Hyperbolic normal stochastic volatility (NSVh) model
Applied Stochastic Processes (FIN 514)

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Motivation 1: 'Normal' is more normal for some asset classes

- Daily price changes may not be proportional to the price level
- Underlying price can be negative: interest rate, inflation rate, spread, etc.
- Two swaption volatility smiles (in implied normal vs BSM volatility):

- Swaptions are quoted and risk-managed under 'normal' volatility (bp vol). Merrill Lynch Option Volatility Estimate (MOVE) index is a weighted average of the 'normal' implied volatility on 1m Treasury options.
- Therefore, a need for stochastic volatility model with normal backbone arises. The literature is rare except for the normal case of SABR model.

Motivation 1: A tale of two backbones

Model	Normal	Lognormal
Reference	Bachelier (1900)	Black and Scholes (1973)
SDE	Arithmetic BM: $dF_t = \sigma dW_t$	Geometric BM: $dF_t/F_t = \sigma dW_t$
Asset class	Interest rate, Inflation, Spread	Equity, FX
Call option price	$(F-K)N(d) + \sigma\sqrt{T}\,n(d)$ $d = \frac{(F-K)}{\sigma\sqrt{T}}$	$F\,N(d_1) - K\,N(d_2)$ $d_{1,2} = \frac{\log(F_0/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$
Volatility conversion	$\sigma_{\text{N}} \approx F_0\,\sigma_{\text{BS}}$	
Digital, $P(F_T > K)$	$N(d)$	$N(d_2)$
Delta $(\partial/\partial F_0)$	$N(d)$	$N(d_1)$
Gamma $(\partial^2/\partial F_0^2)$	$n(d)/\sigma\sqrt{T}$	$n(d_1)/F_0\sigma\sqrt{T}$
Vega $(\partial/\partial\sigma)$	$\sqrt{T}\,n(d)$	$F_0\sqrt{T}\,n(d_1)$
Implied Volatility	Choi et al. (2009)	Jäckel (2015)

Motivation 1: normal SABR model

- Stochastic-alpha-beta-rho (SABR) model (Hagan et al., 2002)

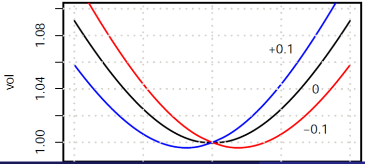
$$\frac{dF_t}{F_t^\beta} = \sigma_t\left(\rho dZ_t + \rho_* dX_t\right) \quad \text{and} \quad \frac{d\sigma_t}{\sigma_t} = \alpha\, dZ_t \quad \text{for} \quad 0 \le \beta \le 1,\ \rho^2 + \rho_*^2 = 1$$

- One of the most popular SV models used in financial engineering.
  - range of backbone choice: $0 \le \beta \le 1$
  - parsimonious and intuitive parameters: ρ and α (and β)
  - asymptotic approximation for the implied vol for $\beta = 0$ (accurate for $\alpha\sqrt{T} \ll 1$) (Hagan et al., 2002)

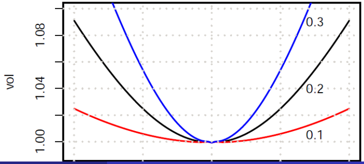
$$\sigma_N = \sigma_0 \left(\frac{\zeta}{\chi(\zeta)}\right)\left(1 + \frac{2-3\rho^2}{24}\alpha^2 T\right)$$

$$\text{where} \quad \zeta = \frac{\alpha}{\sigma_0}(F_0 - K) \quad \text{and} \quad \chi(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2} - \rho + \zeta}{1-\rho}\right)$$

ρ: slope(skew)



α: curvature(smile)



Motivation 2: normal SV for heavy-tailed distribution?

