

Applied Stochastic Processes (FIN 514)

Midterm Exams and Solutions

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2017-18, 2018-19, 2019-20 Academic Years

- **BM** stands for Brownian motion. Assume that B_t , W_t , and Z_t are standard BMs if unless stated otherwise.
 - **RN** and **RV** stand for random number and random variable, respectively.
 - **MC** stands for Monte-Carlo.
 - $P(\cdot)$ and $E(\cdot)$ are probability and expectation, respectively.
 - The PDF and CDF of the standard normal distribution are denoted by $n(z)$ and $N(z)$, respectively.
 - Assume the interest rate and dividend rate are zero in option pricing.
 - **HW** stands for homework and **ME** midterm exam.
1. **[2016ME(StoFin), Generating RNs for correlated BMs]** Throughout this problem, assume that X_t and Y_t are two independent standard BMs.
- (a) Other than the examples we covered in the class, there are many ways to create standard BMs. A linear combination of the two BMs with the coefficients a and b ,

$$W_t = aX_t + bY_t$$

is also a BM. (No need to prove it.) What is the condition for a and b under which W_t is a **standard** BM.

- (b) What is the correlation between X_t and W_t ? We have not defined the correlation of two BMs yet, so simply compute the correlation of the two distributions of the BMs at $t = 1$, i.e, X_1 and W_1 . (In fact, the correlation is same for any time t .) You do not have to use the answer of (a).
- (c) Assume that $\{z_k\}$ for $k = 1, 2, \dots$ is a sequence of standard normal RVs, i.e., $N(0, 1)$, which are generated from computer (e.g., using Box-Muller algorithm). Use $\{z_k\}$ to generate RNs for X_t for a fixed time t .
- (d) Assume that we have two standard BMs, X_t and W_t , which have correlation ρ . How can you generate the pairs of RNs for X_t and W_t for a fixed time t ?

Solution:

(a) $\text{Var}(W_t) = a^2\text{Var}(X_t) + b^2\text{Var}(Y_t) = (a^2 + b^2)t$ should be t . Therefore, $a^2 + b^2 = 1$.

(b)

$$\text{Corr}(W_t, X_t) = \frac{\text{Cov}(X_t, W_t)}{\sqrt{\text{Var}(X_t)\text{Var}(W_t)}} = \frac{at}{\sqrt{t \cdot (a^2 + b^2)t}} = \frac{a}{\sqrt{a^2 + b^2}}$$

(c) $\{\sqrt{t} z_k\}$ is the RNs for X_t .

(d) We can rewrite W_t as $W_t = \rho X_t + \sqrt{1 - \rho^2} Y_t$. Therefore, the random numbers for X_t and W_t can be generated as

$$\begin{aligned} &(\sqrt{t} z_1, \rho\sqrt{t} z_1 + \sqrt{1 - \rho^2}\sqrt{t} z_2) \\ &(\sqrt{t} z_3, \rho\sqrt{t} z_3 + \sqrt{1 - \rho^2}\sqrt{t} z_4) \\ &\quad \dots \\ &(\sqrt{t} z_{2k-1}, \rho\sqrt{t} z_{2k-1} + \sqrt{1 - \rho^2}\sqrt{t} z_{2k}) \end{aligned}$$

2. [2017ME(StoFin), Box–Muller algorithm for generating normal RN] The probability and cumulative distribution functions (PDF and CDF) of exponential RV, Z , are given respectively as

$$f(z) = \lambda e^{-\lambda z}, \quad P(z) = 1 - e^{-\lambda z} \quad \text{for } \lambda > 0, z \geq 0.$$

- (a) If U is a uniform RV, how can you generate the RNs of Z ?
- (b) Let X and Y be two independent standard normal RVs. Show that the squared radius, $Z = X^2 + Y^2$, follows an exponential distribution by computing $P(X^2 + Y^2 < z)$. What is λ ?
- (c) How can you generate the RNs of X and Y from uniform RNs? Hint: introduce another uniform RV, V , and consider the random angle $2\pi V$.

Solution:

- (a) The RN can be generated from the inverse CDF:

$$Z = P^{-1}(U) = -\frac{1}{\lambda} \log(1 - U) \quad \text{or} \quad Z = -\frac{1}{\lambda} \log U,$$

where we use that $1 - U$ is also a uniform RV.

