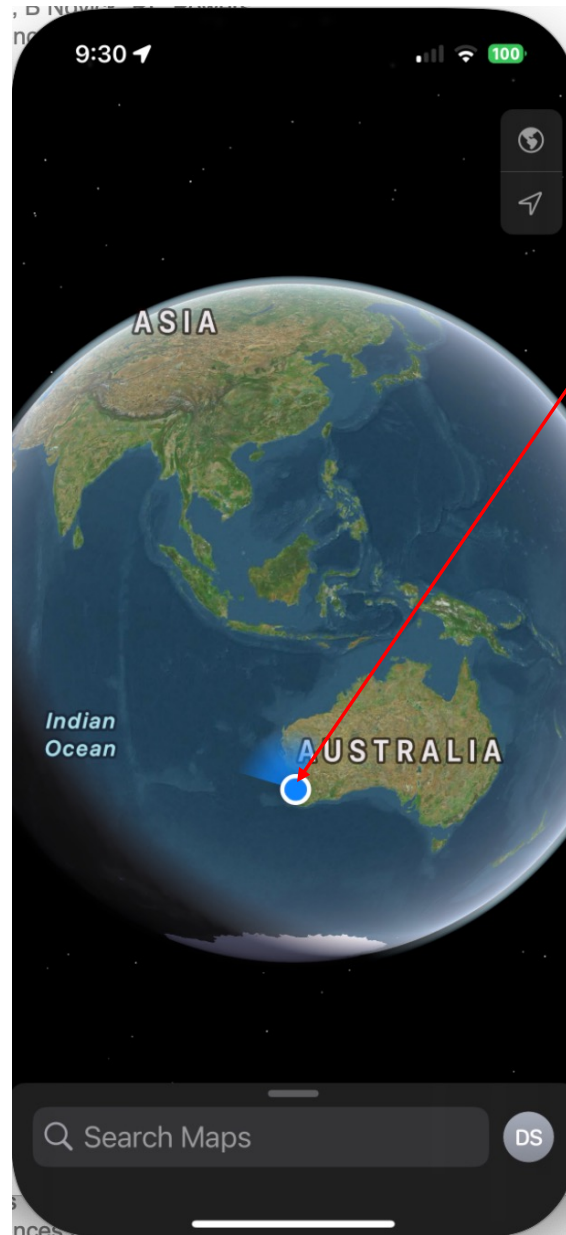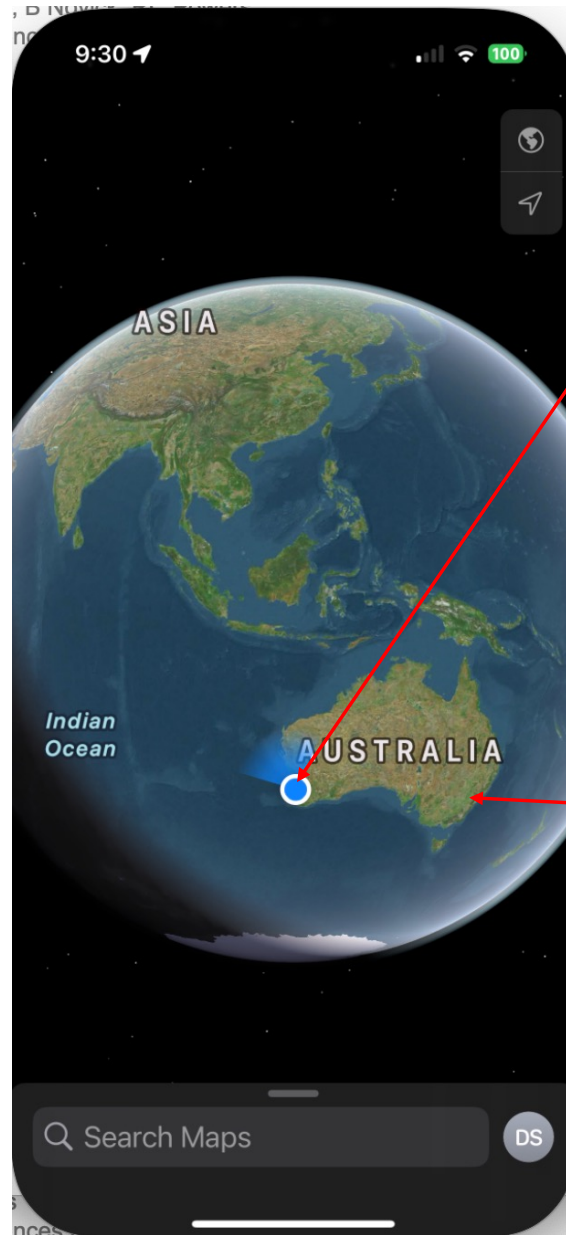# Graphs and networks for clustering and fitting: graph structure
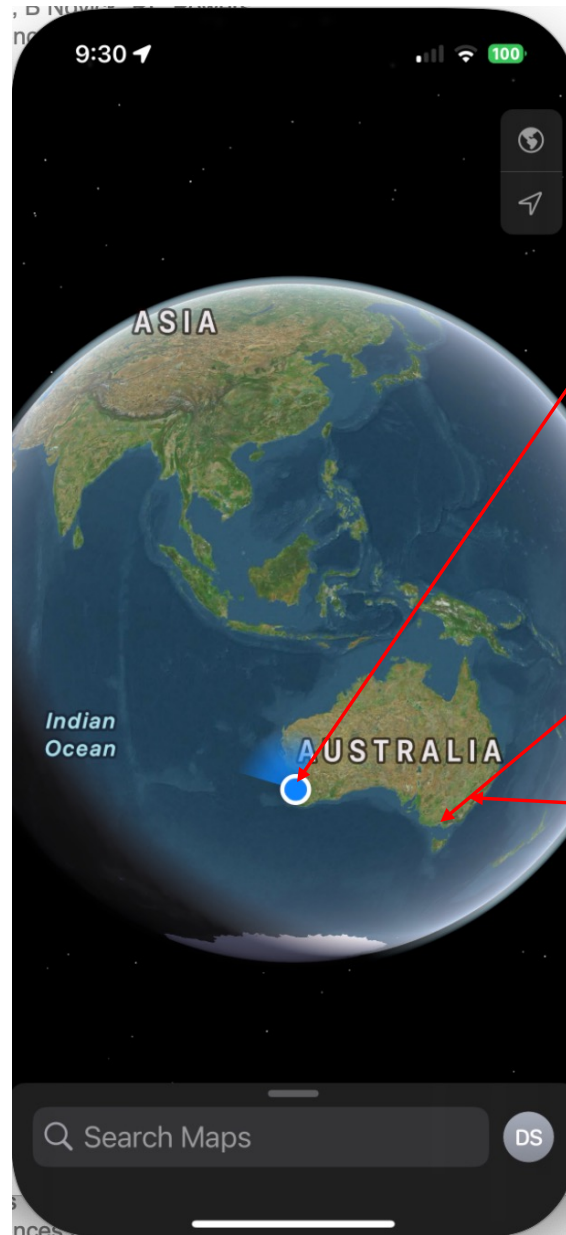
(Harbin, July 2025)

