Applied Stochastic Processes (FIN 514)
Midterm Exams and Solutions

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2017-18 M1, 2018-19 M1, 2019-20 M1, 2020-21 M3

- **BM** stands for Brownian motion. Assume that B_t, W_t, and Z_t are standard BMs if unless stated otherwise.
- **RN** and **RV** stand for random number and random variable, respectively.
- **MC** stands for Monte–Carlo.
- $P(\cdot)$ and $E(\cdot)$ are probability and expectation, respectively.
- The PDF and CDF of the standard normal distribution are denoted by $n(z)$ and $N(z)$, respectively.
- Assume the interest rate and dividend rate are zero in option pricing.
- **HW** stands for homework and **ME** midterm exam.

1. [**2016ME(StoFin), Generating RNs for correlated BMs**] Throughout this problem, assume that X_t and Y_t are two independent standard BMs.
   (a) Other than the examples we covered in the class, there are many ways to create standard BMs. A linear combination of the two BMs with the coefficients a and b,
   $$W_t = aX_t + bY_t$$
   is also a BM. (No need to prove it.) What is the condition for a and b under which W_t is a **standard** BM.
   (b) What is the correlation between X_t and W_t? We have not defined the correlation of two BMs yet, so simply compute the correlation of the two distributions of the BMs at $t = 1$, i.e, X_1 and W_1. (In fact, the correlation is same for any time t.) You do not have to use the answer of (a).
   (c) Assume that $\{z_k\}$ for $k = 1, 2, \cdots$ is a sequence of standard normal RVs, i.e., $N(0, 1)$, which are generated from computer (e.g., using Box–Muller algorithm). Use $\{z_k\}$ to generate RNs for X_t for a fixed time t.
   (d) Assume that we have two standard BMs, X_t and W_t, which have correlation ρ. How can you generate the pairs of RNs for X_t and W_t for a fixed time t?

Solution:

(a) $\mathrm{Var}(W_t) = a^2\mathrm{Var}(X_t) + b^2\mathrm{Var}(Y_t) = (a^2+b^2)t$ should be t. Therefore, $a^2+b^2=1$.

(b)
$$\mathrm{Corr}(W_t, X_t) = \frac{\mathrm{Cov}(X_t, W_t)}{\sqrt{\mathrm{Var}(X_t)\mathrm{Var}(W_t)}} = \frac{at}{\sqrt{t\cdot(a^2+b^2)t}} = \frac{a}{\sqrt{a^2+b^2}}$$

(c) $\{\sqrt{t}\,z_k\}$ is the RNs for X_t.

(d) We can rewrite W_t as $W_t = \rho X_t + \sqrt{1-\rho^2}Y_t$. Therefore, the random numbers for X_t and W_t can be generated as

$$\begin{gathered}(\sqrt{t}\,z_1,\ \rho\sqrt{t}\,z_1 + \sqrt{1-\rho^2}\sqrt{t}\,z_2)\\ (\sqrt{t}\,z_3,\ \rho\sqrt{t}\,z_3 + \sqrt{1-\rho^2}\sqrt{t}\,z_4)\\ \cdots\\ (\sqrt{t}\,z_{2k-1},\ \rho\sqrt{t}\,z_{2k-1} + \sqrt{1-\rho^2}\sqrt{t}\,z_{2k})\end{gathered}$$

2. **[2017ME(StoFin), Box–Muller algorithm for generating normal RN]** The probability and cumulative distribution functions (PDF and CDF) of exponential RV, Z, are given respectively as

$$f(z) = \lambda e^{-\lambda z}, \quad P(z) = 1 - e^{-\lambda z} \quad \text{for} \quad \lambda > 0, z \geq 0.$$

(a) If U is a uniform RV, how can you generate the RNs of Z?

(b) Let X and Y be two independent standard normal RVs. Show that the squared radius, $Z = X^2+Y^2$, follows an exponential distribution by computing $P(X^2+Y^2 < z)$. What is λ?

(c) How can you generate the RNs of X and Y from uniform RNs? Hint: introduce another uniform RV, V, and consider the random angle $2\pi V$.

Solution:

(a) The RN can be generated from the inverse CDF:

$$Z = P^{-1}(U) = -\frac{1}{\lambda}\log(1-U) \quad \text{or} \quad Z = -\frac{1}{\lambda}\log U,$$

where we use that $1-U$ is also a uniform RV.

(b) With the change of variable $r^2 = x^2+y^2$ and radial symmetry,

$$P(X^2+Y^2 < z) = \frac{1}{2\pi}\int_{x^2+y^2<z} e^{-(x^2+y^2)/2}dxdy = \frac{1}{2\pi}\int_{r=0}^{\sqrt{z}} e^{-r^2/2}2\pi r\,dr = 1 - e^{-z/2}.$$

Therefore Z follows an exponential distribution with $\lambda = 1/2$.

