

Graph and Hyper-Graph Structures in Signal, Image Processing and Computer Vision

.....(and Tensors)

(It's all sets of subsets!!!!)

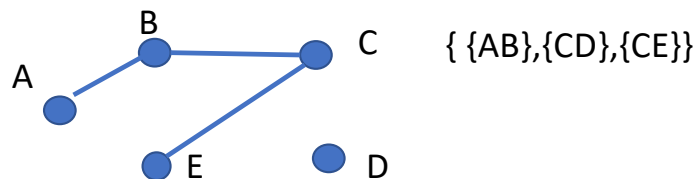
David Suter

Centre for AI and Machine Learning

Edith Cowan University

“It’s all just sets of subsets!!!”

- A graph is just a set of subsets (of vertices.....of size two)

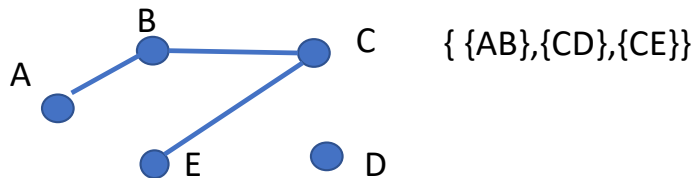


(can represent more: edge weights – strength of association, vertex weights – vertex attributes...)

- The vertex labels are usually “arbitrary”
 - Can use 1,2,3..... (mostly what mathematicians use)
 - If you permute the labels – you have the same graph: “isomorphic”
 - It is actually hard (expensive) in general to check if two graphs are isomorphic
 - But there is a popular test that is quite cheap and if it fails that test then not isomorphic – Weisfeiler-Lehman
 - Weisfeiler disappeared in controversial and mysterious circumstances in 1985 in Chile
 - Even simpler partial test – see if have the same DEGREE SEQUENCE
- The set of all subsets (powerset) is the set of all possible edges – 2^N in size (for N vertices)
- The complete graph has all edges – given the symbol K_N
- The complement graph of graph G has the edges that are “missing in G” and does not have the edges that were in G

Did you understand the previous slide?

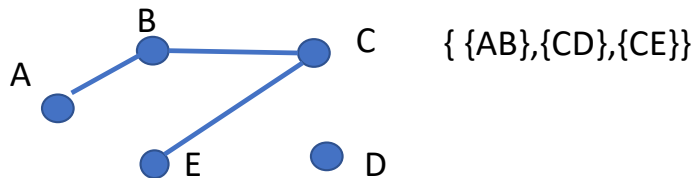
- A graph is just a set of subsets (of vertices.....of size two)



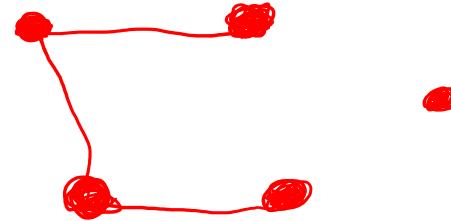
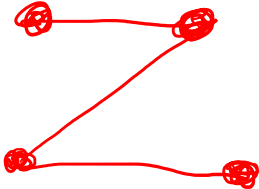
- What is the complement graph?
- How many edges does it have?

Q's

- A graph is just a set of subsets (of vertices.....of size two)



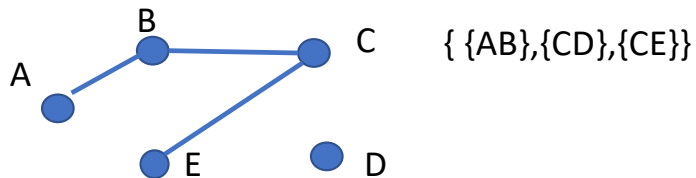
- Are the graphs above and below isomorphic?



- How can you sometimes show two graphs are not isomorphic?

Hypergraphs

- A graph is just a set of subsets (of vertices.....of size two)



(can represent more: edge weights – strength of association, vertex weights – vertex attributes...)

- A (uniform) k-hypergraph is just a set of subsets (of vertices...of size k)
- A hypergraph is just a set of subsets (or vertices...of any size)

Hypergraphs – generalizing from graphs

- A lot of things generalize reasonably readily from graphs (and some do not!)
- For example – independent set of vertices: a set of vertices not containing any hyperedge.

- Give an independent set of the hypergraph:

$\{1,2,3\}, \{4,5,6\}, \{6,8,9\}$. (what is the “ground set”?) (is it a uniform hypergraph?)

- Trivially independent sets
- Non-trivial independent sets
- Maximum independent set?

