Graph Theory

Part Two

Adapted from CS103 Stanford

Outline for Today

- ***Walks, Paths, and Reachability***
  - Walking around a graph.
- ***Graph Complements***
  - Flipping what's in a graph.
- ***The Pigeonhole Principle***
  - Everyone finding a place.

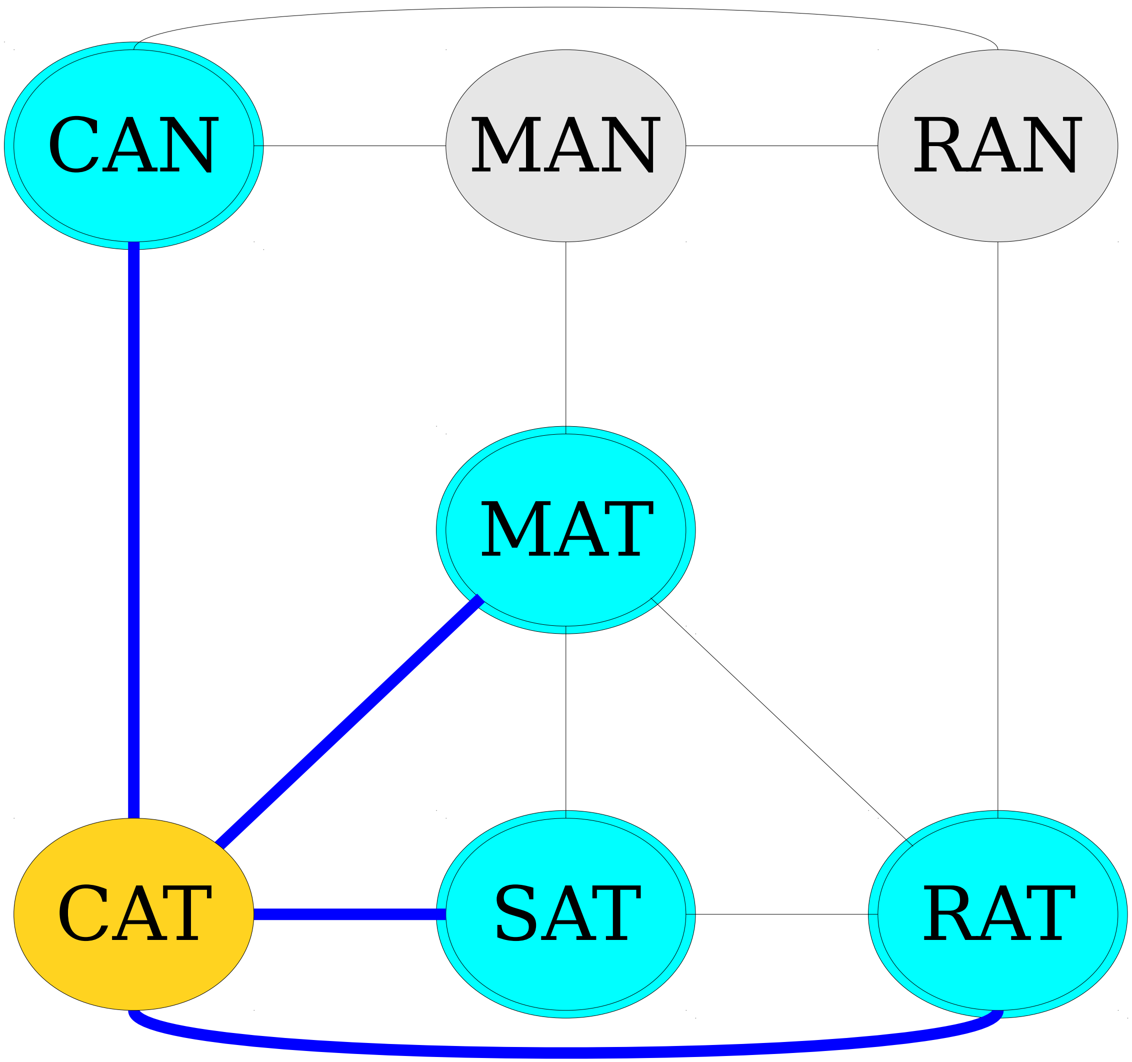
Recap from Last Time

Graphs and Digraphs

- A ***graph*** is a pair $G = (V, E)$ of a set of nodes V and set of edges E.
  - Nodes can be anything.
  - Edges are ***unordered pairs*** (i.e., sets with cardinality 2) of nodes. If $\{u, v\} \in E$, then there's an edge from u to v.

New Stuff!

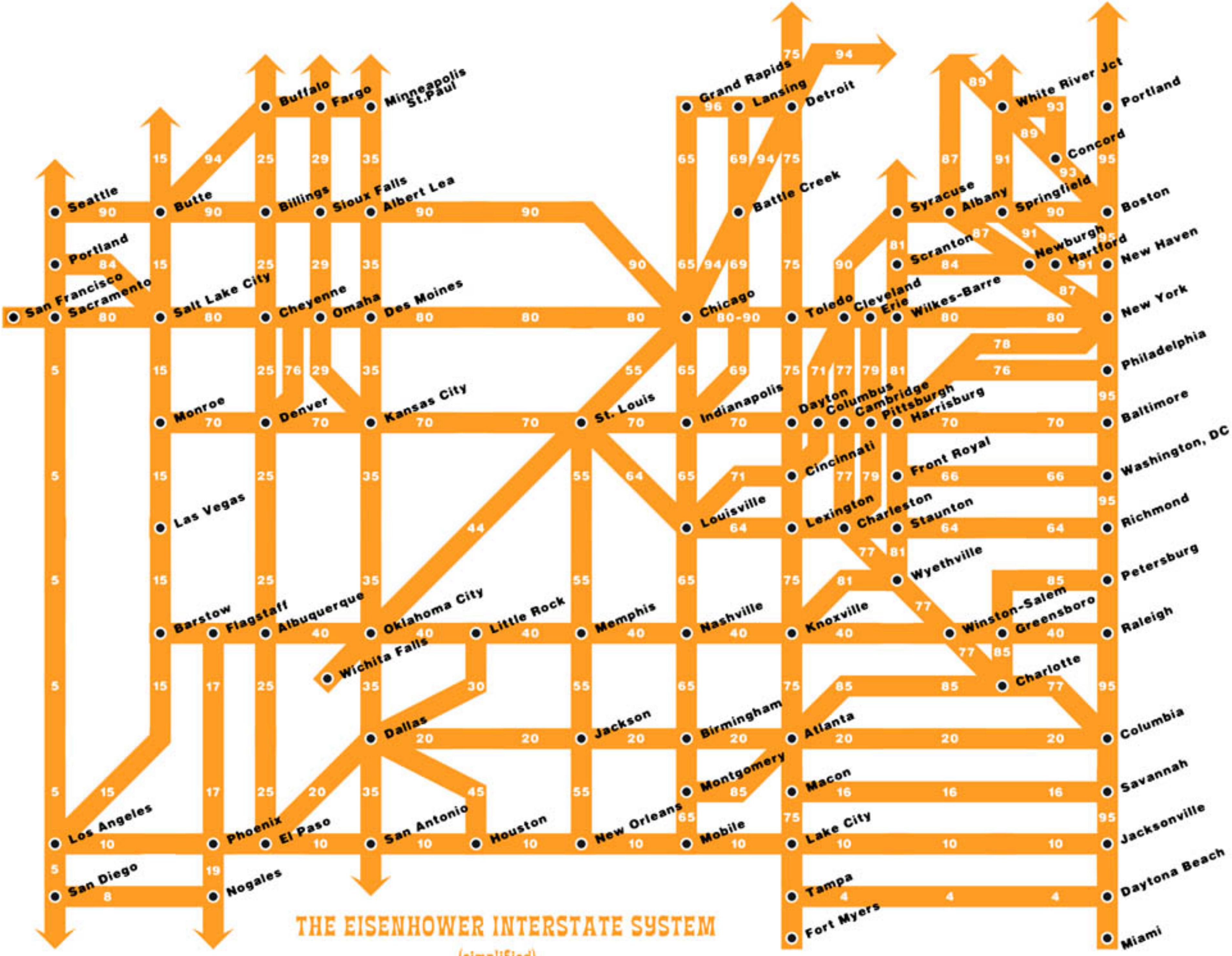
Walks, Paths, and Reachability



Two nodes are called ***adjacent*** if there is an edge between them.

Using our Formalisms

- Let $G = (V, E)$ be an (undirected) graph.
- Intuitively, two nodes are adjacent if they're linked by an edge.
- Formally speaking, we say that two nodes $u, v \in V$ are ***adjacent*** if we have $\{u, v\} \in E$.



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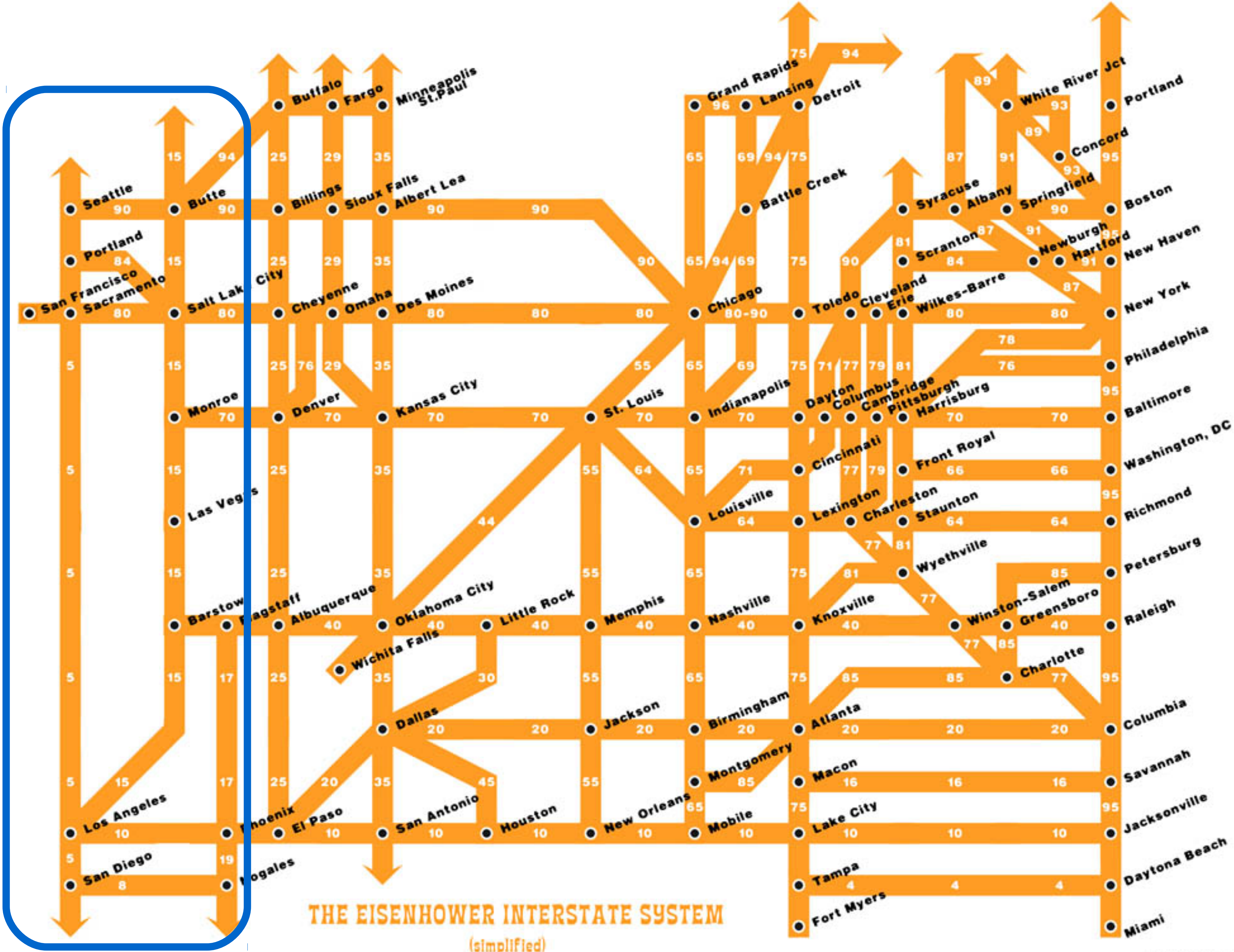

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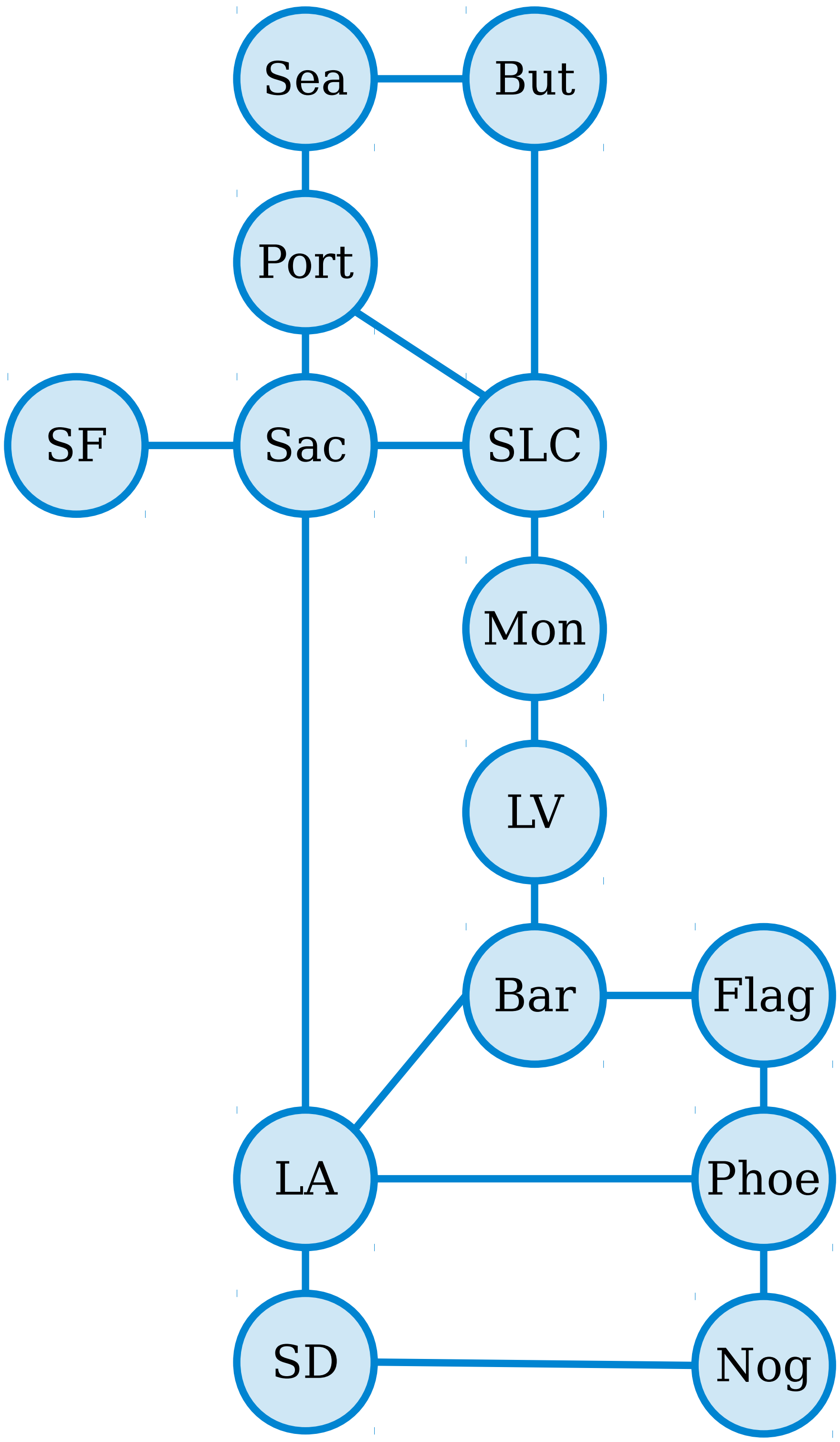
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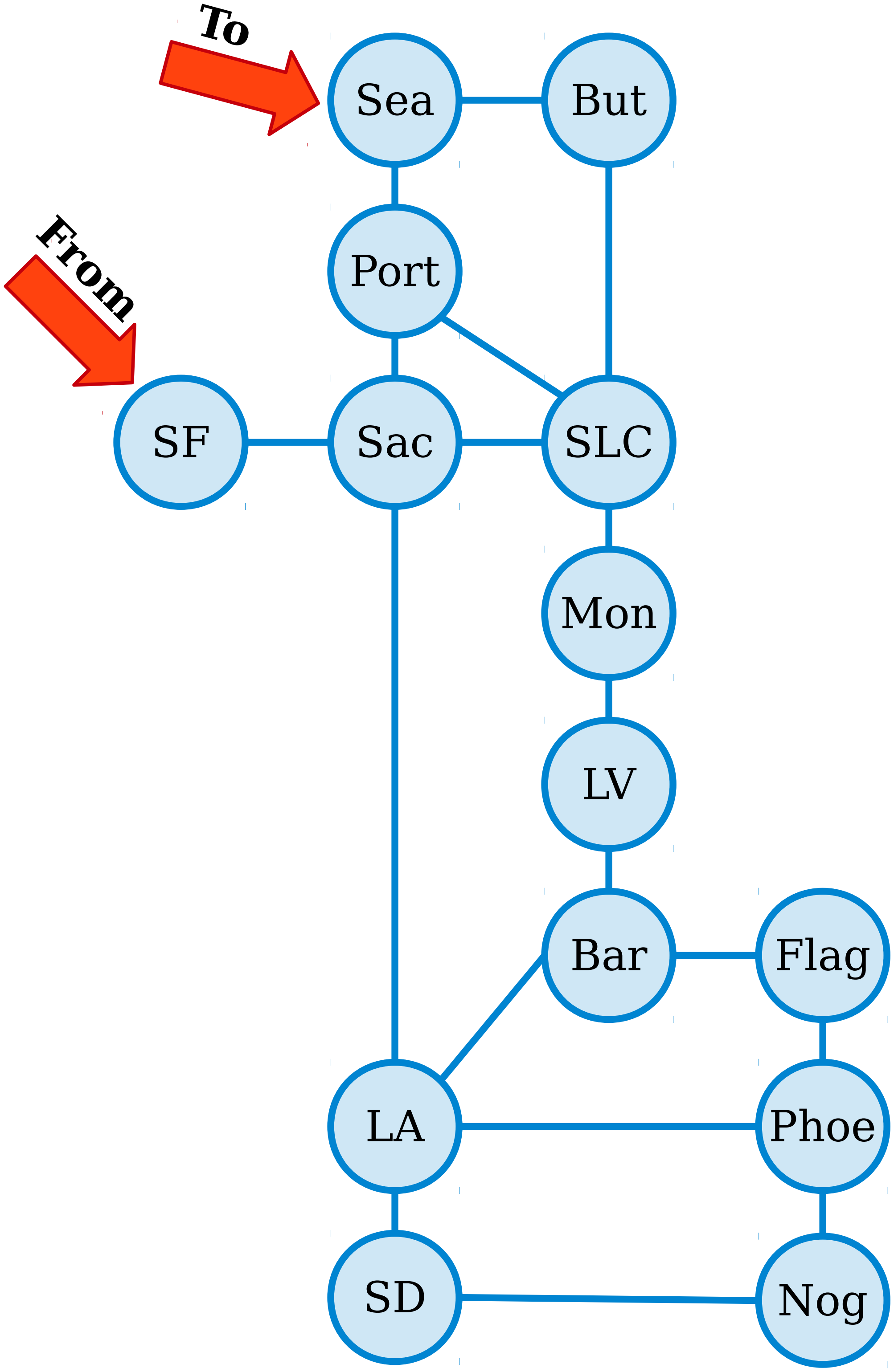
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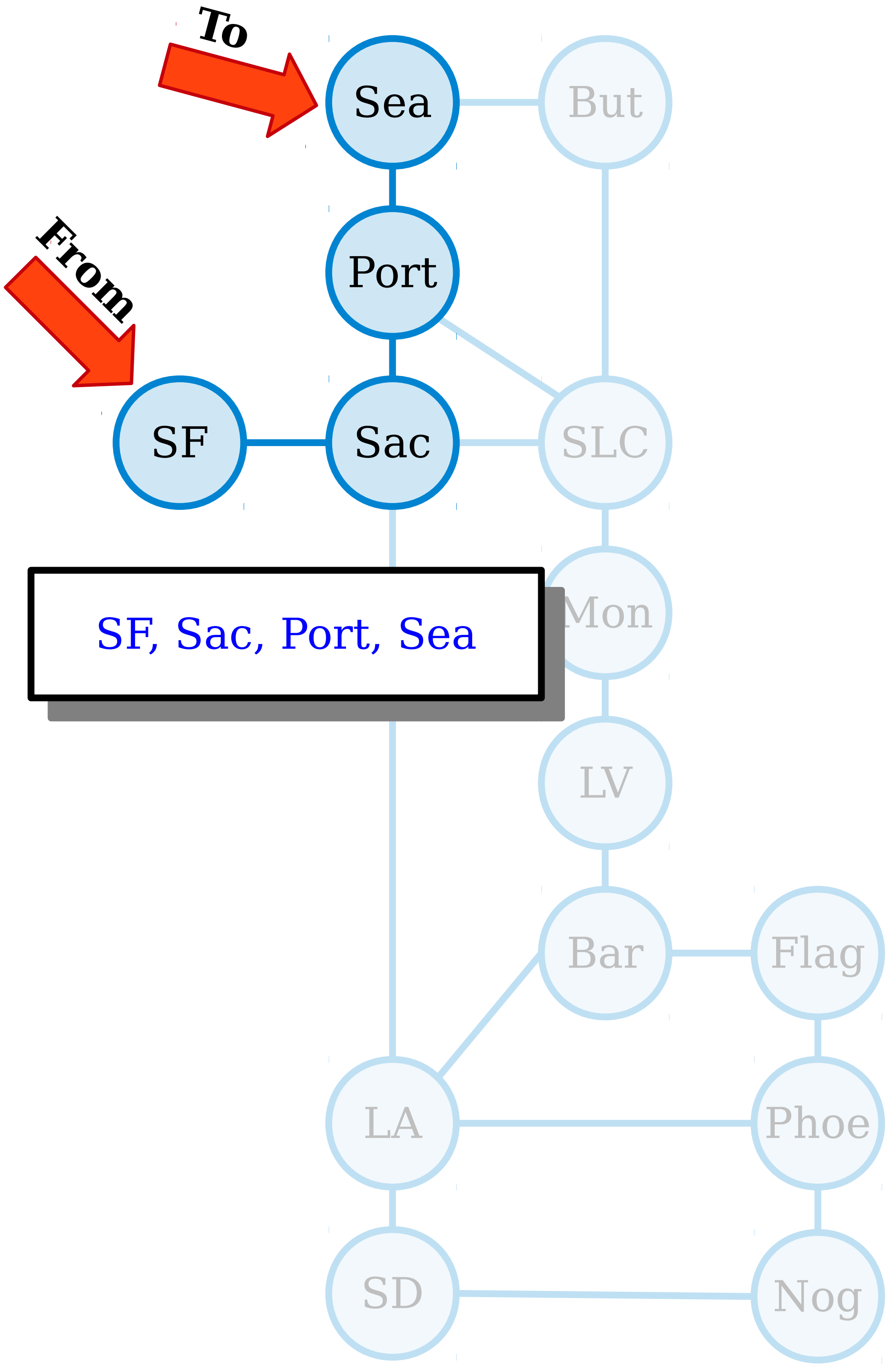
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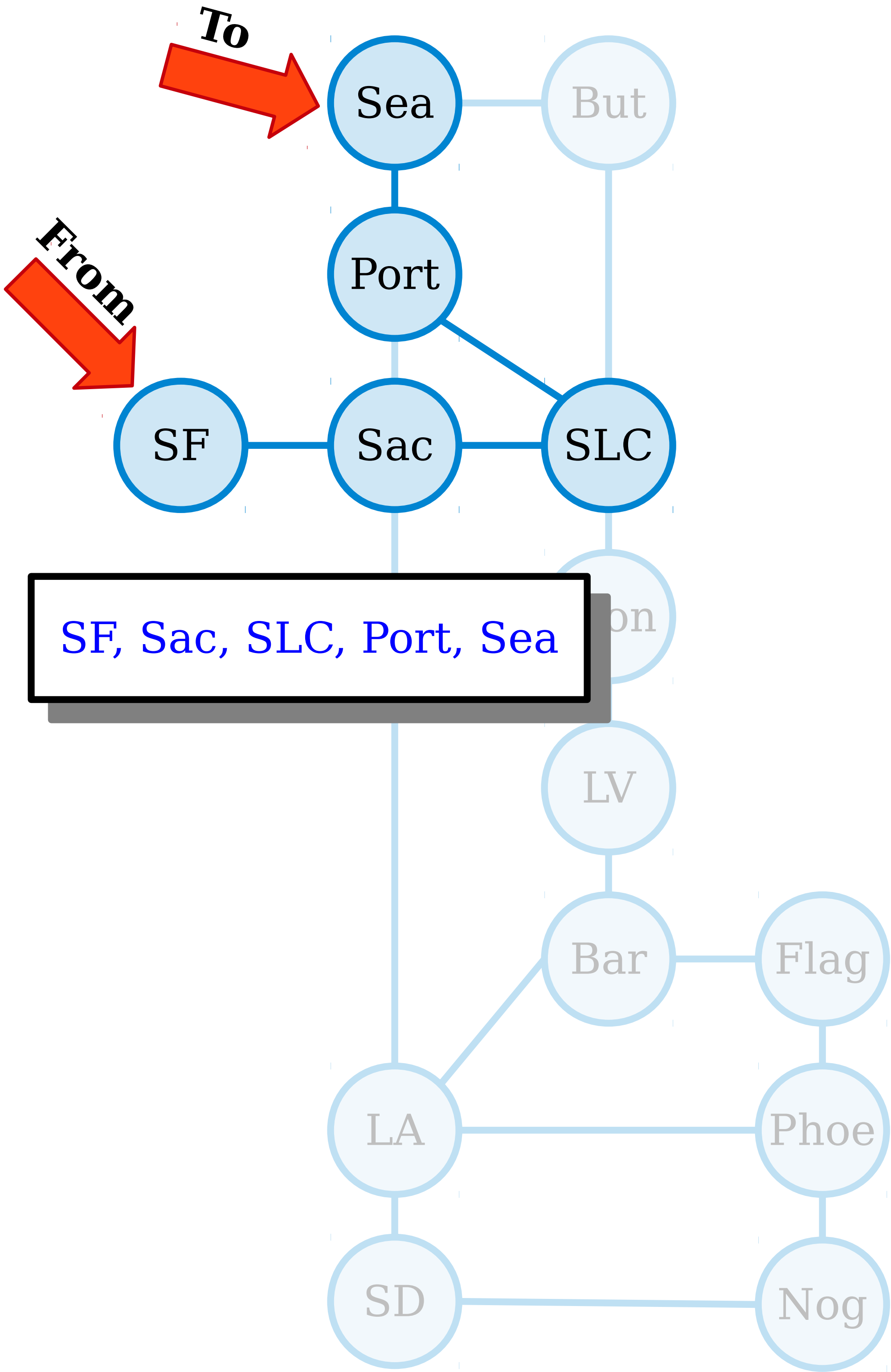

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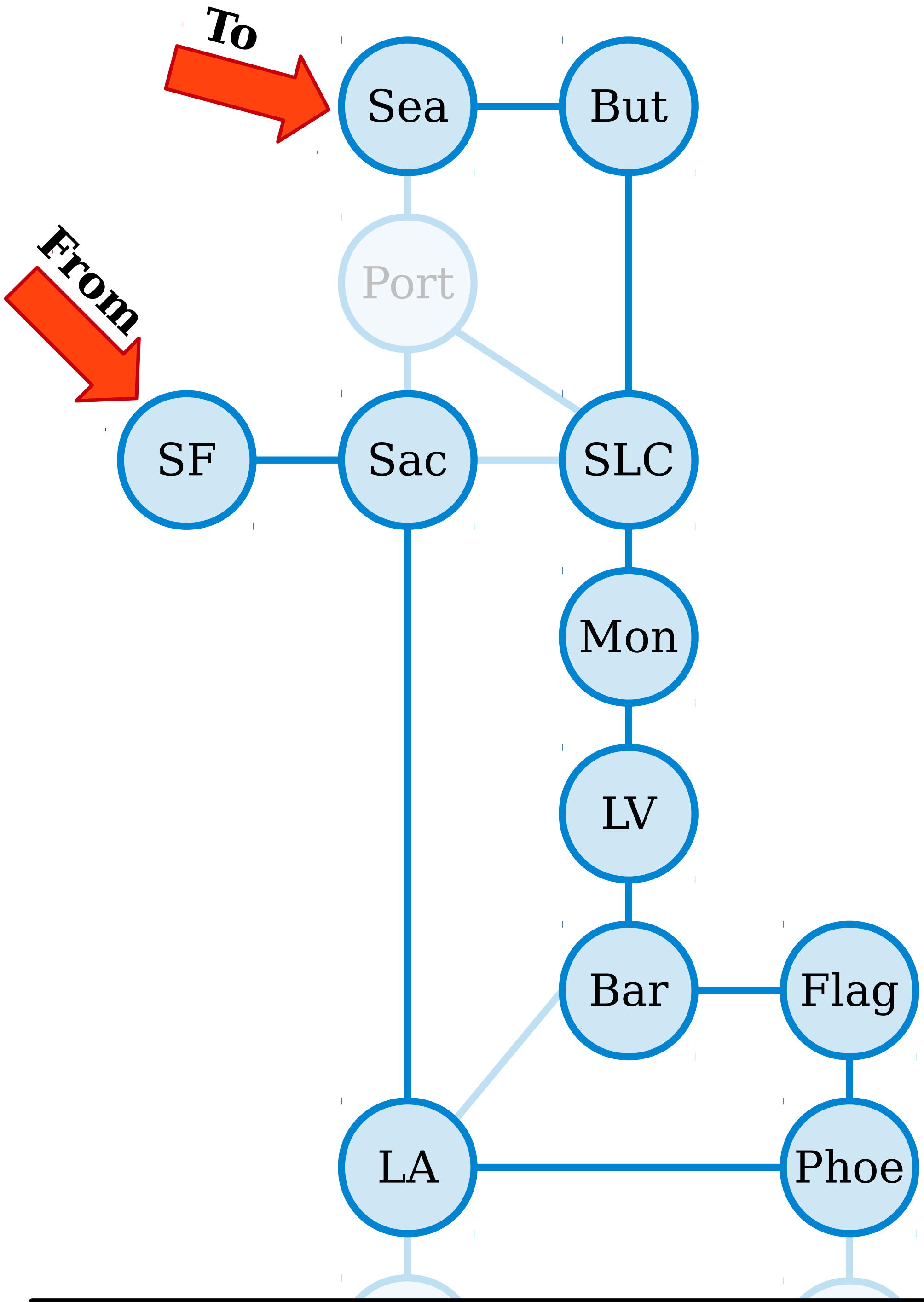
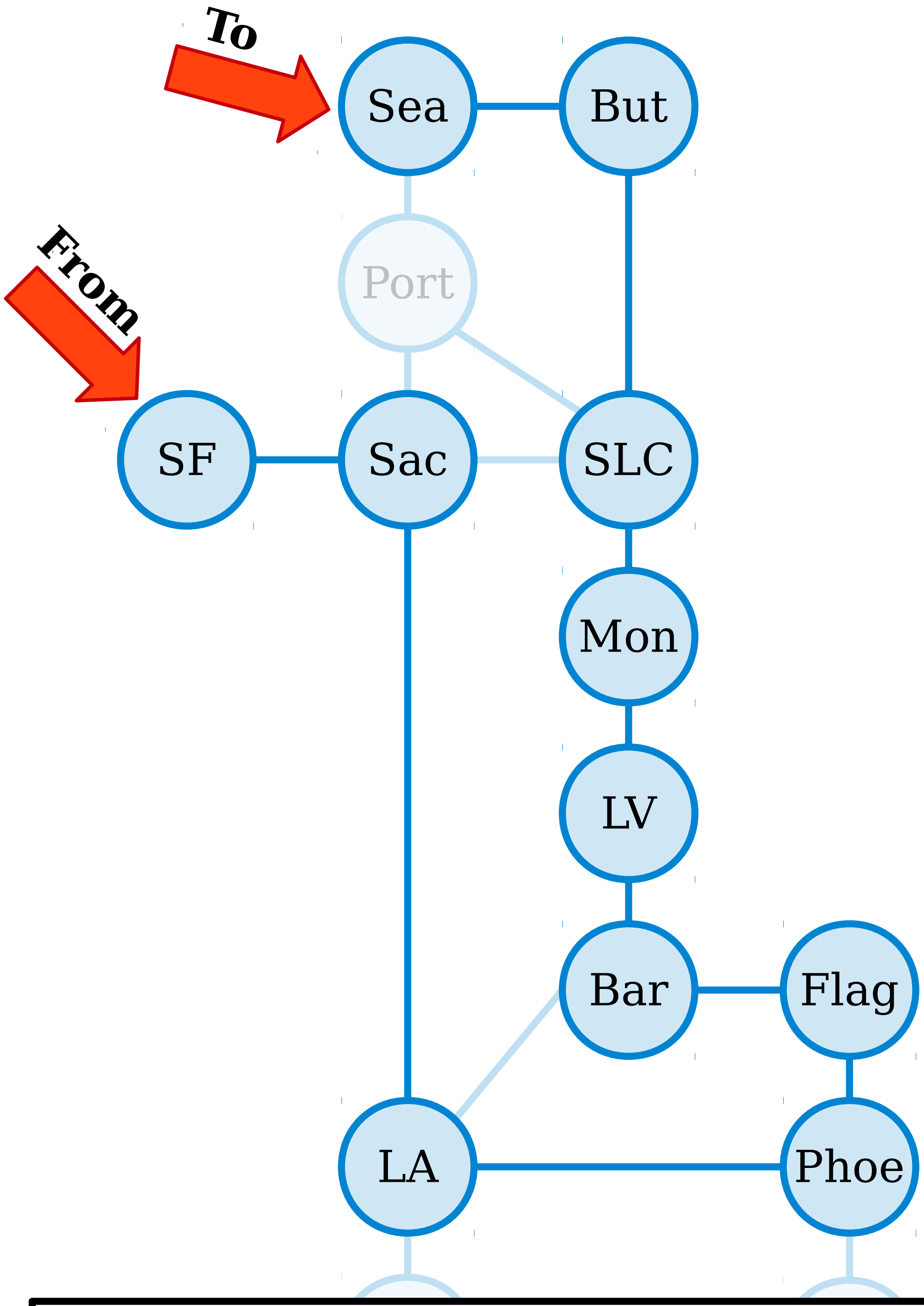


SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



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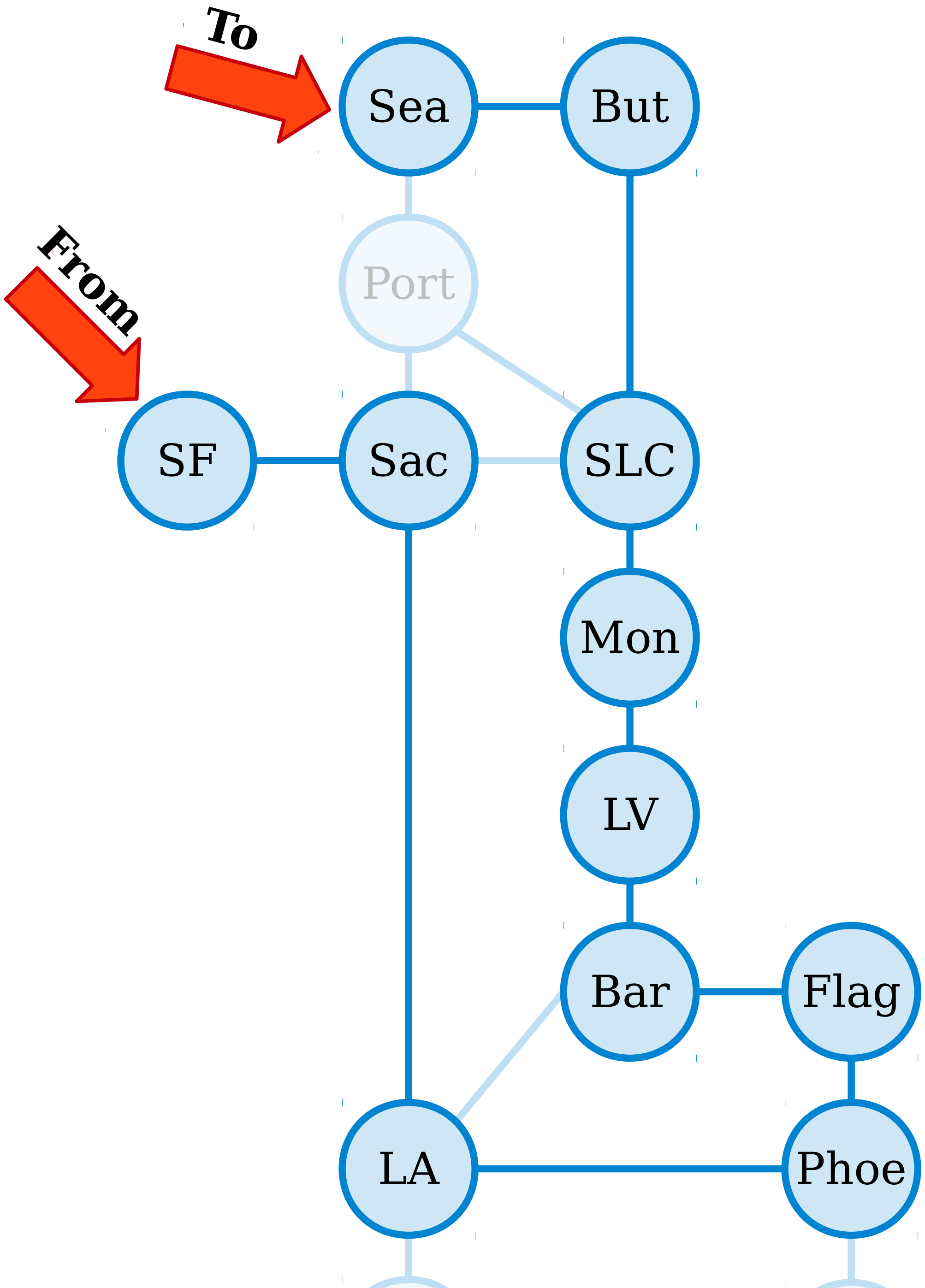

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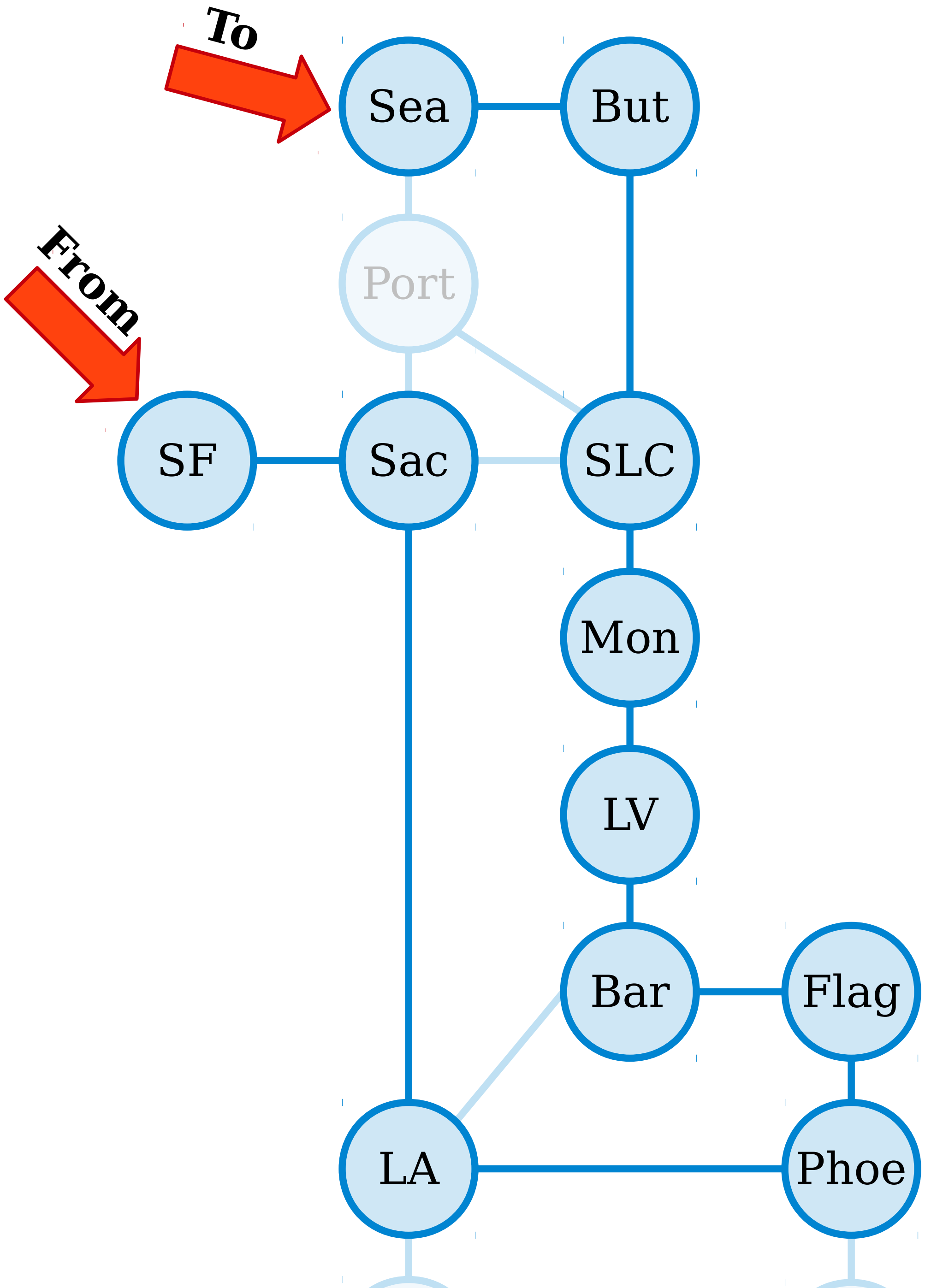

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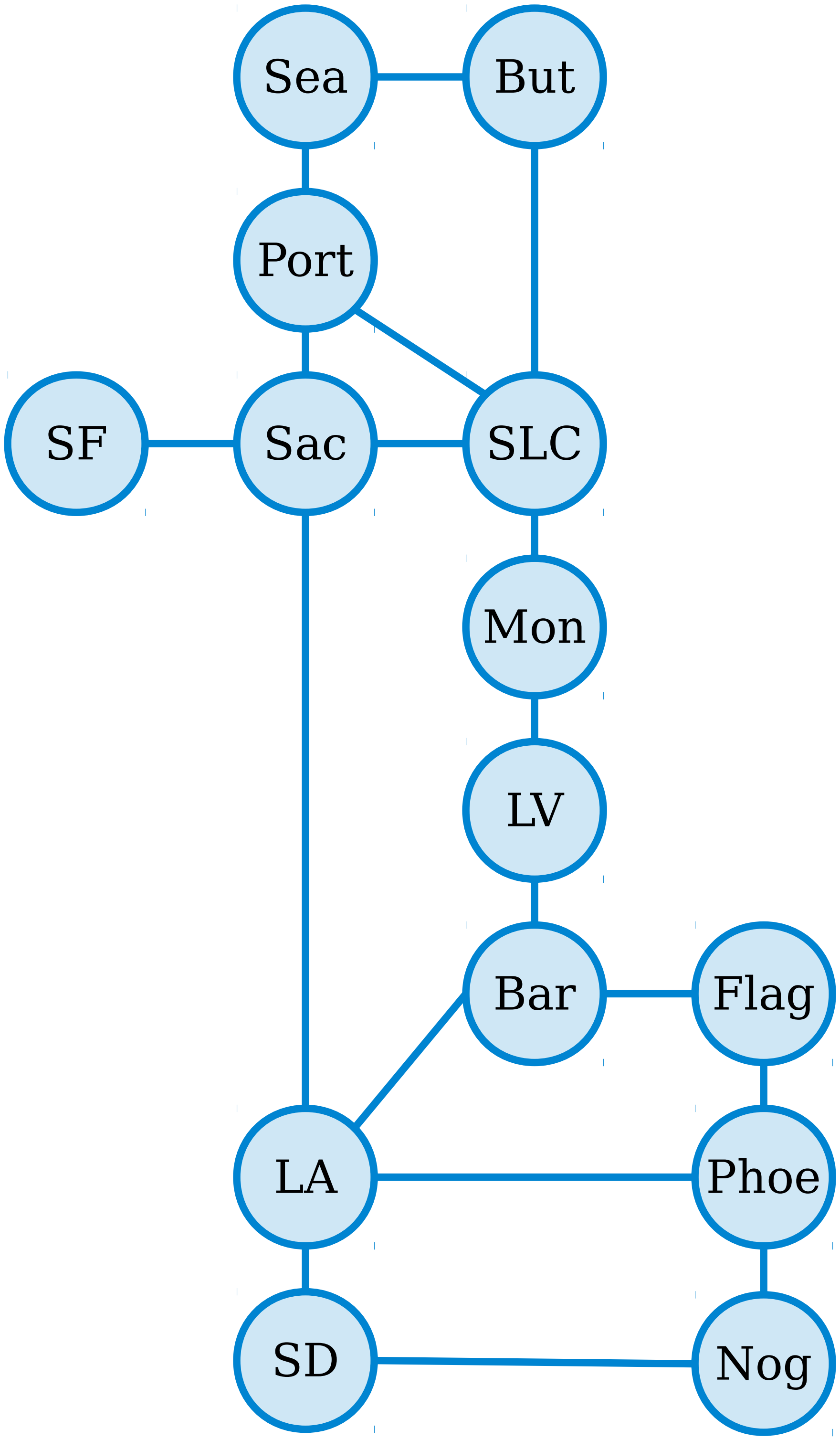
(This walk has length 10, but visits 11 cities.)

SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



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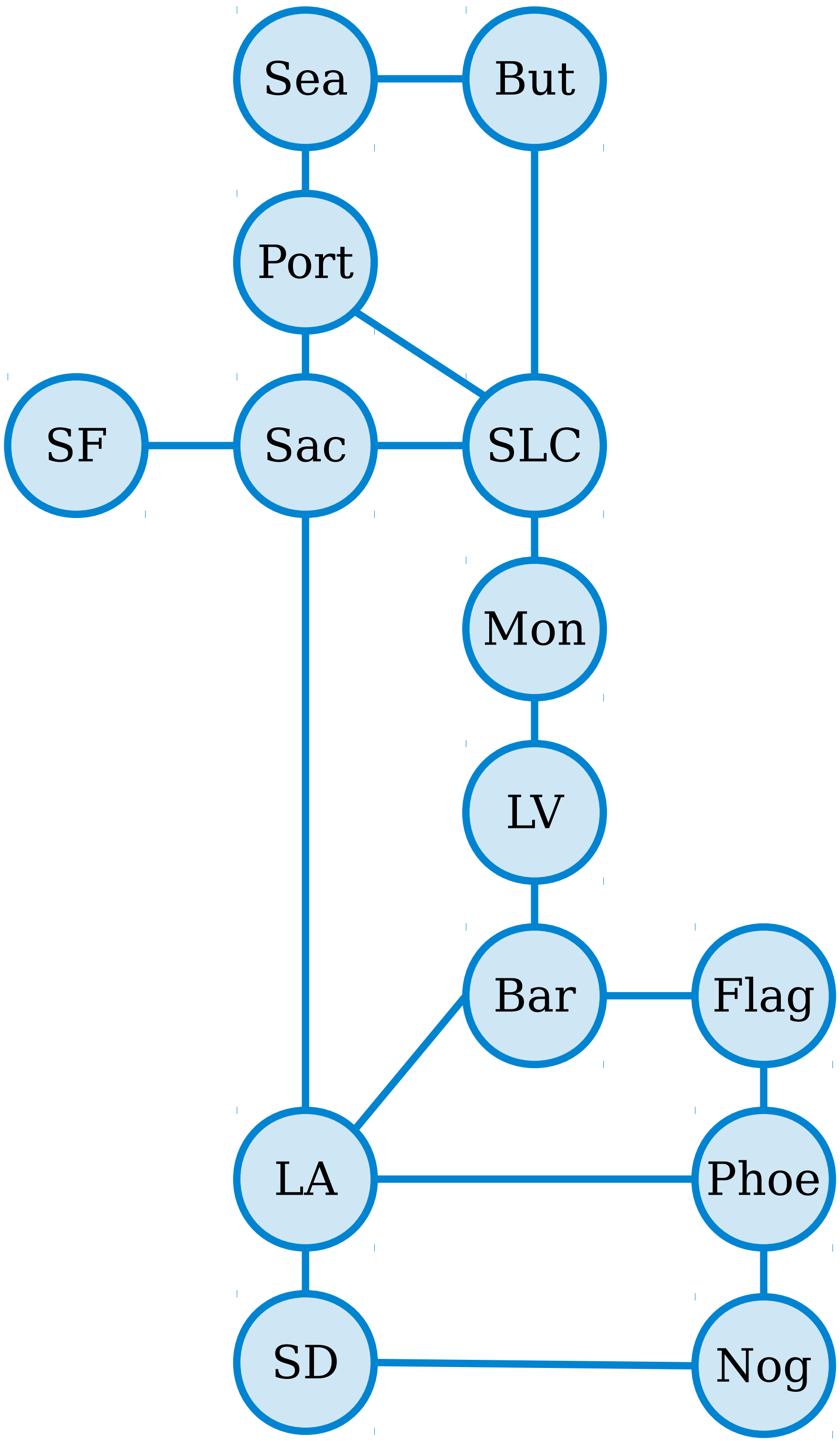

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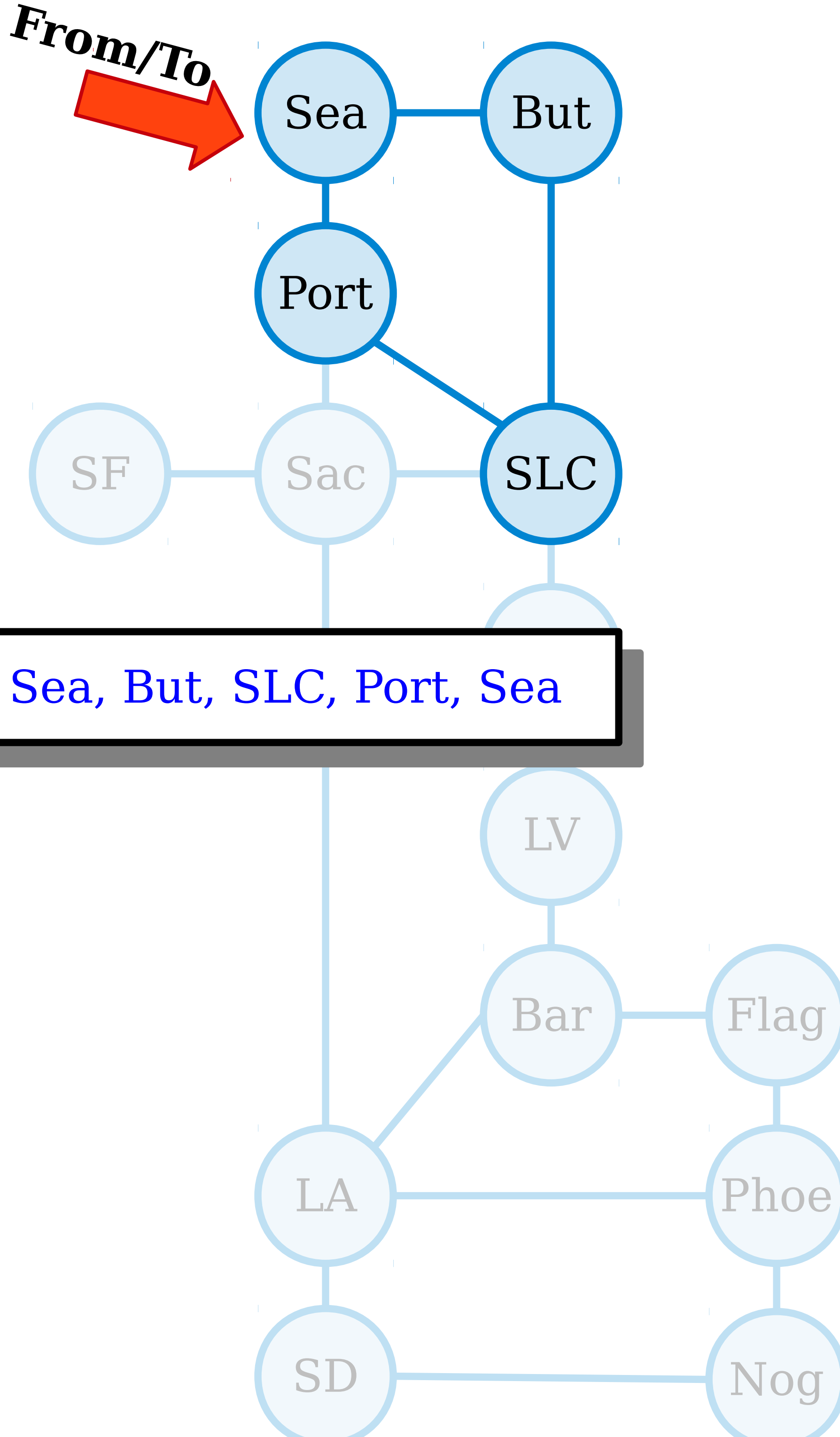
Question:
Is a "staycation" a valid walk? In other words, can a walk be just "SF"?



Sea, But, SLC, Port, Sea

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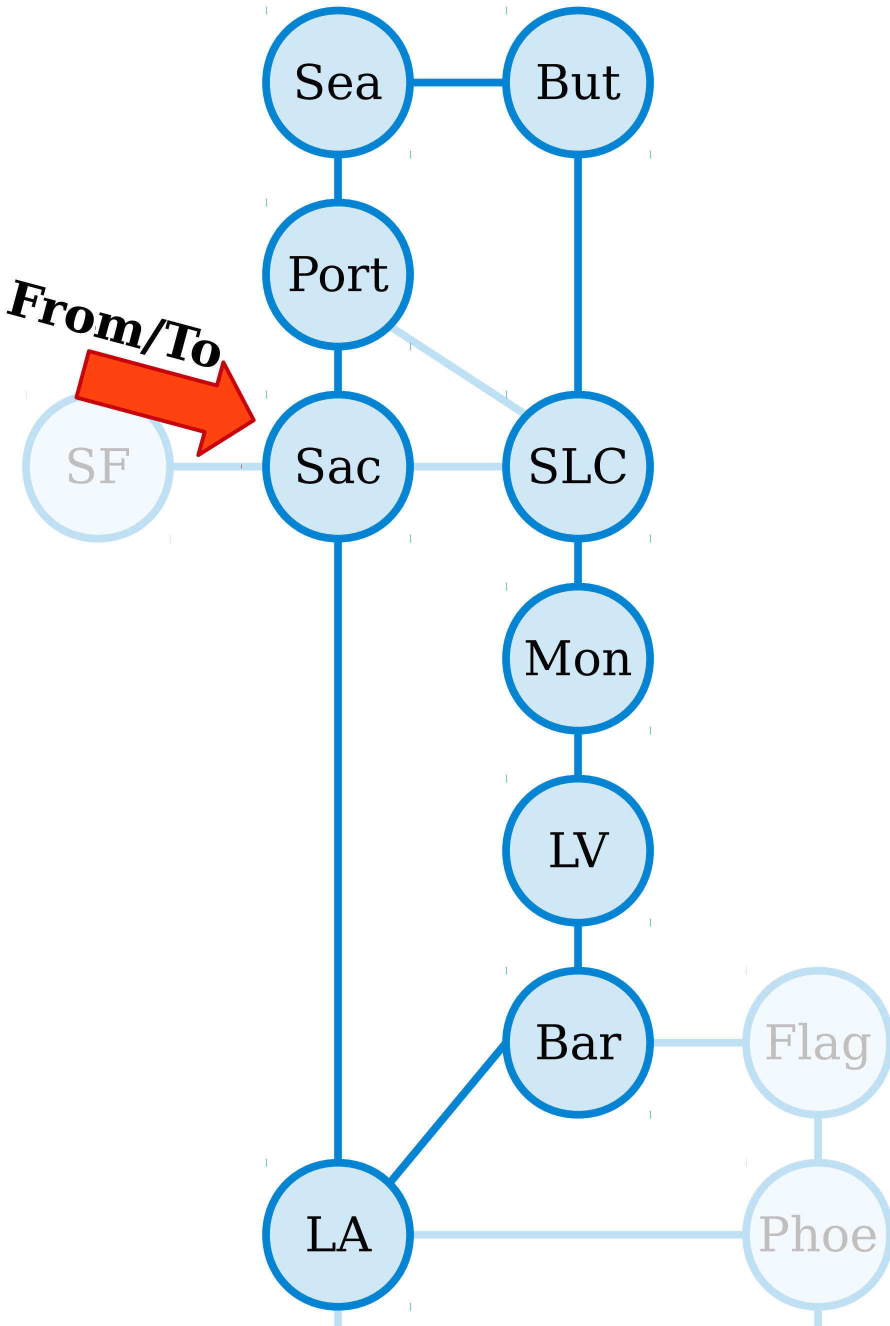

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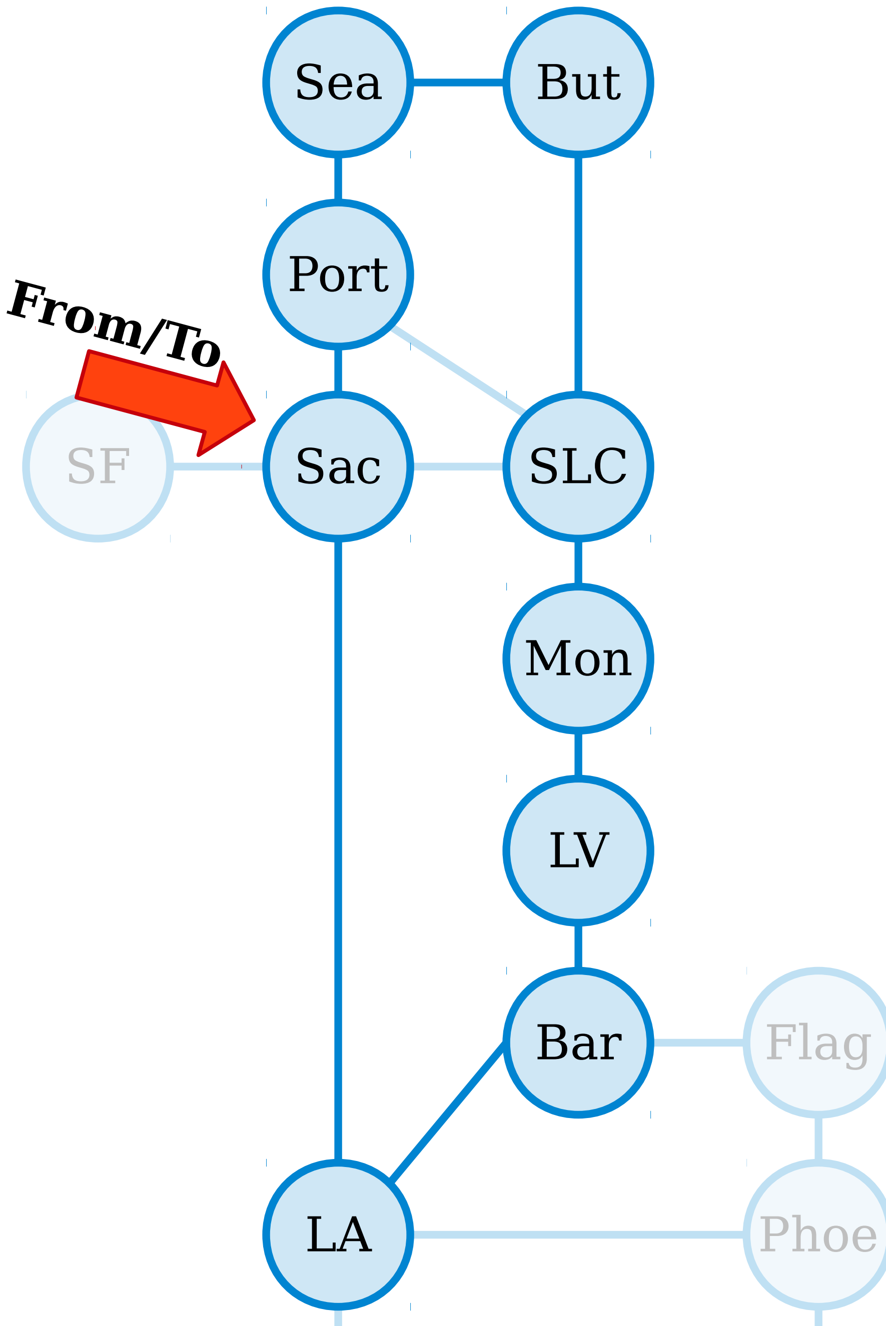
Sac, Port, Sea, But, SLC, Mon, LV, Bar, LA, Sac



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(No “staycation” closed walks, because of this rule.)