Perth, Western Australia

Perth, Western Australia
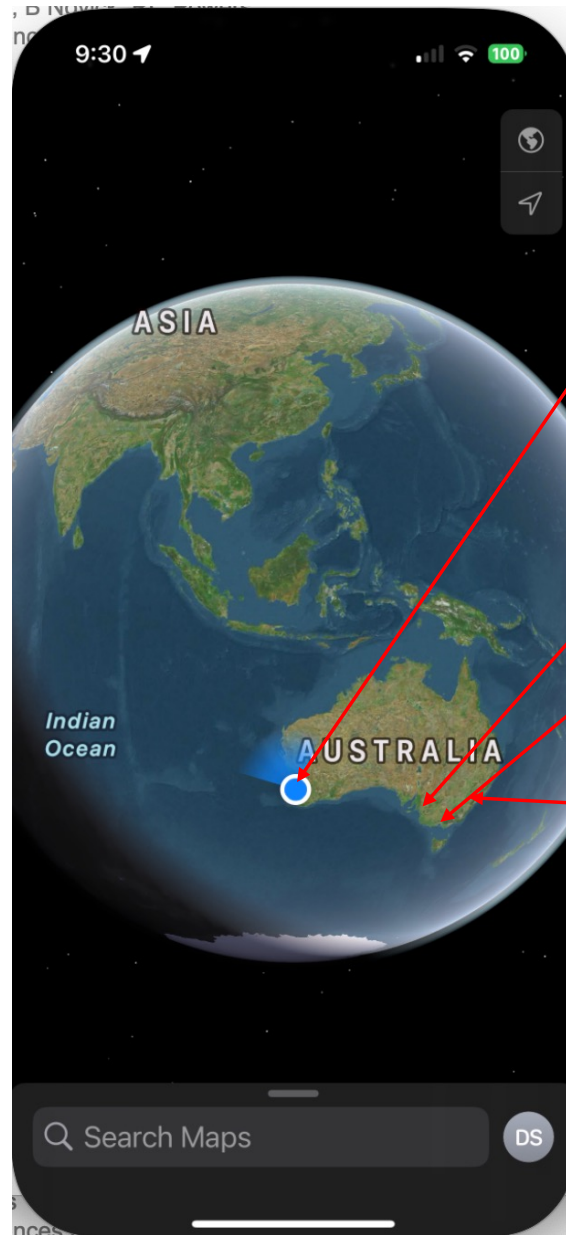
Sydney, New South Wales

Perth, Western Australia

Melbourne, Victoria

Sydney, New South Wales

Perth, Western Australia

Adelaide, South Australia

Melbourne, Victoria

Sydney, New South Wales
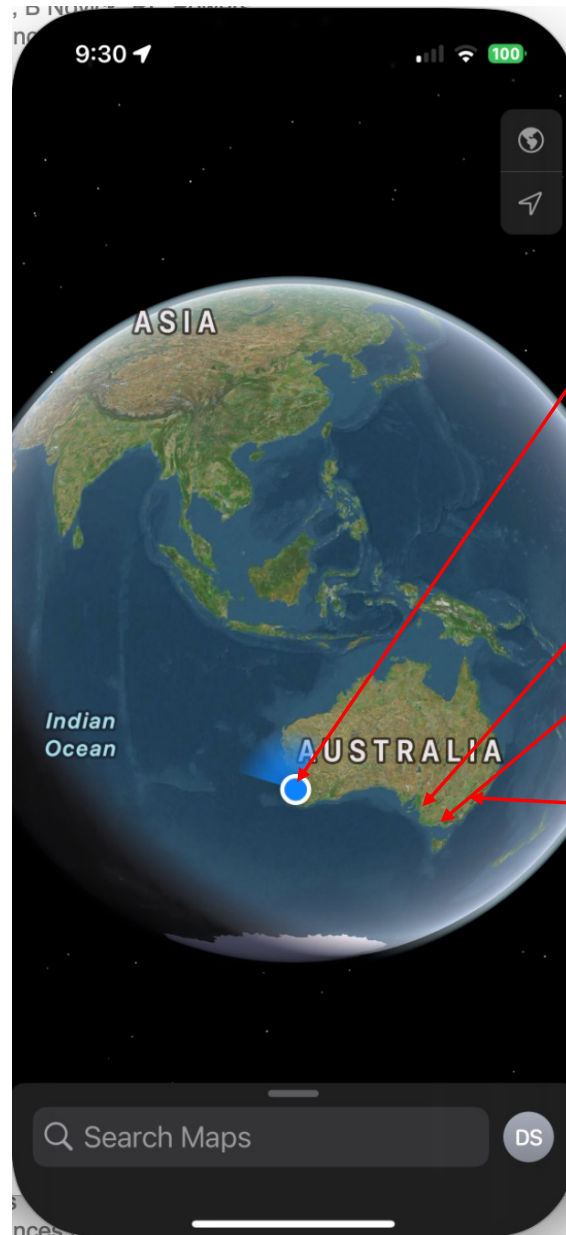
Perth, Western Australia

