# Comparison of chaotic circuits

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| --- | --- |
| Lotka-Volterra |  |
| van der Pol Circuit |  |
| Lorenz System |  |
| Roessler System |  |
| Chua’s Circuit | nonlinear conductor (two NICs)  with partly negative slope |
| Chaotic Diode Circuit |  |
| Chaotic Oscillator  A simple chaotic oscillator for educational purposes |  |
| Colpitts Oscillator | LC oscillator with transistor (orig. vacuum tube) |
| Shinriki Oscillator | two antiparallel Z-diodes and nonlinear conductor (NIC) with partly negative slope |
| Jerk Circuit |  |
| Rikitake System | Two coupled disc dynamos. |
| Chua’s Circuit with Memristor | Replace Chua’s diode with a special memristor |

# OpAmp-Circuits

Algebraic-Differential Equation Systems can be simulated with analog computers   
using operational amplifiers and analog multipliers.

|  |  |
| --- | --- |
| Amplifier/Adder | Input resistance  Output resistance  It is possible to add several inputs. |
| Integrator | Input resistance  Output resistance  It is possible to integrate the sum of several inputs. |
| Analog Multiplier | Functional Block Diagram of AD633  Division by (scaling) inhibits overflow.  Additional summing input Z is omitted.  Negative inputs of X- and Y-amplifiers are connected to ground.  Possible implementations:   * Gilbert cell |

# Lotka-Volterra

[Vano2006]

As a 2-dimensional predator-prey-model no chaos is reported:

We might interpret as number of hares (prey) and as foxed (predator).  
 is the reproduction rate of hares, the deathrate of hares due to foxes.

is the efficiency in growing foxes from hares, the (natural) deathrate of foxes.

A n-dimensional Lotka-Volterra model is defined for ( designates the number of species):

The vector describes the reproduction rates of the species whereas the quadratic matrix describes the competition between species.

The above mentioned predator-prey case with 2 species can be expressed as:

# van der Pol Circuit

[Buscarino2014], [VanDerPol1927]

Balthasar van der Pol reported 1927 strange phenomena about oscillations in a series resonance circuit containing a vacuum electron triode. Due to the nonlinear characteristic of the triode the autonomous circuit is able to maintain periodic oscillations, and with harmonic excitation it is able to produce chaos.



These are the equations of the physical model.

Note the phase shift of the excitation!

|  |  |
| --- | --- |
|  | Alternative Formulation with 2 states: |

Note:

Instead of using a series resonance circuit and deriving a scaled differential equation for the current, we could use an equivalent parallel resonance circuit and derive a scaled differential equation for the voltage.

Initialization:

The physical model has 2 states: and . Current acts as an initial value for the nonlinear resistor.

The analytic equations have 3 states: , and .

The third state has been introduced artificially by first differentiating the voltage equation, generating an equation with second derivative of . Splitting this equation into two first order differential equations, we generate and as states. Calculating capacitor voltage , we get the third state.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. We chose natural eigen frequency as time scale:

We also have to take into account that the analog multiplier divides by to avoid overflow of the output.

After that, none of the computing block should encounter an overflow.  
This leads to the following set of equations:

These equations can easily get implemented as blocks or as an electronic circuit.

Calculating back from per-unit-parameters:

and some assumptions:

we obtain physical parameters:

Investigating the nonlinear resistance of the triode:

Characteristic of the triode :



Zero crossings : with slopes

Extrema: with

Inflection point:

If current is prescribed, voltage can be unambiguously determined.

If is prescribed, in the range i.e. has 2 or 3 possible solutions.

For this application, this restriction has no influence.

Shifting the characteristic up and to the right, it looks like the characteristic of a tunnel (Esaki) diode.

Inversion of the triode characteristic shows hysteretic behavior (split into 2 branches):

# Lorenz System

[Buscarino2014], [Lorenz1963]

Developed 1963 by Edward Lorenz to model atmospheric convection.

is proportional to the rate of convection, to the horizontal temperature variation and to the vertical temperature variation. depicts the Prandtl number, the Rayleigh number and the physical dimensions.

The original parameters were:

leads to a periodic solution.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. This can be compared with calculating per-unit values by dividing by reference values:

We also have to take into account that the analog multiplier divides by to avoid overflow of the output.

After that, none of the computing block should encounter an overflow.  
This leads to the following set of equations:

These equations can easily get implemented as blocks or as an electronic circuit.

# Roessler System

[Roessler1976], [Roessler1979]

A simple system of 3 ordinary nonlinear differential equations to study chaos without physical background.

