AST5220 - Cosmology 2
Milestone 2

Alex Ho

March 15, 2017

The program, plots and the report can be found in the following Github page:

`https://github.com/AHo94/AST5220_Projects/tree/master/Project2`

Mathematics

We start with the differential equation of the optical depth, given as

$$\frac{d\tau}{dx} = -\frac{n_e\sigma_T a}{\mathcal{H}} = -\frac{n_e\sigma_T}{H} \tag{1}$$

Where σ_T is the Thompson cross-section, H the Hubble parameter and n_e the number density for free electrons. To compute the electron density, we define the fractional electron density $X_e \equiv n_e/n_H$. X_e can be found in two different ways

Physical dimensions

Saha equation

One way to determine X_e is by using the Saha equation. The Saha equation is a good approximation if $X_e \approx 1$, that is, for early times of the universe. Saha equation is given as

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{2}$$

First thing to note is that the left hand side is dimensionless, whereas the right hand side has the dimension $(\text{kg K})^{3/2}\text{m}^3$ and the argument inside the exponential (which should be dimensionless) has the dimension J/K. We want to multiply a certain combination of $c, \hbar$ or k_b to turn the right hand side dimensionless. Testing this for different combinations, we get the Saha equation in dimensionless form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b}\left(\frac{m_e T_b k_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/(k_b T_b)} \tag{3}$$

This can be written in the form

$$X_e^2 + BX_e - B = 0, \quad \text{where } B = \frac{1}{n_b}\left(\frac{m_e T_b k_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/(k_b T_b)} \tag{4}$$

This is just a second order equation, which can be easily solved.

Peebles' equation

The second way to determine X_e is to use Peebles' equation. It is a good approximation when $X_e \ll 1$. Peebles' equation is given as

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right] \tag{5}$$

This depends on many other parameters, which I will for now not write down. Once again, the left hand side is dimensionless, but the right hand side has the dimension of seconds, due to the Hubble parameter. The tricky part now is to determine whether $C_r(T_b)$ or $\beta(T_b)$ (and $\alpha^{(2)}(T_b)$) are dimensionless parameters. Let us first check $C_r(T_b)$, which is given as

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)} \tag{6}$$

Where $\Lambda_{2s\to 1s} = 8.227 s^{-1}$, i.e has the unit of s^{-1}. This parameter appears both on the numerator and the denominator, which implies that $C_r(T_b)$ is dimensionless. Now we have to determine the dimensions of Λ_α and $\beta^{(2)}(T_b)$. Looking at the numerator of $C_r(T_b)$, we demand that Λ_α has to have dimension of s^{-1} as well, which ensures that we add two of physical quantities of same dimension [1]. Currently, Λ_α is given as

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{7}$$

which currently has the dimension $\mathrm{J}^3\mathrm{m}^{-3}\mathrm{s}^{-1}$. To fix this, we can multiply with $1/(\hbar c)^3$, so the fixed quantity becomes

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}}\left(\frac{1}{\hbar c}\right)^3 \tag{8}$$

Same reasoning can be applied to $\beta^{(2)}(T_b)$. I will not go into much detail here, but the parameters that are fixed, with respect to their dimensions, are the following:

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/(4k_bT_b)} \tag{9}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\frac{k_b^{3/2}}{\hbar^3}\left(\frac{m_eT_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/(k_bT_b)} \tag{10}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\frac{\hbar^2}{c}\sqrt{\frac{\epsilon_0}{k_bT_b}}\phi_2(T_b) \tag{11}$$

$$\phi_2(T_b) = 0.448\ln[\epsilon_0/(k_bT_b)] \tag{12}$$

Peebles' equation itself remains unchanged.

[1] Let's be honest here. Adding, for instance, 1kg and 1m does not make much sense.