Adelaide, South Australia

Melbourne, Victoria

Sydney, New South Wales

Sydney-Melbourne – about 1,000km 1hr flight, 1 long day drive
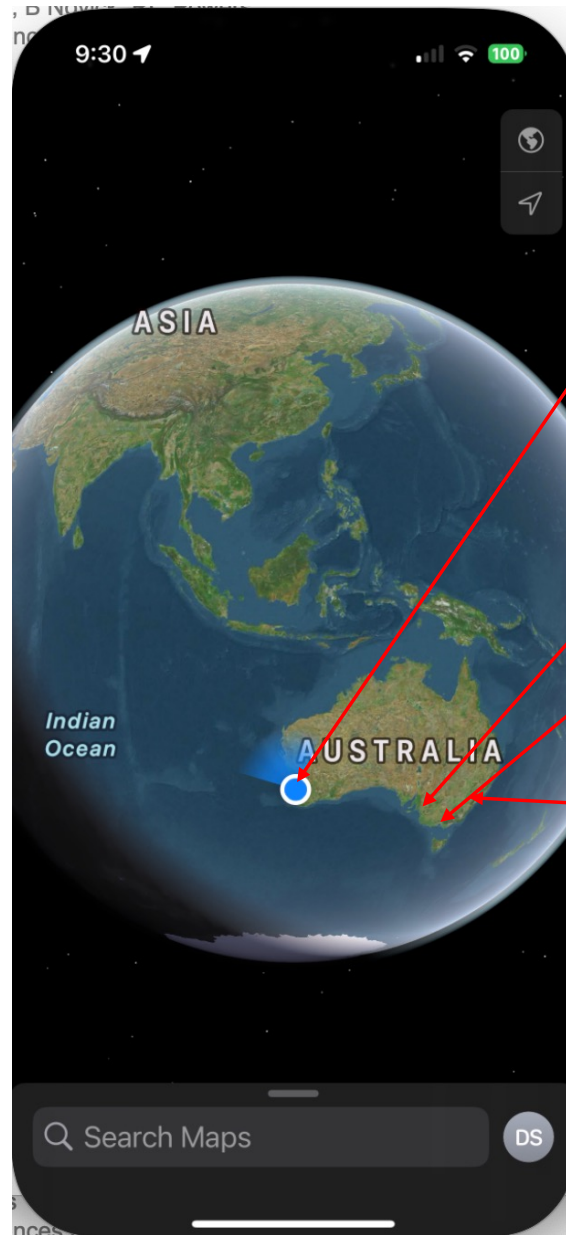
Perth, Western Australia

Adelaide, South Australia

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Melbourne-Adelaide,
Adelaide-Sydney

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About 25 years in Melbourne (La Trobe University, PhD,; 2 years as lecturer; Monash University 18 years...Professor)

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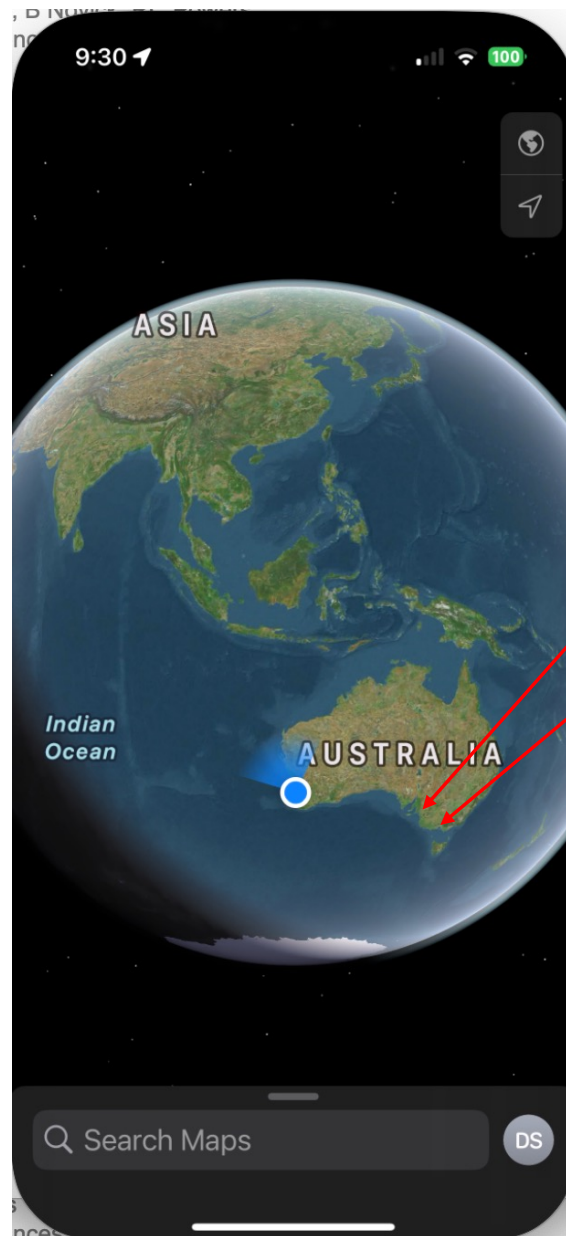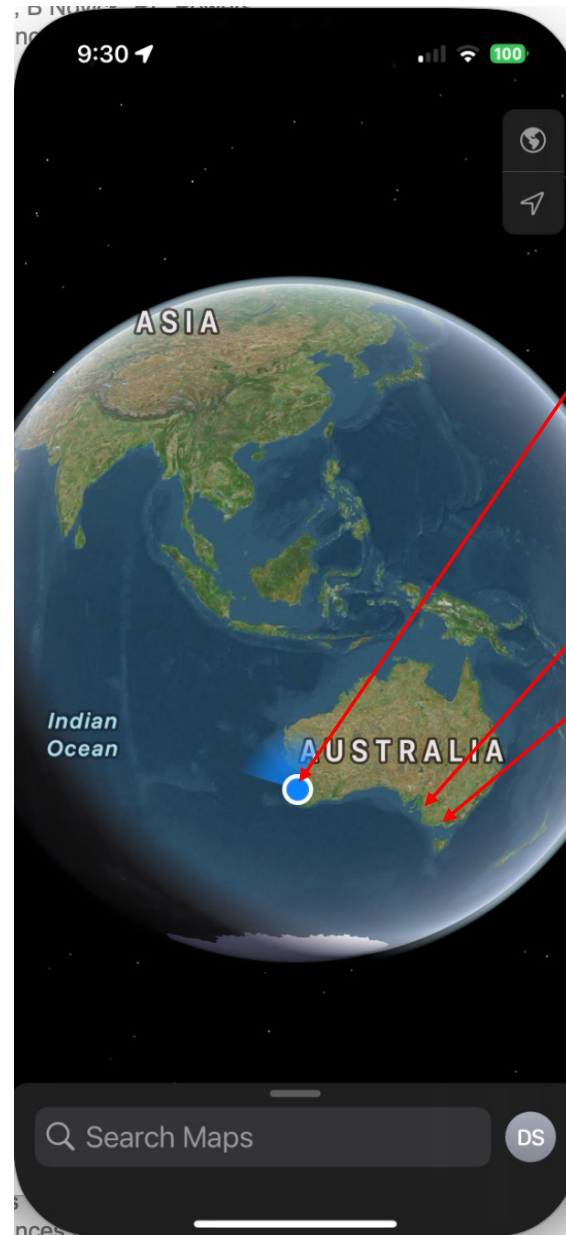
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Go west!!