(c) The RVs, X and Y, can be thought as x- and y-components of $\sqrt{Z}$ with a random angle $2\pi V$. Also, from the results of (a) and (b), the pair (X,Y) is generated by

$$(X,Y) = \sqrt{Z}(\cos(2\pi V), \sin(2\pi V)) = \sqrt{-2\log U}(\cos(2\pi V), \sin(2\pi V))$$

3. **[2017ME, Poisson process]** In Poisson process, the CDF for the arrival time t is given as $F(t) = 1 - e^{-\lambda t}$ for the arrival rate λ.

   (a) From a uniform RV, U, generate RN for the **conditional** arrival time t conditional on that the next arrival is after some time t_0, (i.e., $t > t_0$)

   Solution: The RV for unconditional arrival time t can be simulated as

   $$t = -(1/\lambda)\log U,$$

   where U is a uniform RV. From the memoryless property, t conditional on $t \geq t_0$ can be simulated as

   $$t = t_0 - (1/\lambda)\log U.$$

   (b) Assume that the default of a company follows the Poisson process with the arrival rate λ. In the credit default swap (CDS) on the company, party A pays (to B) premium continuously at the rate p (i.e., pays $p\,dt$ during a time period dt) until the maturity T or the company's default whichever comes first, and party B pays (to A) \$1 when the company defaults. What is the fair premium rate p (which makes the NPVs of both parties equal)? Assume that the risk-free rate is zero, i.e., $r = 0$ (although the problem becomes more interesting if $r > 0$).

   Solution:

   $$\begin{aligned}
   \text{NPV of party A} &= \text{NPV of party B} \\
   \int_0^T 1 \cdot \lambda e^{-\lambda t} dt &= \int_0^T pt \cdot \lambda e^{-\lambda t} dt + pT \cdot e^{-\lambda T} \\
   1 - e^{-\lambda T} &= p\left[-t\,e^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t}\right]_{t=0}^{T} + pT \cdot e^{-\lambda T} \\
   1 - e^{-\lambda T} &= \frac{p}{\lambda}(1 - e^{-\lambda T})
   \end{aligned}$$

   Therefore the fair premium value is $p = \lambda$.

4. **[2019ME, RN generation]** Pareto distribution is defined by the survival function:

   $$S(x) = P(X > x) = \begin{cases} \left(\frac{\lambda}{x}\right)^{\alpha} & (x \geq \lambda) \\ 1 & (x < \lambda). \end{cases}$$

   (a) Find the mean and variance of the distribution. Clearly state the condition that the mean and variance are finite (i.e., not infinite).

   (b) How can you generate the RN following the Pareto distribution from a uniform RN, U ?

   Solution:

(a) Based on the PDF of X,

$$f(x) = \frac{\alpha\lambda^\alpha}{x^{\alpha+1}} \quad \text{for} \quad x \geq \lambda \quad (0 \quad \text{otherwise}),$$

the mean and variance are computed as

$$E(X) = \frac{\alpha\lambda}{\alpha - 1} \quad \text{for} \quad \alpha > 1 \quad (\infty \quad \text{otherwise}),$$

$$\text{Var}(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} \quad \text{for} \quad \alpha > 2 \quad (\infty \quad \text{otherwise}).$$

(b) The CDF is easily invertible. From

$$U = 1 - \left(\frac{\lambda}{X}\right)^\alpha \quad \Rightarrow \quad X = \frac{\lambda}{(1-U)^{1/\alpha}} \quad \text{or} \quad \frac{\lambda}{U^{1/\alpha}}$$

Reference: Pareto Distribution (WIKIPEDIA)

5. **[2020ME, RN generation]** A gamma RV, $X \sim \text{Gamma}(k, \beta)$, is distrusted by the PDF,

$$f_X(x) = \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} \quad \text{for} \quad \Gamma(k) = (k-1)\cdots 2 \cdot 1 \; (\Gamma(1) = 1),$$

where k is a positive integer and $X \geq 0$.

(a) Find the mean and variance of X. **Hint:** $\int_0^\infty f_X(x)dx = 1$ for any k.

(b) How can you generate the RV of $X \sim \text{Gamma}(1, \beta)$?

(c) If $X \sim \text{Gamma}(1, \beta)$, $X' \sim \text{Gamma}(k, \beta)$, and X and X' are independent, find the PDF of $Y = X + X'$.

(d) How can we generate the RV of $\text{Gamma}(k, \beta)$?

