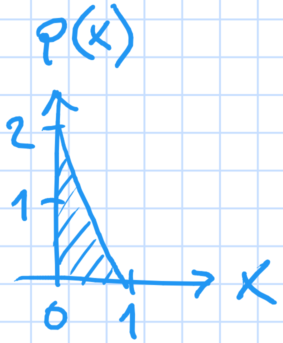
2017-06-03, 9b

$$p(x) = \begin{cases} 2-2x, & 0 \le x \le 1 \\ 0 \end{cases}$$



Check

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^1 2-2x\,dx = \left[2x-x^2\right]_0^1 = 1 \quad \text{ok!}$$

The CDF is:

$$D(X) = \int_0^X 2-2x\,dx = \left[2x-x^2\right]_0^X = 2X-X^2$$

Solve for X:

$$X^2-2X+D(X)=0$$

$$X = 1 \pm \sqrt{1-D(X)}$$, but only $x = 1-\sqrt{1-D(x)}$ fullfills $0 \le X \le 1$

Substitute D(X) with $u_1 \in \mathcal{U}[0,1]$:

answer:

$$\boxed{X_{rnd} = 1-\sqrt{1-u_1}}$$

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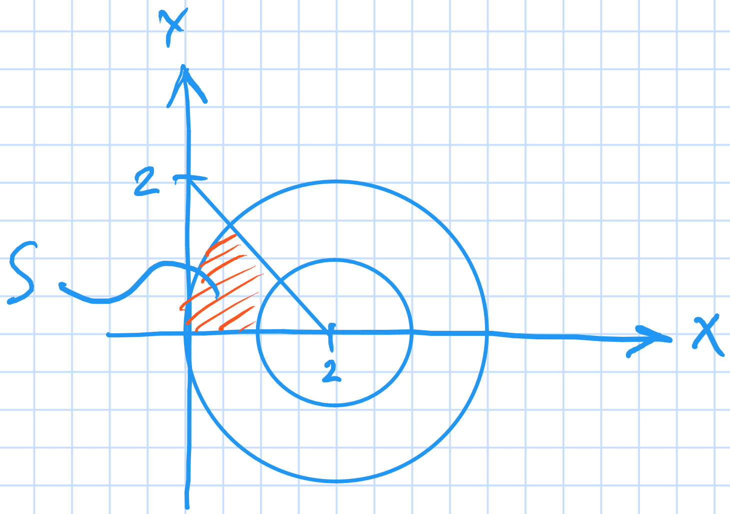
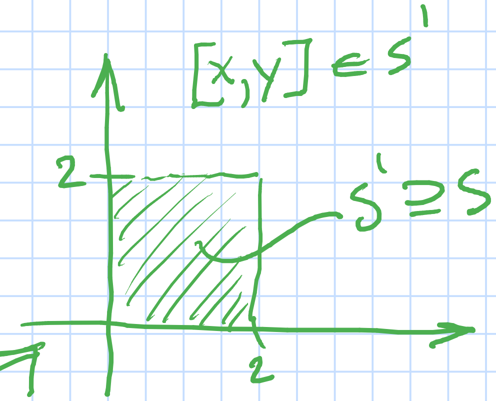
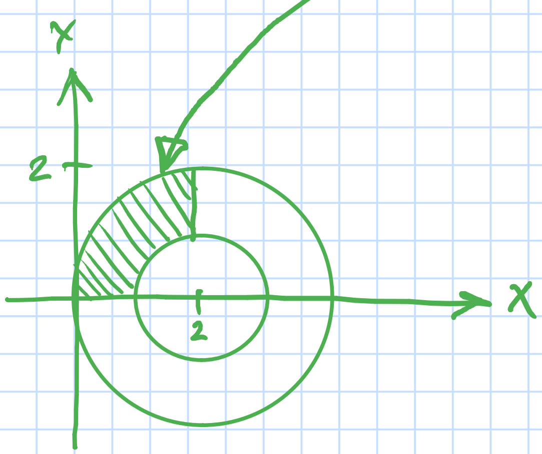
Accept-Reject solution

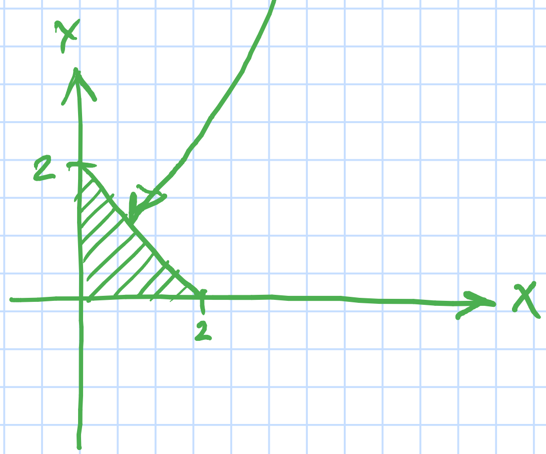




let $\underbrace{x = 2\cdot u_1 \text{ and } y = 2\cdot u_2}$ be your candidate

If $\underbrace{1 \le \sqrt{(x-2)^2+y^2} \le 2}$ and $\underbrace{y \le 2-x}$, then the candidate i accepted as a random number!