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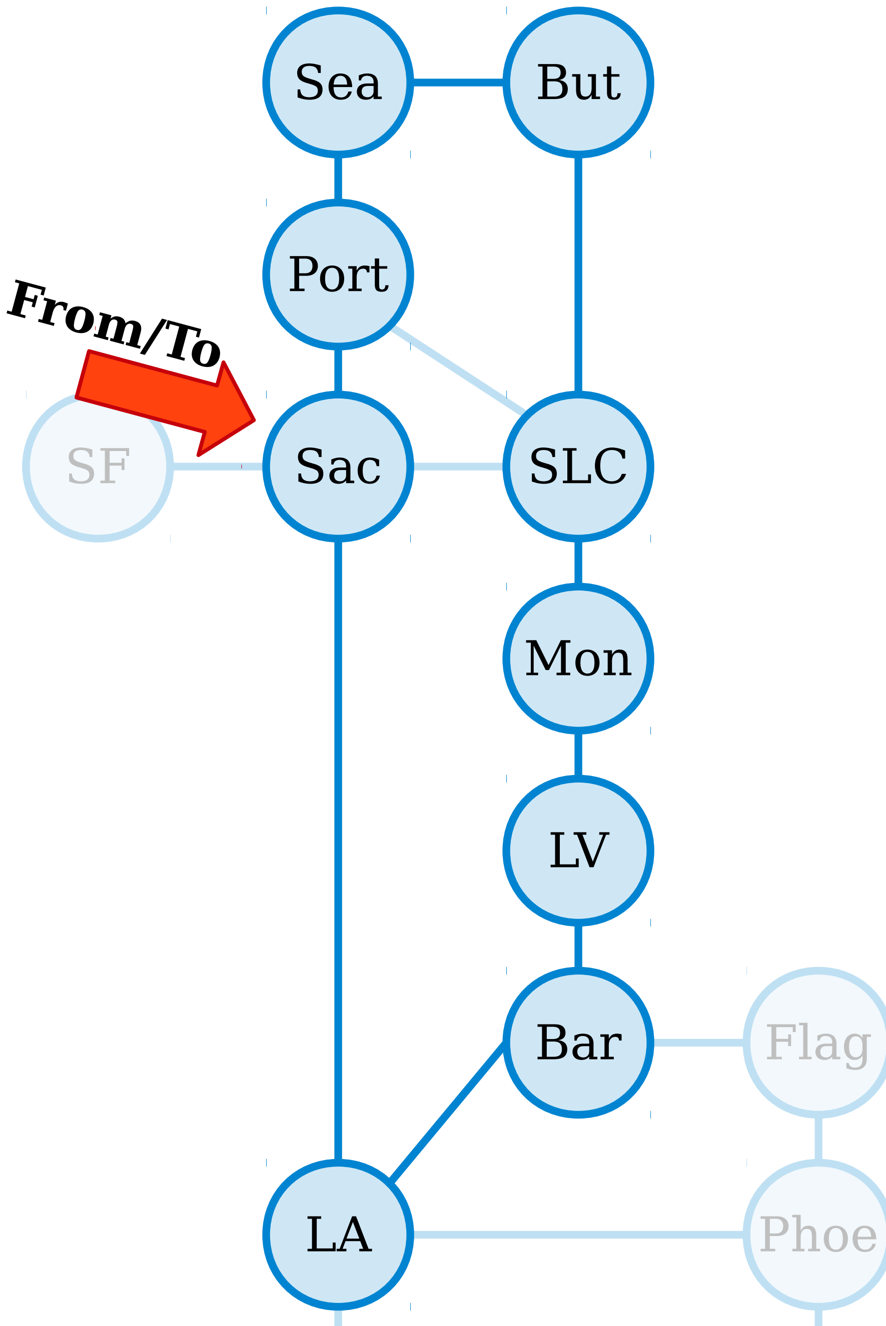

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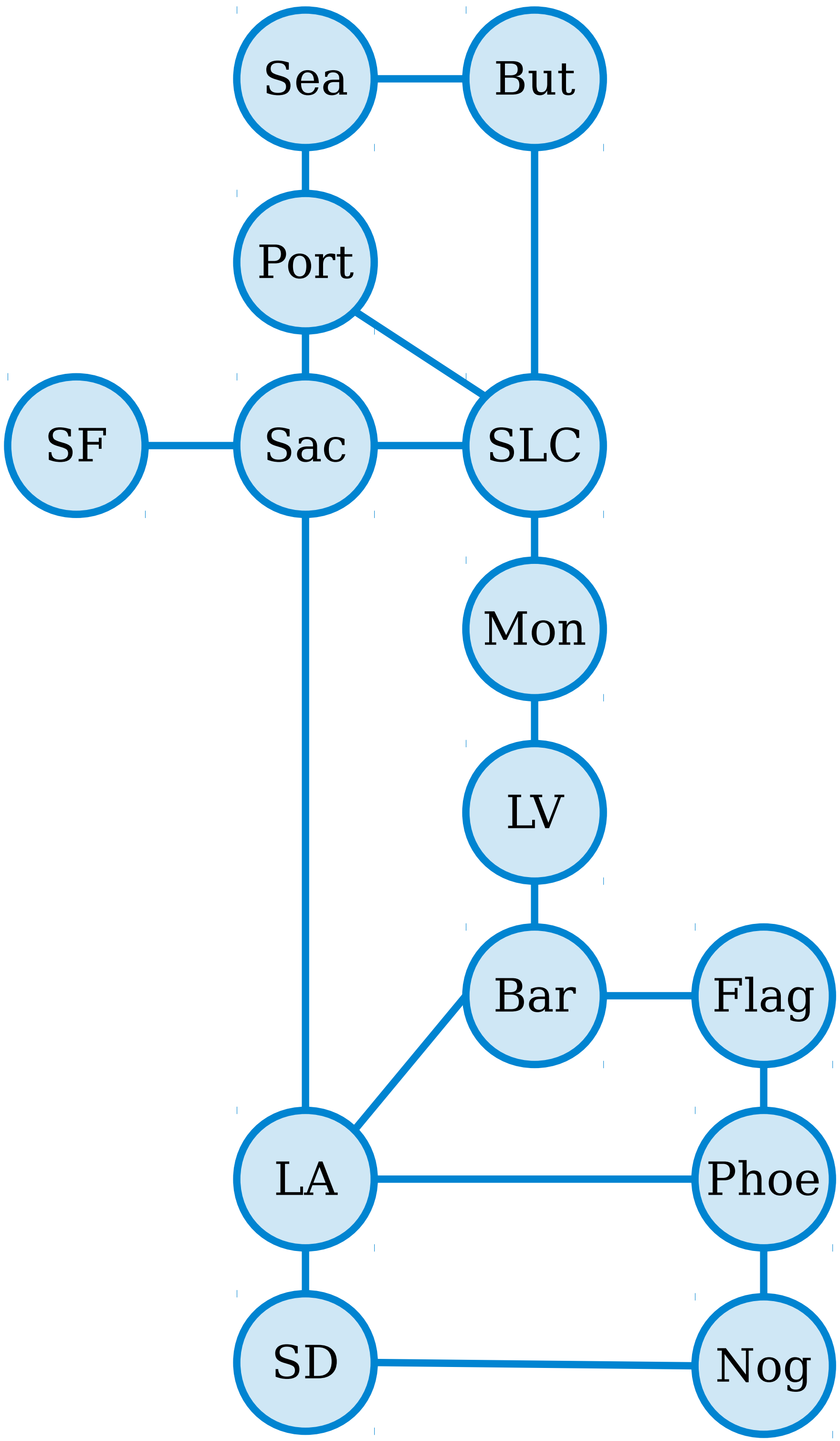
(This closed walk has length nine and visits nine different cities.)

Sac, Port, Sea, But, SLC, Mon, LV, Bar, LA, Sac



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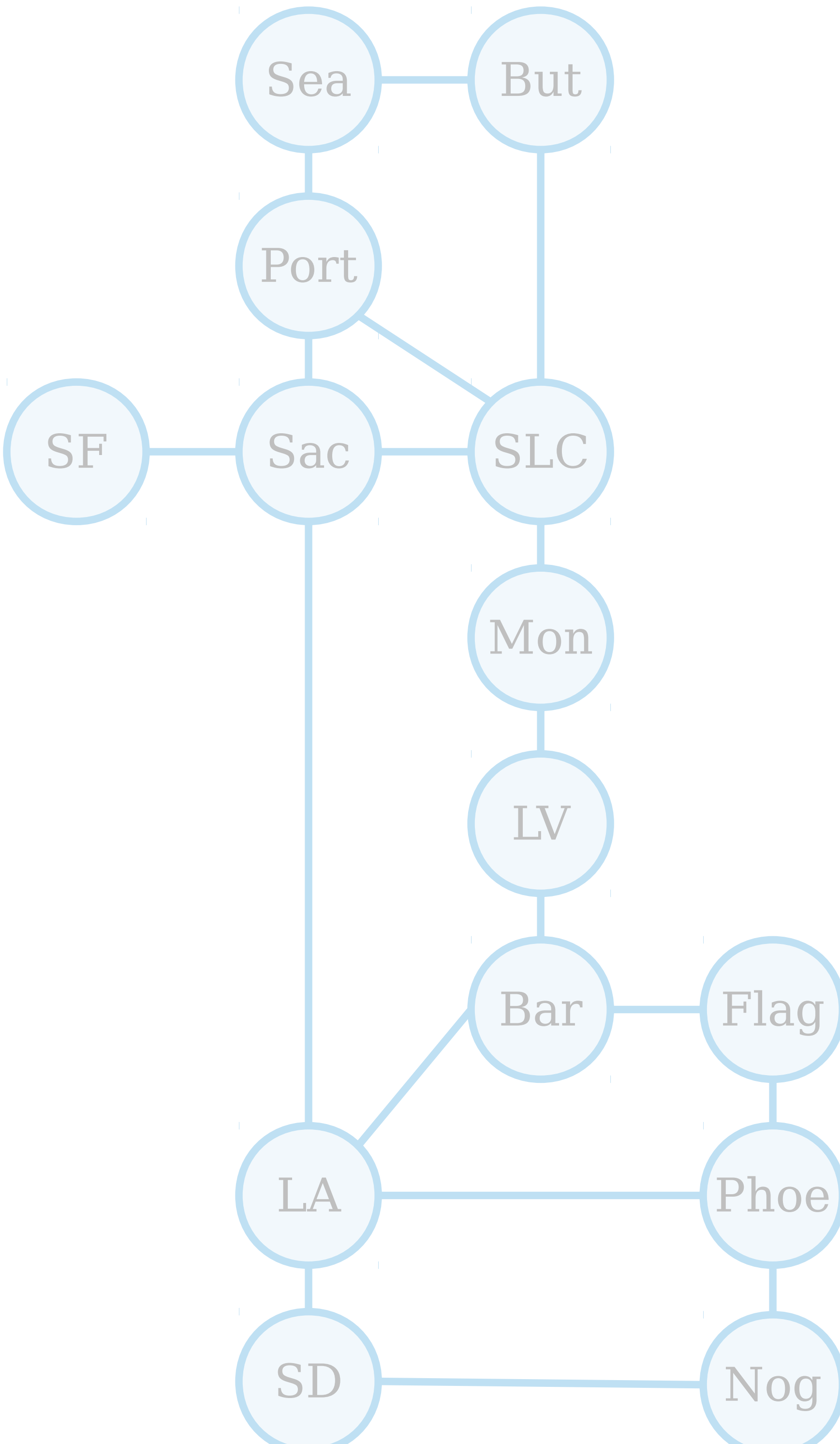

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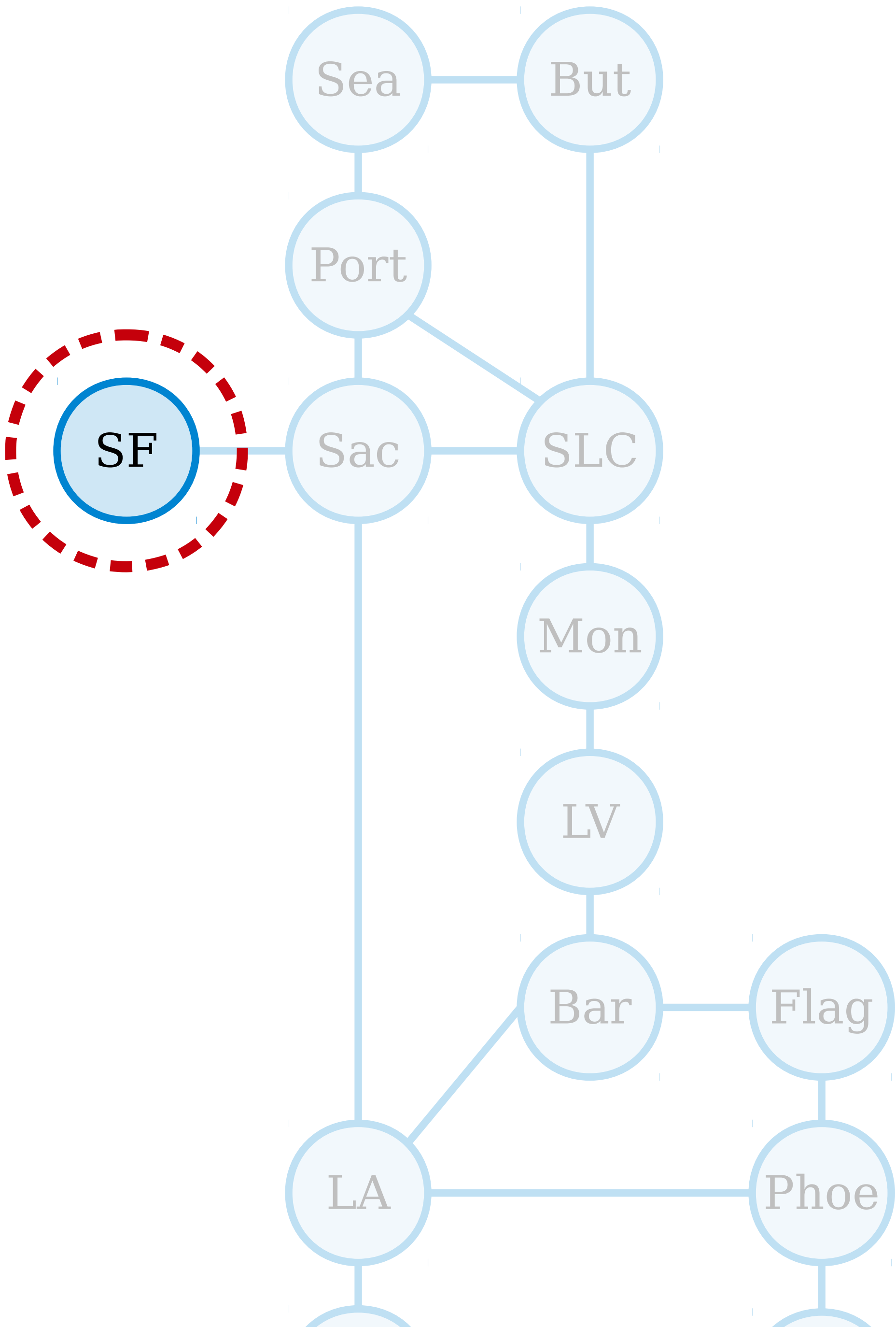

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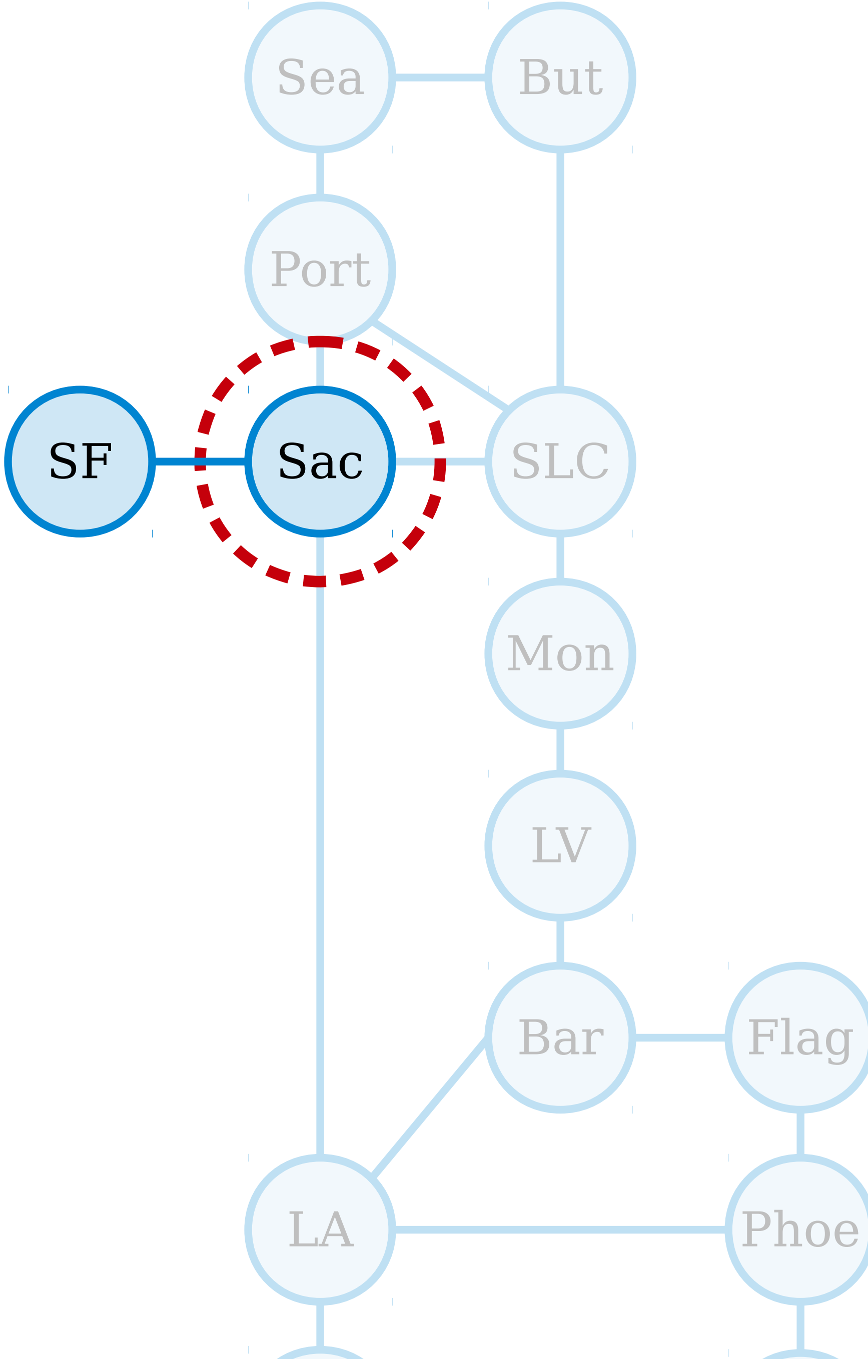

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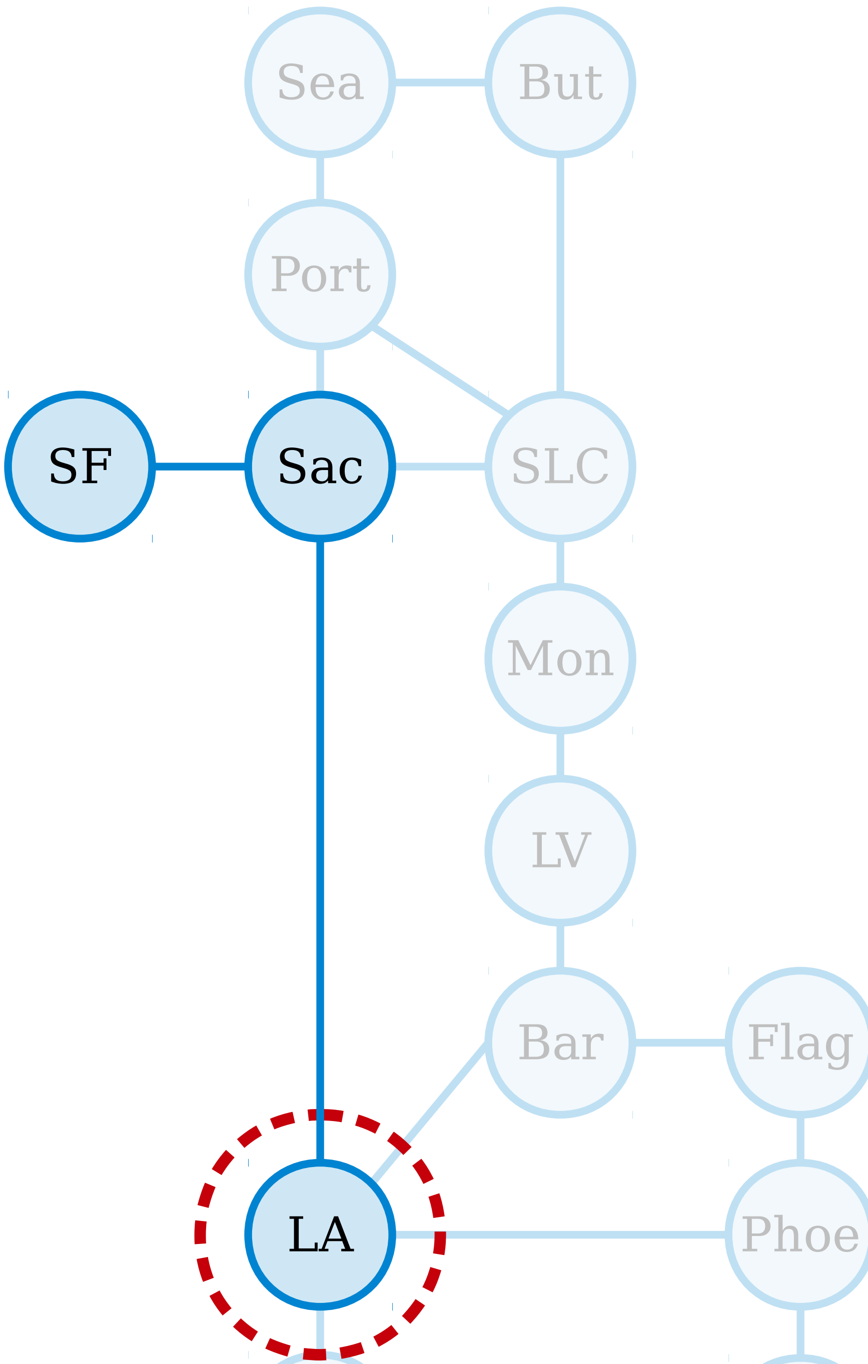

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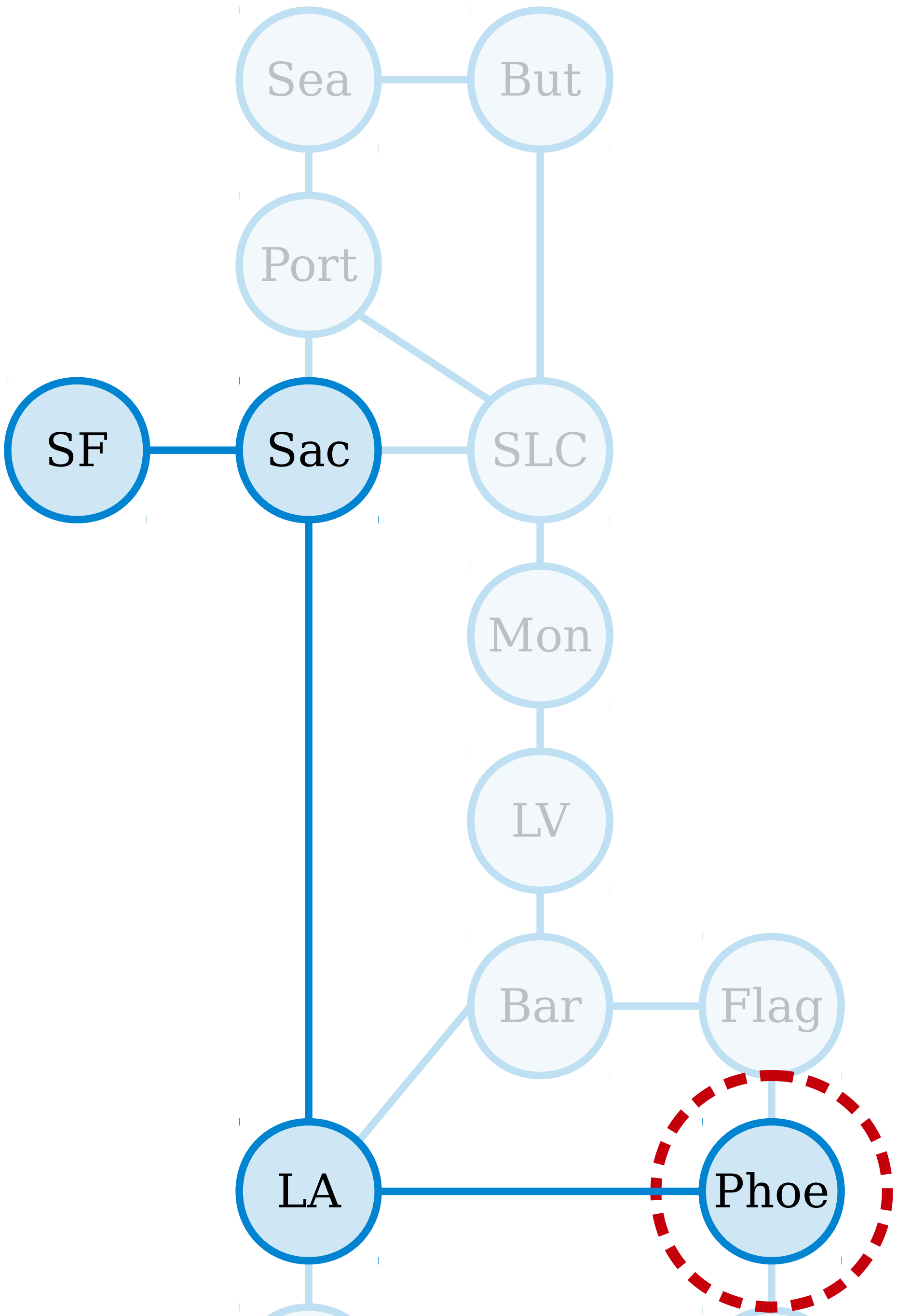

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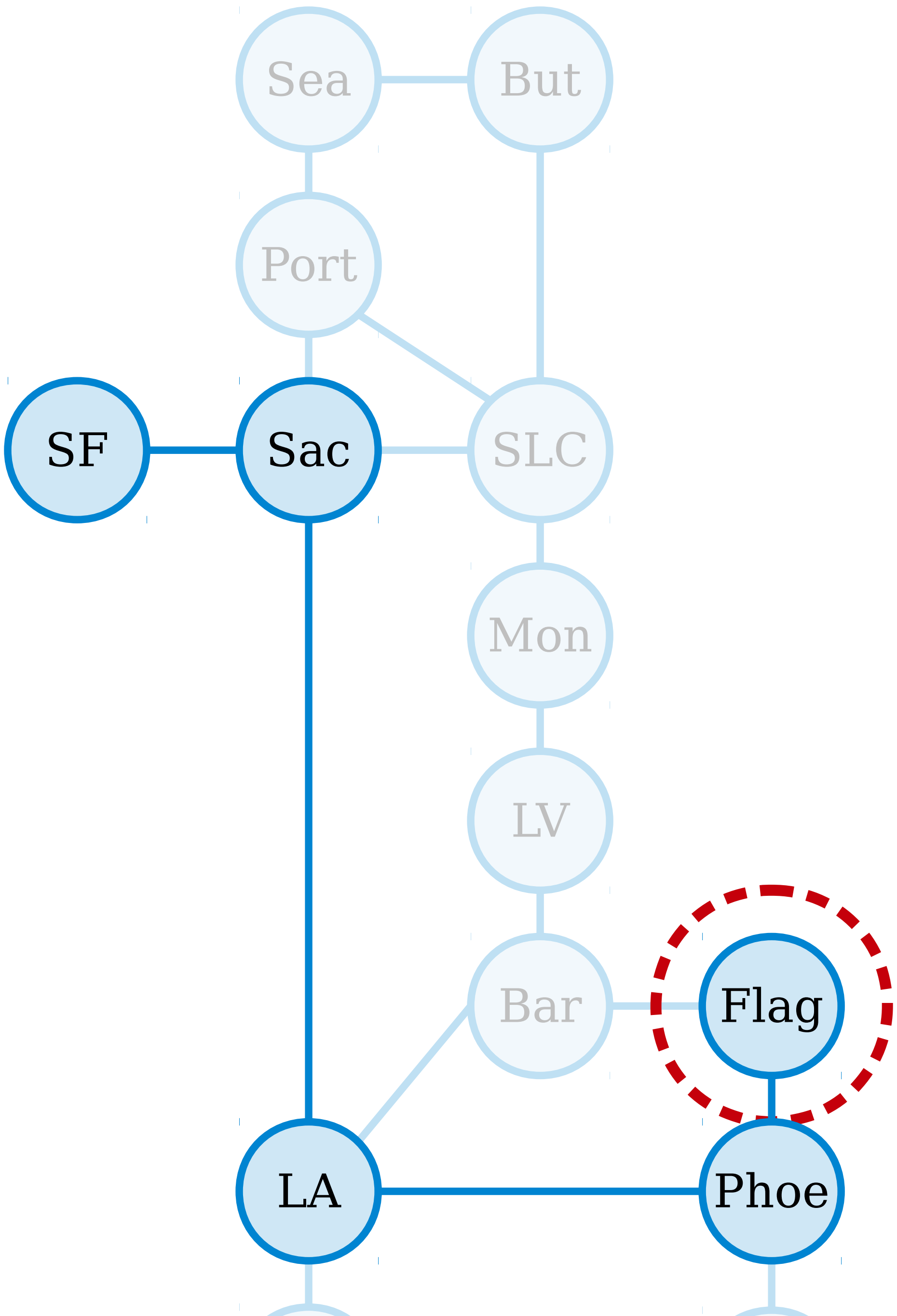
SF, Sac, LA, Phoe



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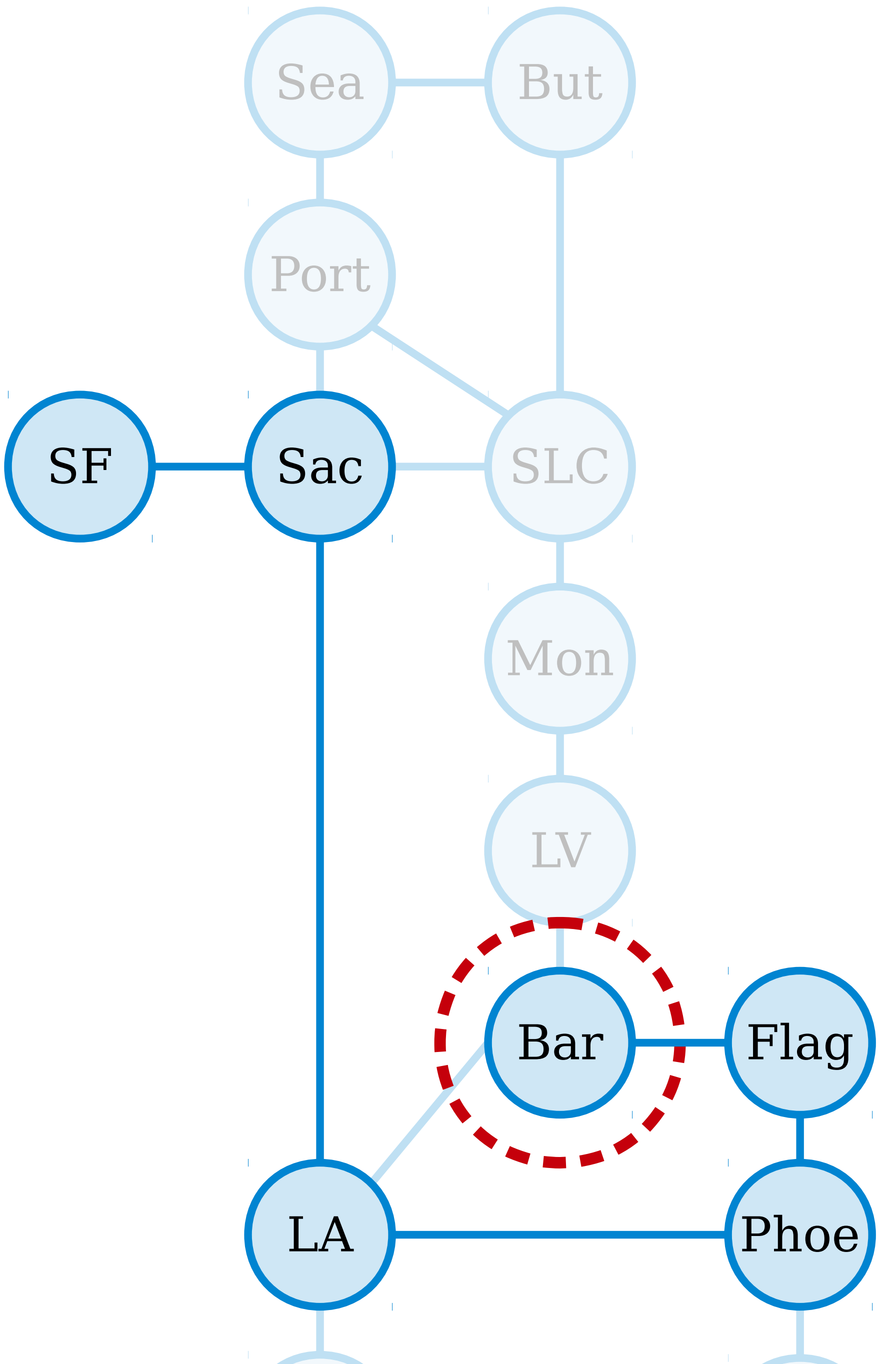
SF, Sac, LA, Phoe, Flag



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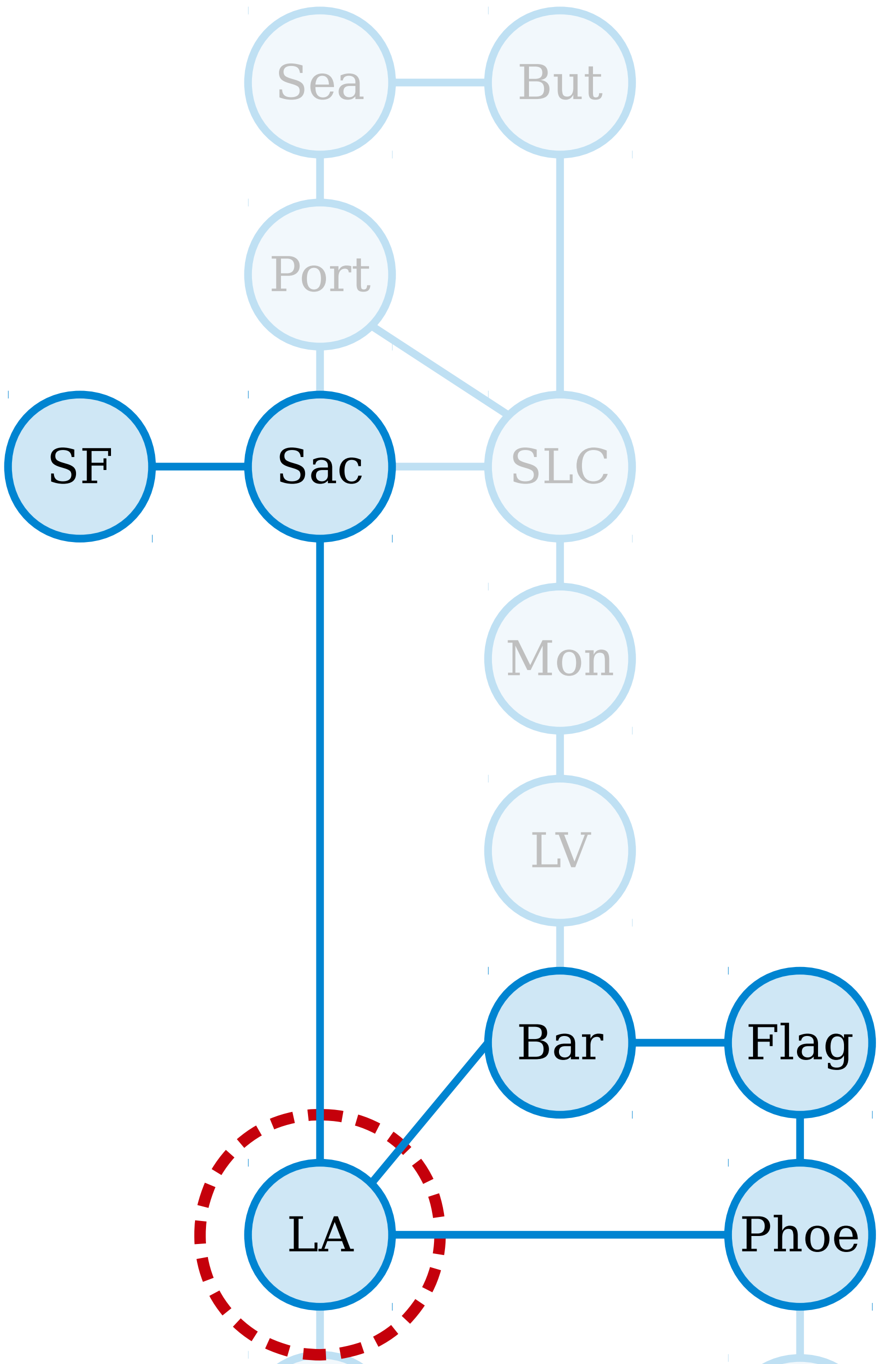

