TBME08 - Exercises

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1 Linear/nonlinear, Static/dynamic system

1.1

Which of the systems below is linear, respectively nonlinear? Motivate.

1. $y(t) + (y(t-2))^2 = 2u(t-2)$
2. $\ddot{y}(t) + 2\dot{y}(t) + 7y(t) = u(t)$
3. $y(t) + 2y(t-2) = u(t-1)$
4. $\dot{y}(t) + \cos(y(t)) = u(t)$

1.2

Which of the following equations describe a dynamic relationship and which a static relationship? Motivate your answer!

1. $y^3(t) = 5u(t)$
2. $\ddot{y}(t) + 0.5y(t) = u(t)$
3. $y(t) - 4u(t-1) + u(t-2) = 0$
4. $\int y(\tau)d\tau + 3y(t) = u(t)$

2 Numerical methods

2.1

A simulation of the system

$$\begin{aligned} \dot{y}(t) &= -10y(t) \\ y(0) &= 1 \end{aligned}$$

is carried out using Euler's explicit method ($x_{n+1} = x_n + hf(x_n)$).

a) Which is the stability region for the system?

b) What does $y(t)$ for the system become at the time instants $t = 0.05$ s and $t = 0.1$ s when the simulation is run with a time step $h = 0.05$ s?

2.2

Consider the system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} x(t)$$

Two different methods are used to simulate the system:

$$\begin{array}{ll} \text{Euler's explicit method:} & x_{n+1} = x_n + hf(x_n) \\ \text{Euler's implicit method:} & x_{n+1} = x_n + hf(x_{n+1}) \end{array}$$

i) Which step sizes should each method use in order to have a stable simulation (the answer should give two different step sizes, one for each method)? Motivate your answer.

ii) Which method is to be preferred?

2.3

Euler's explicit method

$$x_{n+1} = x_n + hf(x_n)$$

is used to simulate a linear system with eigenvalues in $-1 \pm i$.

a) Which is the largest step size that can be used if stability condition should be satisfied.

b) Figure 1 shows the results from two simulations of the system in exercise a).

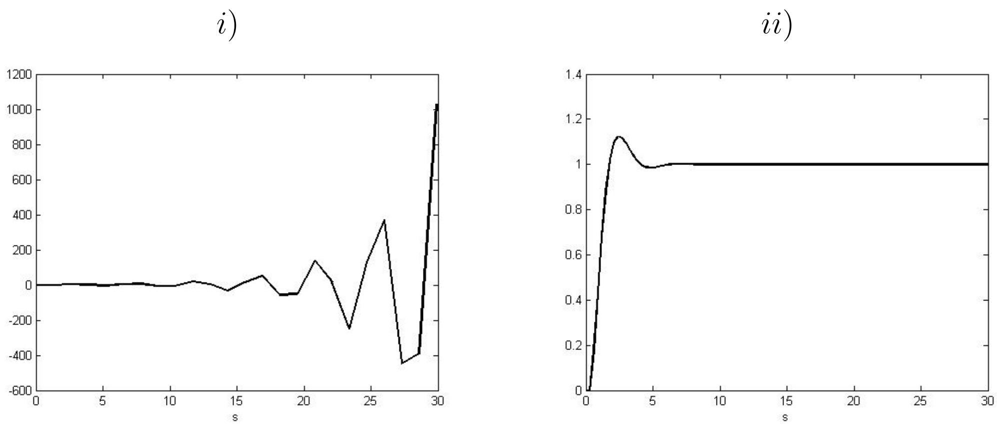

Figure 1: Results of simulations of the system in exercise 2.3.

Both simulations have been run using Euler's explicit method, but two different step sizes, $h = 0.3$ s and $h = 1.3$ s respectively, have been used. Which figure corresponds to a simulation with a step size $h = 0.3$ s and which figure corresponds to $h = 1.3$ s? Motivate your answer.

2.4

Consider the differential equation below

$$\dot{y}(t) = -ay(t) + b$$
$$y(0) = 0$$

with $a = 0.1$ and $b = 2$. The equation is simulated using Euler's explicit method ($x_{n+1} = x_n + hf(x_n)$) with step length $h = 5$ s.

The exact solution for the equation is given by

$$y(t) = \frac{b}{a} - \left(\frac{b}{a} - y_0\right) e^{-at}$$

Compute the local and global error at $t = 10$ s.

