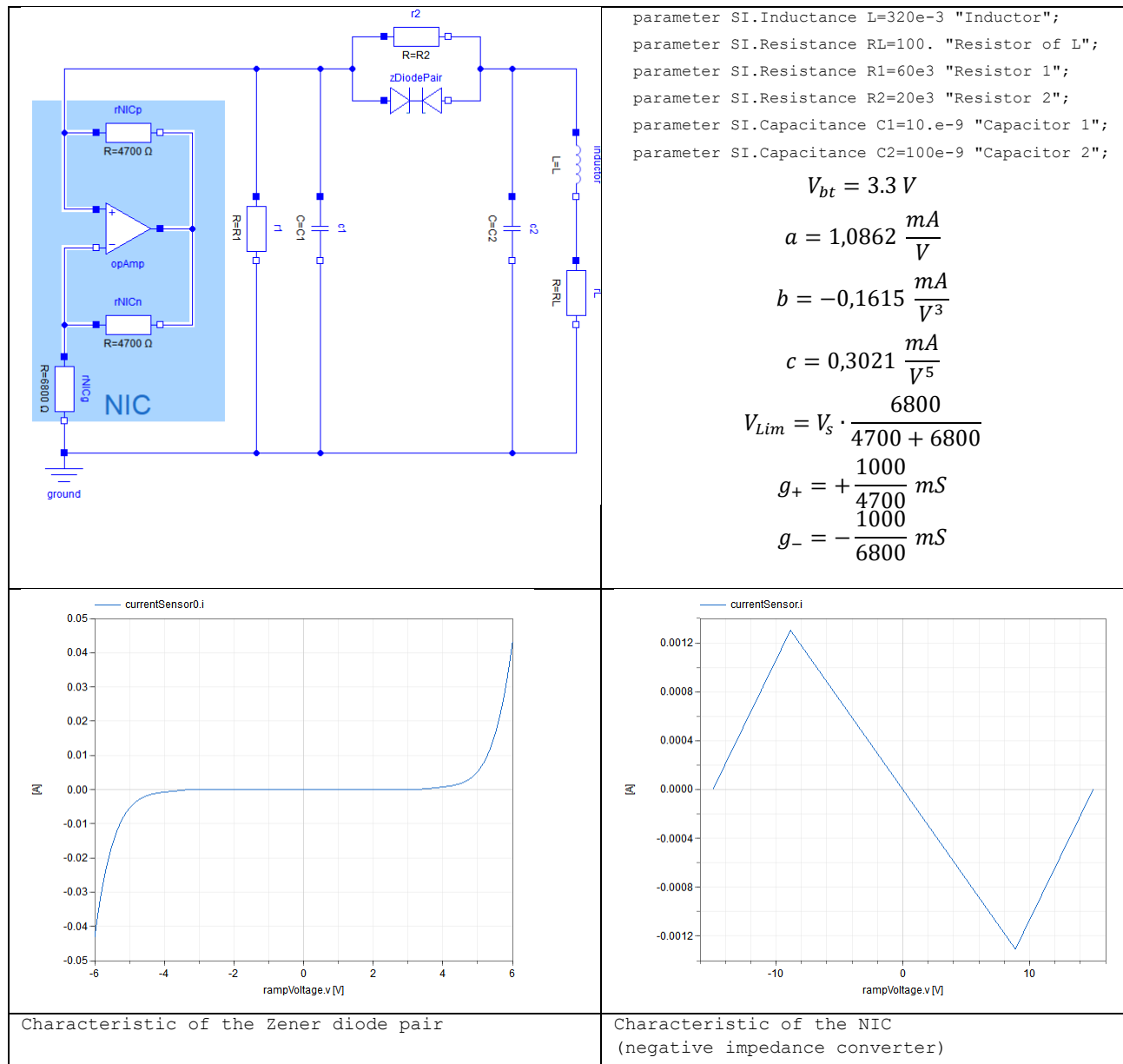
$$i_z = \begin{cases} |v_z| < V_{bt} & 0 \\ |v_z| \geq V_{bt} & \text{sign}(v_z) \cdot [a \cdot (|v_z| - V_{bt}) + b \cdot (|v_z| - V_{bt})^3 + c \cdot (|v_z| - V_{bt})^5] \end{cases}$$

$$C_1 \cdot \frac{dv_1}{dt} = -i_{NIC} - \frac{v_1}{R_1} - i_z$$

$$C_2 \cdot \frac{dv_2}{dt} = i_z - i_L$$

$$v_2 = L \cdot \frac{di_L}{dt} + R_L \cdot i_L$$

$$g_{NIC} = \frac{di_{NIC}}{dv_{NIC}} = \begin{cases} |v_{NIC}| > V_{Lim} & g_+ \\ |v_{NIC}| \leq V_{Lim} & g_- \end{cases}$$