- (b) With the change of variable $r^2 = x^2 + y^2$ and radial symmetry,

$$P(X^2 + Y^2 < z) = \frac{1}{2\pi} \int_{x^2 + y^2 < z} e^{-(x^2 + y^2)/2} dx dy = \frac{1}{2\pi} \int_{r=0}^{\sqrt{z}} e^{-r^2/2} 2\pi r dr = 1 - e^{-z/2}.$$

Therefore Z follows an exponential distribution with $\lambda = 1/2$.

- (c) The RVs, X and Y , can be thought as x - and y -components of \sqrt{Z} with a random angle $2\pi V$. Also, from the results of (a) and (b), the pair (X, Y) is generated by

$$(X, Y) = \sqrt{Z}(\cos(2\pi V), \sin(2\pi V)) = \sqrt{-2 \log U}(\cos(2\pi V), \sin(2\pi V))$$

3. [2017ME, Poisson process] In Poisson process, the CDF for the arrival time t is given as $F(t) = 1 - e^{-\lambda t}$ for the arrival rate λ .

- (a) From a uniform RV, U , generate RN for the **conditional** arrival time t conditional on that the next arrival is after some time t_0 , (i.e., $t > t_0$)

Solution: The RV for unconditional arrival time t can be simulated as

$$t = -(1/\lambda) \log U,$$

where U is a uniform RV. From the memoryless property, t conditional on $t \geq t_0$ can be simulated as

$$t = t_0 - (1/\lambda) \log U.$$

- (b) Assume that the default of a company follows the Poisson process with the arrival rate λ . In the credit default swap (CDS) on the company, party A pays (to B) premium continuously at the rate p (i.e., pays $p dt$ during a time period dt) until the maturity T or the company's default whichever comes first, and party B pays (to A) \$1 when the company defaults. What is the fair premium rate p (which makes the NPVs of both parties equal)? Assume that the risk-free rate is zero, i.e., $r = 0$ (although the problem becomes more interesting if $r > 0$).

Solution:

NPV of party A = NPV of party B

$$\begin{aligned} \int_0^T 1 \cdot \lambda e^{-\lambda t} dt &= \int_0^T p t \cdot \lambda e^{-\lambda t} dt + p T \cdot e^{-\lambda T} \\ 1 - e^{-\lambda T} &= p \left[-t e^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t} \right]_{t=0}^T + p T \cdot e^{-\lambda T} \\ 1 - e^{-\lambda T} &= \frac{p}{\lambda} (1 - e^{-\lambda T}) \end{aligned}$$

Therefore the fair premium value is $p = \lambda$.

4. [2019ME, RN generation] Pareto distribution is defined by the survival function:

$$S(x) = P(X > x) = \begin{cases} \left(\frac{\lambda}{x}\right)^\alpha & (x \geq \lambda) \\ 1 & (x < \lambda). \end{cases}$$

- (a) Find the mean and variance of the distribution. Clearly state the condition that the mean and variance are finite (i.e., not infinite).
- (b) How can you generate the RN following the Pareto distribution from a uniform RN, U ?

Solution:

(a) Based on the PDF of X ,

$$f(x) = \frac{\alpha \lambda^\alpha}{x^{\alpha+1}} \quad \text{for } x \geq \lambda \quad (0 \text{ otherwise}),$$

the mean and variance are computed as

$$E(X) = \frac{\alpha \lambda}{\alpha - 1} \quad \text{for } \alpha > 1 \quad (\infty \text{ otherwise}),$$

$$\text{Var}(X) = \frac{\alpha \lambda^2}{(\alpha - 1)^2(\alpha - 2)} \quad \text{for } \alpha > 2 \quad (\infty \text{ otherwise}).$$

(b) The CDF is easily invertible. From

$$U = 1 - \left(\frac{\lambda}{X}\right)^\alpha \Rightarrow X = \frac{\lambda}{(1 - U)^{1/\alpha}} \quad \text{or} \quad \frac{\lambda}{U^{1/\alpha}}$$

Reference: Pareto Distribution ([WIKIPEDIA](#))

5. **[2019ME, Simulation of multidimensional normal RVs]** Suppose that \mathbf{S}_t is a column vector of three asset prices at time t and that \mathbf{S}_T is distributed as

$$\mathbf{S}_T - \mathbf{S}_0 = \mathbf{L} \mathbf{Z},$$

where \mathbf{Z} is a standard normal RV (column) vector of size 3 and \mathbf{L} is given by

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

(Hint: \mathbf{L} is the lower triangular matrix in Cholesky decomposition.)

