Weighted graphs: putting functions over edges and vertices

Adjacency Matrix

- To every undirected graph, we can associate a (symmetric) matrix A: whose entries A_{ij}=0 if vertex i is connected by an edge to j
- This simple idea open up the study of graphs by studying their associated matrices
- It also suggests generalisations:
  - E.g. $A_{ij}=w_{ij}$ for some wij in (0,1) – with interpretation that it is the strength of the edge. This is attaching a value to an edge – a function mapping edges (pairs of vertices) to values. For loopless graphs – A_{ii}=0.
  - Likewise we might attach values to vertices - a function/signal over the vertices
  - These generalizations give "structured signals" (more "exotic" than a 1-D regularly sampled signal like sound, or even a 2D signal like an image on a rectangular array of pixels) – how would we process/deal with such signals -> Graph Neural Networks

Spectral approaches

- But first let's look at a relatively long-standing consequence of this line of thinking (but revitalized interest recently) – looking at the eigenvalues/vectors of the matrices and what this tells us about the graph/signal on the graph.
- Note: there are MANY matrices on might associate with the graph –

L=A-DI (or DI-A – only a sign change) Laplacian Matrix, D is a diagonal matrix D_{ii} is the degree of vertex I, for example.

- Many things are known about the eigenvalues/vectors of some of these matrices – at least for some classes of graphs, sometimes for all graphs…..

Example facts…

- For a simple graph (undirected, no self-loops) A is symmetric and the eigenvalues are real and can be sorted in descending sequence. The eigenvectors are orthogonal.
- The largest eigenvalue is less than or equal to the maximum degree, the difference between the top 2 eigenvalues is called the spectral gap and is related to “how expanding the graph is” - a topic of very much recent interest (highly expanding graphs are very useful).
- A^k has entries i,j that five the number of walks of length k between vertices i and j. (Counting triangles in a graph is easy…trace(A^3)/3!)

Laplacian Matrix

- L is symmetric positive definite – so all eigenvalues are positive
- It has a smallest eigenvalue of 0 and corresponding eigenvector the “all 1’s vector” – uninteresting…..not particularly useful….
- The number of connected components is the dimension of the nullspace of L (the multiplicity of the 0 eigenvalue)
- “Spectral clustering” is a term that applies to a group of algorithms that use the eigenvalues and eigenvectors of a matrix associated to G (e.g., matrix L). The differ on what the matrix is (for example, affinity matrix – whose extreme binary version is just A, of a lapacian derived from the affinity matrix), which eigenvector/values selected, how to cluster the projection on the eigenvectors etc.

Generalization to Hypergraphs

- A k-uniform hypergaph can be identified with a symmetric k-tensor in exactly the natural generalization of a 2-tensor (matrix) for a graph.
- Unfortunately, two things come into play:
  - The storage requirements go up exponentially in k
  - The theory of eigen-analysis of tensors (even for tensors of dimension 3) is much more complex and less understood
  - There is some recent impressive progress and a lot of research

Generalization to simplicial complexes

- Much less is known – but again, a lot of recent attention to defining Laplacians and the like and how to interpret various results about the underlying structure of simplicial complex captured by these quantities….