- Most heavy-tailed distributions lack dynamics (SDE).
- Heavy-tailed distribution associated with normal SV model?
  E.g., normal/lognormal distribution vs arithmetic/geometric BM

Johnson's distribution family

Johnson (1949) proposed a family of transformation from standard normal distribution:

$$\frac{X-\gamma_X}{\delta_X} = f\left(\frac{Z-\gamma_Z}{\delta_Z}\right) \text{ where } f(x) = \begin{cases} \frac{1}{1+e^{-x}} & \text{for } S_B \text{ (bounded) family} \\ x & \text{for } S_N \text{ (normal) family} \\ \sinh x & \text{for } S_U \text{ (unbounded) family} \\ e^x & \text{for } S_L \text{ (log-normal) family} \end{cases}$$

The S_U (unbounded) Johnson family is an increasingly popular choice for modeling distributions with heavy tail and skewness. (Simonato, 2011; Corlu and Corlu, 2015; Choi and Nam, 2008; Gurrola, 2007; Venkataraman and Rao, 2016)

- PDF/CDF in closed-forms
- wide range of skewness and ex-kurtosis to explain
- trivial random number (RN) draw

Contribution

Hyperbolic normal stochastic volatility (NSVh) model brings together the normal SABR model ($\lambda = 0$) and Johnson's S_U distribution ($\lambda = 1$).

Based on the two generalizations (Alili and Gruet, 1997) of Bougerol's identity, we equip NSVh model with unprecedented analytic tools:

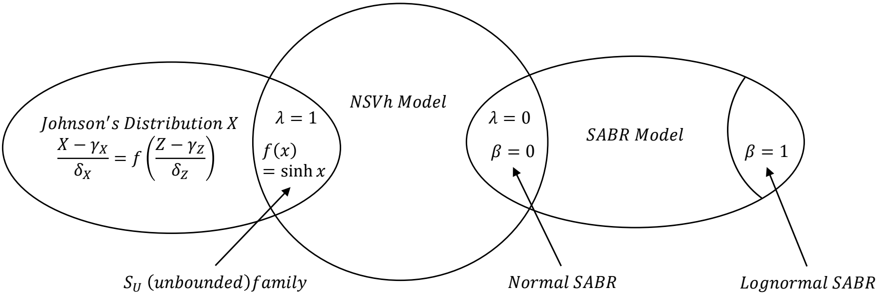
- closed-form (exact and one-step) MC scheme (1 RN draw per 1.5 normal RNs)

For Johnson's S_U distribution ($\lambda = 1$), we provide

- theoretical foundation as a solution of SDE
- intuitive re-parametrization
- possibility for substituting normal SABR ($\lambda = 0$) with various closed-form solutions (option price, VaR, ES, etc).

Contribution

The Overview of the NSVh Model in Relation to Other Previously Known Models and Distributions.



Hyperbolic normal stochastic volatility (NSVh) model

- SDE:

$$dF_t = \sigma_t \left(\rho\, dZ_t^{[\lambda\alpha/2]} + \rho_*\, dX_t\right), \quad \frac{d\sigma_t}{\sigma_t} = \alpha\, dZ_t^{[\lambda\alpha/2]}$$

  where $\rho_* = \sqrt{1-\rho^2}$, $Z_t^{[\mu]} = Z_t + \mu\, t$, and X_t and Z_t are independent BMs.

- The vol process is solved to $\sigma_t = \sigma_0 \exp\left(\alpha Z_t^{[\frac{1}{2}(\lambda-1)\alpha]}\right)$
- The power $(\sigma_t)^{1-\lambda}$ is a martingale:

$$\frac{d(\sigma_t)^{1-\lambda}}{(\sigma_t)^{1-\lambda}} = (1-\lambda)\alpha\, dZ_t \quad (\lambda \neq 1), \quad d(\log \sigma_t) = \alpha\, dZ_t \quad (\lambda = 1).$$

