Comparison of chaotic circuits
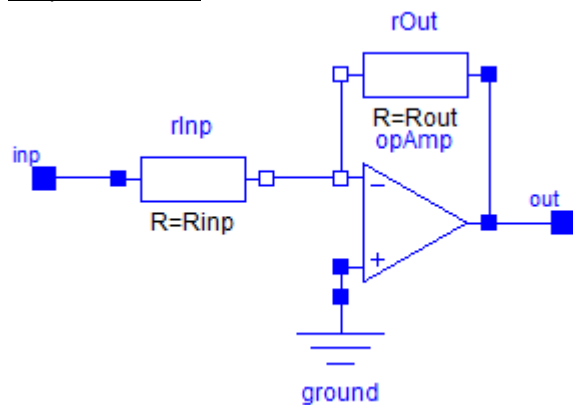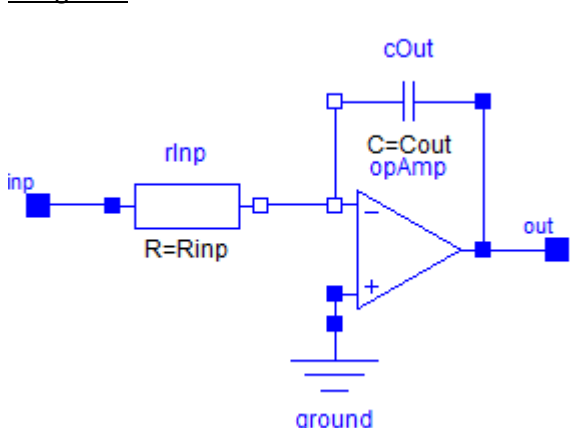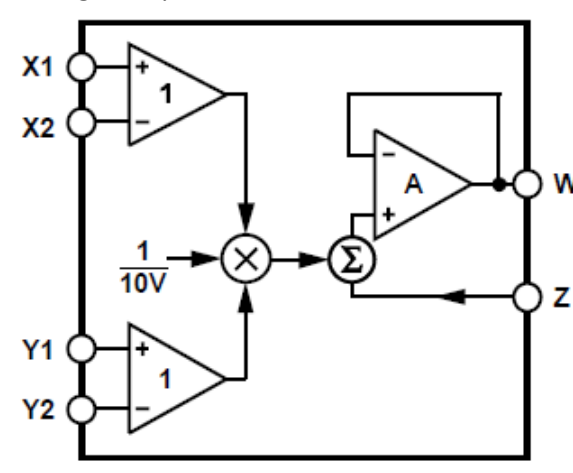
Lotka-Volterra	$$\frac{dx_i}{dt} = r_i \cdot x_i \cdot \left(1 - \sum_{j=1}^{n} a_{ij} \cdot x_j \right)$$
van der Pol Circuit	$$\frac{dx}{d\tau} = y$$ $$y = \mu \cdot \left[1 - \frac{1}{3} \cdot x^2 \right] \cdot x - z + \begin{cases} 0 \\ A \cdot cos(w \cdot \tau - \pi) \end{cases}$$ $$\frac{dz}{d\tau} = x$$
Lorenz System	$$\frac{dx}{dt} = \sigma \cdot (y - x)$$ $$\frac{dy}{dt} = x \cdot (\rho - z) - y$$ $$\frac{dz}{dt} = x \cdot y - \beta \cdot z$$
Roessler System	$$\frac{dx}{dt} = -y - z$$ $$\frac{dy}{dt} = x + a \cdot y$$ $$\frac{dz}{dt} = b + (x - c) \cdot z$$
Chua's Circuit	nonlinear conductor (two NICs) with partly negative slope
Chaotic Diode Circuit (with a light emitting diode)	$$\frac{dx}{dt} = -y$$ $$\frac{dy}{dt} = -z$$ $$\frac{dz}{dt} = -x + a \cdot (e^y - 1) - b \cdot z$$
Chaotic Oscillator A simple chaotic oscillator for educational purposes	$$\frac{dx}{dt} = y$$ $$\frac{dy}{dt} = a \cdot y - x - z$$ $$\frac{dz}{dt} = b + y - c \cdot (e^z - 1)$$
Colpitts Oscillator	LC oscillator with transistor (orig. vacuum tube)
Shinriki Oscillator	two antiparallel Z-diodes and nonlinear conductor (NIC) with partly negative slope
Jerk Circuit	$$\dddot{x} + G(\ddot{x}, \dot{x}, x) = 0$$
Rikitake System	Two coupled disc dynamos.
Simplest Chaotic Circuit	A series resonance circuit with a special memristor

OpAmp-Circuits

Algebraic-Differential Equation Systems can be simulated with analog computers
using operational amplifiers and analog multipliers.

Amplifier/Adder	
	$$\frac{inp.v}{R_{inp}} + \frac{out.v}{R_{out}} = 0$$ $$-out.v = k \cdot inp.v$$ $$k = \frac{R_{out}}{R_{inp}}$$ Input resistance $= R_{inp}$ Output resistance $\rightarrow 0$ It is possible to add several inputs.
Integrator	
	$$\frac{inp.v}{R_{inp}} + C_{out} \cdot \frac{d\ out.v}{dt} = 0$$ $$-out.v = v_0 + \frac{1}{\tau} \cdot \int_0^t inp.v \cdot dt$$ $$\tau = R_{inp} \cdot C_{out}$$ Input resistance $= R_{inp}$ Output resistance $\rightarrow 0$ It is possible to integrate the sum of several inputs.
Analog Multiplier	
	Functional Block Diagram of AD633 Division by $10\ V$ (scaling) inhibits overflow. Additional summing input Z is omitted. Negative inputs of X- and Y-amplifiers are connected to ground. Possible implementations: • Gilbert cell • $y = e^{ln(x_1)+ln(x_2)}$ • $y = \frac{(x_1+x_2)^2-(x_1-x_2)^2}{4}$

Lotka-Volterra

[Vano2006]

As a 2-dimensional predator-prey-model no chaos is reported:

$$\frac{dx}{dt} = r_x \cdot x - d_x \cdot x \cdot y$$
$$\frac{dy}{dt} = d_x \cdot x \cdot e_y \cdot y - d_y \cdot y$$

We might interpret x as number of hares (prey) and y as foxed (predator).

r_x is the reproduction rate of hares, d_x the deathrate of hares due to foxes.

e_y is the efficiency in growing foxes from hares, d_y the (natural) deathrate of foxes.

A n-dimensional Lotka-Volterra model is defined for $2 \leq i \leq n$ (n designates the number of species):

$$\frac{dx_i}{dt} = r_i \cdot x_i \cdot \left(1 - \sum_{j=1}^{n} a_{ij} \cdot x_j\right)$$

The vector $[r]$ describes the reproduction rates of the species whereas the quadratic matrix $[a]$ describes the competition between species.

