Calculating twin width

Slides Adapted from

Twin-Width: Algorithmic Applications and Open Questions

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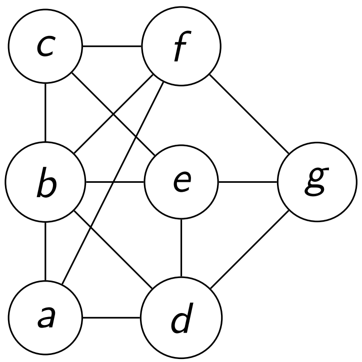
ENS Lyon, LIP

20th January 2025, Solving Problems on Graphs: From Structure to Algorithms

Contraction sequence



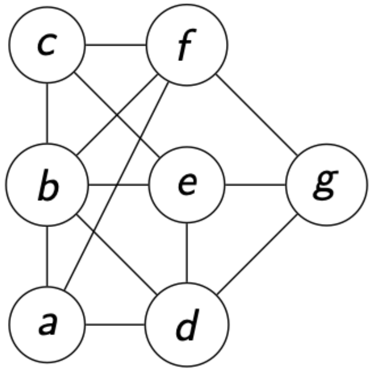
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

A graph is really a (two) colouring of the edges of the complete graph. Colour 1 – edge is present, colour 2 edge is present in the complement graph.
Add a third colour – which will be used to track "disagreement" in neighbours. Colour 3 will be red, colour 1 will be black and colour 2 (as usual) will be "nothing".

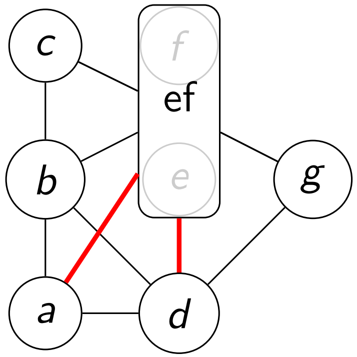
Now lets see how contraction sequences are defined/operate...

Contraction sequence

Starting graph



Pick ANY two vertices – say e and f. Form combined verted (“ef”) and colour edges to “other” vertices by: If e and f had an edge to That vertex then “leave” black. If e and f did not have an edge to that vertex leave “clear” – otherwise colour red.



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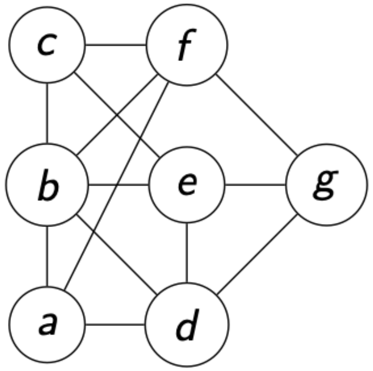

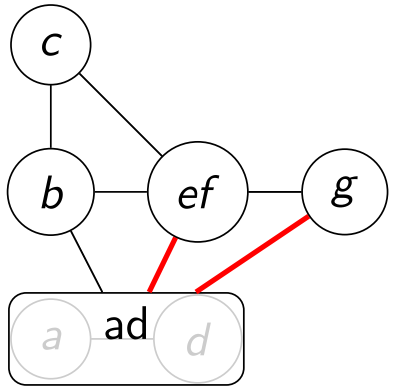

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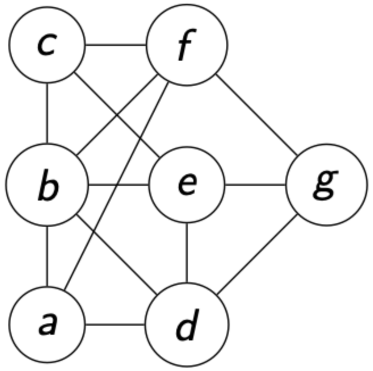

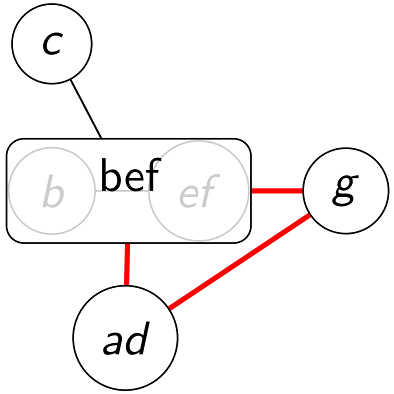

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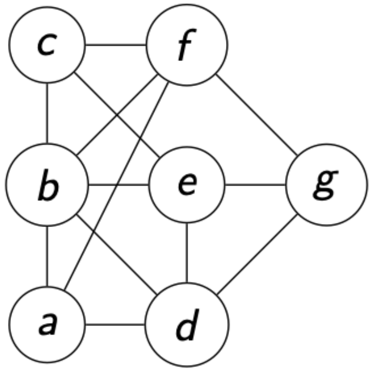

