AI505

Optimization

Bracketing

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Bracketing

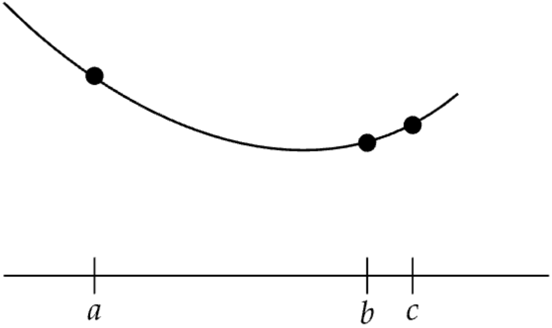
A derivative-free method to identify an interval containing a local minimum and then successively shrinking that interval

Unimodality

There exists a unique optimizer $\boldsymbol{x}^*$ such that f is monotonically decreasing for $\boldsymbol{x} \leq \boldsymbol{x}^*$ and monotonically increasing for $\boldsymbol{x} \geq \boldsymbol{x}^*$

Finding an Initial Bracket

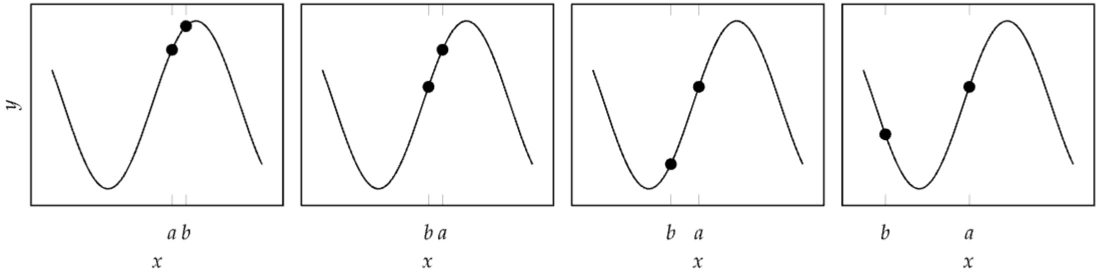
Given a unimodal function, the global minimum is guaranteed to be inside the interval $[a, c]$ if $f(a) > f(b) < (c)$




```
function bracket_minimum(f, x=0; s=1e-2, k=2.0)
    a, ya = x, f(x)
    b, yb = a + s, f(a + s)
    if yb > ya
        a, b = b, a
        ya, yb = yb, ya
        s = -s
    end
    while true
        c, yc = b + s, f(b + s)
        if yc > yb
            return a < c ? (a, c) : (c, a)
        end
        a, ya, b, yb = b, yb, c, yc
        s *= k
    end
end
```

Finding an Initial Bracket

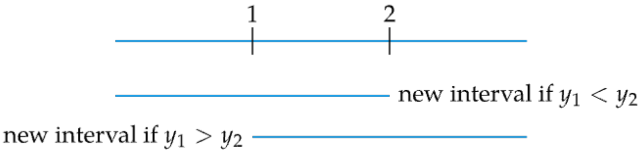
Example of `bracket_minimum` on a function



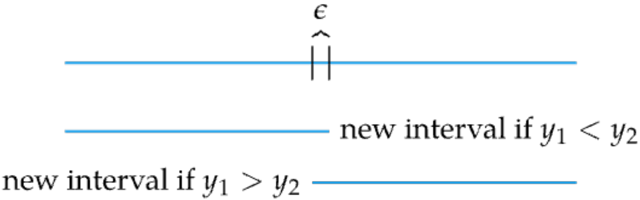
reverses direction between the first and second iteration and expands until a minimum is bracketed in the fourth iteration.

For unimodal functions, when function evaluations are limited, what is the maximal shrinckage we can achieve?

When restricted to only 2 function evaluations (queries) the most we can guarantee to shrink our interval is by just under a factor of 2.