Investigation of the Zener diode pair approximation:

$$i_z = \begin{cases} 0 & |v_z| < V_{bt} \\ \text{sign}(v_z) \cdot [a \cdot (|v_z| - V_{bt}) + b \cdot (|v_z| - V_{bt})^3 + c \cdot (|v_z| - V_{bt})^5] & |v_z| \geq V_{bt} \end{cases}$$

$$\frac{di_z}{dv_z} = \begin{cases} 0 & |v_z| < V_{bt} \\ \text{sign}(v_z) \cdot [a + 3b \cdot (|v_z| - V_{bt})^2 + 5c \cdot (|v_z| - V_{bt})^4] & |v_z| \geq V_{bt} \end{cases}$$

The first derivative  $\frac{di_z}{dv_z}$  is not continuous at  $|v_z| = V_{bt}$ .

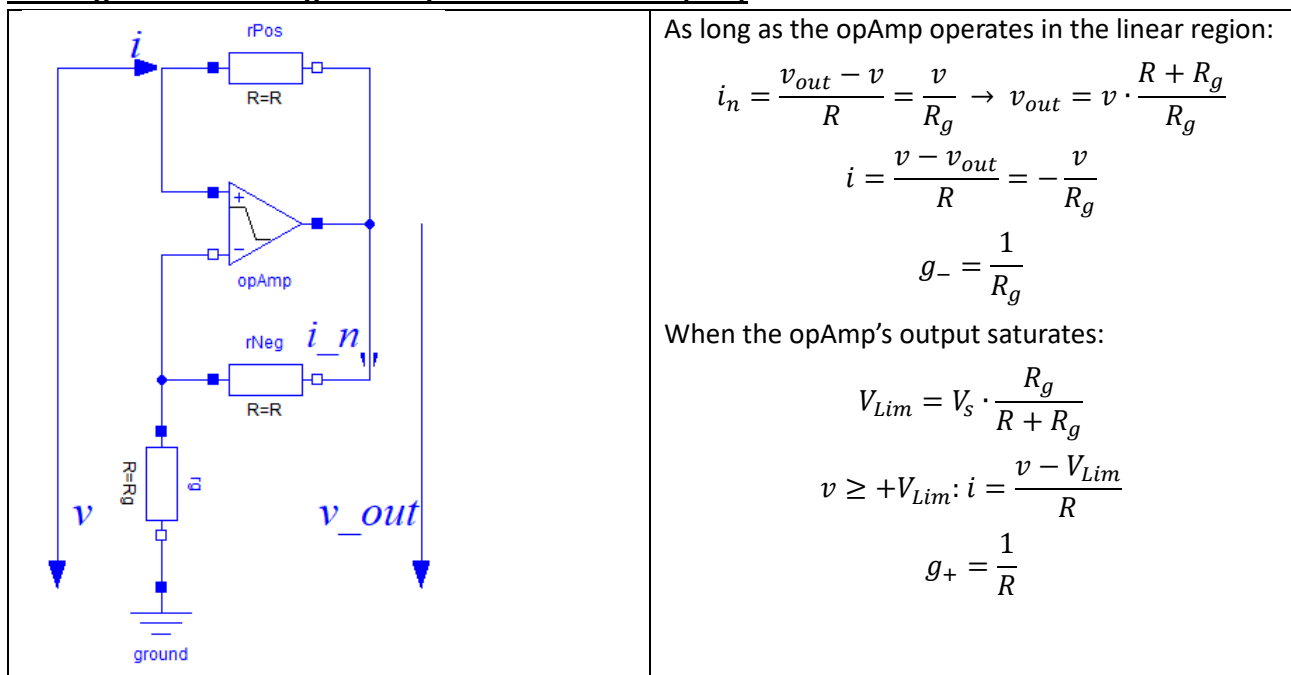
If voltage  $v$  is prescribed, the current  $i$  can be unambiguously determined.

If current  $i$  is prescribed, between  $-V_{bt}$  and  $+V_{bt}$  there is a manifold of solutions for the voltage  $v$ .

For this application, this restriction has no influence.

This restriction could be solved by adaption the approximation, i.e. exchange the horizontal line in the range  $-V_{bt} < v_z < +V_{bt}$  against a characteristic with small constant positive slope and adapt the polynomial approximation to achieve a one times continuously differentiable characteristic.

