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ALMOST EMPTY HEXAGONS

V. A. Koshelev UDC 519.154+519.146

Abstract. In this work, new nontrivial bounds are obtained for the minimum number of points in general position on the plane, among which one certainly finds the set of vertices of a convex hexagon with not more than one point of the initial set inside.

1. Introduction and Result Formulation

In 1935, P. Erdős and G. Szekeres formulated the following problem (see [2, 3]).

First Erdős–Szekeres problem. To find for each integer $n \geq 3$ the minimum positive number $g(n)$ such that it will be possible to select from any set of points on the plane in general position containing at least $g(n)$ points a subset of cardinality n, whose elements are vertices of a convex n-gon.

In 1978, Erdős suggested the following modification of the first problem (see [1]).

Second Erdős–Szekeres problem. To find for each integer $n \geq 3$ the minimum positive number $h(n)$ such that it will be possible to select from any set of points $\mathcal{X}$ on the plane in general position containing at least $h(n)$ points a subset of cardinality n, whose elements are vertices of an empty convex n-gon, i.e., this n-gon contains within it no other points from $\mathcal{X}$.

We recall that a set of points on the plane is in *general position* if none of its three elements lie along a straight line.

These problems are classical in combinatorial geometry and Ramsey theory (see [5, 6, 10, 15]). They are both generalized as follows.

Third Erdős–Szekeres-type problem. Find for any integer $n \geq 3$ and $k \geq 0$ the minimum positive number $h(n, k)$ such that it will be possible to select from any set of points $\mathcal{X}$ on the plane in general position containing at least $h(n, k)$ a subset of cardinality n, whose elements are vertices of a convex n-gon C with the condition $|(C \setminus \partial C) \cap \mathcal{X}| \leq k$, i.e., this n-gon contains within it no more than k other points from $\mathcal{X}$.

The first problem was considered by Erdős and Szekeres in [2]. They proved the existence of $g(n)$ for arbitrary n by demonstrating the upper estimate $g(n) \leq \binom{2n-4}{n-2} + 1$, and they gave the following conjecture: $g(n) = 2^{n-2} + 1$. This conjecture is proved for $n \leq 6$. The case $g(3) = 3$ is obvious here; the equality $g(4) = 5$ was proved by E. Klein in 1935 (see Fig. 1, where all three essentially different ways of placing five points on the plane are displayed); the expression $g(5) = 9$ was obtained by E. Makai (see [2, 3, 10]); the fact $g(6) = 17$ was established rather recently by G. Szekeres and L. Peters in [17]. Moreover, in 1961 Erdős and Szekeres also proved the lower bound $g(n) \geq 2^{n-2} + 1$ (see [3]).

The inequality $g(n) \leq \binom{2n-4}{n-2} + 1$ was repeatedly improved. The strongest result was obtained in 2005 by G. Toth and P. Valtr: $g(n) \leq \binom{2n-5}{n-3} + 1$ (here $n \geq 5$; see [18]). Thus, the Erdős–Szekeres conjecture

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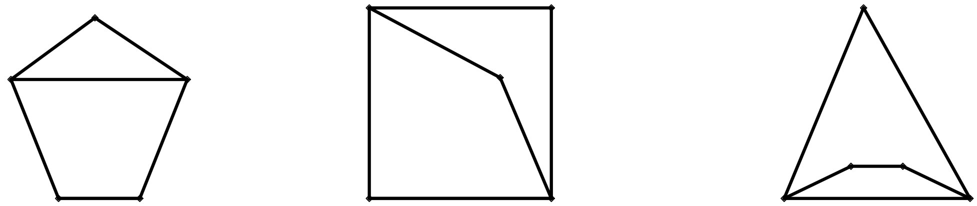

Fig. 1. Any set of five points contains a convex quadrilateral.

is still neither proved nor disproved, and it is only known that

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-3} + 1.$$

The second problem is more deeply understood. Thus, the equalities $h(3) = 3$ and $h(4) = 5$ for it are obvious (see Fig. 1). Expression $h(5) = 10$ was obtained by H. Harborth in 1978 (see [7]). In 1983, J. Horton proved that $h(n)$ does not exist for $n \geq 7$ (see [8]). Actually, Horton proved the nonexistence of $h(n, 0)$ for $n \geq 7$. The question of the existence and value of $h(6)$ (or, which is the same, $h(6, 0)$) has remained open for a long time. Only in 2006 did T. Gerken prove the existence of $h(6)$ by demonstrating the upper estimate $h(6) \leq g(9) \leq \binom{13}{6} + 1 = 1717$ (see [4]). Independently of him, K. Nicholas [11] and Valtr [19] presented their proofs, but their upper estimates are worse and equal to, respectively, $g(25)$ and $g(15)$. In 2007, the upper estimate was improved by the author of this article: $h(6) \leq 463$ (see [9]). Meanwhile, all lower estimates for $h(6)$ were obtained by a computer search. The first of them was obtained by M. Overmars, B. Scholten, and I. Vincent in 1988: $h(6) \geq 27$ (see [14]). The following estimate was obtained in 2001 by Overmars and it is the best one at present: $h(6) \geq 30$ (see [13]). Thus, for $h(6)$ the estimates $30 \leq h(6) \leq 463$ are proved at present.

As is easy to see, for the third problem the inequalities $g(n) \leq h(n, k) \leq h(n)$ are always correct if the appropriate expressions exist. Moreover, $h(n) = h(n, 0) \geq h(n, 1) \geq h(n, 2) \geq \ldots$ and there is k' such that $h(n, k) = g(n)$ for all $k \geq k'$. For small values of n the following results are obvious: $h(3, k) = 3$, $h(4, k) = 5$, $h(5, 0) = 10$, and $h(5, \geq 1) = 9$. The latter result follows from the fact that a convex pentagon with two or three points within it always contains a smaller convex pentagon.

Some results relating to the third problem are obtained in the article by B. Sendov [16]. In this article, with the use of the Horton set (see [8]), through which the existence of $h(7)$ was proved, the nonexistence of $h(n, k)$ is proved for certain values of k where $n > 7$. Similar results are obtained in the article by E. Nyklova [12]; moreover, it is proved there that $h(6, \geq 6) = g(6)$ and the result $h(6, 5) = 19$ is presented.

With respect to the fact that all results for $g(6)$ and $h(6)$ were obtained rather recently, the study of the value $h(6, 1)$ is interesting (values of k other than 1 may not be so interesting with respect to the conjecture set forth below). We managed to estimate the value $h(6, 1)$ much better than the value $h(6, 0)$.

Theorem. *There is the inequality* $h(6, 1) \leq g(7) \leq 127$.

Thus, it appears that at present the estimates $17 \leq h(6, 1) \leq 127$ are proved. We shall note that, if the conjecture of Erdős–Szekeres is true, the equality in the theorem will look like $h(6, 1) \leq g(7) = 33$.

Actually, we suppose that the stronger statement is true.

Conjecture. $h(6, 1) = g(6) = 17$.

We shall note that it follows at once from the conjecture that

$$h(6, 1) = h(6, 2) = h(6, 3) = h(6, 4) = h(6, 5) = 17.$$

The supposed equality $h(6, 5) = 17$ obviously contradicts the result of Nyklova set forth above. The point is that this result was proved inaccurately and there are counterexamples for it.

One may obtain a fuller and more detailed history of Erdős–Szekeres problems, for example, from the survey [10].

2. Theorem Proof Scheme

We say that a (finite) set of points on the plane *contains* a given k-gon if it is possible to select a subset from it whose elements are vertices of that k-gon.

To prove the theorem, we shall ascertain that any set of points on the plain in general position of cardinality $g(7)$ or more contains a convex hexagon with no more than one point inside. Let us fix an arbitrary such set $\mathcal{X}$. Note that $\mathcal{X}$ contains at least one convex heptagon. The inclusion relation in the set of all convex heptagons formed by points of $\mathcal{X}$ is a strict order relation. Therefore, it is always possible to speak of minimum heptagons in $\mathcal{X}$. Let us select one of them and denote the set of its vertices by $\mathcal{H}$. Let $\mathcal{I}' = (\mathrm{conv}(\mathcal{H}) \setminus \mathcal{H}) \cap \mathcal{X}$ be the set of points of $\mathcal{X}$ lying inside the convex hull of $\mathcal{H}$. Either $\mathcal{I}'$ is empty (but then there is nothing to discuss), or the set $\mathrm{conv}(\mathcal{I}')$ is a convex polygon, segment ("digon"), or a point ("monogon"); denote by $\mathcal{I}$ the set of its vertices ($\mathcal{I} = \partial\big(\mathrm{conv}(\mathcal{I}')\big) \cap \mathcal{X}$). If $|\mathcal{I}| > 2$, then we can define $\mathcal{J}' = (\mathrm{conv}(\mathcal{I}) \setminus \mathcal{I}) \cap \mathcal{X}$ as the set of points of $\mathcal{X}$ placed within the convex hull of $\mathcal{I}$. Note that if $\mathcal{J}'$ is not empty, then $\mathrm{conv}(\mathcal{J}')$ is also a convex polygon (in particular, monogon or digon), and, therefore, it is possible to define $\mathcal{J}$ as the set of its vertices. Sets $\mathcal{K}$, $\mathcal{L}$, and so on are drawn by a similar method, provided that the process will terminate at a certain moment (see Fig. 2).

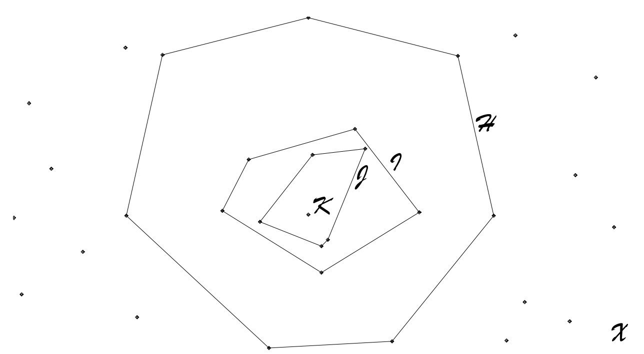

Fig. 2. Defining sets $\mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$.

Put $i = |\mathcal{I}|$, $j = |\mathcal{J}|$,... Let us say that $\mathcal{X}$ has the *type* $(7, i, j, \ldots)$. In particular, in "degenerate" cases the types $(7, 0, 0, \ldots)$, $(7, i, 0, \ldots)$, etc. arise (see Fig. 3).

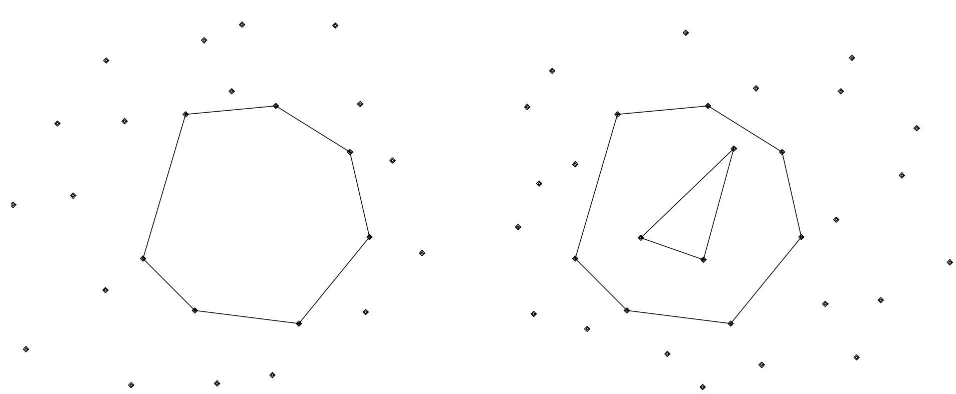

Fig. 3. Set of the types $(7, 0, 0, \ldots)$ and $(7, 3, 0, \ldots)$.