Solution:

(a)

$$E(X) = \int_0^\infty x \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx = \frac{k}{\beta} \int_0^\infty \frac{\beta^{k+1}}{\Gamma(k+1)} x^k e^{-\beta x} dx = \frac{k}{\beta}$$

$$E(X^2) = \int_0^\infty x^2 \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx = \frac{k(k+1)}{\beta^2} \int_0^\infty \frac{\beta^{k+2}}{\Gamma(k+2)} x^{k+1} e^{-\beta x} dx = \frac{k(k+1)}{\beta^2}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{k}{\beta^2}$$

(b) When $k = 1$, X has the same PDF as the exponential distribution with $\lambda = \beta$:

$$f_X(x) = \beta e^{-\beta x}.$$

Therefore, we can generate X by

$$X = -\frac{1}{\beta}\log U \quad \text{or} \quad -\frac{1}{\beta}\log(1-U),$$

where U is a uniform RV.

(c) **Method 1:**

$$\begin{aligned} f_Y(y) &= \int_{x=0}^{y} f_X(y-x) f_{X'}(x)dx = \int_{x=0}^{y} \beta e^{-\beta(y-x)} \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x} dx \\ &= \frac{\beta^{k+1}}{\Gamma(k)} e^{-\beta y} \int_{x=0}^{y} x^{k-1} dx = \frac{\beta^{k+1}}{\Gamma(k)} e^{-\beta y} \frac{y^k}{k} = \frac{\beta^{k+1}}{\Gamma(k+1)} y^k e^{-\beta y}. \end{aligned}$$

Therefore, Y follows $\text{Gamma}(k+1, \beta)$.

Method 2: The MGF of X' is

$$E\left(e^{-tX'}\right) = \int_0^\infty \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-(\beta+t)x} = \frac{\beta^k}{(\beta+t)^k} = (1+t/\beta)^{-k},$$

where we used the hint of (a) for $\beta' = \beta + t$. It follows that the MGF of X is $(1+t/\beta)^{-1}$. Since X and X' are independent,

$$E\left(e^{-tY}\right) = E\left(e^{-tX}\right) E\left(e^{-tX'}\right) = (1+t/\beta)^{-1}(1+t/\beta)^{-k} = (1+t/\beta)^{-(k+1)}.$$

Therefore, we know that $Y \sim \text{Gamma}(k+1, \beta)$.

Method 3: $X' \sim \text{Gamma}(k, \beta)$ is the RV for the k-th arrival time of the Poisson-type events with intensity β. Because the events are memoryless, $X' + X$ is the $(k+1)$-th arrival time and it is $\text{Gamma}(k+1, \beta)$.

(d) From (b), $\text{Gamma}(k, \beta) \sim X_1 + \cdots X_k$, where X_i's are independent Gamma variables following $\text{Gamma}(1, \beta)$. Therefore,

$$X = -\frac{1}{\beta}\log(U_1 \cdots U_k),$$

where U_k are the sequence of uniform RVs.

6. **[2019ME, Simulation of multidimensional normal RVs]** Suppose that $\boldsymbol{S}_t$ is a column vector of three asset prices at time t and that $\boldsymbol{S}_T$ is distributed as

$$\boldsymbol{S}_T - \boldsymbol{S}_0 = \boldsymbol{L}\,\boldsymbol{Z},$$

where $\boldsymbol{Z}$ is a standard normal RV (column) vector of size 3 and $\boldsymbol{L}$ is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

(Hint: L is the lower triangular matrix in Cholesky decomposition.)

(a) Assuming that $T = 5$, what is the normal volatility of each asset?

(b) What is the correlation between the 2nd and 3rd asset?

(c) What is the price of the at-the-money basket call option based on the three assets with equal weight (i.e, 1/3 each)? Assume that the at-the-money option price under the normal volatility σ_{N} is $0.4\,\sigma_{\text{N}}\sqrt{T}$.

Solution: The covariance of the price change is

$$\mathrm{Cov}(\boldsymbol{S}_T - \boldsymbol{S}_0) = \boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T = \begin{pmatrix} 1 & -3 & -2 \\ -3 & 25 & 10 \\ -2 & 10 & 9 \end{pmatrix}$$

(a) The diagonal elements are the variances of assets:

$$1 = \sigma_1^2 T, \quad 25 = \sigma_2^2 T, \quad 9 = \sigma_3^2 T.$$

Therefore, the normal volatilities of the assets are

$$\sigma_1 = \sqrt{1/5}, \quad \sigma_2 = \sqrt{5}, \quad \text{and} \quad \sigma_3 = \sqrt{9/5} = 3/\sqrt{5}.$$

(b) $10/(\sqrt{25}\,\sqrt{9}) = 2/3 \approx 66.6\%$.

(c) From

$$\sigma_{\text{N}}^2 T = \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} = 5 \quad \text{for} \quad \mathbf{w} = [1/3, 1/3, 1/3]^T,$$

the basket option price is $0.4\sqrt{5}$.

7. **[2018ME, Simulation of BM path]** Exotic derivatives often depend on the 'path' of the underlying stock price. Assume that we need to generate the MC paths of standard BM W_t at $t = 1, 3, 5$, and 9. We are going to generate the paths using two approaches, which are eventually same. Assume z_k, for $k = 1, \cdots, 4$ are independent standard normal RV.

