Copula
Applied Stochastic Processes (FIN 514)

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Joint probability distribution

- Two random variables X_1 and X_2 have PDF $f_1(x)$ and $f_2(x)$, and CDF $F_1(x)$ and $F_2(x)$ respectively.

$$F_1(x) = \text{Prob}(X_1 \le x)$$
$$F_2(x) = \text{Prob}(X_2 \le x)$$

- However, the knowledge of the PDFs and CDFs of individual RVs does not tell us how the two RVs are related. We still need to define the joint PDF and CDF:

$$F_{1,2}(x_1, x_2) = \text{Prob}(X_1 \le x_1 \text{ and } X_2 \le x_2)$$

- Note that the definition of $F_{1,2}(x_1, x_2)$ is not related to those of $F_1(x)$ and $F_2(x)$. In two extremes, X_1 and X_2 can be independent or completely correlated, often characterized by the correlation coefficient ρ.

Multivariate normal distribution

- The PDF of multivariate normal variable $\boldsymbol{x}$ (vector) with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$ (matrix) is given as

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^k \det \boldsymbol{\Sigma}}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\right)$$

- For the independent standard normals ($\Sigma = I (\det \Sigma = 1)$ and $\mu = 0$),

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = n(z_1)\cdots n(z_n) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}(z_1^2 + \cdots + z_n^2)\right)$$

- The bivariate case (n=2) is more explicit (see wikipedia).
- In practice, the joint PDF is not often used (remind how we generate correlated normal RNs)
- There are only handful distributions whose joint CDF is known for a given covariance: normal, Student's t, etc.

Joint distribution via copula

- The joint CDF $F_{1,2}(x_1, x_2)$ is *not completely independent* from the individual RNs. For one thing, the function domain has to be same as that of RN: $[0, 1]$ for uniform, $(-\infty, \infty)$ for normal, etc.
- For the CDFs $F_1(x)$ and $F_2(x)$, we know that $F_1(X_1)$ and $F_2(X_2)$ are uniform RNs. (In the same way we generate RNs $X_1 = F_1^{-1}(U)$.)
- So can implicitly define the joint CDF via a copula function $C : [0, 1]^2 \rightarrow [0, 1]$,

$$C(u_1, u_2) = F_{1,2}(x_1 = F_1^{-1}(u_1), x_2 = F_2^{-1}(u_2))$$
$$C(u_1 = F_1(x_1), u_2 = F_2(x_2)) = F_{1,2}(x_1, x_2)$$

  Defining either $F_{1,2}(x_1, x_2)$ or $C(u_1, u_2)$ is equivalent.
- The function C can be understood as a joint CDF on uniform RNs.

Copula: Mix and match

- Copula function, $C(\cdots)$, is a way of defining joint distribution independent from the original RVs.
- Once $C(\cdots)$ is given, the joint distribution of any two RVs can be defined by $F_{1,2}(x_1, x_2) = C(F_1(x_1), F_2(x_2))$
- What are the choices of the copula function, $C(\cdots)$?
- Find forms of $C(\cdots)$ satisfying several mathematical requirements. (next slide)
- Borrow the copulas from several well-known multi-variate distributions whose $F_{\boldsymbol{X}}(\cdots)$ is analytically known: Gaussian copula, Student-t copula, etc.

Copula: Mathematical definition and requirements

Now we generalize to n-dimensional case: $C : [0,1]^n \to [0,1]$. Because C is a joint CDF function, it should satisfy:

- $C(u_1, \cdots, u_{k-1}, 0, u_{k+1}, \cdots, u_n) = 0$
- $C(1, \cdots, 1, u_k, 1, \cdots, 1) = u_k$
- The probability on any hypercube is always non-negative.
  - For $n = 1$, it means $C(u_1^a) \le C(u_1^b)$ if $u_1^a \le u_1^b$.
  - For $n = 2$, the probability over $[u_1^a, u_1^b] \times [u_2^a, u_2^b]$ should be non-negative:

$$0 \le C(u_1^b, u_2^b) - C(u_1^a, u_2^b) - C(u_1^b, u_2^a) + C(u_1^a, u_2^a)$$

  - If $C(u_1, \cdots, u_n)$ is a continuous function, the PDF is non-negative:

$$0 \le c(u_1, \cdots, u_n) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} C(u_1, \cdots, u_n)$$