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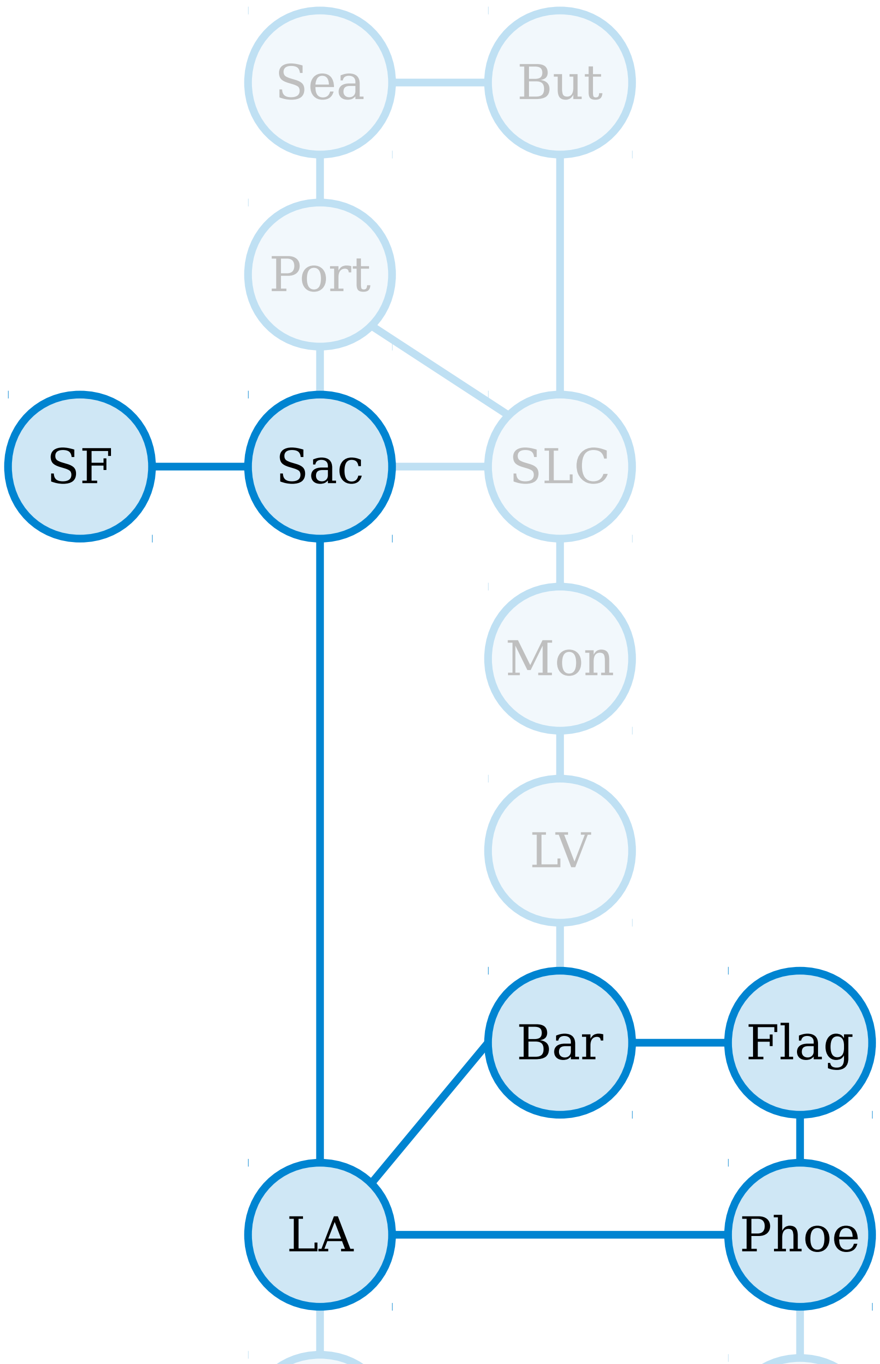

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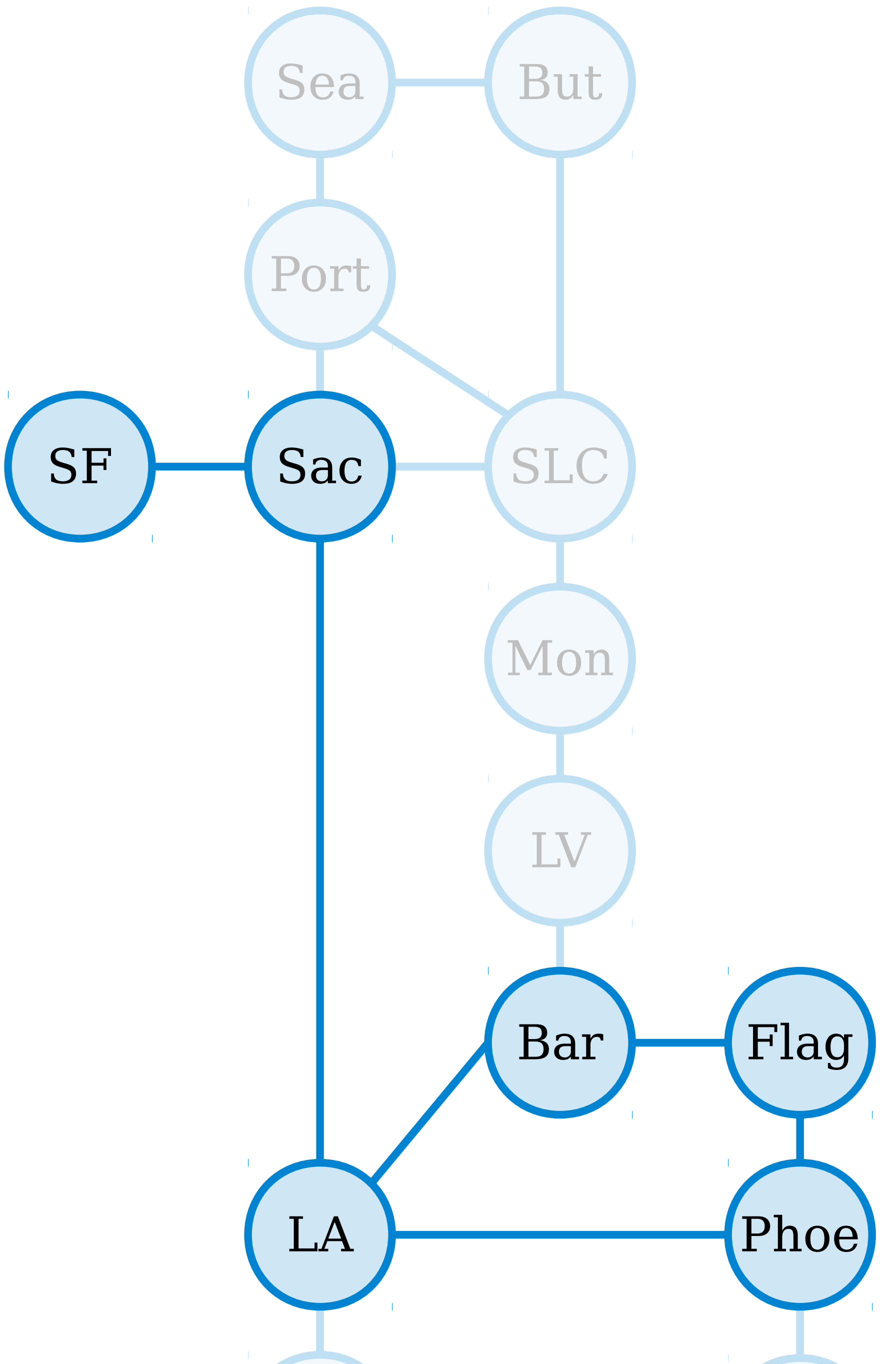

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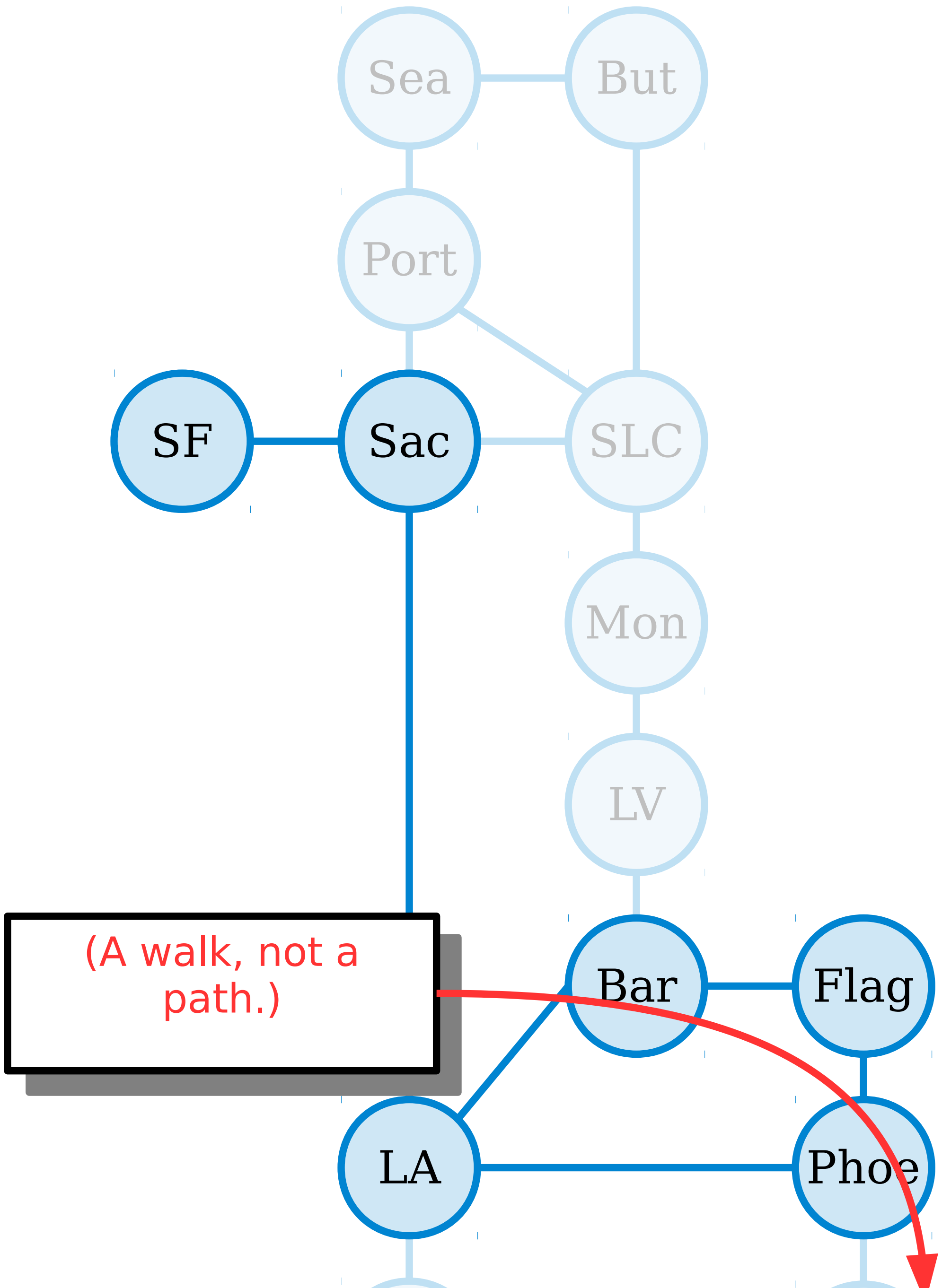
SF, Sac, LA, Phoe, Flag, Bar, LA



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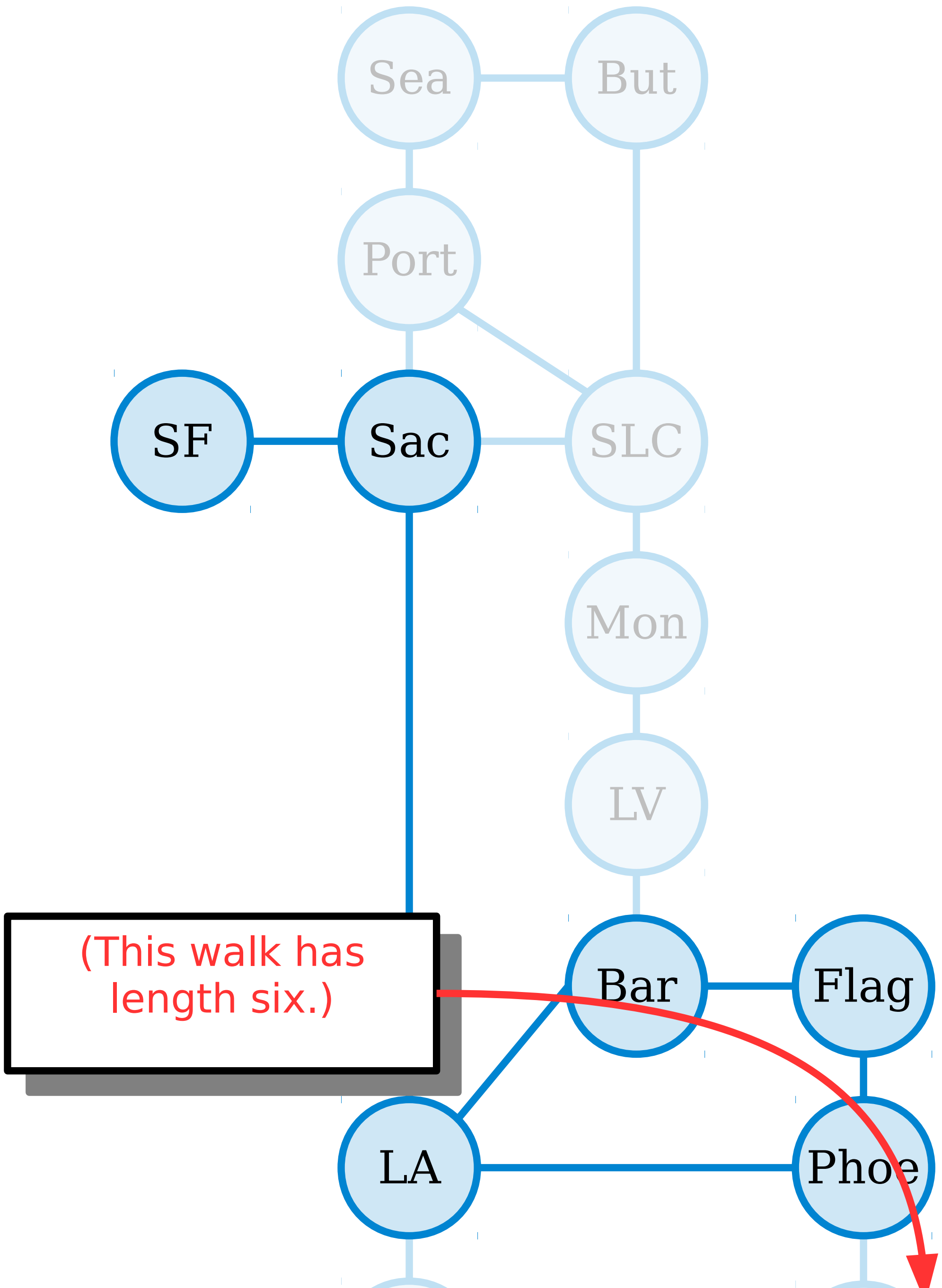

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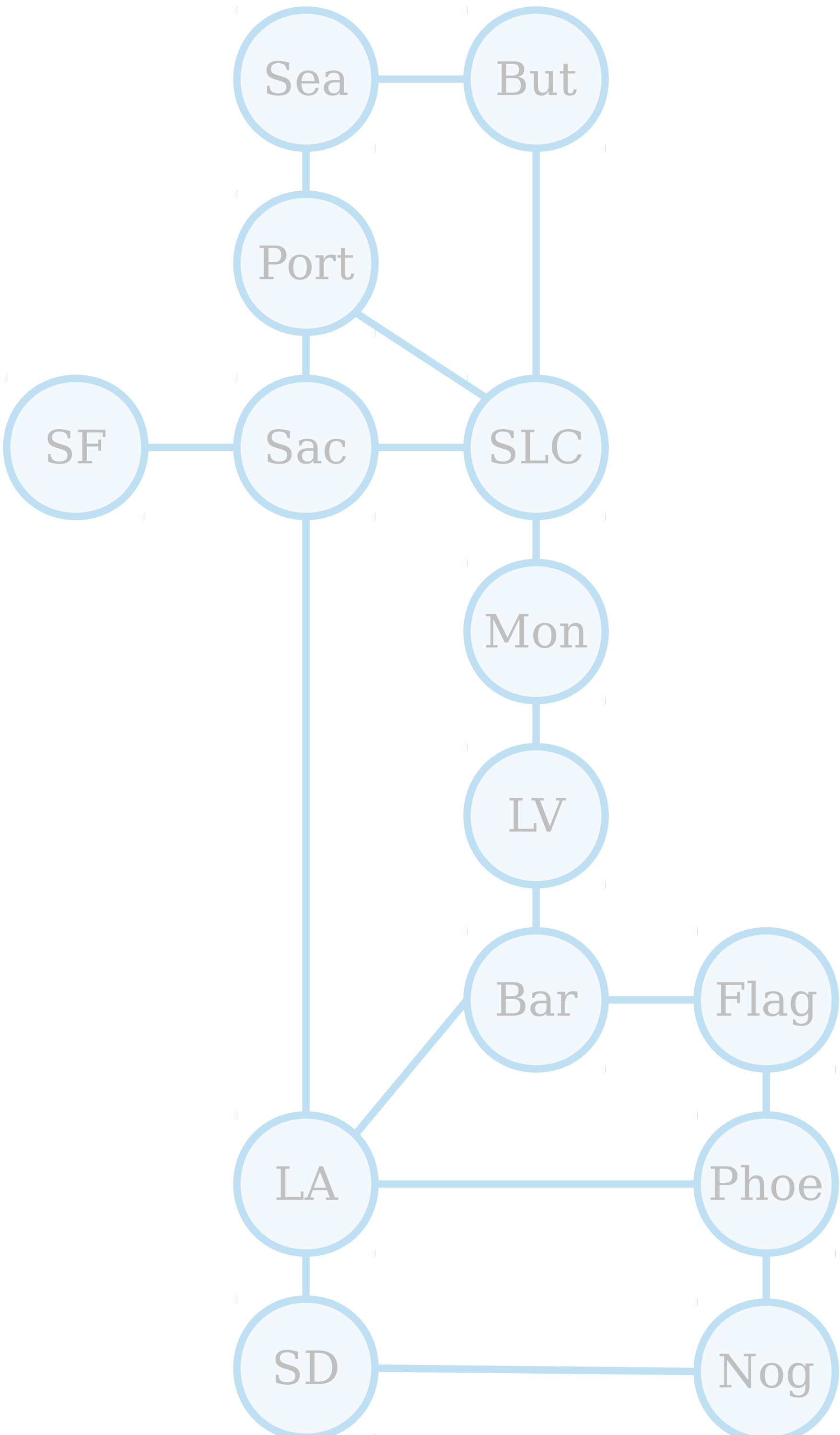

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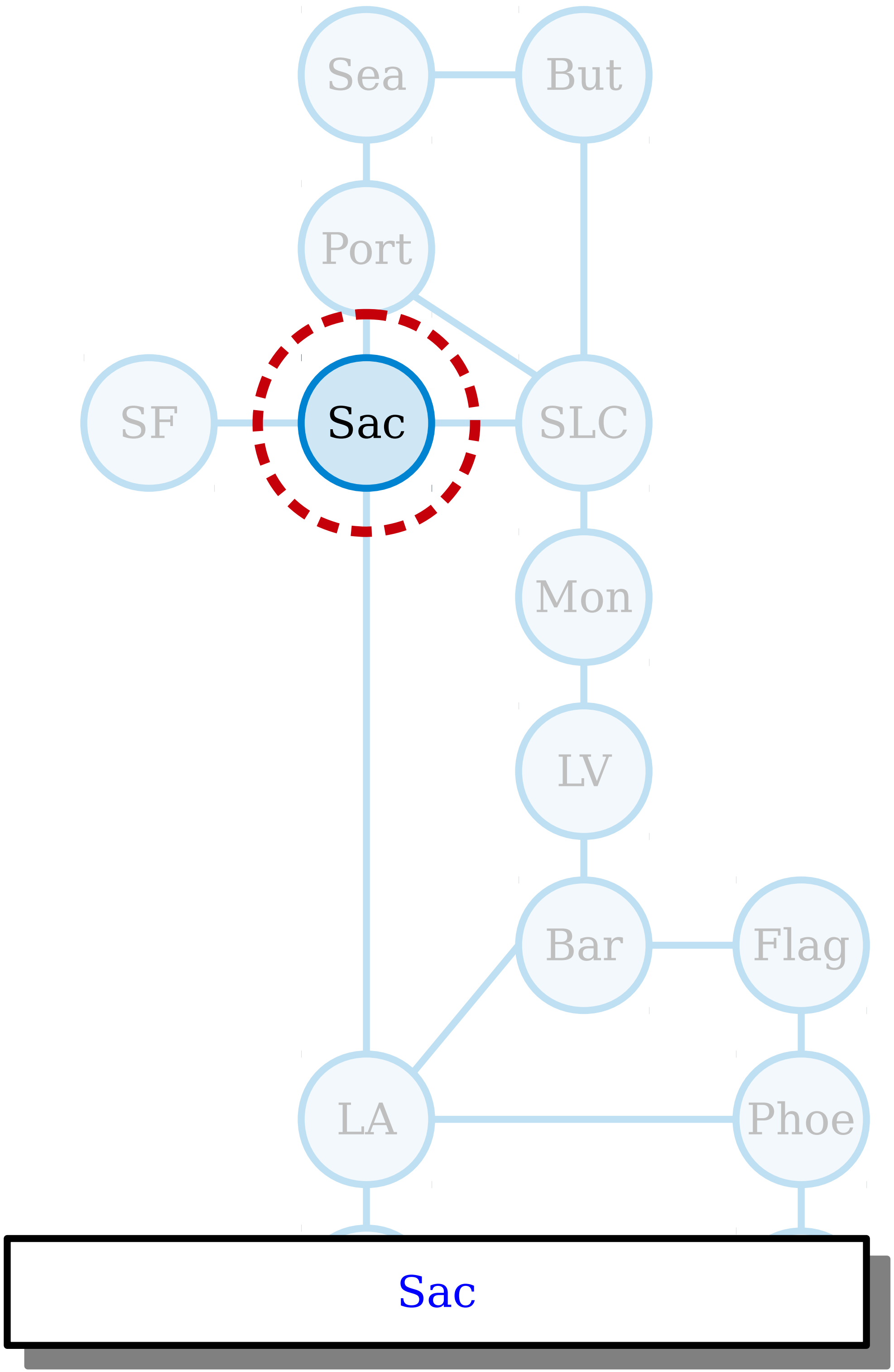

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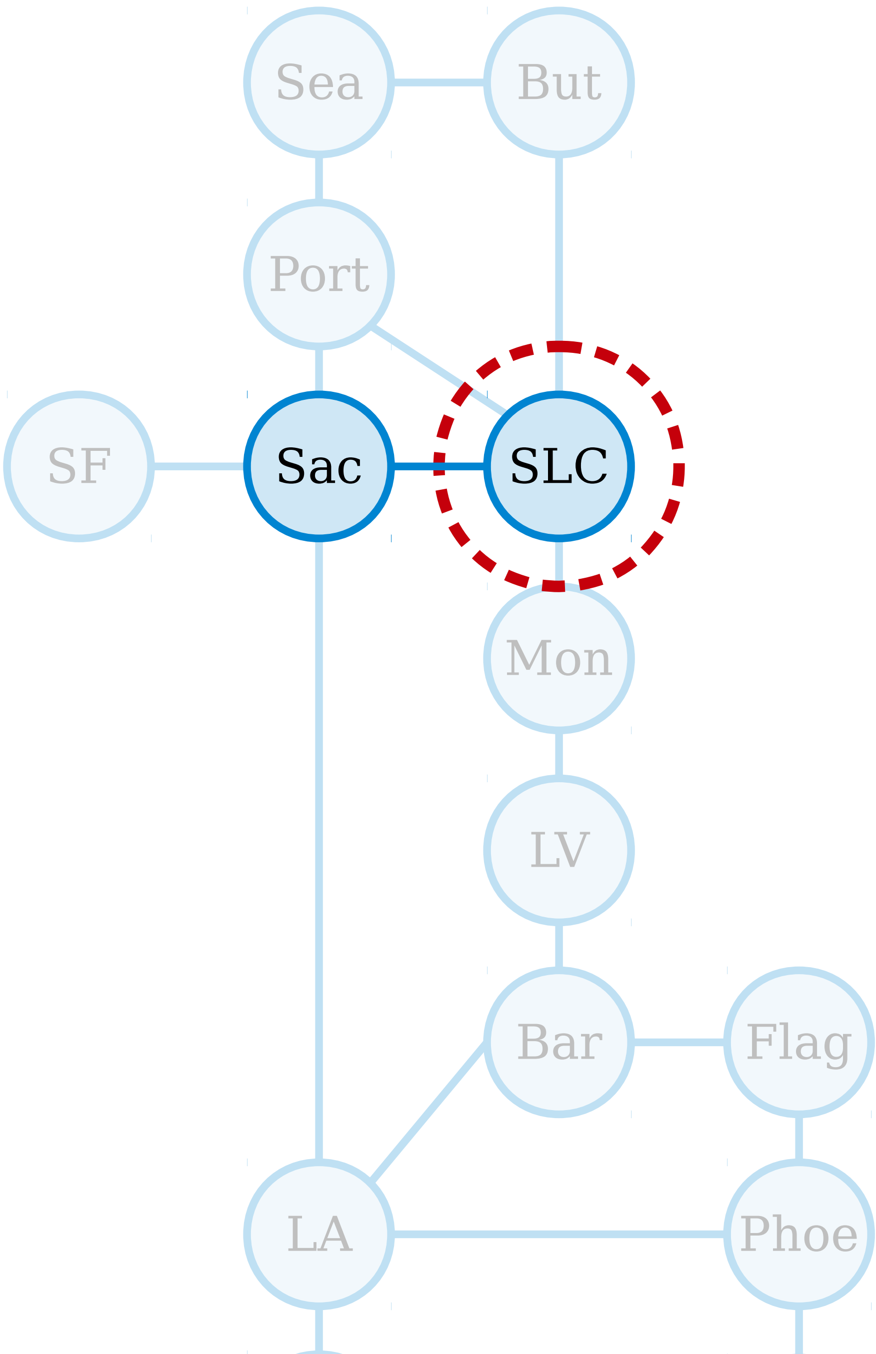

Sac, SLC

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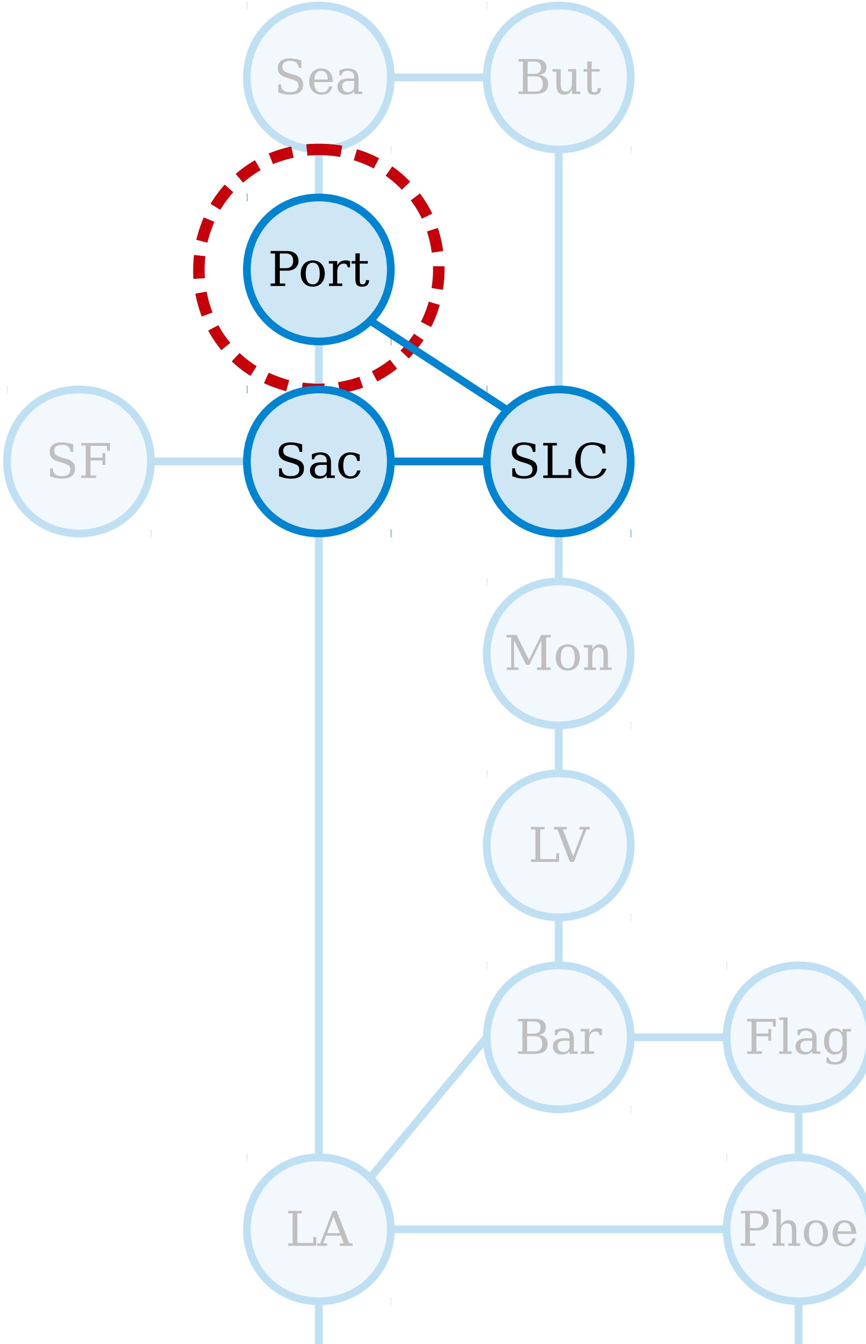