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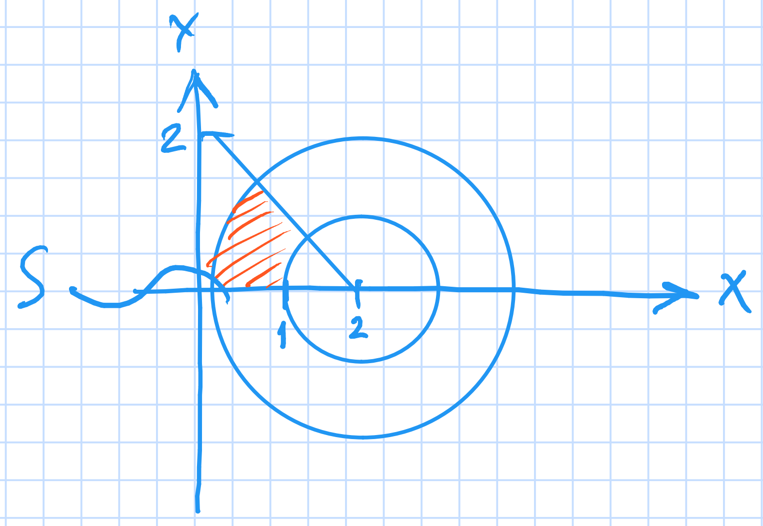
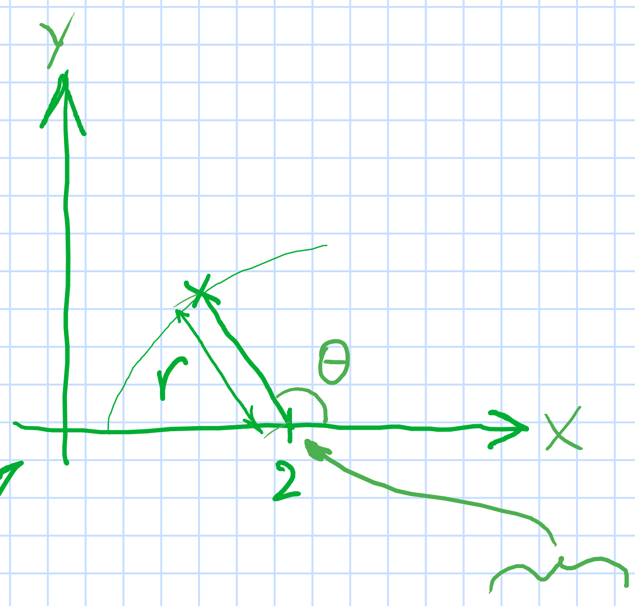
Inversion solution





Use polar coordinates with the origo $[x,y]=[2,0]$ when generating random points.

The probability density for r:

The probability for a point to fall on the surface S increases linear with the radius r (exactly the same as the area which increases with r) $\Rightarrow p(r) \propto r = k_r \cdot r$

Total probability is 1:

$$\int_1^2 k_r \cdot r\,dr = \left[k_r \cdot \frac{r^2}{2}\right]_1^2 = \frac{k_r}{2}\cdot(4-1) = \frac{3}{2}\cdot k_r = 1 \quad \Rightarrow \quad k_r = \frac{2}{3} \quad \Rightarrow \quad p(r) = \frac{2}{3}\cdot r \quad \text{for} \quad 1 \le r \le 2$$

The CDF is then given by:

$$D(R) = \int_1^R \frac{2}{3}\cdot r\,dr = \left[\frac{r^2}{3}\right]_1^R = \frac{1}{3}(R^2-1)$$

Solve the inverse, ie. solve for R:

$$R^2 = 3\cdot D(R)+1 \quad \Rightarrow \quad R = \sqrt{3\cdot D(R)+1}$$

Create an RNG for random radius R by substituting D(R) with $u_1 \in [0,1]$:

$$R_{rnd} = \sqrt{3\cdot u_1 + 1}$$

The probability density for θ:

The probability for θ i uniformly distributed between $\frac{3}{4}\pi$ and π $\Rightarrow p(\theta) = k_\theta$

Total probability is 1:

$$\int_{\frac{3}{4}\pi}^{\pi} k_\theta\,d\theta = \left[k_\theta \cdot \theta\right]_{\frac{3}{4}\pi}^{\pi} = k_\theta\left(\pi - \frac{3}{4}\pi\right) = \frac{k_\theta \cdot \pi}{4} = 1 \quad \Rightarrow \quad k_\theta = \frac{4}{\pi} \quad \Rightarrow \quad p(\theta) = \frac{4}{\pi} \quad \text{for} \quad \frac{3}{4}\pi \le \theta \le \pi$$

The CDF:

$$D(\theta) = \int_{\frac{3}{4}\pi}^{\theta} \frac{4}{\pi}\,d\theta = \left[\frac{4\theta}{\pi}\right]_{\frac{3}{4}\pi}^{\theta} = \frac{4}{\pi}\left(\theta - \frac{3}{4}\pi\right) = \frac{4\theta}{\pi} - 3$$

Solve the inverse:

$$\theta = \frac{\pi}{4}(D(\theta)+3)$$

Substitute $D(\theta)$ with $u_2 \in [0,1]$:

$$\theta_{rnd} = \frac{\pi}{4}\cdot(u_2+3)$$

Switch to cartesian coordinates and translate the origo back to $[0,0]$:

answer:

$$X_{rnd} = 2 + R_{rnd}\cdot\cos\theta_{rnd}$$

$$Y_{rnd} = R_{rnd}\cdot\sin\theta_{rnd}$$

where:

$$R_{rnd} = \sqrt{3\cdot u_1 + 1}$$

$$\theta_{rnd} = \frac{\pi}{4}\cdot(u_2+3)$$