

Scientific computation/Monte Carlo method Applied Stochastic Processes (FIN 514)

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- Most of science (including finance) problems can not be solved analytically. Even analytic/algebraic solutions still requires to be computed!
- We need problem solving methods tailored for computation.
- **Scientific computation** is a branch of (applied)mathematics/computer science to research efficient computation method.
- Scientific computation is also an important part of financial engineering/quantitative finance.

Example: numerical root finding

- Abel (1802-1829) proved that there is no general *algebraic* solution for the roots of a quintic equation ($a_5x^5 + \cdots + a_1x + a_0 = 0$)
- However, we can find the root numerically:

- Bisection

$$x_{n+1} = \frac{a + x_n}{2} \quad \text{or} \quad \frac{x_n + b}{2}$$

- Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Newton-Raphson method

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

- Square root: $x = \sqrt{y}$ [Demo]

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{y}{x_n} \right)$$

Tips and cautions

- In general, carefully translate math formulas to computer implementation
- Avoid repeated evaluation. Special functions, \exp , \sin , \cos , \log , are several times expensive than that of multiplication.
- Even polynomials (Horner's rule):

$$a_0 + a_1x + \cdots + a_nx^n \quad \text{v.s.} \quad a_0 + x(a_1 + x(\cdots (a_{n-1} + x a_n)))$$

- In computer, small difference (machine epsilon) is rounded to 0, which is unexpected and dangerous. For example,

$$f(x) = \sqrt{x+1} - \sqrt{x}$$

we get $f(10^{16}) = 0$. So better use the equivalent same formula,
[Python Demo]

$$f(x) = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

Monte-Carlo (MC) Method

- Even with computer, many problems can not be solved with deterministic (or convergent) method. So we need Monte-Carlo simulation.
- In finance, the focus is on the price of derivative, which is the expected value of stochastic payout

$$E(f(x)) \quad \text{or} \quad E(f(x_1, \dots, x_D))$$

where x, x_1, \dots, x_D are (possibly correlated) D -dimensional random numbers.

- MC is favored in a large dimensions (D) because the cost of MC grows linearly ($\sim D$) whereas the cost of the deterministic method grows exponentially ($\sim e^D$), which is called **the curse of dimensionality**.
- Example: MC computation of π [Python Demo]

Generating Uniform Random Number (RN)

The uniform random number between 0 and 1 is the mother of all RNs. By nature, computer **cannot** generate true RNs. In fact, true RNs are not even desirable as we want to control and reproduce the result. So we generate pseudo-RNs in the following way:

$$X_{n+1} = AX_n + B \pmod C, \quad U_n = X_n/C$$

X_0 is the **seed** of the RN generation. Using a fixed seed, you get the same result. Two popular, but outdated, choices of (A, B, C) are

- $A = 65,539 = 2^{16} + 3, \quad C = 2^{31}, \quad (B = 0)$
- $A = 16,807 = 7^5, \quad C = 2^{31} - 1, \quad (B = 0)$

The Mersenne Twister is the standard pseudo-RN generation method ($C = 2^{19937} - 1$). It is implemented in modern programming languages including python.

Generating Discrete RN

Bivariate RNs, e.g., random walk takes $+1$ and -1 with probabilities p and $1 - p$ respectively can be generated as

$$X_k = \begin{cases} 1 & \text{if } U_k < p \\ -1 & \text{if } U_k > p \end{cases},$$

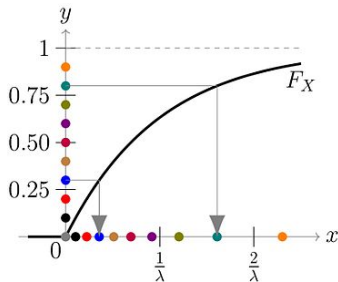
where U_k is uniform RNs. Multiple values can be similarly generated.

Generating RN from a General Distribution

If we know the cumulative distribution function (CDF) of a distribution $P(x)$, the **inverse** CDF gives random numbers following the distribution,

$$X = P^{-1}(U)$$

However, probability distribution with analytically invertible CDF is rare. When inverse CDF is not available, the last resort is numerical root-finding, $P(X) = U$ but this is computationally expensive. Finding efficient sampling method is an important area of research.



Generating Exponential Distribution RNs

Exponential/Poisson distribution

- Distribution for the survival time or arrival time T with intensity λ .
- PDF: $f(t) = \lambda e^{-\lambda t}$, CDF: $F(t) = 1 - e^{-\lambda t}$
- $E(T) = 1/\lambda$, $\text{Var}(T) = 1/\lambda^2$.