, and give periodic results. Changing reveals chaotic results.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. This can be compared with calculating per-unit values by dividing by reference values:

We also have to take into account that the analog multiplier divides by to avoid overflow of the output.

After that, none of the computing block should encounter an overflow.  
This leads to the following set of equations:

These equations can easily get implemented as blocks or as an electronic circuit.

# Chua’s Circuit

[Berkeley], [Buscarino2014], [Chua1983], [Kennedy1993a], [Muthuswamy2009], [Zhong1985]

|  |  |
| --- | --- |
|  | parameter SI.Resistance R=1.9e3 "Resistor";    parameter SI.Inductance L=18e-3 "Inductor";    parameter SI.Resistance RL=14 "Resistance of Inductor";    parameter SI.Capacitance C1=10.e-9 "Capacitor 1";    parameter SI.Capacitance C2=100e-9 "Capacitor 2";    //parameter of Chua's diode    parameter Real k0=15000.0 "No-load amplification ";    parameter SI.Voltage Vs=9 "Supply voltage of opAmps";    parameter SI.Resistance R12=220 "R1 and R2";    parameter SI.Resistance R3=2200 "R3";    parameter SI.Resistance R45=22e3 "R4 and R5";    parameter SI.Resistance R6=3300 "R6"; |

This implementation of Chua’s Diode with opAmps combines two NICs – see NIC (negative impedance converter).

**Chua’s Circuit: Inductor Replacement**

|  |  |
| --- | --- |
| ic  i  i2  v  vc  i1 | OpAmp input currents neglectible  OpAmp differential input voltage neglectible  This TwoPin is *not* a OnePort and !  The ground at the bottom is necessary. |

# Chaotic Diode Circuit

[Pham2016]

|  |  |
| --- | --- |
|  | parameter SI.Current Ids=1e-9 "Diode: saturation current";    parameter SI.Voltage nVt=2\*26e-3   " n\*voltage equ. of temperature";    //time constant and dimensionless parameters    parameter SI.Time Tau=0.1e-3 "Time constant";    parameter Real a=1/5200 "Parameter a = Ra\*Ids/nVt";    parameter Real b=1.5 "Parameter b = R/Rb";    //configuration of the opAmps    parameter SI.Capacitance C=10e-9   "Capacitance of {c1, c2, c3}";    parameter SI.Resistance R=Tau/C "Resistance of {r1,r2,r3,r4}";    parameter SI.Resistance Ra=a\*nVt/Ids "Resistance of ra";    parameter SI.Resistance Rb=R/b "Resistance of rb"; |

# Chaotic Oscillator

[Tamasevicius2005], [Tamasevicius2007]

|  |  |
| --- | --- |
|  | //time constant and dimensionless parameters    parameter SI.Current I0=Vb/R0 "Estimated supply current";    parameter SI.Time Tau=sqrt(L\*C) "Time constant";    parameter SI.Impedance Z=sqrt(L/C) "Characteristic impedance (rho)";    parameter Real k=Z/R\*a + 1 + RL/R "Amplification";    parameter Real a=0.95;    parameter Real b=Z\*I0/nVt;    parameter Real c=Z\*Ids/nVt;    parameter Real e=Cs/C "epsilon";    //components    parameter SI.Resistance R =1.0e3 "Resistor r";    parameter SI.Resistance R1=10.e3 "Resistor r1";    parameter SI.Resistance R2=R1\*(k - 1) "Resistor r2 (tunable)";    parameter SI.Resistance R0=20.e3 "Resistor r0";    parameter SI.Voltage Vb=20 "Supply voltage";    parameter SI.Inductance L=100e-3 "Inductor";    parameter SI.Resistance RL=50 "Resistance of Inductor";    parameter SI.Capacitance C =100e-9 "Capacitor c";    parameter SI.Capacitance Cs=15.e-9 "Capacitor c\*";    //diode    parameter SI.Current Ids=1e-9 "Diode: saturation current";    parameter SI.Voltage nVt=2\*26e-3   "Diode: n \* voltage equivalent of temperature"; |

# Colpitts Oscillator

[Buscarino2014], [Kennedy1994]

|  |  |
| --- | --- |
|  | parameter SI.Resistance RL=35. "Resistance of L";    parameter SI.Inductance L=98.5e-6 "Inductor";    parameter SI.Resistance R2=1000 "Resistor 2";    parameter SI.Capacitance C1=54.e-9 "Capacitor 1";    parameter SI.Capacitance C2=54.e-9 "Capacitor 2";    parameter SI.Voltage Vs=5 "Source Voltage”;  parameter SI.Voltage Vth=0.75 “Transistor threshold voltage”;  parameter SI.Resistance Ron=100   “Small-signal on-resistance of base-emitter junction”;  parameter Real beta=200 “Transistor forward current gain”; |