Note that the type is defined ambiguously. For example, the set in Fig. 4 has the type $(7, 4, 1, \ldots)$ if $A_1A_2 \ldots A_7$ is taken as a minimum convex heptagon in it; the same set has the type $(7, 5, 2, \ldots)$ if the

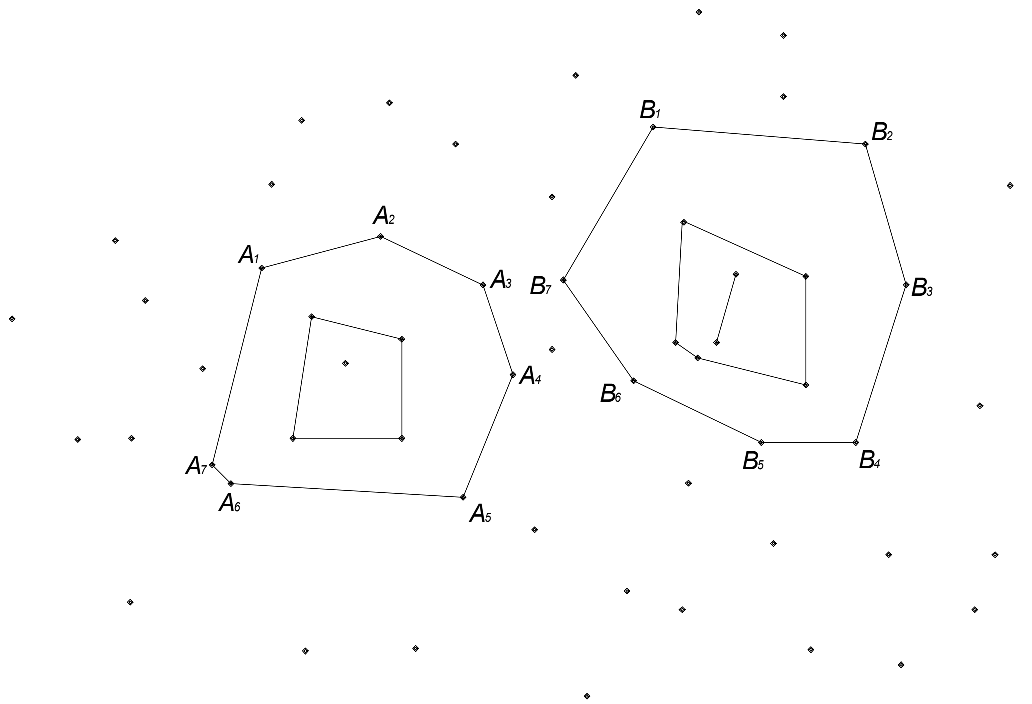

Fig. 4. Ambiguousness in defining the type.

heptagon $B_1B_2\dots B_7$ is considered. Hereafter we will not use the possible variety of convex heptagons in $\mathcal{X}$. When speaking of the set $\mathcal{X}$, we will only mean that there is a convex heptagon in $\mathcal{X}$ with respect to which $\mathcal{X}$ has this type.

To prove the existence of a convex hexagon with no more than one point within, in the set $\mathcal{X}$, we will consider only the subset $\mathcal{H}' = \mathrm{conv}(\mathcal{H}) \cap \mathcal{X}$. The following lemma is true.

Lemma (Valtr [19])**.** *Let $\mathcal{X}$, $\mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ be the same as in the definition set forth above. If $|\mathcal{H}| \geq 7$ and there are no empty convex hexagons in $\mathcal{X}$, then $\mathcal{K} = \varnothing$.*

The expression “the same” from the lemma formulation is, strictly speaking, not quite correct; indeed, up to this moment we supposed that $|\mathcal{H}| = 7$. But it is clear that the definition of convex hulls included in each other (see Fig. 2) may also be given for the cases where $|\mathcal{H}| \neq 7$.

It follows from the lemma that a convex hexagon with no more than one point within (moreover, an empty convex hexagon) exists in any set of the type $(7, i, j, k, \dots)$, where $k > 0$. Therefore, it only remains to consider sets of the type $(7, i, j, 0, 0, \dots)$. Thus, to be brief, we will speak only of cases (configurations) of the “form” $(7, i, j)$. There will be 31 such cases in total, because i and j may change within the following limits: $0 \leq i \leq 2$, and $3 \leq i \leq 6$ and $0 \leq j \leq 6$.

In principle, each of the said 31 cases shall be considered separately, but we shall be able to divide the set of all cases into four classes in order to apply to each class a special method of proving the appropriate statement. Here for all four classes we will somehow determine the existence of a convex hexagon with no more than one point within, in the set $\mathcal{H}'$.

With respect to the above, the further structure of the article is as follows: in the third section, we will introduce some auxiliary definitions and notations; in the fourth section, we will set forth the proof of the theorem, with separate subsection devoted to each class; in the fifth section, we will make some conclusions.

3. Auxiliary Definitions and Notation

In this section, we will introduce some additional geometrical objects that will be essentially used in the following proof.

3.1. Sectors and Prohibited Zones. Let us for three arbitrary points X, Y, and Z on the plane in general position define $P_{XY}(Z)$ as an open half-plane with respect to the line XY, containing the point Z. We shall call a set of consecutive vertices of a convex polygon a *convex chain.* For a given convex chain ABC we will define the 3-*sector* (see Fig. 5 on the left)

$$(ABC) = \big(P_{AB}(C) \cap P_{BC}(A)\big) \setminus \operatorname{conv}(\{A, B, C\}).$$

For a convex chain $ABCD$ we will define the 4-*sector* (see Fig. 5 in the middle)

$$(ABCD) = \big((ABC) \cap (BCD)\big) \setminus \operatorname{conv}(\{A, B, C, D\}).$$

For a convex chain $ABCDE$ we will define the 5-*sector* or *prohibited zone* of a pentagon (see Fig. 5 on the right)

$$(ABCDE) = \big((ABCD) \cap (BCDE)\big) \setminus \operatorname{conv}(\{A, B, C, D, E\}).$$

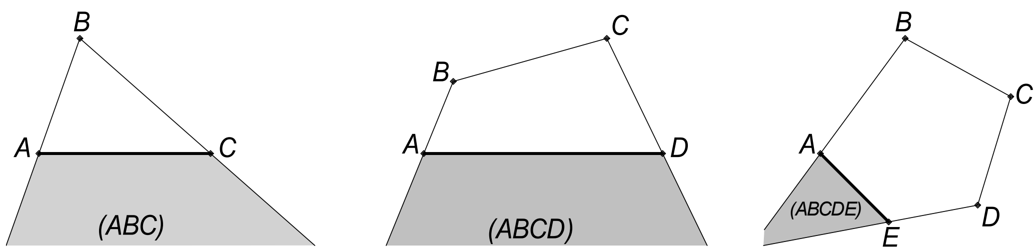

Fig. 5. Defining sectors.

Let us say that a point *is separated* from a given convex polygon by some line containing its side if it is placed in the other half-plane with respect to this line than the polygon itself. Note that any prohibited zone of a pentagon represents a set of points that are separated from it by *only one* direct line (see Fig. 5, where the line AE acts as such line).

Any direct line PQ sets two half-planes. Let us denote one of them by the 2-sector (PQ), and the other by the 2-sector (QP) in accordance with the following principle. If for any point R from a half-plane the points P, Q, and R are placed one after another clockwise, then this half-plane is the 2-*sector* (PQ); otherwise it is the 2-*sector* (QP).

3.2. Placings. Assume that an arbitrary configuration of the type $(7, i, j)$ is given. Mark in it a construction consisting of the intermediate i-gon and the internal j-gon. As we will often need such construction later, let us consider it in more detail. We shall give some classification of relative placements for sets of vertices of an i- and j-gon. Take that $j = 3$, $j = 4$, or $j = 5$, because we will not need other cases.

Assume, first, that $j = 3$. Three lines crossing sides of the triangle divide the plane outside the triangle into six parts. Three of them have the "triangle" form and the other three the "quadrilateral" form. Let us say that the construction consisting of an i-gon and a triangle is an $(i, 3)$-*arrangement of the form* $[a_1, b_1, a_2, b_2, a_3, b_3]$ if the numbers of vertices of the i-gon placed in the triangle parts are a_1, a_2, a_3, and the numbers of vertices of the i-gon placed in the quadrilateral parts are b_1, b_2, b_3 (see Fig. 6). Generally speaking, the notation $[a_1, b_1, a_2, b_2, a_3, b_3]$ is defined ambiguously. For example, we could speak of the same placing and with the same result as of an $(i, 3)$-arrangement of the form $[a_2, b_2, a_3, b_3, a_1, b_1]$, and of an $(i, 3)$-arrangement of the form $[a_1, b_3, a_3, b_2, a_2, b_1]$. But it is not essential for us which of the listed (and other possible) versions is considered. These versions are equivalent for us, and hereafter we will use any of them. It is only important to stress here that every time the version fixed by us will begin with the value in the form a_i, i.e., with the number of vertices in an i-gon in some triangle part of the

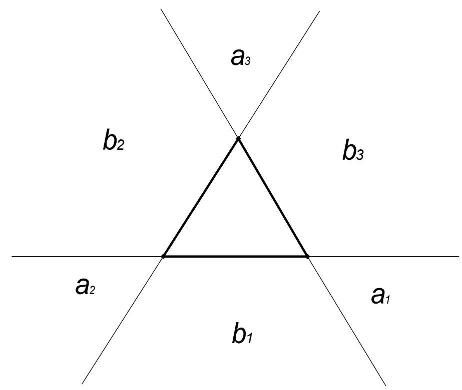

Fig. 6. $(i, 3)$-arrangement of the form $[a_1, b_1, a_2, b_2, a_3, b_3]$.

plane; values in the forms a_i and b_j will alternate in this record and will follow each other in the order of placing the appropriate parts of the plane.

Now assume that $j = 4$. Let us determine the form of a placing as for $j = 3$. It will be a little more difficult here.