- Assuming that $T = 5$, what is the normal volatility of each asset?
- What is the correlation between the 2nd and 3rd asset?
- What is the price of the at-the-money basket call option based on the three assets with equal weight (i.e, 1/3 each)? Assume that the at-the-money option price under the normal volatility σ_N is $0.4 \sigma_N \sqrt{T}$.

Solution: The covariance of the price change is

$$\text{Cov}(\mathbf{S}_T - \mathbf{S}_0) = \mathbf{\Sigma} = \mathbf{L}^T \mathbf{L} = \begin{pmatrix} 1 & -3 & -2 \\ -3 & 25 & 10 \\ -2 & 10 & 9 \end{pmatrix}$$

(a) The diagonal elements are the variances of assets:

$$1 = \sigma_1^2 T, \quad 25 = \sigma_2^2 T, \quad 9 = \sigma_3^2 T.$$

Therefore, the normal volatilities of the assets are

$$\sigma_1 = \sqrt{1/5}, \quad \sigma_2 = \sqrt{5}, \quad \text{and} \quad \sigma_3 = \sqrt{9/5} = 3/\sqrt{5}.$$

(b) $10/(\sqrt{25} \sqrt{9}) = 2/3 \approx 66.6\%$.

(c) From

$$\sigma_N^2 T = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} = 5 \quad \text{for} \quad \mathbf{w} = [1/3, 1/3, 1/3]^T,$$

the basket option price is $0.4\sqrt{5}$.

6. **[2018ME, Simulation of BM path]** Exotic derivatives often depend on the ‘path’ of the underlying stock price. Assume that we need to generate the MC paths of standard BM W_t at $t = 1, 3, 5$, and 9 . We are going to generate the paths using two approaches, which are eventually same. Assume z_k , for $k = 1, \dots, 4$ are independent standard normal RV.
- (a) Using the incremental property of BM, i.e., $W_t - W_s \sim N(0, t - s)$, generate RNs for W_1 , $W_3 - W_1$, $W_5 - W_3$, and $W_9 - W_5$, using z_k ’s. Finally, how can you generate RNs for W_1 , W_3 , W_5 , and W_9 ?
- (b) Now we use covariance matrix approach: Let $\mathbf{\Sigma}$ be the covariance matrix of correlated multivariate normal variables and \mathbf{L} (lower-triangular matrix) be its Cholesky decomposition, which satisfy $\mathbf{\Sigma} = \mathbf{L}\mathbf{L}^T$. Then, the simulation of the normal variables can be obtained as $\mathbf{L}\mathbf{z}$, where \mathbf{z} is the vector of independent standard normal RVs. What is the covariance matrix $\mathbf{\Sigma}$ for our case? (Hint: you may use $\text{Cov}(W_s, W_t) = \min(t, s)$ without proof.)
- (c) From (a) and (b), what is the Cholesky decomposition \mathbf{L} ? Verify that $\mathbf{\Sigma} = \mathbf{L}\mathbf{L}^T$ by direct computation.

Solution:

(a)

$$\begin{array}{ll} W_1 = z_1, & W_1 = z_1, \\ W_3 - W_1 = \sqrt{2}z_2 & \Rightarrow W_3 = z_1 + \sqrt{2}z_2 \\ W_5 - W_3 = \sqrt{2}z_3 & W_5 = z_1 + \sqrt{2}z_2 + \sqrt{2}z_3 \\ W_9 - W_5 = 2z_4 & W_9 = z_1 + \sqrt{2}z_2 + \sqrt{2}z_3 + 2z_4 \end{array}$$

(b)

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix}$$

(c)

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}.$$

$$\mathbf{L}\mathbf{L}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix} = \mathbf{\Sigma}$$

7. **[2018ME, Spread/switch option]** Compute the price of the call option on the spread between two stocks. The payout at maturity T is given as

$$\text{Payout} = \max(S_1(T) - S_2(T), 0).$$

Assume that $S_1(0) = S_2(0) = 100$, $r = q = 0$, $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, and $T = 1$ year. Also assume that the BMs driving the two stocks are correlated by 89%. You may use the following values for $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