Copula: Examples

- Independent Copula:

$$C(u_1, \cdots, u_n) = u_1 \cdots u_n \quad \text{or} \quad c(u_1, \cdots, u_n) = 1$$

- Completely dependent Copula:

$$C(u_1, \cdots, u_n) = \min(u_1, \cdots, u_n)$$

- Gaussian Copula:

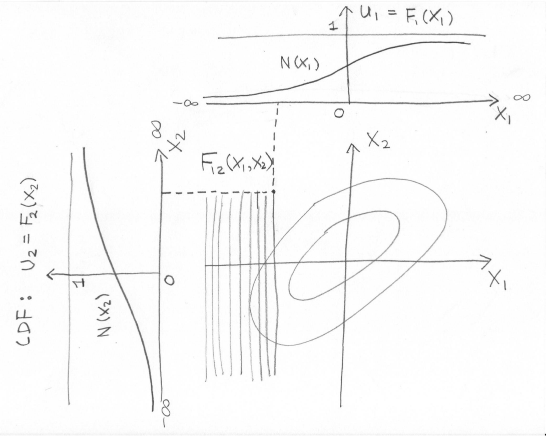
$$C_{\boldsymbol{R}}(u_1, \cdots, u_n) = N_n(N^{-1}(u_1), \cdots, N^{-1}(u_n))$$

  where $N_n(\cdot)$ is the n-dimensional cumulative normal distribution with correlation matrix $\boldsymbol{R}$.

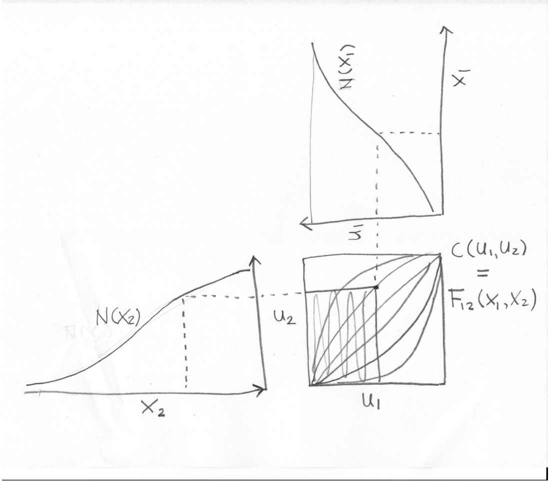
- Others: Clayton, Frank, Gumbel copula. See the copula families in wikipedia.

