Graph Theory

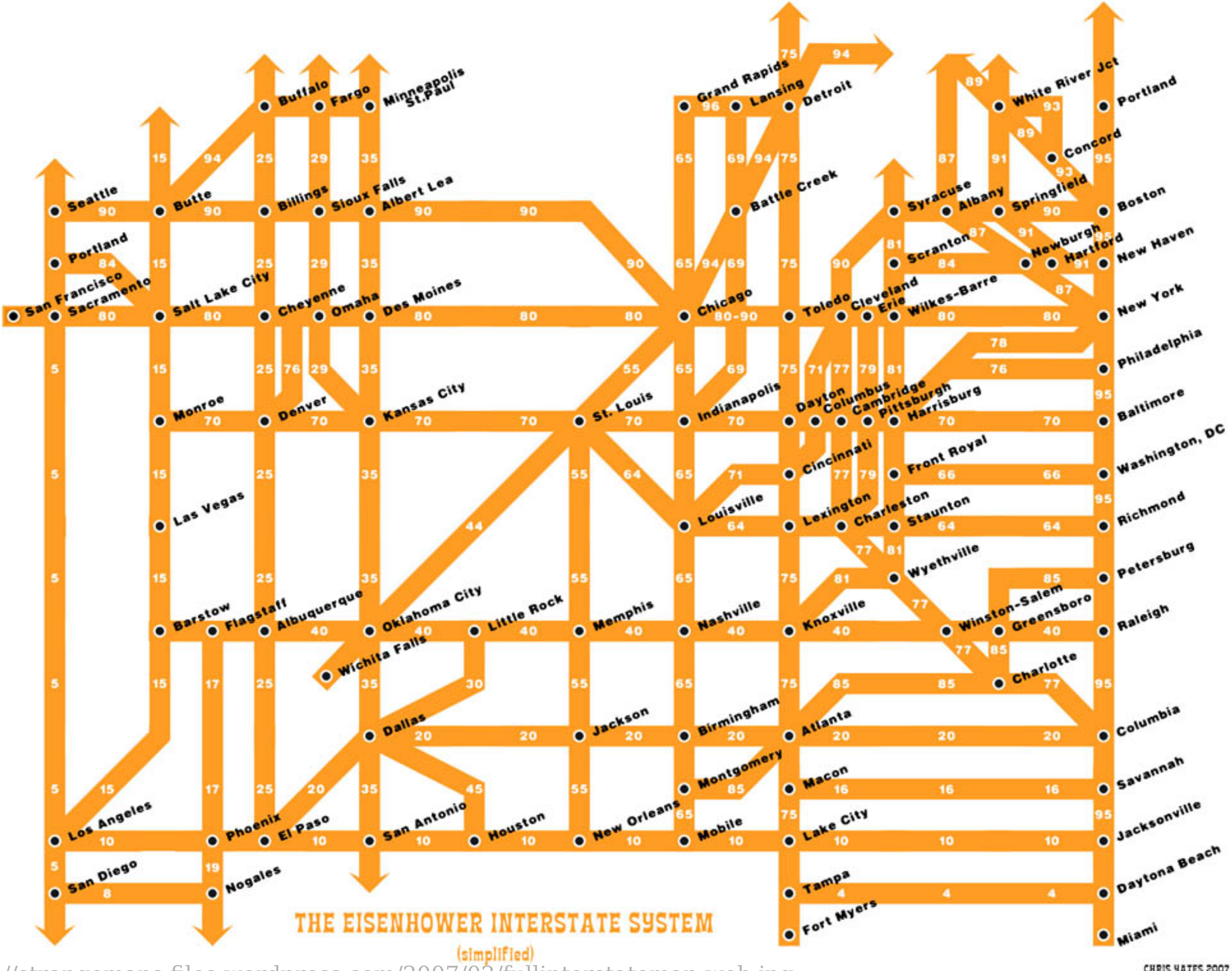
Part One

Slides adapted from CS103 Stanford

Outline for Today

- ***Graphs and Digraphs***
  - Two fundamental mathematical structures.
- ***Independent Sets and Vertex Covers***
  - Two structures in graphs.

Graphs and Digraphs



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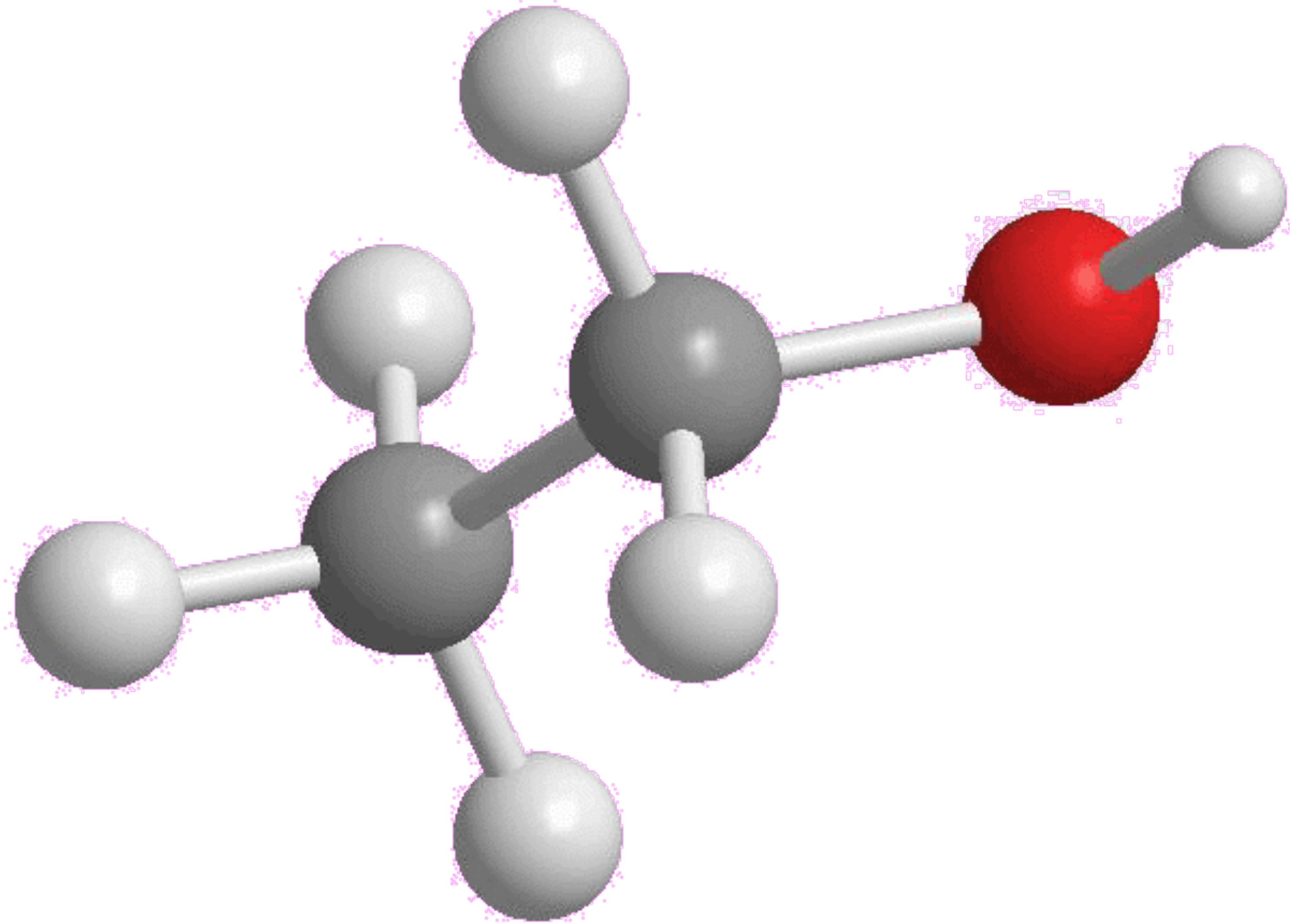
Chemical Bonds



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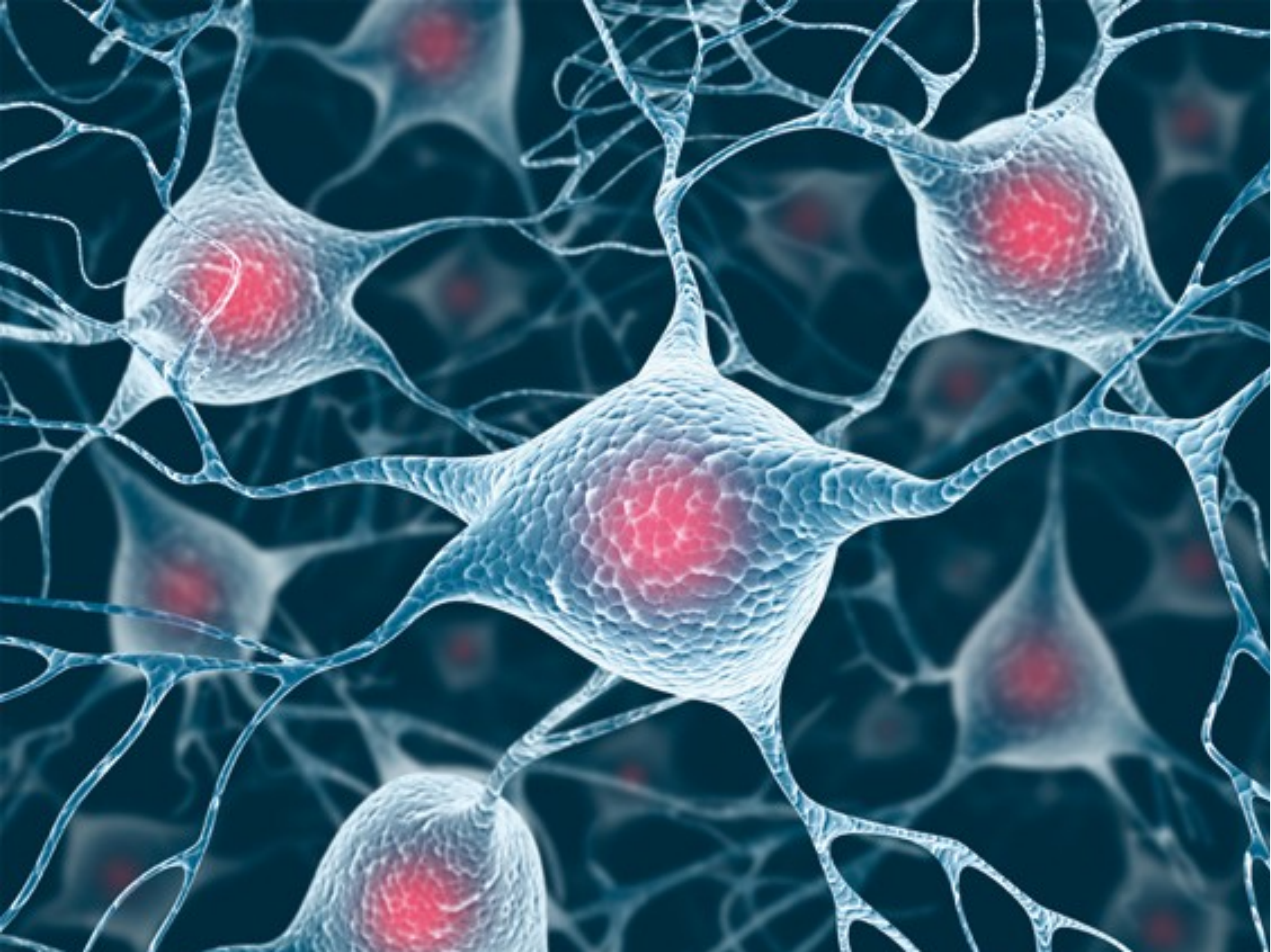

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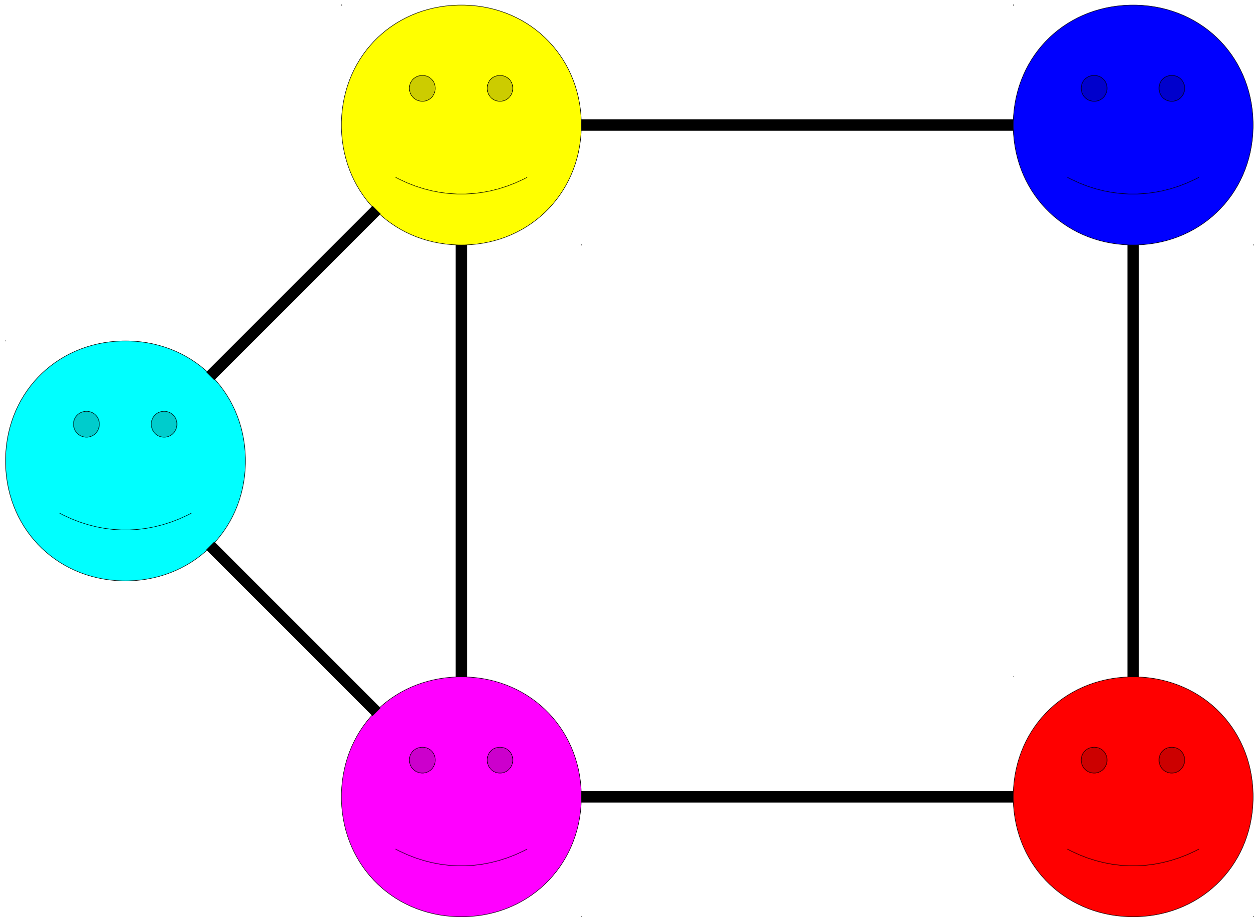
What's in Common

- Each of these structures consists of
  - a collection of objects and
  - links between those objects.
- ***Goal:*** find a general framework for describing these objects and their properties.