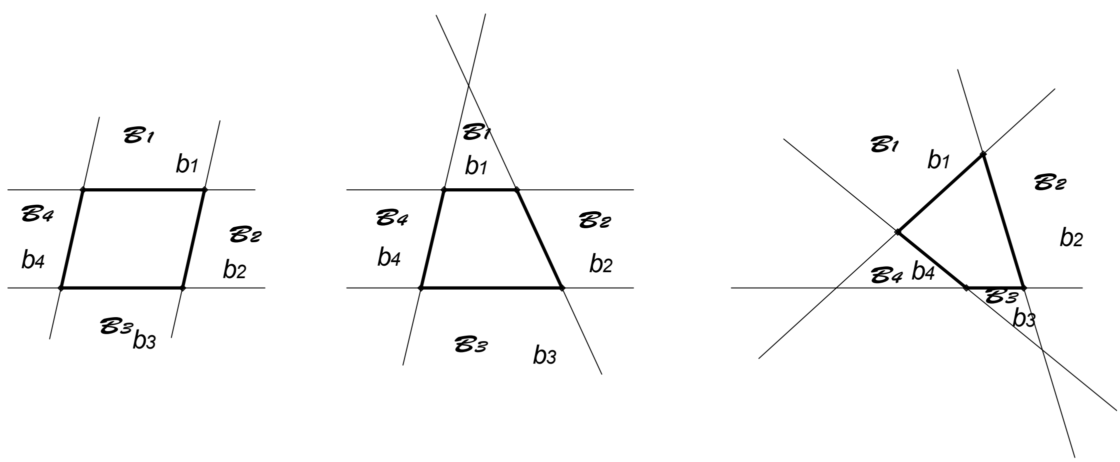

Fig. 7. Partitions of plane and determining sets $\mathcal{B}_i$ and numbers b_i.

Four lines crossing sides of the quadrilateral divide the plane around it into several parts. Three essentially different ways of such partition are possible. They all are displayed in Fig. 7. In the first case, the quadrilateral is a parallelogram; in the second case, it is a trapezoid; and in the third case, the quadrilateral is neither a parallelogram nor a trapezoid. In each of the three cases, we mark the domains in the partition of the plane around the quadrilateral, which are conditionally denoted in Fig. 7 by $\mathcal{B}_1, \ldots, \mathcal{B}_4$. For a given construction consisting of an i-gon and a quadrilateral, we denote by b_k the number of vertices of the i-gon placed in the domain $\mathcal{B}_k$, $k = 1, \ldots, 4$. In each of the three cases, the values $b_1, \ldots, b_4$ are defined unambiguously, up to the succession in which we originally introduce the notations $\mathcal{B}_1, \ldots, \mathcal{B}_4$. Of course, we put in order both domains and numbers of vertices in them (e.g., clockwise).

As the placing of the values $b_1, \ldots, b_4$ for this construction is fixed, let us arrange the values $a_1, \ldots, a_4$ as follows.

If a quadrilateral is a parallelogram, all is trivial (see Fig. 8 on the left): numbers $a_1, \ldots, a_4$ are quantities of vertices of the i-gon placed in the relevant domains.

Let a quadrilateral be a trapezoid. Then, without loss of generality, the numbers $b_1, \ldots, b_4$ may be assumed arranged as in Figs. 7 and 8. Let us consider "additional" domains $\mathcal{A}'$, $\mathcal{A}$, $\mathcal{A}''$, $\mathcal{A}_3$, and $\mathcal{A}_4$ (see

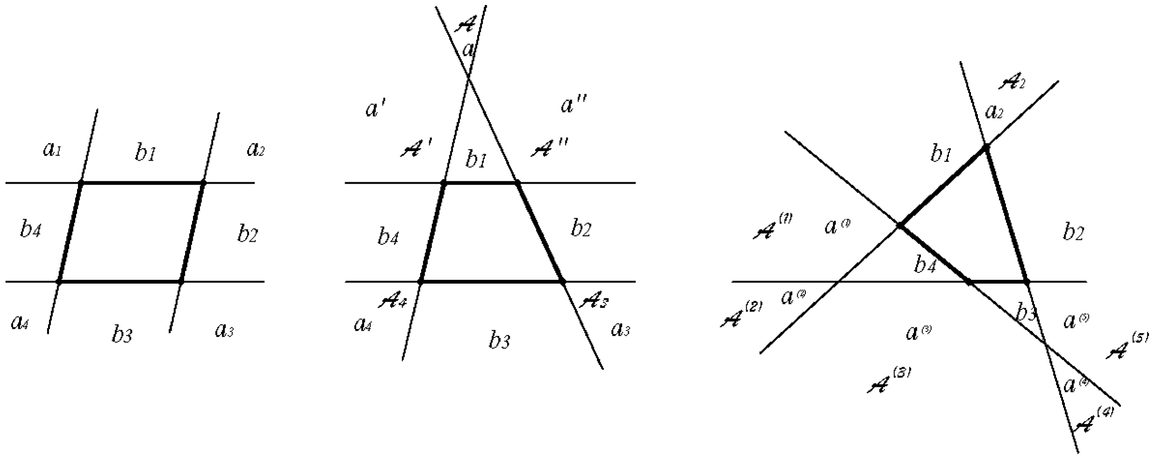

Fig. 8. Partitions of plane and determining sets $\mathcal{A}_i$ and numbers a_i.

Fig. 8 in the middle). Denote by a', a, a'', a_3, and a_4 the numbers of vertices of the i-gon in appropriate domains. Take arbitrary nonnegative integers a_1 and a_2 with the conditions $a_1 + a_2 = a' + a + a''$, $a_1 \le a' + a$, and $a_2 \le a + a''$ and arrange them as shown in Fig. 9 in the middle. In other words, we assign some a_1 of the vertices of the i-gon from the set $\mathcal{A}' \cup \mathcal{A} \cup \mathcal{A}''$ to the domain $\mathcal{A}' \cup \mathcal{A}$, and the remaining a_2 vertices to the domain $\mathcal{A} \cup \mathcal{A}''$. Thus, in determining the form of a placing we ignore the number of vertices of the i-gon in $\mathcal{A}$ and only for certainty distribute its vertices between the domains $\mathcal{A}' \cup \mathcal{A}$ and $\mathcal{A} \cup \mathcal{A}''$. It is of no significance here how exactly we conduct this operation.

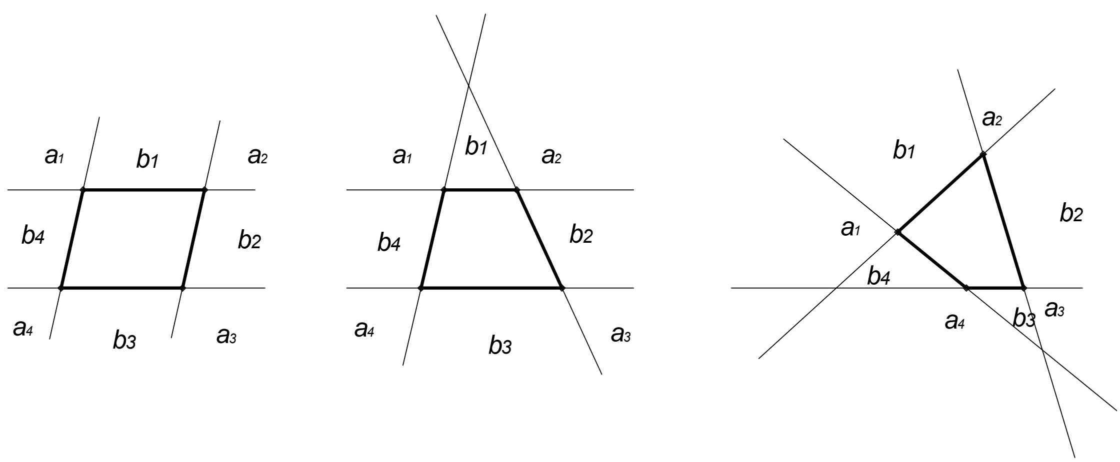

Fig. 9. Finally determining numbers a_i and b_i.

Finally, assume that a quadrilateral is neither a parallelogram nor a trapezoid. This situation is displayed in Fig. 8 on the right. Here, without loss of generality again, additional domains $\mathcal{A}_2, \mathcal{A}^{(1)}, \dots, \mathcal{A}^{(5)}$ arise; there lie, respectively, $a_2, a^{(1)}, \dots, a^{(5)}$ vertices of the i-gon. Select arbitrary nonnegative integers a_1, a_4, and a_3 with the properties $a_1 + a_4 + a_3 = a^{(1)} + \dots + a^{(5)}$, $a_1 \le a^{(1)} + a^{(2)}$, $a_4 \le a^{(2)} + a^{(3)} + a^{(4)}$, and $a_3 \le a^{(4)} + a^{(5)}$, and arrange them as displayed in Fig. 9 on the right. In other words, we assign some a_1 vertices of the i-gon from the set $\mathcal{A}^{(1)} \cup \dots \cup \mathcal{A}^{(5)}$ to the domain $\mathcal{A}^{(1)} \cup \mathcal{A}^{(2)}$; some other a_4 of its vertices are assigned to the domain $\mathcal{A}^{(2)} \cup \mathcal{A}^{(3)} \cup \mathcal{A}^{(4)}$; the remaining a_3 vertices are associated with the domain $\mathcal{A}^{(4)} \cup \mathcal{A}^{(5)}$. Thus, this time we actually pay no attention to the number of vertices of the i-gon in intersections of the sets $\mathcal{A}^{(1)} \cup \mathcal{A}^{(2)}$, $\mathcal{A}^{(2)} \cup \mathcal{A}^{(3)} \cup \mathcal{A}^{(4)}$, and $\mathcal{A}^{(4)} \cup \mathcal{A}^{(5)}$, but for convenience distribute its vertices between these sets. As in the case of a trapezoid, we do not care about the way of such distribution.

As a result, we can say that each construction consisting of an i-gon and a quadrilateral is an $(i, 4)$-*arrangement of the form* $[a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4]$. It is clear that, as in the case of $(i, 3)$-placings, the form is determined ambiguously here. First, there is arbitrariness in the choice of the numbering order of the values a_i, b_i; second, the numbers a_i may assume different values within certain limits. Still, as earlier, we will alternate a_i and b_i, beginning the record of the placing every time with some a_1.

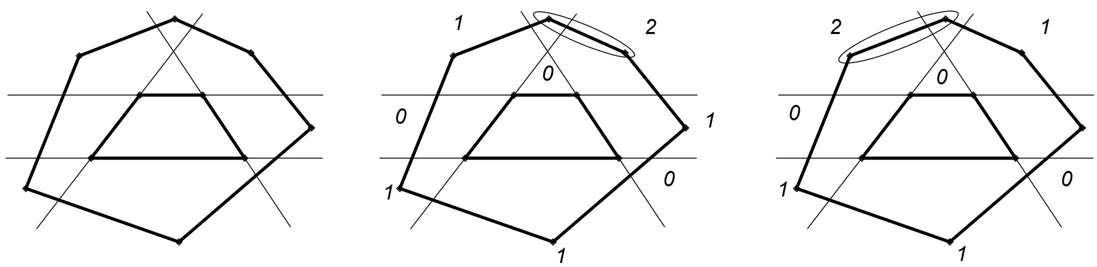

Fig. 10. Example with determining ambiguously the form of placing.

The construction displayed in Fig. 10 on the left consists of a hexagon and a quadrilateral. It may be interpreted, for example, as a $(6, 4)$-placing of the form $[2, 0, 1, 0, 1, 1, 0, 1]$ (see Fig. 10 on the middle). It may also be represented as a $(6, 4)$-placing of the form $[1, 0, 2, 0, 1, 1, 0, 1]$ (see Fig. 10 on the right). There is, of course, a number of other interpretations, but each of them reflects with the same adequacy the information on the relative placement of vertices in the hexagon and the quadrilateral that we will need further.