The CDF function is easily invertible. The RN can be generated with

$$T_k = -\frac{1}{\lambda} \log U_k = \left(-\frac{1}{\lambda} \log(1 - U_k) \right)$$

where U_k is uniform RNs.

Gaussian Distribution

- Given the wide usage, efficient evaluation of Gaussian distribution is critically important.
- How to evaluate CDF? Integration is very expensive!

$$N(z) = \int_{-\infty}^z n(s) ds$$

- An accurate approximation is used instead ($x > 0$):

$$1 - N(x) = 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2/2}, \quad t = \frac{1}{1 + px/\sqrt{2}}$$

where $a_1 = 0.3480242$, $a_2 = -0.0958798$, $a_3 = 0.7478556$, and $p = 0.47047$.

Gaussian RN (Box-Muller Method)

- Reference: Box, George EP, and Mervin E Muller. [A Note on the Generation of Random Normal Deviates](#). The Annals of Mathematical Statistics 29, no. 2 (1958): 610–11.
- Evaluation of the normal inverse CDF, N^{-1} , is expensive.
- There is a brilliant trick of generating Gaussian RN:

For two-dimensional Gaussian random numbers (Z_1, Z_2)

$$\begin{aligned} P(Z_1^2 + Z_2^2 < y) &= \int_{Z_1^2 + Z_2^2 < y} \frac{1}{\sqrt{2\pi}^2} e^{-\frac{1}{2}(z_1^2 + z_2^2)} dz_1 dz_2 \\ &= \frac{2\pi}{2\pi} \int_0^{\sqrt{y}} r e^{-r^2/2} dr = 1 - e^{-y/2} \end{aligned}$$

The squared radius, $Y = Z_1^2 + Z_2^2$, follows exponential distribution and we know the inverse CDF is available:

$$F_Y(y) = 1 - e^{-y/2}, \quad F_Y^{-1}(u) = -2 \log(1 - u).$$

Gaussian RN (Box-Muller Method) cont.

- This means we can accurately sample Y and $R = \sqrt{Y}$:

$$Y = -2 \log(1 - U_1) \quad \text{for a uniform RV, } U_1$$

$$R = \sqrt{Y} = \sqrt{-2 \log U_1} \quad (\text{symmetry: } U_1 \leftrightarrow 1 - U_1)$$

- Then, we extract two normal RNs, Z_1 and Z_2 , as x - and y -axis projections with a random angle $2\pi U_2$:

$$Z_1 = \sqrt{-2 \log U_1} \cos(2\pi U_2), \quad Z_2 = \sqrt{-2 \log U_1} \sin(2\pi U_2).$$

We get two Gaussian RNs from two uniform RNs.

Gaussian RN (Marsaglia-Polar Method)

- Reference: Marsaglia, George, and Thomas A Bray. [A Convenient Method for Generating Normal Variables](#). SIAM Review 6, no. 3 (1964): 260–64.
- The Box-Muller method can be further improved!
- $V_{1,2} = 2U_{1,2} - 1$ so that $V_{1,2}$ is uniform RVs between -1 and 1.
- Take if $0 < W = V_1^2 + V_2^2 < 1$ and reject otherwise so that (V_1, V_2) is uniform random point on the unit circle.
- W has a uniform distribution on $[0, 1]$,

$$P(W < x) = \pi(\sqrt{x})^2 / \pi = x,$$

so W can replace U_1 .

Gaussian RN (Marsaglia-Polar Method) cont.

- Using the trigonometric properties,

$$\left(\frac{V_1}{\sqrt{W}}, \frac{V_2}{\sqrt{W}} \right) = \left(\cos(2\pi U_2), \sin(2\pi U_2) \right)$$

- Finally we obtain

$$Z_1 = \sqrt{-2 \log W} V_1 / \sqrt{W} = V_1 \sqrt{-2(\log W)/W}$$

$$Z_2 = \sqrt{-2 \log W} V_2 / \sqrt{W} = V_2 \sqrt{-2(\log W)/W}$$

- Even after *wasting* 21% of (V_1, V_2) pairs, it is still more efficient by avoiding expensive sin and cos evaluations!