   (a) Using the incremental property of BM, i.e., $W_t - W_s \sim N(0, t-s)$, generate RNs for W_1, $W_3 - W_1$, $W_5 - W_3$, and $W_9 - W_5$, using z_k's. Finally, how can you generate RNs for W_1, W_3, W_5, and W_9?

   (b) Now we use covariance matrix approach: Let $\boldsymbol{\Sigma}$ be the covariance matrix of correlated multivariate normal variables and $\boldsymbol{L}$ (lower-triangular matrix) be its Cholesky decomposition, which satisfy $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$. Then, the simulation of the normal variables can obtained as $\boldsymbol{L}\boldsymbol{z}$, where $\boldsymbol{z}$ is the vector of independent standard normal RVs. What is the covariance matrix $\boldsymbol{\Sigma}$ for our case? (Hint: you may use $\mathrm{Cov}(W_s, W_t) = \min(t, s)$ without proof.)

   (c) From (a) and (b), what is the Cholesky decomposition $\boldsymbol{L}$? Verify that $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$ by direct computation.

Solution:

(a)

$$\begin{aligned} W_1 &= z_1, \\ W_3 - W_1 &= \sqrt{2} z_2 \\ W_5 - W_3 &= \sqrt{2} z_3 \\ W_9 - W_5 &= 2 z_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} W_1 &= z_1, \\ W_3 &= z_1 + \sqrt{2} z_2 \\ W_5 &= z_1 + \sqrt{2} z_2 + \sqrt{2} z_3 \\ W_9 &= z_1 + \sqrt{2} z_2 + \sqrt{2} z_3 + 2 z_4 \end{aligned}$$

(b)

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix}$$

(c)

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}.$$

$$\boldsymbol{L}\boldsymbol{L}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix} = \boldsymbol{\Sigma}$$

8. **[2020ME, Simulation of correlated normal RVs]** The tri-variate normal variable $\boldsymbol{X}$ has the following mean and covariance. How can you simulate RNs for X?

$$\mu = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 16 & 4 \\ -2 & 4 & 9 \end{pmatrix}$$

Solution: First, we obtain the Cholesky decomposition of Σ. We find a lower triangular matrix L such that $LL^T = \Sigma$. After some algebra, we get

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

Therefore, X is simulated by $X = \mu + LZ$ where Z is the independent standard normal random variable of size 3.

9. **[2018ME, Spread/switch option]** Compute the price of the call option on the spread between two stocks. The payout at maturity T is given as

$$\text{Payout} = \max(S_1(T) - S_2(T), 0).$$

Assume that $S_1(0) = S_2(0) = 100$, $r = q = 0$, $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, and $T = 1$ year. Also assume that the BMs driving the two stocks are correlated by 89%. You may use the following values for $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

Solution: We use Margrabe's formula:

$$C = S_1(0)N(d_1) - S_2(0)N(d_2),$$
$$\text{where} \quad d_{1,2} = \frac{\log(S_1(0)/S_2(0))}{\sigma_R\sqrt{T}} \pm \frac{1}{2}\sigma_R\sqrt{T} \quad \text{and} \quad \sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$

we get

$$\sigma_R = \frac{1}{100}\sqrt{400 + 100 - 2 \times 0.89 \times 200} = 12\%,$$
$$d_1 = \frac{\sigma_{\mathrm{R}}}{2} = 0.06, \quad d_2 = -0.06,$$
$$C = S_0 N(d_1) - KN(d_2) = 100N(0.06) + 100(1 - N(0.06)) = 4.8$$

10. **[2017ME, Bessel process]** The distribution of the following RV,

$$Q = \|(Z_1 + \mu_1, \cdots, Z_n + \mu_n)\|^2 = (Z_1 + \mu_1)^2 + \cdots + (Z_n + \mu_n)^2,$$
$$\text{with} \quad \mu = \mu_1^2 + \cdots \mu_n^2$$

where $Z_1, \cdots, Z_n$ are independent standard normal RVs, is defined as *non-central chi square* (χ^2, pronounced as *kai*) distribution with degree n and non-centrality parameter $\mu \geq 0$, denoted by $Q \sim \chi^2(n, \mu)$. Thanks to radial symmetry, the distribution is completely determined by $\mu = \mu_1^2 + \cdots + \mu_n^2$. The χ^2 distribution is an important subject of statistics, so the PDF and CDF is well-known although the computation is still challenging from some cases. The degree n can be generalized to any positive real number (i.e., not only integers).