Perth, Western Australia

Adelaide, South Australia

Melbourne, Victoria

AI/Data Science/Machine Learning

- Finding what information is there (in the data – in the world) and useful

- Using that information to maximum advantage

- This presentation is biased towards information that can (is best?) be modelled as graphs or hypergraphs
  - "networks" "this is related to that/these" "this is connected to that/these"
    - Old-style "knowledge graphs", "scene graphs", Databases,.....,clustering
    - New-style: Graph neural networks, embeddings and then clustering

Conceptual workflow
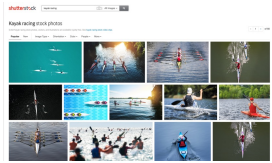
Data

Take measurements/Extract Features
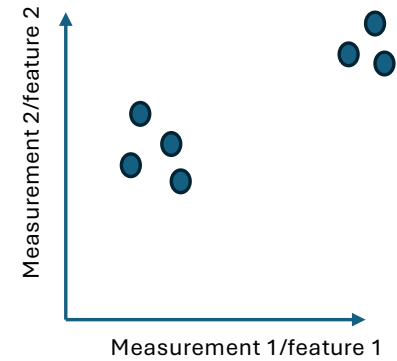
Cluster/form graph

Conceptual workflow
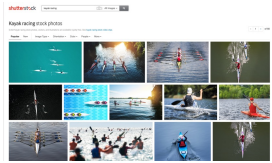
Data

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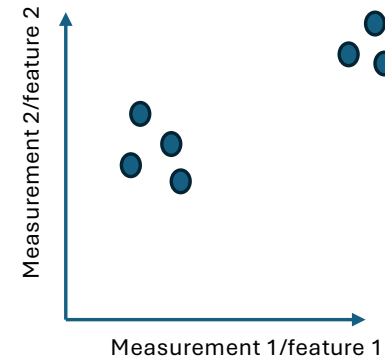
Cluster/form graph

Feature Extraction

Feature embedding/ data embedding

Conceptual workflow
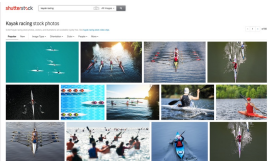
Data

Take measurements/Extract Features

Cluster/form graph

Measurement 2/feature 2

Measurement 1/feature 1

Yes/No/this much... Complete Answer

Feature Extraction

Feature embedding/ data embedding

Clustering/segmentation/ community detection – structure extraction!

Conceptual workflow

Data

Take measurements/Extract Features

Cluster/form graph

Deep learning – learn the feature extraction and even the embedding

Feature Extraction

Feature embedding/
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Measurement 2/feature 2

Measurement 1/feature 1
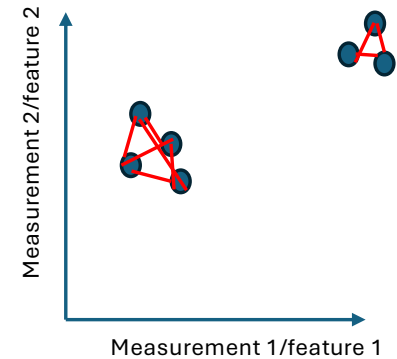
Clustering/segmentation/
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structure extraction!

Yes/No/this
much...
Complete
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Conceptual workflow

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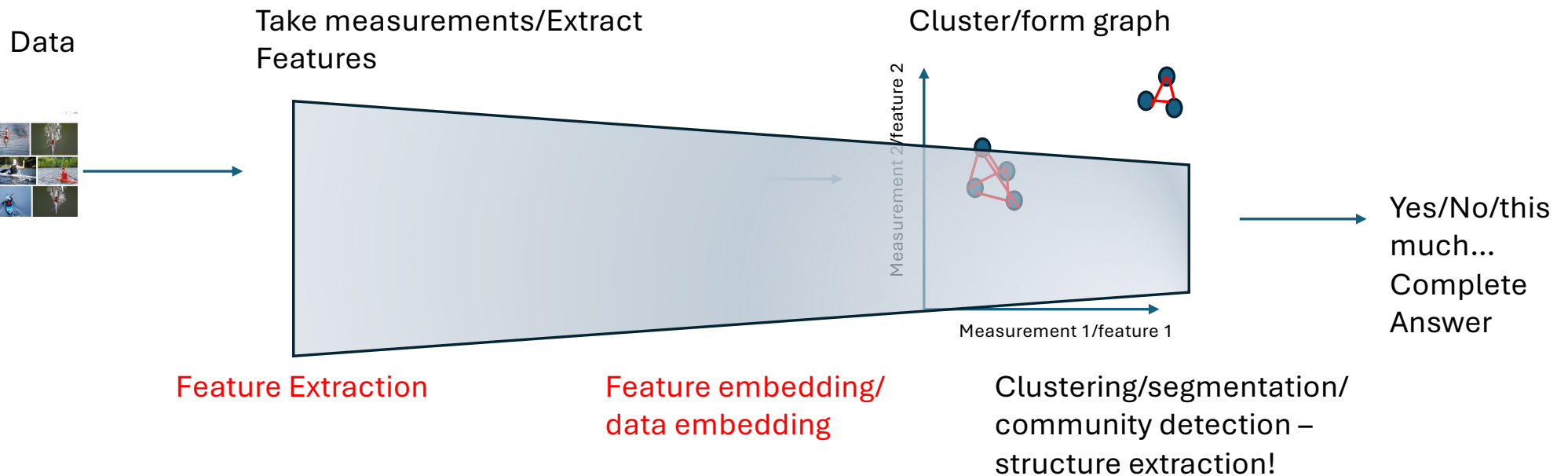
Deep learning – learn the feature extraction and even the embedding – and even the clustering....what metric to use, in what space to embed, etc.