3 DAE

3.1

A system is modelled by the following DAE

$$\dot{x}_1 + a\dot{x}_2 = t$$
$$x_1 + x_2 = \sin(t)$$

a) Which index has the system?

b) How does the expression of $\dot{x}_1$ and $\dot{x}_2$ become?

3.2

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= 2x_1 + x_3 - 2u \\ 0 &= -x_1 - x_2 + u \end{aligned}$$

where x_1, x_2 and x_3 are states and u is the input signal.
Determine the index for the system. Motivate.

3.3

Consider the DAE model

$$\alpha\dot{x}_1 = u - \beta x_1 - \gamma x_2$$
$$\dot{x}_2 + n\dot{x}_3 = \gamma x_1$$
$$x_3 = nx_2$$

where α, β, γ and n are all positive constants. Which index has the system? Motivate.

3.4

Consider the DAE system

$$(1 + x_2^2)\dot{x}_1 = x_3 + u_1$$
$$(1 + x_1^2)\dot{x}_2 = -x_3 + u_2$$
$$x_1 = x_2$$

a) Which index has the system?

b) Show that the system in a) can be written in the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u) \\ x_2 &= f_2(x_1, u, \dot{u}, \ddot{u}, \ldots) \\ x_3 &= f_3(x_1, u, \dot{u}, \ddot{u}, \ldots) \end{aligned}$$

4 System identification

4.1

Which of the following systems is ARX, respectively, ARMAX? Motivate.

1. $y(t) + 2y(t-1) - 3y(t-2) = u(t-3) - 2u(t-4) + e(t) - e(t-1)$
2. $y(t) - 3y(t-2) = u(t) + 6u(t-2) + e(t)$
3. $y(t) = u(t-1) + 3u(t-2) - u(t-3) + e(t)$
4. $y(t) = y(t-2) + u(t-3) + e(t) - 2e(t-1)$

4.2

Consider the system

$$y(t) = \frac{1 + 0.5q^{-1}}{1 + 0.3q^{-1}} u(t-1) + H(q)e(t)$$

where $u(t)$ is the input signal and $e(t)$ white noise. Which conditions must H satisfy in order for the system be in ARX-form, ARMAX-form and OE-form, respectively?

Solutions

1 Linear/nonlinear, Static/dynamic system

1.1

1. NL
2. L
3. L
4. NL

1.2

1. static
2. dynamic
3. dynamic
4. dynamic

2 Numerical methods

2.1

a) Euler's method: $x_{n+1} = x_n + hf(x_n)$ applied to the given differential equation, it gives:

$$y_{n+1} = y_n - 10hy_n$$

the numerical solution is stable if $h < 0.2$.

b) The simulation for the system becomes: $y_{n+1} = (1 - 10 \cdot 0.05)y_n$

$$y_{0.05} = (1 - 10 * 0.05)y(0) = 0.5$$
$$y_{0.1} = (1 - 10 * 0.5)y_{0.05} = 0.25$$

2.2

i) Euler's **explicit** method applied to a stable system is numerically stable when $h\lambda > -2$ that for the given system gives

$$h < \frac{2}{5}$$

Euler's **implicit** method applied to a stable system is numerically stable for

$$h > 0$$

ii) Euler's implicit method is to be preferred as it has a larger stability region.

2.3

a) Euler's explicit method applied to a test equation $\dot{x} = \lambda x,\ \lambda \in \mathbb{C}^-$ becomes

$$x_n = (1 + h\lambda)^n x_0$$

The simulation is numerically stable if $|1 + h\lambda| = |1 + h(-1 \pm i)| < 1$. This implies

$$2h^2 - 2h + 1 < 1 \Leftrightarrow h(h - 1) < 0$$

which gives that $0 < h < 1$ will satisfy the stability condition.

b) We know that the system is stable as the eigenvalues have negative real parts and that the simulation is numerically stable for $h < 1$. Therefore we can say

simulation *i*) is numerically unstable so it corresponds to $h = 1.3$ s.

simulation *ii*) is numerically stable so it corresponds to $h = 0.3$ s.