On the other hand, the **squared** Bessel process with dimension n is defined as

$$X_t = \|(B_{1t}, \cdots, B_{nt})\|^2 = B_{1t}^2 + \cdots + B_{nt}^2$$

where $B_{1t}, \cdots, B_{nt}$ are n independent standard BMs. Therefore the distribution of X_t given X_s ($s < t$) follows a scaled non-central χ^2 distribution,

$$X_t = (t - s)\, Q \quad \text{where} \quad Q \sim \chi^2\left(n,\ \frac{X_s}{t - s}\right)$$

(No need to prove this for the remaining questions. Just use it.)

(a) Show that the **squared** Bessel process satisfies

$$dX_t = 2\sqrt{X_t}\, dW_t + n\, dt$$

(b) Show that the Bessel process defined as $R_t = \sqrt{X_t}$ satisfies

$$dR_t = dW_t + \frac{n-1}{2}\frac{dt}{R_t}.$$

(c) The SDE for the CEV process for $0 < \beta \le 1$ is given as

$$dS_t = \sigma\, S_t^\beta\, dW_t.$$

Show that the CEV process can be reduced to the Bessel process defined in (b). Express the distribution of S_t in terms of S_0 and $Q \sim \chi^2(n, \mu)$. Clearly state the corresponding values for n and μ? (If σ makes the problem difficult for you, you may assume $\sigma = 1$ to solve the problem. But you will get a partial credit.)

Solution:

(a) Taking derivative on X_t, we get

$$dX_t = \sum_{k=1}^{n} \left(2B_{kt}\, dB_{kt} + \frac{1}{2}\cdot 2dt \right) = 2\sqrt{X_t}\, dW_t + n\, dt,$$

where we use $\sum_k B_{kt}\, dB_{kt} = \sqrt{\sum_k B_{kt}^2}\, dW_t = \sqrt{X_t}\, dW_t$ for an independent standard BM W_t.

(b) Applying Itô's lemma,

$$dR_t = \frac{d\,X_t}{2\sqrt{X_t}} - \frac{(dX_t)^2}{8X_t\sqrt{X_t}} = \frac{2R_t\, dW_t + n\, dt}{2R_t} - \frac{(2R_t\, dW_t)^2}{8R_t^3} = dW_t + \frac{n-1}{2}\frac{dt}{R_t}.$$

It also imply that the distribution of R_t given R_s $(s < t)$ follows

$$R_t = \sqrt{(t-s)\, Q} \quad \text{where} \quad Q \sim \chi^2\left(n,\ \frac{R_s^2}{t-s}\right)$$

(c) We apply Itô's lemma to $Y_t = S_t^{1-\beta}/(1-\beta)$:

$$dY_t = S_t^{-\beta} dS_t + \frac{1}{2}(-\beta S_t^{-1-\beta})(dS_t)^2 = \sigma dW_t - \frac{\beta\,\sigma^2}{2(1-\beta)}\frac{dt}{Y_t}.$$

The σ can be absorbed to t by introducing the variance $\tau = \sigma^2 t$,

$$dY_{\tau/\sigma^2} = dW_\tau - \frac{\beta}{2(1-\beta)}\frac{d\tau}{Y_{\tau/\sigma^2}}$$

Therefore $Y_{\tau/\sigma^2}/\tau$ follows χ^2 distribution with $\mu = Y_0^2/\tau$ and

$$n = \frac{1-2\beta}{1-\beta} \quad \text{from} \quad \frac{n-1}{2} = -\frac{\beta}{2(1-\beta)}:$$

$$Y_{\tau/\sigma^2} = \sqrt{\tau Q} \quad \text{where} \quad Q \sim \chi^2\left(\frac{1-2\beta}{1-\beta},\ \frac{S_0^{2(1-\beta)}}{(1-\beta)^2\,\sigma^2\, t}\right).$$

Finally, replacing $\tau = \sigma^2 t$ and $Y_t = S_t^{1-\beta}/(1-\beta)$,

$$\frac{S_t^{1-\beta}}{(1-\beta)} = \sigma\sqrt{t\,Q} \quad \text{or} \quad S_t = \left((1-\beta)^2\sigma^2 t\; Q\right)^{\frac{1}{2(1-\beta)}}$$

Alternatively, the result of the Itô's lemma can be expressed as below by dividing σ:

$$d(Y_t/\sigma) = dW_t - \frac{\beta\,\sigma^2}{2(1-\beta)}\frac{dt}{Y_t/\sigma},$$

which leads to the same answer:

$$\frac{Y_t}{\sigma} = \frac{S_t^{1-\beta}}{\sigma(1-\beta)} = \sqrt{t\,Q}$$

11. **[2017ME, CIR process]** The Cox–Ingersoll–Ross (CIR) process given as

$$dX_t = a(X_\infty - X_t)dt + \sigma\sqrt{X_t}\,dB_t$$

was originally proposed to model the dynamics of interest rate by Cox, Ingersoll, and Ross. The process was also used to model the variance v_t in the Heston stochastic volatility model:

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dZ_t.$$

Applying the similar change of variable used in Ornstein–Uhlenbeck (OU) process, show that the CIR process (either in X_t or v_t) can be represented in terms of the **squared** Bessel process in the 2017ME question above. Clearly state the corresponding dimension n of the squared Bessel process.