Solution: We use Margrabe's formula:

$$C = S_1(0)N(d_1) - S_2(0)N(d_2),$$

$$\text{where } d_{1,2} = \frac{\log(S_1(0)/S_2(0))}{\sigma_R \sqrt{T}} \pm \frac{1}{2} \sigma_R \sqrt{T} \quad \text{and} \quad \sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$

we get

$$\sigma_R = \frac{1}{100} \sqrt{400 + 100 - 2 \times 0.89 \times 200} = 12\%,$$

$$d_1 = \frac{\sigma_R}{2} = 0.06, \quad d_2 = -0.06,$$

$$C = S_0 N(d_1) - K N(d_2) = 100 N(0.06) + 100(1 - N(0.06)) = 4.8$$

8. **[2017ME, Bessel process]** The distribution of the following RV,

$$Q = \|(Z_1 + \mu_1, \dots, Z_n + \mu_n)\|^2 = (Z_1 + \mu_1)^2 + \dots + (Z_n + \mu_n)^2,$$

$$\text{with } \mu = \mu_1^2 + \dots + \mu_n^2$$

where Z_1, \dots, Z_n are independent standard normal RVs, is defined as *non-central chi square* (χ^2 , pronounced as *kai*) distribution with degree n and non-centrality parameter $\mu \geq 0$, denoted by $Q \sim \chi^2(n, \mu)$. Thanks to radial symmetry, the distribution is completely determined by $\mu = \mu_1^2 + \dots + \mu_n^2$. The χ^2 distribution is an important subject of statistics, so the PDF and CDF is well-known although the computation is still challenging from some cases. The degree n can be generalized to any positive real number (i.e., not only integers).

On the other hand, the **squared** Bessel process with dimension n is defined as

$$X_t = \|(B_{1t}, \dots, B_{nt})\|^2 = B_{1t}^2 + \dots + B_{nt}^2$$

where B_{1t}, \dots, B_{nt} are n independent standard BMs. Therefore the distribution of X_t given X_s ($s < t$) follows a scaled non-central χ^2 distribution,

$$X_t = (t - s) Q \quad \text{where} \quad Q \sim \chi^2\left(n, \frac{X_s}{t - s}\right)$$

(No need to prove this for the remaining questions. Just use it.)

- (a) Show that the **squared** Bessel process satisfies

$$dX_t = 2\sqrt{X_t} dW_t + n dt$$

- (b) Show that the Bessel process defined as $R_t = \sqrt{X_t}$ satisfies

$$dR_t = dW_t + \frac{n-1}{2} \frac{dt}{R_t}.$$

- (c) The SDE for the CEV process for $0 < \beta \leq 1$ is given as

$$dS_t = \sigma S_t^\beta dW_t.$$

Show that the CEV process can be reduced to the Bessel process defined in (b). Express the distribution of S_t in terms of S_0 and $Q \sim \chi^2(n, \mu)$. Clearly state the corresponding values for n and μ ? (If σ makes the problem difficult for you, you may assume $\sigma = 1$ to solve the problem. But you will get a partial credit.)

Solution:

- (a) Taking derivative on X_t , we get

$$dX_t = \sum_{k=1}^n \left(2B_{kt} dB_{kt} + \frac{1}{2} \cdot 2dt \right) = 2\sqrt{X_t} dW_t + n dt,$$

where we use $\sum_k B_{kt} dB_{kt} = \sqrt{\sum_k B_{kt}^2} dW_t = \sqrt{X_t} dW_t$ for an independent standard BM W_t .

- (b) Applying Itô's lemma,

$$dR_t = \frac{dX_t}{2\sqrt{X_t}} - \frac{(dX_t)^2}{8X_t\sqrt{X_t}} = \frac{2R_t dW_t + n dt}{2R_t} - \frac{(2R_t dW_t)^2}{8R_t^3} = dW_t + \frac{n-1}{2} \frac{dt}{R_t}.$$

It also imply that the distribution of R_t given R_s ($s < t$) follows

$$R_t = \sqrt{(t-s)Q} \quad \text{where} \quad Q \sim \chi^2\left(n, \frac{R_s^2}{t-s}\right)$$