- The price process F_t is a martingale only when $\lambda = 0$, which is normal SABR model. The case $\lambda \neq 0$ is understood as a perturbation from the baseline case.
- However, using $\bar{F}_T = F_0 + (\sigma_0 \rho/\alpha)\left(e^{\frac{1}{2}\lambda\alpha^2 T} - 1\right)$,

$$F_T - \bar{F}_T = \frac{\rho}{\alpha}\left(\sigma_T - e^{\frac{1}{2}\lambda\alpha^2 T}\sigma_0\right) + \rho_* \int_0^T \sigma_t dX_t$$

- After changes of variables,

$$s = \alpha^2 t \; (S = \alpha^2 T), \quad \tilde{\sigma}_s = \frac{\sigma_t}{\sigma_0}, \quad \text{and} \quad \tilde{F}_s = \frac{\alpha}{\sigma_0}\big(F_t - \bar{F}_T\big),$$

  the canonical form is given as

$$d\tilde{F}_s = \tilde{\sigma}_s(\rho \, dZ_s^{[\lambda/2]} + \rho_* \, dX_s) \quad \text{and} \quad d\tilde{\sigma}_s = \tilde{\sigma}_s \, dZ_s^{[\lambda/2]} \quad (\tilde{\sigma}_0 = 1),$$

$$\tilde{\sigma}_S = \exp\left(Z_S^{[\frac{1}{2}(\lambda-1)]}\right) \quad \text{and} \quad \tilde{F}_S \overset{d}{=} \rho\left(e^{Z_S^{[\frac{1}{2}(\lambda-1)]}} - e^{\frac{1}{2}\lambda S}\right) + \rho_* \, X_{A_S^{[\frac{1}{2}(\lambda-1)]}}.$$

- Distribution parameterized by $(\bar{F}_T, \sigma_0/\alpha, S = \alpha^2 T, \rho, \lambda)$
- Motivated from continuously averaged Asian option, the exponential functional of BM has been an important subject of researches. (Matsumoto and Yor, 2005a,b; Yor, 2012).

$$A_T^{[\mu]} = \int_{t=0}^{T} e^{2Z_t^{[\mu]}} dt$$

  In the context of NSVh (SABR), $A_S^{[\frac{1}{2}(\lambda-1)]} = \int_0^S \tilde{\sigma}_s^2 ds$ is the time-integrated variance under the stochastic volatility.

Main results

Bougerol's identity (Bougerol, 1983)

For the independent BMs, X_t, Z_t and W_t, the followings are equal in law ($\overset{d}{=}$)

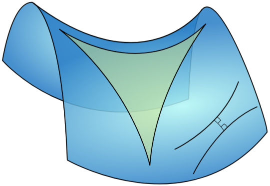
$$\int_0^T e^{Z_t} dX_t \overset{d}{=} X_{A_T^{[0]}} \overset{d}{=} \sinh(W_T) \quad \text{for any fixed time } T.$$

(See the generalized proof in the 2nd extension.)

Our main results are based on two generalization of Bougerol's identity by Alili and Gruet (1997), *An explanation of a generalized Bougerol's identity in terms of hyperbolic Brownian motion*, in *Exponential Funtionals and Principal Values related to Brownian Motion* and Alili et al. (1997).

- BM in hyperbolic geometry (all λ: $Z^{[\mu]}$, $A_T^{[\mu]}$)
- BM with an arbitrary starting point ($\lambda = 1$)

Hyperbolic geometry: 2-d and 3-d Poincaré half-plane



- Geometry with constant negative curvature (e.g., saddle)
- Poincaré disc model [Escher, 1959]: $r < 1$ with ∞ corresponding to $r = 1$.
- Poincaré half-plane model (Dunham, 2012): unit length $\propto z$