Solution: We apply the change of variable, $Y_t = e^{at}X_t$, from the OU process. Then, Y_t satisfy

$$dY_\tau = aX_\infty e^{at}\,dt + \sqrt{X_t}\,\sigma e^{at}\,dB_t = aX_\infty e^{at}\,dt + 2\sqrt{Y_t}\,\frac{\sigma e^{at/2}}{2}\,dB_t.$$

Now we also introduce a new time variable from the variance of the BM,

$$\tau = \int_0^t \left(\frac{\sigma e^{at/2}}{2}\right)^2 ds = \frac{\sigma^2}{4a}(e^{at}-1), \quad d\tau = \frac{\sigma^2 e^{at}}{4}dt$$

Define $\bar{Y}_\tau = Y_t$, then the process $\bar{Y}_\tau$ follows

$$d\bar{Y}_\tau = \frac{4aX_\infty}{\sigma^2}\,d\tau + 2\sqrt{Y_t}\,dB_\tau,$$

which is the squared Bessel process with dimension $n = 4aX_\infty/\sigma^2$. Finally the original process X_t can be expressed in terms of the **squared** Bessel process $\bar{Y}_\tau$ with dimension $n = 4aX_\infty/\sigma^2$:

$$X_t = e^{-at}\,\bar{Y}_{\sigma^2(e^{at}-1)/(4a)}.$$

12. **[2018ME, Euler scheme of the CIR process]** In the Heston stochastic volatility model, the stochastic variance $v(t) = \sigma^2(t)$ follows the SDE:

$$dv(t) = \kappa(\theta - v(t))dt + \nu\sqrt{v(t)}\, dZ_t.$$

We want to MC simulate $v(T)$ for some T by discretizing time as $t_k = (k/N)T$ for $k = 1, \cdots, N$ and $\Delta t = T/N$.

(a) Write the formula to compute $v(t_{k+1})$ from $v(t_k)$. Assume z is a standard normal RV.

(b) Instead of simulating V_t, we may consider simulating $\sigma(t) = \sqrt{v(t)}$. Using Itô's lemma, drive the SDE for σ_t.

(c) From the result of (b), write the formula to update $\sigma(t_{k+1})$ from $\sigma(t_k)$. After replacing $\sigma^2(t)$ with $v(t)$, compare the answer to the result from (a). Are they same?

Solution:

(a)

$$v(t_{k+1}) = v(t_k) + \kappa(\theta - v(t_k))\Delta t + \nu\sqrt{v(t_k)\Delta t}\, z$$

(b) Applying Itô's lemma, we get

$$\begin{aligned}
d\sigma(t) = d\sqrt{v(t)} &= \frac{d\,v(t)}{2\sigma(t)} - \frac{(dv(t))^2}{8\sigma(t)^3} \\
&= \frac{\kappa(\theta - \sigma(t)^2)dt}{2\sigma(t)} + \frac{\nu}{2}dZ_t - \frac{\nu^2 dt}{8\sigma(t)} \\
&= \frac{4\kappa(\theta - \sigma(t)^2) - \nu^2}{8\sigma(t)}dt + \frac{\nu}{2}dZ_t.
\end{aligned}$$

(c) The discretization rule for $\sigma(t)$ is given as

$$\sigma(t_{k+1}) = \sigma(t_k) + \frac{4\kappa(\theta - \sigma(t_k)^2) - \nu^2}{8\sigma(t_t)}\Delta t + \frac{\nu}{2}\sqrt{\Delta t}\, z.$$

By taking the square of both sides,

$$\begin{aligned}
v(t_{k+1}) = \sigma(t_{k+1})^2 &= \left(\sigma(t_k) + \frac{4\kappa(\theta - \sigma(t_k)^2) - \nu^2}{8\sigma(t_k)}\Delta t + \frac{\nu}{2}\sqrt{\Delta t}\, z\right)^2 \\
&= v(t_k) + \frac{4\kappa(\theta - v(t_k)) - \nu^2}{4}\Delta t + \frac{\nu^2}{4}\Delta t\, z^2 + \nu\sqrt{v(t_k)\Delta t}\, z + o(\Delta t) \\
&= v(t_k) + \kappa(\theta - v(t_k))\Delta t + \nu\sqrt{v(t_k)\Delta t}\, z + \frac{\nu^2}{4}\Delta t\,(z^2 - 1),
\end{aligned}$$

where $o(\Delta t)$ is the terms smaller than Δt in order.