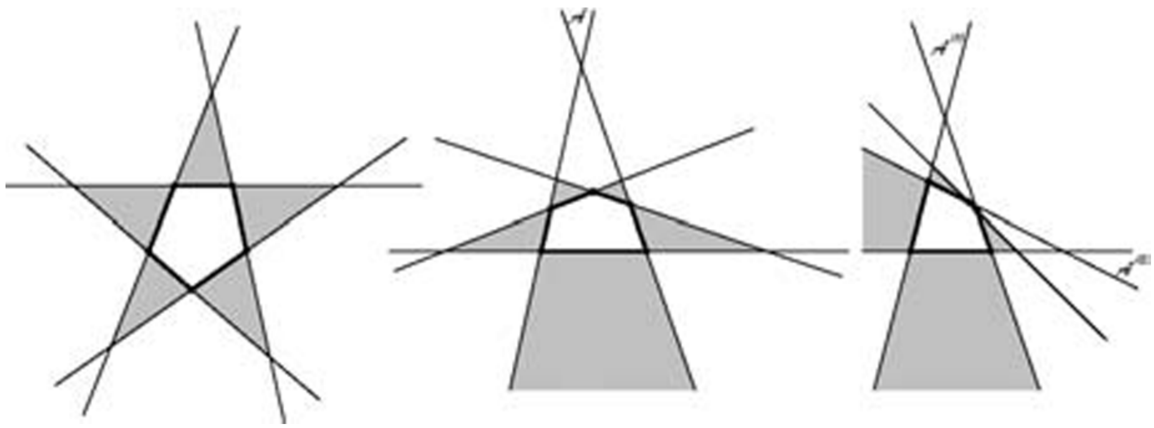

Fig. 11. Partitions of the plane.

Assume, finally, that $j = 5$. Here it is easier to determine the form of a placing than in the last case. There are only three essentially different situations. They are all displayed in Fig. 11. The difference between them is that in the second and third situations there are domains, respectively, $\mathcal{A}$ and $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$ in which each point is separated from the pentagon by exactly four lines containing its sides, and in the first situation there is no such domain. It is easy to see that there are always no more than two such domains. Note that colored in Fig. 11 are the domains in which the points, on the contrary, are separated from the pentagon by only one line intersecting its side. The sense is that the placement of a vertex of the i-gon in the colored domain guarantees the existence in the configuration of a convex hexagon with no more than one point within (empty).

As in all preceding cases, we place the symbols a_i, b_i in each situation according to Fig. 12. Here in the first situation the numbers a_i, b_i unambiguously express the numbers of vertices of the i-gon placed in appropriate domains and in the second and third situations the ambiguousness arising in the domains $\mathcal{A}$, $\mathcal{A}^{(1)}$, and $\mathcal{A}^{(2)}$ is solved by the same inequalities as those we wrote while determining the $(i, 4)$-placing.

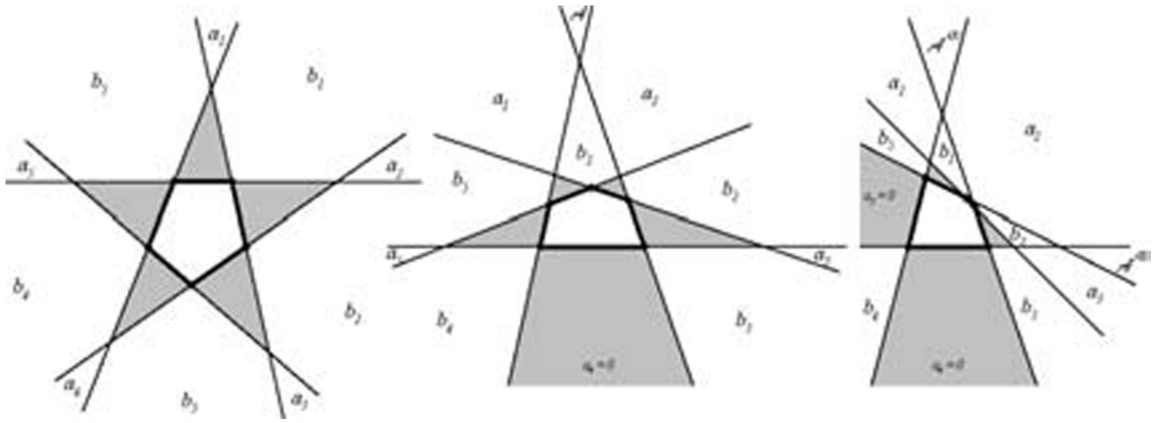

Fig. 12. Partitions of the plane.

Note that, strictly speaking, the meaning of the notation a_i, b_i is not quite the same for an $(i, 5)$-placing and other placings. The point is that before we denoted by b_i the domains in which each point is separated from the j-gon by only one line crossing its side. Now we color such domains. As we said nothing before of the nature of domains with a given number of points within, now we also decided not to change notations.

In any case, now we have the $(j, 5)$-placing of the form $[a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5]$.

4. Proof of the Theorem

In this section, we will consider four classes of cases (see Sec. 2). We will devote a separate subsection to each class and a clause of the relevant subsection to each case. Every time we will prove the existence of an empty convex hexagon in the set $\mathcal{H}' \subset \mathcal{X}$.

4.1. "Trivial" Cases. In this subsection, we will consider the configurations of the following forms:

$$(7, 0, 0),\ (7, 1, 0),\ (7, 6, 0),\ (7, 6, 1),\ (7, i, 6) \quad (3 \le i \le 6).$$

It is obvious that there is a convex hexagon with no more than one point within in each such configuration. It is always a part of some certain set: $\mathcal{H}$, $\mathcal{I}$, or $\mathcal{J}$.

4.2. Cases with $j \le 2$. In this subsection, all rather simple cases will be considered for which $j \le 2$. Cases $(7, 3, 2)$ and $(7, 4, 2)$ are considered in the next subsection.

4.2.1. Configurations of the form $(7, 2, 0)$. Assume that internal points are A and B. The line AB divides the plane into two half-planes and in one of them at least four vertices of the heptagon are placed (see Fig. 13). They, together with the points A and B, form the hexagon we need (moreover, an empty one).

4.2.2. Configurations of the form $(7, 3, 0)$. Assume that the internal triangle is ABC. The line AB divides the plane into two half-planes, and in one of them at least four vertices of the heptagon are placed (see Fig. 14). Together with the points A and B, they form a convex hexagon with no more than one point within.

4.2.3. Configurations of the form $(7, 4, 0)$. Assume that the internal quadrilateral is $ABCD$. Applying the same reasoning as in the preceding clause to the direct line AC, we ascertain that for this configuration there is also a convex hexagon with no more than one point within (see Fig. 15).

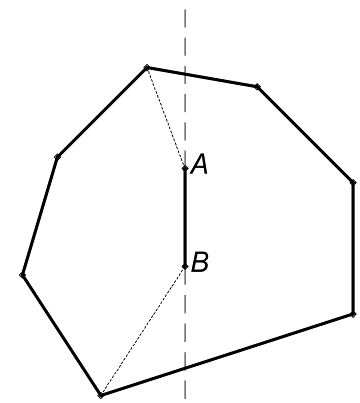

Fig. 13. Configuration of the form $(7, 2, 0)$.

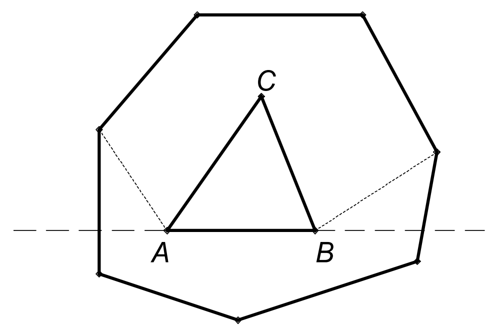

Fig. 14. Configuration of the form $(7, 3, 0)$.

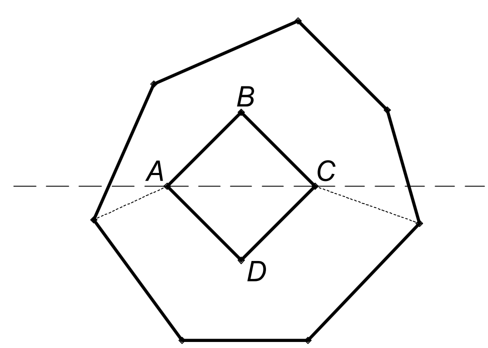

Fig. 15. Configuration of the form $(7, 4, 0)$.

4.2.4. Configurations of the form $(7, 5, 0)$. It is easy to see that if at least one vertex of the heptagon is placed in the colored area (see Fig. 16), it guarantees to us at once the existence of a convex hexagon with no more than one point within. Thus, we may assume that all vertices of the heptagon are placed outside the colored areas. The relative placement of vertices of the heptagon and pentagon represents some analogue of the $(7, 5)$-placing of the form $[a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5]$, for which the condition $\sum(a_i + b_i) = 7$ is fulfilled (see Sec. 3.2). It is analogy we speak of here, because we only used the term "arrangement" with respect to the relative placement of the intermediate i-gon and internal j-gon. This time we speak of an external heptagon and an intermediate i-gon, but this does not change the matter.

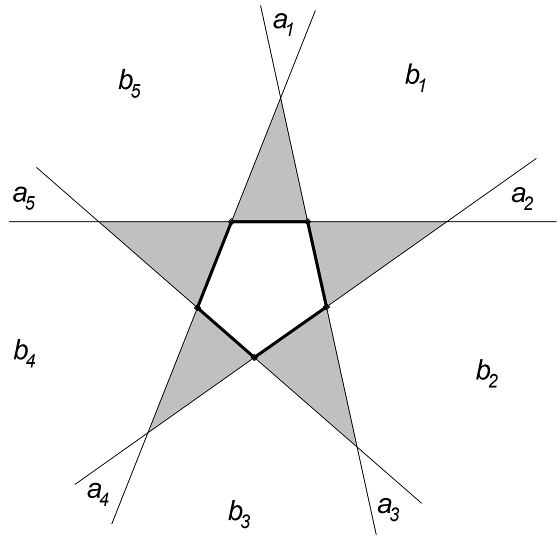

Fig. 16. Configuration of the form $(7, 5, 0)$.

Suppose that in this configuration there is no convex hexagon with no more than one point within. Then the following relations shall be complied with:

$$0 \le b_i \le 1 \quad (1 \le i \le 5),$$
$$a_1 + b_1 + a_2 + b_2 + a_3 \le 3, \quad a_2 + b_2 + a_3 + b_3 + a_4 \le 3, \quad a_3 + b_3 + a_4 + b_4 + a_5 \le 3,$$
$$a_4 + b_4 + a_5 + b_5 + a_1 \le 3, \quad a_5 + b_5 + a_1 + b_1 + a_2 \le 3.$$

Adding them, we obtain that $3\sum(a_i + b_i) \le 20$ — a contradiction with the fact that $\sum(a_i + b_i) = 7$. This means that there is always a convex hexagon with no more than one point within in our construction.

4.2.5. Configuration of the form $(7, 3, 1)$. In this case, all space around the triangle ABC is partitioned into three areas (AXB), (BXC), and (AXC), each of them being a 3-sector (X is an internal point of the triangle) (see Fig. 17). Either one of these areas contains more vertices of the heptagon than marked on the picture, and then there is a convex hexagon with no more than one point within in the configuration (moreover, this hexagon is empty). Or the number of vertices of the heptagon in each of the areas (AXB), (BXC), and (AXC) does not exceed the said value, but that is impossible, as $2 + 2 + 2 = 6 < 7$.