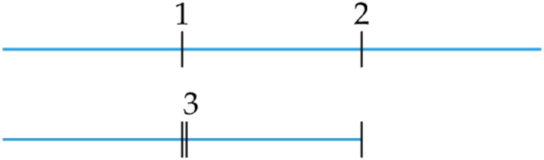


yeilds a factor of 3.



for $\epsilon \to 0$ yields a factor of just less than 2

When restricted to only 3 function evaluations (queries) the most we can guarantee to shrink our interval is by a factor of 3.

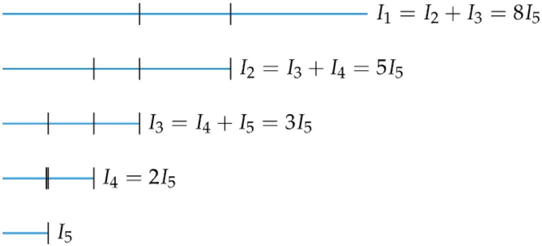


Fibonacci Search

When restricted to n functions evaluations following the previous strategy, we are guaranteed to shrink our interval by a factor of F_{n+1}.

Fibonacci numbers: sum of previous two,
$1, 1, 2, 3, 5, 8, 13, \ldots$

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1, 2 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$



The length of every interval constructed can be expressed in terms of the final interval times a Fibonacci number.
- final, smallest interval has length I_n,
- second smallest interval has length $I_{n-1} = F_3 I_n$
- third smallest interval has length $I_{n-2} = F_4 I_n$,
and so forth.

Fibonacci Search Algorithm

For a unimodal function f in the interval $[a, b]$, we want to shrink the interval within n iterations. (At each iteration we want to shrink by a factor ϕ).

$$b_{k+1} - a_{k+1} = \frac{F_{n-k+1}}{F_{n-k+2}}(b_k - a_k)$$

Therefore:

$$\begin{aligned} b_n - a_n &= \frac{F_2}{F_3}(b_{n-1} - a_{n-1}) \\ &= \frac{F_2}{F_3}\frac{F_3}{F_4}\dots\frac{F_n}{F_{n+1}}(b_1 - a_1) \\ &= \frac{1}{F_{n+1}}(b_1 - a_1) \end{aligned}$$

Closed-form expression (Binet's formula):

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}},$$

$\phi = (1+\sqrt{5})/2 \approx 1.61803$ is the golden ratio.

$$\frac{F_{n+1}}{F_n} = \phi\frac{1 - s^{n+1}}{1 - s^n}, \quad s = (1{-}\sqrt{5})(1{+}\sqrt{5}) \approx -0.382$$

Suppose we have a unimodal function f in the interval $[a, b]$ and a tolerance $\epsilon = 0.01$. Let $k = 1$.

$$\rho = \frac{F_n}{F_{n+1}} = \frac{1 - s^n}{\phi(1 - s^{n+1})}$$

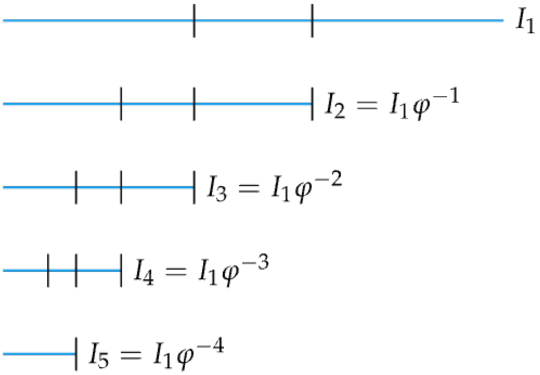
1. $d_k = a_k + \frac{F_{n-k+1}}{F_{n-k+2}}(b_k - a_k)$
2. if $k = n - 1$:

   $$c_k = a_k + \left(1 - \frac{F_{n-k+1}}{F_{n-k+2}}\right)(b_k - a_k)$$

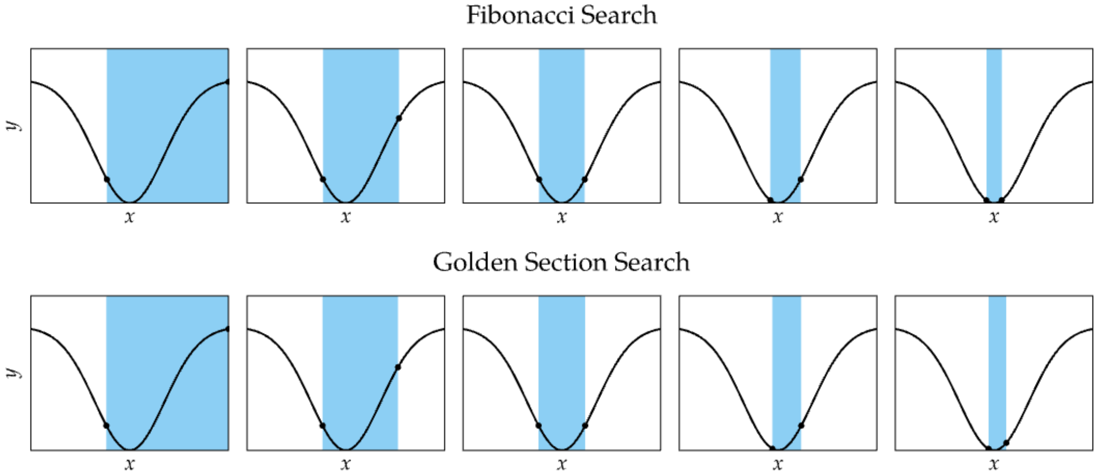
   Otherwise: $c_k = a_k + (1 - \epsilon)(b_k - a_k)$
3. if $f(c_k) < f(d_k)$: $\quad b_{k+1} = d_k, \quad d_{k+1} = c_k, \quad a_{k+1} = a_k$
   otherwise: $a_{k+1} = b_k, \quad b_{k+1} = c_k, \quad d_{k+1} = d_k$
4. $k = k + 1$, go to step 2

Golden Section Search

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi \frac{1 - s^{n+1}}{1 - s^n} = \phi$$