A ***graph*** is a mathematical structure for representing relationships.



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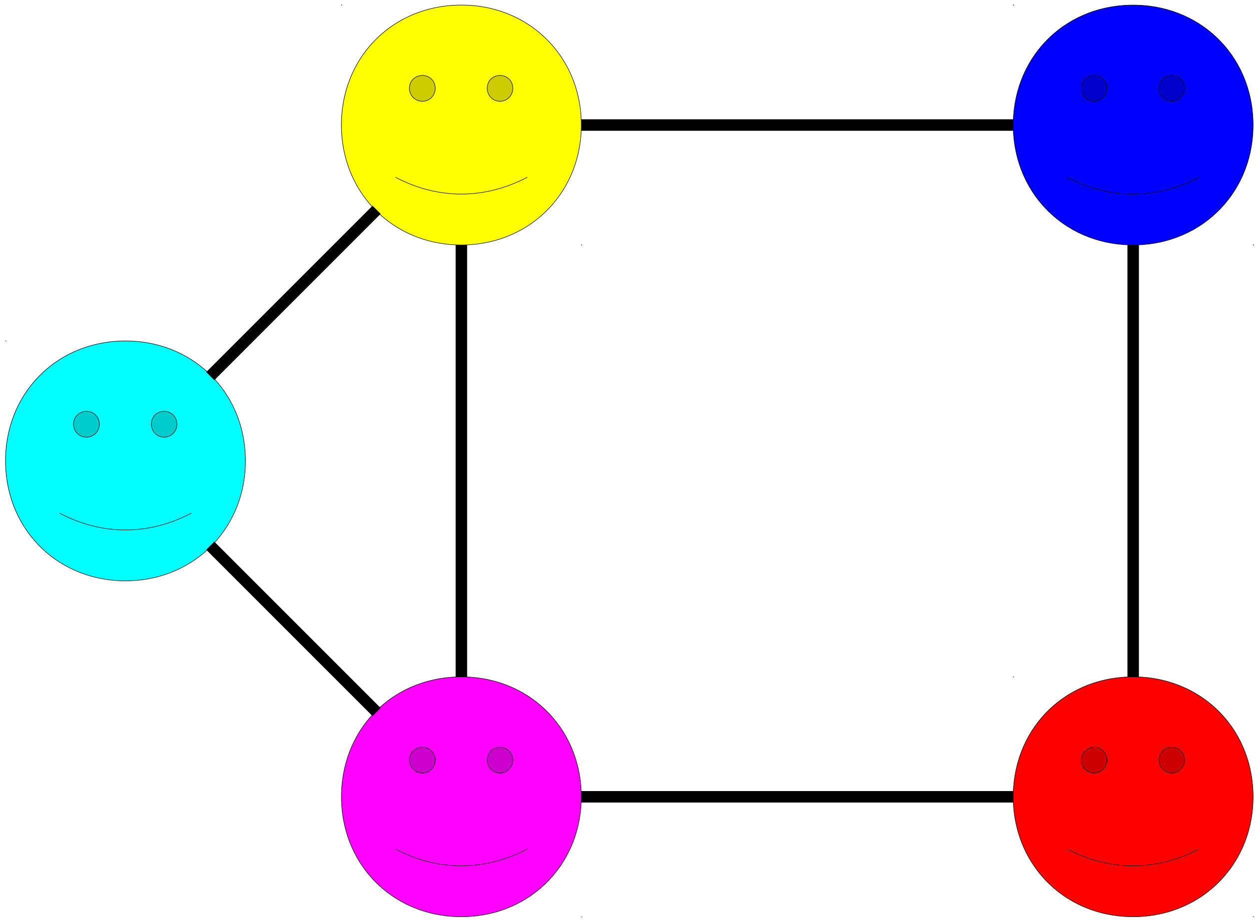

A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

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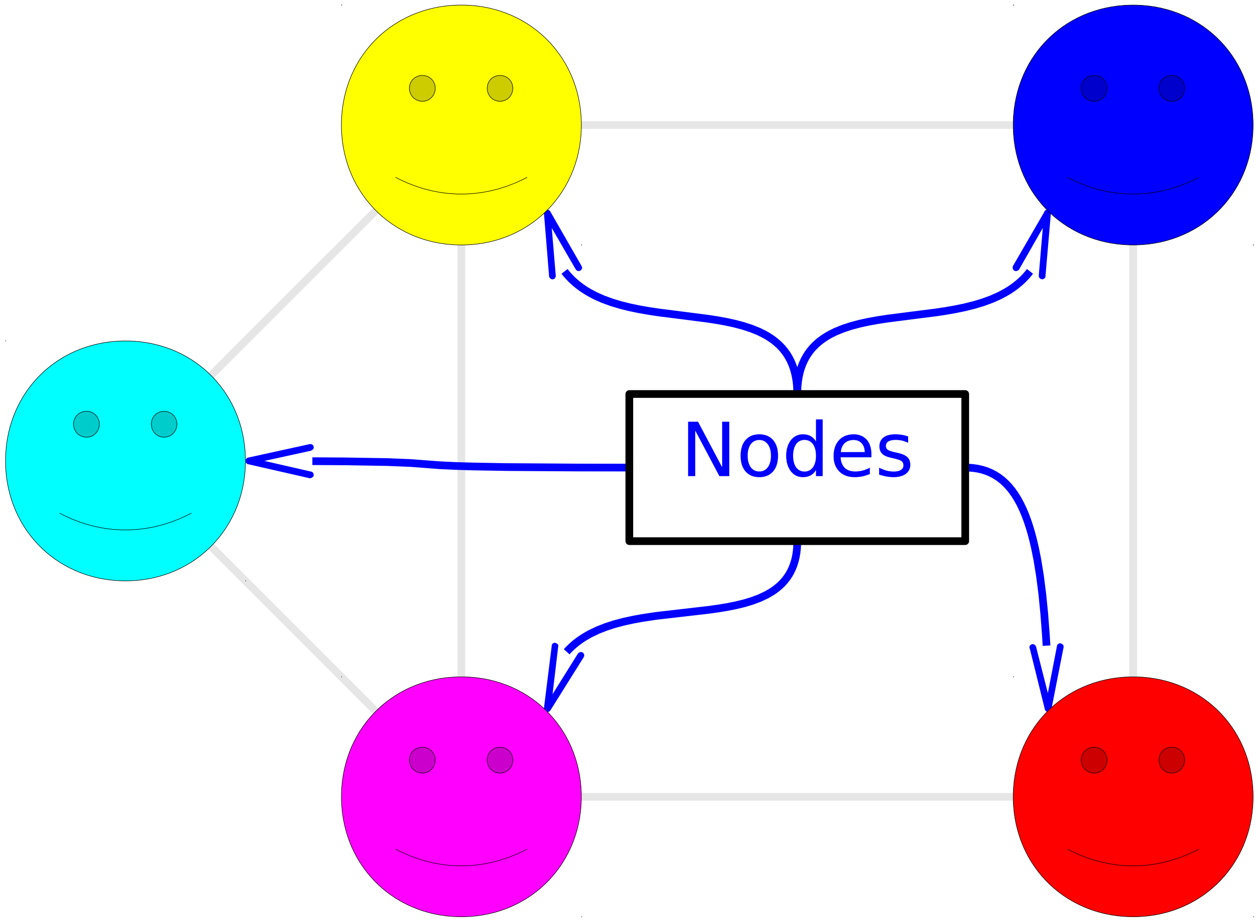

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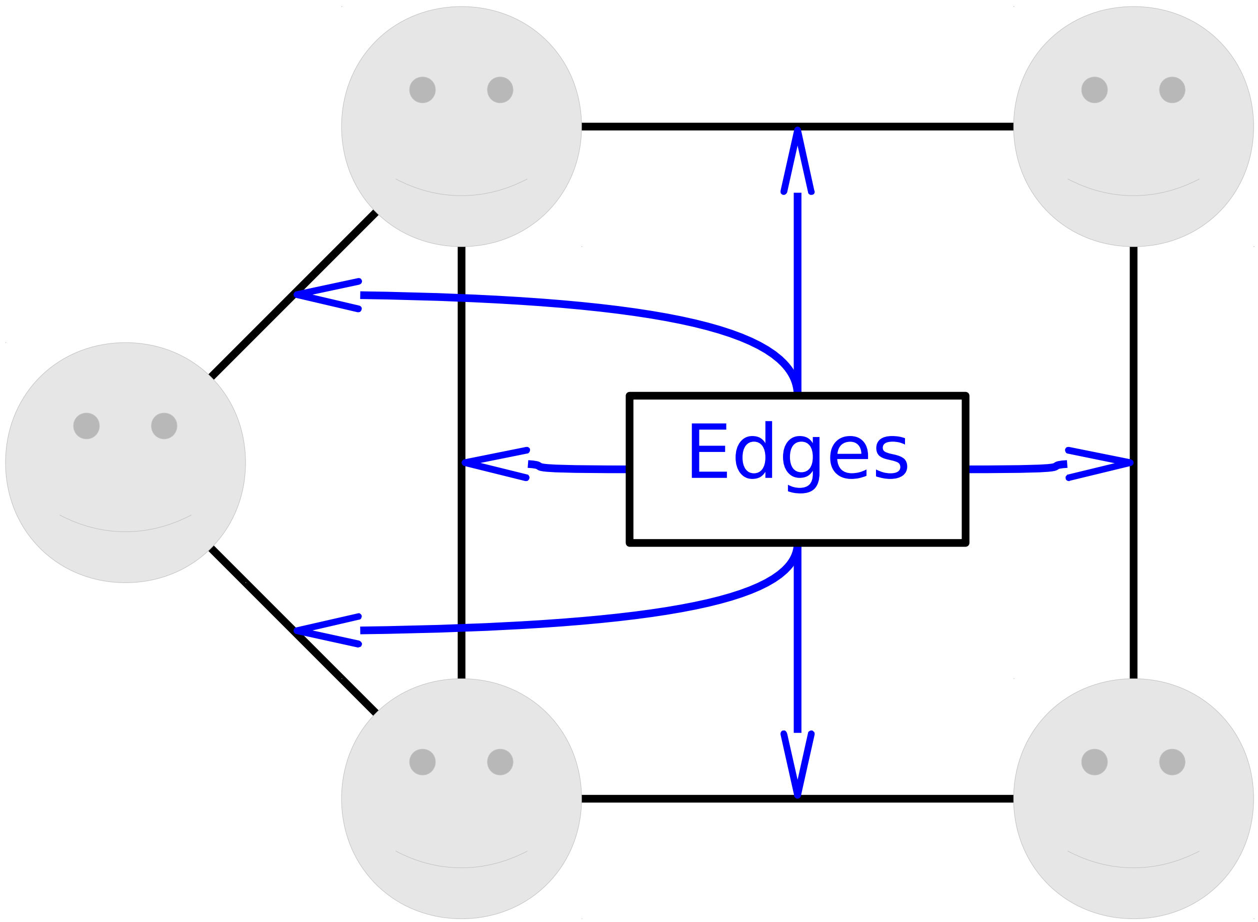