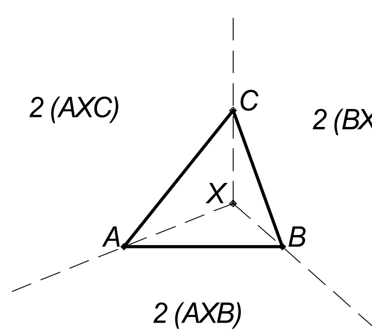

Fig. 17. Configuration of the form $(7, 3, 1)$.

Further similar reasoning will often be used, and in such cases pictures will be represented with indication of areas into which the plane is divided or by which it is covered. In each area the maximum number of vertices of the heptagon conv($\mathcal{H}$) that can be placed inside it without appearance in the construction of a convex hexagon with no more than one point within will be specified. Each time, the sum of such numbers will be less than 7.

4.2.6. Configurations of the form $(7, 4, 1)$. First, we will draw in the quadrilateral $ABCD$ the diagonal BD and fix in which of the triangles, ABD or BDC, the internal point X is placed. Without loss of generality, we will assume that it is placed in the triangle ABD. Further, all space around the quadrilateral $ABCD$ is partitioned into three areas (AXB), (BXD), and (AXD), each of them being a 3-sector (see Fig. 18) and setting the appropriate limitations as to the number of vertices of the heptagon in it. According to the logic of the clause 4.2.5, a convex hexagon with no more than one point within will be found in the configuration.

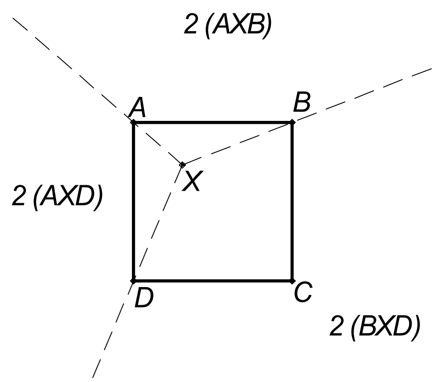

Fig. 18. Configuration of the form $(7, 4, 1)$.

4.2.7. Configurations of the form $(7, 5, 1)$. In this clause, the relative placement of the pentagon $ABCDE$ and the internal point X allows one to draw the partition of the plane around $ABCDE$ into 3-sectors (AXC), (CXD), and (AXD) (see Fig. 19) with the appropriate limitations. According to the logic of Sec. 4.2.5, a convex hexagon with no more than one point within will be found in the configuration.

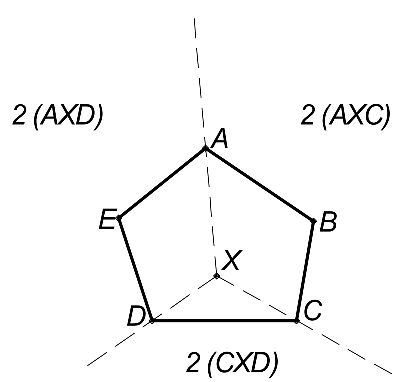

Fig. 19. Configuration of the form $(7, 5, 1)$.

4.2.8. Configurations of the form $(7, 5, 2)$. First, we shall note that the line drawn through two internal points X and Y (of course) crosses the pentagon $ABCDE$ and that in one half-plane related to it there are two vertices and in the other there are three vertices of the pentagon (the case with one and four vertices is trivial and we may disregard it at once). Thus, without loss of generality, we assume that the vertices of our pentagon are divided by the direct line XY into the sets $\{C, D\}$ and $\{A, B, E\}$ (see

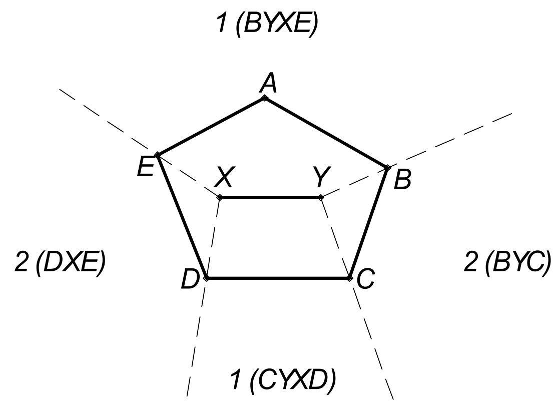

Fig. 20. Configuration of the form $(7, 5, 2)$.

Fig. 20). We also assume that the point X is placed "closer" to the point D and the point Y "closer" to the point C (i.e., that the quadrilateral $CYXD$ is a convex one).

With respect to the suppositions above, the plane around the pentagon is partitioned into the sectors $(BYXE)$, (BYC), $(CYXD)$, and (DXE). In accordance with the logic of Sec. 4.2.5, a convex hexagon with no more than one point within will be found inside.

4.2.9. Configurations of the form $(7, 6, 2)$. First of all, we note that the direct line XY crosses two sides of the hexagon. If these two sides are not opposite (for example, the sides AB and CD), then a convex hexagon with no more than one point within will certainly be found (in this case it is $AXYDEF$). Further, without loss of generality, we will assume that the direct line XY crosses two opposite sides of the hexagon AB and DE. Now it is easy to draw the partition of the plane around the hexagon into sectors (AXB), $(BXYD)$, (DYE), and $(AXYE)$ with the appropriate limitations relating to the number of vertices of the heptagon in them (see Fig. 21). According to the logic of Sec. 4.2.5, a convex hexagon with no more than one point within will be found inside.

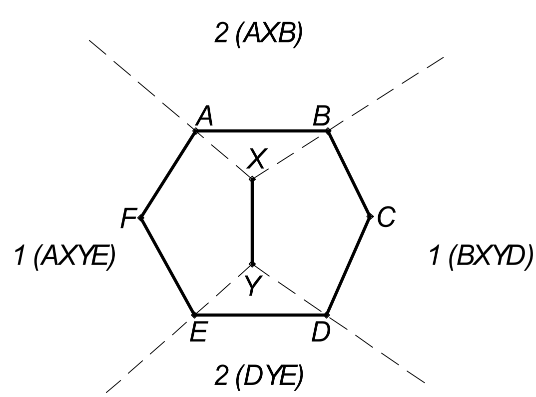

Fig. 21. Configuration of the form $(7, 6, 2)$.

4.3. Cases with Application of Minimality of Heptagon. In this subsection, we will consider configurations of the form

$$(7, 3, \geq 2), \quad (7, 4, \geq 2), \quad (7, 5, \geq 4).$$

The method we use below will allow us to ascertain a statement even stronger than what we seek to obtain. Namely, we will prove the existence of an empty convex hexagon.

To prove the existence of an empty convex hexagon in this configuration, we will use the following statements every time, actually obtained in [4].

Statement 1. *Assume that in any configuration of the form* $(7, i, j)$*,* $j \geq 2$ *is complied with, and assume that* $2 \leq t \leq \min\{i-1, j\}$*. Consider* t *consecutive vertices* $V_1, \ldots, V_t$ *of the polygon* $\mathrm{conv}(\mathcal{J})$*. Denote by* $\mathcal{T}_n$ *the set of vertices of the* i*-gon* $\mathrm{conv}(\mathcal{I})$ *placed in the half-plane with respect to the direct line* $V_n V_{n+1}$ *that contains no other points from* $\mathrm{conv}(\mathcal{J})$ *(if* $j = 2$*, then take any of the two half-planes). If* $\left|\bigcup_{n=1}^{t-1} \mathcal{T}_n\right| < t$*, then there is an empty convex hexagon in the configuration.*

Statement 1′. *Assume that in any configuration of the form* $(7, i, j)$*,* $j \geq 2$ *is complied with and assume that* $2 \leq t \leq \min\{i-1, j\}$*. Consider* t *consecutive vertices* $V_1, \ldots, V_t$ *of the polygon* $\mathrm{conv}(\mathcal{J})$*. Denote by* $\mathcal{T}'_n$ *the set of vertices of the* i*-gon* $\mathrm{conv}(\mathcal{I})$ *placed in the half-plane with respect to the direct line* $V_n V_{n+1}$ *that contains no other points from* $\mathrm{conv}(\mathcal{J})$ *(if* $j = 2$*, take any of the two half-planes). If* $\left|\bigcup_{n=1}^{t-1} \mathcal{T}'_n\right| < t-1$*, then there is an empty convex hexagon in the configuration.*

Proofs of the statements substantially rely on minimality of the heptagon in the configuration of the form $(7, i, j)$.

4.3.1. Configurations of the form $(7, 3, \geq 2)$. We apply Statement 1 with $t = 2 \leq \min\{i-1, j\}$. Let us fix two consecutive vertices of a j-gon $P = V_1$ and $Q = V_2$. We denote by $\mathcal{T}_{PQ}$ the set $\mathcal{T}_1$. According to Statement 1, if $|\mathcal{T}_{PQ}| < 2$, then there is an empty convex hexagon in the configuration and that is all right. Thus, it remains to consider the case $|\mathcal{T}_{PQ}| \geq 2$.

In the said case, let us select two consecutive vertices of the triangle ABC (e.g., B and C) from the set $\mathcal{T}_{PQ}$. Let us consider the sectors $(CPQB)$, (APC), and (AQB), covering the space around the triangle ABC (see Fig. 22). There are no points from $\mathcal{J}$ inside the quadrilateral $CPQB$. But such points may be placed inside the triangles AQB and APC. Assume for example that $P' \in APC$. Then let us replace the sector (APC) by the sector $(AP'C) \supset (APC)$. Let us do the same if $Q' \in AQB$, $P'' \in AP'C$, etc. As a result, we will obtain coverage of the plane around the triangle ABC by one 4-sector and two 3-sectors. Then the limitation of number of vertices in a heptagon displayed in Fig. 22 will arise in each of these sectors. In accordance with the logic of Sec. 4.2.5, there is again an empty convex hexagon in the configuration.

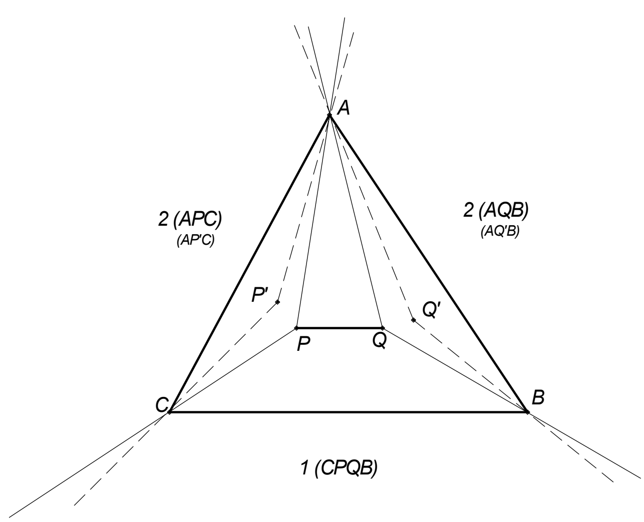

Fig. 22. Configurations of the form $(7, 3, \geq 2)$.