Dimension	$\mathbb{H}_2 = \{(x, z) : z > 0\}$	$\mathbb{H}_3 = \{(x, y, z) : z > 0\}$
Metric $(ds)^2$	$(dx^2 + dz^2)/z^2$	$(dx^2 + dy^2 + dz^2)/z^2$
Volume element dV	$dx\, dz/z^2$	$dx\, dy\, dz/z^3$
Geodesic distance D	$\operatorname{acosh}\left(\frac{(x'-x)^2+z^2+z'^2}{2zz'}\right)$	$\operatorname{acosh}\left(\frac{(x'-x)^2+(y'-y)^2+z'^2+z^2}{2zz'}\right)$
Laplace-Beltrami operator Δ_H	$z^2\,(\partial_x^2 + \partial_z^2)$	$z^2\,(\partial_x^2 + \partial_y^2 + \partial_z^2) - z\,\partial_z$
Standard BM	$dx_t = z_t dX_t$, $dz_t/z_t = dZ_t$	$dx_t = z_t dX_t$, $dy_t = z_t dY_t$, $dz_t/z_t = dZ_t - dt/2$
Heat kernel $p_n(t, D)$ for $n = 2$ or 3 $(\partial_t - \frac{1}{2}\Delta_{\mathbb{H}_n})p_n = 0$	$\frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}} \int_D^\infty ds \frac{se^{-s^2/2t}}{\sqrt{\cosh s - \cosh D}}$	$\frac{1}{(2\pi t)^{3/2}} \frac{D}{\sinh D} e^{-(t^2+D^2)/2t}$

The 1st extension: BM in hyperbolic geometry (all λ)

Proposition 3 in Alili and Gruet (1997)

For the independent BMs X_t and Z_t, the followings are equal in law

$$\int_0^T e^{Z_t^{[\mu]}} dX_t \overset{d}{=} X_{A_T^{[\mu]}} \overset{d}{=} \cos\theta\, \phi\left(Z_T^{[\mu]}, \sqrt{R_T^2 + (Z_T^{[\mu]})^2}\right)$$

$$\text{where} \quad \phi(Z, D) = e^{Z/2}\sqrt{2\cosh D - 2\cosh Z} \quad \text{for} \quad Z \le D.$$

θ is a uniform random angle, R_t is a 2-d Bessel process (the radius of a 2-d BM, $R_t = \sqrt{X_t^2 + Y_t^2}$). The equality in law ($\overset{d}{=}$) even holds conditional on $Z_T^{[\mu]}$.

Sketch of Proof:

- The proof is related to a drifted 3-d hyperbolic BM (x_t, y_t, z_t) started at $(0, 0, 1)$:

$$dx_t = z_t\, dX_t, \quad \text{and} \quad dy_t = z_t\, dY_t \quad (x_0 = y_0 = 0),$$

$$\frac{dz_t}{z_t} = dZ_t + \left(\frac{1}{2} + \mu\right) dt \quad (z_0 = 1).$$

Sketch of Proof

- Let D_t be the hyperbolic distance between (x_t, y_t, z_t) and the starting point $(0, 0, 1)$,

$$D_t = \operatorname{acosh}\left(\frac{1}{2}\left(\frac{x_t^2+y_t^2}{z_t} + z_t + \frac{1}{z_t}\right)\right) \quad \text{or} \quad r_t = \sqrt{x_t^2 + y_t^2} = \phi(Z_t^{[\mu]}, D_t)$$

  $\phi(\cdot)$ is understood as the Euclidean radius of (x_t, y_t).
- For a fixed time T, the hyperbolic distance D_t and Euclidean distance of $(X_T, Y_T, Z_T^{[\mu]})$ from $(0, 0, 0)$ have same distribution conditional on Z_T (or z_T):

$$D_T \stackrel{d}{=} \sqrt{X_T^2 + Y_T^2 + (Z_T^{[\mu]})^2}$$

  We have a proof using $\mathbb{H}_3$ heat kernel, which is more straight-forward than the original proof by Alili and Gruet (1997).
- The result immediately follows

$$\sqrt{x_T^2 + y_T^2} \stackrel{d}{=} \phi\left(Z_T^{[\mu]},\ \sqrt{X_T^2 + Y_T^2 + (Z_T^{[\mu]})^2}\right)$$

$$x_T \stackrel{d}{=} X_{A_T^{[\mu]}} \stackrel{d}{=} \cos\theta\ \phi\left(Z_T^{[\mu]},\ \sqrt{X_T^2 + Y_T^2 + (Z_T^{[\mu]})^2}\right).$$