The optical depth

The right hand side of equation (1) has the dimension s m^{-1}. We want the optical depth to be dimensionless so that the visibility function $g(x)$ (as well as the exponential in $g(x)$) remains dimensionless. This is easily fixed by multiplying with c on the right hand side, that is

$$\frac{d\tau}{dx} = -\frac{n_e \sigma_T c}{H} \tag{13}$$

Numerics

Overflows in Peebles' equation

When we do the computation of the Peebles' equation, we will quickly run into overflow problems. This is especially true in the $\beta^{(2)}(T_b)$ term, where the exponential explodes because $\epsilon_0/(k_b T_b)$ is of order $\approx 10^4$. Instead, we can write out the expression of $\beta^{(2)}(T_b)$ to include the exponential in $\beta(T_b)$, which we write into a single exponential. That is

$$\begin{aligned}\beta^{(2)}(T_b) &= \alpha^{(2)}(T_b)\frac{k_b^{3/2}}{\hbar^3}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/(k_b T_b)} e^{3\epsilon_0/(4k_b T_b)} \\ &= \alpha^{(2)}(T_b)\frac{k_b^{3/2}}{\hbar^3}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/(4k_b T_b)} \end{aligned} \tag{14}$$

The exponential should now no longer give any problems with overflows.

The program

The program is a continuation of the program from the previous milestone, with some extras and small changes to the old code.

The function `Get_eta` has been replaced with a new function `Cubic_spline`, which does the exact same thing, but applies for any function $f(x)$. The function `Spline_DoubleDerivative`, from milestone 1, has also been replaced with a new function, `Spline_derivative`, which applies to any derivatives and also uses natural spline boundary condition. Like the previous milestone, the interpolation is done by using Scipy's `interpolate` functions.

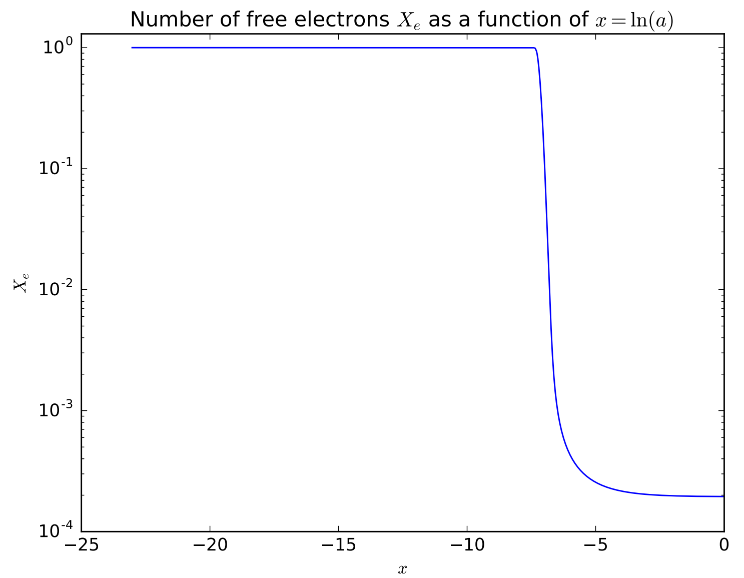
Calculating X_e is done in its own function `Calculate_Xe`. The Saha equation and Peebles' equation has also been split into two functions. When solving Saha equation, we assume that the second order equation takes the form $X_e^2 + BX_e - B = 0$. From this, we use Numpy's function `roots` to solve this second order equation. Peebles' equation is solved as a first order differential equation, using Scipy's `odeint` function, and the initial condition of X_e is the last calculated X_e value from the Saha equation. Both these methods has a lot of constant terms which can be pre-calculated outside the loops (calculated as global constants), which will reduce the overall computation time.

Once we have calculated X_e, we can compute $n_e = X_e n_H$. Note that the number density of the electrons is a function of x, that is $n_e = n_e(x)$. Each value of n_e thus have their corresponding value of x. We have to keep this in mind when calculating τ. We only know the initial condition of τ, which is $\tau(0) = 0$, i.e, zero today. Because of this, we will have to solve τ "backwards" in time, so we will have to assign a new array for x, which we call `x_tau`. This array is the same as `x_eta`, but in reverse order. While calculating τ, we will have to find the n_e value, which corresponds to the correct x value. The calculated τ array is now in reversed order, with respect to `x_eta`, so we will have to reverse the whole array before we calculate the visibility function g.

When we have calculated τ, we can calculate the visibility function $\tilde{g} = -\tau' \exp(-\tau)$. Note that τ' is the interpolated values of τ, and not the one from equation (13). The interpolated values are calculated using the spline functions, explained above. With the interpolated values of τ' and computed values of τ, we use the function `Visibility_func` to compute $\tilde{g}$. The number of points `n_eta` has been increased from 1000 to 3000 to give a better resolution to the visibility function. However, too many points will give some instabilities to the interpolated optical depth τ'.

All sanity checks are done with respect to the results from Callin, in reference [1].