# Shinriki Oscillator

[Lueck1995], [Shinriki1981]

|  |  |
| --- | --- |
|  | parameter SI.Inductance L=320e-3 "Inductor";    parameter SI.Resistance RL=100. "Resistor of L";    parameter SI.Resistance R1=60e3 "Resistor 1";    parameter SI.Resistance R2=20e3 "Resistor 2";    parameter SI.Capacitance C1=10.e-9 "Capacitor 1";    parameter SI.Capacitance C2=100e-9 "Capacitor 2"; |
|  |  |
| Characteristic of the Zener diode pair | Characteristic of the NIC (negative impedance converter) |

Investigation of the Zener diode pair approximation:

The first derivative is not continuous at .

If voltage is prescribed, the current can be unambiguously determined.

If current is prescribed, between and there is a manifold of solutions for the voltage .

For this application, this restriction has no influence.

This restriction could be solved by adaption the approximation, i.e. exchange the horizontal line in the range against a characteristic with small constant positive slope and adapt the polynomial approximation to achieve a one times continuously differentiable characteristic.

# NIC (negative impedance converter)

|  |  |
| --- | --- |
|  | As long as the opAmp operates in the linear region:  When the opAmp’s output saturates: |

If voltage is prescribed, the current can be unambiguously determined.

If current is prescribed, in the range between the zero crossings the voltage has 3 possible solutions.

For this application, this restriction has no influence.

# Jerk Circuit

[Buscarino2014], [Sprott2011]

The name of the system stems from the third derivative of , which – in a mechanical system – is the derivative of acceleration called jerk. The Jerk equation has been investigated in different versions.

The version implemented here uses a diode as described in the mentioned publications:

is modeled using the Shockley equation of a diode:

This leads to a system of 3 ordinary differential equations with one nonlinearity:

The values stay pretty inside a practicable range for a normal voltage supply.

Using 4 operational amplifiers and an acceleration factor of 1000, the circuit can be implemented as follows.  
The parameter A influences only the feedback resistor at opAMp\_z:

|  |  |
| --- | --- |
|  | parameter Real A=0.3 "Parameter to be varied";  parameter SI.Resistance R=1e3 "Resistance";  parameter SI.Capacitance C=1e-6 "Capacitance";  parameter SI.Current Ids=1e-12 "Sat.current";  parameter SI.Voltage nVt=26e-3 " voltage equ.";  A=0.3 for periodic results  A=1.0 for chaotic results |

# Rikitake System

[Rikitake1958], [Rikitake1973]

The system proposed by Rikitake has been used to explain irregular reversals of the Earth’s magnetic ﬁeld.

2 identical magnetically coupled disc dynamos ( covers the losses ):

is the magnetic flux in machine 1 excited by current ,   
 is the magnetic flux in machine 2 excited by current .

The circuit could be implemented using two series excited DC machines   
with the armature current of the other machine as excitation current.

The equations of motion have identical right hand sides:

Mechanical and electrical time constant:

used to scale the variables:

lead to the scaled equations:

The states stay within a range that needs no scaling when implemented with an opAmp-circuit.

# Chua’s Circuit with Memristor

[Muthuswamy2010a]

Muthuswamy suggested 2010 a special memristor function to replace Chua’s diode:

|  |  |
| --- | --- |
|  | is choosen to scale the system. |

# Memristor models

[Biolek2009], [Biolek2015], [Chua1971], [Chua1977], [Itoh2008], [Joglekar2009], [Majetta2012], [Maurer2014], [Muthuswamy2010], [Muthuswamy2010a], [Oguz2018], [Strukov2008], [Yakopcic2011]

|  |  |
| --- | --- |
| **Charge-controlled memristor**  State = charge:  which remembers the history of current. |  |
| **Flux-controlled memristor**  State = magnetic flux:  which remembers the history of voltage. |  |

|  |  |
| --- | --- |
| [Majetta2012] Charge-controlled memristor  (HP) drift model: | is the dopant mobility:  is a window-function.  [Joglekar2009]  [Biolek2009] |

|  |  |
| --- | --- |
| [Muthuswamy2010] Charge-controlled memristor |  |

Appendix B.3 of [Muthuswamy2010] demonstrates the implementation of this special memristor   
using operational amplifiers and analog multipliers.

|  |  |
| --- | --- |
| [Muthuswamy2010a] Flux-controlled memristor |  |

Fig. 6 of [Muthuswamy2010a] demonstrates the implementation of this special memristor   
using operational amplifiers and analog multipliers.

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