Closed-form price transition and MC scheme (all λ)

The joint transition of price and volatility from time 0 to S

$$\tilde{\sigma}_S = \exp\left(Z_S^{[\frac{1}{2}(\lambda-1)]}\right),$$

$$\tilde{F}_S \overset{d}{=} \rho\left(e^{Z_S^{[\frac{1}{2}(\lambda-1)]}} - e^{\frac{1}{2}\lambda S}\right) + \rho_* \cos\theta\, \phi\left(Z_S^{[\frac{1}{2}(\lambda-1)]}, \sqrt{R_S^2 + (Z_S^{[\frac{1}{2}(\lambda-1)]})^2}\right).$$

MC simulation

The transition is simulated with three standard normal RNs, X_1, Y_1 and Z_1,

$$(Z_S, R_S^2,\ \cos\theta) \overset{d}{=} \left(Z_1\sqrt{S},\ (X_1^2 + Y_1^2)S,\ \frac{X_1 \text{ (or } Y_1)}{\sqrt{X_1^2 + Y_1^2}}\right),$$

- Exact and one-step jump from $s = S_1$ to $s = S_2$: no Euler method, no bias!
- One RN draw for 1.5 normal RNs.
- For normal SABR, more efficient than the scheme of Cai et al. (2017).

The 2nd extension: an arbitrary starting point ($\lambda = 1$)

Proposition 4 in Alili and Gruet (1997)

$$e^{Z_T}\sinh(a) + \int_0^T e^{Z_t} dX_t \stackrel{d}{=} \sinh(a + W_T) \quad \text{for any fixed time } T.$$

Remark: The original Bougerol's identity is just a special case with $a = 0$.

Following the proof in Alili et al. (1997), the two processes,

$$P_t = \sinh(W_t + a) \quad \text{and} \quad Q_t = e^{Z_t}\left(\sinh(a) + \int_0^t e^{-Z_s} dX_s\right),$$

are equal in law as they have the same SDE with the same initial condition $P_0 = Q_0 = \sinh(a)$,

$$dP_t = \frac{1}{2}P_t dt + \sqrt{1 + P_t^2}\, dW_t \quad \text{and}$$
$$dQ_t = \frac{1}{2}Q_t dt + dX_t + Q_t\, dZ_t \stackrel{d}{=} \frac{1}{2}Q_t dt + \sqrt{1 + Q_t^2}\, dW_t.$$

The invariance of BM under time reversal $Z_T - Z_{T-s} \leftrightarrow Z_s$ completes the proof.

Closed-form price transition and MC scheme ($\lambda = 1$)

Price transition given by Johnson's S_U distribution

$$\tilde{F}_S \stackrel{d}{=} \rho_* \sinh\left(W_S + \operatorname{atanh}\rho\right) - \rho\, e^{S/2}$$
$$= \sinh(W_S) + \rho \left(\cosh(W_S) - e^{S/2}\right).$$

- Johnson's S_U distribution is the solution of NSVh with $\lambda = 1$.
- S_U re-parametrized with intuitive SV parameters:

$$\delta_Z = \frac{1}{\sqrt{S}}, \quad \frac{\gamma_Z}{\delta_Z} = -\operatorname{atanh}\rho, \quad \delta_X = \frac{\sigma_0 \rho_*}{\alpha}, \text{ and } \gamma_X = \bar{F}_T - \delta_X e^{S/2}\left(\frac{\rho}{\rho_*}\right).$$

- Two limiting cases: lognormal (S_L) for $\rho = \pm 1$ and normal (S_N) as $S \to 0$.
- Clear split between heavy-tail ($\sinh$) and skewness ($\cosh$) components.
- Closed-form simulation from $s = 0$ to S. One RN draw per 1 normal RN.
- Level of analytic tractability same as that of BSM model.
- The distributions are similar for different λ if the parameters are calibrated to the same observations (e.g., moments or volatility smiles)

