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SHORT COMMUNICATIONS

Computer Solution of the Almost Empty Hexagon Problem

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1. INTRODUCTION AND STATEMENT OF RESULTS

In 1935, Erdős and Szekeres stated the following problem (see [1] and [2]).

The first Erdős–Szekeres problem. *For any integer $n \geq 3$, find a minimal positive number $g(n)$ such that any set of at least $g(n)$ points in general position in the plane has a subset of cardinality n whose elements are the vertices of a convex n-gon.*

In 1978, Erdős proposed the following modification of the first problem (see [3]).

The second Erdős–Szekeres problem. *For any integer $n \geq 3$, find a minimal positive number $h(n)$ such that any set X of at least $h(n)$ points in general position in the plane has a subset of cardinality n whose elements are the vertices of an empty convex n-gon, i.e., of a convex n-gon containing no points of X inside.*

Recall that a set points is *in general position* if no three points in this set are collinear.

The problems mentioned above are classical problems of combinatorial geometry and Ramsey theory (see [4]–[7]). Both of them are generalized as follows.

The third Erdős–Szekeres problem. *For any integers $n \geq 3$ and $k \geq 0$, find a minimal positive number $h(n,k)$ such that any set $\mathscr{X}$ of at least $h(n,k)$ points in general position in the plane has a subset of cardinality n whose elements are the vertices of a convex n-gon C satisfying the condition $|(C \setminus \partial C) \cap \mathscr{X}| \leq k$, i.e., containing at most k points of $\mathscr{X}$ inside.*

Concerning the first Erdős–Szekeres problem, it is known that

$$g(3) = 3, \qquad g(4) = 5, \qquad g(5) = 9, \qquad g(6) = 17$$

(see [4] and [8]). The last relation, which was proved much later than the others, is due to efforts of G. Szekeres, B. McKay, and L. Peters. They developed a computer algorithm which performs an exhaustive search over all 17-point sets in the plane and finds a convex hexagon in each set. The proof of the theorem has taken 1500 hours of computer time. A brief description of the algorithm is given in the next section.

In the general case, the best upper and lower bounds are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-3} + 1.$$

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The second problem has been studied more deeply in some respects. The following results have been obtained (see [4] and [9]):

$$h(3) = 3, \qquad h(4) = 5, \qquad h(5) = 10, \qquad 30 \le h(6) \le 463;$$

for $n \ge 7$, $h(n)$ does not exist.

For the third problem, as is easy to see, we have

$$g(n) \le h(n,k) \le h(n)$$

for all n and k for which the quantities under consideration are defined. Moreover,

$$h(n) = h(n,0) \ge h(n,1) \ge h(n,2) \ge \cdots ,$$

and there exists a k' such that $h(n,k) = g(n)$ for all $k \ge k'$. For small n, the following relations obviously hold:

$$h(3,k) = 3, \qquad h(4,k) = 5, \qquad h(5,0) = 10, \qquad h(5, \ge 1) = 9.$$

The last result equality holds because a convex pentagon containing two or more points inside always contains a smaller convex pentagon.

Some results on the third problem were obtained in Sendov's paper [10], where the nonexistence of $h(n,k)$ for certain k and any $n > 7$ was proved by using the set constructed by Horton in [11] to prove the nonexistence of $h(7)$. Nyklová obtained similar results (see [12]); moreover, she proved that $h(6, \ge 6) = g(6)$ and claimed that $h(6,5) = 19$. The proof of the last equality given in [12] contains gaps; the theorems stated below completely disprove it.

At present, the best known values of k for which $h(n,k)$ does not exist are those found by the author in [13] and [14]; they are $\binom{n-7}{(n-7)/2} - 1$ for odd n and $\binom{n-8}{(n-8)/2} - 1$ for even n.

It is also interesting to find values of k for which $h(n,k) = g(n)$ and $h(n,k) > g(n)$. The author has proved the inequality

$$h\left(n, \binom{(n-3)}{\lceil (n-3)/2 \rceil} - \left\lceil \frac{n}{2} \right\rceil\right) > 2^{n-2} + 1$$

(see [13] and [14]). The right-hand side of this inequality contains $2^{n-2} + 1$ rather than $g(n)$, because the conjecture about the coincidence of these quantities has not been proved.

Among recent results for fixed n, let us mention the inequality

$$h(6,1) \le g(7) \le 127$$

(see [15]). It is clear from its proof that the bound 127 can be significantly improved. Using a new algorithm based on the Szekeres–McKay–Peters algorithm (see [8]) and a powerful computational cluster, we have succeeded in proving the following theorem.

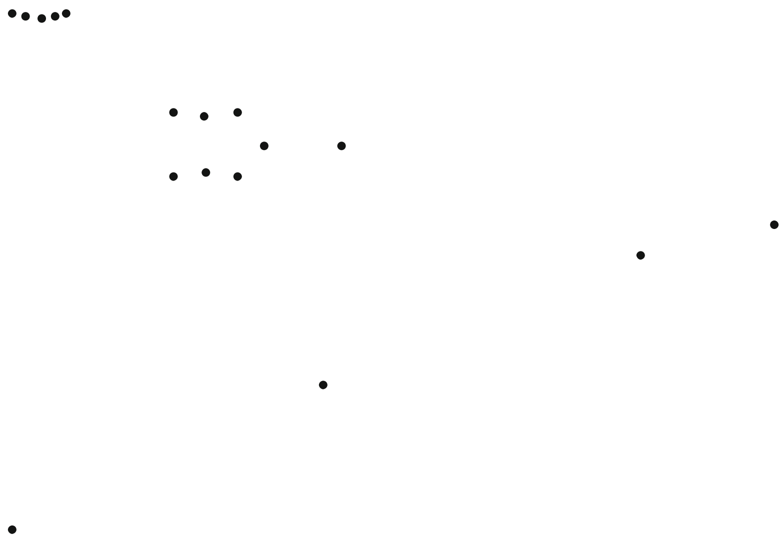
Theorem 1. *The exact equality* $h(6,2) = 17$ *holds.*

Corollary 1. *The exact equality* $h(6, \ge 2) = 17$ *holds.*

Theorem 2. *The exact equality* $h(6,1) = 18$ *holds.*

Thanks to Theorems 1 and 2, we now know all the exact values of $h(n,k)$ for $n \le 6$, except only $h(6,0) = h(6)$.