Sac, SLC, Port

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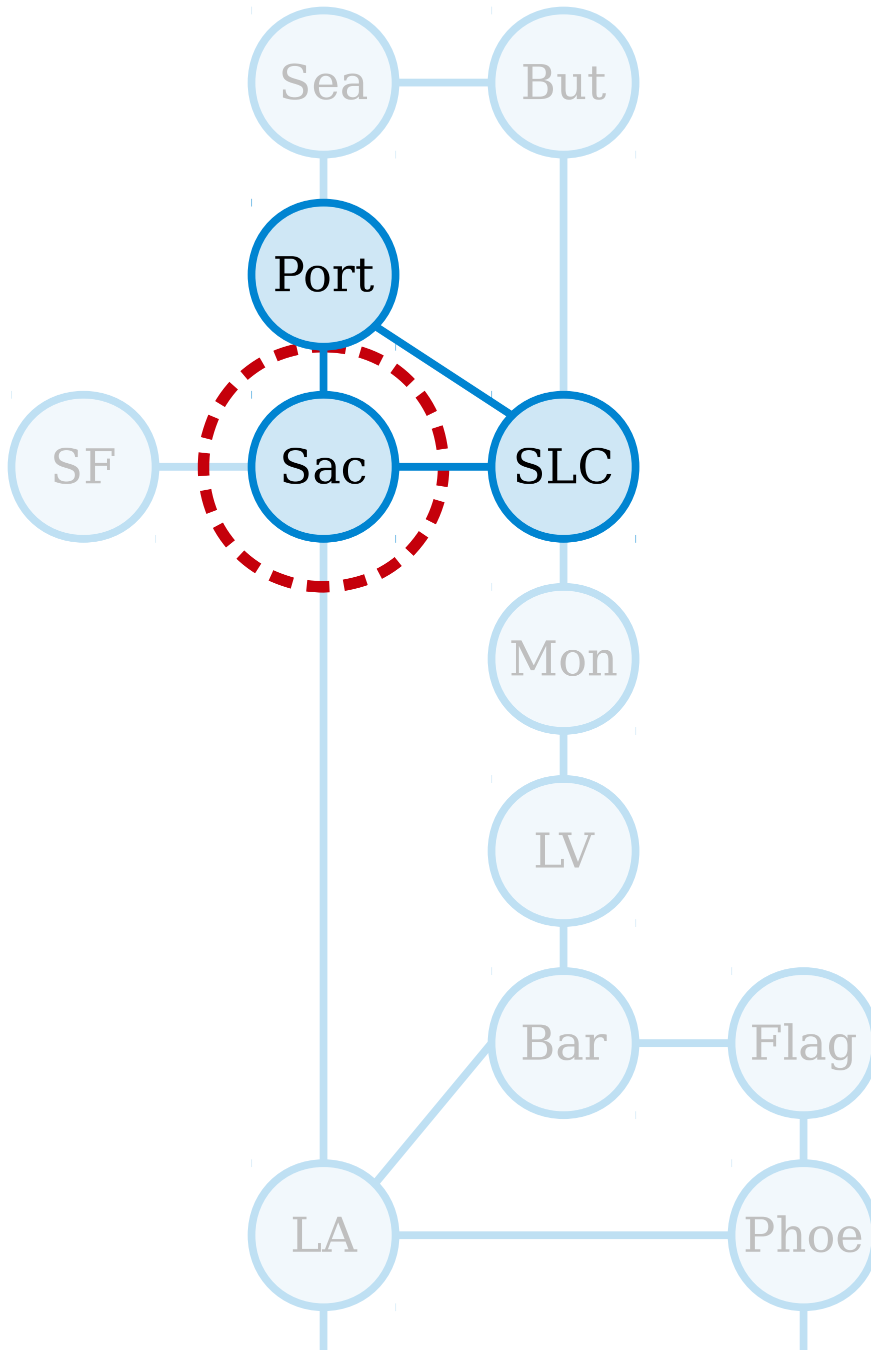

Sac, SLC, Port, Sac

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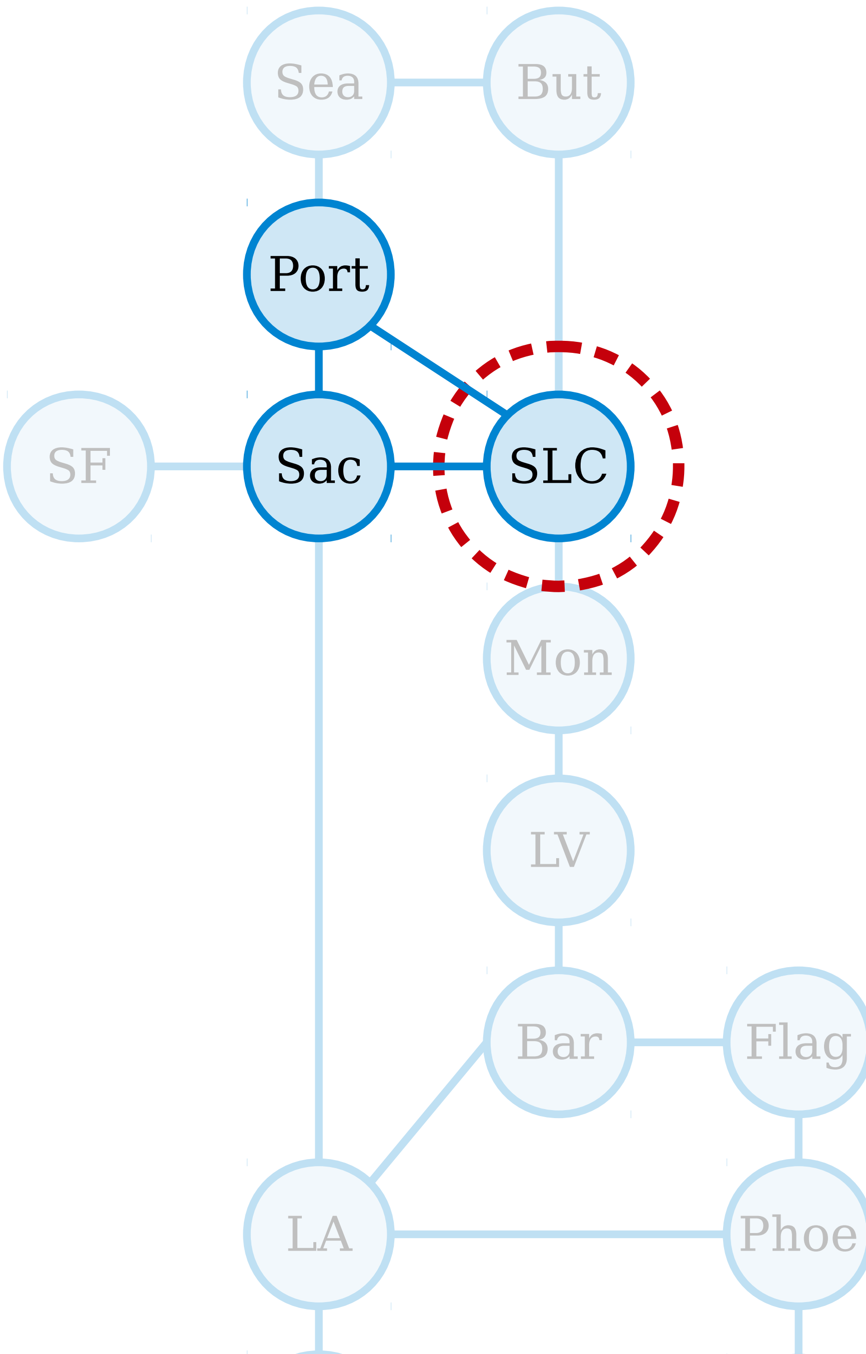

Sac, SLC, Port, Sac, SLC

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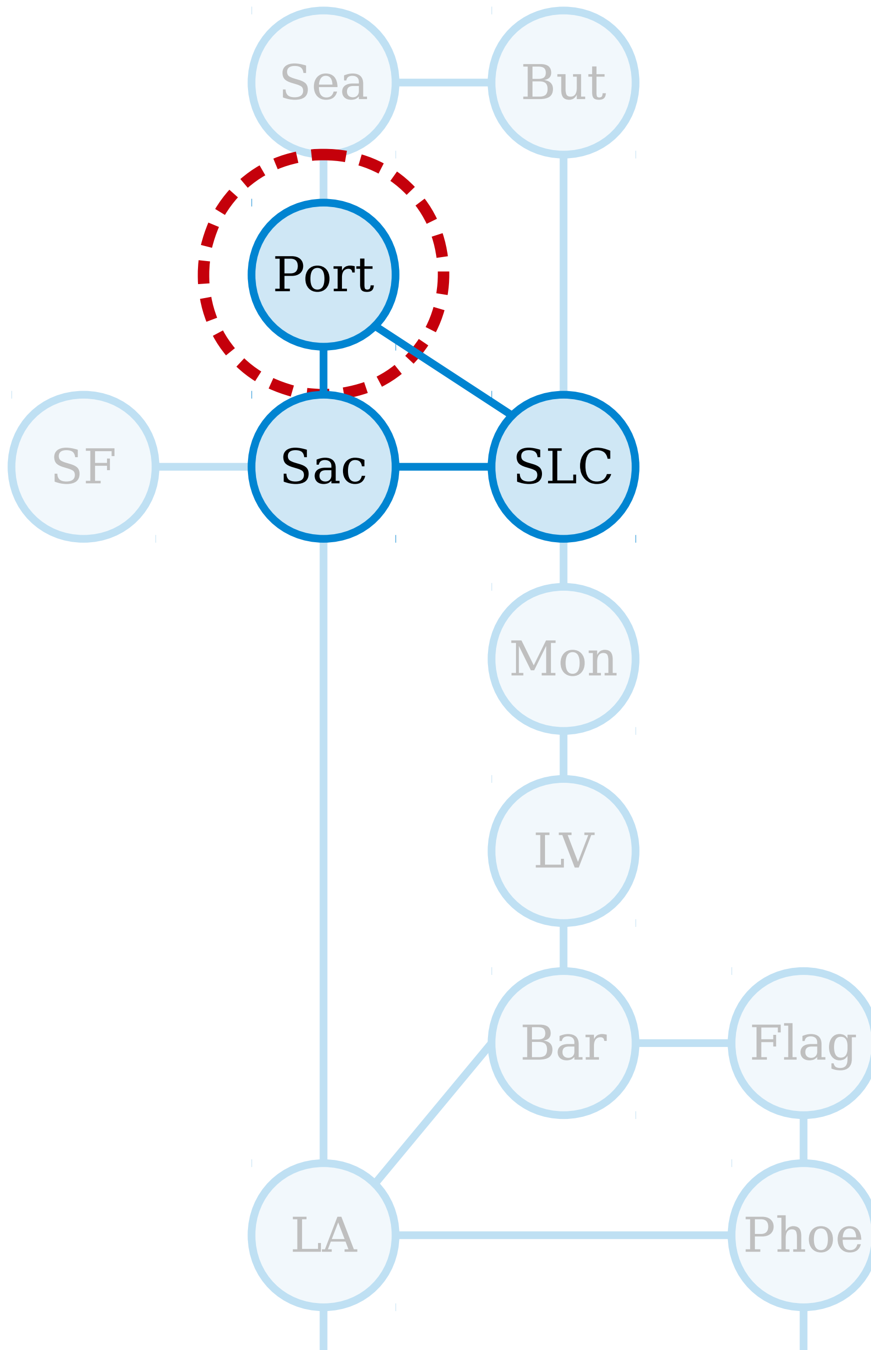

Sac, SLC, Port, Sac, SLC, Port

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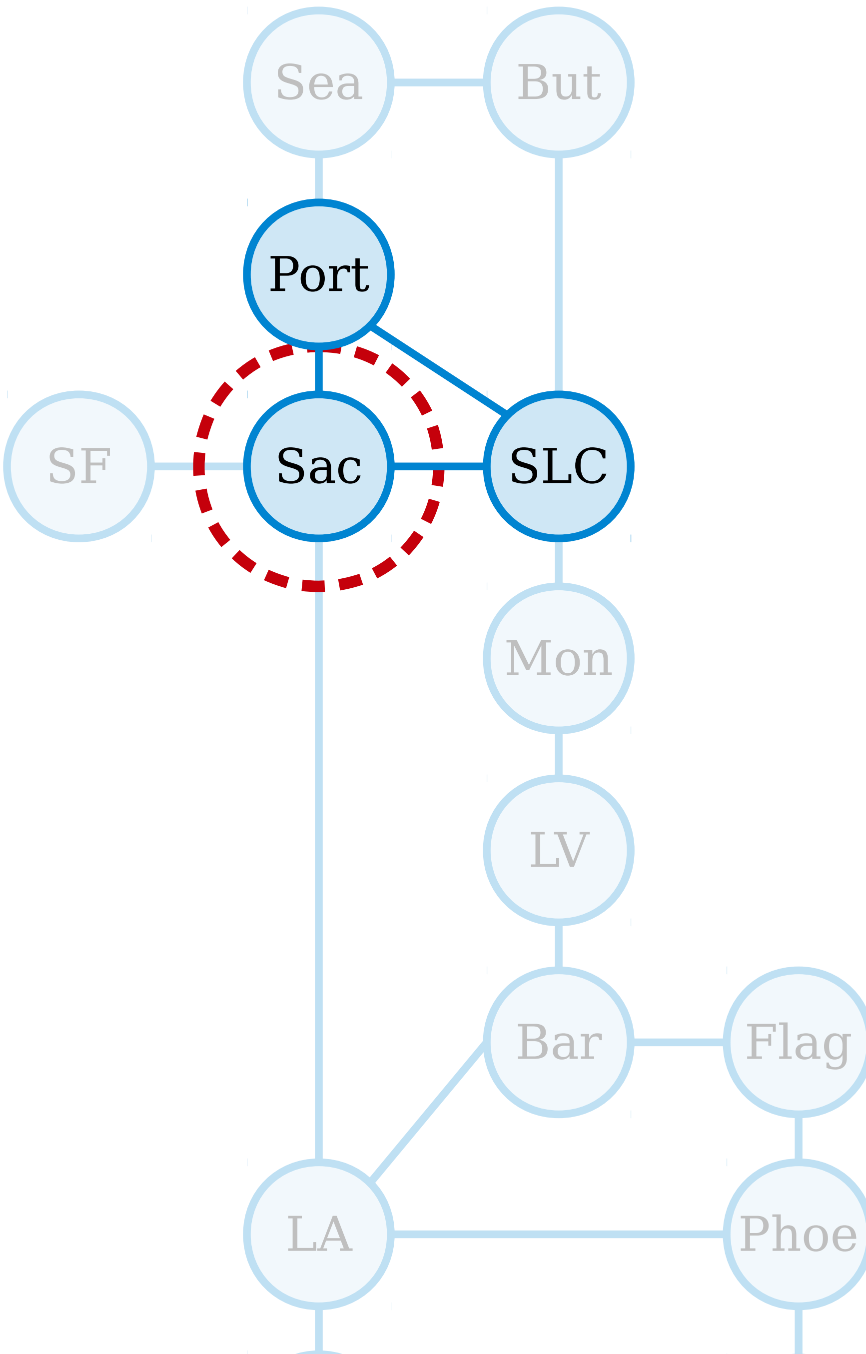

Sac, SLC, Port, Sac, SLC, Port, Sac

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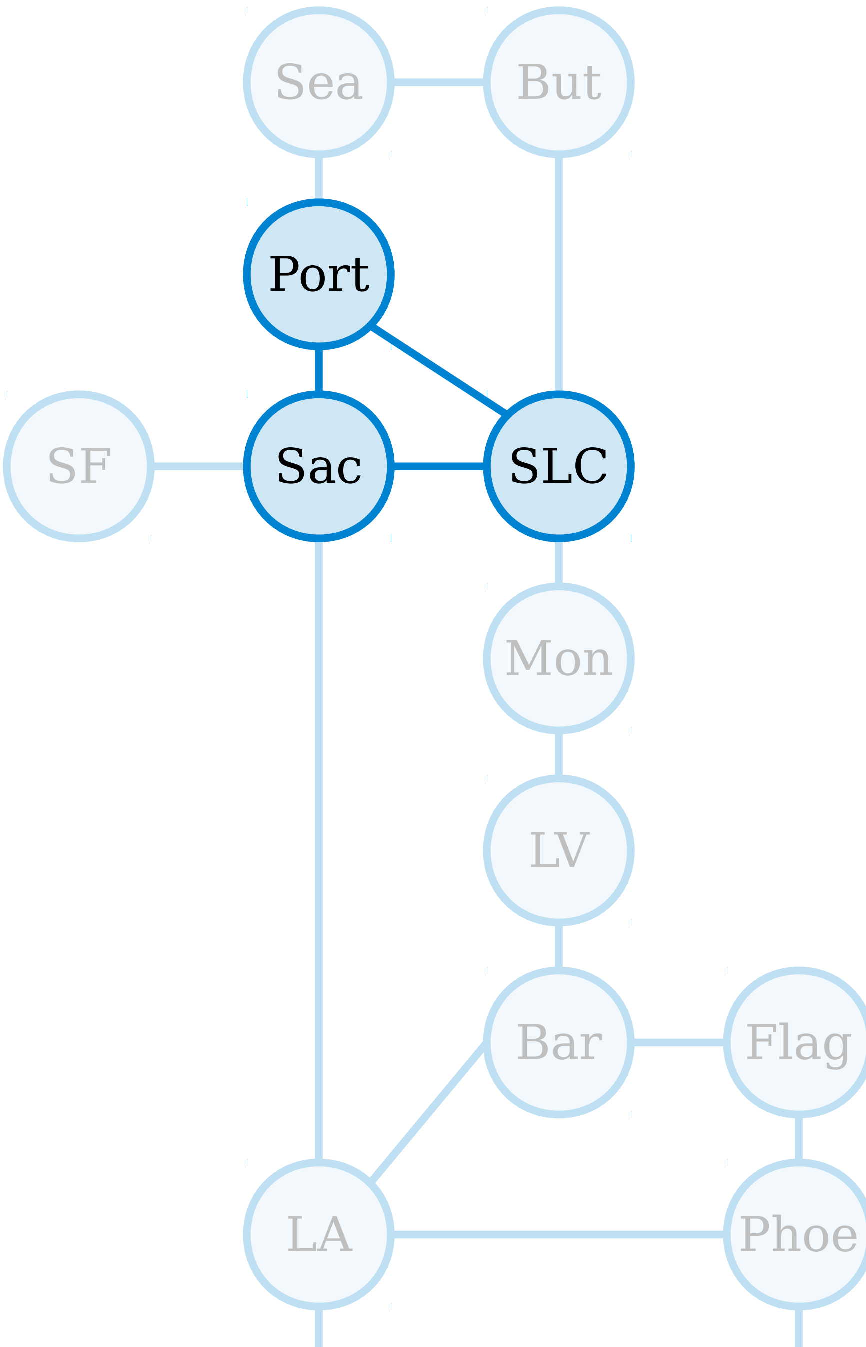

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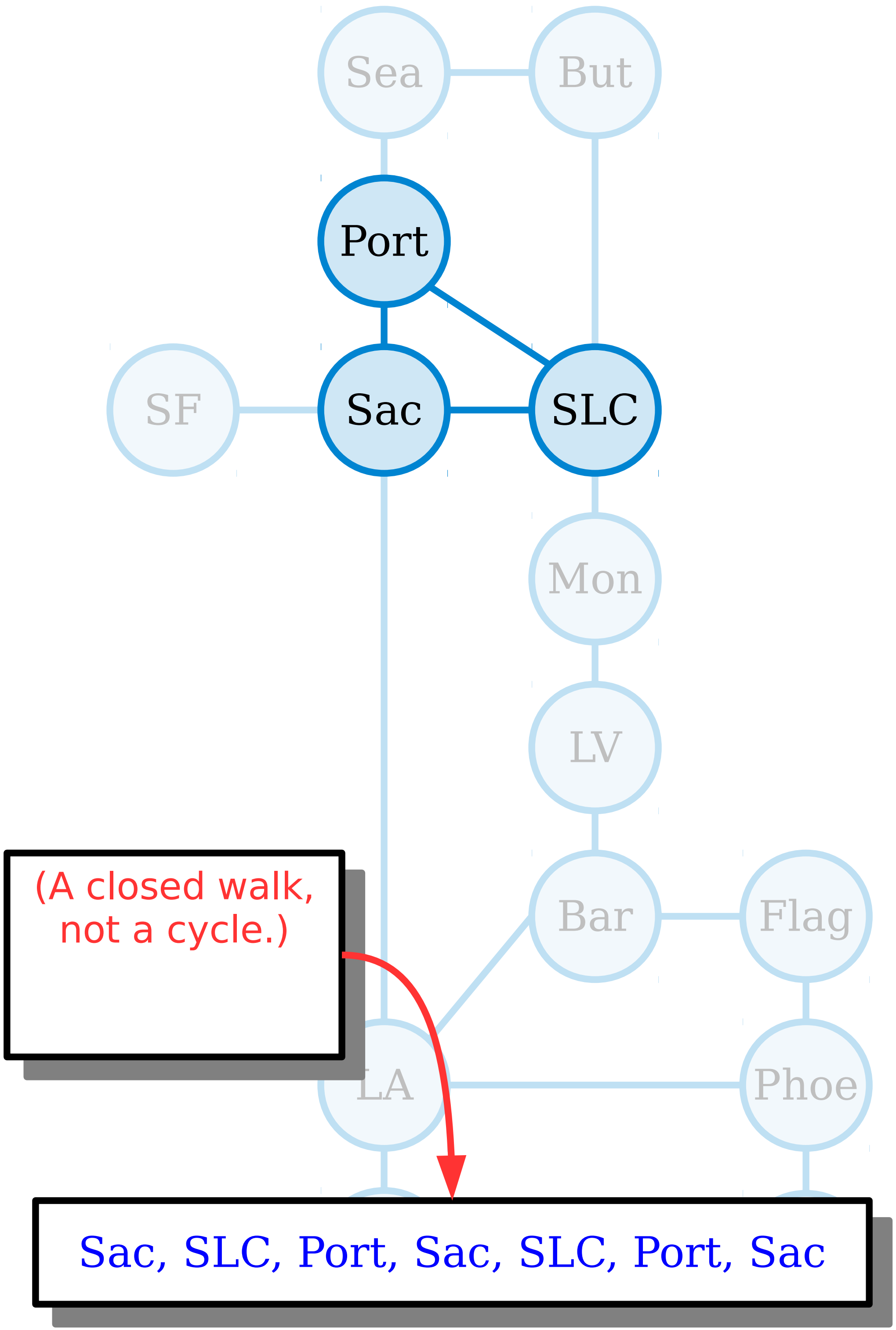

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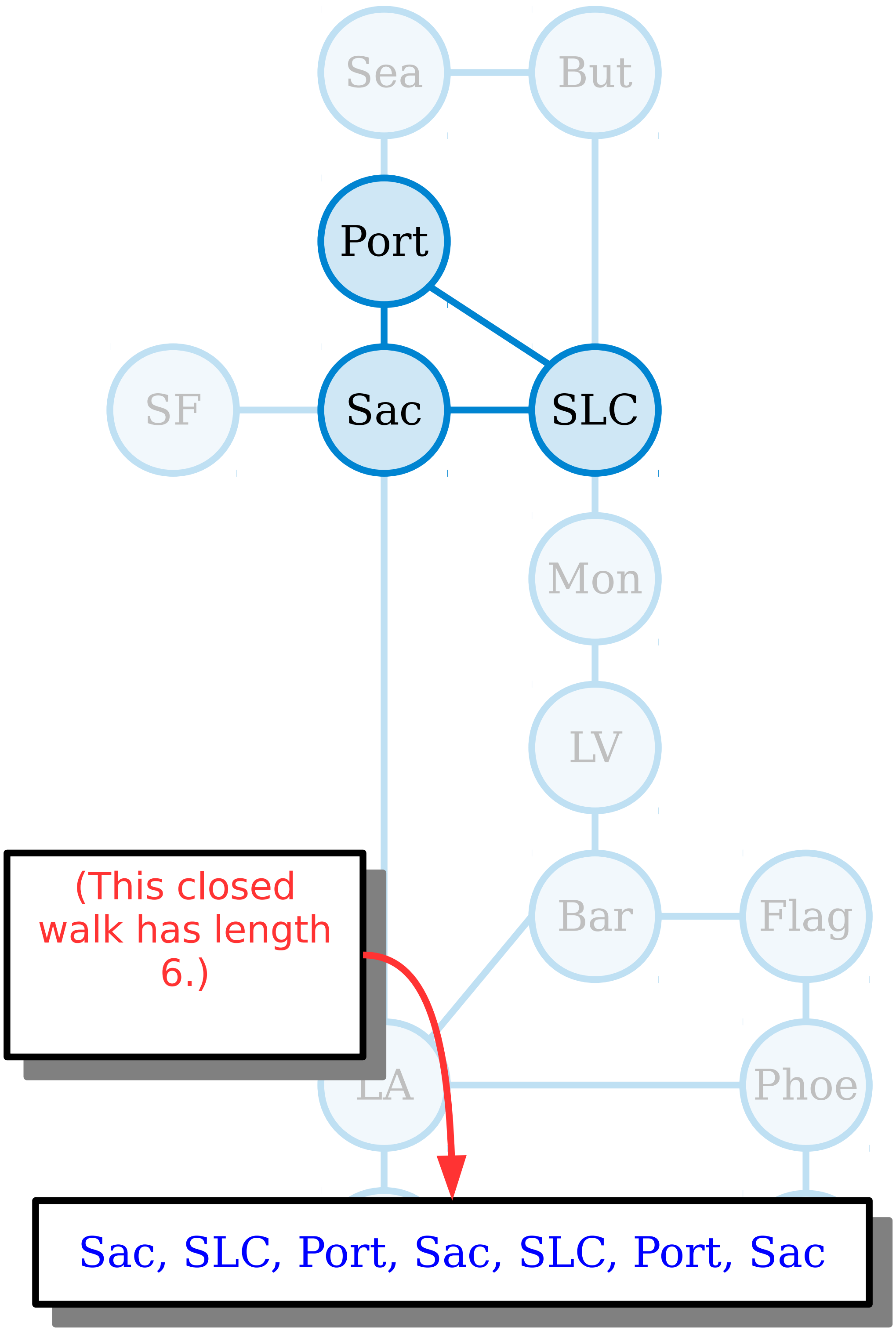

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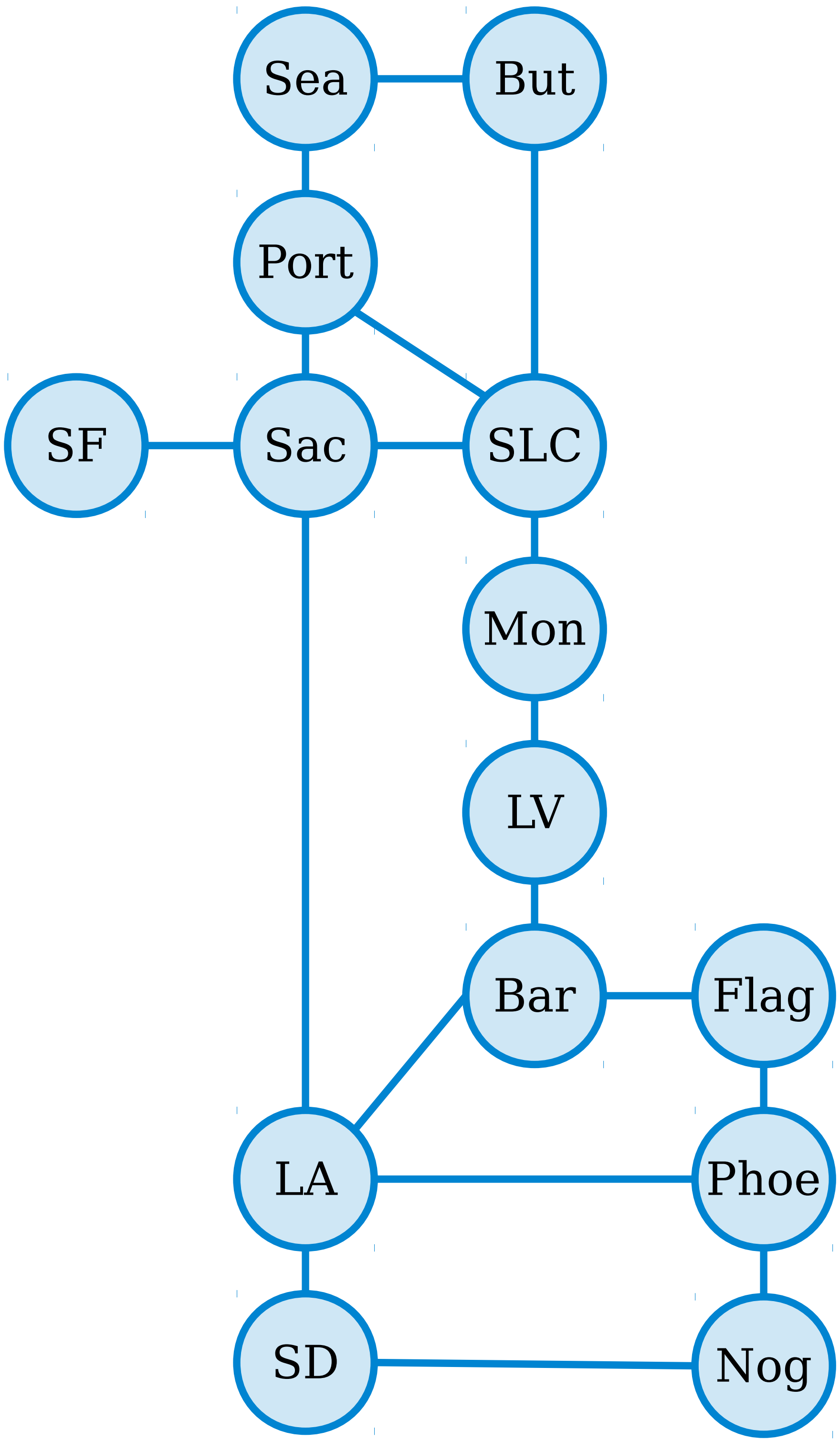

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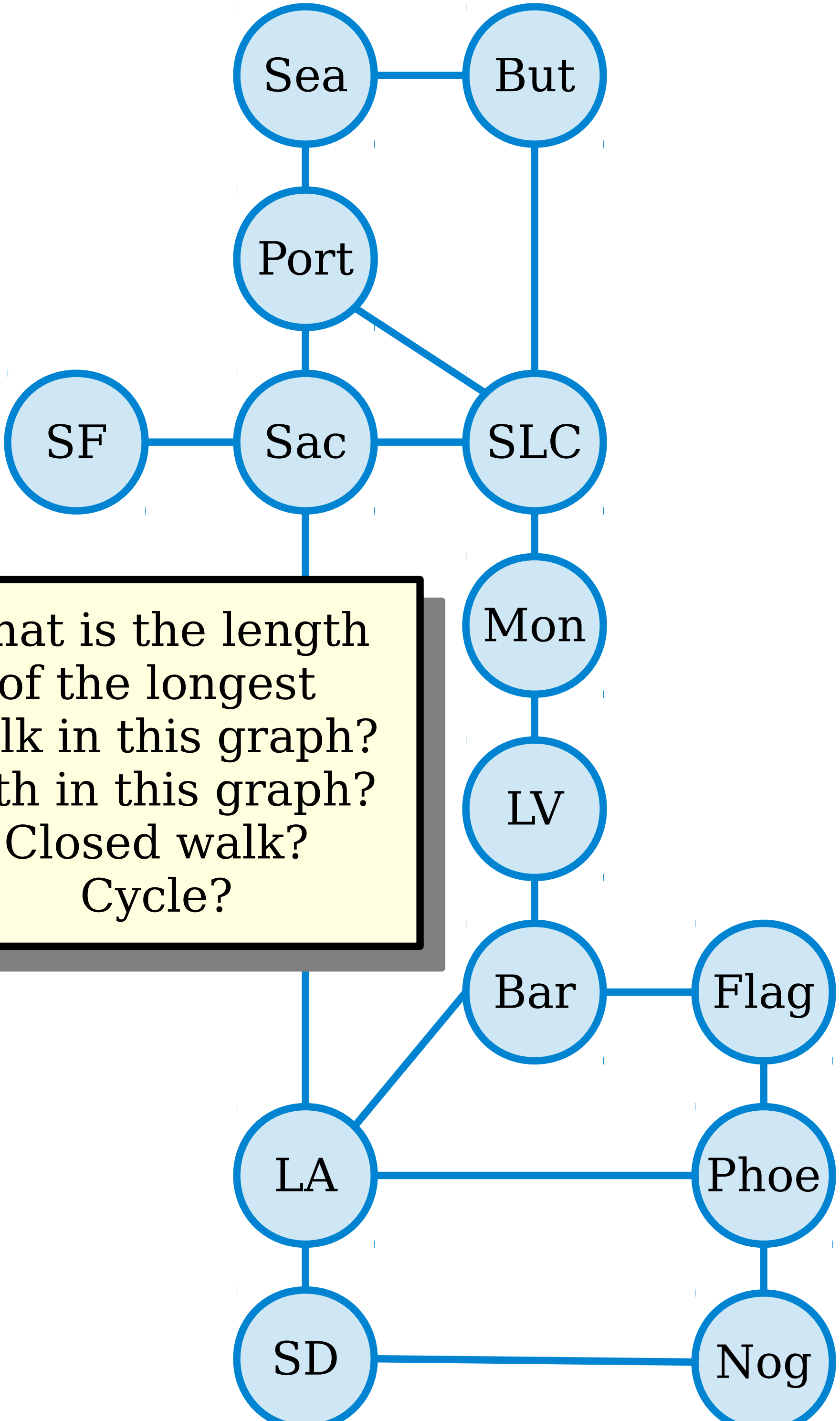

What is the length of the longest walk in this graph? Path in this graph? Closed walk? Cycle?