- (c) We apply Itô's lemma to $Y_t = S_t^{1-\beta}/(1-\beta)$:

$$dY_t = S_t^{-\beta} dS_t + \frac{1}{2}(-\beta S_t^{-1-\beta})(dS_t)^2 = \sigma dW_t - \frac{\beta \sigma^2}{2(1-\beta)} \frac{dt}{Y_t}.$$

The σ can be absorbed to t by introducing the variance $\tau = \sigma^2 t$,

$$dY_{\tau/\sigma^2} = dW_\tau - \frac{\beta}{2(1-\beta)} \frac{d\tau}{Y_{\tau/\sigma^2}}$$

Therefore $Y_{\tau/\sigma^2}/\tau$ follows χ^2 distribution with $\mu = Y_0^2/\tau$ and

$$n = \frac{1-2\beta}{1-\beta} \quad \text{from} \quad \frac{n-1}{2} = -\frac{\beta}{2(1-\beta)} :$$

$$Y_{\tau/\sigma^2} = \sqrt{\tau Q} \quad \text{where} \quad Q \sim \chi^2 \left(\frac{1-2\beta}{1-\beta}, \frac{S_0^{2(1-\beta)}}{(1-\beta)^2 \sigma^2 t} \right).$$

Finally, replacing $\tau = \sigma^2 t$ and $Y_t = S_t^{1-\beta}/(1-\beta)$,

$$\frac{S_t^{1-\beta}}{(1-\beta)} = \sigma \sqrt{t Q} \quad \text{or} \quad S_t = ((1-\beta)^2 \sigma^2 t Q)^{\frac{1}{2(1-\beta)}}$$

Alternatively, the result of the Itô's lemma can be expressed as below by dividing σ :

$$d(Y_t/\sigma) = dW_t - \frac{\beta \sigma^2}{2(1-\beta)} \frac{dt}{Y_t/\sigma},$$

which leads to the same answer:

$$\frac{Y_t}{\sigma} = \frac{S_t^{1-\beta}}{\sigma(1-\beta)} = \sqrt{t Q}$$

9. [2017ME, CIR process] The Cox–Ingersoll–Ross (CIR) process given as

$$dX_t = a(X_\infty - X_t)dt + \sigma \sqrt{X_t} dB_t$$

was originally proposed to model the dynamics of interest rate by Cox, Ingersoll, and Ross. The process was also used to model the variance V_t in the Heston stochastic volatility model:

$$dv_t = \kappa(\theta - v_t)dt + \nu \sqrt{v_t} dZ_t.$$

Applying the similar change of variable used in Ornstein–Uhlenbeck (OU) process, show that the CIR process (either in X_t or V_t) can be represented in terms of the **squared** Bessel process in the 2017ME question above. Clearly state the corresponding dimension n of the squared Bessel process.

Solution: We apply the change of variable, $Y_t = e^{at} X_t$, from the OU process. Then, Y_t satisfy

$$dY_\tau = aX_\infty e^{at} dt + \sqrt{X_t} \sigma e^{at} dB_t = aX_\infty e^{at} dt + 2\sqrt{Y_t} \frac{\sigma e^{at/2}}{2} dB_t.$$

Now we also introduce a new time variable from the variance of the BM,

$$\tau = \int_0^t \left(\frac{\sigma e^{at/2}}{2} \right)^2 ds = \frac{\sigma^2}{4a} (e^{at} - 1), \quad d\tau = \frac{\sigma^2 e^{at}}{4} dt$$

Define $\bar{Y}_\tau = Y_t$, then the process \bar{Y}_τ follows

$$d\bar{Y}_\tau = \frac{4aX_\infty}{\sigma^2} d\tau + 2\sqrt{\bar{Y}_\tau} dB_\tau,$$

which is the squared Bessel process with dimension $n = 4aX_\infty/\sigma^2$. Finally the original process X_t can be expressed in terms of the **squared** Bessel process \bar{Y}_τ with dimension $n = 4aX_\infty/\sigma^2$:

$$X_t = e^{-at} \bar{Y}_{\sigma^2(e^{at}-1)/(4a)}.$$

10. **[2018ME, Euler scheme of the CIR process]** In the Heston stochastic volatility model, the stochastic variance $v(t) = \sigma^2(t)$ follows the SDE:

$$dv(t) = \kappa(\theta - v(t))dt + \nu\sqrt{v(t)} dZ_t.$$

We want to MC simulate $v(T)$ for some T by discretizing time as $t_k = (k/N)T$ for $k = 1, \dots, N$ and $\Delta t = T/N$.

