Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol * will be tackled in class. The remaining exercises are left for self training after the exercise class. Some exercises are from the text book and the number is reported. They have the solution at the end of the book.

Exercises

Exercise 1 * (3.6)

Suppose we have a unimodal function defined on the interval $[1, 32]$. After three function evaluations of our choice, will we be able to narrow the optimum to an interval of at most length 10? Why or why not? How much more can we reduce with one further evaluation?

Exercise 2 $^+$ (3.1)

Give an example of a problem when Fibonacci search can be applied while the bisection method not.

Exercise 3 $^+$ (3.4)

Suppose we have $f(x) = x^2/2 - x$. Apply the bisection method to find an interval containing the minimizer of f starting with the interval $[0, 1000]$. Execute three steps of the algorithm (you can do this by hand or in Python).

Exercise 4 $^+$ (3.5)

Suppose we have a function $f(x) = (x + 2)^2$ on the interval $[0, 1]$. Is 2 a valid Lipschitz constant for f on that interval?

Exercise 5 $^+$ (4.1)

Find examples where each of the four termination conditions would not work individually, showing the importance of having more than one.

Exercise 6 $^+$ (4.2)

The first Wolfe condition requires

$$f(\boldsymbol{x}_k + \alpha \boldsymbol{d}_k) \leq f(\boldsymbol{x}_k) + \beta \alpha \nabla \boldsymbol{d}_k f(\boldsymbol{x}_k)$$

What is the maximum step length α that satisfies this condition, given that $f(\boldsymbol{x}) = 5 + x_1^2 + x_2^2$, $\boldsymbol{x}_k = [-1, -1]$, $\boldsymbol{d} = [1, 0]$, and $\beta = 10^{-4}$?

Exercise 7 *

The steepest descent algorithm is a Descent Direction Iteration method that moves along $d_k = -\nabla f(\boldsymbol{x}_k)$ at every step. Program steepest descent algorithms using the backtracking line search. Use them to minimize the Rosenbrock function. Set the initial step length $\alpha_0 = 1$ and print the step length used by each method at each iteration. First, try the initial point $x_0 = [1.2, 1.2]$ and then the more difficult starting point $x_0 = [-1.2, 1]$.

Consider implementing and comparing also other ways for solving the line search problem and the conjugate gradient.

Exercise 8 *

Descent direction methods may use search directions other than the steepest descent mentioned in the previous exercise. In general, which descent direction guarantees to produce a decrease in f?

Exercise 9 *

Show that the positive definiteness of a matrix implies symmetry.

Exercise 10 + (5.1)

Compute the gradient of $\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}$ when A is symmetric.

Exercise 11 * (5.2)

Apply one step of gradient descent to $f(x) = x^4$ from $x_0 = 1$ with both a unit step factor and with exact line search.

Exercise 12 * (5.7)

In conjugate gradient descent, what is the descent direction at the first iteration for the function $f(x, y) = x^2 + xy + y^2 + 5$ when initialized at $[x, y] = [1, 1]$? What is the resulting point after two steps of the conjugate gradient method?