2.4

The approximated solution given by Euler's explicit method is

$$y_{n+1} = y_n + h\,(2 - 0.1y_n) = (1 - 0.1h)\,y_n + 2h$$

that gives

$t = 0$ s $\quad y_0 = 0$
$t = 5$ s $\quad y_5 = (1 - 0.1h)y_0 + 2 \cdot 5 = 10$
$t = 10$ s $\quad y_{10} = (1 - 0.1h)\,y_1 + 2 \cdot 5 = 15$

Exact solution: $y(10) = 20 - 20e^{-1} = 12.64$

Global error: $\quad E_{10} = |y(10) - y_{10}| = 2.35$

Local error: $\quad e_{10} = |y(10) - z_{10}| = 1.29$

where $z_{10} = y(5) + h\,(-0.1y(5) + 2) = 13.93$

3 DAE

3.1

Generalized state

$$z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \dot{z} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

If I derivate the algebraic equation, I get

$$\begin{array}{l} \dot{x}_1 = \frac{a\cos t - t}{a-1} \\ \dot{x}_2 = \frac{t - \cos t}{a-1} \end{array}$$

therefore INDEX=1

3.2

The generalized state is given by

$$z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

we see that we need to compute $\dot{x}_3$ in order to express $\dot{z}$ explicitly. We start by differentiating the algebraic part obtaining

$$-\dot{x}_1 - \dot{x}_2 + \dot{u} = 0$$

We replace $\dot{x}_1$ and $\dot{x}_2$ from the derivative part and we get

$$-2x_1 - 2x_3 + 2u + \dot{u}$$

By differentiating this equation we obtain

$$2\dot{x}_3 = -2x_3 + 2\dot{u} + \ddot{u}$$

Resulting in INDEX=2

3.3

The generalized state is given by $z = [x_1\, x_2\, x_3]'$. In order to express $\dot{z}$ explicitly we need $\dot{x}_2$ and $\dot{x}_3$. We start by differentiating the algebraic part

$$\dot{x}_3 = n\dot{x}_2$$

and we replace it in the second equation

$$\dot{x}_2 + n^2\dot{x}_2 = \gamma x_1 \quad \Rightarrow \quad \dot{x}_2 = \frac{\gamma}{1+n^2}x_1$$

and consequently

$$\dot{x}_3 = n\frac{\gamma}{1+n^2}x_1$$

Therefore INDEX=1.

3.4

a) Given the DAE system

$$(1+x_2^2)\dot{x}_1 = x_3 + u_1 \tag{1}$$
$$(1+x_1^2)\dot{x}_2 = -x_3 + u_2 \tag{2}$$
$$x_1 = x_2 \tag{3}$$

the generalized state is given by $z = [x_1\ x_2\ x_3]'$. It is not possible to explicitly formulate $\dot{z}$ immediately ($\dot{x}_3$ is not explicitly defined) so we start by differentiating the algebraic equation

$$\dot{x}_1 = \dot{x}_2$$

Replacing $\dot{x}_1$ and $\dot{x}_2$ with (1) and (2) and making some computations it gives

$$x_3 = \frac{1}{2}(u_2 - u_1) \tag{4}$$

Differentiating (4) it gives

$$\dot{x}_3 = \frac{1}{2}(\dot{u}_2 - \dot{u}_1)$$

and we can say that INDEX = 2.

b) Using the algebraic relationship (3) in (1) and (2) we can write

$$\begin{array}{lllll} \dot{x}_1 & = & \frac{u_1+u_2}{2(1+x_1^2)} & = & f_1(x_1, u) \\ x_2 & = & x_1 & = & f_2(x_1) \\ x_3 & = & \frac{u_2-u_1}{2} & = & f_3(u) \end{array}$$

4 System identification

4.1

1. ARMAX
2. ARX
3. ARX
4. ARMAX

4.2

With respect to the given model

$$y(t) = \frac{1 + 0.5q^{-1}}{1 + 0.3q^{-1}} q^{-1} u(t) + H(q)e(t)$$

we have that

$$\begin{array}{lll}
\text{OE:} & H(q) = 1 & \\
\text{ARX:} & H(q) = \dfrac{1}{1 + 0.3q^{-1}} & \\
\text{ARMAX:} & H(q) = \dfrac{C(q)}{1 + 0.3q^{-1}} & \begin{array}{l}\text{with } C(q) \text{ any polynomial of the form} \\ C(q) = 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}\end{array}
\end{array}$$