The above mentioned predator-prey case with 2 species can be expressed as:

$$[r] = \begin{bmatrix} r_x \\ -d_y \end{bmatrix}$$

$$[a] = \begin{bmatrix} 0 & \dfrac{d_x}{r_x} \\ \dfrac{d_x \cdot e_y}{d_y} & 0 \end{bmatrix}$$

Lotka-Volterra

van der Pol Circuit

[Buscarino2014], [VanDerPol1927]

Balthasar van der Pol reported 1927 strange phenomena about oscillations in a series resonance circuit containing a vacuum electron triode. Due to the nonlinear characteristic of the triode the autonomous circuit is able to maintain periodic oscillations, and with harmonic excitation it is able to produce chaos.
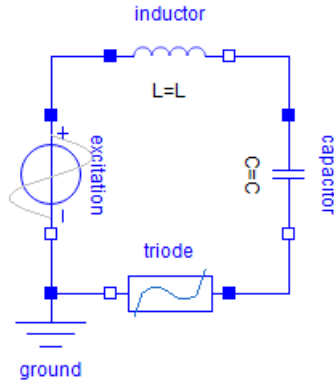
$$i = C \cdot \frac{dv_C}{dt} \rightarrow v_C = v_{C0} + \frac{1}{C} \cdot \int_{t_0}^{t} i \cdot dt'$$

$$L \cdot \frac{di}{dt} + \tilde{R}(i) \cdot i + v_C = \begin{cases} 0 \\ \hat{V} \cdot cos(\omega \cdot t - \pi) \end{cases}$$

$$\tilde{R}(i) = -R_0 \cdot \left[1 - \frac{1}{3} \cdot \left(\frac{i}{I_0}\right)^2\right]$$

These are the equations of the physical model.
Note the phase shift of the excitation!

$$L \cdot \frac{d^2 i}{dt^2} - R_0 \cdot \left[1 - \left(\frac{i}{I_0}\right)^2\right] \cdot \frac{di}{dt} + \frac{1}{C} \cdot i = \begin{cases} 0 \\ \hat{V} \cdot \omega \cdot sin(\omega \cdot t) \end{cases}$$

$$\omega_0 = \frac{1}{\sqrt{L \cdot C}}$$

$$\mu = R_0 \cdot \sqrt{\frac{C}{L}}$$

$$A = \frac{\omega_0 \cdot C \cdot \hat{V}}{I_0}$$

$$w = \frac{\omega}{\omega_0}$$

$$\tau = \omega_0 \cdot t$$

$$x = \frac{i}{I_0}$$

$$y = \frac{dx}{d\tau} = \frac{1}{\omega_0 \cdot I_0} \cdot \frac{di}{dt}$$

$$\frac{I_0}{\omega_0 \cdot C} \cdot z = v_C = v_{C0} + \frac{1}{C} \cdot \int_{t_0}^{t} i \cdot dt'$$

$$z = z_0 + \int_{\tau_0}^{\tau} x \cdot d\tau'$$

$$\frac{d^2 x}{d\tau^2} - \mu \cdot [1 - x^2] \cdot \frac{dx}{d\tau} + x = \begin{cases} 0 \\ A \cdot w \cdot sin(w \cdot \tau) \end{cases}$$ $$\frac{dx}{d\tau} = y$$ $$\frac{dy}{d\tau} = \mu \cdot [1 - x^2] \cdot y - x + \begin{cases} 0 \\ A \cdot w \cdot sin(w \cdot \tau) \end{cases}$$ $$\frac{dz}{d\tau} = x$$	Alternative Formulation with 2 states: $$\frac{dx}{d\tau} = y$$ $$y = \mu \cdot \left[1 - \frac{1}{3} \cdot x^2\right] \cdot x - z + \begin{cases} 0 \\ A \cdot cos(w \cdot \tau - \pi) \end{cases}$$ $$\frac{dz}{d\tau} = x$$

Note:

Instead of using a series resonance circuit and deriving a scaled differential equation for the current, we could use an equivalent parallel resonance circuit and derive a scaled differential equation for the voltage.

$$v = L \cdot \frac{di_L}{dt} \rightarrow i_L = i_{L0} + \frac{1}{L} \cdot \int_{t_0}^{t} v \cdot dt'$$

$$i_L + \tilde{G}(v) \cdot v + C \cdot \frac{dv}{dt} = i_e$$

Initialization:

The physical model has 2 states: i and v_C. Current i acts as an initial value for the nonlinear resistor.

The analytic equations have 3 states: x, y and z.

The third state has been introduced artificially by first differentiating the voltage equation, generating an equation with second derivative of i. Splitting this equation into two first order differential equations, we generate i and $\frac{di}{dt}$ as states. Calculating capacitor voltage v, we get the third state.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. We chose natural eigen frequency ω_0 as time scale:

$$x' = \frac{x}{k_x}$$
$$y' = \frac{y}{k_y}$$
$$z' = \frac{z}{k_z}$$
$$t' = \frac{\tau}{\omega_0}$$

We also have to take into account that the analog multiplier divides by V_S to avoid overflow of the output. After that, none of the computing block should encounter an overflow.

This leads to the following set of equations:

$$\frac{1}{\omega_0} \cdot \frac{dx'}{dt'} = \frac{k_y}{k_x} \cdot y'$$

$$y' = \mu \cdot \frac{k_x \cdot V_s}{k_y} \cdot \left[1 - \frac{k_x^2 \cdot V_s}{3} \cdot \frac{x'^2}{V_s} \right] \cdot \frac{x'}{V_s} - \frac{k_z}{k_y} \cdot z' + \begin{cases} 0 \\ \frac{A}{k_y} \cdot cos(w \cdot \omega_0 \cdot t' - \pi) \end{cases}$$

$$\frac{1}{\omega_0} \cdot \frac{dz'}{dt'} = \frac{k_x}{k_z} \cdot x'$$

These equations can easily get implemented as blocks or as an electronic circuit.

Calculating back from per-unit-parameters:

$$\mu = 0.2, w = 1.15, A = [0..1]$$

and some assumptions:

$$C = \frac{100}{2\pi} \, \mu F, \omega_0 = 2\pi \cdot 1000 \, \frac{rad}{s}, I_0 = 0.5 \, A$$

we obtain physical parameters:

$$L = \frac{1}{\omega_0^2 \cdot C} = \frac{10}{2\pi} \, mH$$

$$R_0 = \mu \cdot \sqrt{\frac{L}{C}} = \frac{\mu}{\omega_0 \cdot C} = 2 \, \Omega$$

$$\hat{V} = A \cdot \frac{I_0}{\omega_0 \cdot C} = [0 \cdots 5] \, V$$

Investigating the nonlinear resistance of the triode:

$$v_R = -R_0 \cdot I_0 \cdot \left[\left(\frac{i}{I_0} \right) - \frac{1}{3} \cdot \left(\frac{i}{I_0} \right)^3 \right] = -R_0 \cdot i \cdot \left[1 - \frac{1}{3} \cdot \left(\frac{i}{I_0} \right)^2 \right]$$

$$\frac{v_R}{i} = -R_0 \cdot \left[1 - \frac{1}{3} \cdot \left(\frac{i}{I_0} \right)^2 \right]$$

$$\frac{dv_R}{di} = -R_0 \cdot \left[1 - \left(\frac{i}{I_0} \right)^2 \right]$$

$$x = \frac{i}{I_0}$$

$$y = \frac{v_R}{R_0 \cdot I_0}$$

$$y = -\left(x - \frac{x^3}{3} \right) = -x \cdot \left(1 - \frac{x^2}{3} \right)$$

$$\frac{y}{x} = -\left(1 - \frac{x^2}{3} \right)$$

$$\frac{dy}{dx} = -(1 - x^2)$$

Characteristic of the triode $y(x)$:
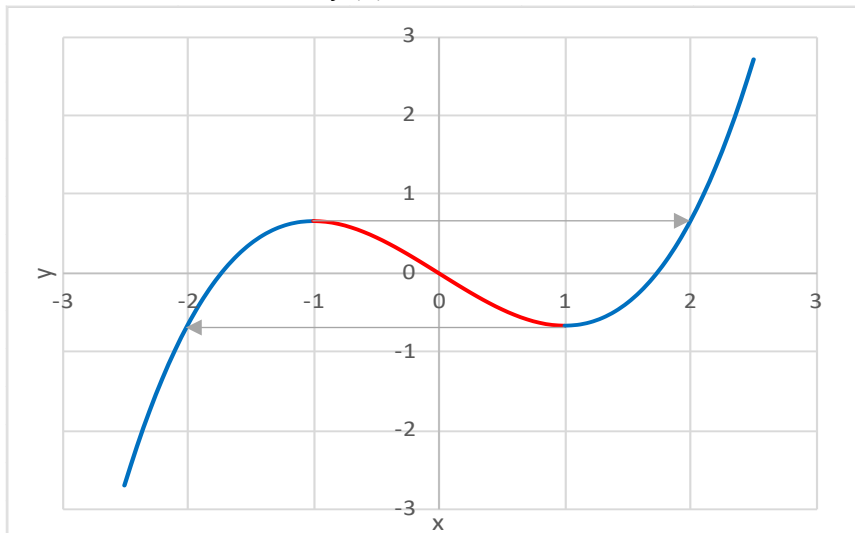
Zero crossings $y = 0$: $x = \{-\sqrt{3}, 0, +\sqrt{3}\}$ with slopes $\frac{dy}{dx} = \{+2, -1, +2\}$

Extrema: $x = \{-1, +1\}$ with $x = \left\{ +\frac{2}{3}, -\frac{2}{3} \right\}$

Inflection point: $[x, y] = [0, 0]$

If current x is prescribed, voltage y can be unambiguously determined.