### Investigation of the negative impedance converter (NIC)



If voltage  $v$  is prescribed, the current  $i$  can be unambiguously determined.

If current  $i$  is prescribed, in the range between the zero crossings the voltage  $v$  has 3 possible solutions.

For this application, this restriction has no influence.

## Jerk Circuit

[Buscarino2014], [Sprott2011]

The name of the system stems from the third derivative of  $x$ , which – in a mechanical system – is the derivative of acceleration called jerk. The Jerk equation has been investigated in different versions.

$$\ddot{x} + G(\ddot{x}, \dot{x}, x) = 0$$

The version implemented here uses a diode as described in the mentioned publications:

$$G(\ddot{x}, \dot{x}, x) = A \cdot \ddot{x} + f(\dot{x}) + x$$

$$\ddot{x} + A \cdot \ddot{x} + f(\dot{x}) + x = 0$$

$f(\dot{x})$  is modeled using the Shockley equation of a diode:

$$f(\dot{x}) = R \cdot I_S \cdot \left( e^{\frac{\dot{x}}{nV_t}} - 1 \right)$$

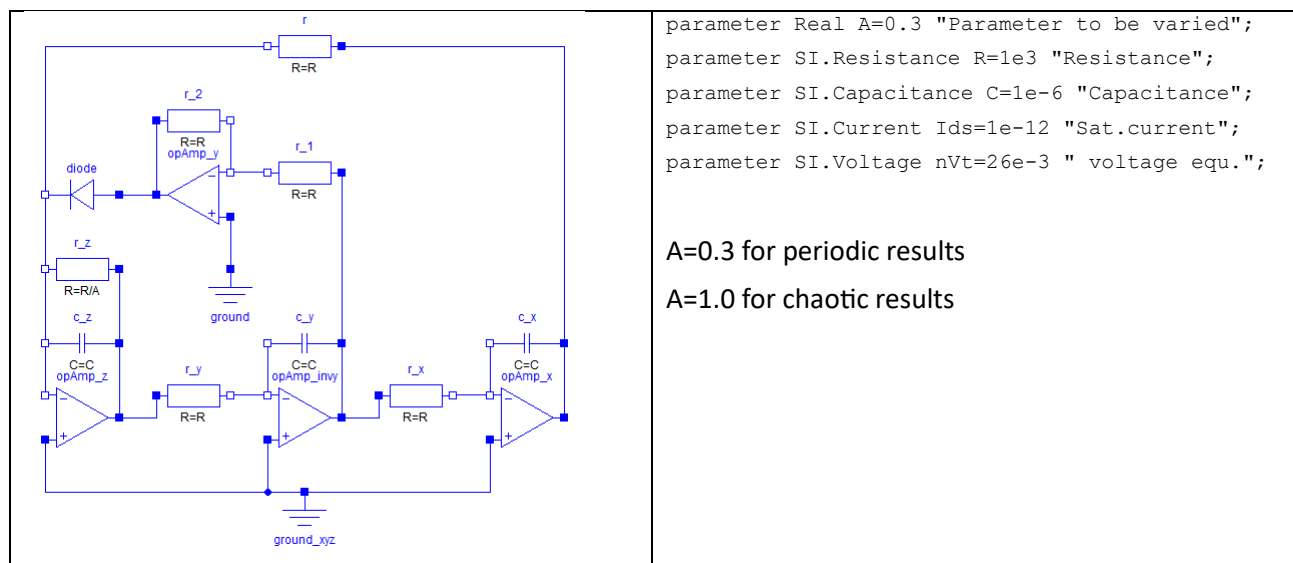
This leads to a system of 3 ordinary differential equations with one nonlinearity:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -A \cdot z - x - f(y)\end{aligned}$$

The values stay pretty inside a practicable range for a normal voltage supply.

Using 4 operational amplifiers and an acceleration factor of 1000, the circuit can be implemented as follows.

The parameter  $A$  influences only the feedback resistor at opAMP\_z:





## Rikitake System

[Rikitake1958], [Rikitake1973]

The system proposed by Rikitake has been used to explain irregular reversals of the Earth's magnetic field.  
2 identical magnetically coupled disc dynamos ( $\tau \cdot \omega$  covers the losses  $R \cdot i^2$ ):

$$\begin{aligned} L \cdot \frac{di_1}{dt} + R \cdot i_1 &= M \cdot i_2 \cdot \omega_1 \\ J \cdot \frac{d\omega_1}{dt} &= \tau - M \cdot i_2 \cdot i_1 \\ L \cdot \frac{di_2}{dt} + R \cdot i_2 &= M \cdot i_1 \cdot \omega_2 \\ J \cdot \frac{d\omega_2}{dt} &= \tau - M \cdot i_1 \cdot i_2 \end{aligned}$$

$\phi_{12} = M \cdot i_2$  is the magnetic flux in machine 1 excited by current  $i_2$ ,

$\phi_{21} = M \cdot i_1$  is the magnetic flux in machine 2 excited by current  $i_1$ .