Plots



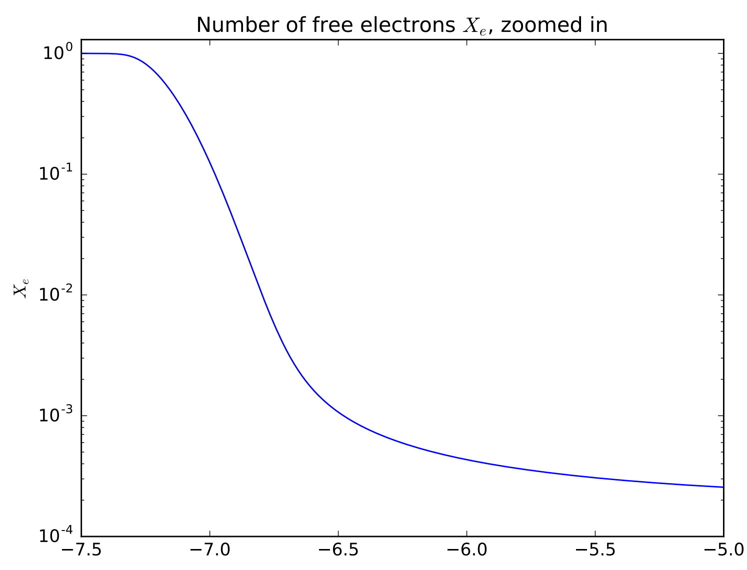


Figure 1: The number of free electrons as a function of $x = \ln a$. The bottom image is a zoomed in segment from $x = -7.5$ to $x = -5$ and it serves as a sanity check with respect to figure 1 in Callin.

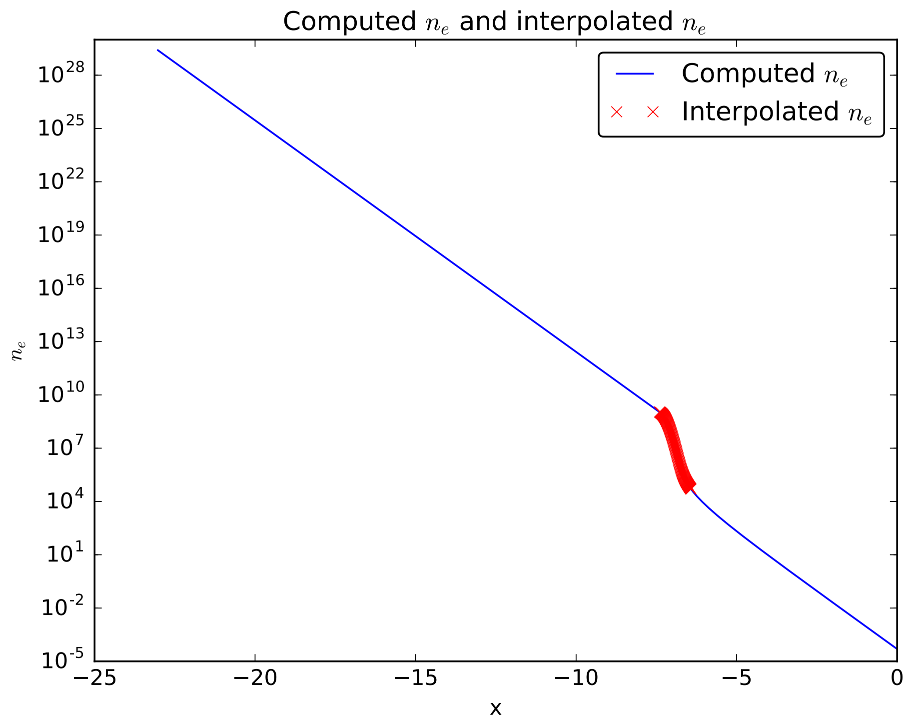


Figure 2: Plot of the number density of the electrons. This plot also serves as a sanity check of the interpolation, to see whether it works or not. We see that the interpolation does its job quite nicely. The interpolated segment is in the time when recombination started to the end of recombination.