- Write the formula to compute $v(t_{k+1})$ from $v(t_k)$. Assume z is a standard normal RV.
- Instead of simulating V_t , we may consider simulating $\sigma(t) = \sqrt{v(t)}$. Using Itô's lemma, drive the SDE for σ_t .
- From the result of (b), write the formula to update $\sigma(t_{k+1})$ from $\sigma(t_k)$. After replacing $\sigma^2(t)$ with $v(t)$, compare the answer to the result from (a). Are they same?

Solution:

(a)

$$v(t_{k+1}) = v(t_k) + \kappa(\theta - v(t_k))\Delta t + \nu\sqrt{v(t_k)}\Delta t z$$

(b) Applying Itô's lemma, we get

$$\begin{aligned} d\sigma(t) &= d\sqrt{v(t)} = \frac{dv(t)}{2\sigma(t)} - \frac{(dv(t))^2}{8\sigma(t)^3} \\ &= \frac{\kappa(\theta - \sigma(t)^2)dt}{2\sigma(t)} + \frac{\nu}{2}dZ_t - \frac{\nu^2 dt}{8\sigma(t)} \\ &= \frac{4\kappa(\theta - \sigma(t)^2) - \nu^2}{8\sigma(t)}dt + \frac{\nu}{2}dZ_t. \end{aligned}$$

(c) The discretization rule for $\sigma(t)$ is given as

$$\sigma(t_{k+1}) = \sigma(t_k) + \frac{4\kappa(\theta - \sigma(t_k)^2) - \nu^2}{8\sigma(t_k)}\Delta t + \frac{\nu}{2}\sqrt{\Delta t} z.$$

By taking the square of both sides,

$$\begin{aligned} v(t_{k+1}) &= \sigma(t_{k+1})^2 = \left(\sigma(t_k) + \frac{4\kappa(\theta - \sigma(t_k)^2) - \nu^2}{8\sigma(t_k)}\Delta t + \frac{\nu}{2}\sqrt{\Delta t} z \right)^2 \\ &= v(t_k) + \frac{4\kappa(\theta - v(t_k)) - \nu^2}{4}\Delta t + \frac{\nu^2}{4}\Delta t z^2 + \nu\sqrt{v(t_k)}\Delta t z + o(\Delta t) \\ &= v(t_k) + \kappa(\theta - v(t_k))\Delta t + \nu\sqrt{v(t_k)}\Delta t z + \frac{\nu^2}{4}\Delta t (z^2 - 1), \end{aligned}$$

where $o(\Delta t)$ is the terms smaller than Δt in order.

This result is differ from (a) by the two terms in red above. Even after ignoring $o(\Delta t)$, the term $\nu^2\Delta t(z^2 - 1)/4$ remains. So the two discretization methods are different. The discretization method we applied to $v(t)$ and $\sigma(t)$ (that we learned from class) is called Euler-Maruyama method ([WIKIPEDIA](#)). The discretization for $v(t)$ derived via $\sigma(t)$ is called Milstein method ([WIKIPEDIA](#)). If we apply Milstein method to $v(t)$, we directly get the same result. Milstein method is known to be more accurate than Euler-Maruyama method.

11. **[2019ME, Euler and Milstein Schemes of GARCH model]** The variance process for the GARCH diffusion model is given by

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t dZ_t$$

and you want to simulate v_t using time-discretization scheme.

- (a) What is the Euler and Milstein schemes for v_t ? Explicitly write down the expression for $v_{t+\Delta t} - v_t$ using standard normal RV Z_1 .
- (b) The SDE for v_t tells us that v_t cannot go negative. However, in the MC simulation with the time-discretization scheme, v_t sometimes go negative. To avoid this problem, it is better simulate $w_t = \log v_t$ instead. Derive the SDE for w_t .
- (c) What is the Euler and Milstein schemes for w_t ?

Solution:

- (a) The Euler and Milstein schemes for v_t is given by

$$v_{t+\Delta t} - v_t = \kappa(\theta - v_t)\Delta t + \nu v_t Z_1 \sqrt{\Delta t} + \boxed{\frac{\nu^2}{2} v_t (Z_1^2 - 1) \Delta t},$$

where the boxed term is only for the Milstein scheme.