Conceptual workflow
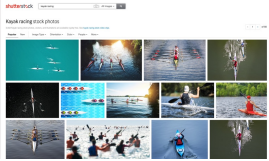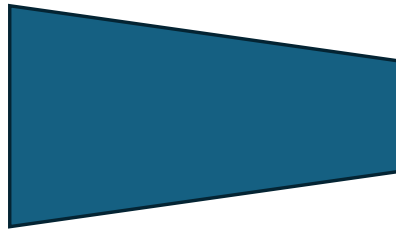
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Deep learning – learn the feature extraction and even the embedding

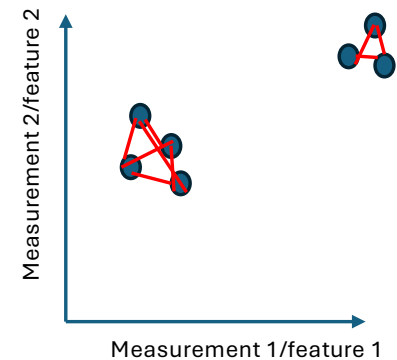Feature Extraction

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Clustering/segmentation/ community detection – structure extraction!
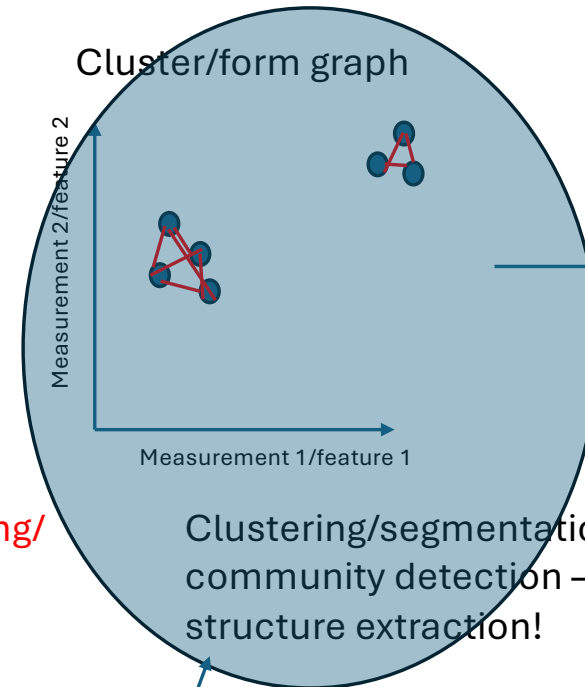
This Talk

Networks (graphs) Under a Microscope

Networks (graphs)
Under a Microscope
....
And through a telescope

Why – what things can we discover??

- It's like a zoo....there are "families" of graphs, just like there are "families" of animals: the bird family, the reptile family, the mammal family.....

- There are hierarchies you can see in these families (just like sub-families of birds, subfamilies of reptiles, etc.

- These hierarchies of families of graphs tell us many things
  - Expressivity of a graph family – what set of problems are these families describing/capturing the characteristics of?
  - Algorithmic (shared characteristics that one can exploit for efficient algorithms)

Why – what things can we discover??

- It's an exciting time! "Graph theory" (structural graph theory) has been developing for decades – and even in some sence tracing back a couple of centuries..BUT

- There seem to be some rapid recent advances making the "picture" more clear/powerful/useful.

In particular...the understanding of the structure of SPARSE graphs (and even the concepts around what are meaningful ways to define "sparsity") has been very much advanced in the last 5 years or so.

Hierarchy of "complexity"/niceness – for `Sparse' graphs

Nowhere dense

Locally bounded
expansion

Locally excluding
a minor

Bounded expansion

Excluding a
topological minor

Excluding a minor

Locally bounded
treewidth

Bounded
genus

Planar

Bounded degree

Bounded treewidth

Outerplanar

Bounded treedepth

Forests

Star forests

Linear (path)
forests

Slightly modified from diagram of Felix Reidl – circa 2020

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Linear (path) forests

Slightly modified from diagram of Felix Reidl – circa 2020

Very, Very, Very nice – trivial to deal with

For example, max clique size is 2; finding a maximal clique is equiv to merely finding an edge

Hierarchy of "complexity"/niceness – for `Sparse' graphs

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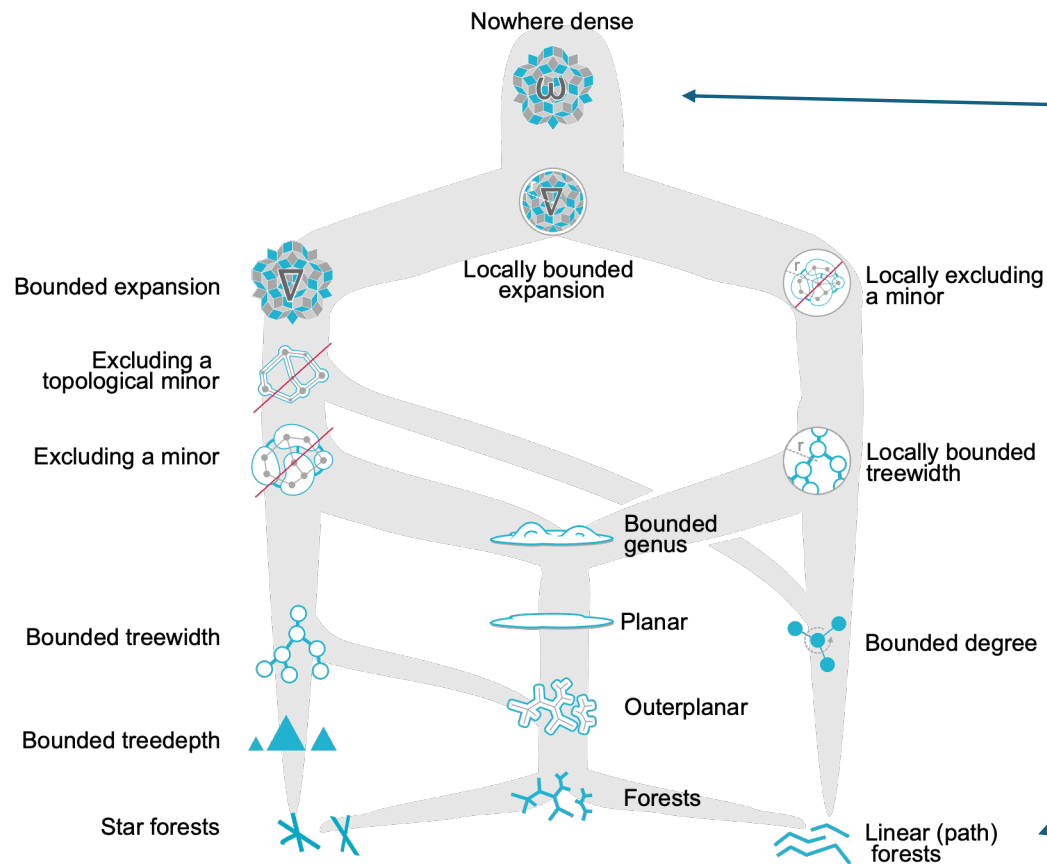
Still – "nice" – fixed parameter tractable for example

Very, Very, Very nice – trivial to deal with

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Slightly modified from diagram of Felix Reidl – circa 2020

Still – "nice" – fixed parameter tractable for example

Very, Very, Very nice – trivial to deal with

"Nicety"
(more properties to exploit)