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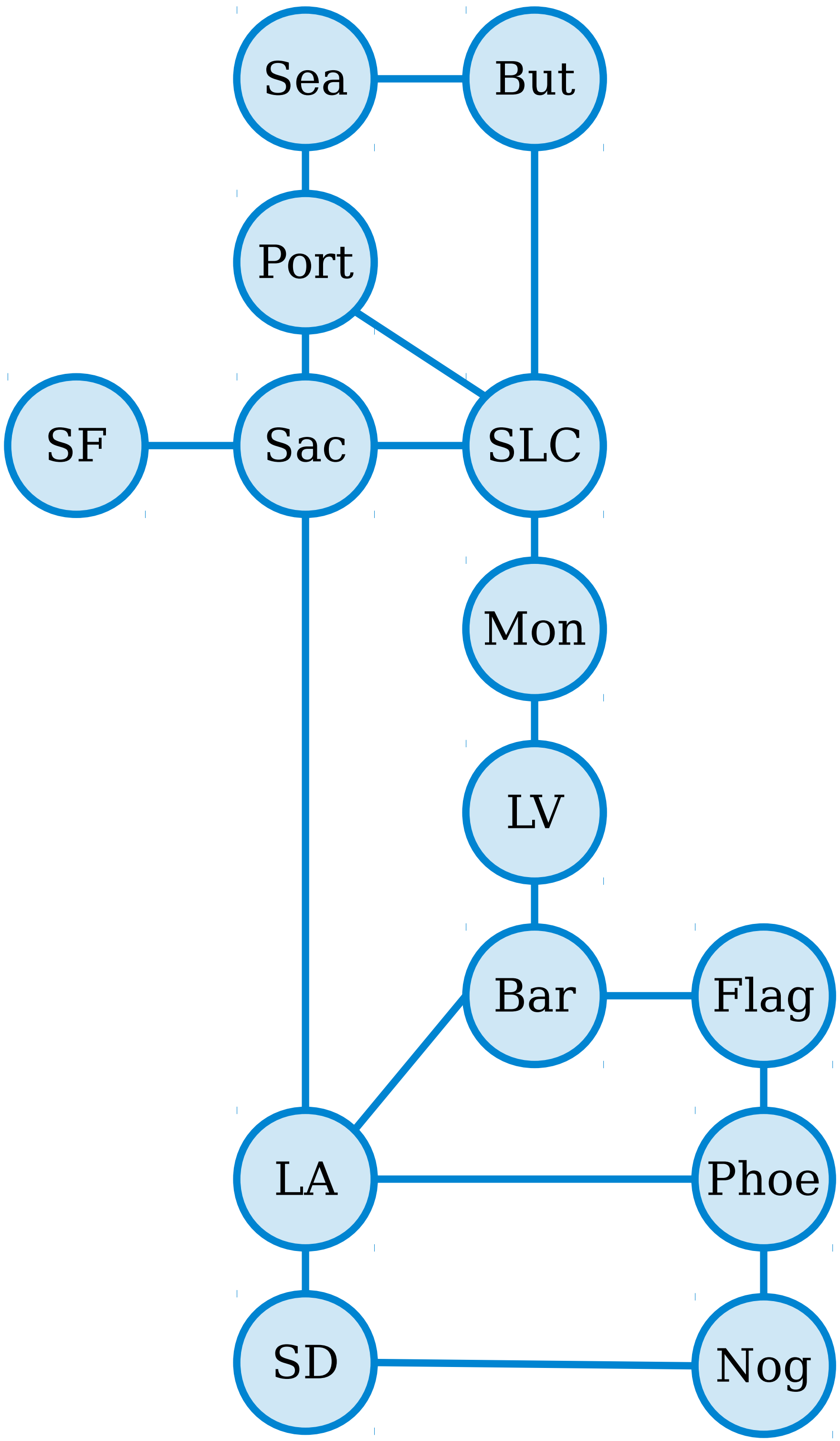

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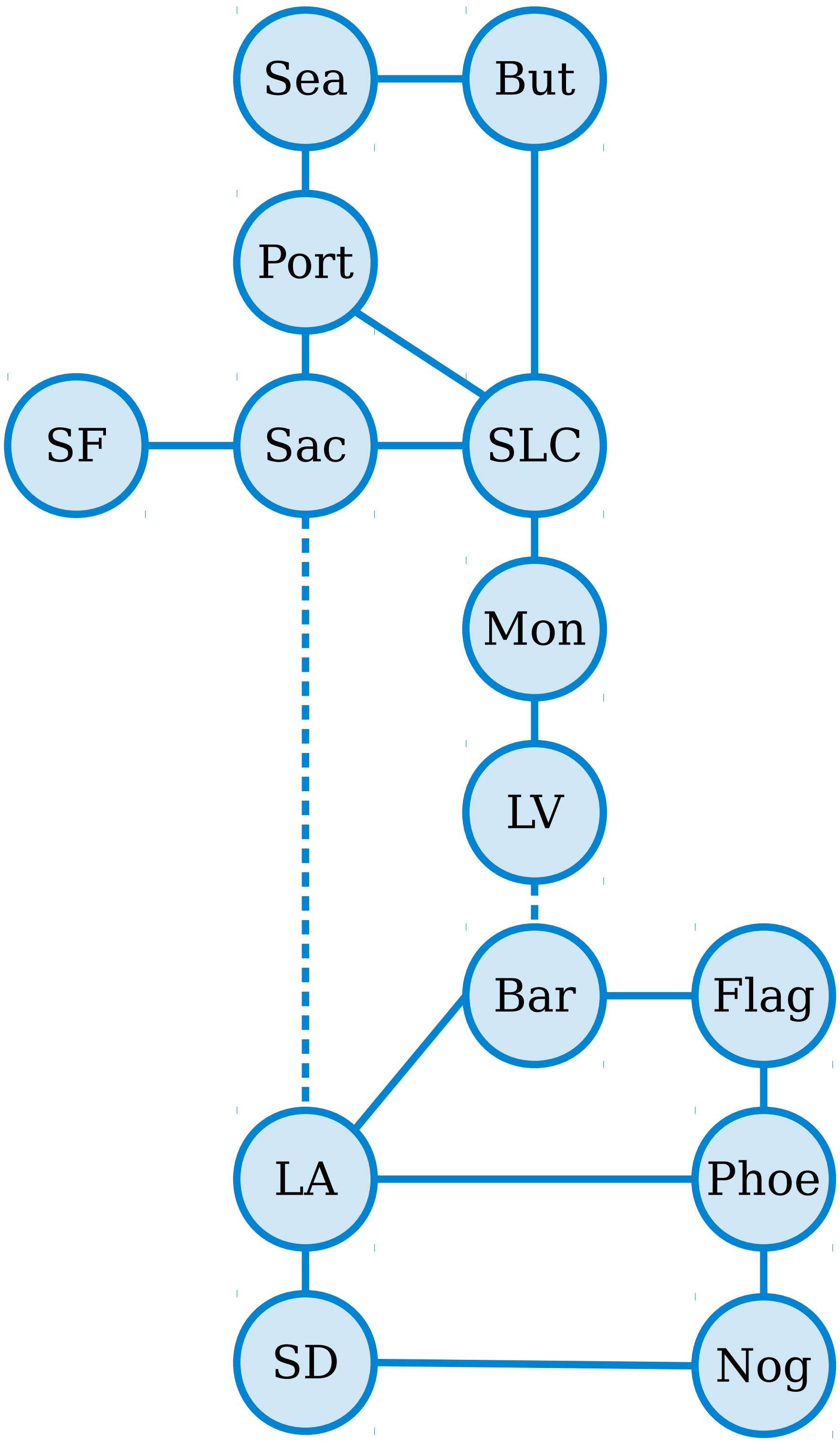

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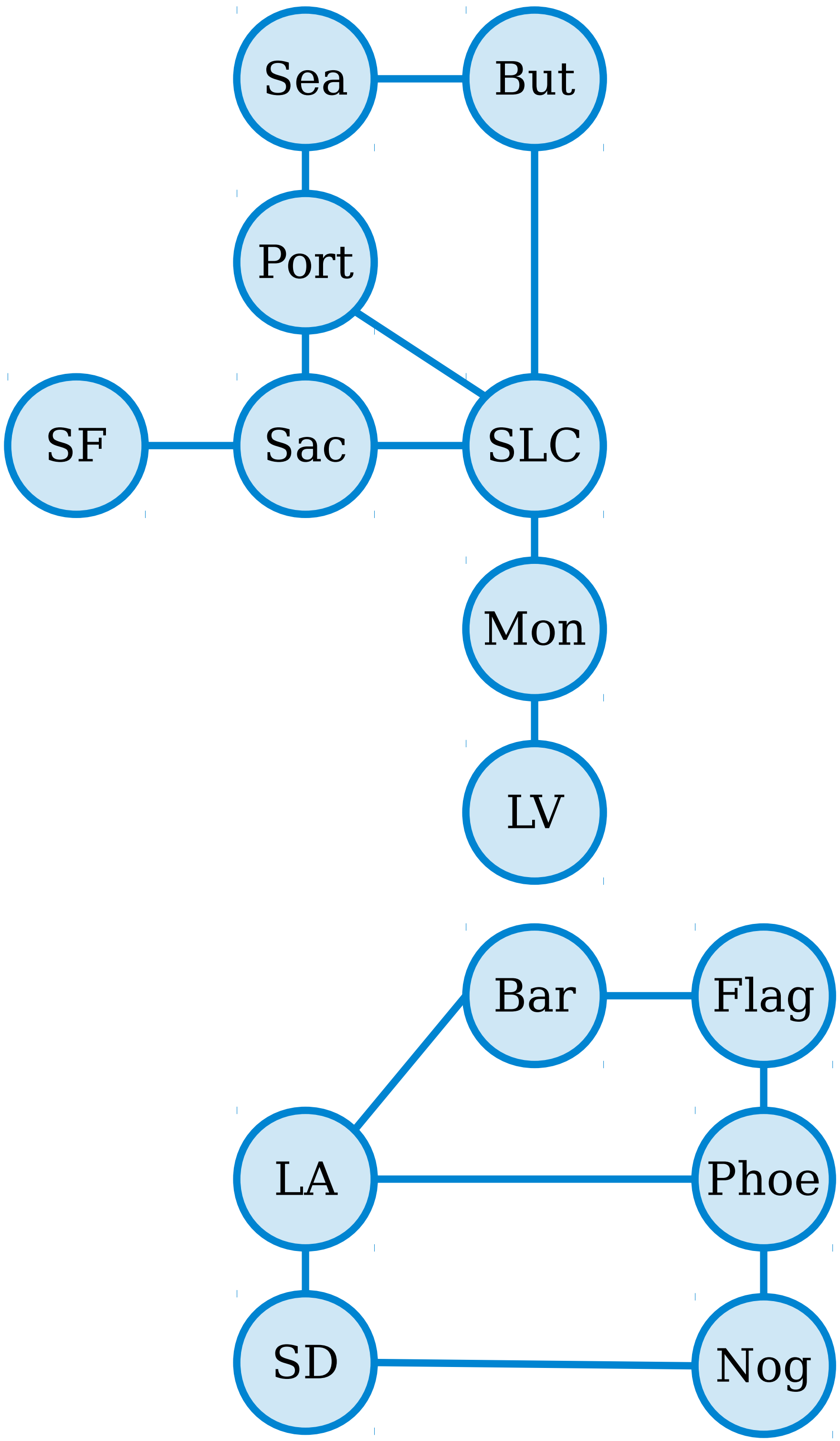

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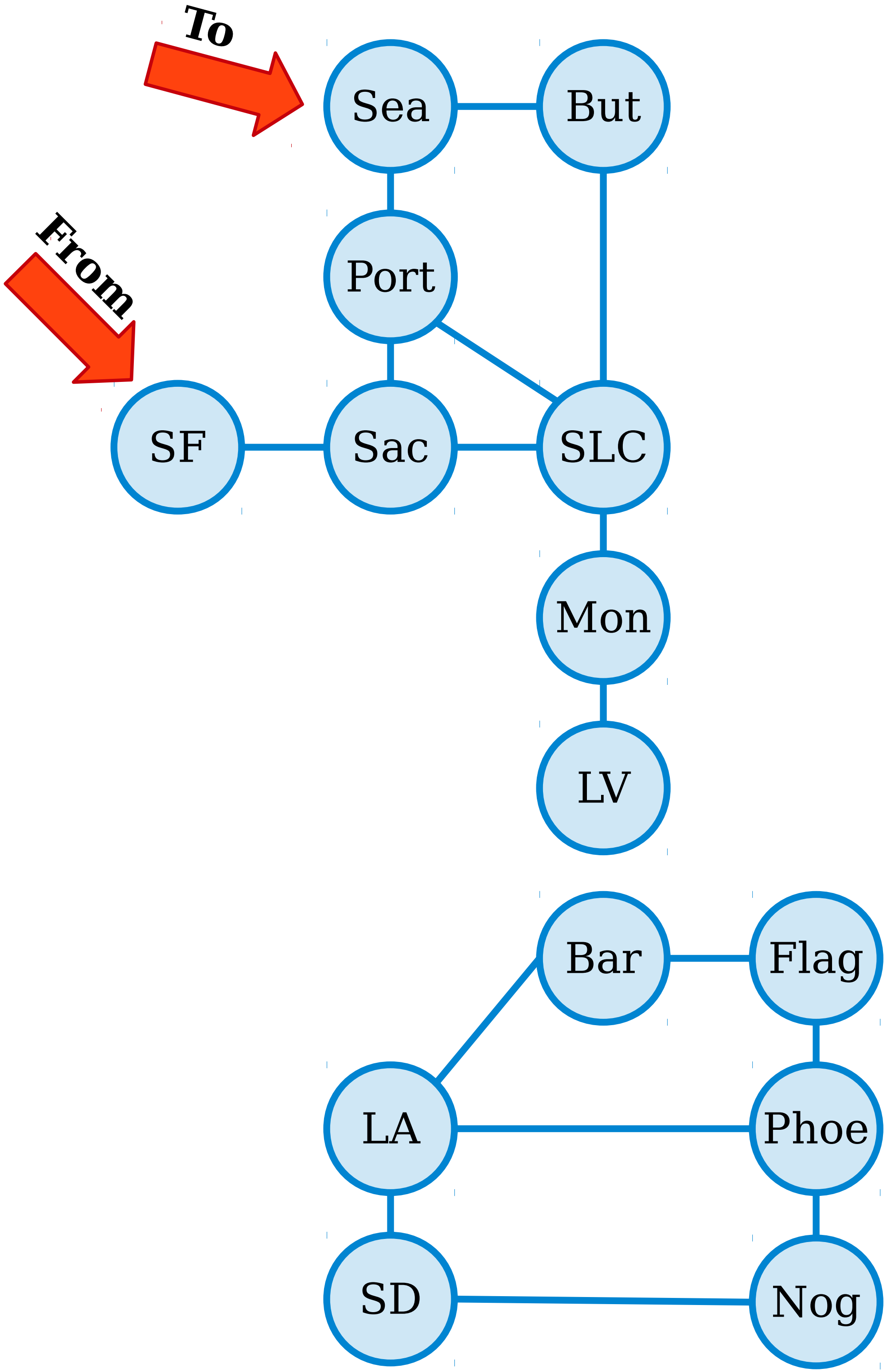

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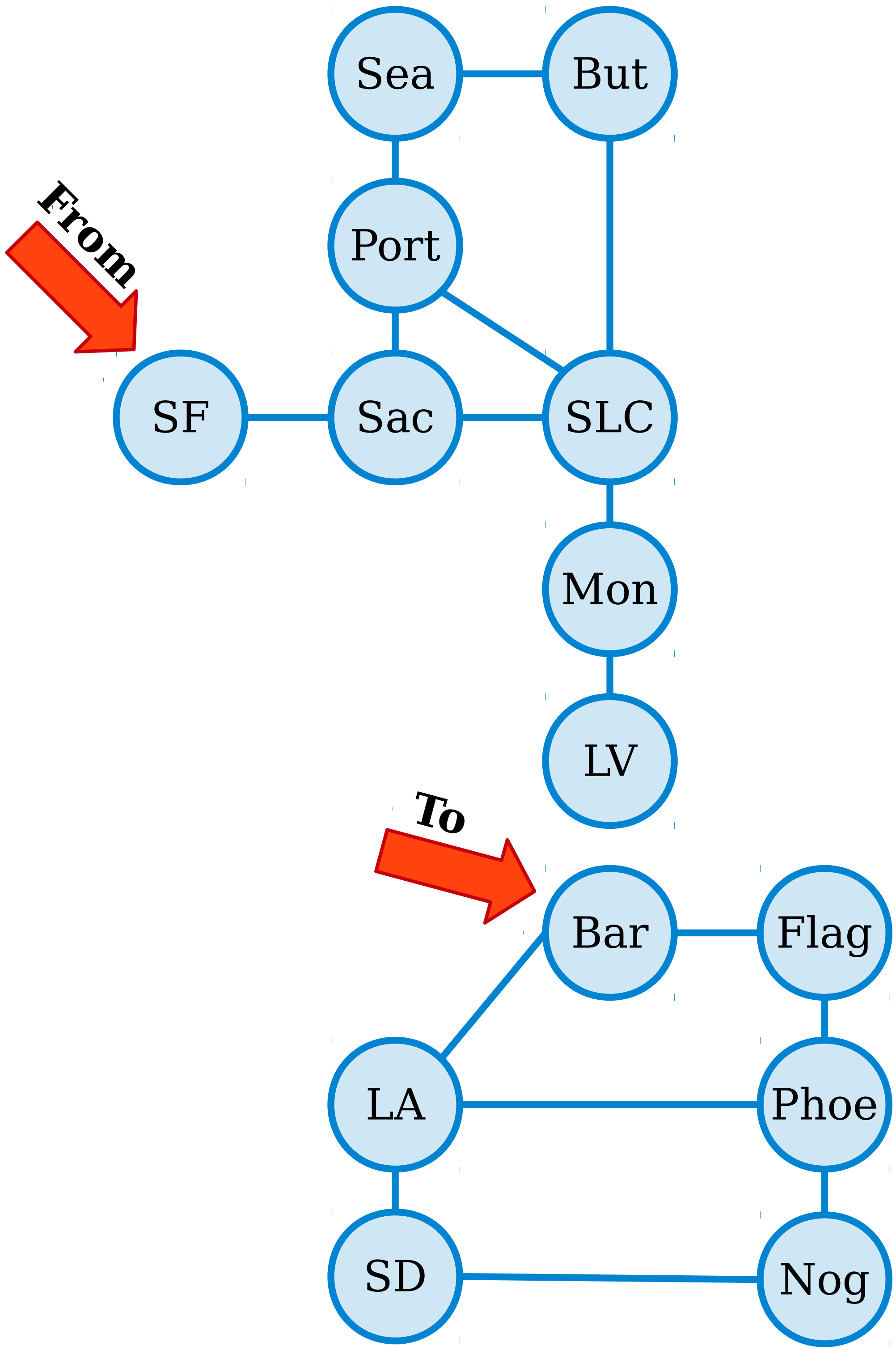

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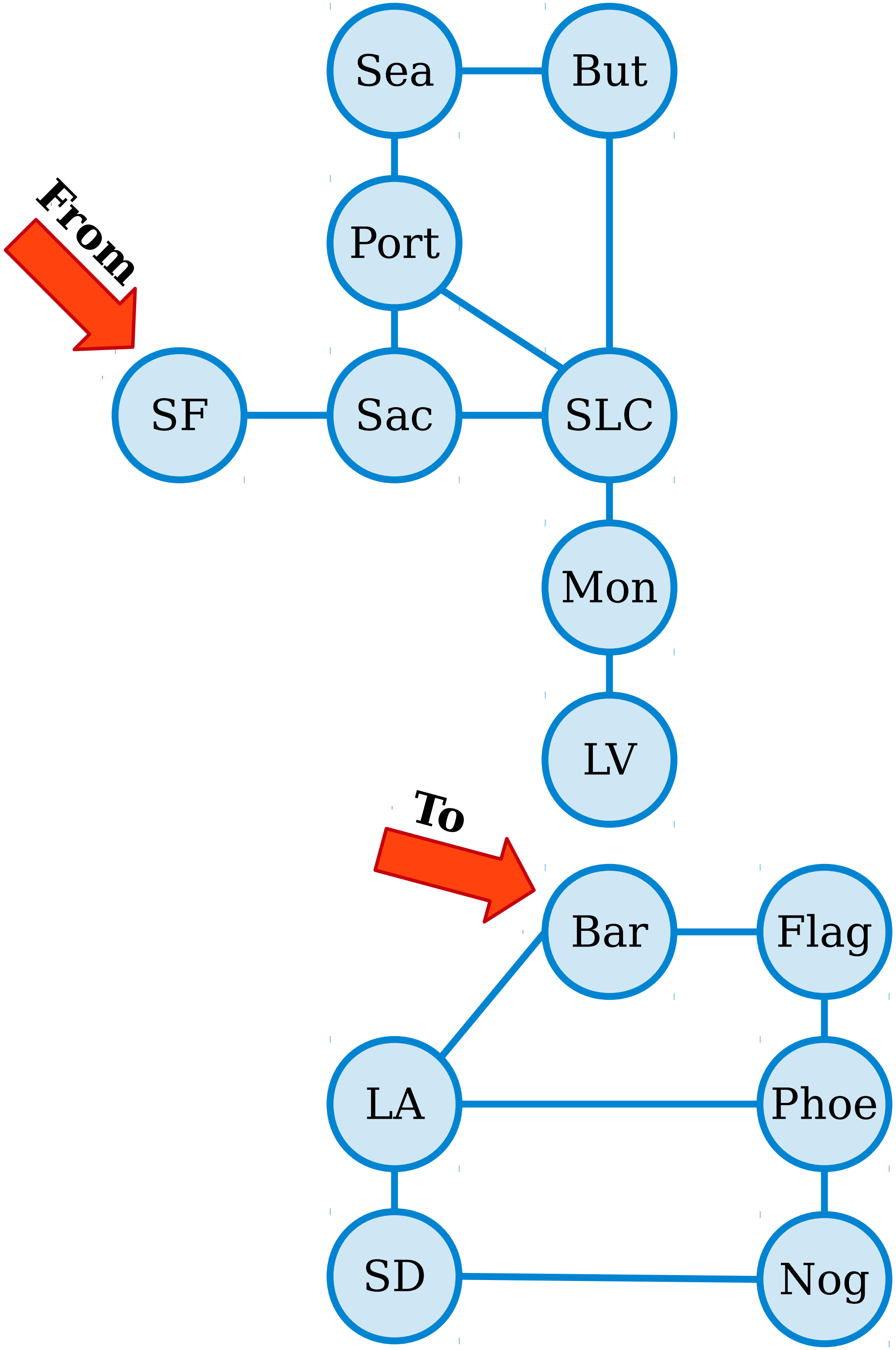

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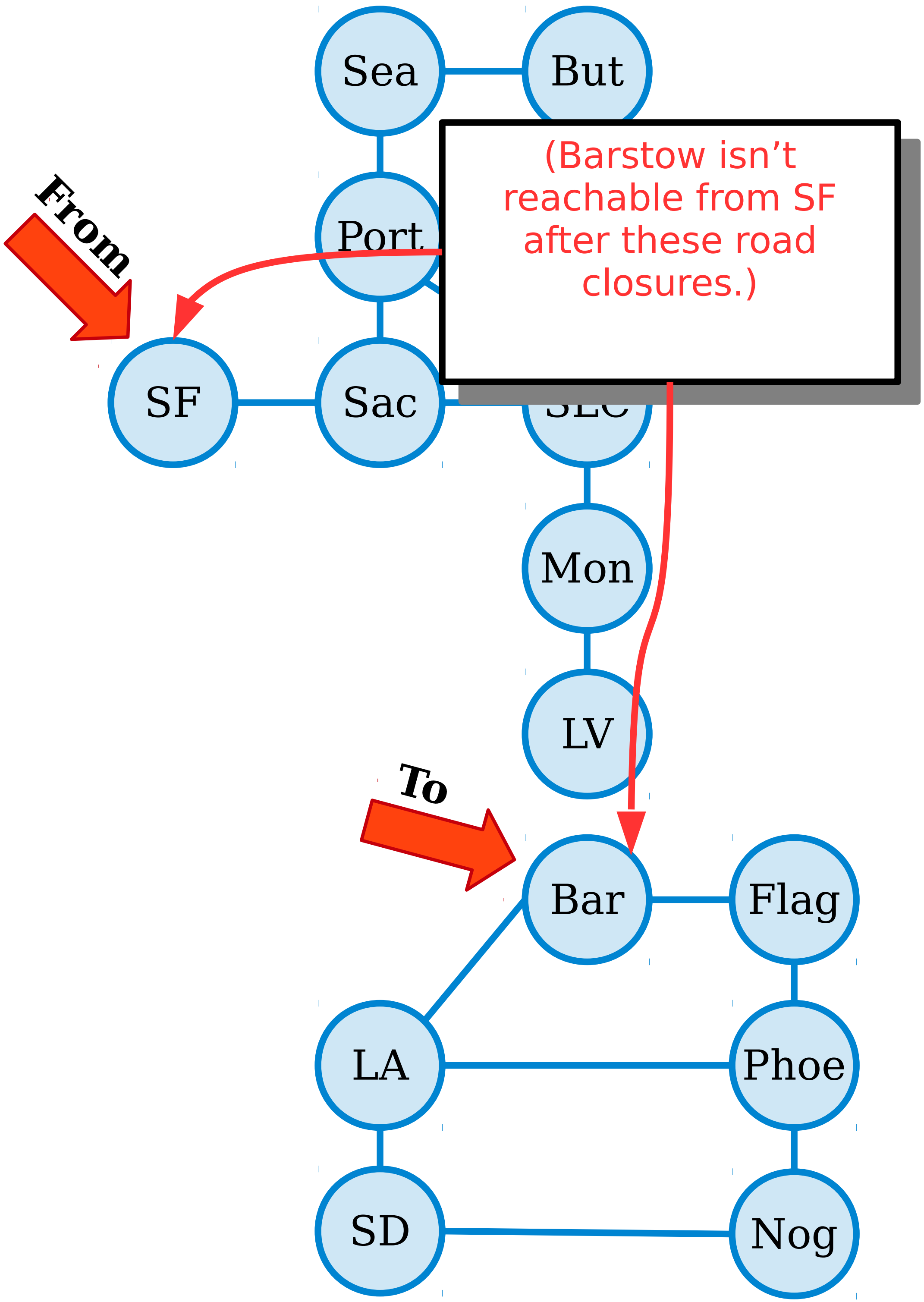

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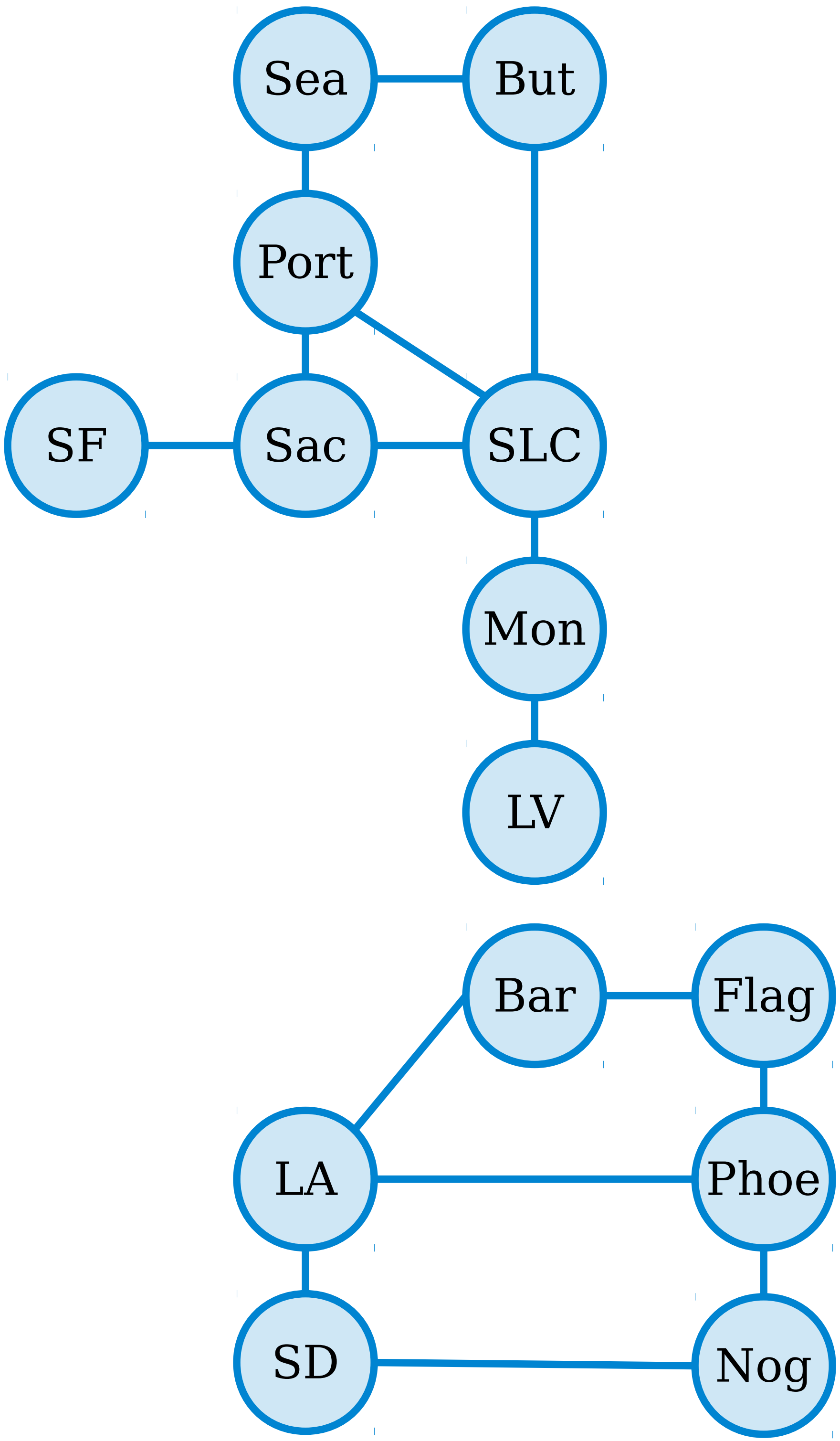