If y is prescribed, in the range $-2 \le x \le +2$ i.e. $-\frac{2}{3} \le y \le +\frac{2}{3} x$ has 2 or 3 possible solutions.

For this application, this restriction has no influence.

Shifting the characteristic up and to the right, it looks like the $i(v)$ characteristic of a tunnel (Esaki) diode.

Inversion of the triode characteristic shows hysteretic behavior (split into 2 branches):

$$x \ge +1 : y + \frac{2}{3} = \frac{(x-1)^3}{3} + (x-1)^2$$

$$x \le -1 : y - \frac{2}{3} = \frac{(x+1)^3}{3} - (x+1)^2$$

Lorenz System

[Buscarino2014], [Lorenz1963]

Developed 1963 by Edward Lorenz to model atmospheric convection.

x is proportional to the rate of convection, y to the horizontal temperature variation and z to the vertical temperature variation. σ depicts the Prandtl number, ρ the Rayleigh number and β the physical dimensions.

The original parameters were: $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$

$$\frac{dx}{dt} = \sigma \cdot (y - x)$$
$$\frac{dy}{dt} = x \cdot (\rho - z) - y$$
$$\frac{dz}{dt} = x \cdot y - \beta \cdot z$$

$\beta = \frac{1}{3}$ leads to a periodic solution.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. This can be compared with calculating per-unit values by dividing by reference values:

$$x' = \frac{x}{k_x}$$
$$y' = \frac{y}{k_y}$$
$$z' = \frac{z}{k_z}$$
$$t' = \frac{t}{\tau}$$

We also have to take into account that the analog multiplier divides by V_S to avoid overflow of the output. After that, none of the computing block should encounter an overflow.

This leads to the following set of equations:

$$\frac{1}{\tau} \cdot \frac{dx'}{dt'} = -\sigma \cdot x' + \sigma \cdot \frac{k_y}{k_x} \cdot y'$$
$$\frac{1}{\tau} \cdot \frac{dy'}{dt'} = \rho \cdot \frac{k_x}{k_y} \cdot x' - y' - \frac{k_x \cdot k_z \cdot V_S}{k_y} \cdot \frac{x' \cdot z'}{V_S}$$
$$\frac{1}{\tau} \cdot \frac{dz'}{dt'} = \frac{k_x \cdot k_y \cdot V_S}{k_z} \cdot \frac{x' \cdot y'}{V_S} - \beta \cdot z'$$

These equations can easily get implemented as blocks or as an electronic circuit.

Roessler System
[Roessler1976], [Roessler1979]

A simple system of 3 ordinary nonlinear differential equations to study chaos without physical background.

$$\frac{dx}{dt} = -y - z$$
$$\frac{dy}{dt} = x + a \cdot y$$
$$\frac{dz}{dt} = b + (x - c) \cdot z$$

$a = 0.2$, $b = 0.2$ and $c = 1$ give periodic results. Changing $c = 5.7$ reveals chaotic results.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. This can be compared with calculating per-unit values by dividing by reference values:

$$x' = \frac{x}{k_x}$$
$$y' = \frac{y}{k_y}$$
$$z' = \frac{z}{k_z}$$
$$t' = \frac{t}{\tau}$$

We also have to take into account that the analog multiplier divides by V_S to avoid overflow of the output. After that, none of the computing block should encounter an overflow.

This leads to the following set of equations:

$$\frac{1}{\tau} \cdot \frac{dx'}{dt} = -\frac{k_y}{k_x} \cdot y' - \frac{k_z}{k_x} \cdot z'$$
$$\frac{1}{\tau} \cdot \frac{dy'}{dt} = \frac{k_x}{k_y} \cdot x' + a \cdot y'$$
$$\frac{1}{\tau} \cdot \frac{dz'}{dt} = \frac{b}{k_z} + k_x \cdot V_S \cdot \frac{x' \cdot z'}{V_S} - c \cdot z'$$

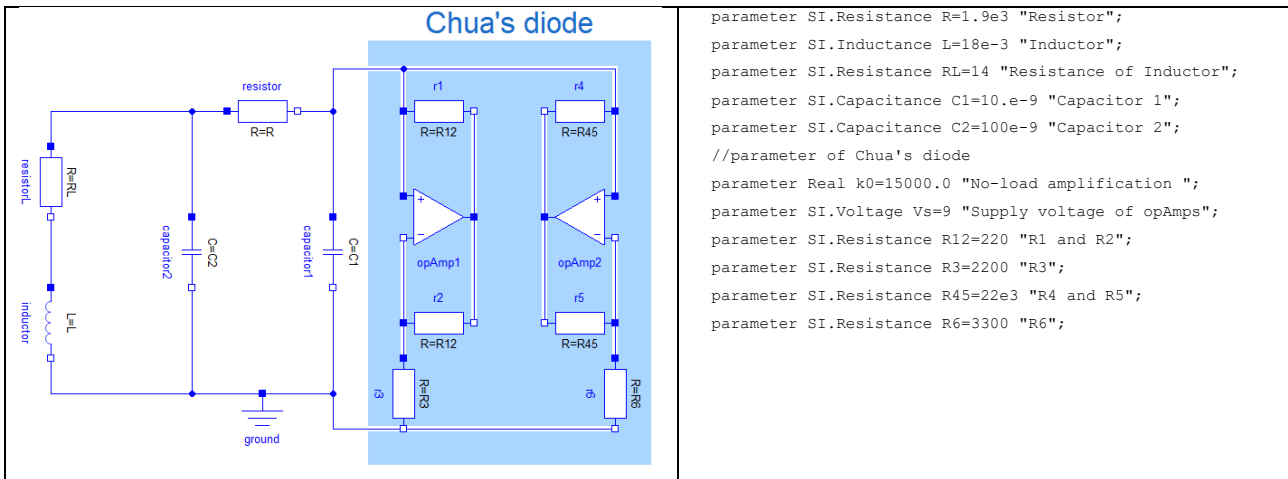These equations can easily get implemented as blocks or as an electronic circuit.