Hierarchy of "complexity"/niceness – for `Sparse' graphs

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Hierarchy of "complexity"/niceness – for `Sparse' graphs

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Expressivity/Complexity

Still – "nice" – fixed parameter tractable for example

Very, Very, Very nice – trivial to deal with

"Nicety"

Where do my problems/data sit?
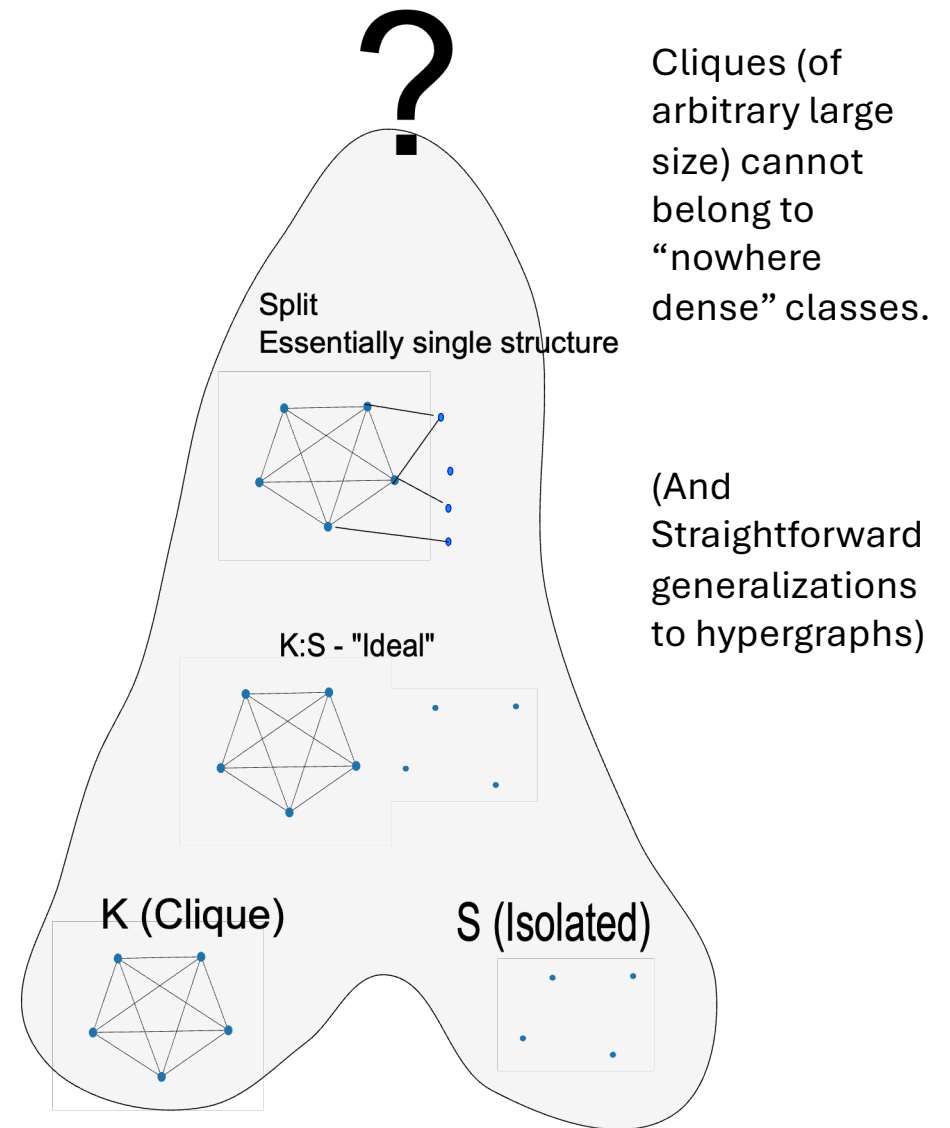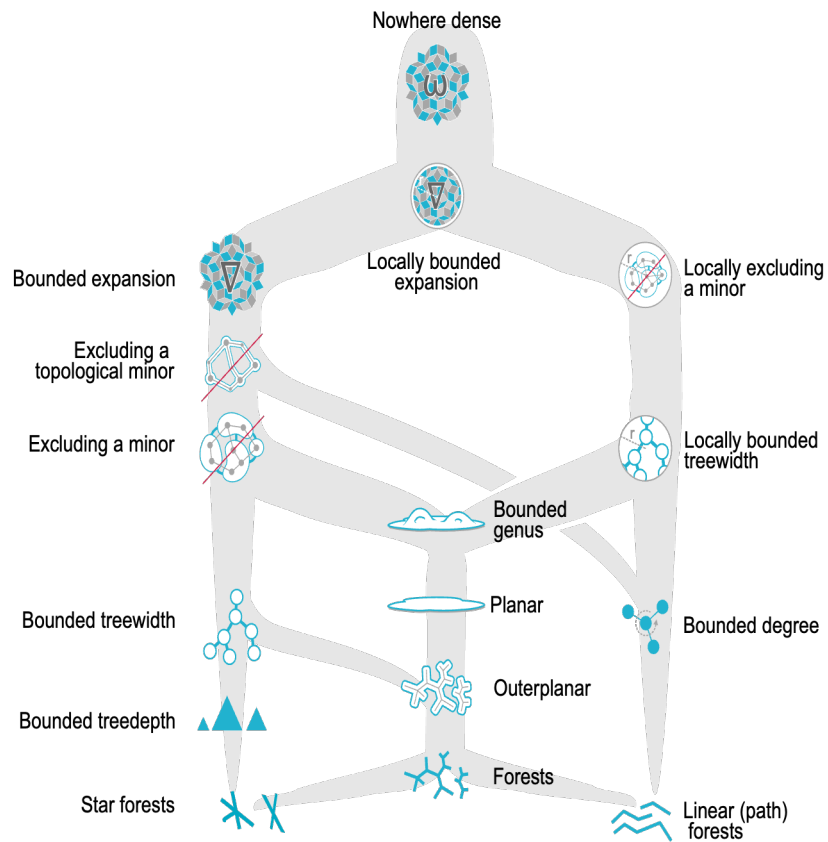
But what problems can be described in such a "sub-zoo"??

- Many!

But what problems can be described in such a "sub-zoo"??

- Many!
- ....

But what problems can be described in such a "sub-zoo"??

- Many!
- ………..

- But Not enough! Not all. Not even the main clustering problems
- That isn't to say you can do clustering on sparse graphs….but it is to say that "most" clustering problems aren't captured by the sparse graph classes.
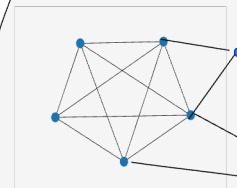
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Linear (path) forests

?