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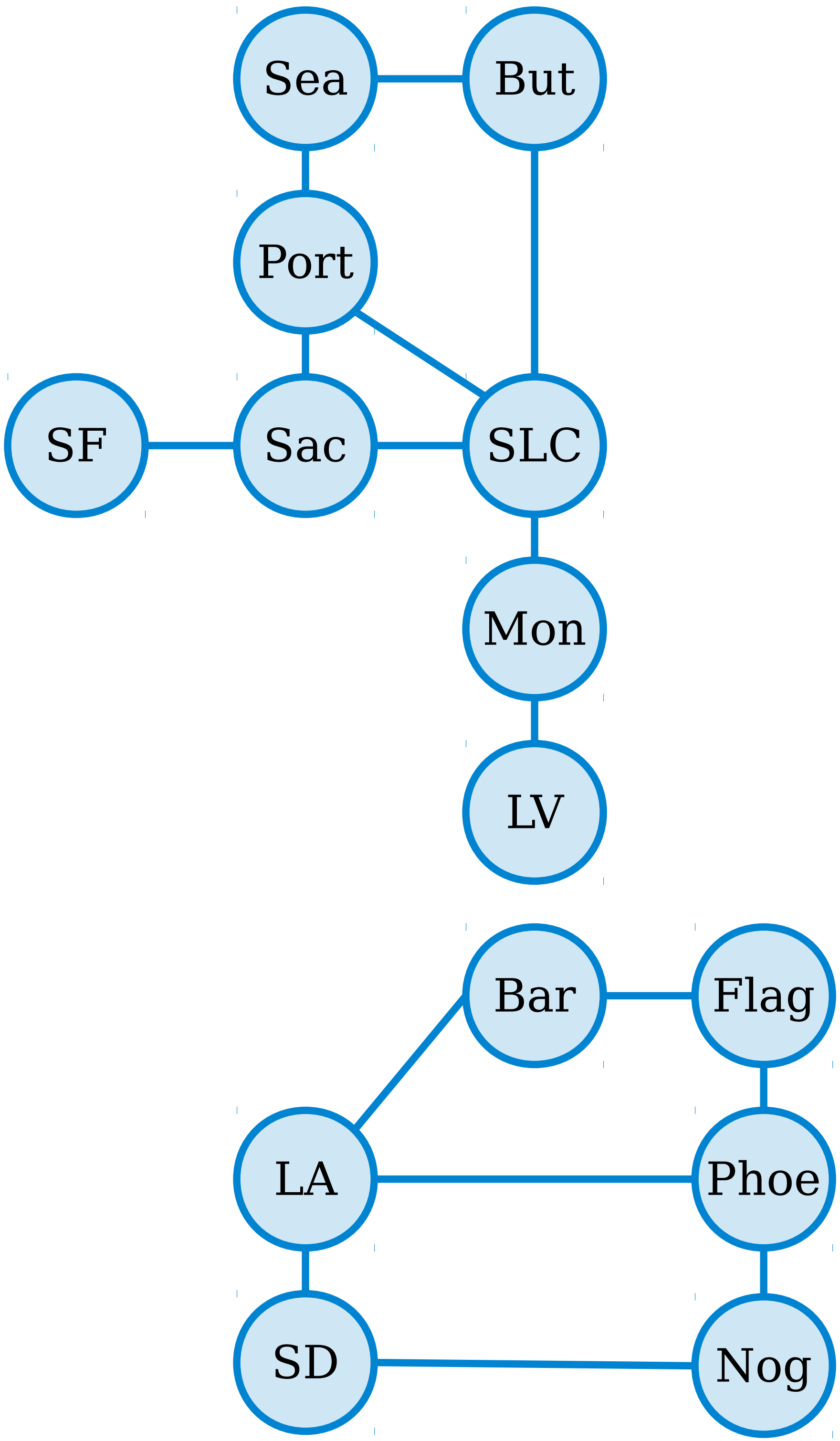

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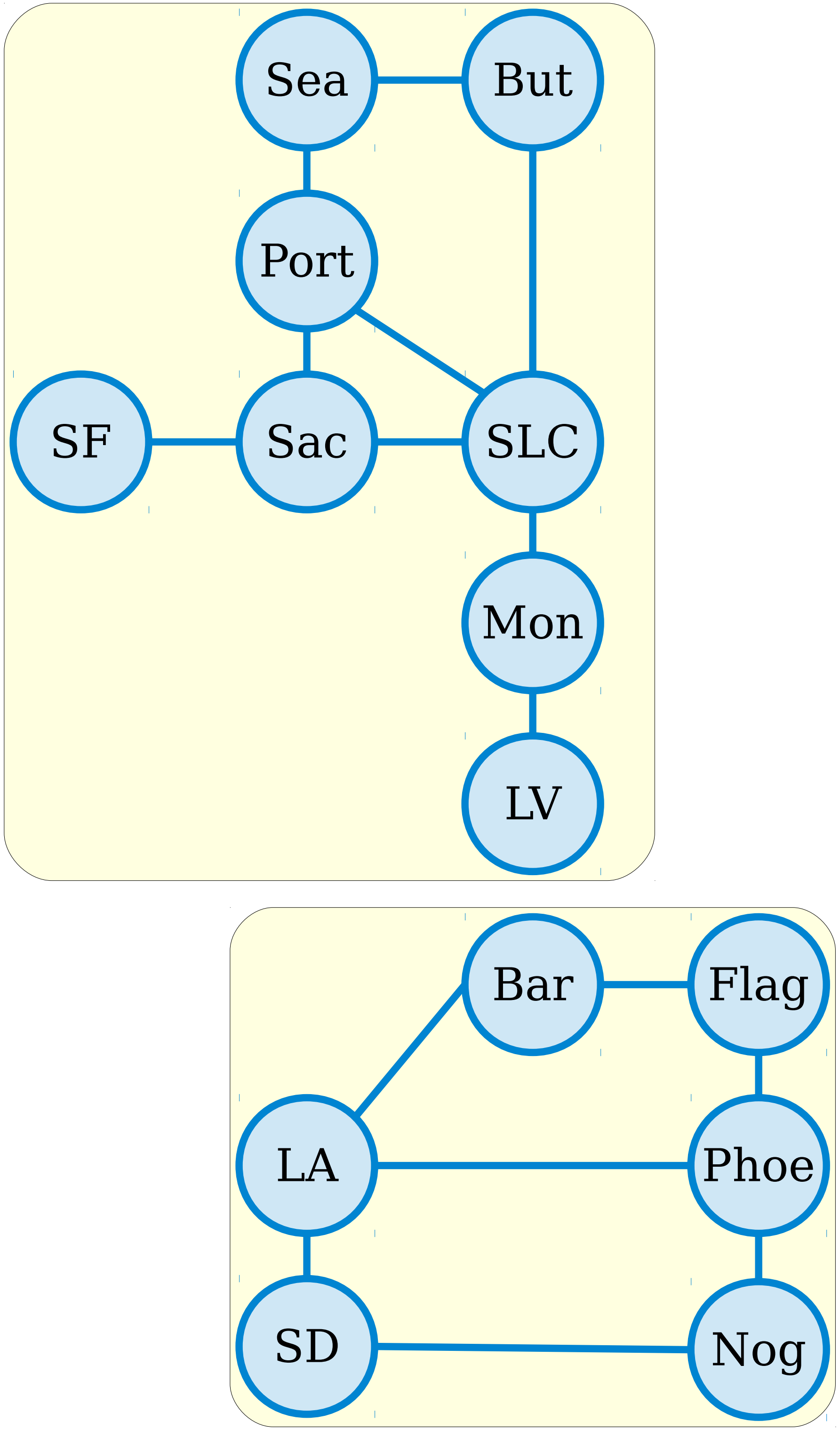
(This graph is not connected.)



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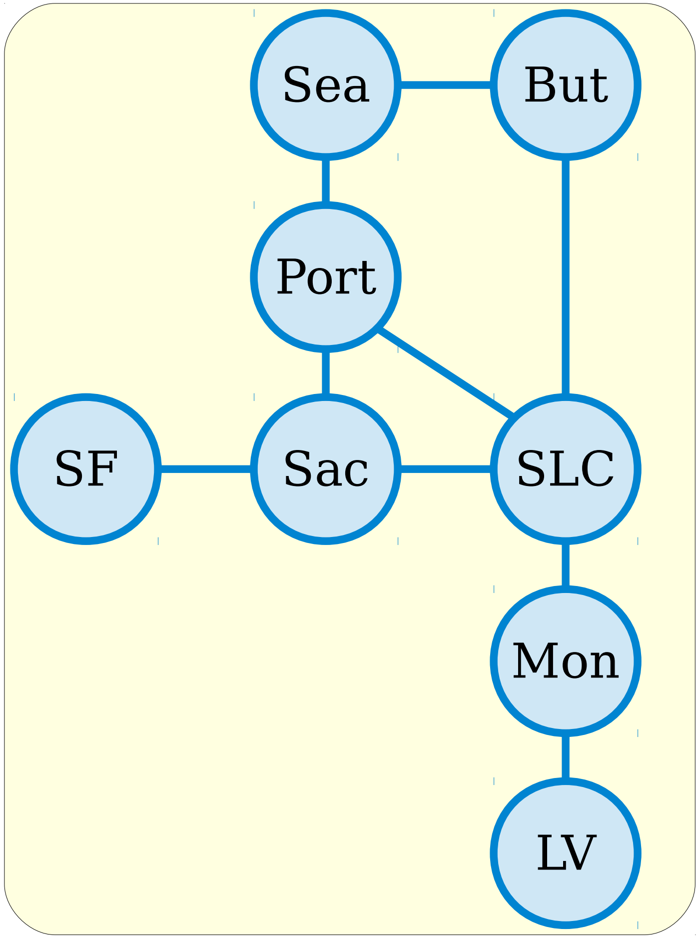

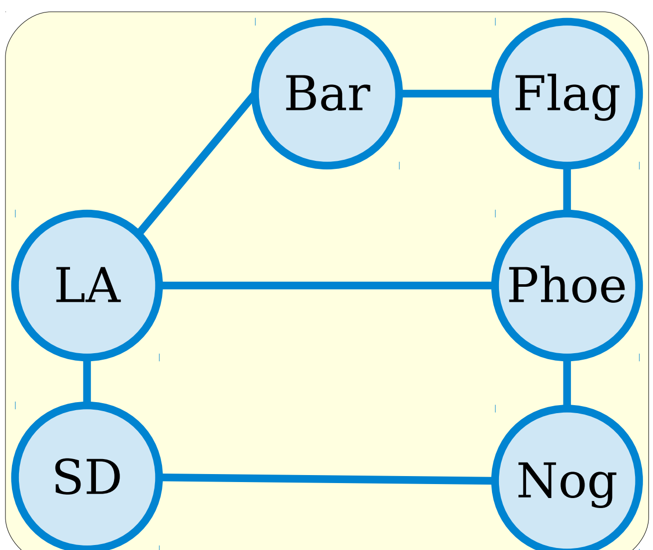

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A ***connected component*** (or ***CC***) of G is a maximal set of mutually reachable nodes.

Travelling Salesman Problem

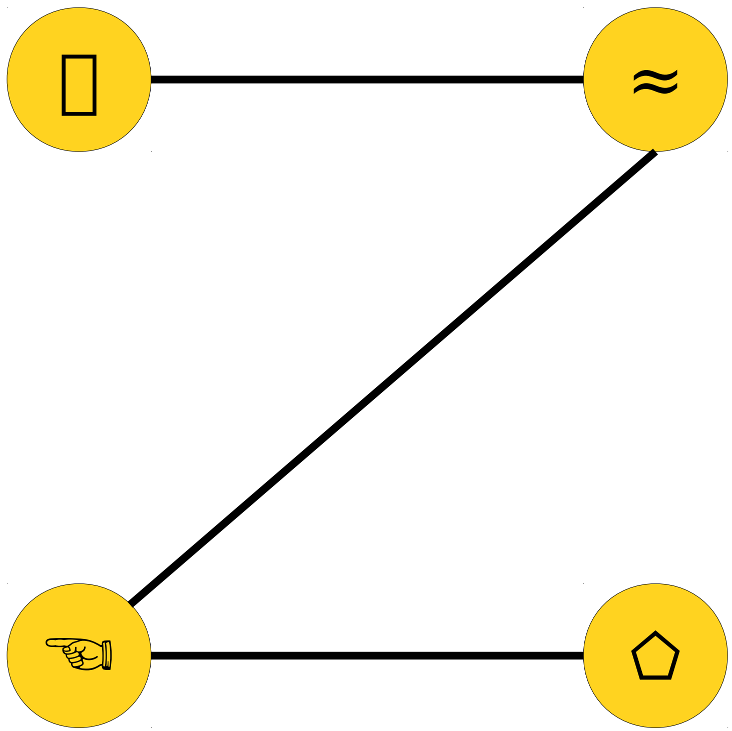
Given a set of vertices (towns/cities/shops/facilities) what is the shortest tour (visit each vertex at least once). This is usually a hard problem.

So we have met several (usually hard) problems: Maximum Independent Set Minimum Vertex Cover, Travelling Salesman….

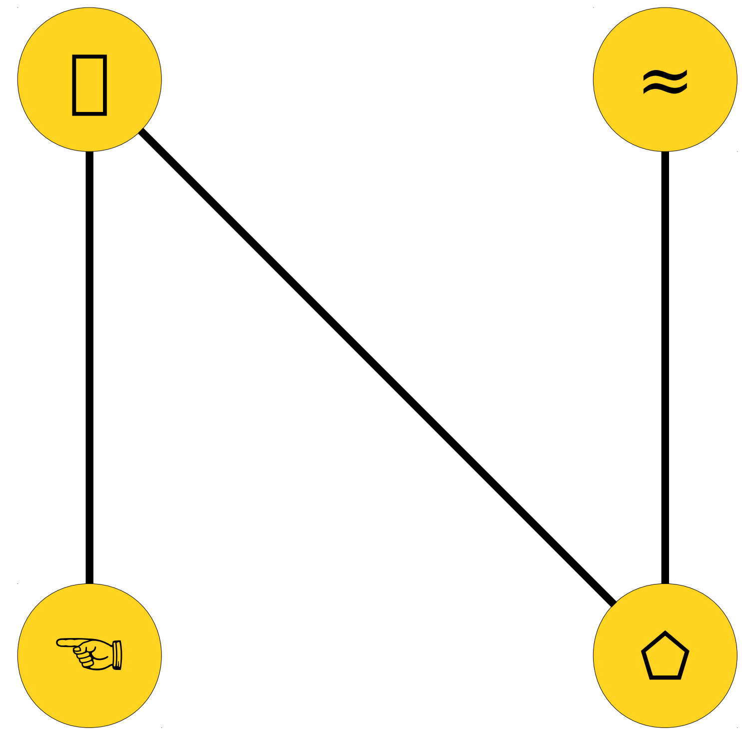
Indeed, there is a whole class (NP-complete) https://en.wikipedia.org/wiki/List_of_NP-complete_problems

(Note - there are often different versions of these problems - e.g., decision problem (given a set of vertices is it the minimum vertex cover), the search problem (given a graph - find the minimum vertex cover)).

Graph Complements



Graph G



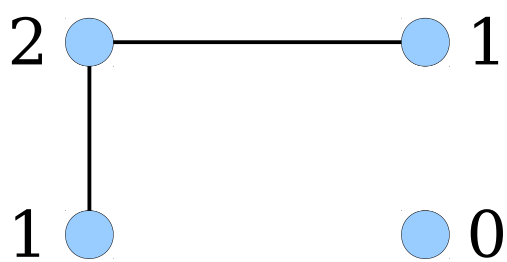
Graph G^c

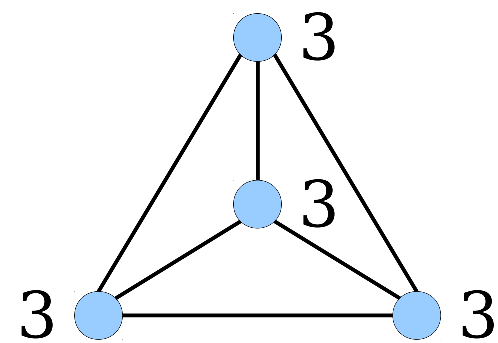
Let $G = (V, E)$ be an undirected graph.
The ***complement of G*** is the graph $G^c = (V, E^c)$, where
$E^c = \{ \{u, v\} \mid u \in V, v \in V, u \neq v, \text{ and } \{u, v\} \notin E \}$

Theorem: For any graph $G = (V, E)$, at least one of G and G^c is connected.

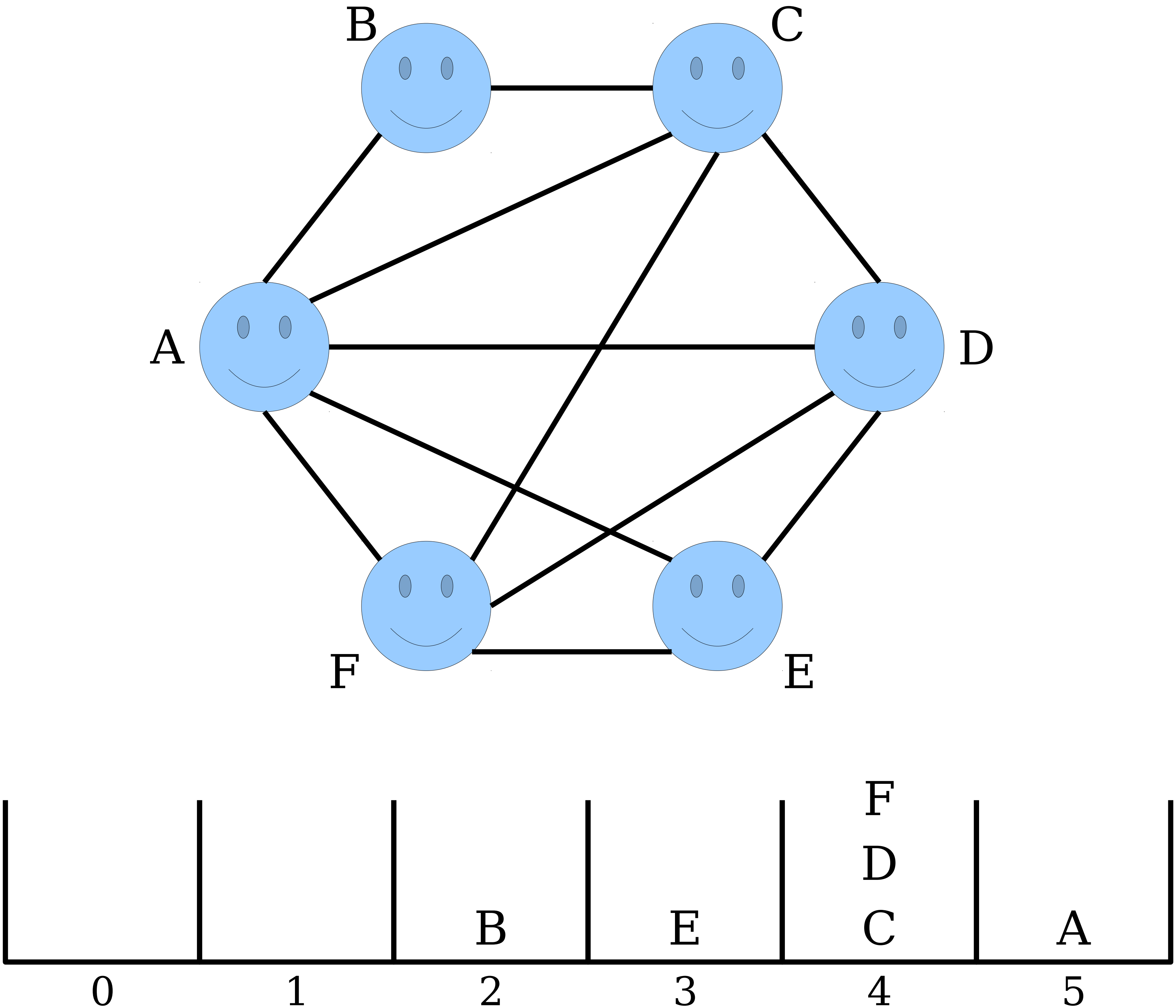
Degrees

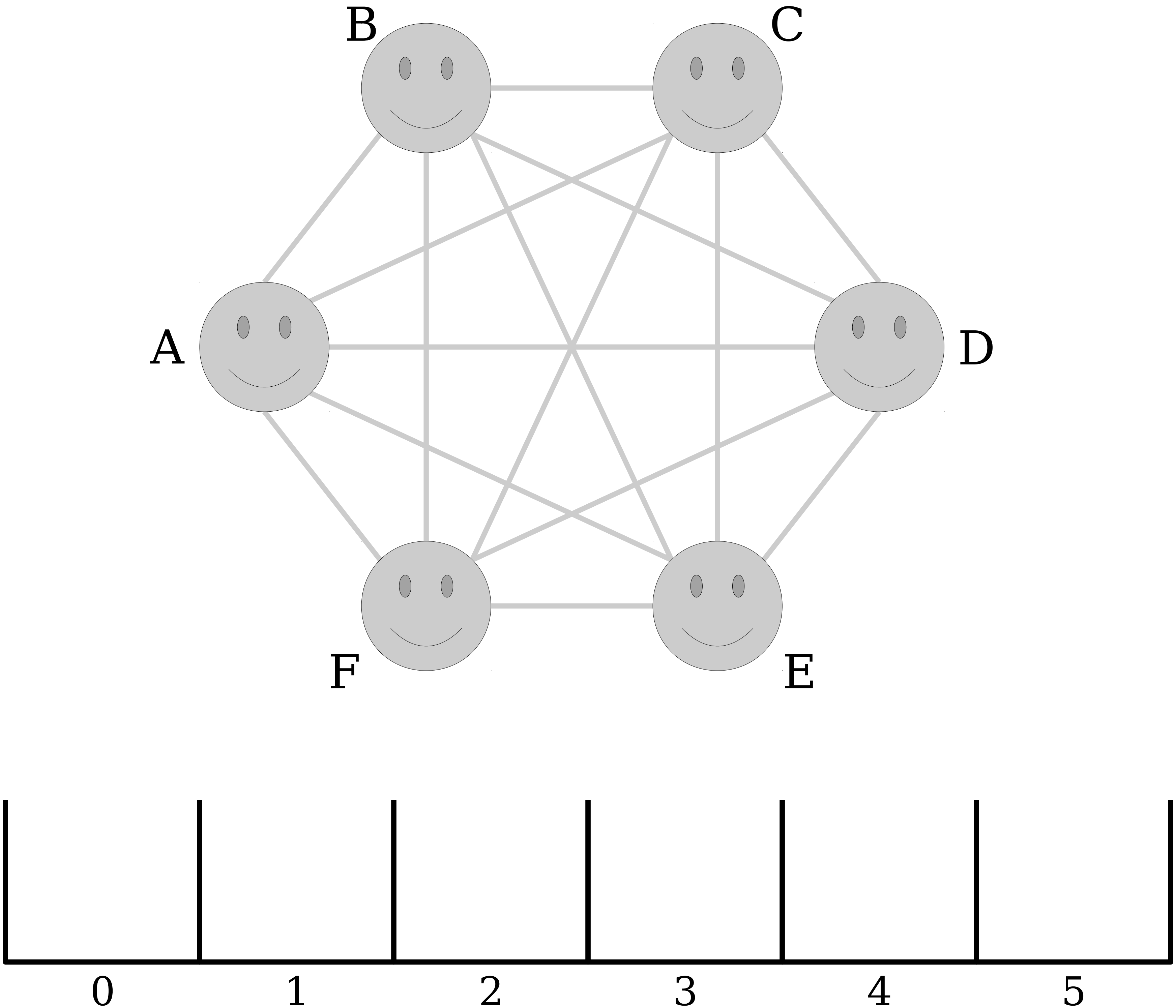
- The ***degree*** of a node *v* in a graph is the number of nodes that *v* is adjacent to.





- ***Theorem:*** Every graph with at least two nodes has at least two nodes with the same degree.
  - Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.


B
C
A
D
F
E
F
D
B
E
C
A
0
1
2
3
4
5


B
C
A
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0
1
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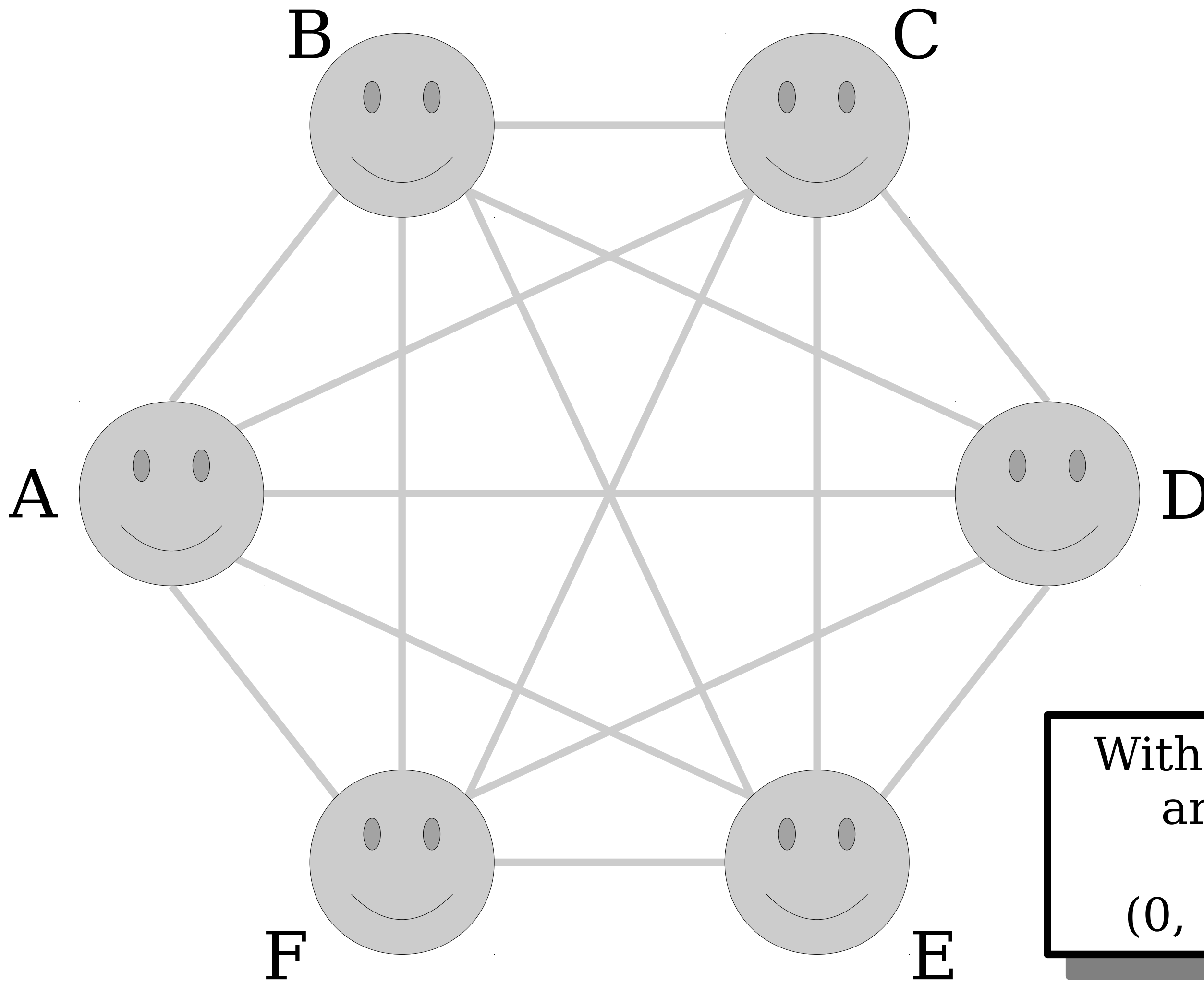
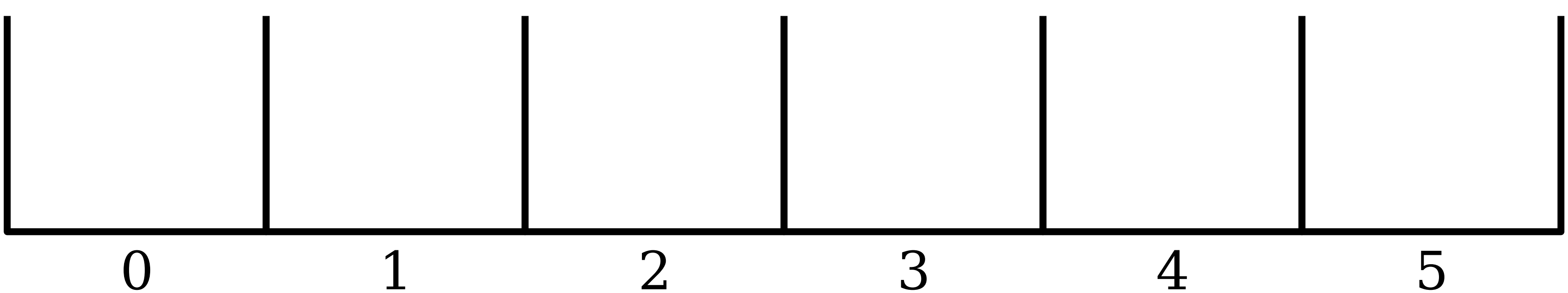
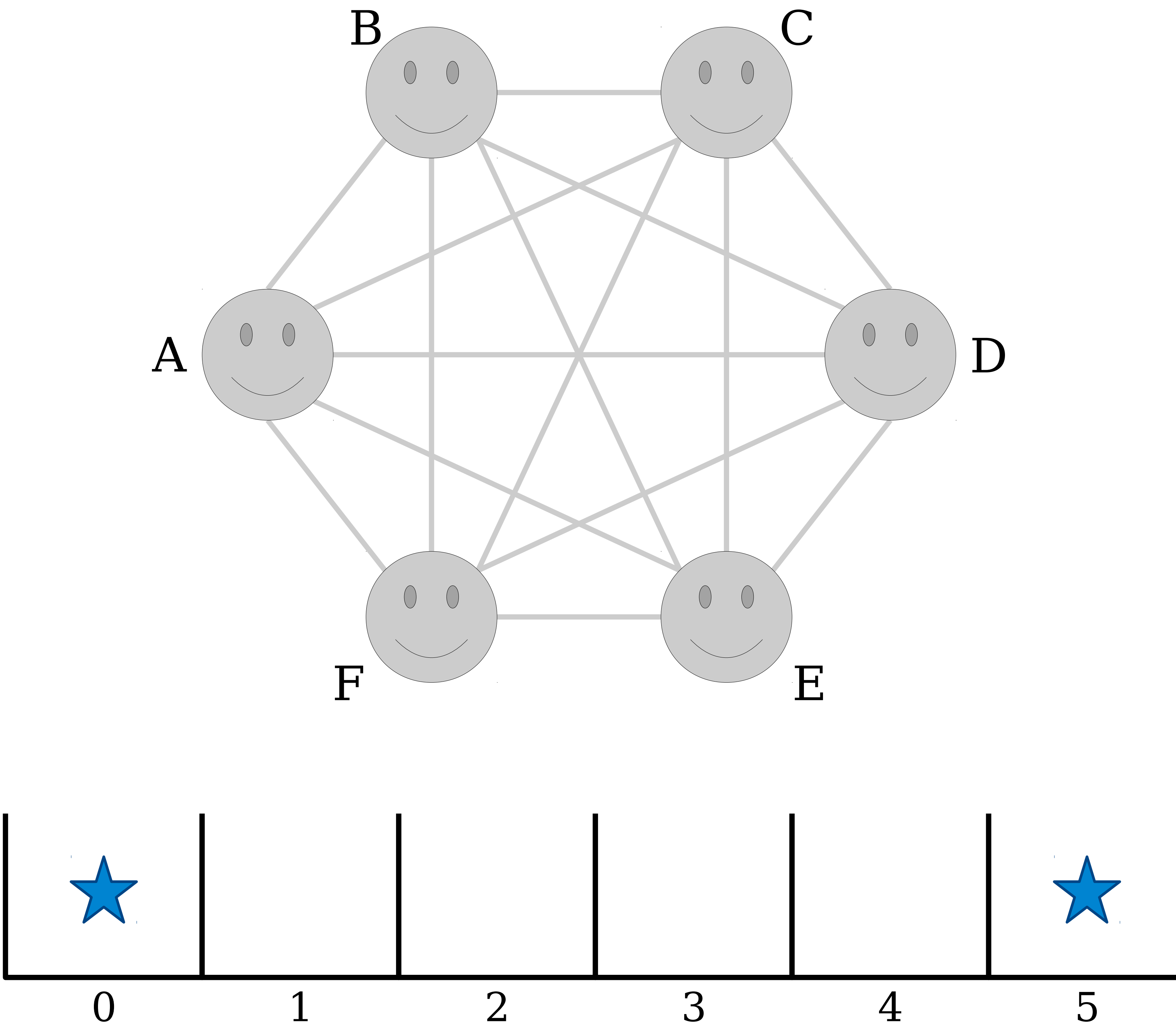
5

