Stochastic Volatility Models and Simulation
Applied Stochastic Processes (FIN 514)

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Stochastic Volatility (SV) Models

- The price process (martingale):

$$\frac{dS_t}{S_t^\beta} = \sigma_t dW_t = \sigma_t(\rho dZ_t + \rho_* dX_t), \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2}.$$

  BSM-base: $\beta = 1$, normal-base: $\beta = 0$.
  For the models except SABR, the base model is BSM (i.e., $\beta = 1$).

- The (stochastic) volatility process may vary:

$$d\sigma_t = a(t, \sigma_t)dt + b(t, \sigma_t)dZ_t,$$

- The correlation between the two Brownian motions:

$$dW_t\, dZ_t = \rho\, dt.$$

  The correlation explains the *leverage effect*: equity volatility increases as price goes down.

Various SV models

- SABR model (?):

$$d\sigma_t = \nu\sigma_t \, dZ_t.$$

- ? model (CIR process):

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dZ_t.$$

- 3/2 model (??):

$$dv_t = \kappa v_t(\theta - v_t)dt + \nu v_t^{3/2} \, dZ_t.$$

- GARCH diffusion model (relatively new):

$$dv_t = \kappa(\theta - v_t)dt + \nu v_t \, dZ_t.$$

- OU volatility model (?):

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu \, dZ_t.$$

Simulation scheme for the SV models

- Time discretization (Euler/Milstein):
  - Almost no restriction but computationally expensive.
- Conditional MC:
  - Can skip the simulation of price S_t. (Simulate volatility σ_t only)
  - The final price S_T should be expressed by σ_T and $V_T = \int_0^T \sigma_t^2 dt$.
- Exact Simulation:
  - No need for time-discretization: jump from $t = 0$ to T.
  - The conditional MC condition + should be able to sample σ_T and V_T.
  - σ_T follows a well-known distribution and the (conditional) Laplace transform of $V_T \,|\, v_T$ should be analytically available (Heston, 3/2, SABR)

    $$E(e^{-sV_T} | v_T = x) = f(s, x)$$

    If Laplace transform is known, the CDF of $V_T \,|\, v_T$ can be obtained by (numerical) inverse-transform although it may be computationally expensive.
  - There are other lucky cases too: Normal SABR (**?**).

Conditional and Exact MC

Conditional MC:

1) Simulate path of σ_t for $(0 \le t \le T)$
2) Obtain σ_T and V_T (trapezoidal / Simpson's rule)

Exact MC:

1) Sample v_T from the (well-known) distribution
2) Sample V_T from the (numerical) CDF of $V_T \,|\, v_T$.

In Common:

3) Obtain $E(S_T|v_T, V_T)$ and effective volatility (usually $\rho_* \sqrt{V_T/T}$)

4-1) Price sampling: draw normal / log-normal distribution

4-2) Option price: Bachelier / BSM option price formula with $S_0 := E(S_T|v_T)$ and $\sigma := \rho_* \sqrt{V_T/T}$.
Then, average over the simulations.

Heston model (conditional MC)

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dZ_t \quad (v_t = \sigma_t^2)$$

Integrating v_t,

$$v_T - v_0 = \kappa(\theta T - V_T) + \nu\int_0^T \sqrt{v_t}dZ_t$$
$$\int_0^T \sigma_t\, dZ_t = \frac{1}{\nu}\Big(v_T - v_0 - \kappa(T\theta - V_T)\Big).$$

S_T is expressed by v_T and V_T (conditional MC possible)!

$$\log\left(\frac{S_T}{S_0}\right) = \frac{\rho}{\nu}\big(v_T - v_0 - \kappa(\theta - V_T)\big) - \frac{1}{2}V_T + \rho_*\sqrt{V_T}\,X_1$$
$$E(S_T \mid v_T, V_T) = S_0 \exp\left(\frac{\rho}{\nu}\big(v_T - v_0 - \kappa(\theta - V_T)\big) - \frac{\rho^2}{2}V_T\right)$$

Heston model (exact MC)

- It is known that v_T is distributed as a noncentral chi-square distribution, $\textsc{ncx2}(\delta, \lambda)$:

$$v_T \;=\; \frac{\nu^2(1-e^{-\kappa T})}{4\kappa}\textsc{ncx2}(\delta,\lambda) = \frac{e^{-\kappa T/2}}{2\phi(\kappa)}\textsc{ncx2}(\delta,\lambda),$$

where the degrees of freedom δ and the noncentrality λ are

$$\delta = \frac{4\kappa\theta}{\nu^2},\quad \lambda = \frac{4v_0\kappa\, e^{-\kappa T}}{\nu^2(1-e^{-\kappa T})} = 2v_0 e^{-\kappa T/2}\phi(\kappa) \text{ for } \phi(\kappa) = \frac{2\kappa/\nu^2}{\sinh(\kappa T/2)}$$