Theorem 2 is illustrated by the example of the 17-point set given in the figure, which contains no convex hexagon with at most one point inside. Moreover, according to Theorem 1, this set must contain convex hexagons with at most two points inside (in the example given in the figure, there is a hexagon containing precisely two points inside). It is easy to see that this is indeed so.



(0, 0)
(0, 506)
(13, 503)
(29, 501)
(42, 503)
(53, 506)
(158, 346)
(158, 409)
(188, 405)
(190, 350)
(221, 346)
(221, 409)
(247, 376)
(305, 142)
(323, 376)
(616, 269)
(747, 299)

Figure: An example of a 17-point set not containing a convex hexagon with at most one point inside.

2. BRIEF DESCRIPTION OF COMPUTATIONS

The set of all arrangements of finitely many points in the plane has the cardinality of the continuum; therefore, an exhaustive search cannot be performed over this set, and some other solution is required.

For convenience, we assume that all points in the sets under consideration have different x coordinates. Any triple of points in each set has either clockwise or counterclockwise orientation (the points are passed in the order of increasing numbers). To each triple we assign a bit indicating its orientation. The bit vector thus obtained (it has length $\binom{17}{3} = 680$ for 17 points and $\binom{18}{3} = 816$ for 18 points) is called the *signature* of the set under consideration. Clearly, knowing the signature of a set, we can find all convex polygons in this set and the number of points inside each polygons. For example, a set contains a convex hexagon if four bits in its signature take certain values. Thus, to prove both theorems, it suffices to search all signatures (their number is finite) and find the required subset in each of them. On the other hand, the set of all signatures has very large cardinality.

The algorithm performs an *exhaustive* search for a signature corresponding to a point set not containing the required hexagon (i.e., it tries to find a counterexample). If no signature is found when the algorithm terminates, then both theorems are valid. The algorithm fills the bit array in a certain order, until all elements are assigned bits, i.e., until a complete signature is constructed. Note that even before the array is completely filled, the presence of convex hexagons and some elementary *geometric* conditions (which determine the possibility of the existence of point sets with given signature) can be checked for the elements already filled. If a partially filled array contains values determining the presence of the required convex hexagon, then the array is not filled further.

The algorithm has yet another interesting feature. In fact, filling the bit array terminates when all special elements chosen in advance are filled (the number of such elements is 139 for Theorem 1 and 140 for Theorem 2). After this, special verification procedures, called the "one-bit test" and the "two-bit test," are applied, which sift out fillings that surely cannot be completed to signatures such that the corresponding point sets might be counterexamples (see [8]). The authors of the original version of the algorithm give the following data for the theorem $g(6) = 17$ in their paper [8]: the number of partially filled arrays was 20312212; the number of arrays left after the one-bit test was 23339; after the two-bit test, no arrays remained. For Theorem 1, the number of partially filled arrays was 19698264093, the number of arrays left after the one-bit test was 824389378, and the two-bit test eliminated all arrays. For Theorem 2, there were 26665261949 arrays, 1836007604 of them remained after the one-bit test, and none remained after the two-bit test.

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REFERENCES

1. P. Erdős and G. Szekeres, Compositio Math. **2**, 463 (1935).
2. P. Erdős and G. Szekeres, Ann. Univ. Sci. Budapest. Eötvös Sec. Math. **3–4**, 53 (1961).
3. P. Erdős, Austral. Math. Soc. Gaz. **5** (2), 52 (1978).
4. W. Morris and V. Soltan, Bull. Amer. Math. Soc. (N. S.) **37** (4), 437 (2000).
5. F. P. Ramsey, Proc. London Math. Soc. (2) **30**, 264 (1930).
6. R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey Theory*, in *Wiley-Intersci. Ser. Discrete Math. Optim.* (Wiley, New York, 1990).
7. M. Hall, Jr., *Combinatorial Theory* (Blaisdell, Waltham, Mass., 1967; Mir, Moscow, 1970).
8. G. Szekeres and L. Peters, ANZIAM J. **48** (2), 151 (2006).
9. V. A. Koshelev, Dokl. Ross. Akad. Nauk **415** (6), 734 (2007) [Russian Acad. Sci. Dokl. Math. **76** (1), 603 (2007)].
10. B. Kh. Sendov, Fundam. Prikl. Mat. **1** (2), 491 (1995).
11. J. D. Horton, Canad. Math. Bull. **26** (4), 482 (1983).
12. H. Nyklová, Studia Sci. Math. Hungar. **40** (3), 269 (2003).
13. V. A. Koshelev, Mat. Zametki (in press).
14. V. A. Koshelev, *On the Erdős–Szekeres Problem and Related Problems*, arXiv: `math. CO/0910.2700`.
15. V. A. Koshelev, Fundam. Prikl. Mat. **14** (6), 91 (2008) [J. Math. Sci. (N. Y.) **164** (1), 60 (2010)].