Hypergraphs – generalizing from graphs

- A lot of things generalize reasonably readily from graphs (and some do not!)
- For example – vertex cover: a set of vertices “touching every edge” (sometimes called “hitting set”)
- Give a hitting set/vertex cover for the 3-hypergraph
 $\{1,2,3\}, \{2,3,6\}, \{1,2,5\}$

Hypergraphs – generalizing from graphs

- Drawing a hypergraph

Gets messy very quickly! But not so much an issue because pretty well only uninteresting (small) graphs can be meaningfully drawn anyway...no-one would try to draw the graph of the internet!

Hypergraphs – generalizing from graphs

- Drawing a hypergraph

Hypergraphs – generalizing from graphs

- Just as graphs are usually defined to exclude loops and parallel edges (SOMETIMES called simple graphs to emphasize these exclusions) – Hypergraphs usually exclude edges that are subsets of other edges, multisets (multisets are a generalization of sets that allow repeated edges...).
- Sometimes the term “clutter” is used when we forbid edges that are proper subsets of other edges.

Simplicial Complexes – graphical realization

- Simplices
 - “nothing” (empty set) – is a simplex in some definitions
 - Mainly included simply to make a definition “consistent” – see later
 - Single vertex – 0-dimensional simplex
 - Single edge between two vertices – 1-dimensional simplex
 - “Triangle” of 3 edges – 2-dimensional simplex - *with the interior*
 - 3-dimensional simplex.....etc.....

$$2^4 = 16$$

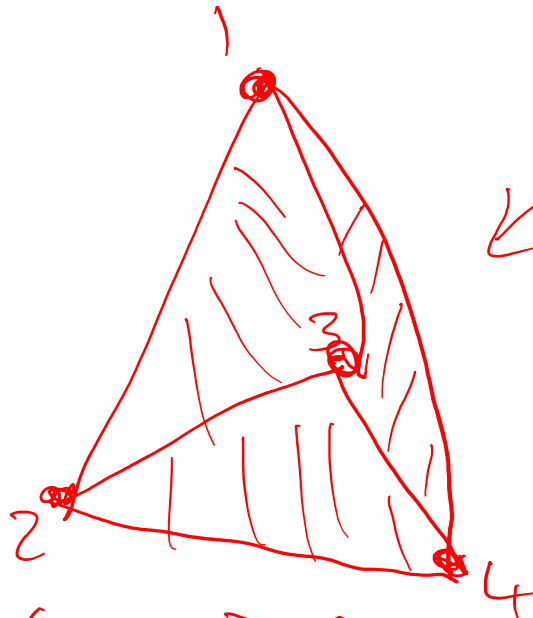
$$C_4^4 = 1$$

$$C_3^4 = 4$$

$$C_2^4 = 6$$

$$C_1^4 = 4$$

$$C_0^4 = 1$$

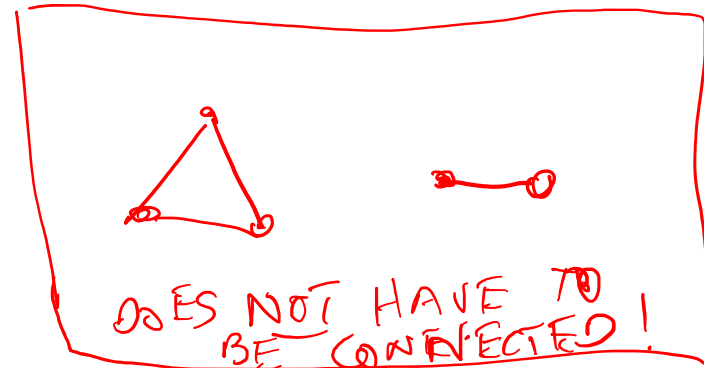
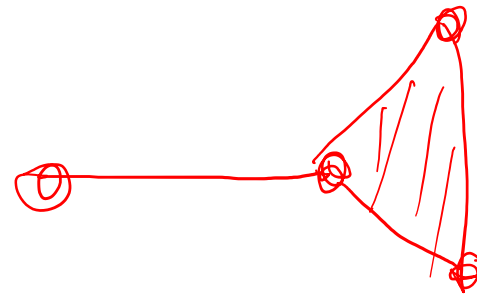
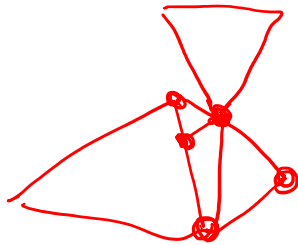
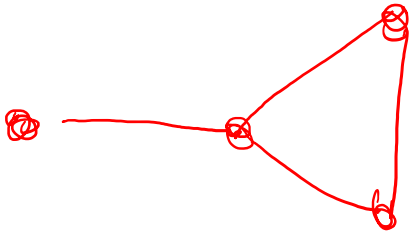


AND
INTERIOR
SHADED
(INCLUDED)

$\{1, 2, 3, 4\} : \{1, 2, 3\} \{1, 2, 4\} \{2, 3, 4\} \{1, 3, 4\} : \{1, 2\} \{1, 3\} \{1, 4\} \{2, 3\} \{2, 4\} \{3, 4\} : \{1\} \{2\} \{3\} \{4\} : \emptyset$

Simplicial Complex

- “Simplices sharing vertices”



Simplicial Complex – abstract definition – it's all sets of subsets!