Standard library is available for drawing $\textsc{ncx2}$ random number.
- The conditional Laplace transform of V_T is also known (**?**).
- Reference: **?** and **?** (improvement)

SABR model (conditional MC)

$$\frac{d\sigma_t}{\sigma_t} = \nu\, dZ_t \quad \Rightarrow \quad \sigma_T = \sigma_0 \exp\left(-\frac{1}{2}\nu^2 T + \nu Z_T\right)$$

Integrating σ_t,

$$\nu \int_0^T \sigma_t dZ_t = \sigma_T - \sigma_0 = \sigma_0 \exp\left(-\frac{1}{2}\nu^2 T + \nu Z_T\right) - \sigma_0$$

S_T is expressed by σ_T and V_T (conditional MC possible) !

$$S_T = S_0 + \frac{\rho}{\nu}(\sigma_T - \sigma_0) + \rho_* \sqrt{V_T}\, X_1$$

$$\log\left(\frac{S_T}{S_0}\right) = \frac{\rho}{\nu}(\sigma_T - \sigma_0) - \frac{1}{2}V_T + \rho_* \sqrt{V_T}\, X_1$$

SABR Model (exact MC)

- σ_T is distributed by a log-normal distribution. Sampling is trivial.
- The conditional Laplace transform of $1/V_T$ is also known:

$$E\left(e^{-s/V_T}\middle| v_T\right) = \exp\left(-\frac{\phi_x(s)^2 - x^2}{2T}\right)$$

where $\phi_x(s) = \text{acosh}(se^{-x} + \cosh(x))$ and $v_T = \exp(\nu x)$

- From above, we can sample $1/V_T$ and get V_T.
- Reference: **?**

3/2 model (conditional MC)

$$dv_t = \kappa v_t(\theta - v_t)dt + \nu\, v_t^{3/2}\, dZ_t.$$

The change of variable, $x_t = 1/v_t$ yields (a good Itô calculus exercise!)

$$dx_t = -\frac{dv_t}{v_t^2} + \frac{(dv_t)^2}{v_t^3} = (\kappa + \nu^2 - \kappa\theta\, x_t)dt - \nu\sqrt{x_t}\, dZ_t.$$

This is same as v_t in Heston model with new parameters:

$$\nu' = -\nu, \quad \kappa' = \kappa\theta, \quad \text{and} \quad \theta' = (\kappa + \nu^2)/\kappa\theta$$

$$(\nu = -\nu', \quad \kappa = \theta'\kappa' - \nu'^2, \quad \text{and} \quad \nu' = \kappa'/(\kappa'\theta' - \nu'^2))$$

We can S_T as a function of V_T and v_T (conditional MC possible)!

$$d\log(x_t) = \left(\frac{\kappa + \nu^2/2}{x_t} - \kappa\theta\right) dt - \frac{\nu}{\sqrt{x_t}}\, dZ_t$$

$$\int_0^T \frac{1}{\sqrt{x_t}}\, dZ_t = \frac{1}{\nu}\left(\log\left(\frac{x_0}{x_T}\right) + (\kappa + \nu^2/2)V_T - \kappa\theta T\right),$$

$$\log\left(\frac{S_T}{S_0}\right) = \frac{\rho}{\nu}\left(\log\left(\frac{v_T}{v_0}\right) - \kappa\left(T\theta - \left(1 + \frac{\nu^2}{2\kappa}\right)V_T\right)\right) - \frac{1}{2}V_T + \rho_*\sqrt{V_T}\, X_1$$

3/2 model (exact MC)

- From Heston model, $1/v_T$ is distributed as a noncentral chi-square distribution, $\textsc{ncx2}(\delta', \lambda')$ where the degrees of freedom δ' and the noncentrality λ' are

$$\delta' = \frac{4\kappa'\theta'}{\nu^2}, \quad \lambda = \frac{4\kappa'\, e^{-\kappa' T}}{v_0 \nu^2 (1 - e^{-\kappa' T})}.$$

  Standard library is available for drawing $\textsc{ncx2}$ random number.
- The conditional Laplace transform of V_T is also known.
- Reference: **?**

OUSV model (conditional MC). 2019 exam question

Let $U_T = \int_0^T \sigma_t \, dt$ and $V_T = \int_0^T \sigma_t^2 \, dt$.