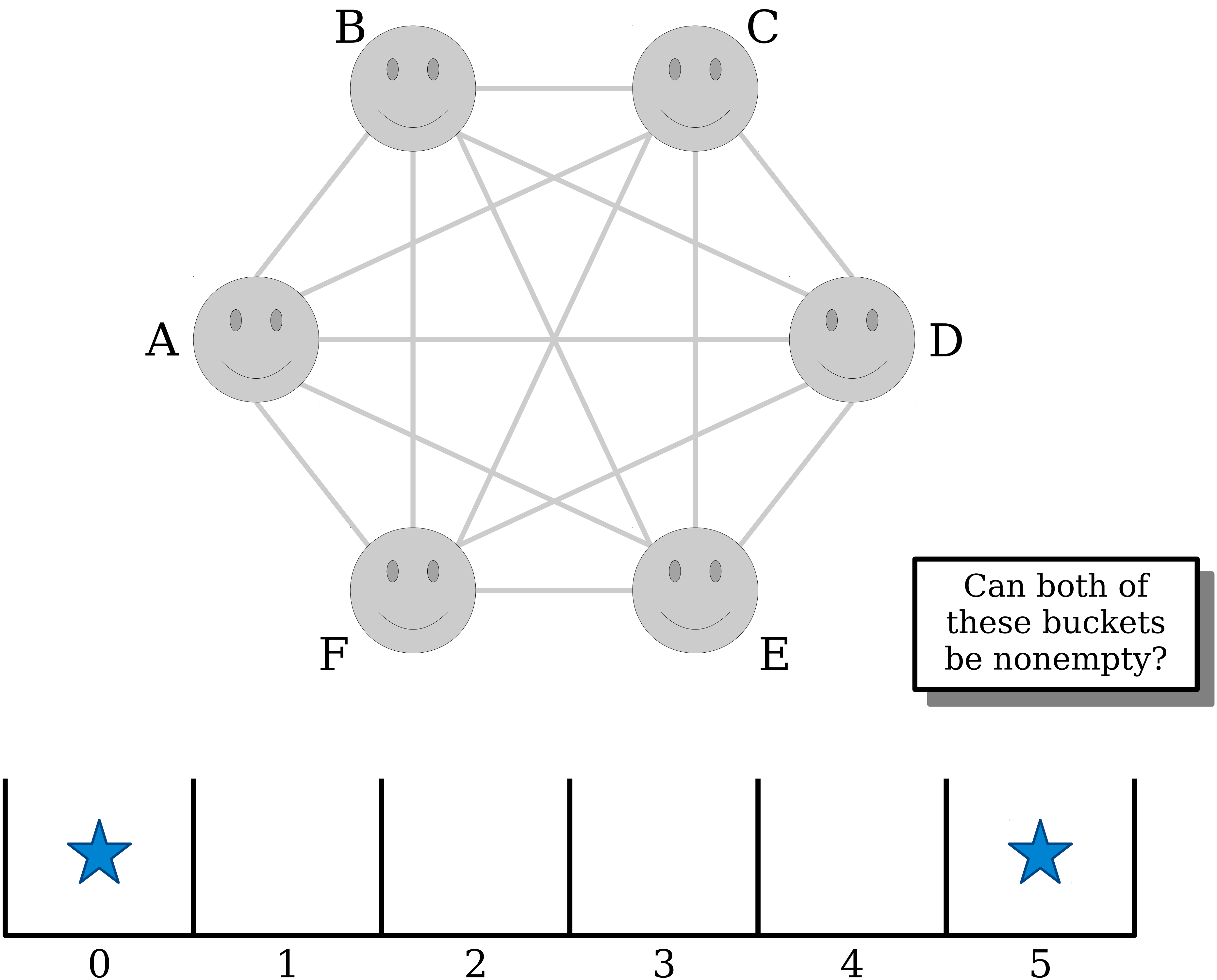


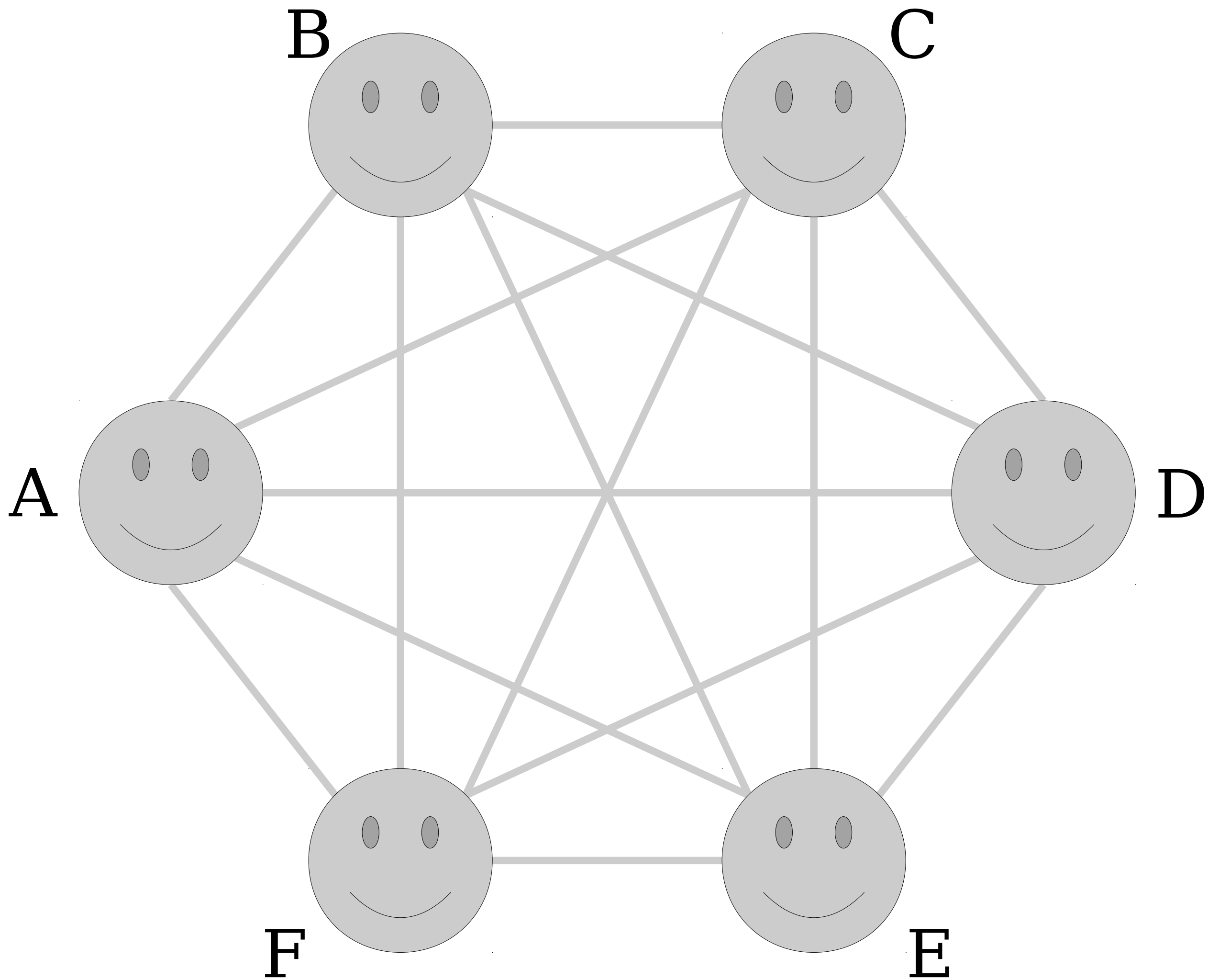
With n nodes, there are n possible degrees $(0, 1, 2, \ldots, n - 1)$

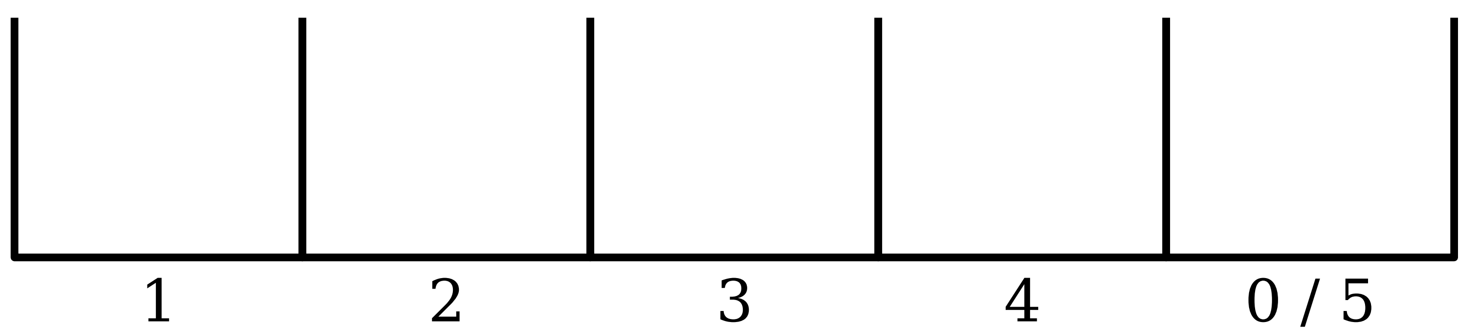




B
C
A
D
F
E
0
1
2
3
4
5


B
C
A
D
F
E
Can both of these buckets be nonempty?
0
1
2
3
4
5


B
C
A
D
F
E

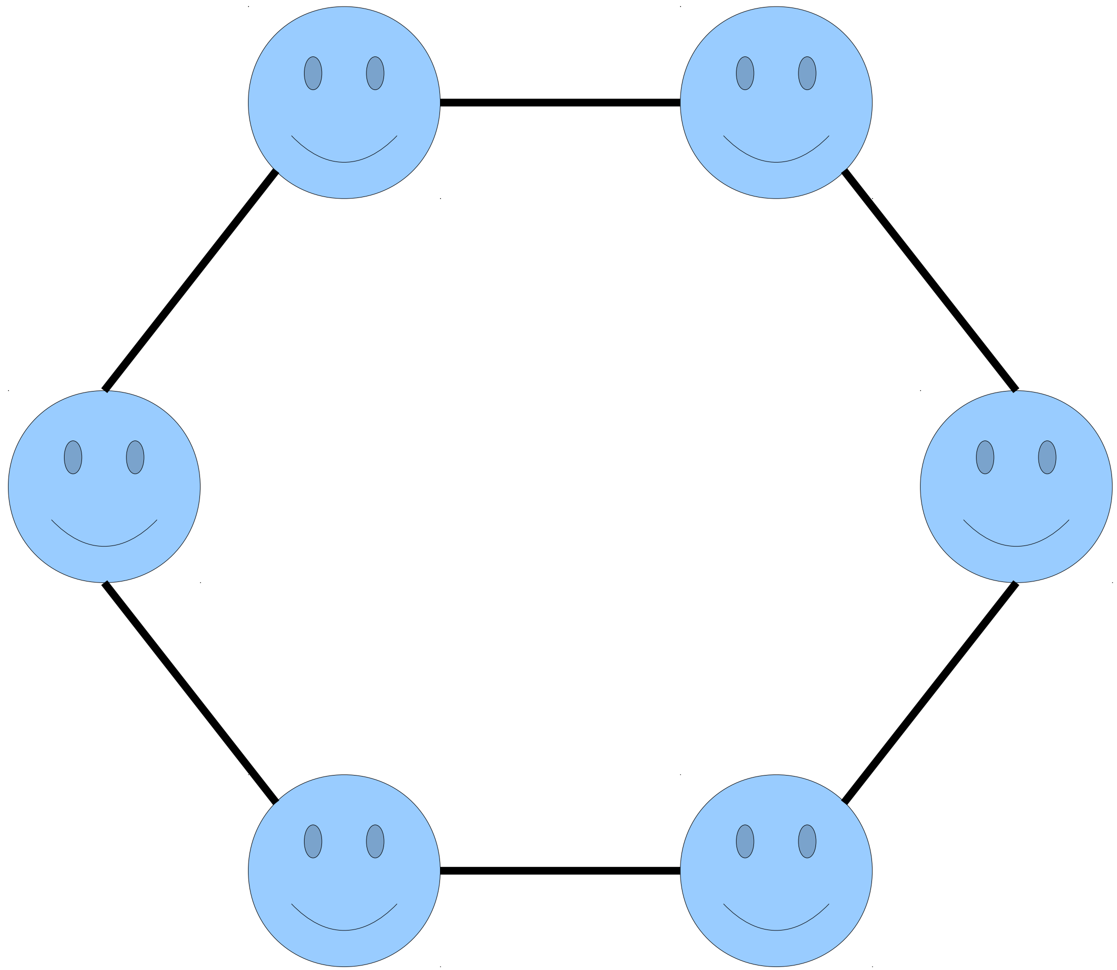

1
2
3
4
0 / 5

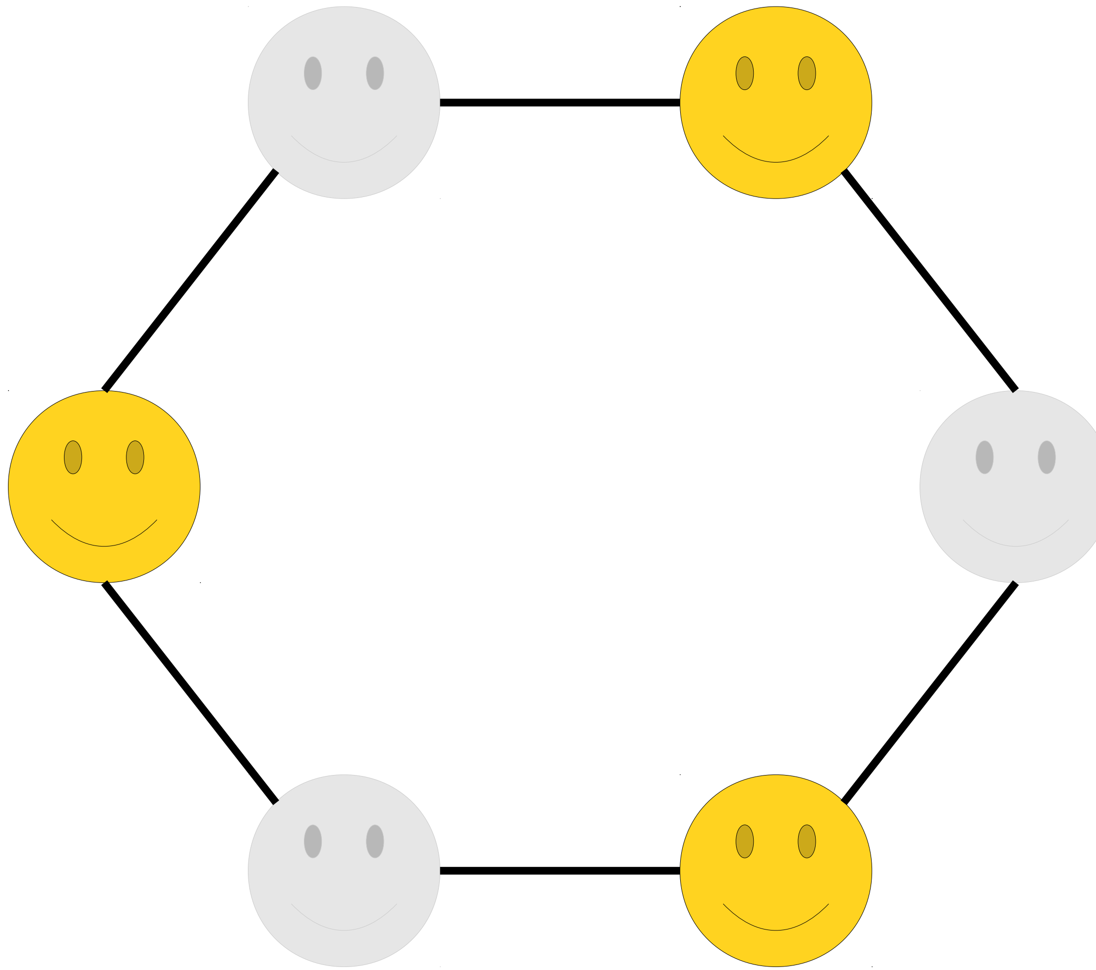
Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

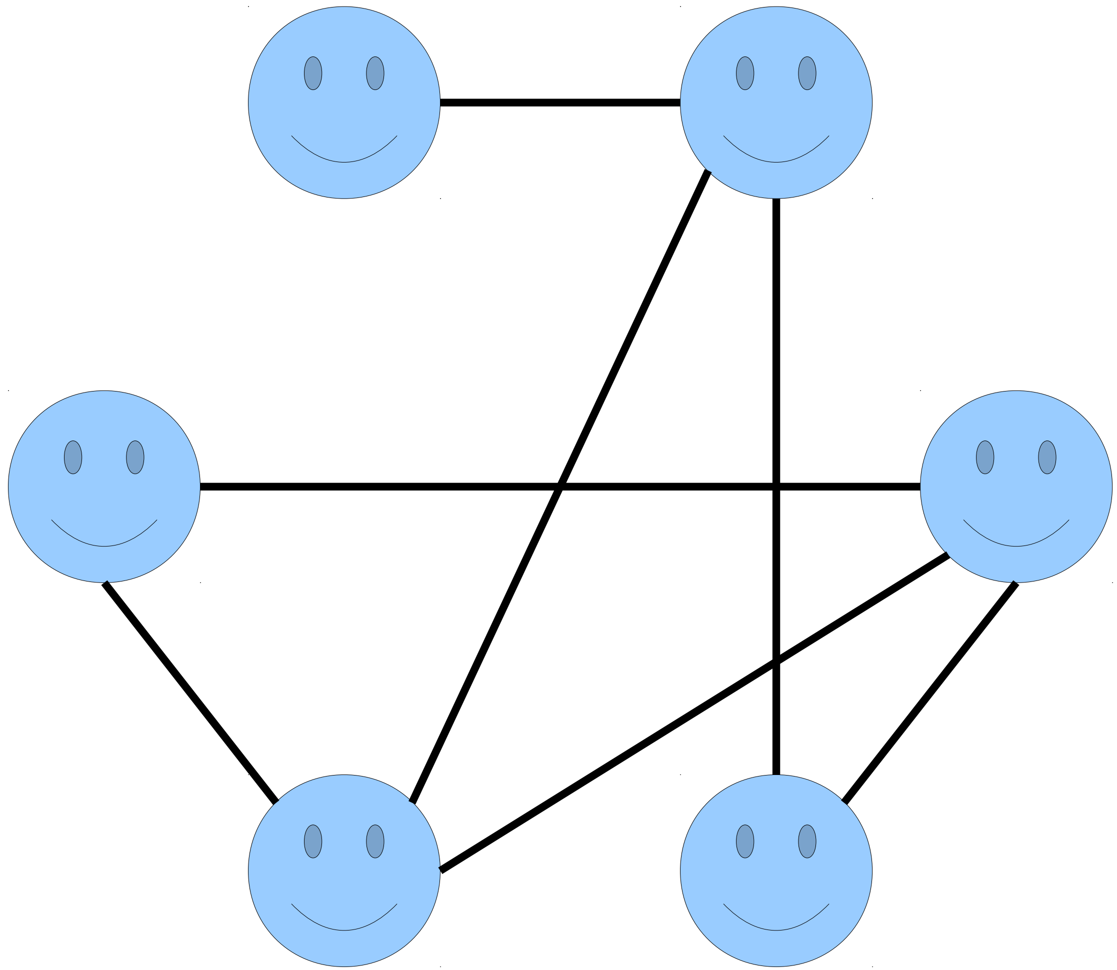
An Application: Friends and Strangers

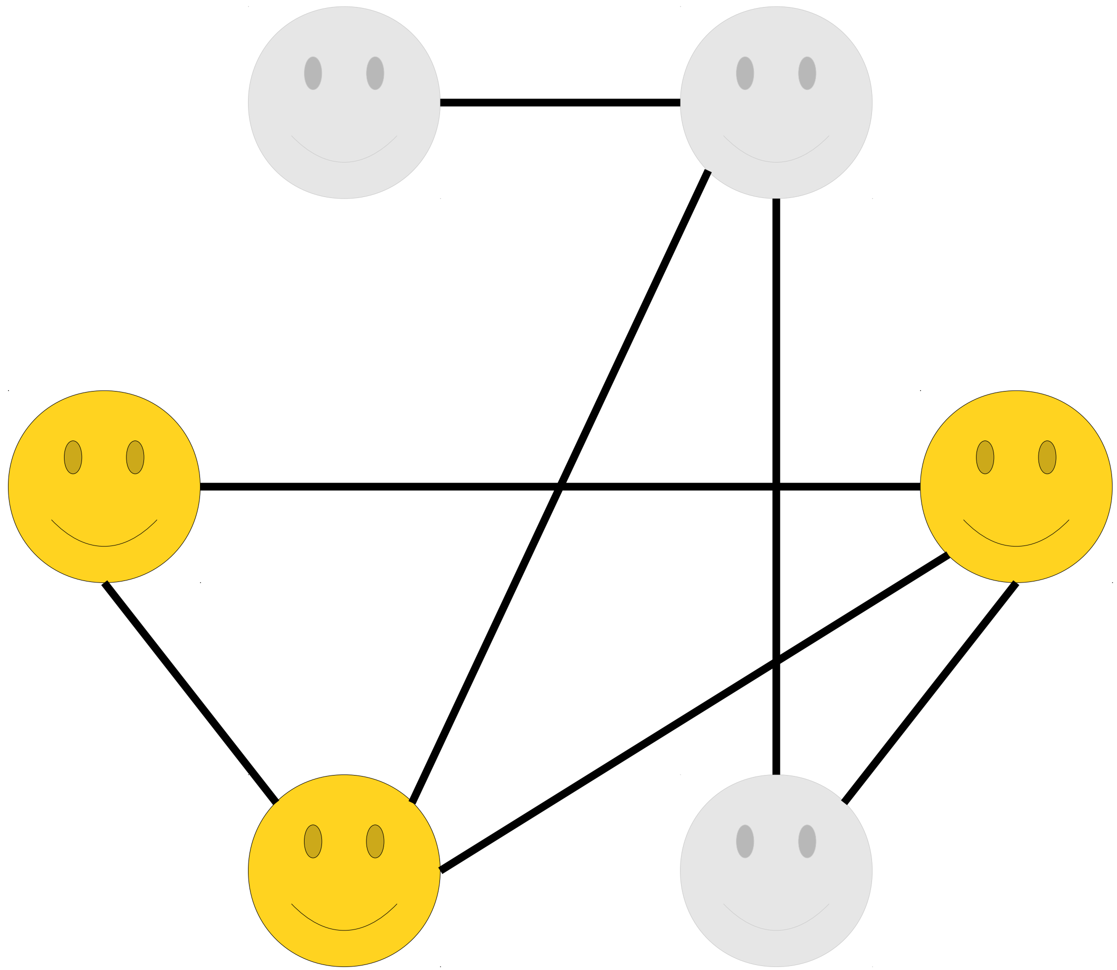
Friends and Strangers

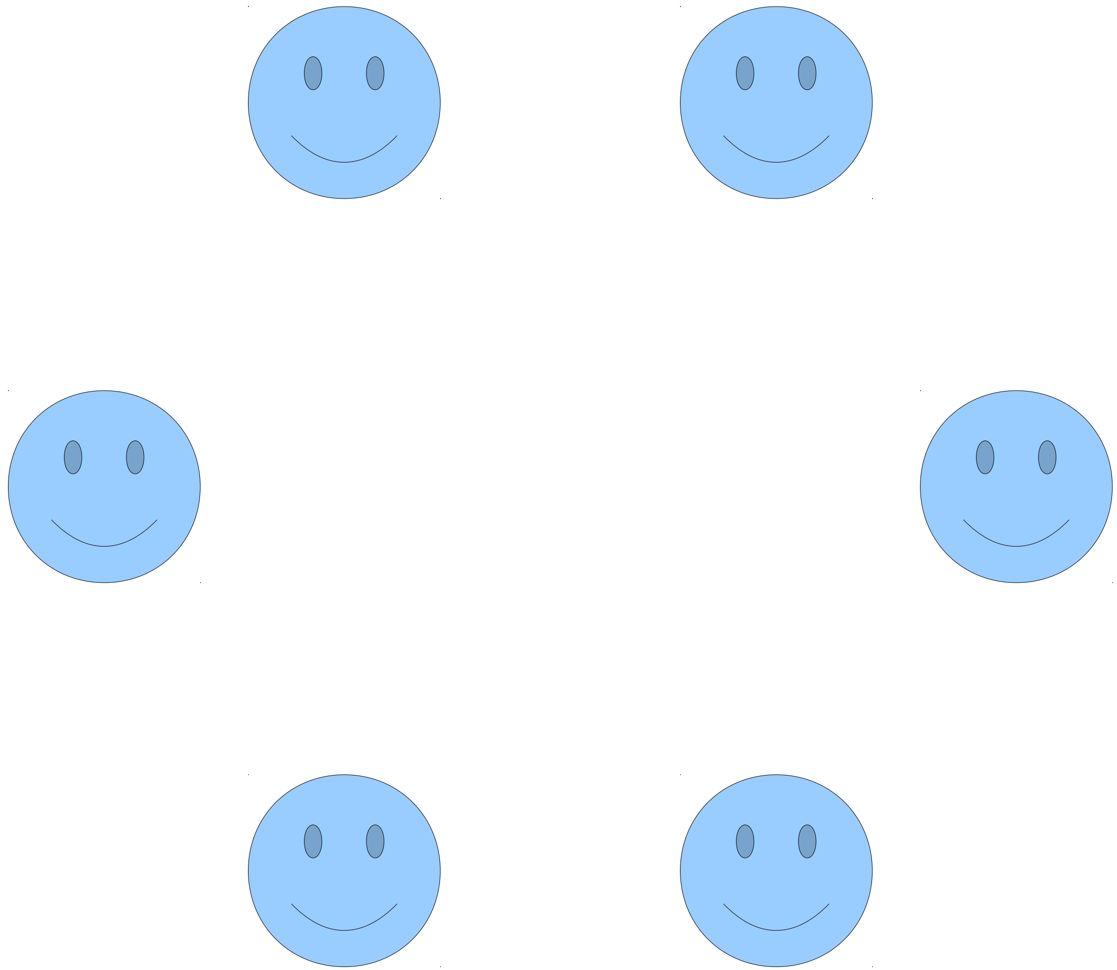
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

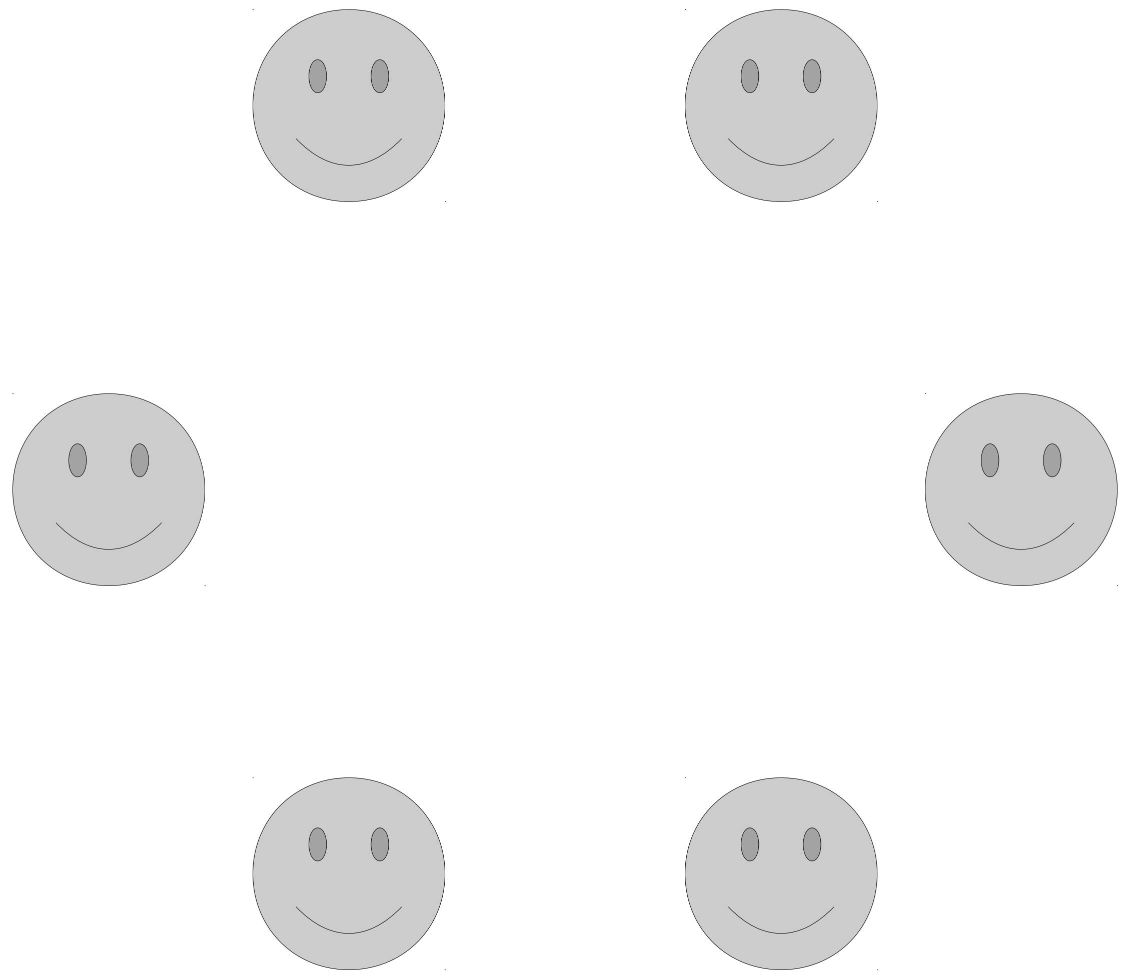


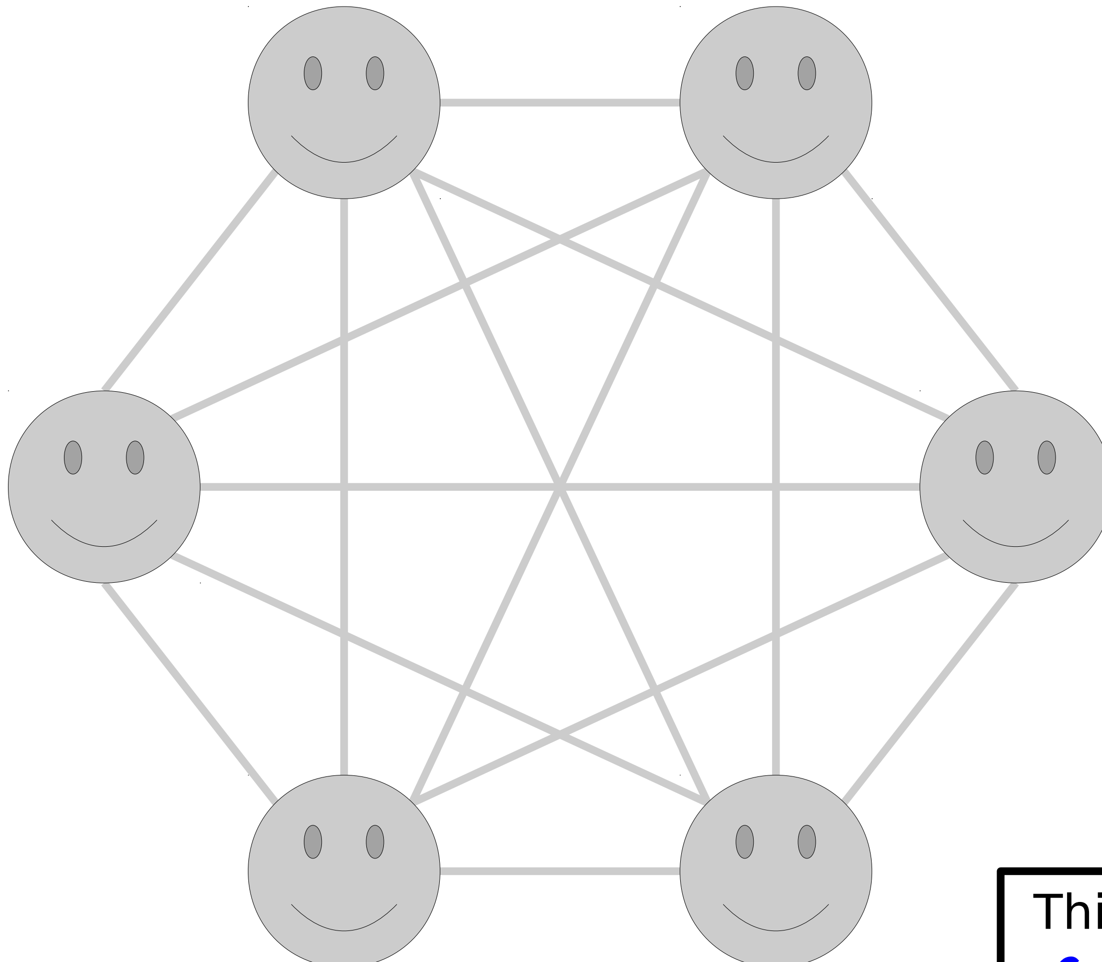












This graph is called a ***6-clique***, by the way.

Friends and Strangers Restated

- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

  Theorem: Any 6-clique whose edges are colored red and blue contains a red triangle or a blue triangle (or both).

- How can we prove this?