Chua's Circuit

[Berkeley], [Buscarino2014], [Chua1983], [Kennedy1993a], [Muthuswamy2009], [Zhong1985]

$$L \cdot \frac{di_L}{dt} = v_2 - R_L \cdot i_L$$

$$C_2 \cdot \frac{dv_2}{dt} = -i_L - \frac{v_2 - v_1}{R}$$

$$C_1 \cdot \frac{dv_1}{dt} = -i_{NL} + \frac{v_2 - v_1}{R}$$

$$-i_{NL}(v_1) = \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b \cdot (v_1 + V_e) - G_a \cdot V_e \\ -V_e < v_1 < +V_e \rightarrow G_a \cdot v_1 \\ +V_e < v_1 < +\infty \rightarrow G_b \cdot (v_1 - V_e) + G_a \cdot V_e \end{cases}$$

$$-\frac{i_{NL}}{v_1} = \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b - (G_a - G_b) \cdot \frac{V_e}{v_1} \\ -V_e < v_1 < +V_e \rightarrow G_a \\ +V_e < v_1 < +\infty \rightarrow G_b + (G_a - G_b) \cdot \frac{V_e}{v_1} \end{cases}$$

$$-\frac{di_{NL}}{dv_1} = \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b \\ -V_e < v_1 < +V_e \rightarrow G_a \\ +V_e < v_1 < +\infty \rightarrow G_b \end{cases}$$

$$\tau_L \cdot \dot{v}_{RL} = v_2 - v_{RL}$$

$$\tau_2 \cdot \dot{v}_2 = +v_1 - v_2 - \frac{R}{R_L} \cdot v_{RL}$$

$$\tau_1 \cdot \dot{v}_1 = -v_1 + v_2 + R \cdot g \cdot v_1$$

$$g(v_1) = \begin{cases} |v_1| > V_e \rightarrow G_b + (G_a - G_b) \cdot \frac{V_e}{|v_1|} \\ |v_1| < V_e \rightarrow G_a \end{cases}$$


```
parameter SI.Resistance R=1.9e3 "Resistor";
parameter SI.Inductance L=18e-3 "Inductor";
parameter SI.Resistance RL=14 "Resistance of Inductor";
parameter SI.Capacitance C1=10.e-9 "Capacitor 1";
parameter SI.Capacitance C2=100e-9 "Capacitor 2";
//parameter of Chua's diode
parameter Real k0=15000.0 "No-load amplification ";
parameter SI.Voltage Vs=9 "Supply voltage of opAmps";
parameter SI.Resistance R12=220 "R1 and R2";
parameter SI.Resistance R3=2200 "R3";
parameter SI.Resistance R45=22e3 "R4 and R5";
parameter SI.Resistance R6=3300 "R6";
```

This implementation of Chua's Diode with opAmps combines two NICs (**Fehler! Verweisquelle konnte nicht gefunden werden.**).

Chua's Circuit: Inductor Replacement
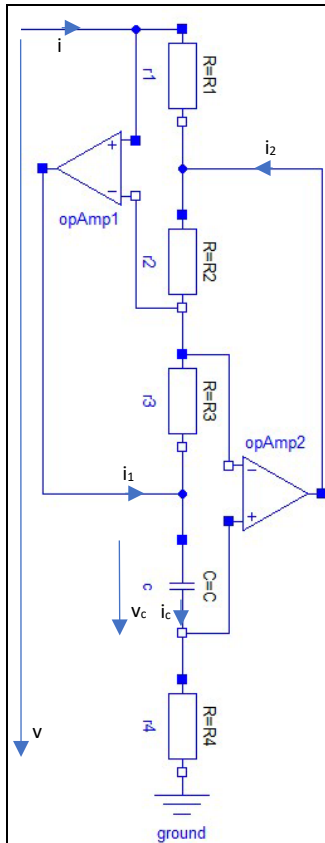
OpAmp input currents neglectible

OpAmp differential input voltage neglectible

$$R_1 \cdot i + R_2 \cdot (i + i_2) = 0$$
$$R_3 \cdot (i + i_2) + v_c = 0$$
$$i + i_2 + i_1 = i_c = C \cdot \frac{dv_c}{dt}$$

$$i + i_2 = -\frac{R_1}{R_2} \cdot i$$
$$v_c = \frac{R_1 \cdot R_3}{R_2} \cdot i$$
$$i_c = C \cdot \frac{R_1 \cdot R_3}{R_2} \cdot \frac{di}{dt}$$
$$v = R_4 \cdot i_c = C \cdot \frac{R_1 \cdot R_3 \cdot R_4}{R_2} \cdot \frac{di}{dt}$$

This TwoPin is *not* a OnePort $i_c \neq i$ and $i_1 + i_2 \neq 0$!

The ground at the bottom is necessary.

$$R_1 = 100 \ \Omega$$
$$R_2 = 1 \ k\Omega$$
$$R_3 = 1 \ k\Omega$$
$$R_4 = 1{,}8 \ k\Omega$$
$$C = 100 \ nF$$
$$L = C \cdot \frac{R_1 \cdot R_3 \cdot R_4}{R_2} = 18 \ mH$$

Chaotic Diode Circuit (with a light emitting diode)

[Pham2016]

$$C \cdot \frac{dv_1}{dt} = -\frac{v_2}{R}$$

$$C \cdot \frac{dv_2}{dt} = -\frac{v_3}{R}$$

$$C \cdot \frac{dv_3}{dt} = -\frac{v_1}{R} - \frac{v_3}{R_b} - \frac{v_4}{R}$$

$$\frac{v_4}{R_a} = -I_{ds} \cdot \left(e^{\frac{v_2}{nV_t}} - 1 \right)$$

$$\tau = R \cdot C$$

$$\tau \cdot \frac{dv_1}{dt} = -v_2$$

$$\tau \cdot \frac{dv_2}{dt} = -v_3$$

$$\tau \cdot \frac{dv_3}{dt} = -v_1 - \frac{R}{R_b} \cdot v_3 + R_a \cdot I_{ds} \cdot \left(e^{\frac{v_2}{nV_t}} - 1 \right)$$

$$a = \frac{R_a \cdot I_{ds}}{nV_t}$$

$$b = \frac{R}{R_b}$$

$$\tau \cdot \dot{x}_1 = -x_2$$

$$\tau \cdot \dot{x}_2 = -x_3$$

$$\tau \cdot \dot{x}_3 = -x_1 + a \cdot (e^{x_2} - 1) - b \cdot x_3$$


```
parameter SI.Current Ids=1e-9 "Diode: saturation current";

parameter SI.Voltage nVt=2*26e-3
" n*voltage equ. of temperature";

//time constant and dimensionless parameters

parameter SI.Time Tau=0.1e-3 "Time constant";

parameter Real a=1/5200 "Parameter a = Ra*Ids/nVt";

parameter Real b=1.5 "Parameter b = R/Rb";

//configuration of the opAmps

parameter SI.Capacitance C=10e-9
"Capacitance of {c1, c2, c3}";

parameter SI.Resistance R=Tau/C "Resistance of {r1,r2,r3,r4}";

parameter SI.Resistance Ra=a*nVt/Ids "Resistance of ra";

parameter SI.Resistance Rb=R/b "Resistance of rb";
```

Chaotic Oscillator

[Tamasevicius2005], [Tamasevicius2007]

$$i_L = C \cdot \frac{dv_C}{dt}$$

$$L \cdot \frac{di_L}{dt} = \left(k - 1 - \frac{R_L}{R} \right) \cdot R \cdot i_L - v_C - v_{C^*}$$

$$k = 1 + \frac{R_2}{R_1}$$

$$C^* \cdot \frac{dv_{C^*}}{dt} = I_0 + i_L - I_{DS} \cdot \left(e^{\frac{v_{C^*}}{nV_t}} - 1 \right)$$

$$I_0 \approx \frac{V_b}{R_0}$$

$$\tau = \sqrt{L \cdot C}$$

$$Z = \sqrt{\frac{L}{C}}$$

$$a = \left(k - 1 - \frac{R_L}{R} \right) \cdot \frac{R}{Z}$$

$$b = \frac{Z \cdot I_0}{nV_t}$$

$$c = \frac{Z \cdot I_{DS}}{nV_t}$$

$$e = \frac{C^*}{C}$$

$$\tau \cdot \frac{\dot{v}_C}{nV_t} = \frac{Z \cdot i_L}{nV_t}$$

$$\tau \cdot \frac{Z \cdot \dot{i}_L}{nV_t} = \left(k - 1 - \frac{R_L}{R} \right) \cdot \frac{R}{Z} \cdot \frac{Z \cdot i_L}{nV_t} - \frac{v_C}{nV_t} - \frac{v_{C^*}}{nV_t}$$