- A downward closed set of sets...
 - If A is in the simplicial complex then so is B for any B contained in A (usually including the empty set)
 - Exercise – check this is true for the simplicies given
- A non-empty subset of simplicial complex is given the name “a face”.
- For a non-empty simplicial complex, there is always one or more (non-empty) maximal faces.

Simplicial Complex – abstract definition – it's all sets of subsets!

- A downward closed set of sets...
- So only need to specify the maximal faces....

Simplicial Complex – abstract definition – it's all sets of subsets!

- A downward closed set of sets...
- A simplicial complex with all maximal faces the same size is called a pure or homogeneous simplicial complex
- We can take the k -skeleton of a simplicial complex – all faces of size k in the complex
- A 1-skeleton of a simplicial complex is a graph
- A k -skeleton of a simplicial complex is a k -uniform hypergraph

Examples!

Aside: Clutters and simplicial complexes...

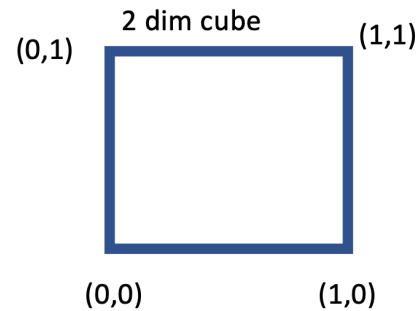
- In some sense a clutter (generalization of hypergraphs that forbids edges that are (proper) subsets of other edges) – are the opposite of simplicial complexes *because* the latter enforces that proper subsets are in the collection of sets.

So far, graphs, hypergraphs, simplicial complexes...one more structure – Boolean Cube

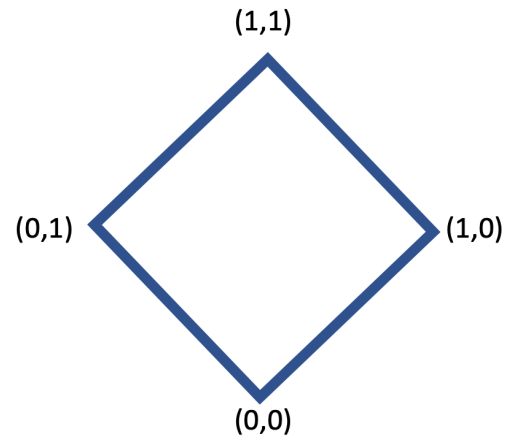
- Again, it all sets of subsets.
- The vertices of the Boolean Cube can be identified with each and every one of the subsets of some ground set. Usually, we use $1\dots N$ as the ground set.
- We usually encode the membership in the subset by “1-hot” encoding. Bit k is 1 if k is in the subset, otherwise bit k is 0. This gives the unit Boolean Cube vertices.

So far, graphs, hypergraphs, simplicial complexes...one more structure – Boolean Cube

- 1-dim Boolean cube is boring...the interval $[0,1]$
- 2-dim Boolean cube



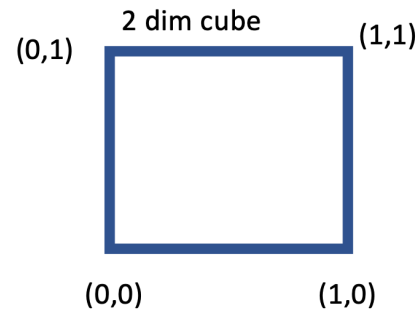
Can represent all subsets
of 2 element ground set



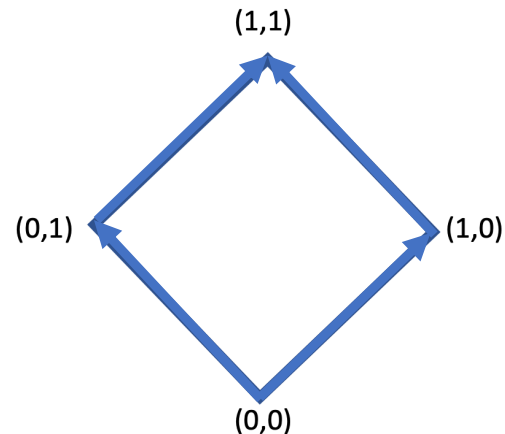
Note: we are only talking
about the vertices. The edges
we draw make a graph...but
we don't want to focus on that!