Split
Essentially single structure

K:S - "Ideal"

K (Clique)

S (Isolated)

Cliques (of arbitrary large size) cannot belong to "nowhere dense" classes.

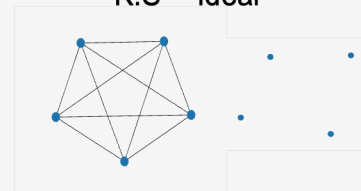(And Straightforward generalizations to hypergraphs)

Cliques (of arbitrary large size) cannot belong to "nowhere dense" classes.

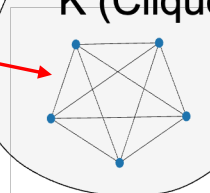(And Straightforward generalizations to hypergraphs)

Split
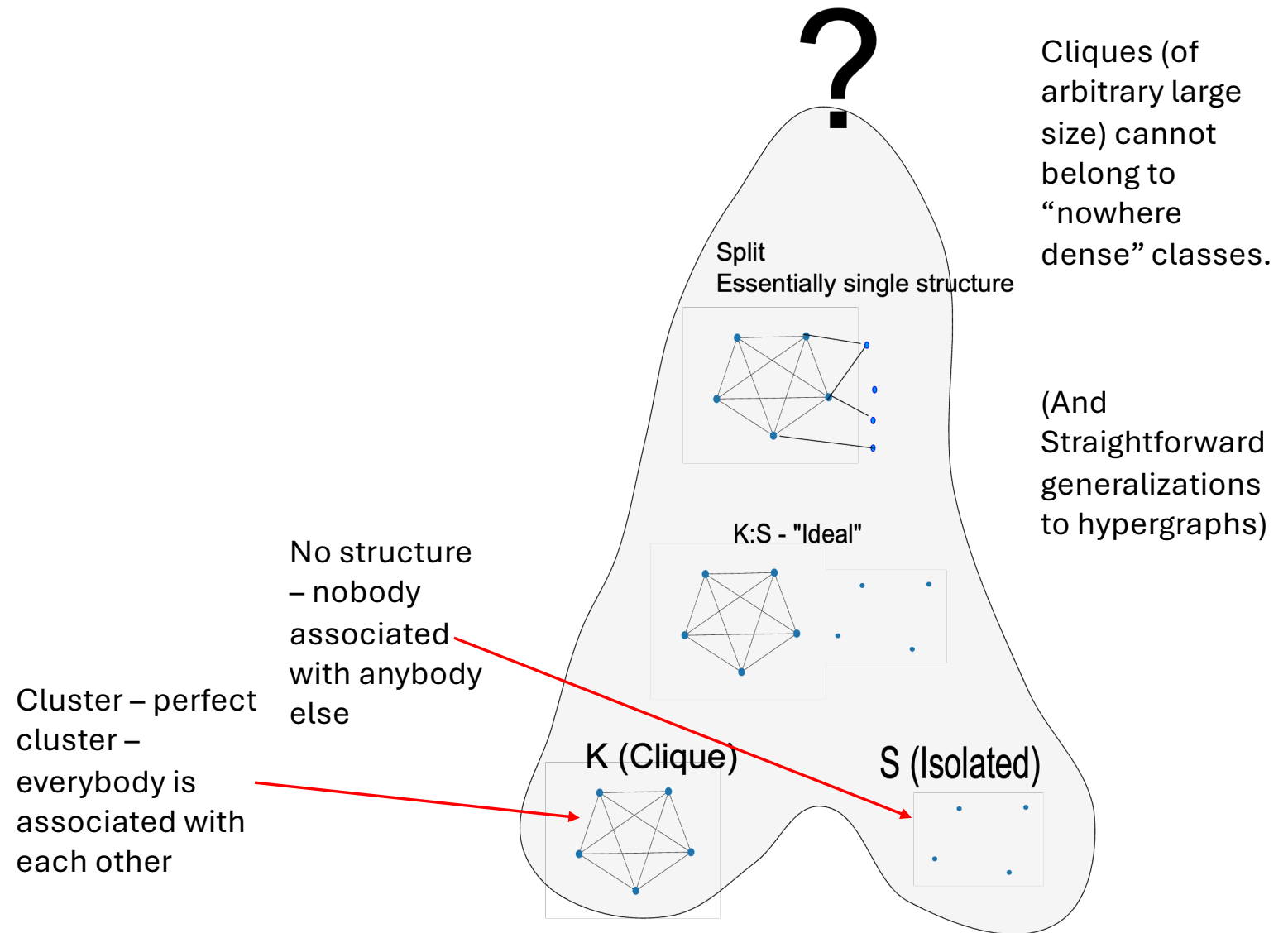Essentially single structure

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S (Isolated)

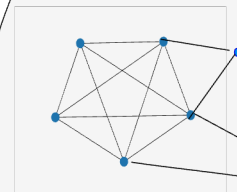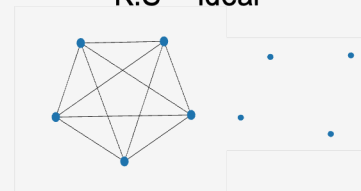Cluster – perfect cluster – everybody is associated with each other

?

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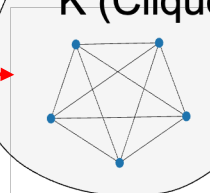(And Straightforward generalizations to hypergraphs)

Split
Essentially single structure

K:S - "Ideal"

No structure – nobody associated with anybody else

Cluster – perfect cluster – everybody is associated with each other

K (Clique)

S (Isolated)

?

Cliques (of arbitrary large size) cannot belong to "nowhere dense" classes.

Split
Essentially single structure

(And Straightforward generalizations to hypergraphs)

K:S - "Ideal"

K (Clique)

S (Isolated)

"Building Blocks"
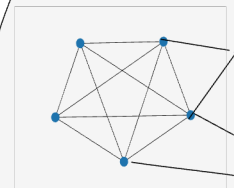By themselves model only the "trivial" situations:
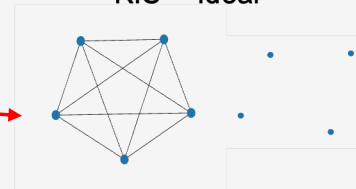Perfect – structure and nothing else/no structure and nothing else

?

Cliques (of arbitrary large size) cannot belong to "nowhere dense" classes.

