Solutions to the TBME08 exam, 20170603, part 1

1. a) nonlinear - dynamic
   b) linear - dynamic
   c) nonlinear - dynamic
   d) nonlinear - dynamic

2. a) The system can be written as

$$\begin{aligned} \dot{x}_1 + x_4 &= 4u \\ 2\dot{x}_4 + x_1 &= 3u \\ 3\dot{x}_2 + x_3 &= 2u \\ 3x_2 + 2x_4 &= u \end{aligned} \tag{1}$$

   We want to write the equation in explicit form, i.e. $\dot{x}$ as a function of x, input and possibly derivatives of the input. We see that

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -x_4 + 4u \\ -\frac{1}{3}x_3 + \frac{2}{3}u \\ \\ -\frac{1}{2}x_1 + \frac{3}{2}u \end{bmatrix}$$

   The components $\dot{x}_1$, $\dot{x}_2$ and $\dot{x}_4$ are already given explicitly but not $\dot{x}_3$. This means that the index is larger than 0.
   We differentiate the algebraic part: $3\dot{x}_2 + 2\dot{x}_4 = \dot{u}$.
   We replace $\dot{x}_2$ and $\dot{x}_4$ with the expressions given in (1) and after some computations we obtain

$$-x_3 - x_1 = -5u + \dot{u} \tag{2}$$

   We differentiate equation (2): $-\dot{x}_3 - \dot{x}_1 = -5\dot{u} + \ddot{u}$.
   We replace $\dot{x}_1$ with its expression given in (1) resulting in

$$\dot{x}_3 = x_4 - 4u + 5\dot{u} - \ddot{u}$$

   Now $\dot{x}$ is expressed explicitly

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -x_4 + 4u \\ -\frac{1}{3} + \frac{2}{3}u \\ x_4 - 4u + 5\dot{u} - \ddot{u} \\ -\frac{1}{2}x_1 + \frac{3}{2}u \end{bmatrix}$$

   We have differentiated twice in order to express $\dot{x}$ explicitly so INDEX = 2.

   b) INDEX of a DAE is a measure of its complexity. It is important to know the INDEX when the DAE is going to be simulated.
   c) ODE has INDEX 0.

3. Given the differential equation $\dot{y}(t) = -2y(t) + 3$, the approximated solution with Euler explicit method becomes $y_{n+1} = (1-2h)\,y_n + 3h$ that gives at time $t = 0.2$ s $y_{t=0.2} = 0.54$.

   Exact solution is given by: $y(0.2) = 0.4945$.

   Global error at time $t = 0.2$ s is $E = y(0.2) - y_{t=0.2} = 0.0455$.

4. a) ARX: $A(q) = 1 + 2q^{-1}, B(q) = q^{-1} + 3q^{-3}, C(q) = 1$

   b) OE: $B(q) = q^{-1} + 3q^{-3}, F(q) = 1 + 2q^{-1}, C(q) = 1, D(q) = 1$

   c) ARMAX: $A(q) = 1 + 2q^{-1}, B(q) = q^{-1} + 3q^{-3}, C(q) = q^{-1}$

   d) BJ: $B(q) = q^{-1} + 3q^{-3}, F(q) = 1 + 2q^{-1} + q^{-2}, C(q) = q^{-1}, D(q) = 1 + 2q^{-1}$

5. a) The model for the system is

$$\begin{aligned} i(t) &= \frac{1}{R}u_C(t) + C\frac{du_C(t)}{dt} \\ u(t) &= ri(t) + u_C(t) \end{aligned}$$

   It is a first order differential equation so the state has one component

$$x_1(t) = u_C(t)$$

   State-space description becomes

$$\begin{aligned} \dot{x_1}(t) &= -\frac{1}{RC}x_1(t) + \frac{1}{C}i(t) \\ y(t) &= x_1(t) + ri(t) \end{aligned}$$

   where we can say

$$A = -\frac{1}{RC}, \quad B = \frac{1}{C}, \quad C = 1, \quad D = r$$

   b) The simulation of the model with Euler's explicit method gives

$$\begin{aligned} &x_{1,n+1} = x_{1,n} + h\left(-\tfrac{1}{RC}x_{1,n} + \tfrac{1}{C}i_n\right) = \left(1 - \tfrac{h}{RC}\right)x_{1,n} + \tfrac{h}{C}i_n \\ &y_n = x_{1,n} + ri_n \end{aligned}$$

$$\begin{aligned} &n=0 \quad t=0 \quad x_{1,0} = u_C(0) = 75 \\ &n=1 \quad t=0.1 \quad x_{1,0.1} = \left(1 - \frac{0.1}{2\cdot 0.25}\right)x_{1,0} + \frac{0.1}{0.25}100 = 100 \\ &n=2 \quad t=0.2 \quad x_{1,0.2} = \left(1 - \frac{0.1}{2\cdot 0.25}\right)x_{1,0.1} + \frac{0.1}{0.25}100 = 120 \\ &\qquad\qquad\qquad y_{0.2} = x_{1,0.2} + 0.2\cdot 100 = 140 \end{aligned}$$