4.3.2. Configurations of the form $(7, 4, \geq 2)$. We apply Statement 1 with $t = 2 \leq \min\{i-1, j\}$. Let us fix three consecutive vertices of a j-gon $P = V_1$, $Q = V_2$, and $R = V_3$ (here $P = R$ if $j = 2$). Denote by $\mathcal{T}_{PQ}$ the set $\mathcal{T}_1$, and by $\mathcal{T}_{QR}$ the set $\mathcal{T}_2$. According to Statement 1, if $|\mathcal{T}_{PQ}| < 2$ or $|\mathcal{T}_{QR}| < 2$, then there is an empty convex hexagon in the configuration and that is all right. Thus, it remains to consider the case $|\mathcal{T}_{PQ}| \geq 2$, $|\mathcal{T}_{QR}| \geq 2$.

Note that if $P = R$ ($j = 2$), then the conditions $|\mathcal{T}_{PQ}| \geq 2$, $|\mathcal{T}_{QR}| \geq 2$ mean that we have the situation displayed in Fig. 23 on the left, and that is all right again (clause 4.2.5).

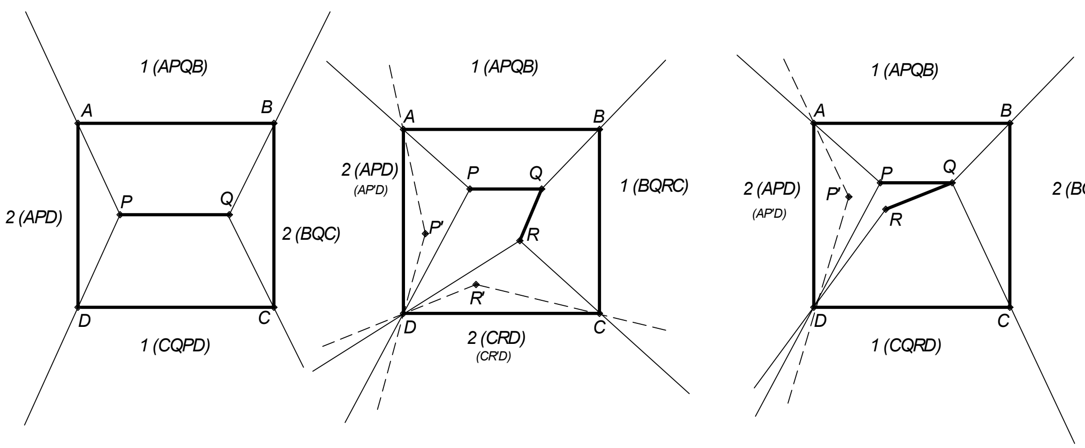

Fig. 23. Configurations of the form $(7, 4, \geq 2)$.

Now let us apply Statement 1 with $t = 3 \leq \min\{i-1, j\}$, $j \geq 3$. In accordance therewith, if $|\mathcal{T}_{PQ} \cup \mathcal{T}_{QR}| < 3$, then there is an empty convex hexagon in the configuration. Let us suppose, therefore, that $|\mathcal{T}_{PQ} \cup \mathcal{T}_{QR}| \geq 3$.

The above-mentioned conditions mean that, with respect to the placement of vertices of the quadrilateral $ABCD$, without loss of generality, two situations are possible: either $\{A, B\} \subset \mathcal{T}_{PQ}$ or $\{B, C\} \subset \mathcal{T}_{QR}$ (see Fig. 23 in the middle), or $\{A, B\} \subset \mathcal{T}_{PQ}$ and $\{C, D\} \subset \mathcal{T}_{QR}$ (see Fig. 23 on the right).

In the first situation, we must consider the sectors $(APQB)$, $(BQRC)$, (CRD), and (APD), covering the plane around the quadrilateral $ABCD$. Further, if necessary, the last two sectors can be replaced by $(CR'D)$ and $(AP'D)$ in accordance with the logic of the preceding clause (see Fig. 23). In this case, it is quite possible that $R' = P$ or that $P' = R$. Thus, in any case it appears that the plane around the quadrilateral $ABCD$ is covered by two 3-sectors and two 4-sectors with the appropriate limitations of the number of vertices of the heptagon in each of them (see Fig. 23).

In the second situation, we have the other selection of sectors: $(APQB)$, (BQC), $(CQRD)$, and (APD). As earlier, if necessary, we can replace the sector (APD) by the sector $(AP'D)$ (or the sector (ARD), if $R \in APD$) and obtain the needed coverage again.

In both situations, we apply the logic of Sec. 4.2.5 and find an empty convex hexagon in this configuration.

Note that, in both situations we have just considered, an equal number of 3- and 4-sectors is used. This will be the case further, too.

4.3.3. Configurations of the form $(7, 5, \geq 4)$. Actually in this clause we have only two cases to consider: $(7, 5, 4)$ and $(7, 5, 5)$.

Let us fix five consecutive vertices of a j-gon $P = V_1$, $Q = V_2$, $R = V_3$, $S = V_4$, and $T = V_5$ (here $P = T$ if $j = 4$). Denote by $\mathcal{T}'_{PQ}$ the set $\mathcal{T}'_1$, by $\mathcal{T}'_{QR}$ the set $\mathcal{T}'_2$, by $\mathcal{T}'_{RS}$ the set $\mathcal{T}'_3$, and by $\mathcal{T}'_{ST}$ the set $\mathcal{T}'_4$.

According to Statement 1$'$ when $t = 2$, if at least one of the equalities

$$|\mathcal{T}'_{PQ}| < 1, \quad |\mathcal{T}'_{QR}| < 1, \quad |\mathcal{T}'_{RS}| < 1, \quad |\mathcal{T}'_{ST}| < 1$$

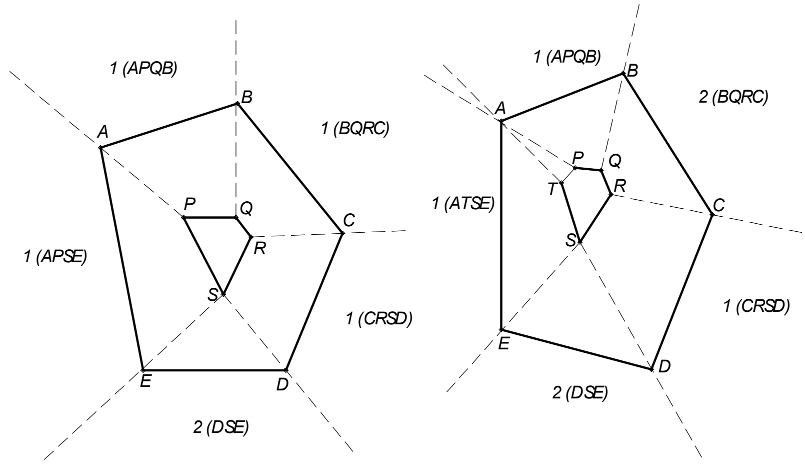

Fig. 24. Configurations of the form $(7, 5, \geq 4)$.

is true, then there is an empty convex hexagon and that is all right. Thus, it remains to consider the case

$$|\mathcal{T}'_{PQ}| \geq 1, \quad |\mathcal{T}'_{QR}| \geq 1, \quad |\mathcal{T}'_{RS}| \geq 1, \quad |\mathcal{T}'_{ST}| \geq 1. \tag{1}$$

First, let us consider the situation where $P = T$. Let us apply Statement $1'$ with $t = 3$ and $t = 4$. It appears that

$$|\mathcal{T}'_{PQ} \cup \mathcal{T}'_{QR}| \geq 2, \quad |\mathcal{T}'_{QR} \cup \mathcal{T}'_{RS}| \geq 2, \quad |\mathcal{T}'_{RS} \cup \mathcal{T}'_{ST}| \geq 2, \tag{2}$$

$$|\mathcal{T}'_{PQ} \cup \mathcal{T}'_{QR} \cup \mathcal{T}'_{RS}| \geq 3, \quad |\mathcal{T}'_{QR} \cup \mathcal{T}'_{RS} \cup \mathcal{T}'_{ST}| \geq 3. \tag{3}$$

Further, let us demonstrate that

$$|\mathcal{T}'_{PQ} \cup \mathcal{T}'_{QR} \cup \mathcal{T}'_{RS} \cup \mathcal{T}'_{ST}| \geq 4. \tag{4}$$

Suppose the opposite is true. Then, with respect to Statement $1'$,

$$|\mathcal{T}'_{PQ} \cup \mathcal{T}'_{QR} \cup \mathcal{T}'_{RS} \cup \mathcal{T}'_{ST}| = 3.$$

Let us denote consecutive sides of the pentagon by $\mathcal{L}_1, \ldots, \mathcal{L}_5$ (e.g., $\mathcal{L}_1 = AB$, and the numbering goes clockwise). Let us order similarly the sides of the quadrilateral assuming that $\mathcal{M}_1 = PQ$ and so on up to $\mathcal{M}_4$ (we move clockwise).

We appeal again to Statement $1'$ and note that the union of any three sets of the type $\mathcal{T}'_{PQ}$ has cardinality 3, and, therefore, all such sets coincide with each other. We may assume, without loss of generality, that they have one of the the following two forms:

$$\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_4\}, \quad \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}.$$

It is clear that each set of the type $\mathcal{T}'_{PQ}$ contains consecutive sides of the pentagon. This observation together with Statement $1'$ immediately leads us to the contradiction in the first situation considered above. Indeed, any $\mathcal{T}'$ is either a subset in $\{\mathcal{L}_1, \mathcal{L}_2\}$ or a subset in $\{\mathcal{L}_4\}$. With respect to Statement $1'$, this is only possible when $\{\mathcal{L}_1, \mathcal{L}_2\}$ contains no more than two and $\{\mathcal{L}_4\}$ no more than one set $\mathcal{T}'$. A contradiction.

It remains to study the second situation. Denote by $\mathcal{M}_i$ the last (by numbering) side of the quadrilateral such that the side $\mathcal{L}_3$ will be placed in the set $\mathcal{T}'$ corresponding to it. Then five different situations in placement of sides $\mathcal{M}_i$, $\mathcal{M}_{i+1}$ are possible (we take $i+1$ modulo 5, if necessary), and they are all displayed in Fig. 25. Here in the first case the line crossing $\mathcal{M}_{i+1}$ crosses only the side $\mathcal{L}_3$, in the second case it crosses the sides $\mathcal{L}_3$ and $\mathcal{L}_5$, etc. In the first three cases, we have a contradiction with the assumption that $\mathcal{L}_4$ and $\mathcal{L}_5$ do not belong to any $\mathcal{T}'$. In the last two cases, we obtain a contradiction with Statement 1.

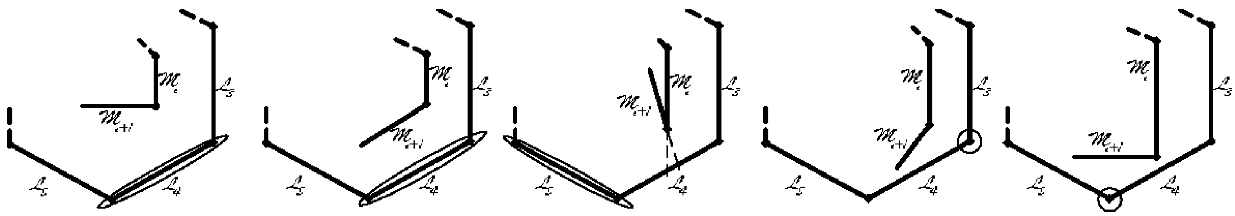

Fig. 25. Five different situations in placement of sides.