The circuit could be implemented using two series excited DC machines with the armature current of the other machine as excitation current.

The equations of motion have identical right hand sides:

$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} \rightarrow \omega_1 - \omega_2 = \Delta\omega = \text{const.}$$

Mechanical and electrical time constant:

$$\begin{aligned} T_m &= \frac{J}{\tau} \cdot \frac{R}{M} \\ T_e &= \frac{L}{R} \\ \mu &= \sqrt{\frac{T_m}{T_e}} = \sqrt{\frac{J}{\tau} \cdot \frac{R^2}{L \cdot M}} \end{aligned}$$

used to scale the variables:

$$\begin{aligned} i_{1,2} &= x_{1,2} \cdot \sqrt{\frac{\tau}{M}} \\ \omega_1 &= z \cdot \sqrt{\frac{\tau}{J} \cdot \frac{L}{M}} \rightarrow \omega_2 = (z - \Delta) \cdot \sqrt{\frac{\tau}{J} \cdot \frac{L}{M}} \\ t &= t' \cdot \sqrt{T_m \cdot T_e} = t' \cdot \sqrt{\frac{J}{\tau} \cdot \frac{L}{M}} \end{aligned}$$

lead to the scaled equations:

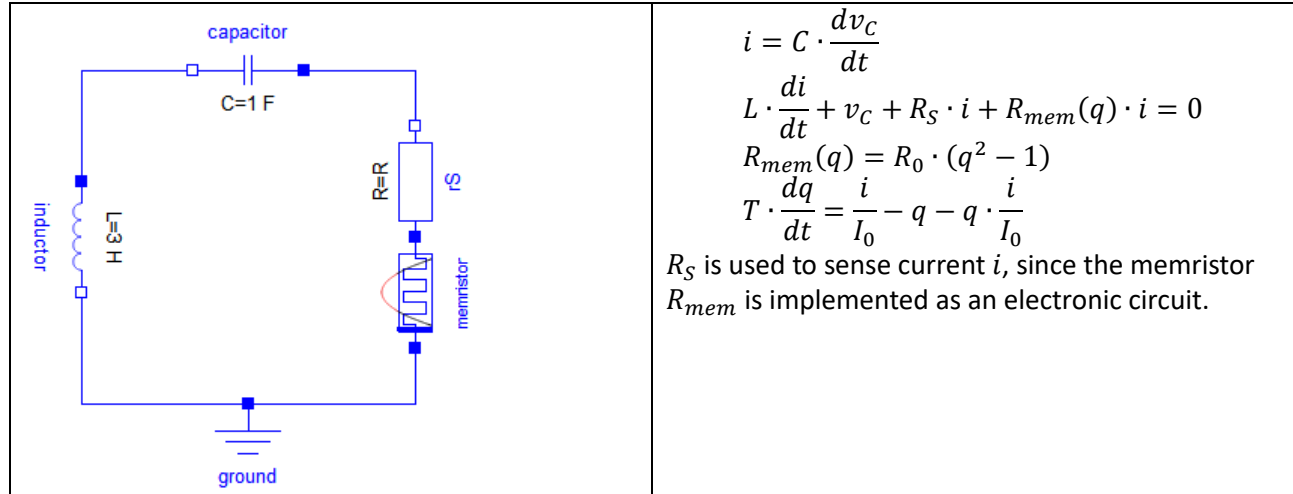
$$\begin{aligned} \frac{dx_1}{dt'} &= -\mu \cdot x_1 + x_2 \cdot z \\ \frac{dx_2}{dt'} &= -\mu \cdot x_2 + x_1 \cdot (z - \Delta) \\ \frac{dz}{dt'} &= 1 - x_1 \cdot x_2 \end{aligned}$$

The states stay within a range that needs no scaling when implemented with an opAmp-circuit.

### Simplest Chaotic Circuit

[Muthuswamy2010]

The memristor drift model - either linear or with window function  $fw(x)$  – doesn't include a negative resistance  $R_{mem}$  in the characteristic. Although the memristor itself is discussed controversially [Abraham2018], Muthuswamy and Chua suggested 2010 a special memristor function to construct an autonomous resonance circuit which shows chaotic results.