$$\tau \cdot e \cdot \frac{\dot{v}_{C^*}}{nV_t} = \frac{Z \cdot I_0}{nV_t} + \frac{Z \cdot i_L}{nV_t} - \frac{Z \cdot I_{DS}}{nV_t} \cdot \left(e^{\frac{v_{C^*}}{nV_t}} - 1 \right)$$


```
//time constant and dimensionless parameters
parameter SI.Current I0=Vb/R0 "Estimated supply current";
parameter SI.Time Tau=sqrt(L*C) "Time constant";
parameter SI.Impedance Z=sqrt(L/C) "Characteristic impedance (rho)";
parameter Real k=Z/R*a + 1 + RL/R "Amplification";
parameter Real a=0.95;
parameter Real b=Z*I0/nVt;
parameter Real c=Z*Ids/nVt;
parameter Real e=Cs/C "epsilon";
//components
parameter SI.Resistance R =1.0e3 "Resistor r";
parameter SI.Resistance R1=10.e3 "Resistor r1";
parameter SI.Resistance R2=R1*(k - 1) "Resistor r2 (tunable)";
parameter SI.Resistance R0=20.e3 "Resistor r0";
parameter SI.Voltage Vb=20 "Supply voltage";
parameter SI.Inductance L=100e-3 "Inductor";
parameter SI.Resistance RL=50 "Resistance of Inductor";
parameter SI.Capacitance C =100e-9 "Capacitor c";
parameter SI.Capacitance Cs=15.e-9 "Capacitor c*";
//diode
parameter SI.Current Ids=1e-9 "Diode: saturation current";
parameter SI.Voltage nVt=2*26e-3
"Diode: n * voltage equivalent of temperature";
```

Colpitts Oscillator

[Buscarino2014], [Kennedy1994]

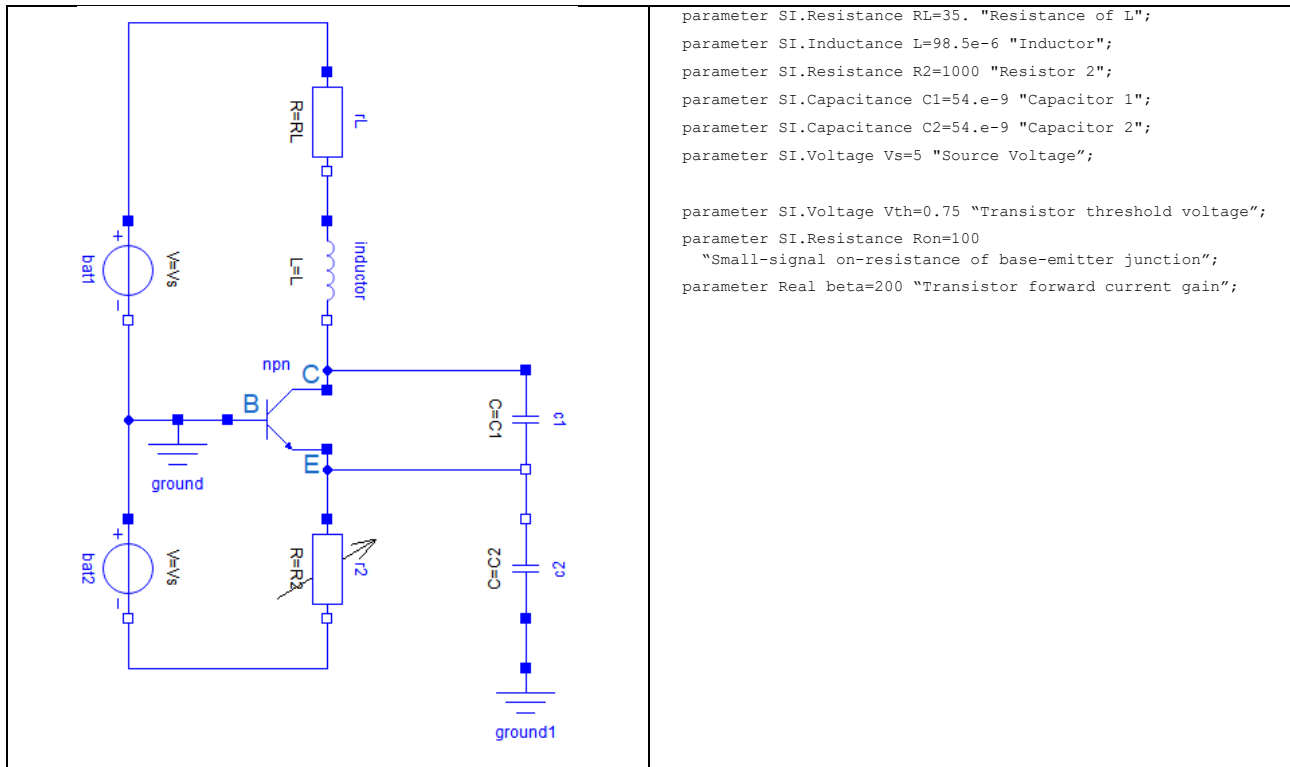$$C_1 \cdot \frac{dv_1}{dt} = i_L - \beta \cdot i_B$$

$$C_2 \cdot \frac{dv_2}{dt} = -\frac{V_{s-} + v_2}{R_2} - i_L - i_B$$

$$L \cdot \frac{di_L}{dt} = V_{s+} - v_1 + v_2 - R_L \cdot i_L$$

$$i_B = \begin{cases} v_2 = v_{BE} \leq V_{th} \rightarrow & 0 \\ v_2 = v_{BE} > V_{th} \rightarrow & \dfrac{v_2 - V_{th}}{R_{on}} \end{cases}$$


```
parameter SI.Resistance RL=35. "Resistance of L";
parameter SI.Inductance L=98.5e-6 "Inductor";
parameter SI.Resistance R2=1000 "Resistor 2";
parameter SI.Capacitance C1=54.e-9 "Capacitor 1";
parameter SI.Capacitance C2=54.e-9 "Capacitor 2";
parameter SI.Voltage Vs=5 "Source Voltage";

parameter SI.Voltage Vth=0.75 "Transistor threshold voltage";
parameter SI.Resistance Ron=100
  "Small-signal on-resistance of base-emitter junction";
parameter Real beta=200 "Transistor forward current gain";
```

