Graph and Hyper-Graph Structures in Signal, Image Processing and Computer Vision

…..(and Tensors)

(It’s all sets of subsets!!!!)

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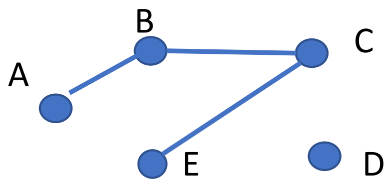
“It’s all just sets of subsets!!!”

- A graph is just a set of subsets (of vertices…….of size two)



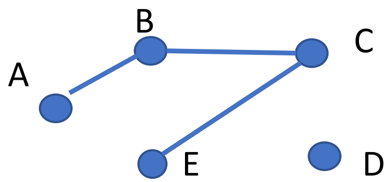
{ {AB},{CD},{CE}}

(can represent more: edge weights – strength of association, vertex weights – vertex attributes…)

- The vertex labels are usually “arbitrary”
  - Can use 1,2,3……. (mostly what mathematicians use)
  - If you permute the labels – you have the same graph: “isomorphic”
    - It is actually hard (expensive) in general to check if two graphs are isomorphic
      - But there is a popular test that is quite cheap and if it fails that test then not isomorphic – Weisfeiler-Lehman
        - Weisfeiler disappeared in controversial and mysterious circumstances in 1985 in Chile
      - Even simpler partial test – see if have the same DEGREE SEQUENCE
- The set of all subsets (powerset) is the set of all possible edges – 2^N in size (for N vertices)
- The complete graph has all edges – given the symbol K_N
- The complement graph of graph G has the edges that are “missing in G” and does not have the edges that were in G

Did you understand the previous slide?

- A graph is just a set of subsets (of vertices……of size two)



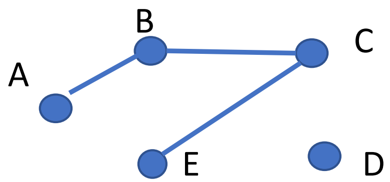
{ {AB},{CD},{CE}}

- What is the complement graph?

- How many edges does it have?

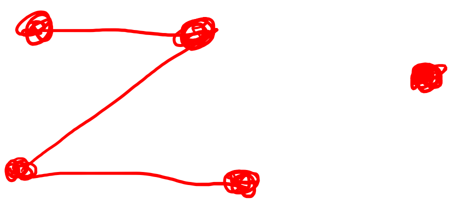
Q's

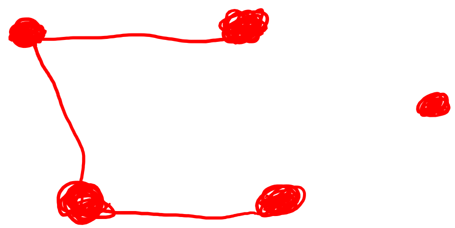
- A graph is just a set of subsets (of vertices…….of size two)



{ {AB},{CD},{CE}}

- Are the graphs above and below isomorphic?

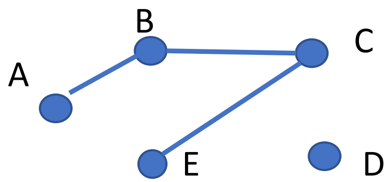




- How can you sometimes show two graphs are not isomorphic?

Hypergraphs

- A graph is just a set of subsets (of vertices…….of size two)



{ {AB},{CD},{CE}}

(can represent more: edge weights – strength of association, vertex weights – vertex attributes…)

- A (uniform) k-hypergraph is just a set of subsets (of vertices…of size k)

- A hypergraph is just a set of of subsets (or vertices…of any size)

Hypergraphs – generalizing from graphs

- A lot of things generalize reasonably readily from graphs (and some do not!)
- For example – independent set of vertices: a set of vertices not containing any hyperedge.
- Give an independent set of the hypergraph:

{1,2,3}, {4,5,6},{6,8,9}. (what is the “ground set”?)(is it a uniform hypergraph?)

  - Trivially independent sets
  - Non-trivial independent sets
  - Maximum independent set?

Hypergraphs – generalizing from graphs

- A lot of things generalize reasonably readily from graphs (and some do not!)
- For example – vertex cover: a set of vertices “touching every edge” (sometimes called “hitting set”)
- Give a hitting set/vertex cover for the 3-hypergraph

{1,2,3},{2,3,6},{1,2,5}

Hypergraphs – generalizing from graphs

- Drawing a hypergraph

Gets messy very quickly! But not so much an issue because pretty well only uninteresting (small) graphs can be meaningfully drawn anyway…no-one would try to draw the graph of the internet!