- (b) Applying Itô's lemma, we obtain

$$\begin{aligned} dw_t &= \frac{dv_t}{v_t} - \frac{1}{2} \frac{(dv_t)^2}{v_t^2} = \kappa \left(\frac{\theta}{v_t} - 1 \right) dt + \nu dZ_t - \frac{\nu^2}{2} dt \\ &= (\kappa \theta e^{-w_t} - \kappa - \nu^2/2) dt + \nu dZ_t. \end{aligned}$$

- (c) The Euler and Milstein scheme is same for w_t and they are given by

$$w_{t+\Delta t} - w_t = (\kappa \theta e^{-w_t} - \kappa - \nu^2/2) \Delta t + \nu Z_1 \sqrt{\Delta t}.$$

So it is better to simulate w_t first and obtain $v_t = e^{w_t}$, which is always positive. Also note that the Milstein scheme for v_t in (a) can be recovered by the Taylor expansion of e^x :

$$\begin{aligned} v_{t+\Delta t} &= v_t \exp(w_{t+\Delta t} - w_t) \\ &= v_t \left(1 + \left(\kappa \theta e^{-w_t} - \kappa - \frac{\nu^2}{2} \right) \Delta t + \nu Z_1 \sqrt{\Delta t} + \frac{\nu^2}{2} Z_1^2 \Delta t + o(\Delta t) \right) \\ v_{t+\Delta t} - v_t &= \kappa(\theta - v_t)\Delta t + \nu v_t Z_1 \sqrt{\Delta t} + \frac{\nu^2}{2} v_t (Z_1^2 - 1) \Delta t, \end{aligned}$$

12. **[2019ME, Conditional MC Simulation of OUSV]** We are going to formulate the conditional MC simulation for the Ornstein–Uhlenbeck stochastic volatility (OUSV) model. The processes for the price and volatility under the OUSV model are respectively given by

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma_t dW_t = \sigma_t (\rho dZ_t + \rho_* dX_t) \quad \text{for } \rho_* = \sqrt{1 - \rho^2}, \\ d\sigma_t &= \kappa(\theta - \sigma_t)dt + \nu dZ_t, \end{aligned}$$

where X_t and Z_t are independent standard BMs.

- (a) Derive the SDE for σ_t^2 .
(b) Based on the answer of (a), express S_T in terms of (σ_T, U_T, V_T) where U_T and V_T are give by

$$U_T = \int_0^T \sigma_t dt \quad \text{and} \quad V_T = \int_0^T \sigma_t^2 dt.$$

What are $E(S_T)$ and the BS volatility of S_T conditional on the triplet (σ_T, U_T, V_T) ?

Solution:

- (a) Using Itô's lemma,

$$d\sigma_t^2 = (\nu^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\nu\sigma_t dZ_t.$$

- (b) Integrating the result of (a),

$$\sigma_t^2 - \sigma_0^2 = \nu^2 T + 2\kappa(\theta U_T - V_T) + 2\nu \int_0^T \sigma_t dZ_t.$$

Therefore,

$$\begin{aligned} \log\left(\frac{S_T}{S_0}\right) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \frac{1}{2} V_T \\ &= \frac{\rho}{2\nu} (\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2} T - \frac{\rho\kappa\theta}{\nu} U_T + \left(\frac{\rho\kappa}{\nu} - \frac{1}{2}\right) V_T + \rho_* \sqrt{V_T} X_1 \end{aligned}$$

and we obtain

$$\begin{aligned} E(S_T) &= S_0 \exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2} V_T\right) \\ &= S_0 \exp\left(\frac{\rho}{2\nu} (\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2} T - \frac{\rho\kappa\theta}{\nu} U_T + \left(\frac{\rho\kappa}{\nu} - \frac{\rho_*^2}{2}\right) V_T\right) \\ \text{Vol}(S_T) &= \rho_* \sqrt{V_T/T}. \end{aligned}$$

Reference: Li, C., Wu, L., 2019. **Exact simulation of the Ornstein–Uhlenbeck driven stochastic volatility model**. European Journal of Operational Research 275, 768–779. <https://doi.org/10.1016/j.ejor.2018.11.057>