This result is differ from (a) by the two terms in red above. Even after ignoring $o(\Delta t)$, the term $\nu^2\Delta t\,(z^2 - 1)/4$ remains. So the two discretization methods are different. The discretization method we applied to $v(t)$ and $\sigma(t)$ (that we learned from class) is called Euler-Maruyama method (WIKIPEDIA). The discretization for $v(t)$ derived via $\sigma(t)$ is called Milstein method (WIKIPEDIA). If we apply Milstein method to $v(t)$, we directly get the same result. Milstein method is known to be more accurate than Euler-Maruyama method.

13. **[2019ME, Euler and Milstein Schemes of GARCH model]** The variance process for the GARCH diffusion model is given by

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t\, dZ_t$$

and you want to simulate v_t using time-discretization scheme.

(a) What is the Euler and Milstein schemes for v_t? Explicitly write down the expression for $v_{t+\Delta t} - v_t$ using standard normal RV Z_1.

(b) The SDE for v_t tells us that v_t cannot go negative. However, in the MC simulation with the time-discretization scheme, v_t sometimes go negative. To avoid this problem, it is better simulate $w_t = \log v_t$ instead. Derive the SDE for w_t.

(c) What is the Euler and Milstein schemes for w_t?

Solution:

(a) The Euler and Milstein schemes for v_t is given by

$$v_{t+\Delta t} - v_t = \kappa(\theta - v_t)\Delta t + \nu v_t Z_1 \sqrt{\Delta t} + \boxed{\frac{\nu^2}{2} v_t (Z_1^2 - 1)\Delta t}\,,$$

where the boxed term is only for the Milstein scheme.

(b) Applying Itô's lemma, we obtain

$$\begin{aligned} dw_t &= \frac{dv_t}{v_t} - \frac{1}{2}\frac{(dv_t)^2}{v_t^2} = \kappa\left(\frac{\theta}{v_t} - 1\right) dt + \nu dZ_t - \frac{\nu^2}{2} dt \\ &= \left(\kappa\theta e^{-w_t} - \kappa - \nu^2/2\right) dt + \nu dZ_t. \end{aligned}$$

(c) The Euler and Milstein scheme is same for w_t and they are given by

$$w_{t+\Delta t} - w_t = \left(\kappa\theta e^{-w_t} - \kappa - \nu^2/2\right) \Delta t + \nu Z_1 \sqrt{\Delta t}.$$

So it is better to simulate w_t first and obtain $v_t = e^{w_t}$, which is always positive. Also note that the Milstein scheme for v_t in (a) can be recovered by the Taylor expansion of e^x:

$$\begin{aligned} v_{t+\Delta t} &= v_t\ \exp\left(w_{t+\Delta t} - w_t\right) \\ &= v_t \left(1 + \left(\kappa\theta e^{-w_t} - \kappa - \frac{\nu^2}{2}\right) \Delta t + \nu Z_1 \sqrt{\Delta t} + \frac{\nu^2}{2} Z_1^2 \Delta t + o(\Delta t)\right) \\ v_{t+\Delta t} - v_t &= \kappa(\theta - v_t)\Delta t + \nu v_t Z_1 \sqrt{\Delta t} + \frac{\nu^2}{2} v_t (Z_1^2 - 1)\Delta t\ , \end{aligned}$$

14. **[2020ME, Euler and Milstein Schemes]** The stochastic differential equation for the constant-elasticity-of-variance (CEV) model is given by

$$dS_t = \sigma S_t^\beta dW_t \quad (0 \le \beta \le 1).$$

Find the Euler and Milstein schemes for obtaining $S_{t+\Delta t}$ from S_t.

Solution: For a standard normal RV, W_1, the Milstein scheme for the CEV model is given by

$$\begin{aligned}S_{t+\Delta t} &= S_t + \sigma S_t^{\beta} W_1\sqrt{\Delta t} + \frac{\sigma S_t^{\beta}\cdot\sigma\beta S_t^{\beta-1}}{2}(W_1^2-1)\Delta t\\ &= S_t + \sigma S_t^{\beta} W_1\sqrt{\Delta t} + \frac{1}{2}\sigma^2\beta S_t^{2\beta-1}(W_1^2-1)\Delta t.\end{aligned}$$

15. **[2019ME, Conditional MC Simulation of OUSV]** We are going to formulate the conditional MC simulation for the Ornstein–Uhlenbeck stochastic volatility (OUSV) model. The processes for the price and volatility under the OUSV model are respectively given by

$$\frac{dS_t}{S_t} = \sigma_t dW_t = \sigma_t(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2},$$
$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu\, dZ_t,$$

where X_t and Z_t are independent standard BMs.

(a) Derive the SDE for σ_t^2.

(b) Based on the answer of (a), express S_T in terms of (σ_T, U_T, V_T) where U_T and V_T are give by

$$U_T = \int_0^T \sigma_t\, dt \quad \text{and} \qquad V_T = \int_0^T \sigma_t^2\, dt.$$

What are $E(S_T)$ and the BS volatility of S_T conditional on the triplet (σ_T, U_T, V_T)?