Shinriki Oscillator

[Lueck1995], [Shinriki1981]
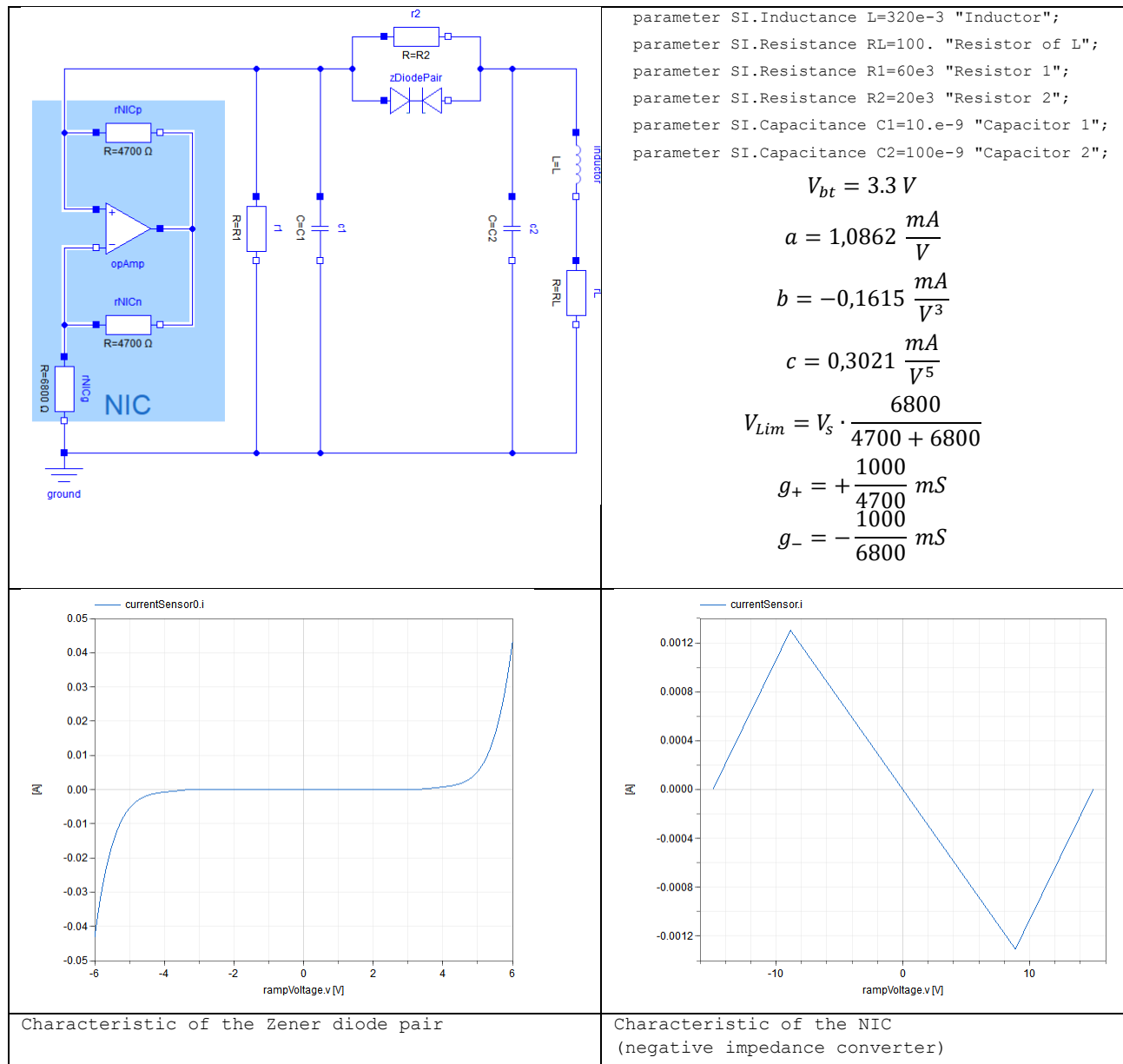
$$i_z = \begin{cases} |v_z| < V_{bt} & 0 \\ |v_z| \geq V_{bt} & sign(v_z) \cdot [a \cdot (|v_z| - V_{bt}) + b \cdot (|v_z| - V_{bt})^3 + c \cdot (|v_z| - V_{bt})^5] \end{cases}$$

$$C_1 \cdot \frac{dv_1}{dt} = -i_{NIC} - \frac{v_1}{R_1} - i_z$$

$$C_2 \cdot \frac{dv_2}{dt} = i_z - i_L$$

$$v_2 = L \cdot \frac{di_L}{dt} + R_L \cdot i_L$$

$$g_{NIC} = \frac{di_{NIC}}{dv_{NIC}} = \begin{cases} |v_{NIC}| > V_{Lim} & g_+ \\ |v_{NIC}| \leq V_{Lim} & g_- \end{cases}$$


```
parameter SI.Inductance L=320e-3 "Inductor";
parameter SI.Resistance RL=100. "Resistor of L";
parameter SI.Resistance R1=60e3 "Resistor 1";
parameter SI.Resistance R2=20e3 "Resistor 2";
parameter SI.Capacitance C1=10.e-9 "Capacitor 1";
parameter SI.Capacitance C2=100e-9 "Capacitor 2";
```

$$V_{bt} = 3.3 \, V$$

$$a = 1{,}0862 \, \frac{mA}{V}$$

$$b = -0{,}1615 \, \frac{mA}{V^3}$$

$$c = 0{,}3021 \, \frac{mA}{V^5}$$

$$V_{Lim} = V_s \cdot \frac{6800}{4700 + 6800}$$

$$g_+ = +\frac{1000}{4700} \, mS$$

$$g_- = -\frac{1000}{6800} \, mS$$

Characteristic of the Zener diode pair

Characteristic of the NIC
(negative impedance converter)

Investigation of the Zener diode pair approximation:

$$i_z = \begin{cases} |v_z| < V_{bt} & 0 \\ |v_z| \geq V_{bt} & sign(v_z) \cdot [a \cdot (|v_z| - V_{bt}) + b \cdot (|v_z| - V_{bt})^3 + c \cdot (|v_z| - V_{bt})^5] \end{cases}$$

$$\frac{di_z}{dv_z} = \begin{cases} |v_z| < V_{bt} & 0 \\ |v_z| \geq V_{bt} & sign(v_z) \cdot [a + 3b \cdot (|v_z| - V_{bt})^2 + 5c \cdot (|v_z| - V_{bt})^4] \end{cases}$$

The first derivative $\frac{di_z}{dv_z}$ is not continuous at $|v_z| = V_{bt}$.

If voltage v is prescribed, the current i can be unambiguously determined.

If current i is prescribed, between $-V_{bt}$ and $+V_{bt}$ there is a manifold of solutions for the voltage v.

For this application, this restriction has no influence.

This restriction could be solved by adaption the approximation, i.e. exchange the horizontal line in the range $-V_{bt} < v_z < +V_{bt}$ against a characteristic with small constant positive slope and adapt the polynomial approximation to achieve a one times continuously differentiable characteristic.

Investigation of the negative impedance converter (NIC):

As long as the opAmp operates in the linear region:

$$i_n = \frac{v_{out} - v}{R} = \frac{v}{R_g} \rightarrow v_{out} = v \cdot \frac{R + R_g}{R_g}$$

$$i = \frac{v - v_{out}}{R} = -\frac{v}{R_g}$$

$$g_- = \frac{1}{R_g}$$

When the opAmp's output saturates:

$$V_{Lim} = V_s \cdot \frac{R_g}{R + R_g}$$

$$v \geq +V_{Lim}: i = \frac{v - V_{Lim}}{R}$$

$$g_+ = \frac{1}{R}$$

If voltage v is prescribed, the current i can be unambiguously determined.

If current i is prescribed, in the range between the zero crossings the voltage v has 3 possible solutions.

For this application, this restriction has no influence.

Jerk Circuit

[Buscarino2014], [Sprott2011]

The name of the system stems from the third derivative of x, which – in a mechanical system – is the derivative of acceleration called jerk. The Jerk equation has been investigated in different versions.

$$\dddot{x} + G(\ddot{x}, \dot{x}, x) = 0$$

The version implemented here uses a diode as described in the mentioned publications:

$$G(\ddot{x}, \dot{x}, x) = A \cdot \ddot{x} + f(\dot{x}) + x$$
$$\dddot{x} + A \cdot \ddot{x} + f(\dot{x}) + x = 0$$

$f(\dot{x})$ is modeled using the Shockley equation of a diode:

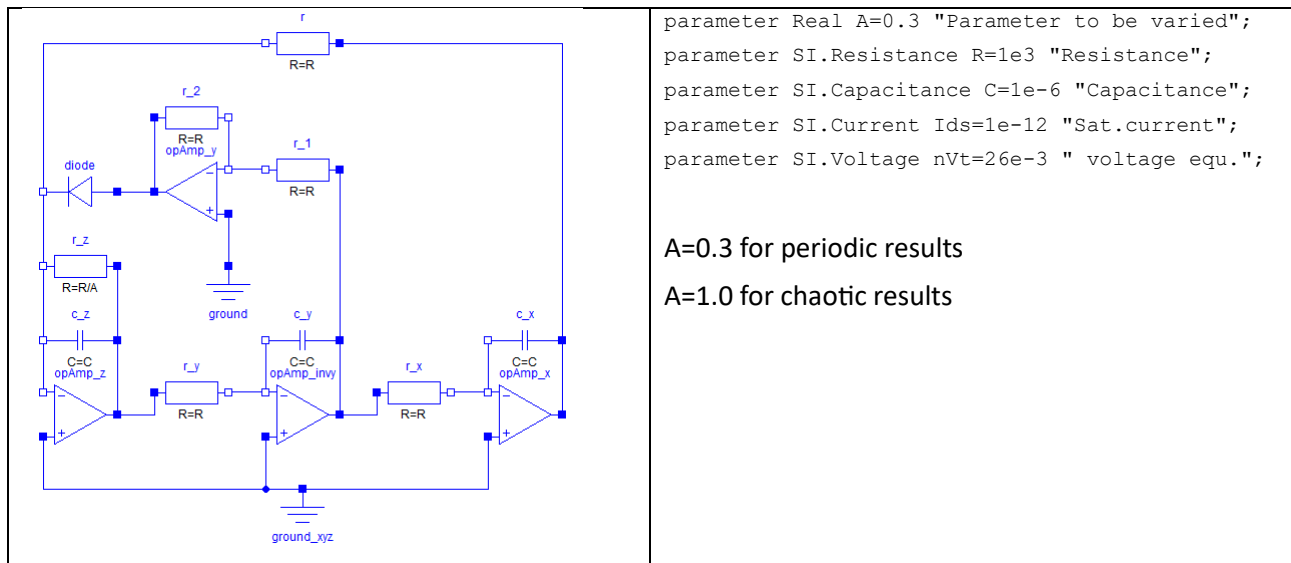$$f(\dot{x}) = R \cdot I_S \cdot \left(e^{\frac{\dot{x}}{nV_t}} - 1 \right)$$

This leads to a system of 3 ordinary differential equations with one nonlinearity:

$$\dot{x} = y$$
$$\ddot{x} = \dot{y} = z$$
$$\dddot{x} = \ddot{y} = \dot{z}$$
$$\dot{z} = -A \cdot z - x - f(y)$$

The values stay pretty inside a practicable range for a normal voltage supply.

Using 4 operational amplifiers and an acceleration factor of 1000, the circuit can be implemented as follows. The parameter A influences only the feedback resistor at opAMp_z:


```
parameter Real A=0.3 "Parameter to be varied";
parameter SI.Resistance R=1e3 "Resistance";
parameter SI.Capacitance C=1e-6 "Capacitance";
parameter SI.Current Ids=1e-12 "Sat.current";
parameter SI.Voltage nVt=26e-3 " voltage equ.";
```

A=0.3 for periodic results

A=1.0 for chaotic results

Rikitake System

[Rikitake1958], [Rikitake1973]

The system proposed by Rikitake has been used to explain irregular reversals of the Earth's magnetic field. 2 identical magnetically coupled disc dynamos ($\tau \cdot \omega$ covers the losses $R \cdot i^2$):

$$L \cdot \frac{di_1}{dt} + R \cdot i_1 = M \cdot i_2 \cdot \omega_1$$

$$J \cdot \frac{d\omega_1}{dt} = \tau - M \cdot i_2 \cdot i_1$$

$$L \cdot \frac{di_1}{dt} + R \cdot i_2 = M \cdot i_1 \cdot \omega_2$$

$$J \cdot \frac{d\omega_2}{dt} = \tau - M \cdot i_1 \cdot i_2$$

$\phi_{12} = M \cdot i_2$ is the magnetic flux in machine 1 excited by current i_2,
$\phi_{21} = M \cdot i_1$ is the magnetic flux in machine 2 excited by current i_1.
The circuit could be implemented using two series excited DC machines
with the armature current of the other machine as excitation current.
The equations of motion have identical right hand sides:

$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} \rightarrow \omega_1 - \omega_2 = \Delta\omega = const.$$

Mechanical and electrical time constant:

$$T_m = \frac{J}{\tau} \cdot \frac{R}{M}$$

$$T_e = \frac{L}{R}$$

$$\mu = \sqrt{\frac{T_m}{T_e}} = \sqrt{\frac{J}{\tau} \cdot \frac{R^2}{L \cdot M}}$$

used to scale the variables:

$$i_{1,2} = x_{1,2} \cdot \sqrt{\frac{\tau}{M}}$$

$$\omega_1 = z \cdot \sqrt{\frac{\tau}{J} \cdot \frac{L}{M}} \rightarrow \omega_2 = (z - \Delta) \cdot \sqrt{\frac{\tau}{J} \cdot \frac{L}{M}}$$

$$t = t' \cdot \sqrt{T_m \cdot T_e} = t' \cdot \sqrt{\frac{J}{\tau} \cdot \frac{L}{M}}$$

lead to the scaled equations:

$$\frac{dx_1}{dt'} = -\mu \cdot x_1 + x_2 \cdot z$$

$$\frac{dx_2}{dt'} = -\mu \cdot x_2 + x_1 \cdot (z - \Delta)$$

$$\frac{dz}{dt'} = 1 - x_1 \cdot x_2$$

The states stay within a range that needs no scaling when implemented with an opAmp-circuit.

Simplest Chaotic Circuit

[Muthuswamy2010]

The memristor drift model - either linear or with window function $fw(x)$ – doesn't include a negative resistance R_{mem} in the characteristic. Although the memristor itself is discussed controversially [Abraham2018], Muthuswamy and Chua suggested 2010 a special memristor function to construct an autonomous resonance circuit which shows chaotic results.
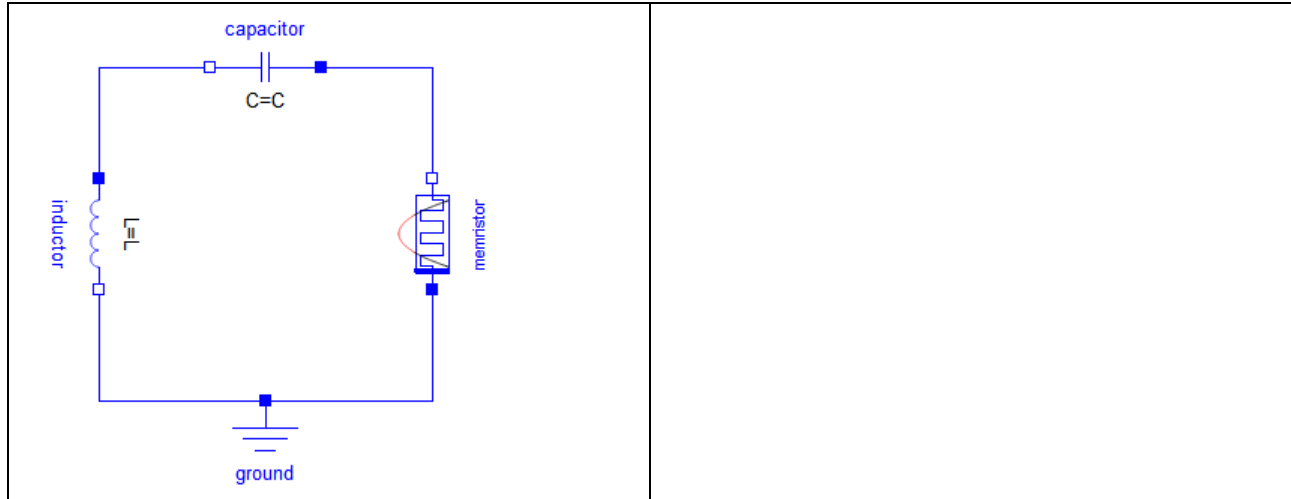
Memristor models

[Biolek2009], [Biolek2015], [Chua1971], [Chua1977], [Joglekar2009], [Majetta2012], [Maurer2014], [Muthuswamy2010], [Oguz2018], [Yakopcic2011]

In general: q is an internal state, ideally charge: $\frac{dq}{dt} \sim i$	$v = R_{mem} \cdot i$ $R_{mem} = f(q)$ $\frac{dq}{dt} = g(i, q)$
	$R_{mem} = R_{on} \cdot q + R_{off} \cdot (1 - q)$ $q = \frac{w}{D}$ $\frac{dq}{dt} = \frac{\mu_v \cdot R_{on}}{D^2} \cdot i \cdot fw(q)$ [Joglekar2009] $\qquad fw(q) = 1 - (2q - 1)^{2p}$ [Biolek2009] $\qquad fw(q) = 1 - \left(q - \begin{cases} 1 & -i \geq 0 \\ 0 & -i < 0 \end{cases}\right)^{2p}$
[Muthuswamy2010] $\qquad \alpha = I_0$ $\qquad \beta = R_0$	$R_{mem} = R_0 \cdot (q^2 - 1)$ $\frac{dq}{dt} = i - q \cdot I_0 - q \cdot i = i \cdot (1 - q) - I_0 \cdot q$

Appendix B.3 of [Muthuswamy2010] demonstrates the implementation of this special memristor using 4 operational amplifiers and 2 multipliers.

