LINKÖPING UNIVERSITY
Department of Biomedical Engineering

Lecture Notes

Submitted with the written examination in

TBME08 Biomedical Modeling and Simulation

Computation of the INDEX for a linear DAE

$\boxed{E\dot{z} + Fz = Gu}$

E, F : square matrices
dim$E = n \times n$
rank$(E) = r, \quad r < n$

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \dot{z} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} z = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$

E_1, F_1, G_1 : have r rows
E_1 : full rank

By rearranging the rows of E:

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{z} + \begin{bmatrix} F_1 \\ \tilde{F}_2 \end{bmatrix} z = \begin{bmatrix} G_1 \\ \tilde{G}_2 \end{bmatrix} u$$

Computation of the INDEX for a linear DAE

By derivating the lower part the system becomes

$$\begin{bmatrix} E_1 \\ \tilde{F}_2 \end{bmatrix} \dot{z} + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} z = \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ \tilde{G}_2 \end{bmatrix} \dot{u}$$

If the matrix $\begin{bmatrix} E_1 \\ \tilde{F}_2 \end{bmatrix}$ has full rank then the system can be expressed explicitly. In this case the system has INDEX=1.

Otherwise the procedure is repeated.

Euler method

$$\frac{x_{n+1} - x_n}{h} \approx \dot{x}(t_n) = f(t_n, x_n)$$

$$h = t_{n+1} - t_n$$

it gives

$$\boxed{x_{n+1} = x_n + hf(t_n, x_n)}$$

Runge-Kutta method, order 4 - RK4

$$\boxed{x_{n+1} = x_n + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)}$$

where

$$\begin{aligned} k_1 &= f(t_n, x_n) \\ k_2 &= f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}hk_1) \\ k_3 &= f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}hk_2) \\ k_4 &= f(t_n + h, x_n + hk_3) \end{aligned}$$

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Stability of numerical methods

Euler:

$$\frac{x_{n+1} - x_n}{h} \approx \dot{x}(t_n) = f(t_n, x_n) \qquad h = t_{n+1} - t_n$$

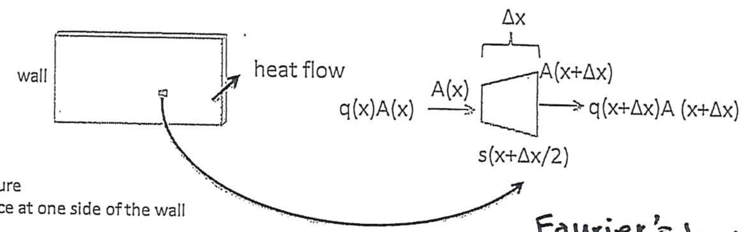
$$\boxed{x_{n+1} = x_n + hf(t_n, x_n)}$$

The method is numerically stable if:

stable system $x_n \to 0$ when $n \to \infty$

UNstable system $x_n \to \infty$ when $n \to \infty$

Example: Heat Conduction



T: temperature
s: heat source at one side of the wall
q: heat flux
k: thermal conductivity
l: wall thickness
A: area normal to the heat flow direction
q bar: prescribed flux
T bar: prescribed temperature

Fourier's law:

$$q = -k \frac{dT}{dx}$$

Source: Jacob Fish

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Non-uniformly Distributed RNG

- Inversion
  - An *rnd* (random number) is produced using the invers of the cumulative distribution function (*cdf*).
  - *cdf* is defined as

$$D(x) = P(X \leq x) \equiv \int_{-\infty}^{x} P(\xi) d\xi$$

where *P(x)*, the probability density function (*pdf*), is normalized to

$$\int_{-\infty}^{\infty} P(x) dx = 1$$

Non-uniformly Distributed RNG

- Accept-reject
  - Numerical method (no analytical solution).
  - Based on "pairs of *rnd*":
    - Let $x_{rnd} = rnd_1$ — rnd_1 is $U(x_{min}, x_{max})$
    - If $P(x_{rnd}) \geq rnd_2$: — rnd_2 is $U(0, P_{max})$
      - \+ is true, accept x_{rnd}
      - \+ is fals, reject and try again with two new *rnd*