Various analytics for NSVh $\lambda = 1$

- PDF, $f_{\lambda=1}(x)$, and CDF, $F_{\lambda=1}(x)$:

$$f_{\lambda=1}(x) = \frac{n(d)}{\rho_* \sigma_0 \sqrt{T}\sqrt{1+\xi^2}}, \quad F_{\lambda=1}(x) = N(-d)$$

$$\text{where} \quad \xi = \frac{\alpha}{\rho_* \sigma_0}(\bar{F}_T - x) - \frac{\rho}{\rho_*} e^{S/2}, \quad d = \frac{1}{\sqrt{S}}\left(\operatorname{asinh}\xi + \operatorname{atanh}\rho\right)$$

- The forward price of the call/put ($\pm$) option struck at K:

$$V_\pm = \frac{\sigma_0\, e^{S/2}}{2\alpha}\Big((1+\rho)N(d+\sqrt{S}) - (1-\rho)N(d-\sqrt{S}) - 2\rho N(d)\Big) \pm \big(\bar{F}_T - K\big)\, N(\pm d)$$

$$\text{where} \quad d = \frac{1}{\sqrt{S}}\left(\operatorname{asinh}\left(\frac{\alpha}{\rho_*\sigma_0}(\bar{F}_T - K) - \frac{\rho}{\rho_*} e^{S/2}\right) + \operatorname{atanh}\rho\right)$$

- Value-at-Risk (VaR):

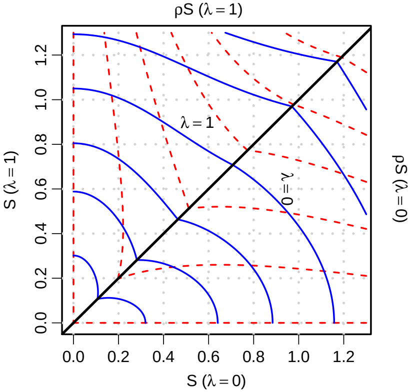
$$\text{VaR}(q) = \bar{F}_T + \frac{\sigma_0}{\alpha}\left(\rho_*\, \sinh\left(\sqrt{S} Z_q + \operatorname{atanh}\rho\right) - \rho\, e^{S/2}\right) \quad \text{for} \quad Z_q = N^{-1}(q)$$

- Expected Shortfall (ES):

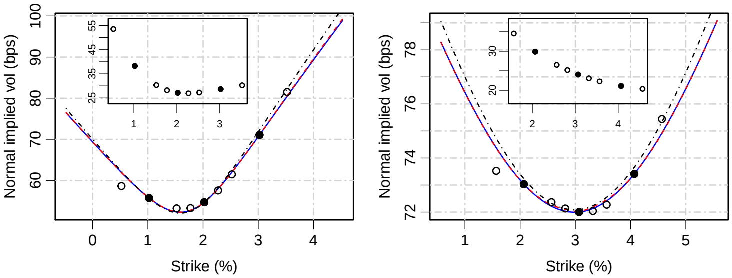
$$\text{ES}(q) = \bar{F}_T + \frac{\sigma_0 e^{S/2}}{2\alpha q}\Big((1+\rho)N(Z_q - \sqrt{S}) - (1-\rho)N(Z_q + \sqrt{S}) - 2\rho q\Big) \text{ for } Z_q = N^{-1}(q)$$

Moments of NSVh distribution

Contours of skewness and ex-kurtosis for N-SABR ($\lambda = 0$) and S_U ($\lambda = 1$)



- Like S_U, NSVh generates all possible combinations of skewness and positive ex-kurtosis. In particular, ex-kurtosis is unbounded for a given skewness.
- The moments, skewness and kurtosis have closed-forms for all λ.
- The moment matching can be simplified to 1-d root finding for $\lambda = 1$ (Tuenter, 2001) and $\lambda = 0$. The parameters can be quickly converted between $\lambda = 0$ and 1 for the same moments.