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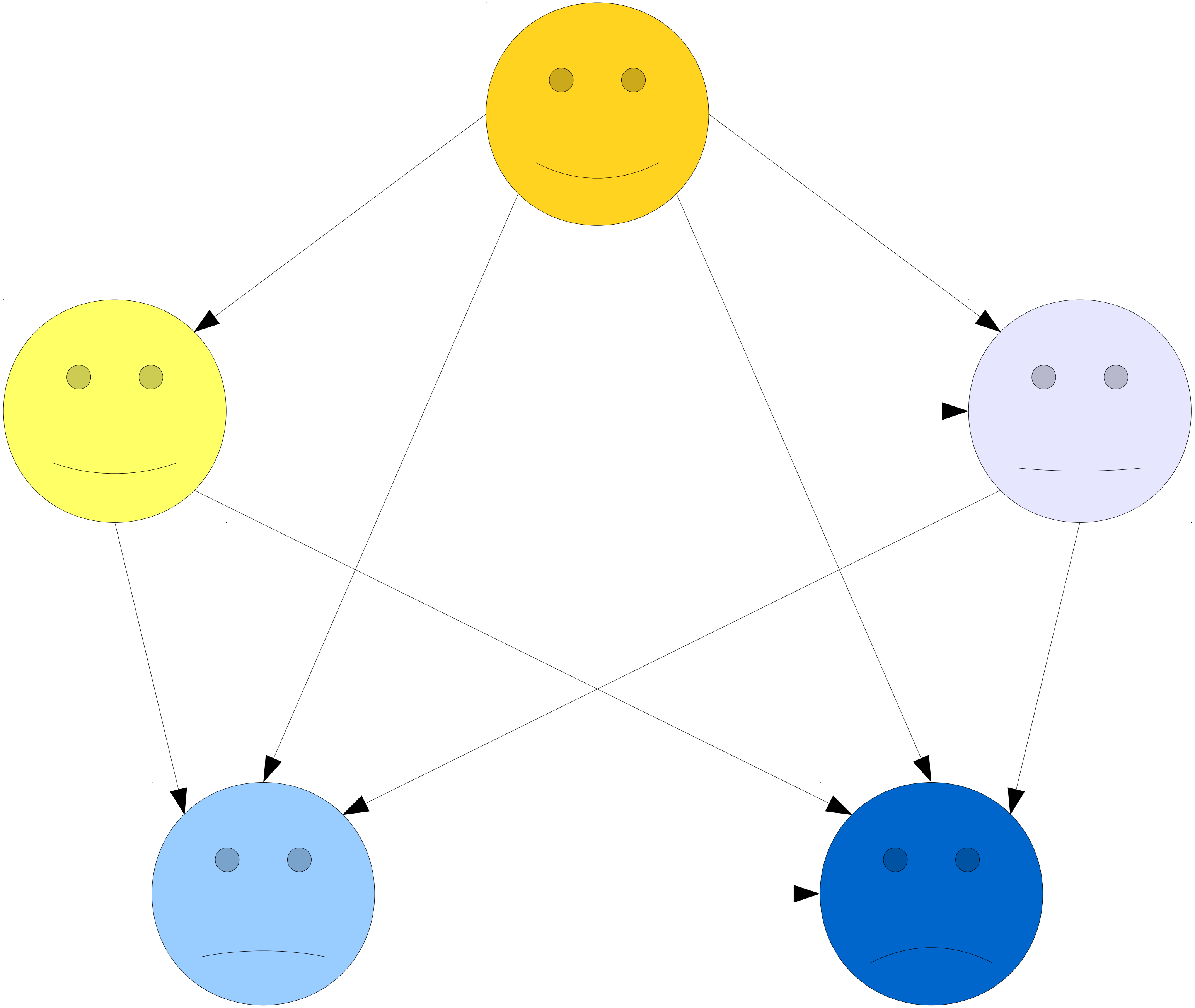
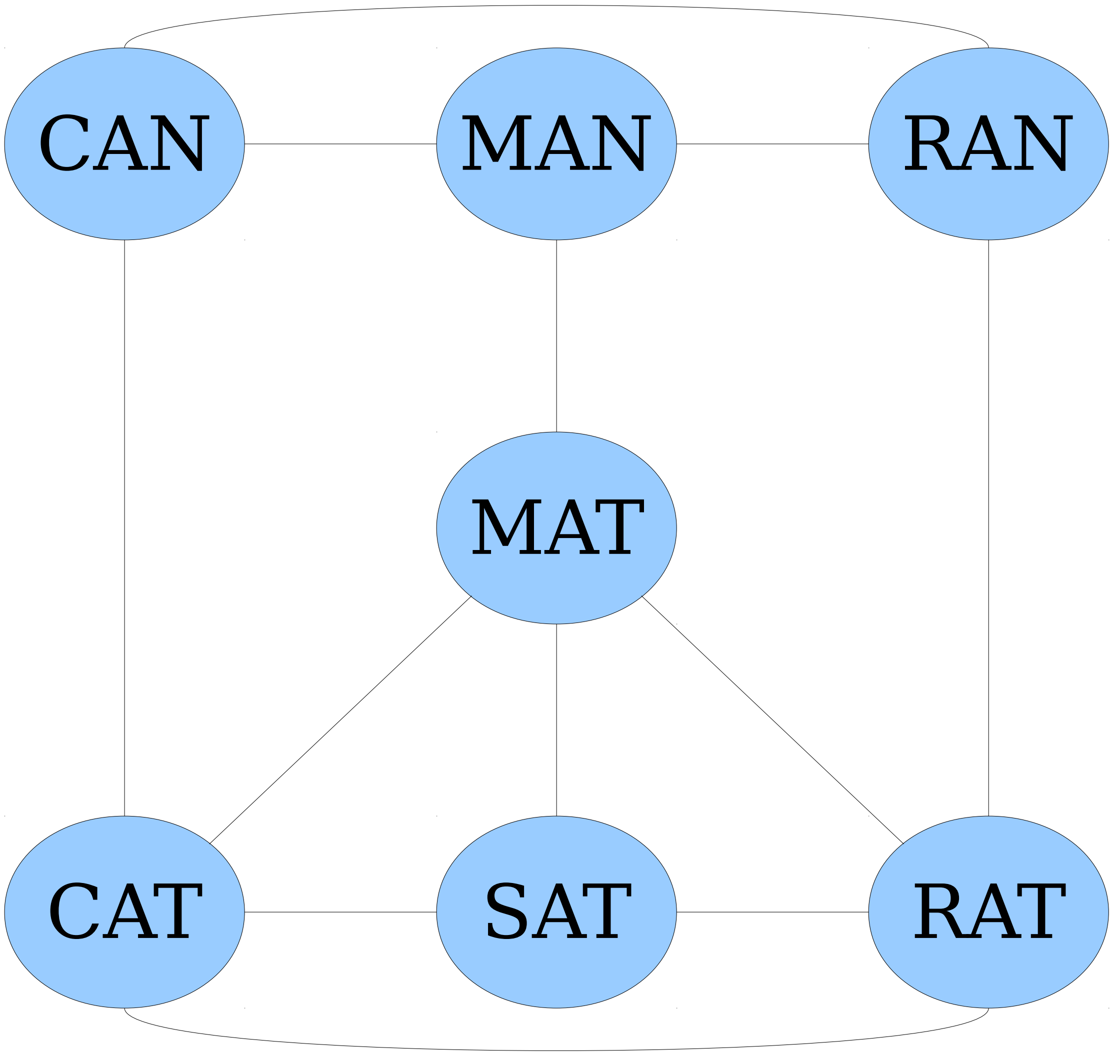
Some graphs are ***directed***.



Some graphs are ***undirected***.



Graphs and Digraphs

- An ***undirected graph*** is one where edges link nodes, with no endpoint preferred over the other.
- A ***directed graph*** (or ***digraph***) is one where edges have an associated direction.

- Unless specified otherwise:

☞ ***"Graph" means "undirected graph"*** ☜

Formalizing Graphs

- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *unordered* pairs of nodes drawn from V.
  - An ***unordered pair*** is a set with cardinality two.
- a ***directed graph*** (or ***digraph***) is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *ordered* pairs of nodes drawn from V.

- An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$.
- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
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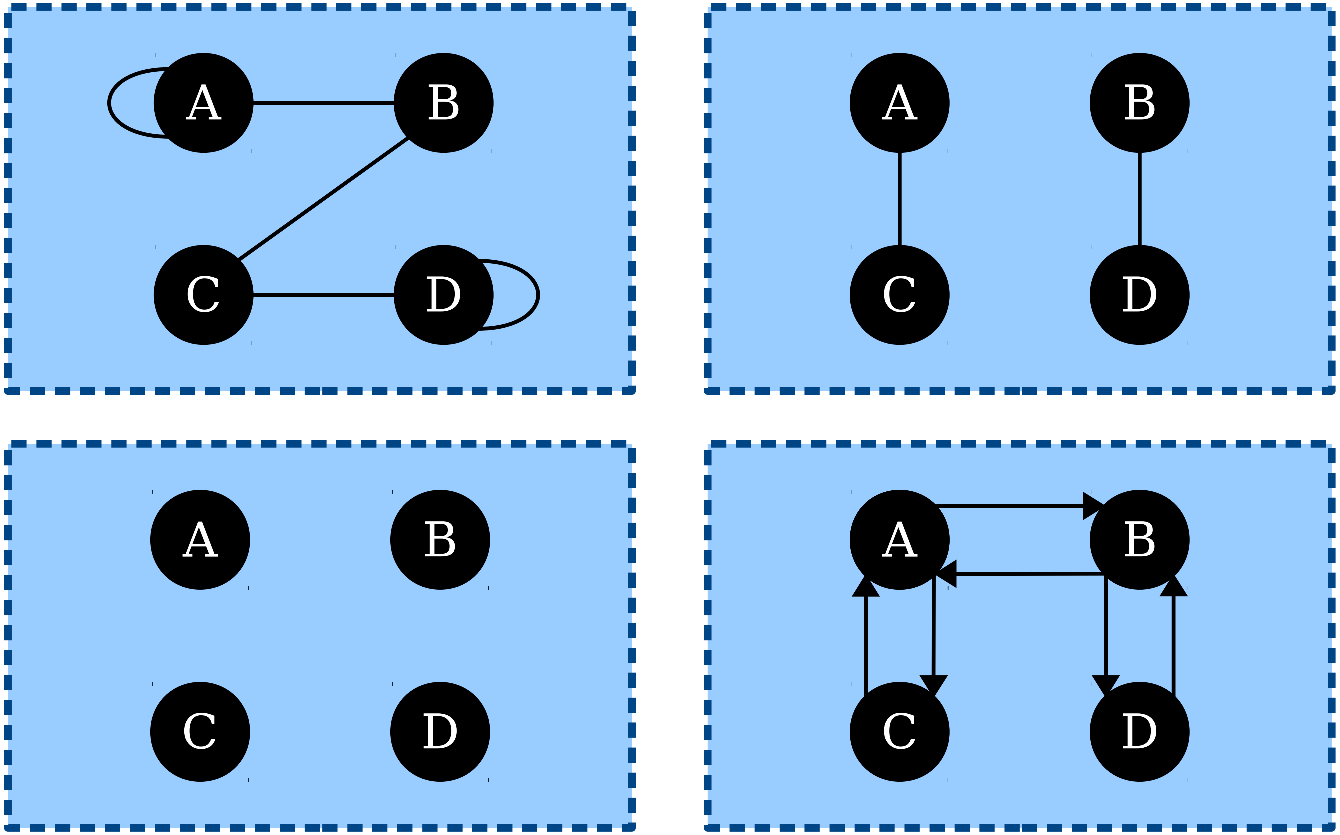
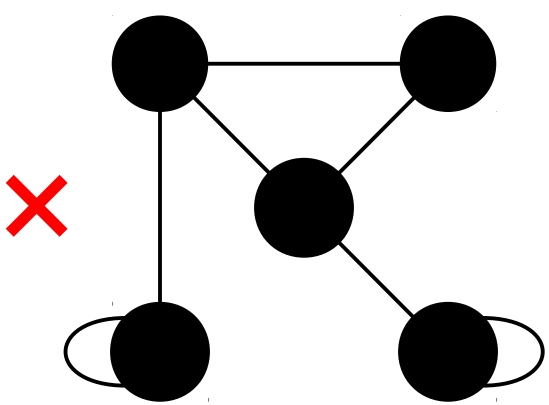
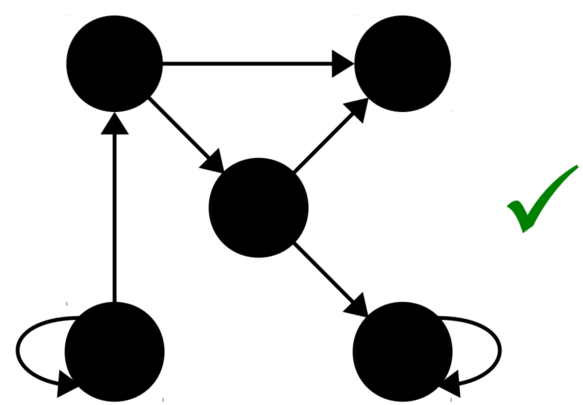


How many of these drawings are of valid undirected graphs?

Self-Loops

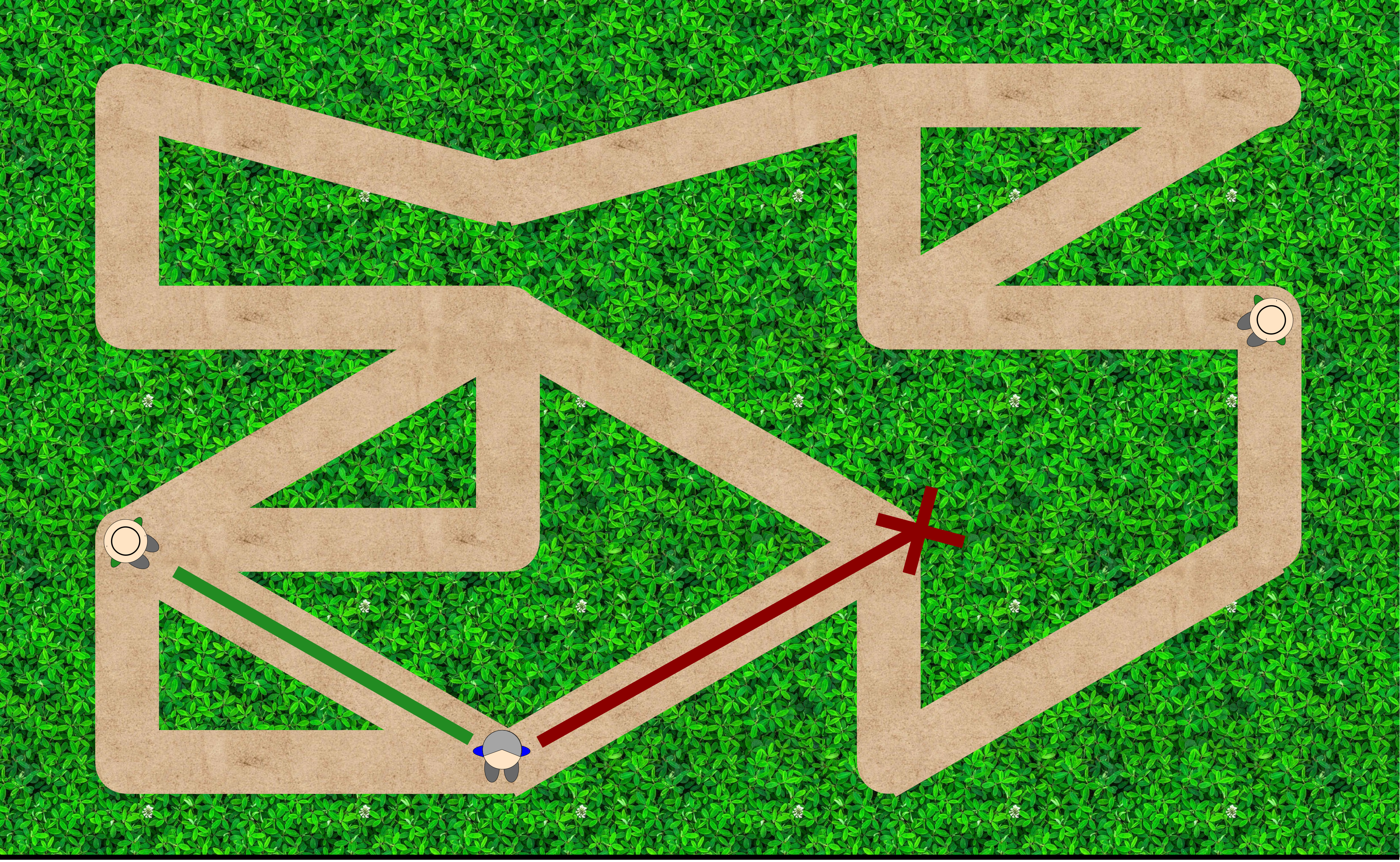
- An edge from a node to itself is called a ***self-loop***.
- In (undirected) graphs, self-loops are generally not allowed.
  - Can you see how this follows from the definition?
- In digraphs, self-loops are generally allowed unless specified otherwise.