(And Straightforward generalizations to hypergraphs)

Split
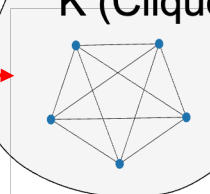Essentially single structure

K:S - "Ideal"

Realistic – but "ideal" data is a mix of structure and "background" and the structure is "pure" and the background "pure" and no interaction/association between structure and background

K (Clique)

S (Isolated)

"Building Blocks"
By themselves model only the "trivial" situations:
Perfect – structure and nothing else/no structure and nothing else

"not ideal" but still relatively ideal and still single structure with some background – and only some "accidental" or minor associations between the two.

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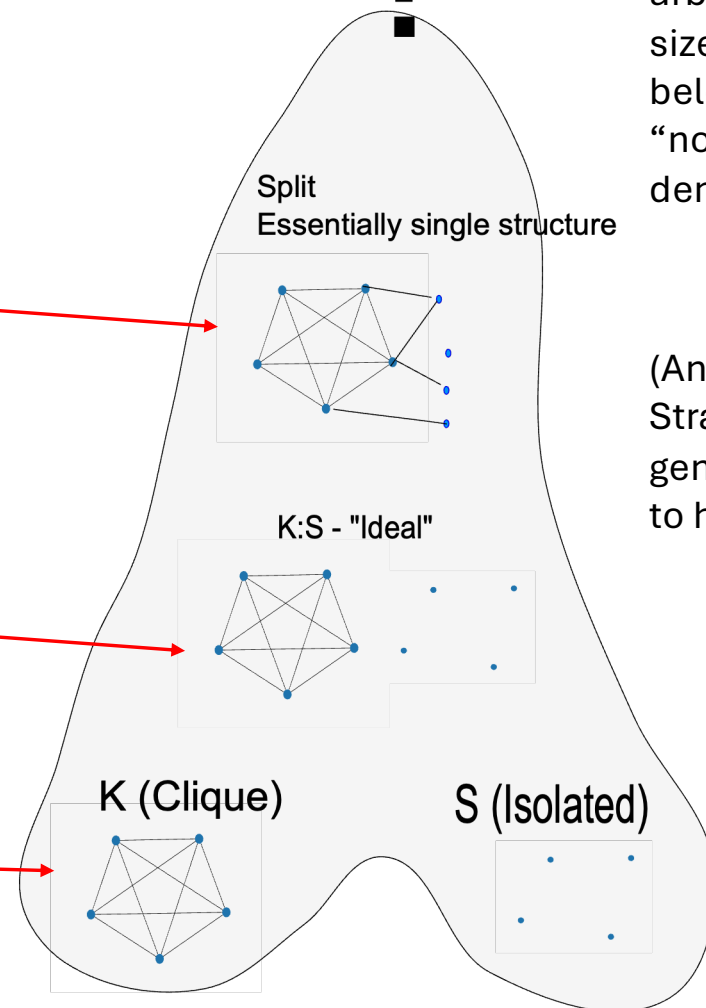
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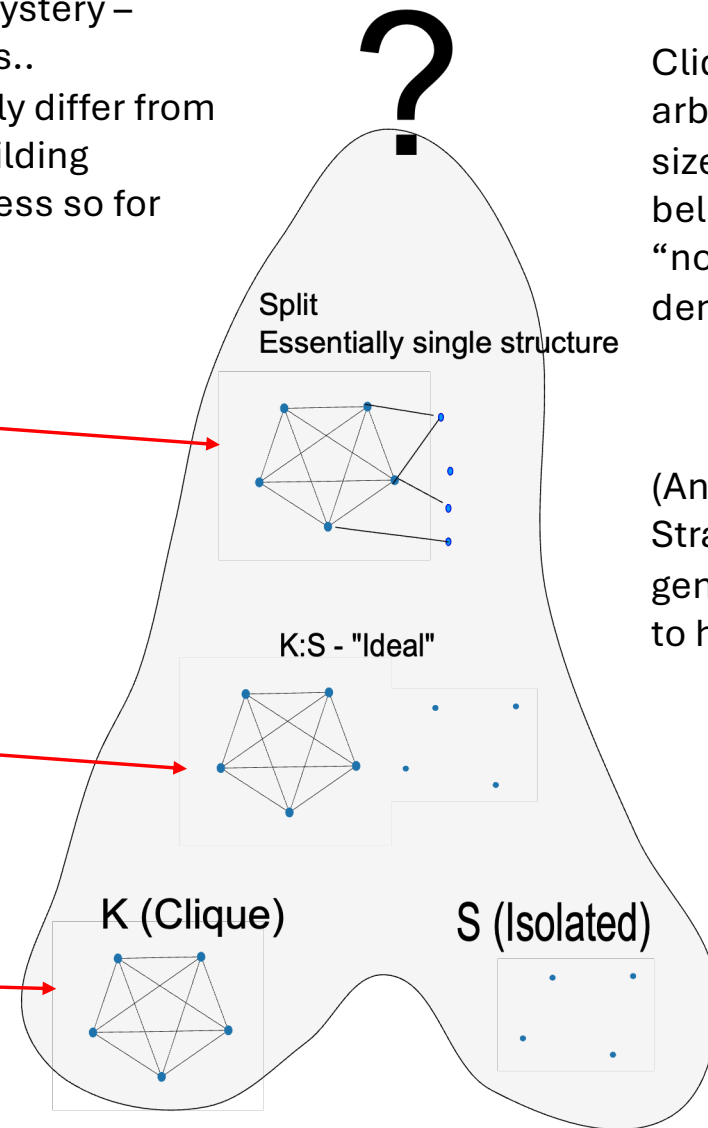(And Straightforward generalizations to hypergraphs)

Much above this....somewhat a mystery – especially in the hypergraph cases..
AND the relevant "actors" will likely differ from problem type to problem type. Building blocks/base common...less and less so for classes "up the tree"

?

Cliques (of arbitrary large size) cannot belong to "nowhere dense" classes.

Split
Essentially single structure

"not ideal" but still relatively ideal and still single structure with some background – and only some "accidental" or minor associations between the two.

(And Straightforward generalizations to hypergraphs)

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S (Isolated)

"Building Blocks"
By themselves model only the "trivial" situations:
Perfect – structure and nothing else/no structure and nothing else

It's an exciting time for RESEARCH!

- "Doing the same thing" achieved for sparse classes but now in classes more suited to covering the entire space of clustering problems. (realm of Mathematicians (graph theorists, logicians, combinatorics, topology,.....; and the realm of theoretical computer scientists)

- Working out the implications of each (small ......or large) advance. – realm of more applied people as well as the "theorists"