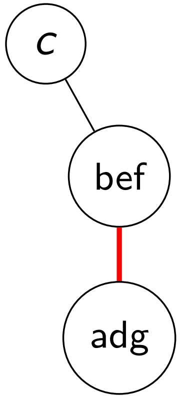

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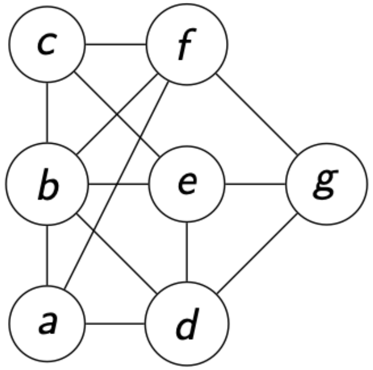

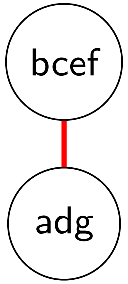

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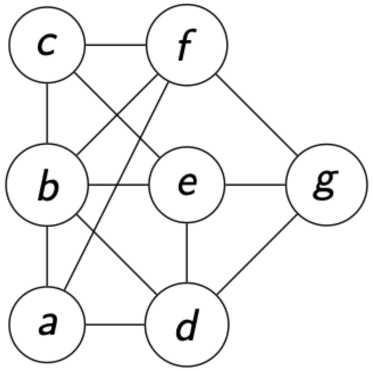

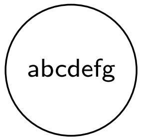

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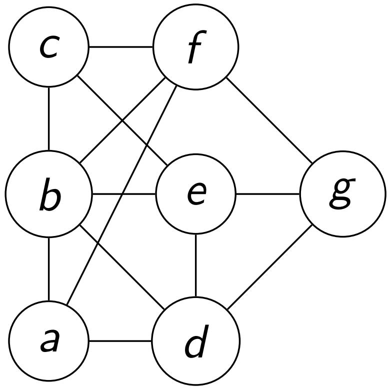
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

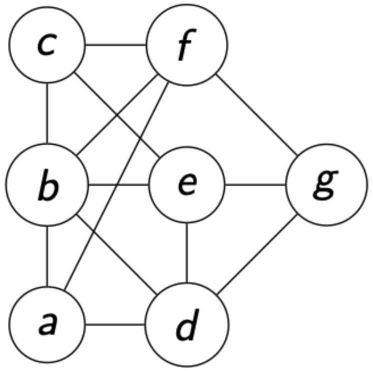


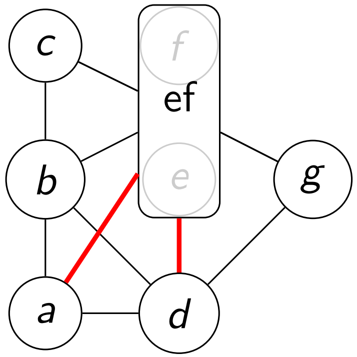
Maximum red degree $= 0$

overall maximum red degree = 0

Contraction sequence

Starting graph





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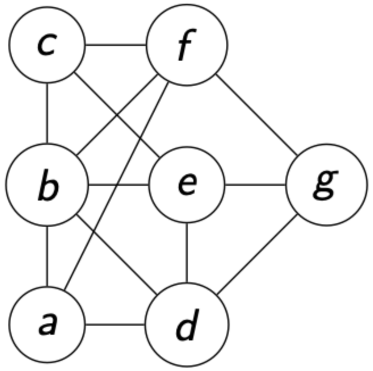
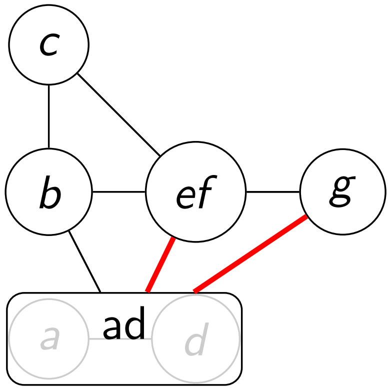
Maximum red degree 2
Overall maximum red degree 2