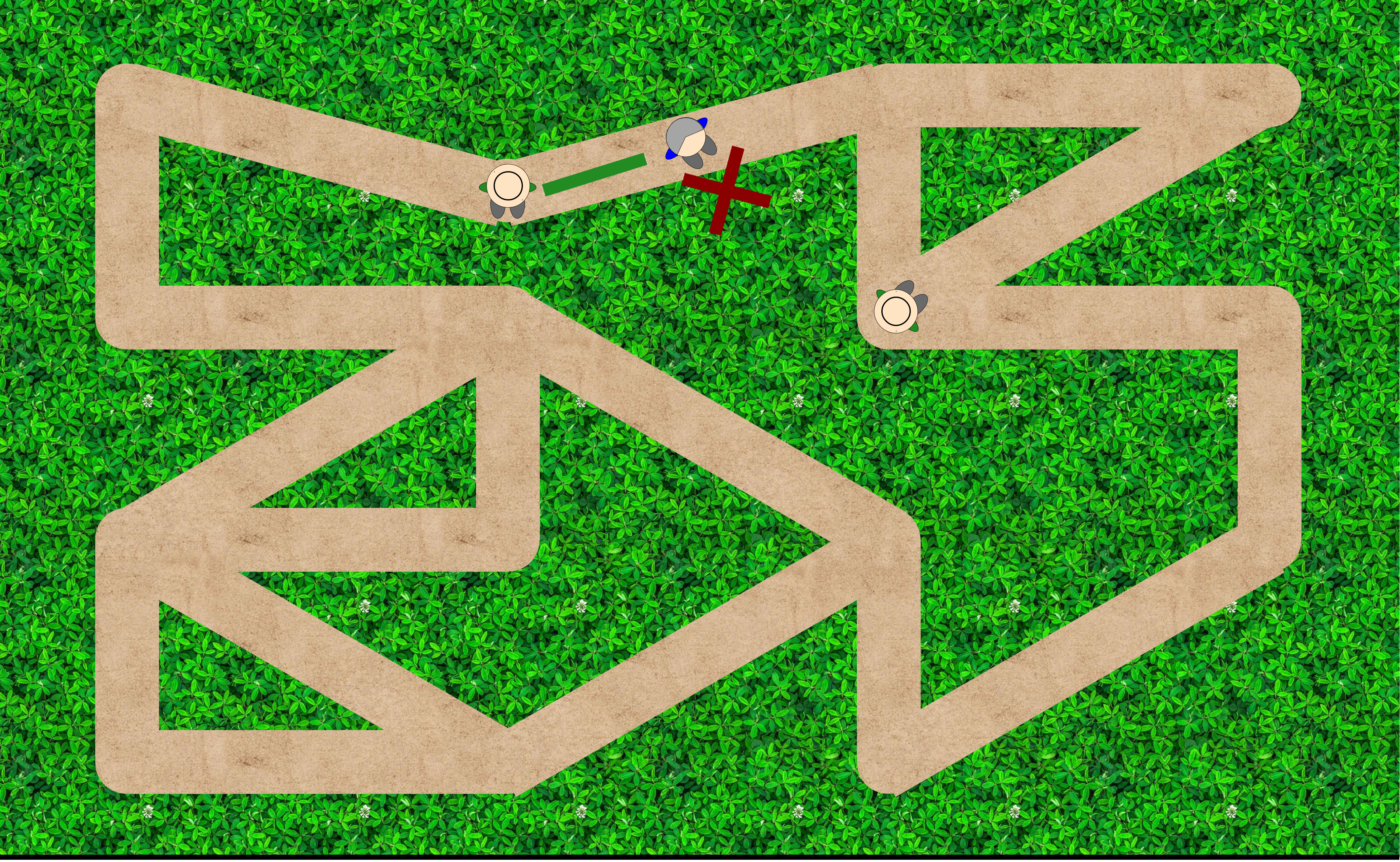


Independent Sets and Vertex Covers

Two Motivating Problems



Place park rangers in these forest trails so that
a hiker anywhere on a trail can see a park ranger.



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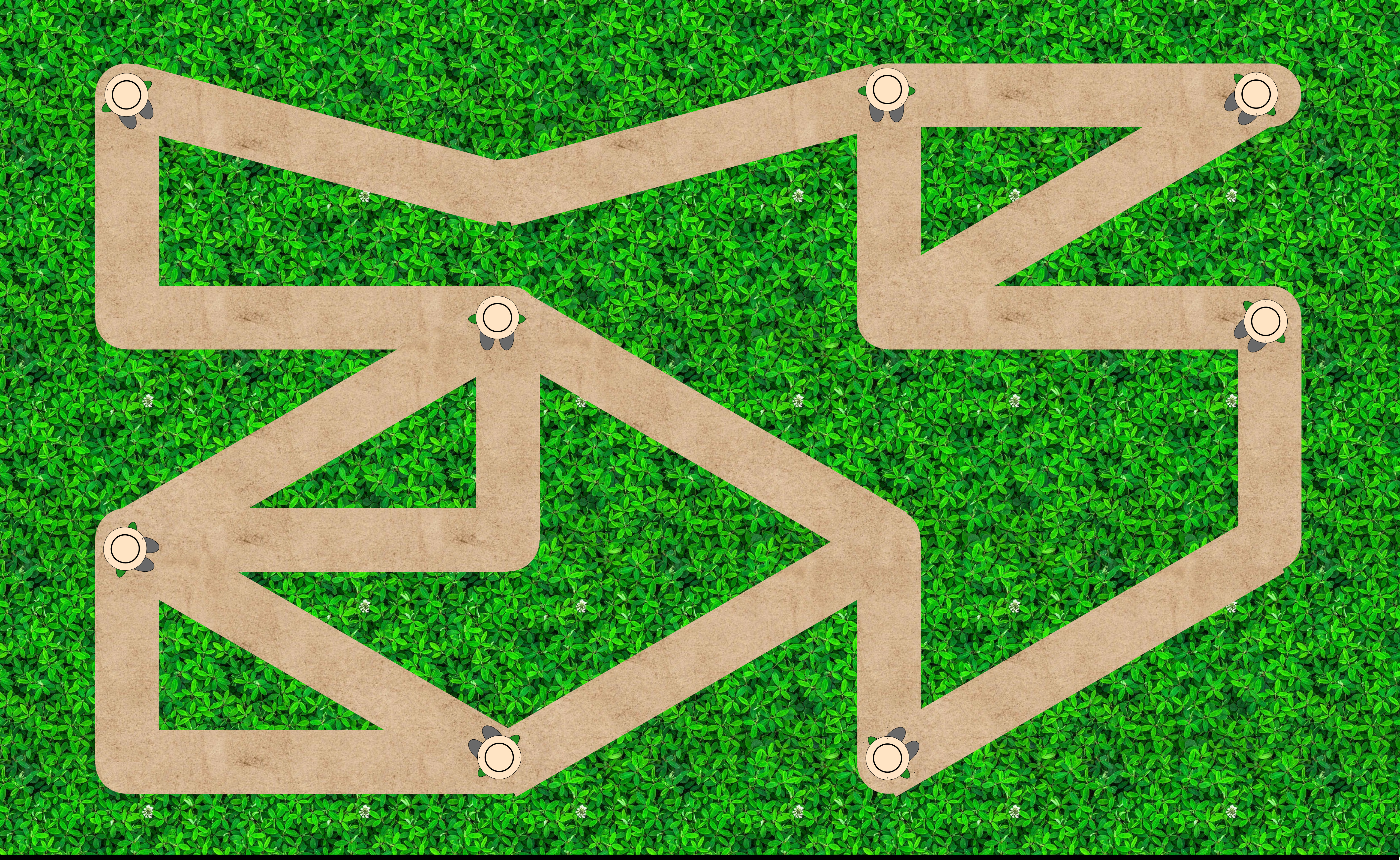

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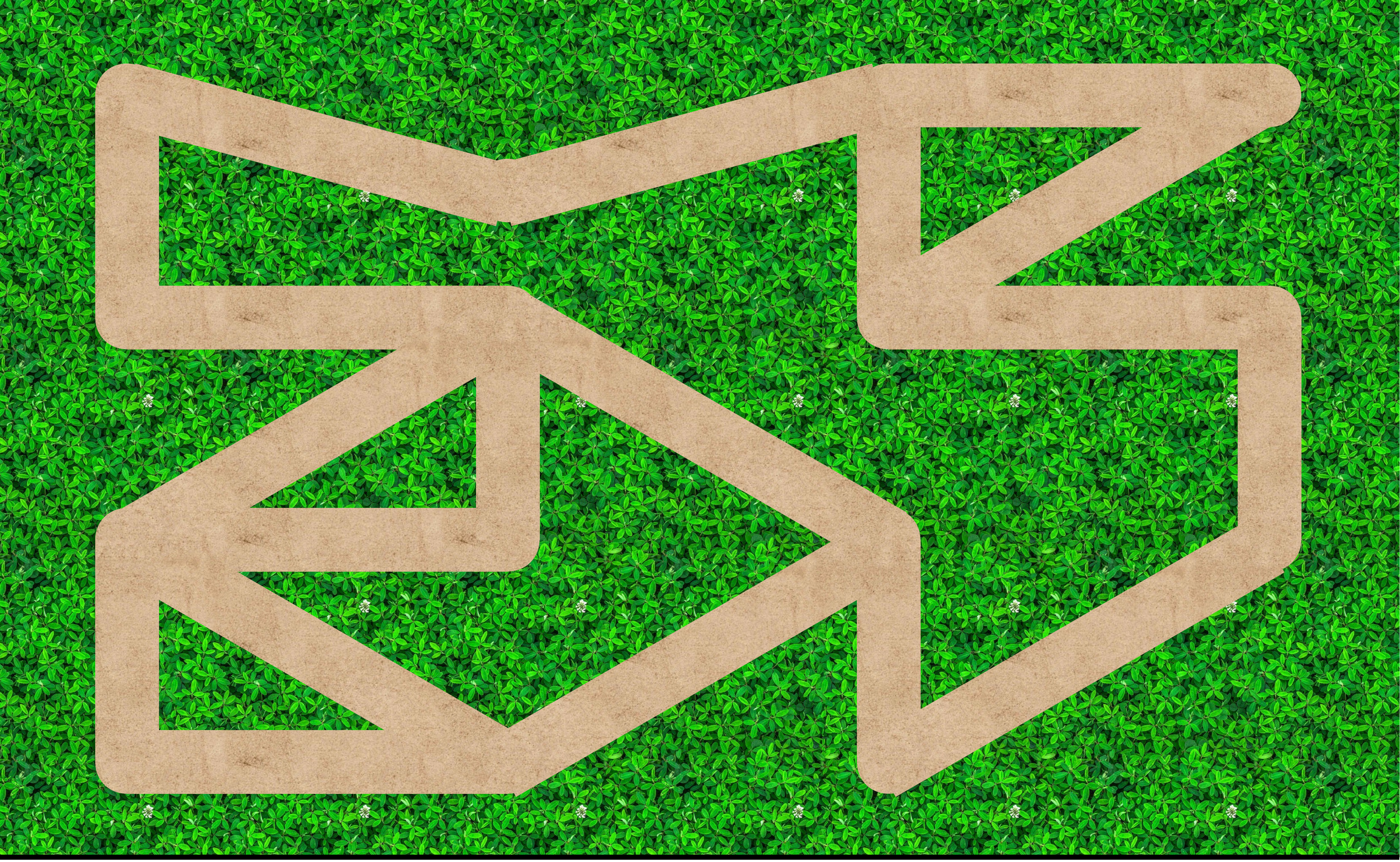