Thus, we have the aggregate of four sets $\mathcal{T}'_{PQ}$, $\mathcal{T}'_{QR}$, $\mathcal{T}'_{RS}$, and $\mathcal{T}'_{ST}$ meeting the system of inequalities (1)–(4). Let us apply to it the theorem of P. Hall [6] and find a system of distinct representatives consisting of sides $\mathcal{L}_{PQ}$, $\mathcal{L}_{QR}$, $\mathcal{L}_{RS}$, and $\mathcal{L}_{ST}$. The pairs of sides $(PQ, \mathcal{L}_{PQ}), \dots, (ST, \mathcal{L}_{ST})$ cause four 4-sectors displayed in Fig. 24 on the left. To cover the whole plane around the pentagon $ABCDE$, one more 3-sector has to be added to these 4-sectors (see Fig. 24 on the left, 3-sector (DSE)). With respect to the clause 4.2.5, there is certainly an empty convex hexagon in the configuration.

Now if $P \neq T$, then the picture formally looks as displayed in Fig. 24 on the right. But its form (i.e., the existence of four 4-sectors and one 3-sector on it) is conditioned by quite the same grounds as those we applied above. The final part of the proof is obvious.

Now let us apply the Hall theorem to find four 4-sectors and one 3-sector covering the plane around the pentagon as shown in Fig. 24 on the left. With respect to Sec. 4.2.5, there is certainly an empty convex hexagon in the configuration.

4.4. Individual Cases. In this subsection, the remaining cases will be considered. These are the cases of the form $(7,6,3)$, $(7,6,4)$, $(7,6,5)$, and $(7,5,3)$. An individual approach is required for each case, and it is also necessary to apply a detailed search of several possibilities.

4.4.1. Configurations of the form $(7,6,3)$. A hexagon $ABCDEF$ with a triangle XYZ inside form a $(6,3)$-placing of the form $[a_1, b_1, a_2, b_2, a_3, b_3]$, for which the equality $\sum(a_i + b_i) = 6$ is true. Note

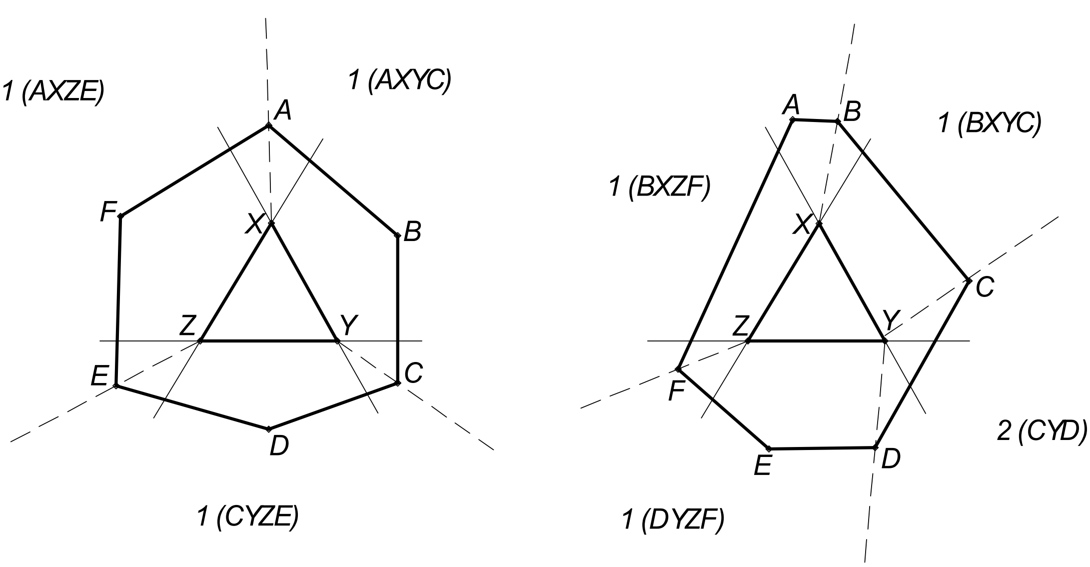

Fig. 26. Configuration of the form $(7,6,3)$.

that if even one of the inequalities (see Fig. 6):

$$a_1 + b_1 + a_2 \le 3, \quad a_2 + b_2 + a_3 \le 3, \quad a_3 + b_3 + a_1 \le 3,$$
$$b_1 + a_2 + b_2 \le 3, \quad b_2 + a_3 + b_3 \le 3, \quad b_3 + a_1 + b_1 \le 3,$$
$$b_i \le 2 \quad (1 \le i \le 3)$$

is not true, then there is certainly a convex hexagon with no more than one point within. If all inequalities are true, then, it is easy to see, the first six of them are transformed to equalities. It follows from this that $a_1 = b_2$, $a_2 = b_3$, and $a_3 = b_1$. Thus, it only remains to study two placings, namely $[1, 1, 1, 1, 1, 1]$ and $[2, 1, 0, 2, 1, 0]$ (see Fig. 26). For each of them it is easy to draw the required partition of the plane around the hexagon $ABCDEF$ to sectors (see Fig. 26). In accordance with the logic of Sec. 4.2.5, we come to the conclusion that a convex hexagon with no more than one point within will be always found.

4.4.2. Configurations of the form $(7, 6, 4)$. Consider a hexagon $ABCDEF$ with a quadrilateral $PQRS$ as a $(6, 4)$-placing $[a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4]$, for which $\sum(a_i + b_i) = 6$. Note that if even one of the equalities (see Fig. 9)

$$b_1 + a_2 + b_2 \le 2, \quad b_2 + a_3 + b_3 \le 2, \quad b_3 + a_4 + b_4 \le 2, \quad b_4 + a_1 + b_1 \le 2,$$
$$b_1 + b_3 \le 1, \quad b_2 + b_4 \le 1$$

is not true, then there is certainly a convex hexagon with no more than one point within. If all inequalities are true, then it follows from the first four of them that $a_1 + a_3 \ge 2$ and $a_2 + a_4 \ge 2$. Moreover, the case $a_1 = a_2 = a_3 = a_4 = 1$ is impossible as, again, it leads to a failure in some of the inequalities. Thus, at least one of a_i equals 2.

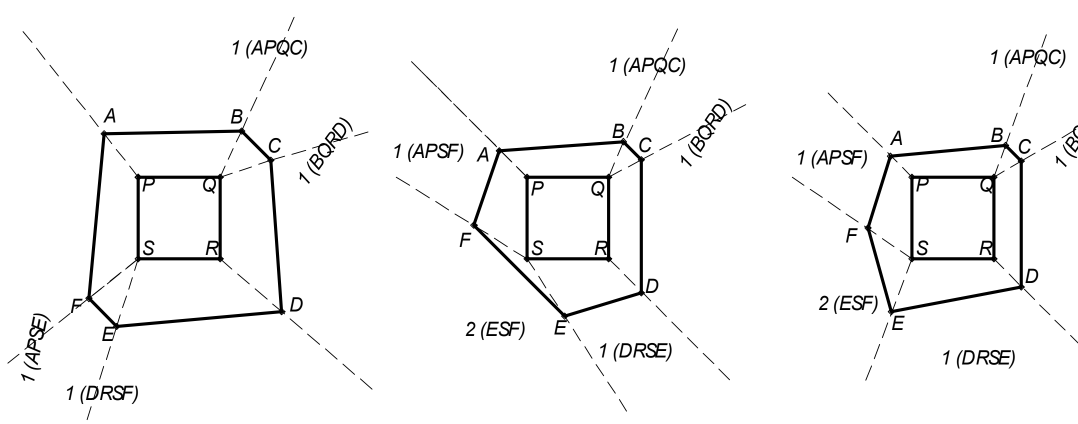

Fig. 27. Configuration of the form $(7, 6, 4)$.

Without loss of generality, we will assume that $a_1 = 2$; then (according to the appropriate inequality) $b_1 = b_4 = 0$. Note further that if the inequalities $a_1 + b_1 + a_2 \le 3$ and $a_1 + b_4 + a_4 \le 3$ are not true, then there is again a convex hexagon with no more than one point within. These inequalities together with $a_2 + a_4 \ge 2$ allow us to conclude that $a_2 = a_4 = 1$. Thus, it remains to consider placings of the form $[2, 0, 1, ?, ?, ?, 1, 0]$. There are only three such placings ($[2, 0, 1, 0, 2, 0, 1, 0]$, $[2, 0, 1, 1, 0, 1, 1, 0]$, and $[2, 0, 1, 1, 1, 0, 1, 0]$), and for each of them a coverage of the plain around the hexagon $ABCDEF$ by sectors with the appropriate limitations may be drawn (see Fig. 27). According to the logic of Sec. 4.2.5, we come to the conclusion that there is always a convex hexagon with no more than one point within in this configuration.

4.4.3. Configurations of the form $(7,6,5)$. A hexagon $ABCDEF$ together with a pentagon $PQRST$ form a $(6,5)$-placing of the form $[a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5]$, for which $\sum(a_i + b_i) = 6$. It is enough to consider the cases where the following inequalities are true (see Fig. 12)

$$0 \le b_i \le 1 \quad (1 \le i \le 5),$$
$$a_1 + b_1 + a_2 + b_2 + a_3 \le 3, \quad a_2 + b_2 + a_3 + b_3 + a_4 \le 3,$$
$$a_3 + b_3 + a_4 + b_4 + a_5 \le 3, \quad a_4 + b_4 + a_5 + b_5 + a_1 \le 3,$$
$$a_5 + b_5 + a_1 + b_1 + a_2 \le 3$$

(if even one of them is not true, then there is certainly a convex hexagon with no more than one point within). It follows from these inequalities that $3\sum a_i + 2\sum b_i \le 15$, and, therefore, $5 \ge \sum b_i \ge 3$ and, accordingly, $1 \le \sum a_i \le 3$. Let us consider in detail all possible values for the sum $\sum b_i$.

If $\sum b_i = 5$, then the placing $[?, 1, ?, 1, ?, 1, ?, 1, ?, 1]$, without loss of generality, may be uniquely completed to $[1, 1, 0, 1, 0, 1, 0, 1, 0, 1]$. If $\sum b_i = 4$, then the placing $[?, 1, ?, 1, ?, 1, ?, 1, ?, 0]$, without loss of generality, may be completed (with respect to inequalities) in two ways $[1, 1, 0, 1, 0, 1, 0, 1, 1, 0]$ or $[1, 1, 0, 1, 0, 1, 1, 1, 0, 0]$. If $\sum b_i = 3$, then we have two possible placings: $[?, 1, ?, 1, ?, 1, ?, 0, ?, 0]$

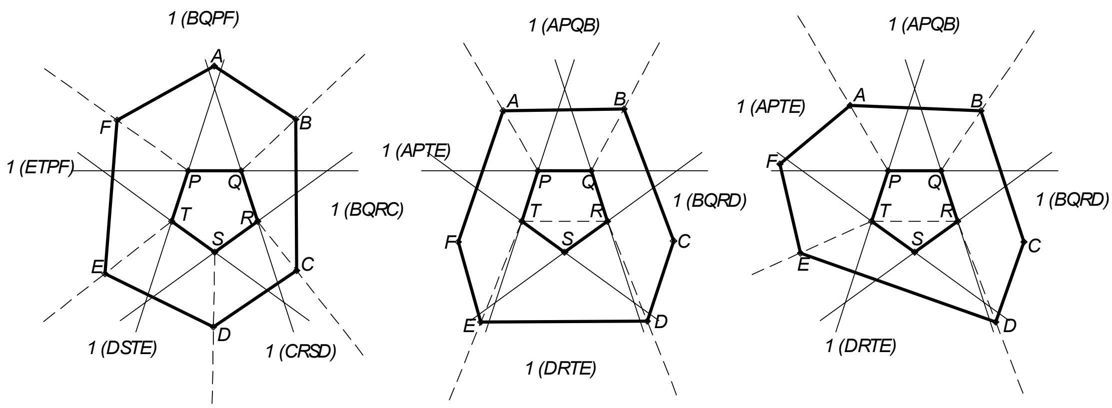

Fig. 28. Configurations of the form $(7,6,5)$.