Solution:

(a) Using Itô's lemma,

$$d\sigma_t^2 = (\nu^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\nu\sigma_t dZ_t.$$

(b) Integrating the result of (a),

$$\sigma_t^2 - \sigma_0^2 = \nu^2 T + 2\kappa(\theta U_T - V_T) + 2\nu\int_0^T \sigma_t dZ_t.$$

Therefore,

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho\int_0^T \sigma_t dZ_t + \rho_*\int_0^T \sigma_t dX_t - \frac{1}{2}V_T\\ &= \frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{1}{2}\right)V_T + \rho_*\sqrt{V_T}\,X_1\end{aligned}$$

and we obtain

$$\begin{aligned}E(S_T) &= S_0\exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2}V_T\right)\\ &= S_0\exp\left(\frac{\rho}{2\nu}(\sigma_T^2-\sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{\rho^2}{2}\right)V_T\right)\\ \mathrm{Vol}(S_T) &= \rho_*\sqrt{V_T/T}.\end{aligned}$$

Reference: Li, C., Wu, L., 2019. **Exact simulation of the Ornstein–Uhlenbeck driven stochastic volatility model**. European Journal of Operational Research 275, 768–779. https://doi.org/10.1016/j.ejor.2018.11.057

16. **[2020ME, Conditional MC Simulation]** We are going to formulate the conditional MC simulation for the GARCH diffusion model. The SDEs for the price and volatility under the GARCH diffusion model are given by

$$\frac{dS_t}{S_t} = \sqrt{v_t}(\rho dZ_t + \rho_* dX_t) \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2},$$
$$dv_t = \kappa(\theta - v_t)dt + \nu v_t\, dZ_t$$

where X_t and Z_t are independent standard BMs.

(a) Derive the SDE for $\sigma_t = \sqrt{v_t}$.

(b) Based on the answer of (a), express S_T in terms of σ_T, Y_T, U_T, V_T, and a standard normal RV X_1, where Y_T, U_T and V_T are given by

$$Y_T = \int_0^T \frac{1}{\sigma_t}\, dt, \quad U_T = \int_0^T \sigma_t\, dt \quad \text{and} \quad V_T = \int_0^T \sigma_t^2\, dt.$$

(c) What are $E(S_T)$ and the BS volatility of S_T conditional on the quadruplet $(\sigma_T, Y_T, U_T, V_T)$?

Solution:

(a) Using Itô's lemma,

$$\begin{aligned}
d\sigma_t = d\sqrt{v_t} &= \frac{1}{2}\frac{dv_t}{\sqrt{v_t}} - \frac{1}{8}\frac{(dv_t)^2}{v_t\sqrt{v_t}} \\
&= \frac{1}{2}\kappa\left(\frac{\theta}{\sigma_t} - \sigma_t\right)dt + \frac{\nu}{2}\sigma_t dZ_t - \frac{\nu^2}{8}\sigma_t dt \\
&= \frac{1}{2}\left(\frac{\kappa\theta}{\sigma_t} - \left(\kappa + \frac{\nu^2}{4}\right)\sigma_t\right)dt + \frac{\nu}{2}\sigma_t dZ_t
\end{aligned}$$

(b) Integrating the result of (a),

$$\sigma_T - \sigma_0 = \frac{1}{2}\left(\kappa\theta\, Y_T - \left(\kappa + \frac{\nu^2}{4}\right)U_T\right) + \frac{\nu}{2}\int_0^T \sigma_t dZ_t$$
$$\int_0^T \sigma_t dZ_t = \frac{2}{\nu}(\sigma_T - \sigma_0) - \left(\frac{\kappa\theta}{\nu}\, Y_T - \left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right)U_T\right)$$

Therefore,

$$\begin{aligned}
\log\left(\frac{S_T}{S_0}\right) &= \rho\int_0^T \sigma_t dZ_t + \rho_*\int_0^T \sigma_t dX_t - \frac{1}{2}V_T \\
&= \frac{2\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho\kappa\theta}{\nu}\, Y_T + \rho\left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right)U_T - \frac{1}{2}V_T + \rho_*\sqrt{V_T}\, X_1.
\end{aligned}$$

(c) Accordingly, we obtain

$$
\begin{aligned}
E(S_T|\sigma_T, Y_T, U_T, V_T) &= S_0 \exp\left(E\left(\log\left(\frac{S_T}{S_0}\right)\right) + \frac{\rho_*^2}{2} V_T\right) \\
&= S_0 \exp\left(\frac{2\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho\kappa\theta}{\nu} Y_T + \rho\left(\frac{\kappa}{\nu} + \frac{\nu}{4}\right) U_T - \frac{\rho^2}{2} V_T\right) \\
\sigma_{\text{BS}} &= \rho_* \sqrt{V_T/T}.
\end{aligned}
$$