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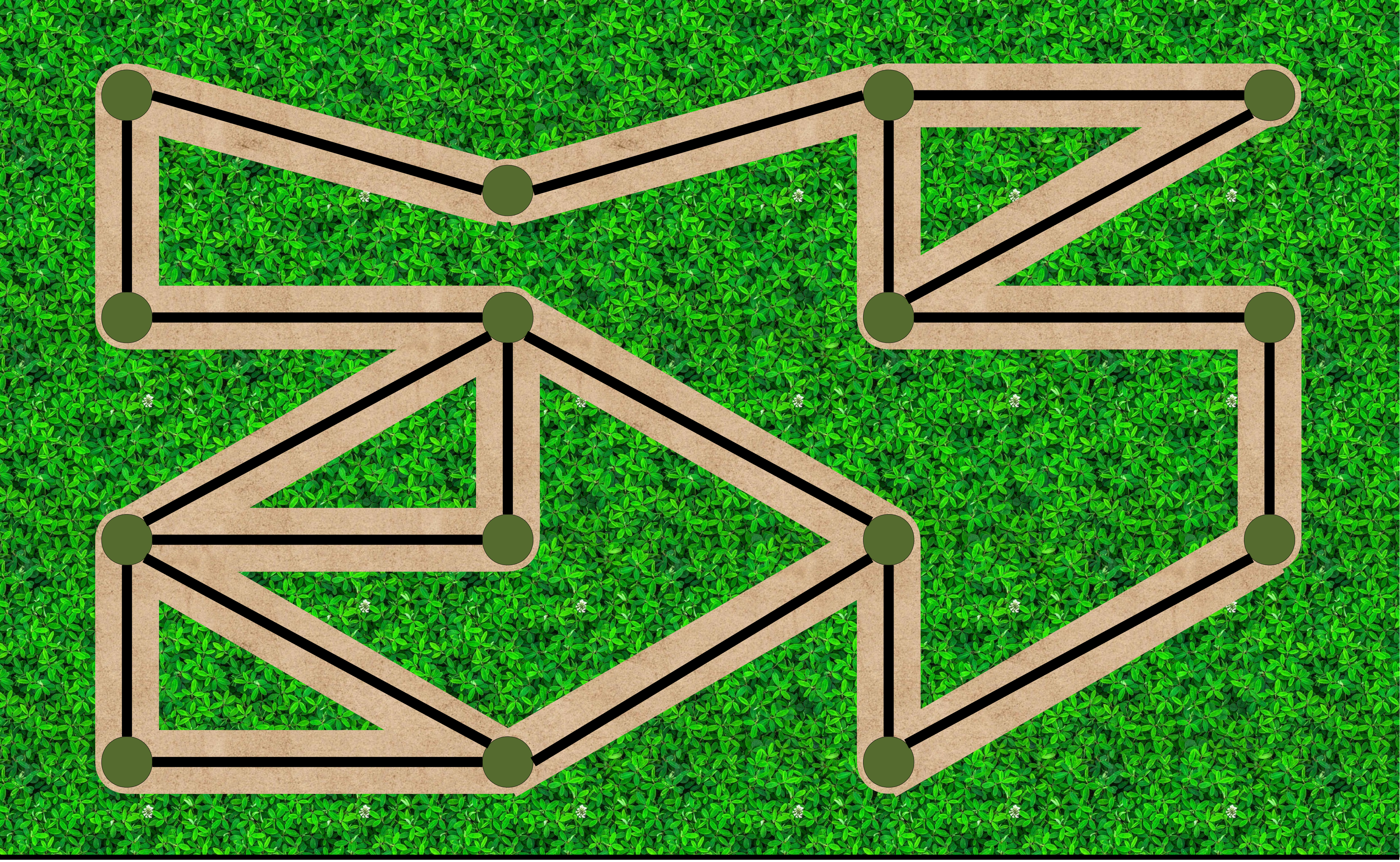

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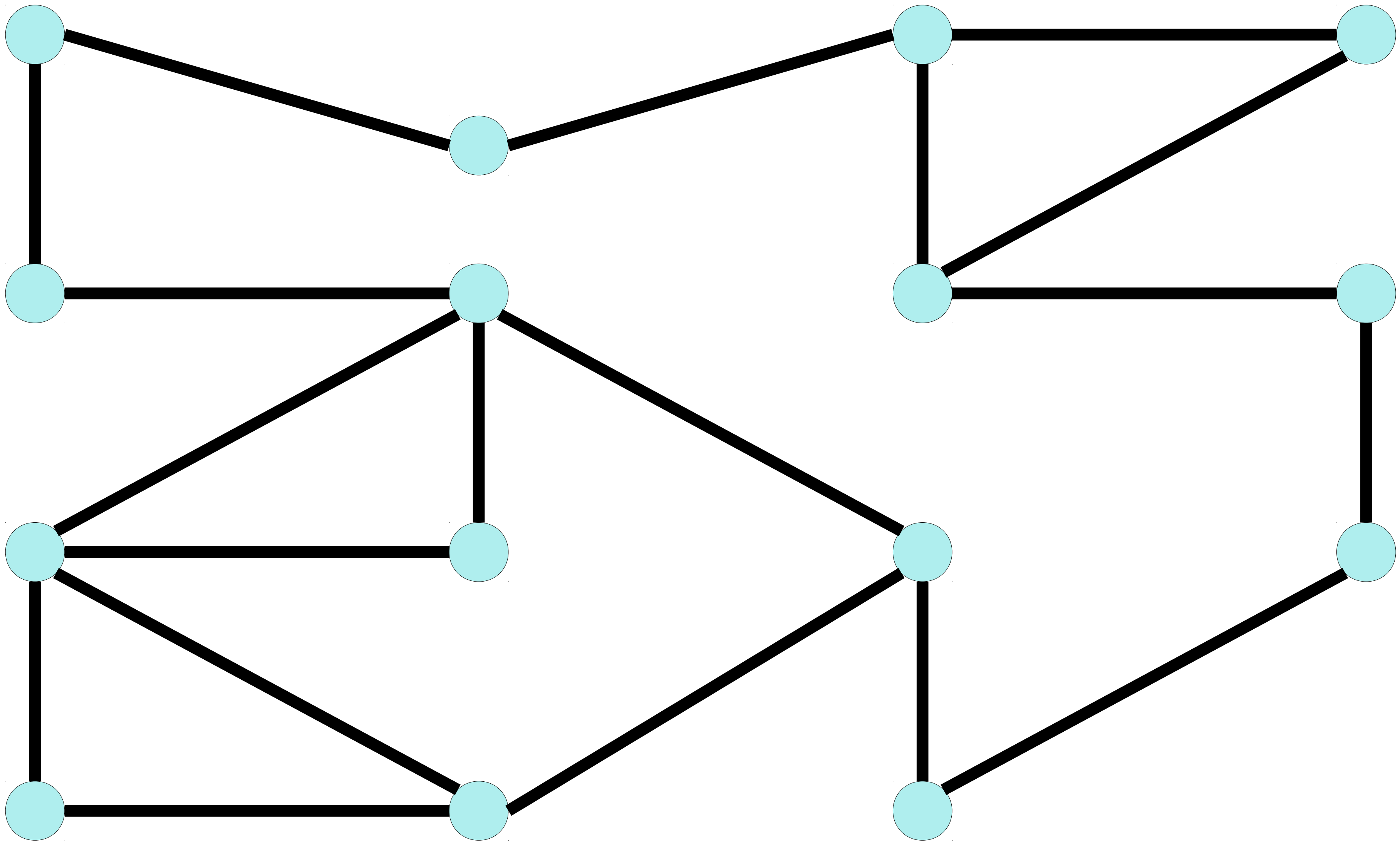

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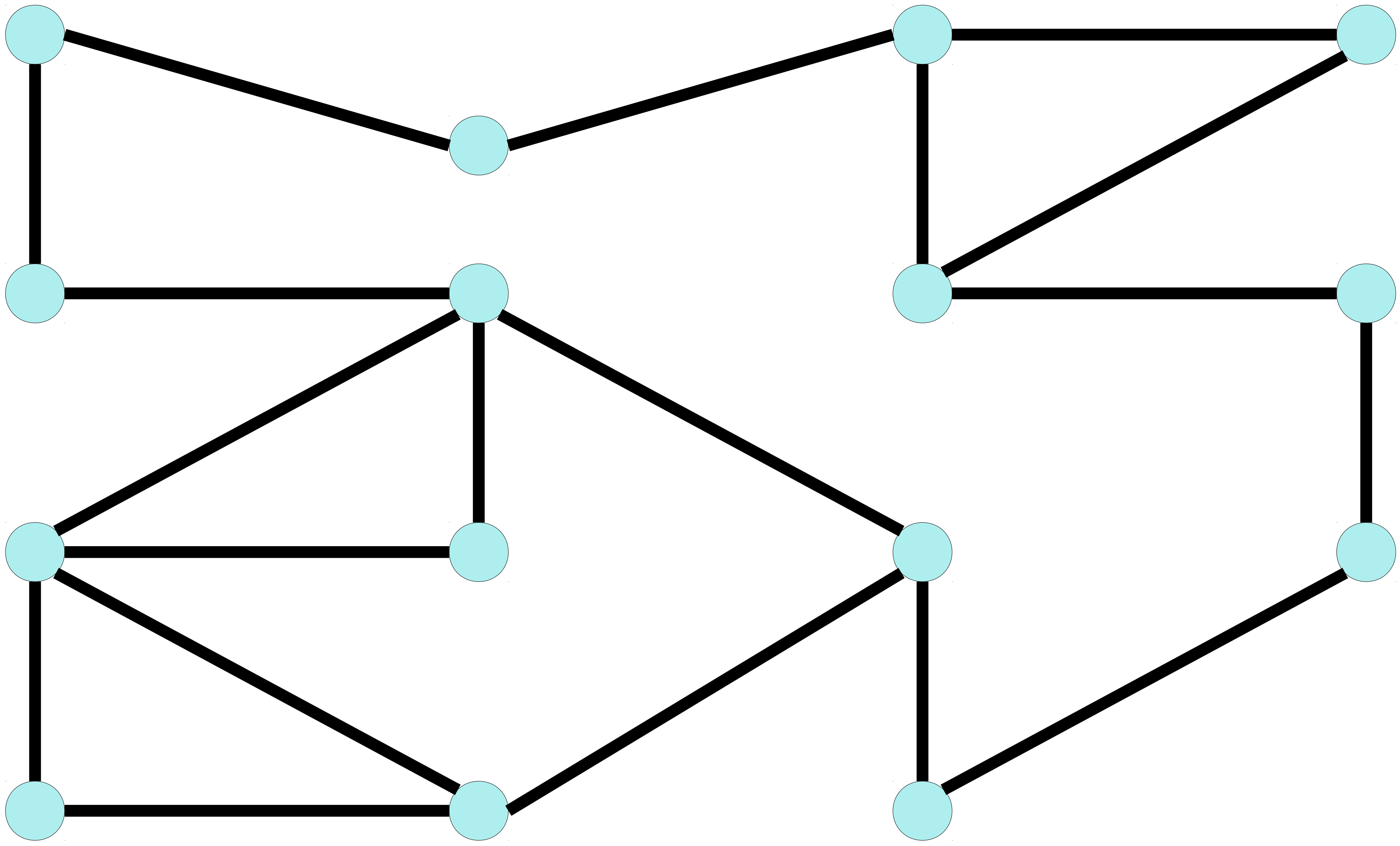
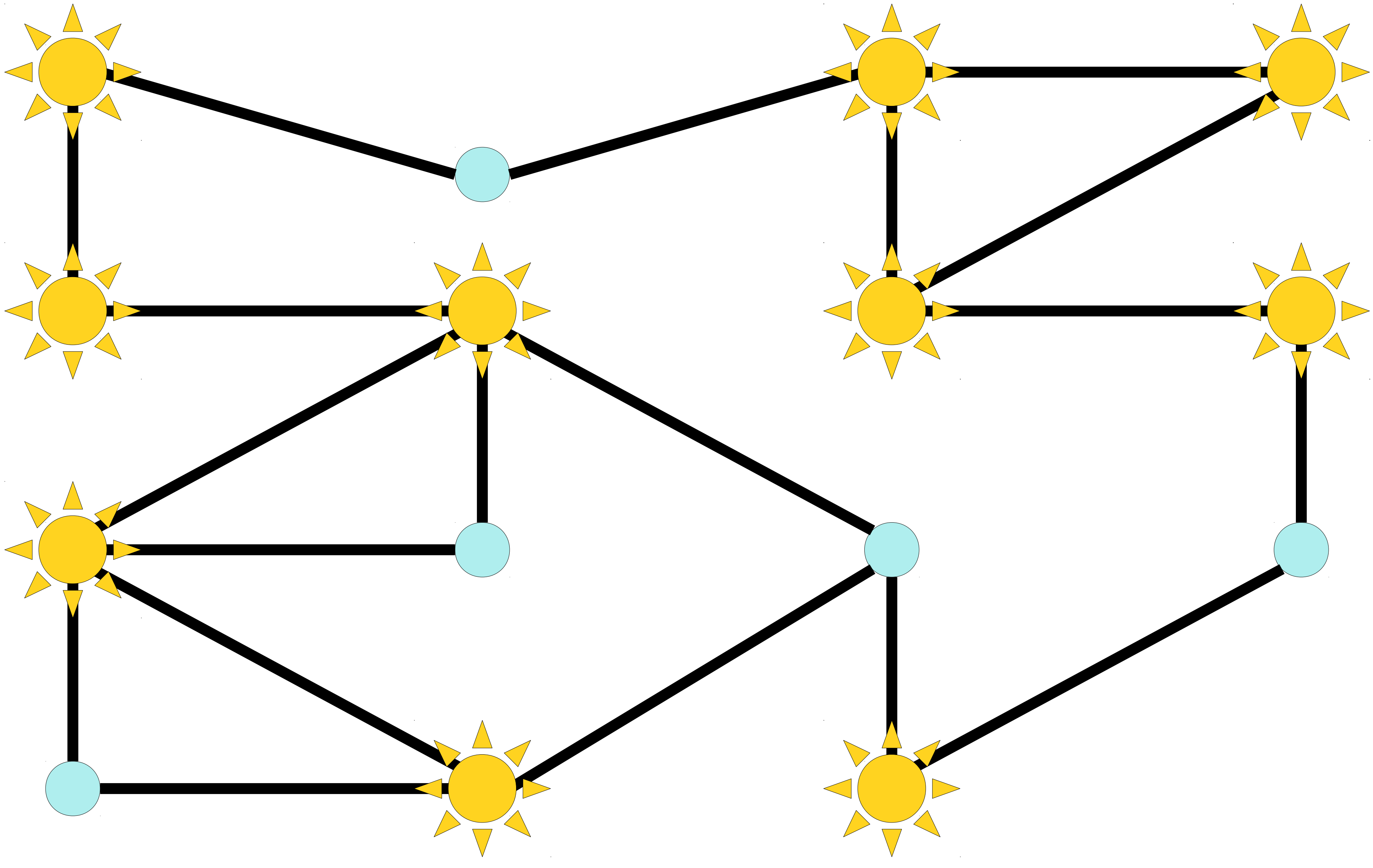


Choose at least one endpoint of each edge.