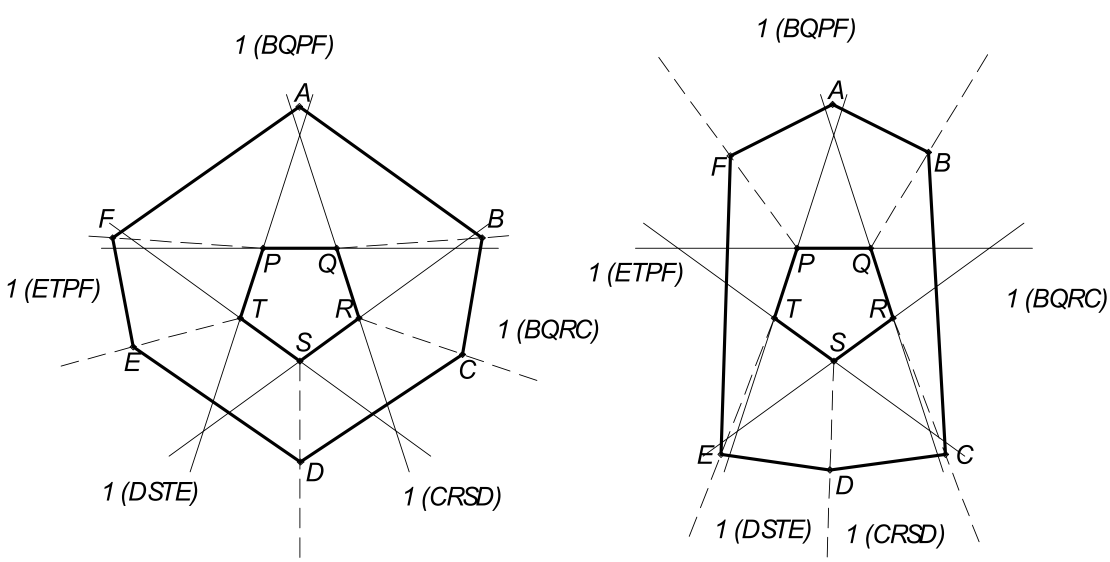

Fig. 29. Configurations of the form $(7,6,5)$.

and $[?, 1, ?, 1, ?, 0, ?, 1, ?, 0]$, each of which, without loss of generality, may be uniquely completed: $[1, 1, 0, 1, 0, 1, 1, 0, 1, 0]$ and $[0, 1, 1, 1, 0, 0, 1, 1, 1, 0]$, respectively. All five placings are displayed in Figs. 28 and 29, and for each of them sectors are specified with the appropriate limitations as regards the number of vertices of the heptagon in them. According to the logic of the clause 4.2.5, we conclude that a convex hexagon with no more than one point within will be found in any position of points.

4.4.4. Configurations of the form $(7, 5, 3)$. First of all, we note that the construction consisting of the pentagon $ABCDE$ and the triangle PQR, provided that there is no convex hexagon with no more than one point within in it (the same inequalities as in the case $(7, 6, 3)$ will be true), may be a $(5, 3)$-placing of only one of the following seven forms up to equivalence of writing (see Sec. 3.2):

$$[2, 1, 0, 2, 0, 0],\ [2, 1, 0, 1, 1, 0],\ [2, 0, 1, 2, 0, 0],\ [2, 0, 1, 1, 1, 0],$$
$$[1, 1, 1, 1, 1, 0],\ [1, 1, 0, 2, 1, 0],\ [3, 0, 0, 2, 0, 0].$$

The proof of this fact is confined to the simple search of cases, which we will not do here. For more clarity we will only represent in Fig. 30 schemes of each of the $(5, 3)$-placings listed above (compare Sec. 3.2): here a_1 is the number of points in the upper part of the scheme and the writing $[a_1, b_1, \ldots]$ is made counter-clockwise.

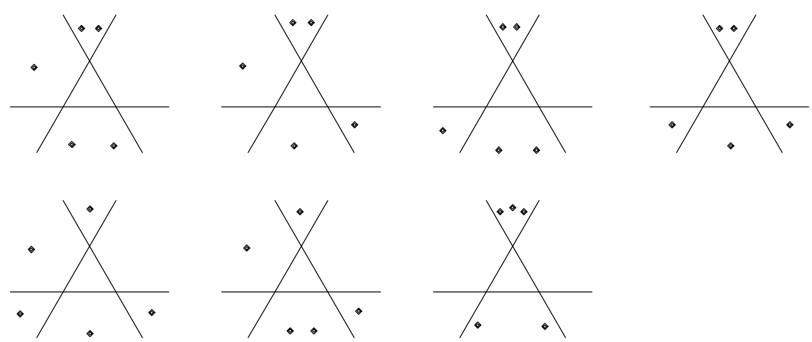

Fig. 30. Schemes of all possible placings for configurations of the form $(7, 5, 3)$.

In Figs. 31–33 all seven situations are displayed and the appropriate coverages of a plane around a pentagon by sectors with limitations are specified. According to the logic of Sec. 4.2.5, a convex hexagon with no more than one point within will always be found in the configuration. The theorem is proved.

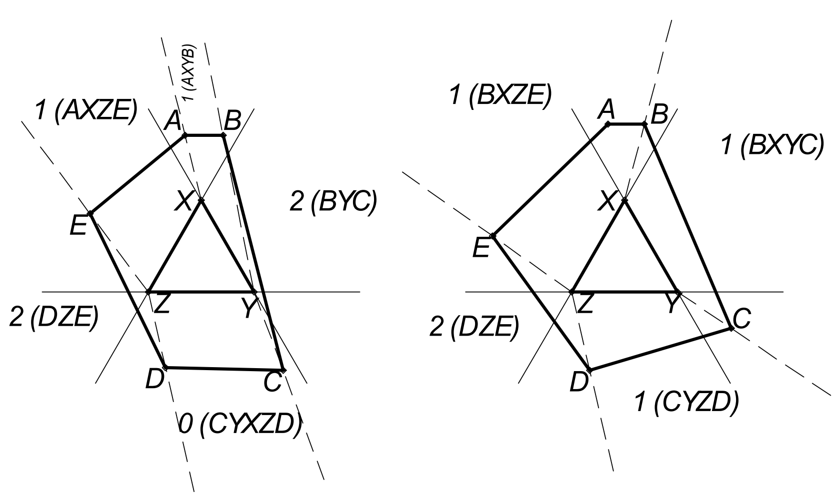

Fig. 31. Configurations of the form $(7, 5, 3)$.

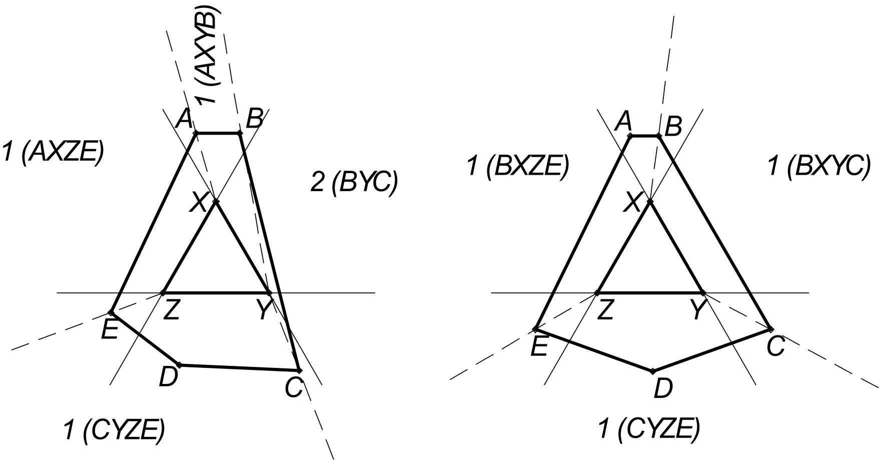

Fig. 32. Configurations of the form $(7, 5, 3)$.

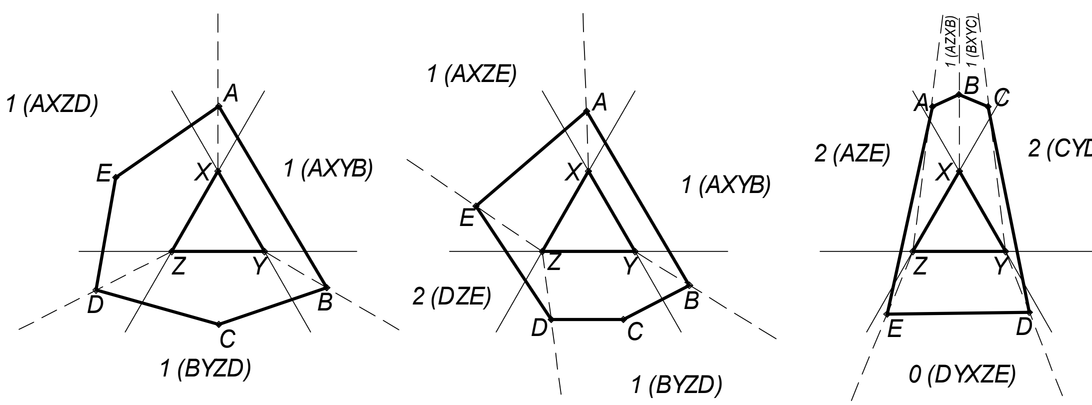

Fig. 33. Configurations of the form $(7, 5, 3)$.

5. Conclusions

In the context of studying $h(6)$, the issue of interest is the following: In which of the cases we analyzed is the existence of a convex and empty hexagon guaranteed? One may deduce from the proof of the theorem that there is certainly an empty convex hexagon in all cases of Sec. 4.3 (minimality), in the cases $(7, 2, 0)$ and $(7, 3, 1)$ (Sec. 4.2), and in the following trivial cases (Sec. 4.1): $(7, 0, 0)$, $(7, 6, 0)$, and $(7, i, 6)$ $(3 \le i \le 6)$ — altogether 18 of the considered cases. Moreover, according to the lemma of Valtr, there is an empty convex hexagon in all configurations of the type $(7, i, j, k, \ldots)$, where $k > 0$. In all other cases (altogether 13), the existence of an empty convex hexagon is not guaranteed, i.e., some position of points always exists in the appropriate configurations where there is no empty convex hexagon.

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V. A. Koshelev
Moscow Institute of Physics and Technology, Moscow, Russia
E-mail: `koshelev@mccme.ru`