- Swaption smiles for 1y1y and 10y10y in normal volatility observed on March 2017. The indistinguishable solid lines (blue and red) are from NSVh with $\lambda = 0$ and 1 while the dotted lines (black) are from Hagan's approximation.
- Calibrated parameters:

Parameters	1y1y ($T = 1$)		10y10y ($T = 10$)	
	$\lambda = 0$	$\lambda = 1$	$\lambda = 0$	$\lambda = 1$
ρ (%)	35.280	32.244	1.811	1.580
α (%)	64.634	62.181	23.535	22.196
σ_0 (%)	0.532	0.477	0.689	0.609
$\bar{F}_T$ (%)	2.0221		3.0673	

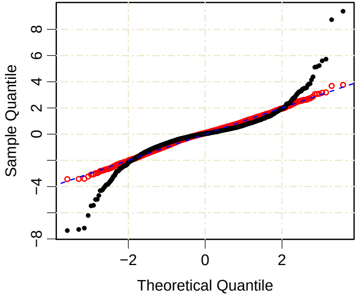
Empirical study 1 (cont'd): Vanilla option price

- Vanilla Options Pricing Tested Against the Parameters for 10y10y Swaption. Analytic prices (P_{ANA}) are computed from Hagan's approximation and exact S_U analytics and MC price (P_{MC}) is shown relative to P_{ANA} with the standard deviation. The MC simulation with 10^6 paths is repeated 100 times. Prices are in the unit of the annuity of the underlying swap.

$K-\bar{F}_T$ (bps)	$\lambda=0$		$\lambda=1$	
	P_{ANA}	$P_{\text{MC}}-P_{\text{ANA}}$	P_{ANA}	$P_{\text{MC}}-P_{\text{ANA}}$
-200	2.275E-2	-4.1E-5 $\pm$ 1.8E-5	2.274E-2	6.4E-7 $\pm$ 1.8E-5
-100	1.506E-2	-2.1E-5 $\pm$ 1.6E-5	1.506E-2	6.0E-7 $\pm$ 1.6E-5
0	9.083E-3	-1.2E-5 $\pm$ 1.3E-5	9.083E-3	5.3E-7 $\pm$ 1.3E-5
100	5.108E-3	-2.6E-5 $\pm$ 1.1E-5	5.108E-3	3.3E-7 $\pm$ 1.1E-5
200	2.807E-3	-4.8E-5 $\pm$ 8.8E-6	2.804E-3	3.9E-7 $\pm$ 9.1E-6
300	1.567E-3	-6.0E-5 $\pm$ 7.0E-6	1.559E-3	3.7E-7 $\pm$ 7.3E-6

Empirical study 2: Distribution of stock index return

- Daily returns of S&P 500 index (left) and China 300 ETF index (right) since 2005.
- Fit $\lambda = 0$ (N-SABR) and $\lambda = 1$ (S_U) distributions to the first 4 moments.
- Normal probability Q-Q plots of raw (black) and asinh-transformed (red) sample quantile.
- Distributions with $\lambda = 0$ and $\lambda = 1$ indistinguishable.



Conclusion

Hyperbolic normal stochastic volatility (NSVh) model brings together the normal SABR model ($\lambda = 0$) and Johnson's S_U distribution ($\lambda = 1$).

Based on the two generalizations (Alili and Gruet, 1997) of Bougerol's identity, we equip NSVh model with unprecedented analytic tools:

- closed-form (exact and one-step) MC scheme (1 RN draw per 1.5 normal RNs)

For Johnson's S_U distribution ($\lambda = 1$), we provide

- theoretical foundation as a solution of SDE
- intuitive re-parametrization
- possibility for substituting normal SABR ($\lambda = 0$) with various closed-form solutions (option price, VaR, ES, etc).

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