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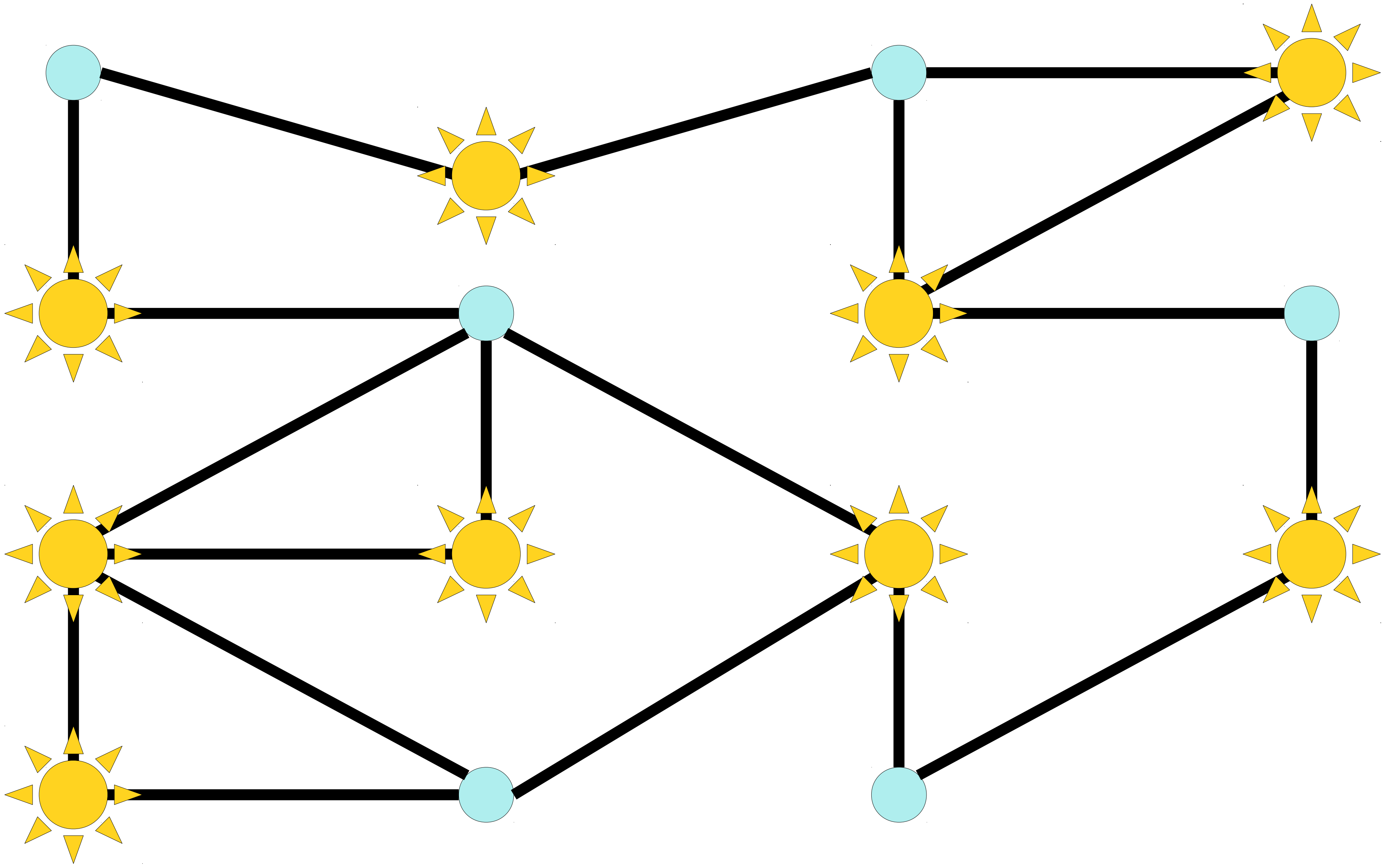

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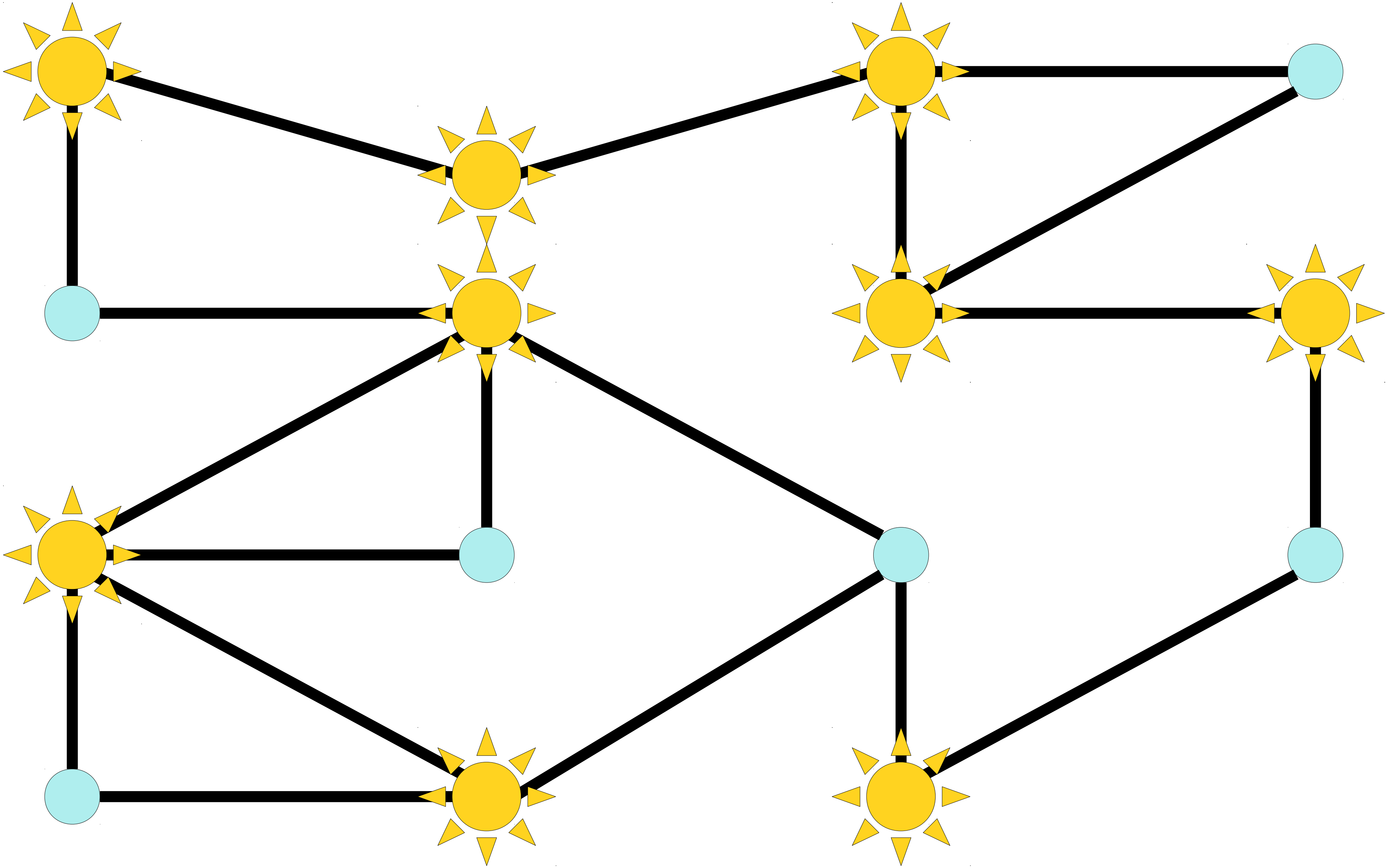


Choose at least one endpoint of each edge.

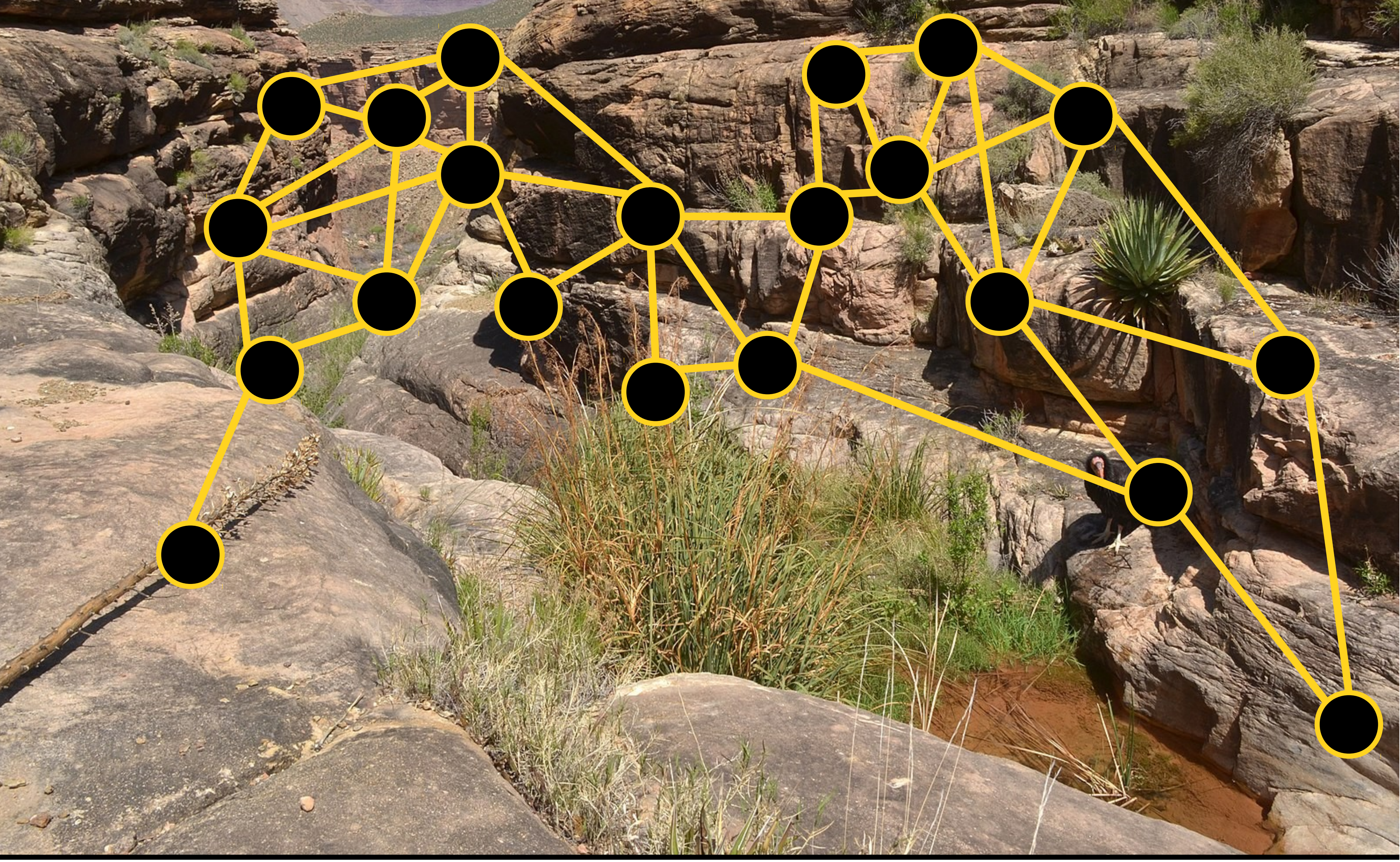
Vertex Covers

- Let $G = (V, E)$ be an undirected graph. A ***vertex cover*** of G is a set $C \subseteq V$ such that the following statement is true:

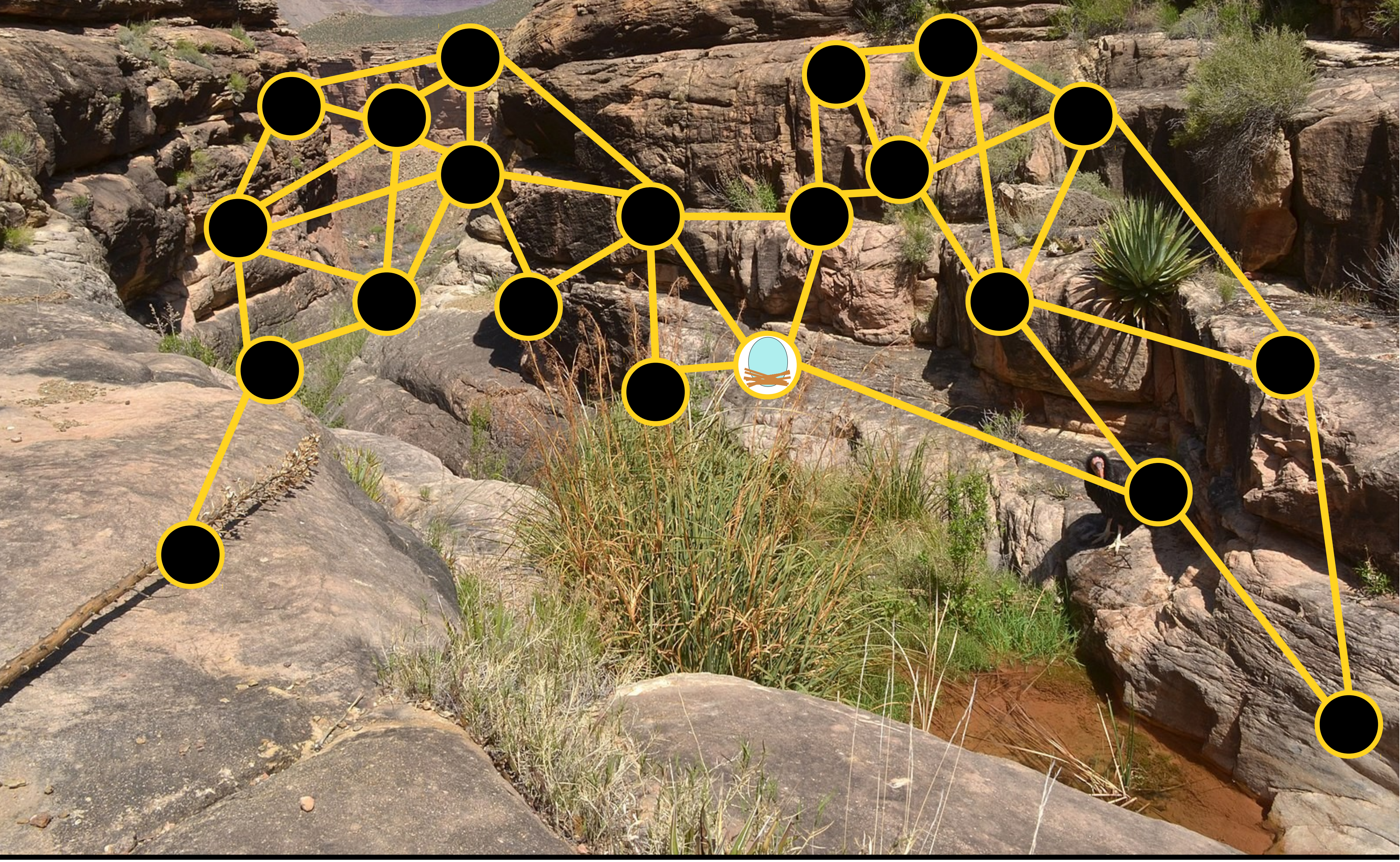
$$\forall x \in V.\ \forall y \in V.\ (\{x, y\} \in E \to (x \in C \lor y \in C))$$

("Every edge has at least one endpoint in C.")