Hypergraphs – generalizing from graphs

- Drawing a hypergraph

Hypergraphs – generalizing from graphs

- Just as graphs are usually defined to exclude loops and parallel edges (SOMETIMES called simple graphs to emphasize these exclusions) – Hypergraphs usually exclude edges that are subsets of other edges, multisets (mutisets are a generalization of sets that allow repeated edges…).
- Sometimes the term “clutter” is used when we forbid edges that are proper subsets of other edges.

Simplicial Complexes – graphical realization

- Simplices
  - “nothing” (empty set) – is a simplex in some definitions
    - Mainly included simply to make a definition “consistent” – see later
  - Single vertex – 0-dimensional simplex
  - Single edge between two vertices – 1-dimensional simplex
  - “Triangle” of 3 edges – 2-dimensional simplex - *with the interior*
  - 3-dimensional simplex……etc……

$2^4 = 16$

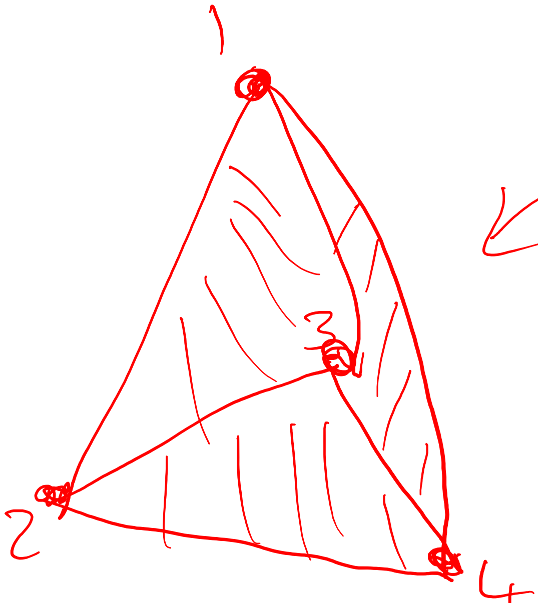
$C^4_4 = 1$

$C^4_3 = 4$

$C^4_2 = 6$

$C^4_1 = 4$

$C^4_0 = 1$



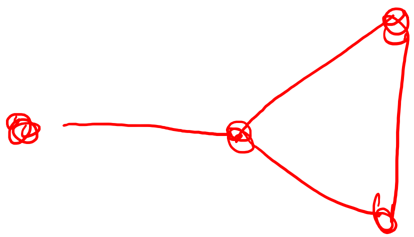
AND INTERIOR SHADED (INCLUDED)

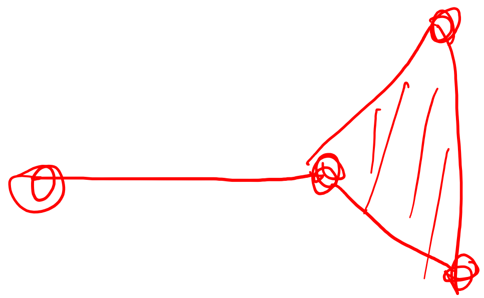
$\{1,2,3,4\}$ | $\{1,2,3\}$ $\{1,2,4\}$ $\{2,3,4\}$ $\{1,3,4\}$ | $\emptyset$

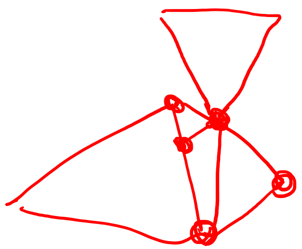
$\{1,2\}$ $\{1,3\}$ $\{1,4\}$ $\{2,3\}$ $\{2,4\}$ $\{3,4\}$ | $\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ | $\{\}$

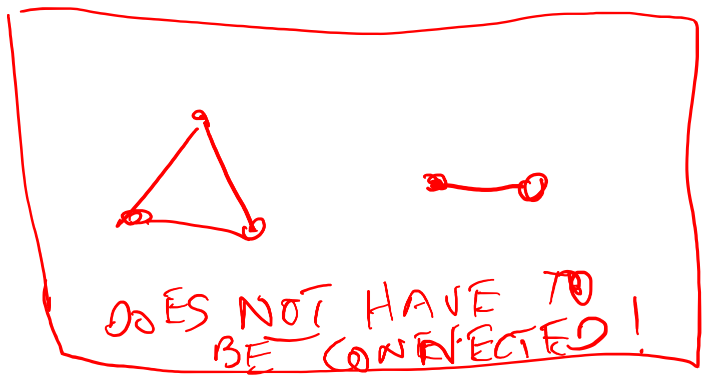
Simplicial Complex

- “Simplices sharing vertices”









Simplicial Complex – abstract definition – it's all sets of subsets!