Contraction sequence

Starting graph





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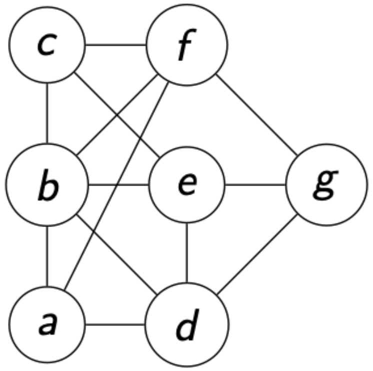

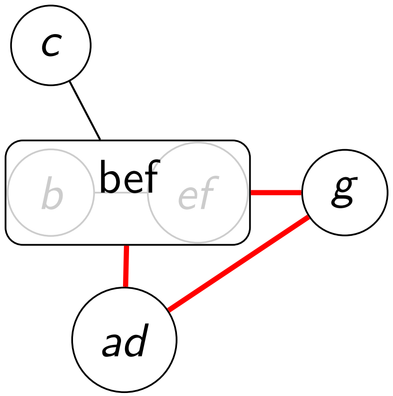

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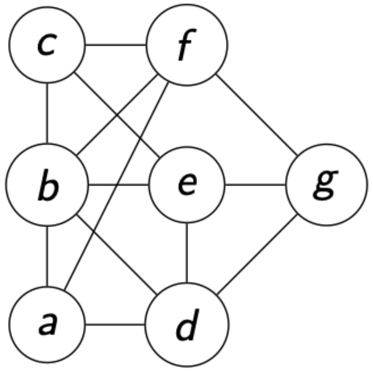
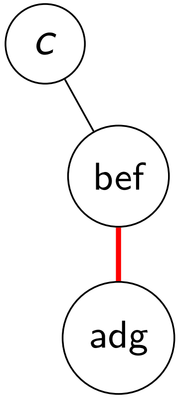
Maximum red degree 2
Overall maximum red degree 2

Contraction sequence

Starting graph





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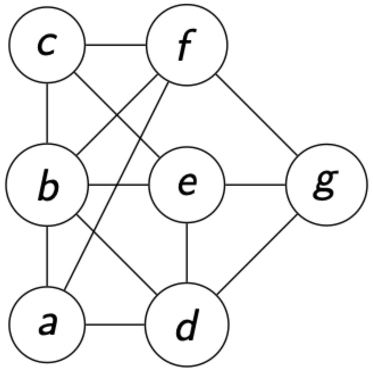
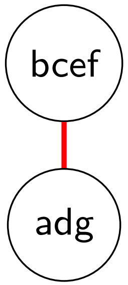
Maximum red degree 1
Overall maximum red degree 2

Contraction sequence

Starting graph





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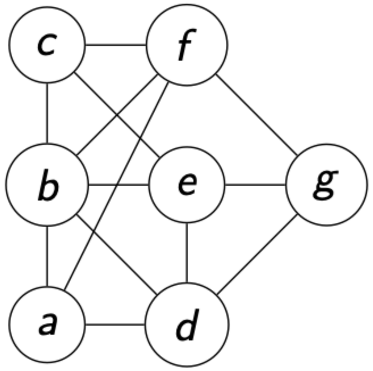

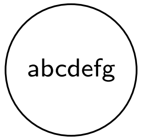

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Maximum red degree 0
Overall maximum red degree 2

But you have to check all possible contraction sequences (in the worst case) – some graphs might be easier! (If you are clever and know how to exploit properties.

Use is theoretical – what you can prove this captures about a graph class

Theorem (B., Geniet, Kim, Thomassé, Watrigant '20 & '21)

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width or clique-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size,*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
- *strong products of two bounded twin-width classes, one with bounded degree,*
- *(given) first-order transductions of the above.*

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Random graphs and graphs of largest twin-width

Theorem (Ahn, Chakraborti, Hendrey, Kim & Oum '22)

Almost surely $tww(G(n, \frac{1}{2})) = \frac{n}{2} - \frac{\sqrt{3n \log n}}{2} \pm o(\sqrt{n \log n})$.

Theorem (Ahn, Chakraborti, Hendrey, Kim & Oum '22)

For any $p \in [\frac{726 \ln n}{n}, \frac{1}{2}]$, $tww(G(n, p)) = \Theta(n\sqrt{p})$.

Let $P(q)$ be the Paley graphs on q vertices.

Theorem (Ahn, Hendrey, Kim & Oum '21)

$tww(P(q)) = \frac{q-1}{2}$.

(all the above have unbounded twin width as n-> infinity)

Open Question

Is there an n*-vertex graph of twin-width at least* $\frac{n}{2}$*?*