- Intuitively speaking, a vertex cover is a set formed by picking at least one endpoint of each edge in the graph.
- Vertex covers have applications to placing streetlights/benches/security guards, as well as in gene sequencing, machine learning, and combinatorics.



Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



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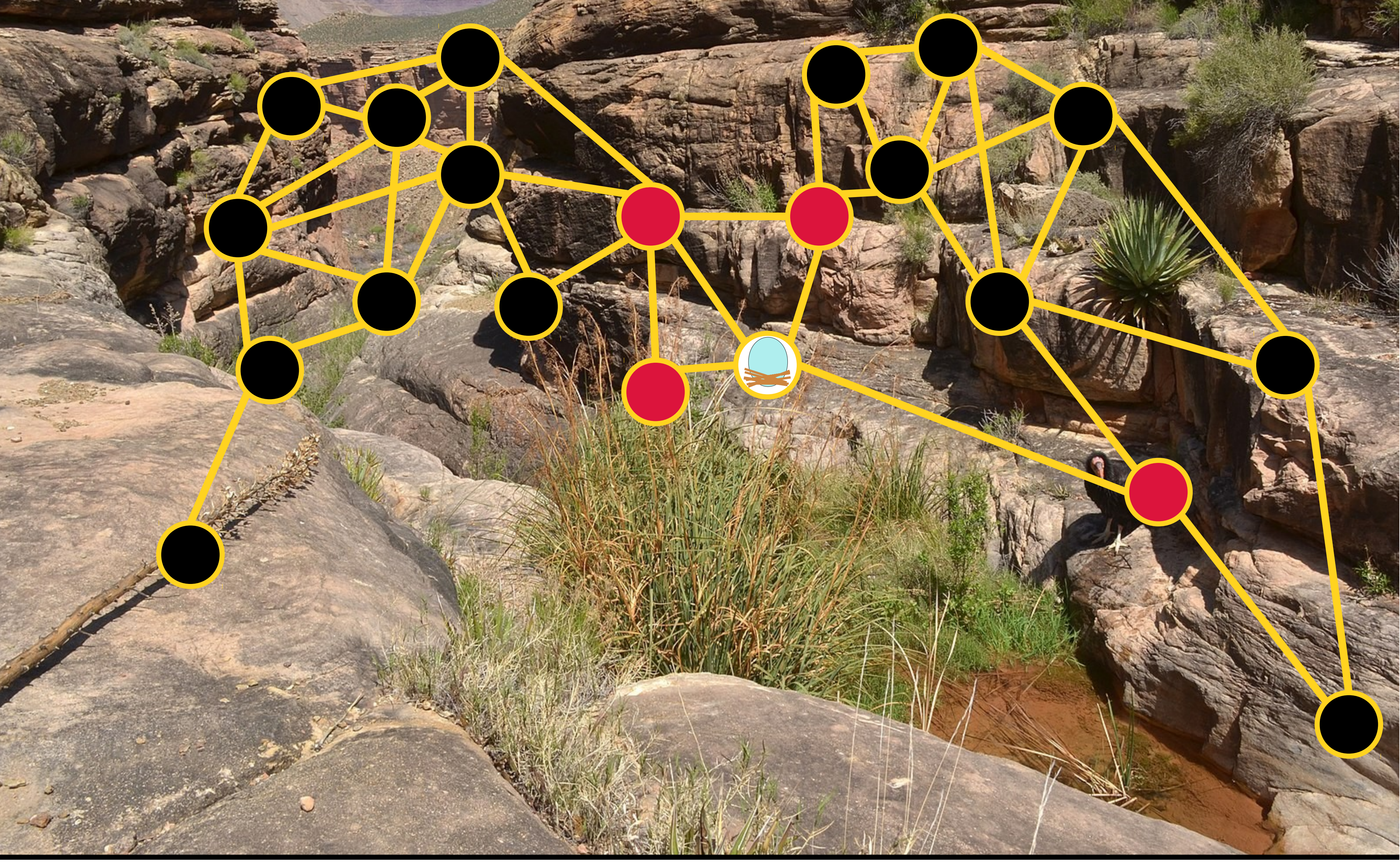

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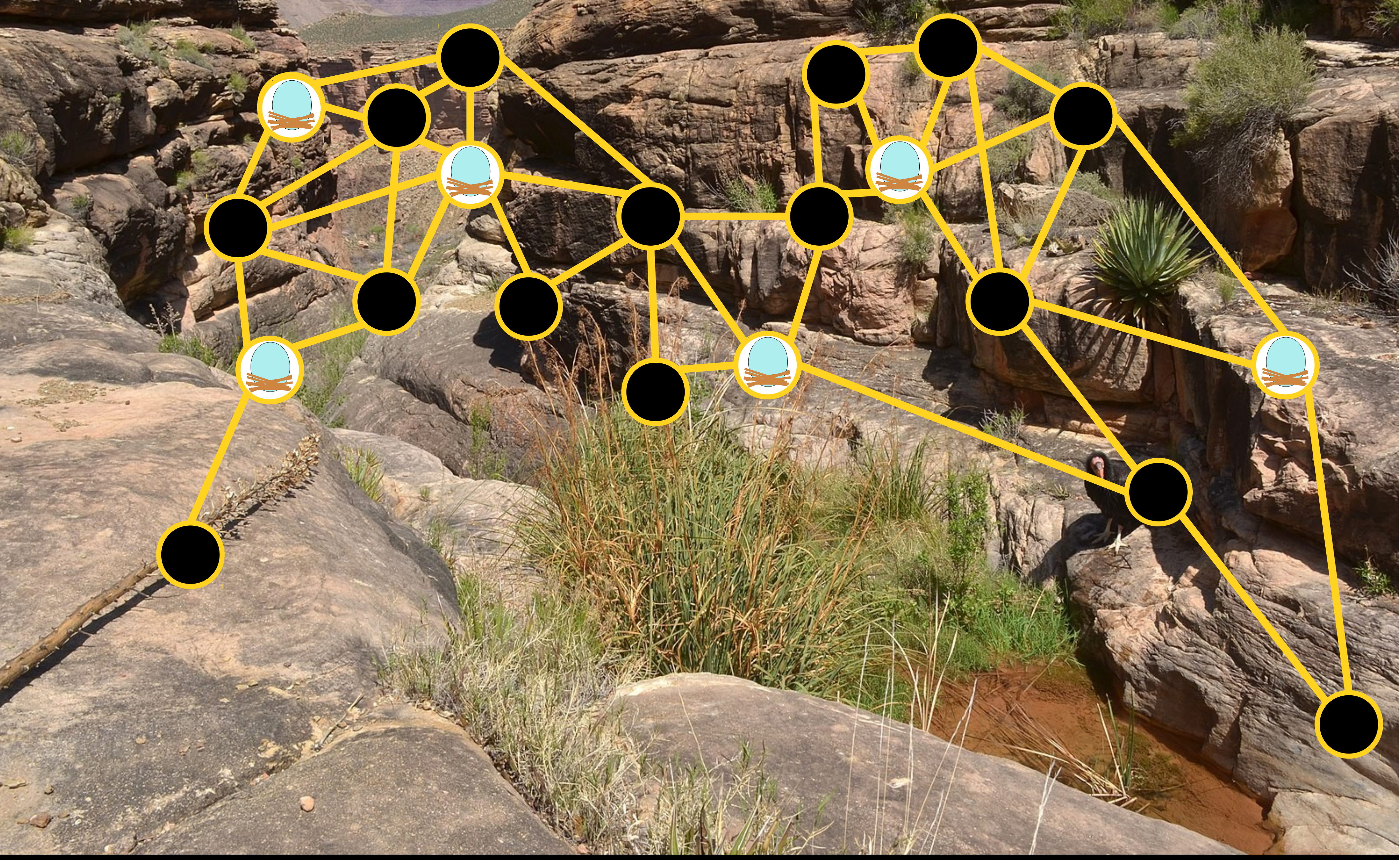

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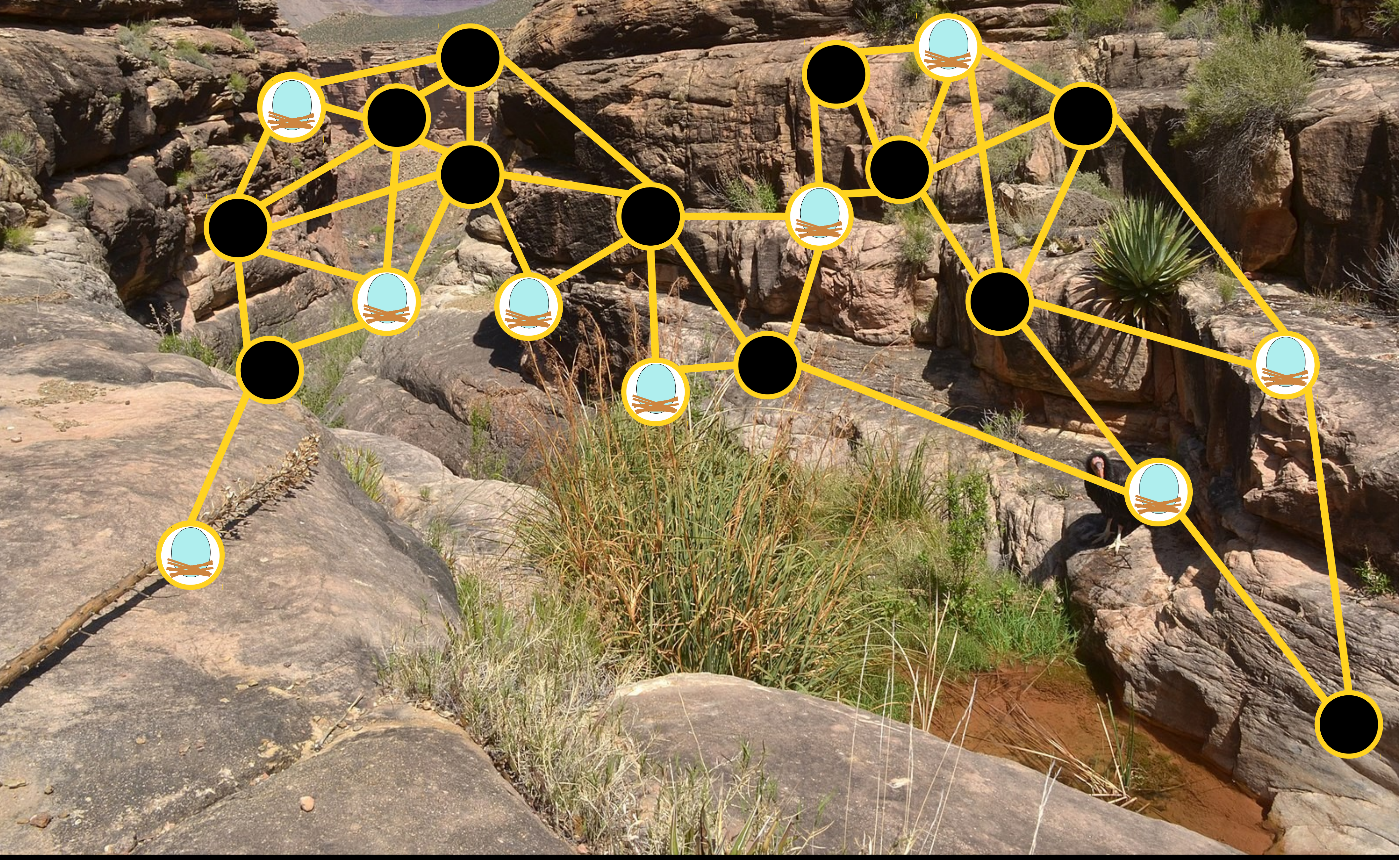