Copula in figures

Joint distribution function: $F_{1,2}(x_1, x_2)$

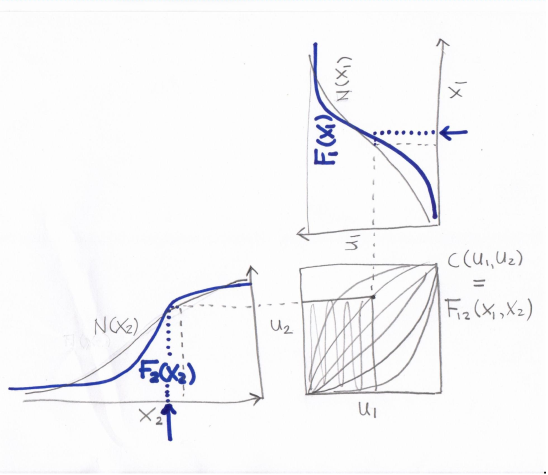


Copula function: $C(u_1, u_2)$

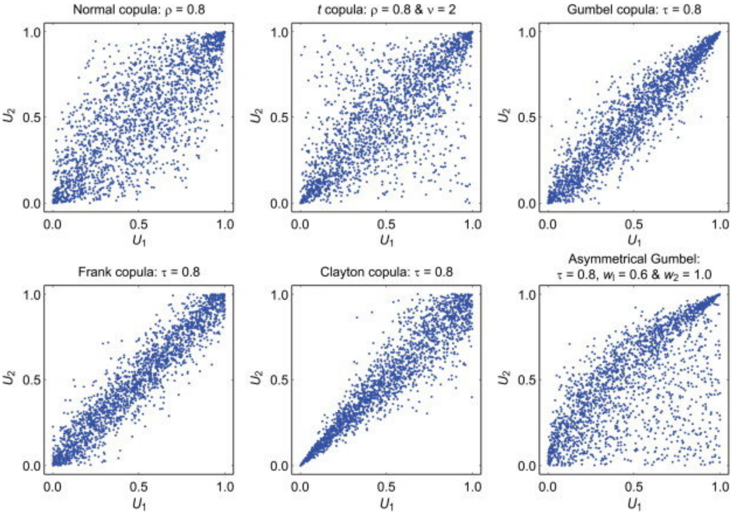


Copula in figures

Copula function: $C(u_1, u_2)$



Copula Examples



RN generation from Gaussian copula

- Imagine that two CDFs, $F_1(x_1)$ and $F_2(x_2)$, are given and we want to generate joint RNs, e.g., (X_1, X_2) in order to evaluate an expectation, $E\left[g(X_1, X_2)\right]$, (e.g., a price of a derivative).
- As long as we generate joint uniform RNs (U_1, U_2), we can transform them to $(X_1, X_2) = (F_1^{-1}(U_1), F_2^{-1}(U_2))$.
- We borrow Gaussian variables to generate (U_1, U_2):
  - Generate pairs of independent normal RNs: $(Z_1,\ Z_2)$.
  - Correlate the normal RNs: $(Z_1', Z_2') = (Z_1,\ \rho Z_1 + \sqrt{1-\rho^2} Z_2)$
  - Generate the joint uniform RNs: $(U_1, U_2) = (N(Z_1'),\ N(Z_2'))$
  - Generate the original RNs: $(X_1, X_2) = (F_1^{-1}(U_1),\ F_2^{-1}(U_2))$
- Finally Monte-Carlo method is applied as

$$E\left[g(X_1, X_2)\right] = \frac{1}{N}\sum_{k=1}^{N} g(X_1^{(k)}, X_2^{(k)})$$

Case 1: spread option under SV models

- We want to price a spread option, i.e., $E(S_{1T} - S_{2T} - K)^+$ using MC.
- If two stocks S_1 and S_2 follow GBMs, we know how to correlate them (HW2) since the the distribution is transformed from normal RVs. However, GBMs may not be right distributions due to the volatility smile.
- If the stocks follow SV models, creating a joint distribution is not easy. So we use copula.
- We first build discrete CDFs for $S_1(T)$ and $S_2(T)$ from the call prices at the series of strikes, $K_j = S_0 + j\Delta K$ for $j = 0, \pm 1, \pm 2, \cdots$.

$$F(K_j) = -\frac{\partial}{\partial K} C(K) \approx \frac{C(K_{j-1}) - C(K_{j+1})}{2\Delta K}$$

- The discrete inverse CDF is the interpolation from the inversed pairs, $(F(K_j), K_j)$.

Case 2: Collateralized debt obligation (CDO)

- A COD is a bond backed by a pool of (housing) loans.
- Naturally the joint distribution of the default of the underlying loans are important. So Gaussian copula is used as a standard way of pricing CDOs.
- While the underlying loans are sub-primes (below investment grade `BBB-`), the super-senior *tranche* of CDO got `AAA` credit rating as the pools were considered *diversified*. The correlation was usually estimated from historical data.
- In financial crisis, however, the correlation across all assets significantly increased: when a bond defaults, the others do so. So the pool is not really diversified.
- The use of copula is criticized as one reason behind the financial criss in 2008–9. Copula in general can not capture the dynamic changes of the correlation over time.