Scaling:

$$\omega_0 = \frac{1}{\sqrt{L \cdot C}}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$\tau = \omega_0 \cdot t$$

$$x = \frac{v_C}{R_0 \cdot I_0}$$

$$y = \frac{i}{I_0}$$

$$z = q$$

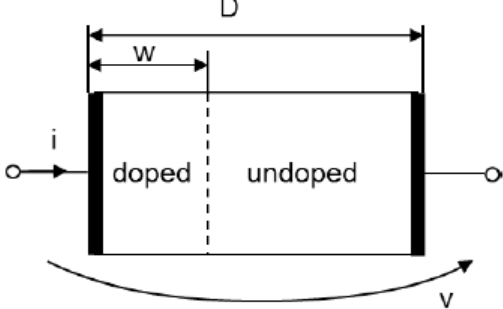
$$\frac{dx}{d\tau} = \frac{Z_0}{R_0} \cdot y$$

$$\frac{dy}{d\tau} = -\frac{R_0}{Z_0} \cdot x - \frac{R_0}{Z_0} \cdot \left( \frac{R_S}{R_0} + z^2 - 1 \right) \cdot y$$

$$\omega_0 \cdot T \cdot \frac{dz}{d\tau} = y - z - z \cdot y$$

### Memristor models

[Biolek2009], [Biolek2015], [Chua1971], [Chua1977], [Itoh2008], [Joglekar2009], [Majetta2012], [Maurer2014], [Muthuswamy2010], [Muthuswamy2010a], [Oguz2018], [Strukov2008], [Yakopcic2011]

<p>Charge-controlled memristor:  <math>q</math> is the dimensionless internal state,              ideally meant as charge: <math>\frac{dq}{dt} \sim i</math>              “Remembers” history of current</p>	$v = R_{mem} \cdot i$ $R_{mem} = r(q)$ $\frac{dq}{dt} = f(i, q)$
<p>Flux-controlled memristor:  <math>\phi</math> is the dimensionless internal state,              ideally meant as magnetic flux: <math>\frac{d\phi}{dt} \sim v</math>              “Remembers” history of voltage</p>	$i = G_{mem} \cdot v$ $G_{mem} = g(\phi)$ $\frac{d\phi}{dt} = f(v, \phi)$
<p>Drift model: <math>q = \frac{w}{D}</math></p> 	$R_{mem} = R_{on} \cdot q + R_{off} \cdot (1 - q)$ $\frac{dq}{dt} = \frac{\mu_v \cdot R_{on}}{D^2} \cdot i \cdot fw(q)$ <p><math>\mu_v</math> is the dopant mobility: <math>[\mu_v] = \frac{m^2}{V \cdot s}</math>              [Joglekar2009]</p> $fw(q) = 1 - (2q - 1)^{2p}$ <p>[Biolek2009]</p> $fw(q) = 1 - \left( q - \begin{cases} 1 & -i \geq 0 \\ 0 & -i < 0 \end{cases} \right)^{2p}$

[Muthuswamy2010]

$$v = \beta \cdot (x^2 - 1) \cdot i$$

$$\frac{dx}{dt} = i - \alpha \cdot x - x \cdot i$$

$$R_{mem} = R_0 \cdot (q^2 - 1)$$

$$q = \sqrt{1 + \frac{R_{mem}}{R_0}}$$

$$T \cdot \frac{dq}{dt} = \frac{i}{I_0} - q - q \cdot \frac{i}{I_0}$$

Appendix B.3 of [Muthuswamy2010] demonstrates the implementation of this special memristor using operational amplifiers and analog multipliers.

Steady state characteristic of this special memristor:



$$\frac{dq}{dt} = 0 \rightarrow q = \frac{\frac{i}{I_0}}{1 + \frac{i}{I_0}}$$

$$v_{mem} = R_0 \cdot (q^2 - 1) \cdot i$$

$$= -R_0 \cdot I_0 \cdot \frac{\left(1 + 2 \cdot \frac{i}{I_0}\right) \cdot \frac{i}{I_0}}{\left(1 + \frac{i}{I_0}\right)^2}$$

$$x = \frac{i}{I_0}$$

$$y = \frac{v_{mem}}{R_0 \cdot I_0}$$

$$y = -\frac{(1 + 2 \cdot x) \cdot x}{(1 + x)^2}$$

$$x = -1: y \rightarrow -\infty$$

$$x = \left\{-\frac{1}{2}, 0\right\}: y = 0$$

$$x = -\frac{1}{3}: y = \frac{1}{4} \text{ max.}$$

$$x \rightarrow \pm\infty: y \rightarrow -2$$

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