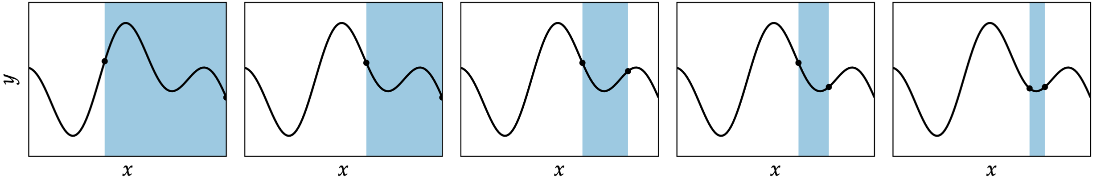


Comparison

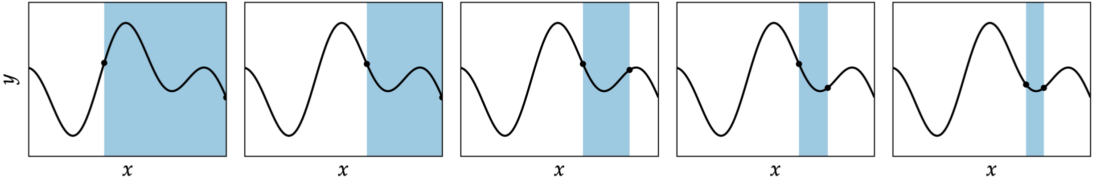


Comparison

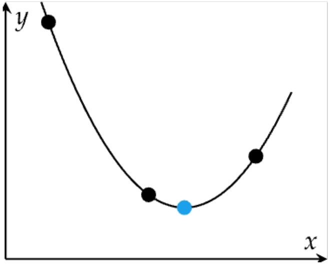
Fibonacci Search

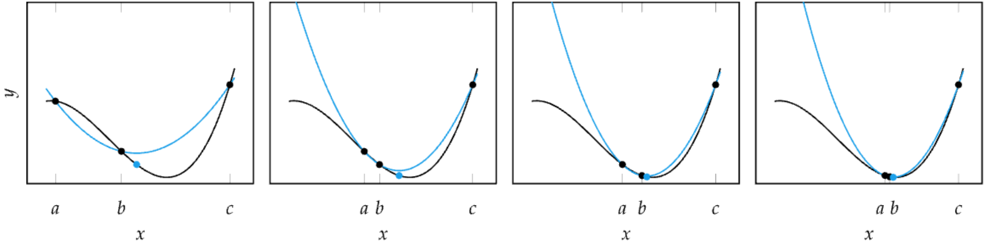


Golden Section Search



Quadratic Fit Search





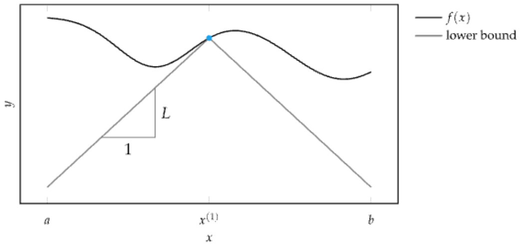
Shubert-Piyavskii Method

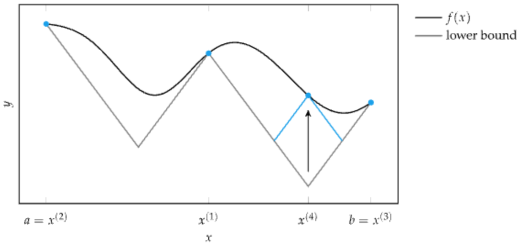
- The Shubert-Piyavskii method is guaranteed to find the global minimum of any bounded function

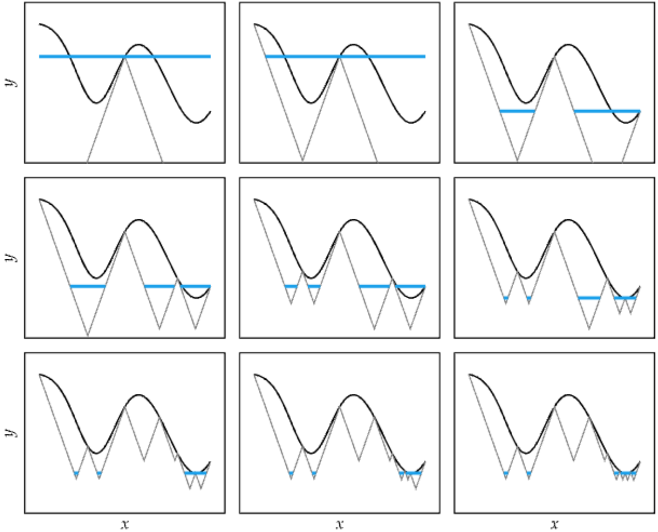
- but requires that the function be Lipschitz continuous

- A function is **Lipschitz continuous** if there is an upper bound on the magnitude of its derivative. A function f is Lipschitz continuous on $[a, b]$ if there exists an $\ell > 0$ such that:

$$|f(x) - f(y)| \leq \ell|x - y|, \quad \forall x, y \in [a, b]$$

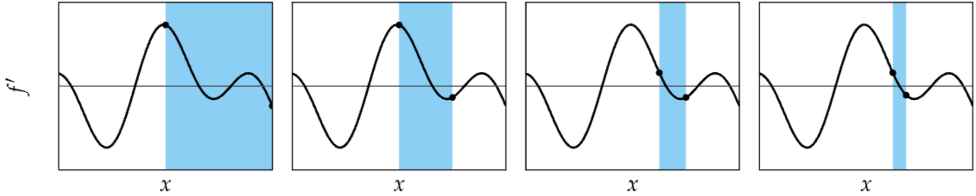





y
y
y
x
x
x

Bisection Method

- Used in root-finding methods
- When applied to $f'(x)$, can be used to find minimum



Summary

- Many optimization methods shrink a bracketing interval, including Fibonacci search, golden section search, and quadratic fit search
- The Shubert-Piyavskii method outputs a set of bracketed intervals containing the global minima, given the Lipschitz constant
- Root-finding methods like the bisection method can be used to find where the derivative of a function is zero