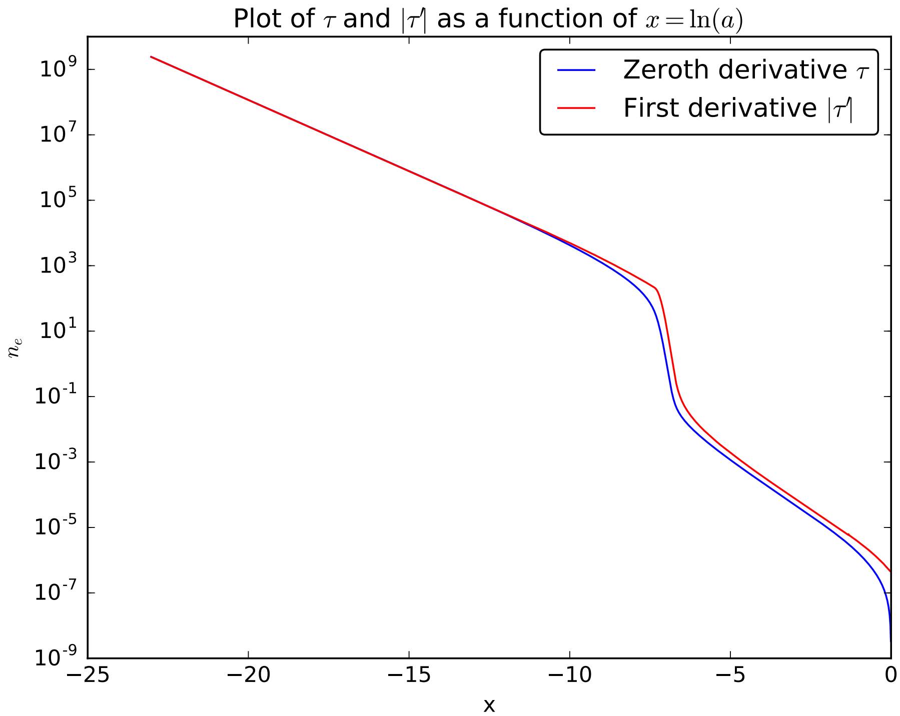


Figure 3: Plot of the computed optical depth τ (blue) and its interpolated derivative $|\tau'|$ (red).

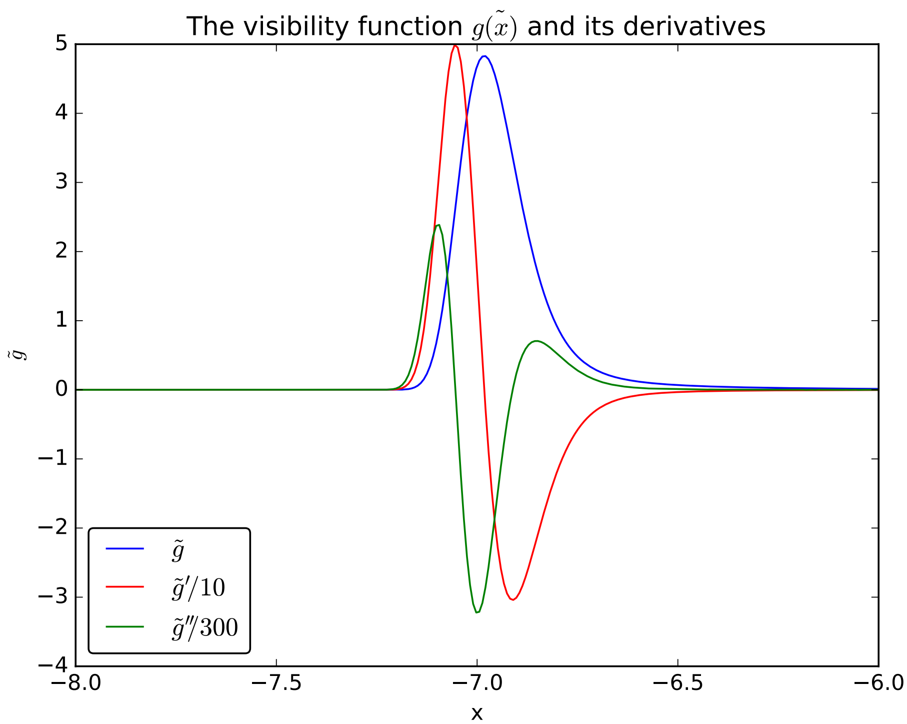


Figure 4: Plot of the visibility function $\tilde{g}(x)$ and its interpolated derivatives. We note the peak of the visibility function around the point $x = -7$.

The code

References

[1] P. Callin, `https://arxiv.org/abs/astro-ph/0606683`.