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu \, dZ_t.$$

$$d\sigma_t^2 = 2\sigma_t d\sigma_t + (d\sigma_t)^2 = (\nu^2 + 2\kappa(\theta\sigma_t - \sigma_t^2))dt + 2\nu\sigma_t dZ_t$$

$$\sigma_T^2 - \sigma_0^2 = \nu^2 T + 2\kappa(\theta U_T - V_T) + 2\nu \int_0^T \sigma_t dZ_t$$

$$\int_0^T \sigma_t dZ_t = \frac{1}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\nu}{2}T - \frac{\kappa\theta}{\nu}U_T + \frac{\kappa}{\nu}V_T$$

S_T is expressed by v_T and V_T (conditional MC possible)!

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \rho \int_0^T \sigma_t dZ_t + \rho_* \int_0^T \sigma_t dX_t - \frac{1}{2}V_T \\ &= \frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{1}{2}\right)V_T + \rho_*\sqrt{V_T}\, X_1\end{aligned}$$

$$S_0 := E(S_T) = S_0 \exp\left(\frac{\rho}{2\nu}(\sigma_T^2 - \sigma_0^2) - \frac{\rho\nu}{2}T - \frac{\rho\kappa\theta}{\nu}U_T + \left(\frac{\rho\kappa}{\nu} - \frac{\rho^2}{2}\right)V_T\right)$$

$$\sigma_{\text{BS}} := \rho_*\sqrt{V_T/T}.$$

Project Suggestion: *Almost* Exact MC (by Choi)

- General scheme:
  - Implementing existing paper is OK:
  - Improving Euler / Milstein scheme? or exact simulation?
  - Or try something new (see below):
- Simulation for GARCH diffusion:

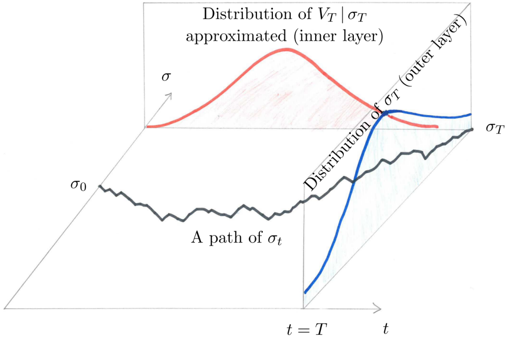
$$dv_t = \kappa(\theta - v_t)dt + \nu v_t \, dZ_t.$$

  - Currently, there is no easy way to solve the SDE.
  - Conditional MC possible? Option pricing with conditional MC?
  - How to express S_T as a function of v_T and V_T (or something else)?
  - Exact simulation possible?
- *Almost* Exact MC (by Choi)

Project Suggestion: *Almost* Exact MC (by Choi)

The illustration of the proposed double layer approximation method:

1. The outer layer distribution of σ_T (in blue) is typically known
2. The inner layer distribution of $V_T|\sigma_T$ (in red) is approximated as well-known distributions such as log-normal or inverse Gaussian.



Project Suggestion: *Almost* Exact MC (by Choi)

Drawback of the exact simulation methods:

- Inverse of the Laplace transform, $E(e^{-sV_T}|v_T) = f(s, v_T)$, is complicated.
- Drawing random number from the numerical CDF is also slow.

How can we simplify this step with some approximation?

- Approximate $V_T|v_T$ with a well-known distribution by matching the first two moments , $M_1 = E(V_T|v_T)$ and $M_2 = E(V_T^2|v_T)$.
- The RN sampling should be easy from the approximate distribution.

Almost Exact MC: Candidates for distributions

- Log-normal (LN):

$$Y \sim \mu \exp(\sigma Z - \sigma^2/2) \quad \text{for} \quad Z \sim N(0,1)$$

The parameters (μ, σ) can be obtained from the two moments:

$$\mu = M_1 \quad \text{and} \quad \sigma = \sqrt{\log(M_2/M_1^2)}.$$

- Inverse-Gaussian:

$$f_{\mathrm{IG}}(x \,|\, \gamma, \delta) = \frac{\delta}{\sqrt{2\pi x^3}} \,\exp\left(-\frac{(\gamma x - \delta)^2}{2x}\right) \quad \text{for} \quad \gamma \ge 0,\; \delta > 0.$$

How to determine (γ, μ) from M_1 and M_2?

- The sampling methods for LN and IG are available. See **?** (WIKIPEDIA) for IG.

Almost Exact MC: How to obtain M_1 and M_2?

Keep in mind for a random variable $X \geq 0$, the MGF and Laplace transform are same:

$$M_X(-s) = E(e^{-sX}) = \int_{x=0}^{\infty} e^{-sX} f_X(x) dx = f(s)$$

$$f(s) = 1 - M_1 s + \frac{1}{2} M_2 s^2 + \cdots,$$

where $M_1 = E(V_T | v_T)$ and $M_2 = E(V_T^2 | v_T)$.

- Numerical method: **?**
- Analytic method (from Taylor's expansion or etc):
    - SABR: well-known **?**.
    - GARCH: **?**(?)
    - Heston, 3/2, OU?

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