So far, graphs, hypergraphs, simplicial complexes...one more structure – Boolean Cube

- 1-dim Boolean cube is boring...the interval $[0,1]$
- 2-dim Boolean cube



Can represent all subsets
of 2 element ground set

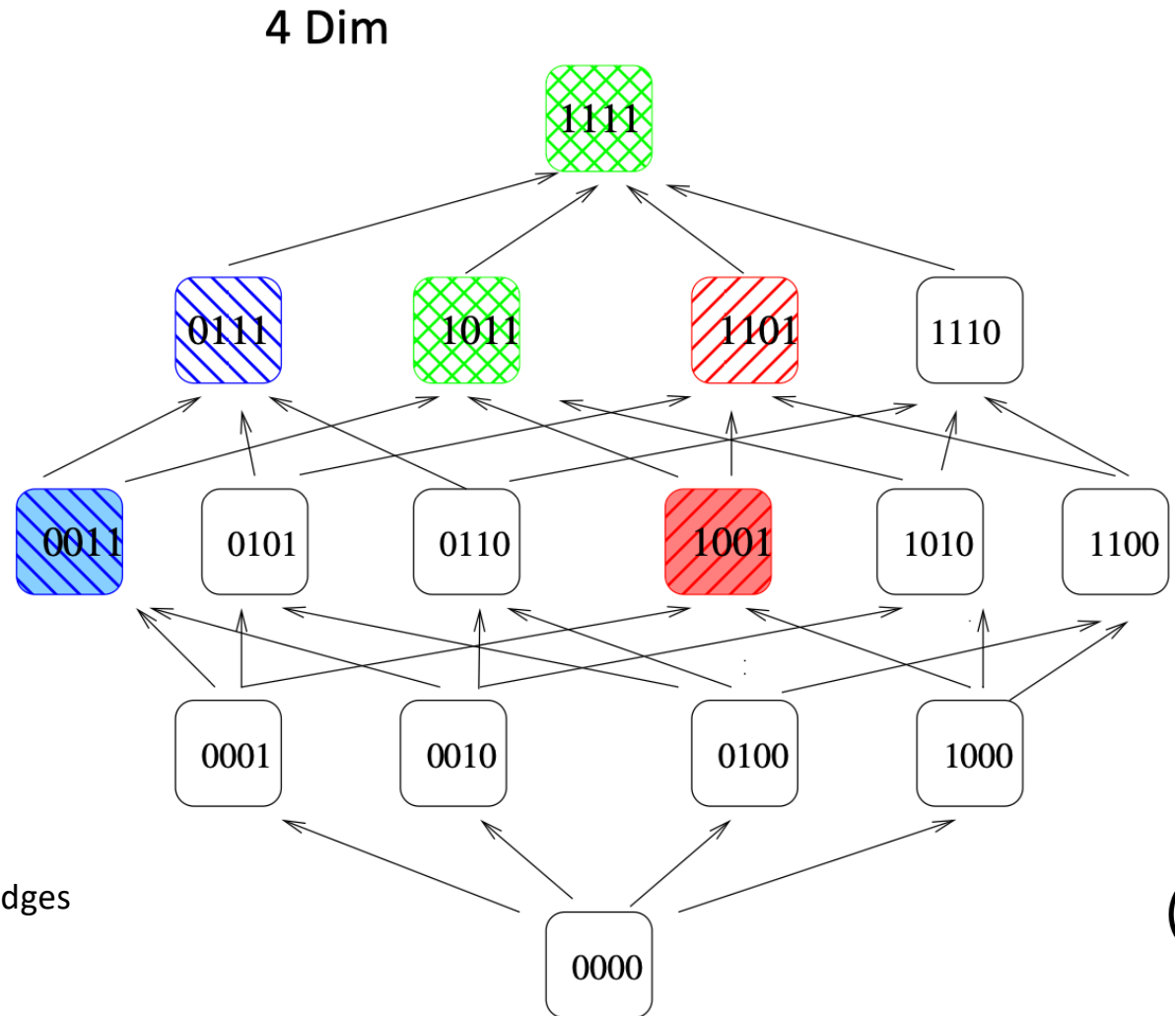


Note: but if we give these
edges a direction – we get a
directed graph – Hasse Diagram
of subset inclusion

Boolean Cube

- 1-dim Boolean cube $[0,1]$
- 2-dim Boolean cube
- 3-dim Boolean Cube
- 4-dim Boolean Cube
(ignore the colours for now)

Hasse Diagram – directed
Boolean Cube – undirected edges



N-dim - Boolean cube – vertices all subsets of $1\dots N$

- Set of vertices contain all (hyper)graphs on N vertices
- Also contain all simplicial complexes on N -vertices (essentially, because simplicial complexes are downward closed, a simplicial complex divides the Boolean cube into two (usually unequal in size) “halves”) – the bottom “half” (the simplicial complex itself) – the top “half” (the complement of the simplicial complex).
- Boolean cube has 2^N vertices.