References

[Abraham2018]   Isaac Abraham,
"The case for rejecting the memristor as a fundamental circuit element",
Scientific Reports 8, Article No.10971, 2018.

[Berkeley]   Berkeley Electrical Engineering and Computer Science,
Nonlinear Electronics Lab: "Working with Chaos".

[Biolek2009]   Zdeněk Biolek, Dalibor Biolek, Viera Biolková,
"SPICE Model of Memristor with Nonlinear Dopant Drift",
Radioengineering, Vol.18, No.2, 2009, pp. 210-214.

[Biolek2015]   Zdeněk Biolek, Dalibor Biolek, Viera Biolková,
"Differential Equations of Ideal Memristors",
Radioengineering, Vol.24, No.2, 2015, pp. 369-377.

[Buscarino2014]   Arturo Buscarino, Luigi Fortuna, Mattia Frasca and Gregorio Sciuto,
"A Concise Guide to Chaotic Electronic Components",
Springer Briefs in Applied Sciences and Technology 2014,
ISBN 978-3-319-05899-3, ISSN 2191-530X,
DOI 10.1007/978-3-319-05900-6.

[Chua1971]   Leon O. Chua,
"Memristor-The Missing Circuit Element",
IEEE Transactions on Circuit Theory Vol.18, No.5, 1971, pp. 507-519.

[Chua1977]   Leon O. Chua, and Sung Mo Kang,
"Memristive devices and systems",
Proceedings of the IEEE Vol.64, No.2, 1976, pp. 209-223.

[Chua1983]   Leon O. Chua, C.W. Wu, A. Huang, and Guo-Qun Zhong,
"A universal circuit for studying and generating chaos. I. Routes to chaos",
IEEE Transactions on CAS I 40 (1993)10, pp. 732-744.

[Joglekar2009]   Yogesh N Joglekar and Stephen J Wolf,
"The elusive memristor: properties of basic electrical circuits",
European Journal of Physics Vol.30, No. 4, 2009, pp. 661-675.

[Kennedy1993a]   Michael Peter Kennedy,
"Three Steps to Chaos - Part I: Evolution",
IEEE Transactions on CAS I 40 (1993)10, pp. 640-656.

[Kennedy1994]   Michael Peter Kennedy,
"Chaos in the Colpitts Oscillator",
IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications,
Vol.41, No.11, 1994, ISSN 1057-7122.

[Lueck1995]   Stefan Lück,
"Ein Praktikumsversuch zum Thema: Chaos in einfachen physikalischen Systemen"
(German), Zulassungsarbeit zum ersten Staatsexamen, University of Bayreuth,
Lehrstuhl für Experimentalphysik II, 1995.

[Lorenz1963]   Edward N. Lorenz,
"Deterministic Nonperiodic Flow",
Journal of the Atmospheric Sciences, Vol.20, 1963, pp. 130-142.

[Majetta2012]   Kristin Majetta, Christoph Clauss, Torsten Schmidt,
"Towards a Memristor Model Library in Modelica",
Proceedings of the 9th International MODELICA Conference, 2012, Munich.

[Maurer2014]   Christoph Maurer,
                  "Zur Memristik in neuromorphen Schaltkreisen." (German),
                  Master Thesis 2014, Humboldt-Universität zu Berlin.

[Muthuswamy2009] Bharathwaj Muthuswamy,
                  "A Synthetic Inductor Implementation of Chua's Circuit",
                  Technical Report No. UCB/EECS-2009-20, University of California at Berkeley.

[Muthuswamy2010] Bharathwaj Muthuswamy, and Leon O. Chua,
                  "Simplest Chaotic Circuit",
                  International Journal of Bifurcation and Chaos, Vol.20, No.5, 2010, pp. 1567-1580.

[Oguz2018]   Yasin Oğuz,
                  "Mathematical Modeling of Memristors",
                  Chapter 9 in: Alex James, Memristor and Memristive Neural Networks, Intechopen
                  2018, ISBN 978-953-51-3948-5.

[Pham2016]   Viet-Thanh Pham, Christos Volos, Sajad Jafari, Xiong Wang, and
                  Tomasz Kapitaniak,
                  "A simple chaotic circuit with a light-emitting diode",
                  Optoelectronics and Advanced Materials – Rapid Communications Vol. 10, No. 9-10,
                  September-October 2016, pp. 640 - 646.

[Rikitake1958]   Tsuneji Rikitake,
                  "Oscillations of a system of disk dynamos",
                  Mathematical Proceedings of the Cambridge Philosophical Society, Vol.54, No.1,
                  January 1958, 89 – 105.

[Rikitake1973]   Tsuneji Rikitake,
                  " Non-steady Geomagnetic Dynamo Models",
                  Geophysical Journal International, Vol.35, No.1-3, December 1973, 277–284.

[Roessler1976]   Otto E. Roessler,
                  "An equation for continuous chaos",
                  Physics Letters A Vol. 57, No. 5, July 1976, pp. 397-398.

[Roessler1979]   Otto E. Roessler,
                  "An equation for hyperchaos",
                  Physics Letters A Vol. 71, No. 2-3, April 1979, pp. 155-157.

[Shinriki1981]   Masanon Shinriki, Masahiro Yamamoto, Shinsaku Mori,
                  "Multimode oscillations in a modified Van Der Pol oscillator containing a positive
                  nonlinear conductance",
                  Proceedings of the IEEE Vol. 69, No. 3, March 1981, pp. 394 - 395.

[Sprott2011]   J.C. Sprott,
                  "A New Chaotic Jerk Circuit",
                  IEEE Transactions on Circuits and Systems II: Express Briefs Vol. 58, No. 4, April 2011,
                  pp. 240 - 253.

[Tamasevicius2005] A. Tamaševičius, G. Mykolaitis, V. Pyragas and K. Pyragas,
                  " A simple chaotic oscillator for educational purposes",
                  European Journal of Physics 26(1):61-63 (Nov. 2005).

[Tamasevicius2007] A. Tamaševičius, T. Pyragiene, K. Pyragas, S. Bumeliene and M. Meskauskas,
                  " Numerical Treatment of Educational Chaos Oscillator",
                  International Journal of Bifurcation and Chaos Vol 17, No. 10 (2007), pp. 3657-3661.

[VanDerPol1927]   B. van der Pol and J. van der Mark,
                  "Frequency Demultiplication"",
                  Nature 120(1927), pp. 363-364

[Vano2006]   J.A.Vano, J.C.Wildenberg, M.B.Anderson, J.K.Noel, J.C.Sprott,
                  "Chaos in low-dimensional Lotka–Volterra models of competition",
                  Nonlinearity 19 (2006) 10, pp. 2391-2404.

[Yakopcic2011]   Chris Yakopcic, Tarek M. Taha, Guru Subramanyam, Robinson E. Pino, and Stanley
                  Rogers,
                  "A Memristor Device Model",
                  IEEE Electron Device Letters Vol.32, No.10, 2011, pp. 1436-1438.

[Zhong1985]   Guo-Qun Zhong and F. Ayrom,
                  "Periodicity and Chaos in Chua's Circuit",
                  IEEE Transactions on CAS I 40 (1993)10, pp. 732-744.