Exercise 2.2

Test equation: $\dot{x}(t) = \lambda x(t)$ λ real negative

in this case $\lambda = -1$, $\lambda = -5$

$$\Rightarrow f(x) = \lambda x(t)$$

Euler explicit method $x_{n+1} = x_n + h\lambda x_n = (1+h\lambda)x_n$

$$x_0$$
$$x_1 = (1+h\lambda)x_0$$
$$x_2 = (1+h\lambda)x_1 = (1+h\lambda)^2 x_0$$
$$\Rightarrow \text{in general } x_n = (1+h\lambda)^n x_0$$

the numerical method applied to the test equation is numerically stable if $\lim_{n\to\infty} x_n = 0$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (1+h\lambda)^n x_0 = 0 \Rightarrow |1+h\lambda| < 1$$

$$|1+h\lambda| < 1 \Rightarrow \begin{cases} 1+h\lambda < 1 & \Rightarrow h\lambda < 0 \text{ always true} \\ 1+h\lambda > -1 & \Rightarrow h\lambda > -2 \end{cases}$$

$\lambda = -1$ $\quad -h > -2 \Rightarrow h < 2$

$\lambda = -5$ $\quad -5h > -2 \Rightarrow h < \frac{2}{5}$ $\quad \Rightarrow \boxed{h < \frac{2}{5}}$

I take the values of h that are valid for both λ

Euler implicit method

$$x_{n+1} = x_n + h\lambda x_{n+1}$$

$$(1 - h\lambda)x_{n+1} = x_n \Rightarrow x_{n+1} = \frac{1}{1-h\lambda} x_n$$

$$x_0$$

$$x_1 = \frac{1}{1-h\lambda} x_0$$

$$x_2 = \frac{1}{1-h\lambda} x_1 = \left(\frac{1}{1-h\lambda}\right)^2 x_0$$

$\Rightarrow$ in general $\quad x_n = \dfrac{1}{(1-h\lambda)^n} x_0$

the numerical approximation is numerically stable if $\lim\limits_{n\to\infty} x_n = 0$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{(1-h\lambda)^n} x_0 = 0 \Rightarrow (1-h\lambda)^n \xrightarrow[n\to\infty]{} \infty$$

$$\Rightarrow |1-h\lambda| > 1$$

$$|1-h\lambda| > 1 \Rightarrow \begin{cases} 1-h\lambda > 1 \Rightarrow -h\lambda > 0 & \text{always} \\ \quad \text{or} & \\ 1-h\lambda < -1 \Rightarrow -h\lambda < -2 & \text{never} \end{cases}$$

$$\Rightarrow \boxed{h > 0}$$