Reducing MC variance

- Anti-thetic method; (π by MC)
 - By symmetry, U is a uniform random number, so is $1 - U$
 - By symmetry, Z is a Gaussian random number, so is $-Z$
- Importance sampling method: ($P(Z > 30)$ by MC)
 - Original probability = Modified probability \times Correction
- Low-discrepancy sequence
 - Quasi-random numbers evenly distributed over $(0, 1)$.
- Control Variate: for $g(x)$ similar to $f(x)$ and known $E(g(X))$,

$$E_{CV}(f(X)) = E_{MC}(f(X)) + \left(E(g(X)) - E_{MC}(g(X)) \right)$$

[Python Demo]

Generation of correlated Gaussian RNs ($N = 2$)

- We want to generate two standard Gaussian RNs, W_1 and W_2 , correlated by ρ ,

$$\frac{E(W_1 W_2)}{E(W_1)E(W_2)} = E(W_1 W_2) = \rho.$$

- We can achieve that by separating correlate and de-correlated parts:

$$W_1 = Z_1, \quad W_2 = \rho Z_1 + \sqrt{1 - \rho^2} Z_2,$$

where Z_1 and Z_2 are independent and standard RNs.

- You can easily prove that (see StoFin mid-term exam):

$$E(W_2^2) = 1 \quad \text{and} \quad E(W_1 W_2) = \rho$$

- You'll see this trick in spread option and stochastic differential equations for stochastic volatility models (Heston and SABR model)

Covariance/correlation matrix

For N -dimensional samples, $X = (X_1, \dots, X_N)^T$, with mean 0 and stdev σ_k , the covariance matrix Σ is given as

$$\Sigma_{ij} = \text{Cov}(X_i, X_j) = E((X_i - \bar{X}_i)(X_j - \bar{X}_j)) \quad (\Sigma_{ii} = \sigma_i^2).$$

The correlation matrix R is similarly defined as

$$R_{ij} = \text{Corr}(X_i, X_j) = \frac{\Sigma_{ij}}{\sigma_i \sigma_j} \quad (R_{ii} = 1)$$

- If $\{X_i\}$ are independent standard normals, $\Sigma = I$
- Symmetric: $\Sigma = \Sigma^T$ ($\Sigma_{ij} = \Sigma_{ji}$)
- Positive-definite: $\mathbf{a}^T \Sigma \mathbf{a} \geq 0$ for any (row) vector \mathbf{a} :

$$\text{Var}(\mathbf{a}X) = \text{Var}(a_1 X_1 + \dots + a_N X_N) = \mathbf{a} \Sigma \mathbf{a}^T \geq 0$$

- For a linear combination, $\mathbf{L}X$, $\text{Cov}(\mathbf{L}X) = \mathbf{L} \Sigma \mathbf{L}^T$.

Generation of correlated Gaussian RNs (N in general)

- $X = (X_i)^T$ with covariance matrix Σ : we want to find such that $X = LZ$ where $Z = (Z_i)^T$ is independent standard normals.
- Then, L should satisfy the covariance matrix condition:

$$\Sigma = \text{Var}(LZ) = LIL^T = LL^T$$

Therefore, L should be a **square-root matrix** of Σ : $L = \sqrt{\Sigma}$.

- The square-matrix L is not unique. Cholesky decomposition (lower triangular matrix) is a numerically stable and fast solution. [Python Demo]
- The 2 asset case ($\rho = \rho_{12}$):

$$L = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}, \quad \Sigma = LL^T = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

MC of Brownian Motion(s)

- Standard BM: $W_T \sim \sqrt{T}Z_1$ where Z_1 is a standard normal RV.
- Arithmetic Brownian motion (Bachelier model)

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$$dF_t = \sigma_N dW_t \Rightarrow F_T \sim F_0 + \sigma_N \sqrt{T} Z_1$$

- From $t = S$ to $t = T$: $F_T \sim F_S + \sigma_N \sqrt{T - S} Z_1$
- For N assets: $\mathbf{F}_T = \mathbf{F}_0 + \mathbf{L} \sqrt{T} Z$, where $\mathbf{L}\mathbf{L}^T = \Sigma$
- Geometric Brownian motion (BSM model)

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$$\frac{dF_t}{F_t} = \sigma dW_t \Rightarrow F_T \sim F_0 \exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T} Z_1\right)$$

- From $t = S$ to $t = T$: $F_T \sim F_S \exp\left(-\frac{1}{2}\sigma^2(T - S) + \sigma\sqrt{(T - S)} Z_1\right)$
- For N assets: $F_k(T) = F_k(0) \exp\left(-\frac{1}{2}T\Sigma_{kk} + \mathbf{L}_{k*}\sqrt{T} Z\right)$, where $\mathbf{L}\mathbf{L}^T = \Sigma$ and \mathbf{L}_{k*} is the k -th row vector of \mathbf{L} .
- [Python Demo]