- A downward closed set of sets…
  - If A is in the simplicial complex then so is B for any B contained in A (usually including the empty set)
  - Exercise – check this is true for the simplicies given
- A non-empty subset of simplicial complex is given the name "a face".
- For a non-empty simplicial complex, there is always one or more (non-empty) maximal faces.

Simplicial Complex – abstract definition – it’s all sets of subsets!

- A downward closed set of sets…
- So only need to specify the maximal faces….

Simplicial Complex – abstract definition – it's all sets of subsets!

- A downward closed set of sets…
- A simplicial complex with all maximal faces the same size is called a pure or homogeneous simplicial complex
- We can take the k-skeleton of a simplicial complex – all faces of size k in the complex
- A 1-skeleton of a simplicial complex is a graph
- A k-skeleton of a simplicial complex is a k-uniform hypergraph

Examples!

Aside: Clutters and simplicial complexes…

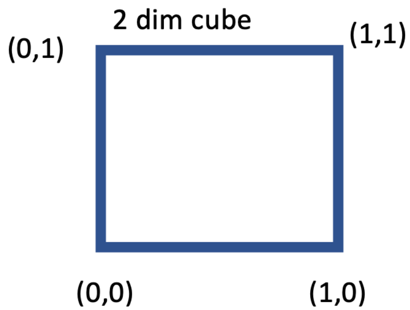
- In some sense a clutter (generalization of hypergraphs that forbids edges that are (proper) subsets of other edges) – are the opposite of simplicial complexes *because* the latter enforces that proper subsets are in the collection of sets.

So far, graphs, hypergraphs, simplicial complexes…one more structure – Boolean Cube

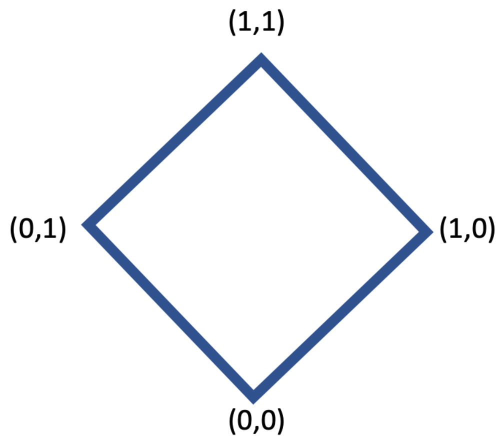
- Again, it all sets of subsets.
- The vertices of the Boolean Cube can be identified with each an every one of the subsets of some ground set. Usually, we use 1…N as the ground set.
- We usually encode the membership in the subset by “1-hot” encoding. Bit k is 1 if k is in the subset, otherwise bit k is 0. This gives the unit Boolean Cube vertices.

So far, graphs, hypergraphs, simplicial complexes…one more structure – Boolean Cube

- 1-dim Boolean cube is boring…the interval [0,1]
- 2-dim Boolean cube



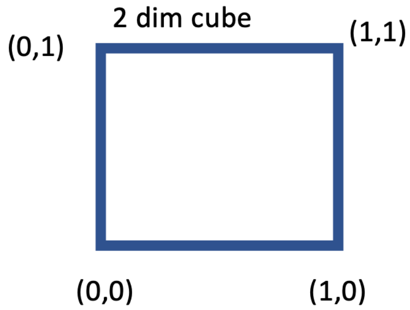
Can represent all subsets of 2 element ground set



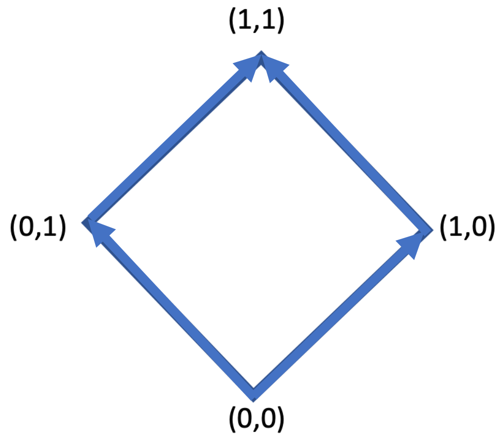
Note: we are only talking about the vertices. The edges we draw make a graph…but we don't want to focus on that!

So far, graphs, hypergraphs, simplicial complexes…one more structure – Boolean Cube

- 1-dim Boolean cube is boring…the interval [0,1]
- 2-dim Boolean cube



Can represent all subsets of 2 element ground set



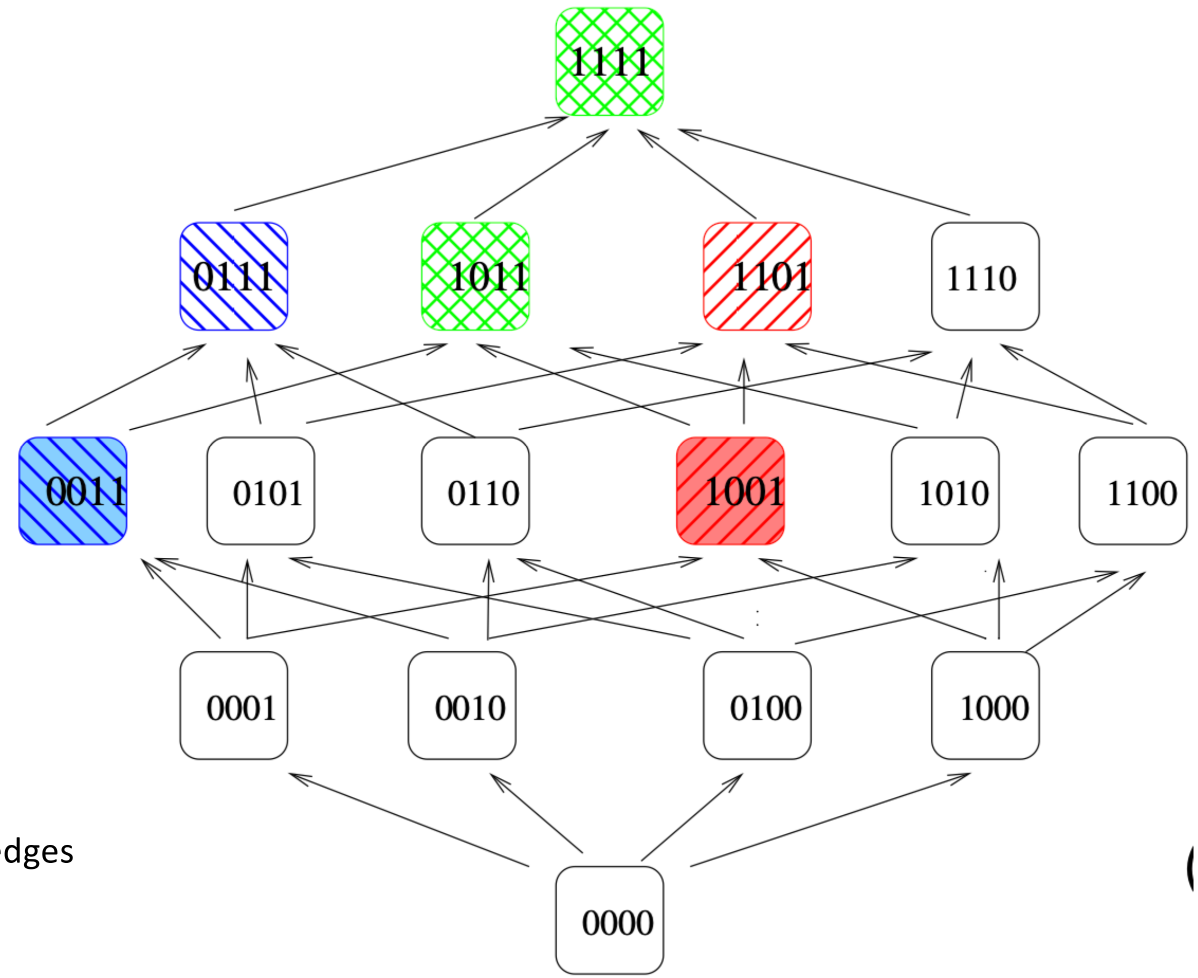
Note: but if we give these edges a direction – we get a directed graph – Hasse Diagram of subset inclusion

Boolean Cube

- 1-dim Boolean cube [0,1]
- 2-dim Boolean cube
- 3-dim Boolean Cube
- 4-dim Boolean Cube

(ignore the colours for now)

4 Dim



Hasse Diagram – directed
Boolean Cube – undirected edges

N-dim - Boolean cube – vertices all subsets of 1…N

- Set of vertices contain all (hyper)graphs on N vertices
- Also contain all simplicial complexes on N-vertices (essentially, because simplicial complexes are downward closed, a simplicial complex divides the Boolean cube into two (usually unequal in size) “halves”) – the bottom “half” (the simplicial complex itself) – the top ”half” (the complement of the simplicial complex).
- Boolean cube has 2^N vertices.