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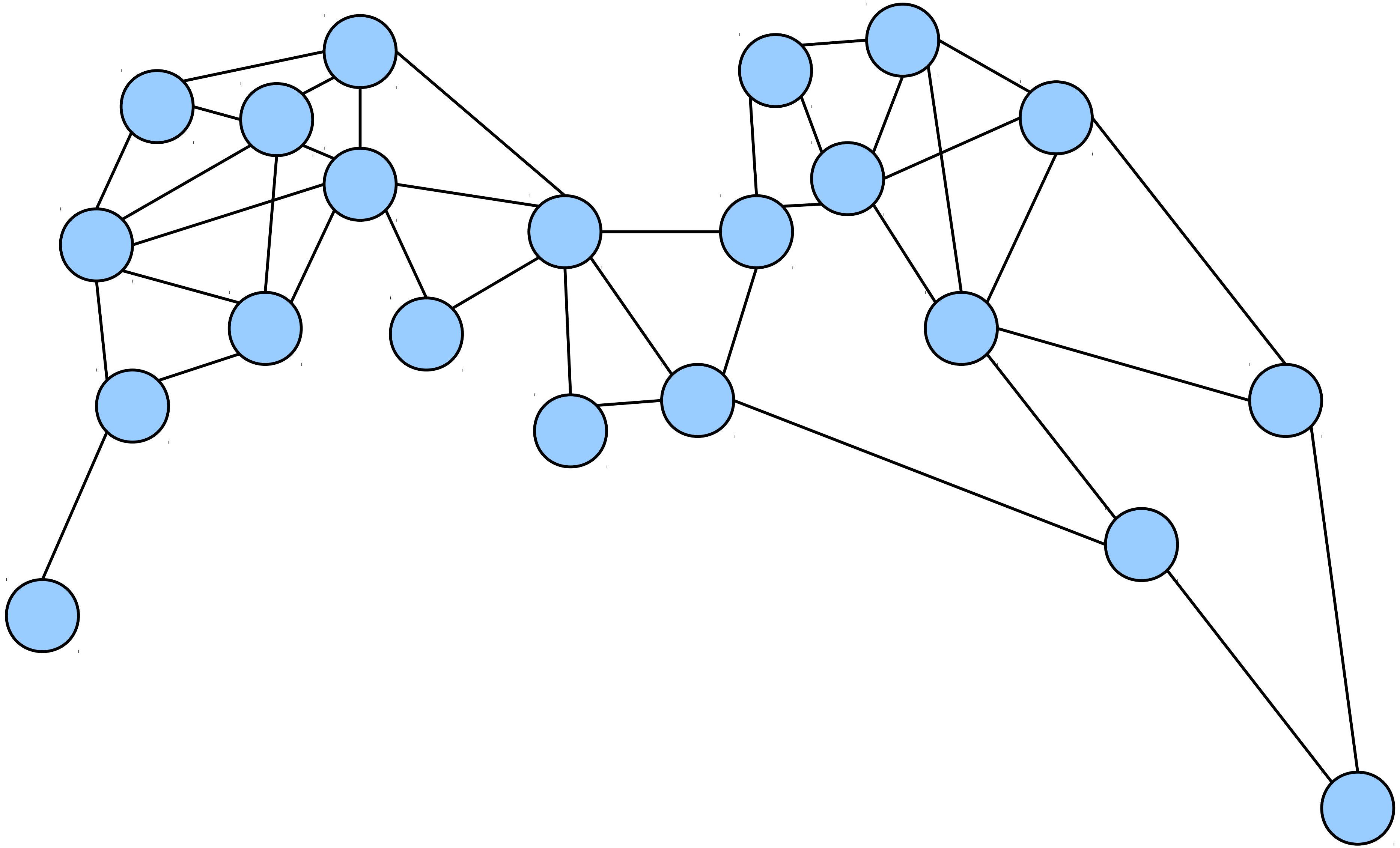

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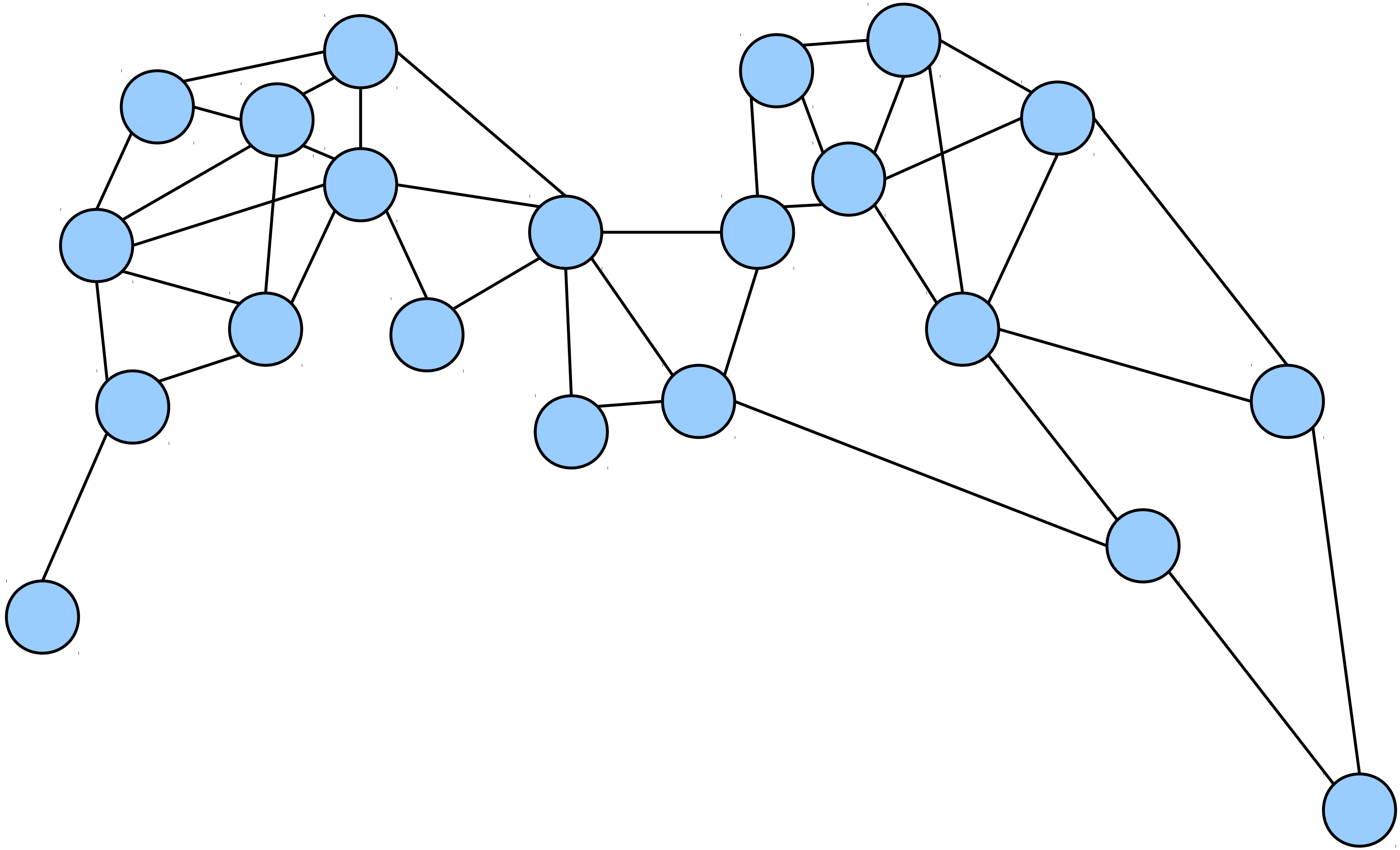


Choose a set of nodes, no two of which are adjacent.

Independent Sets

- If $G = (V, E)$ is an (undirected) graph, then an ***independent set*** in G is a set $I \subseteq V$ such that

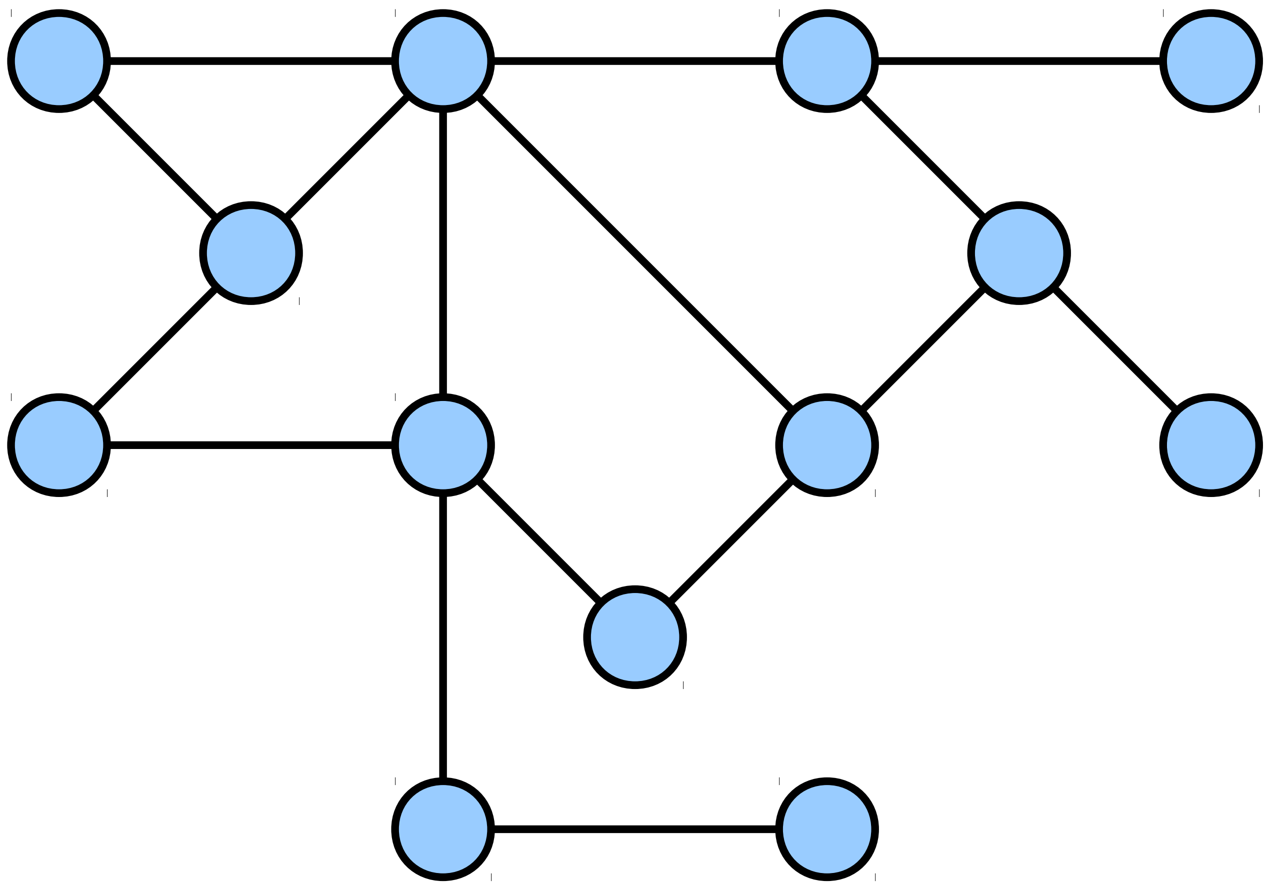
$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E.$$

("No two nodes in I are adjacent.")

- Independent sets have applications to resource optimization, conflict minimization, error-correcting codes, cryptography, and more.

Constraint Optimization with Independent Set and Vertex Cover

What is the *smallest* Independent Set for this graph?



$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E.$$

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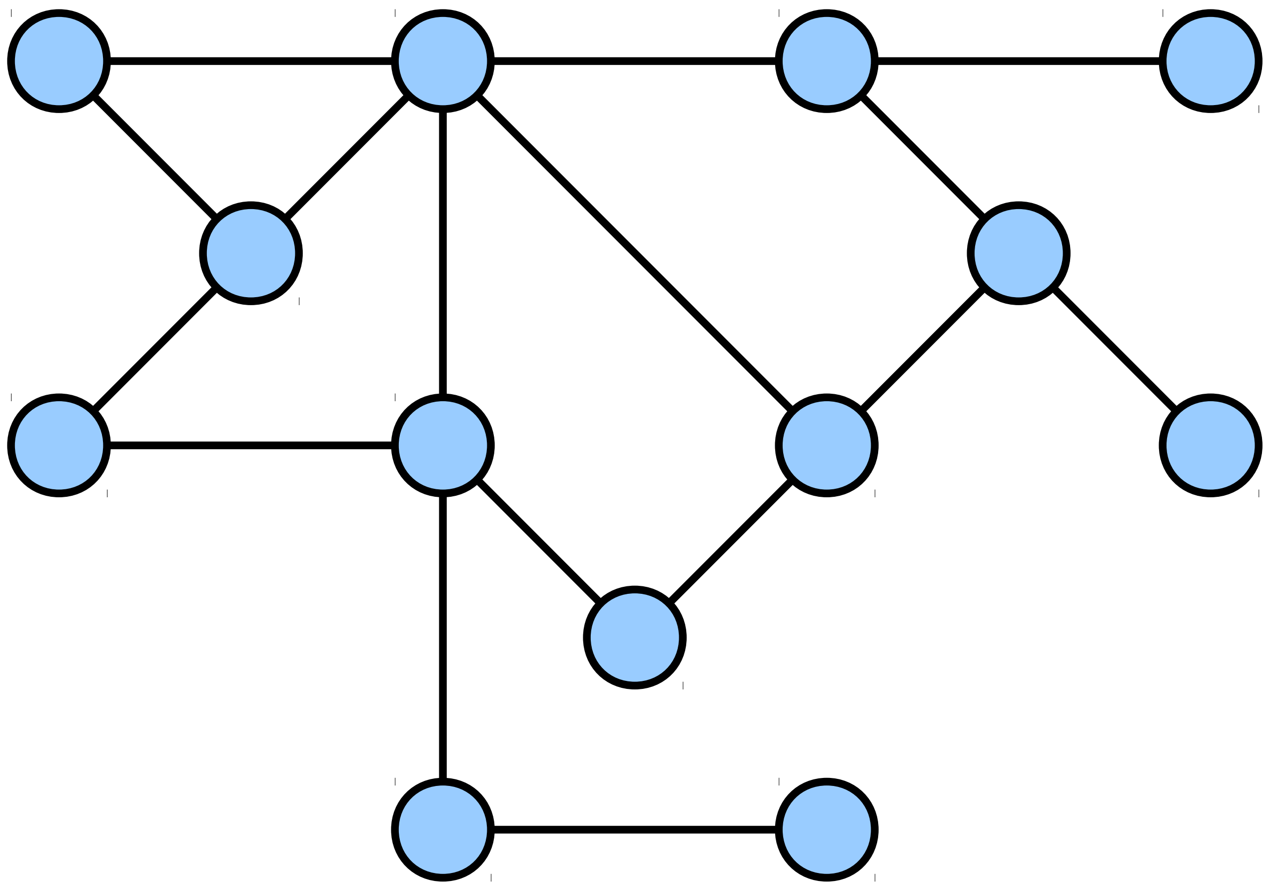
What is the *largest* Vertex Cover for this graph?



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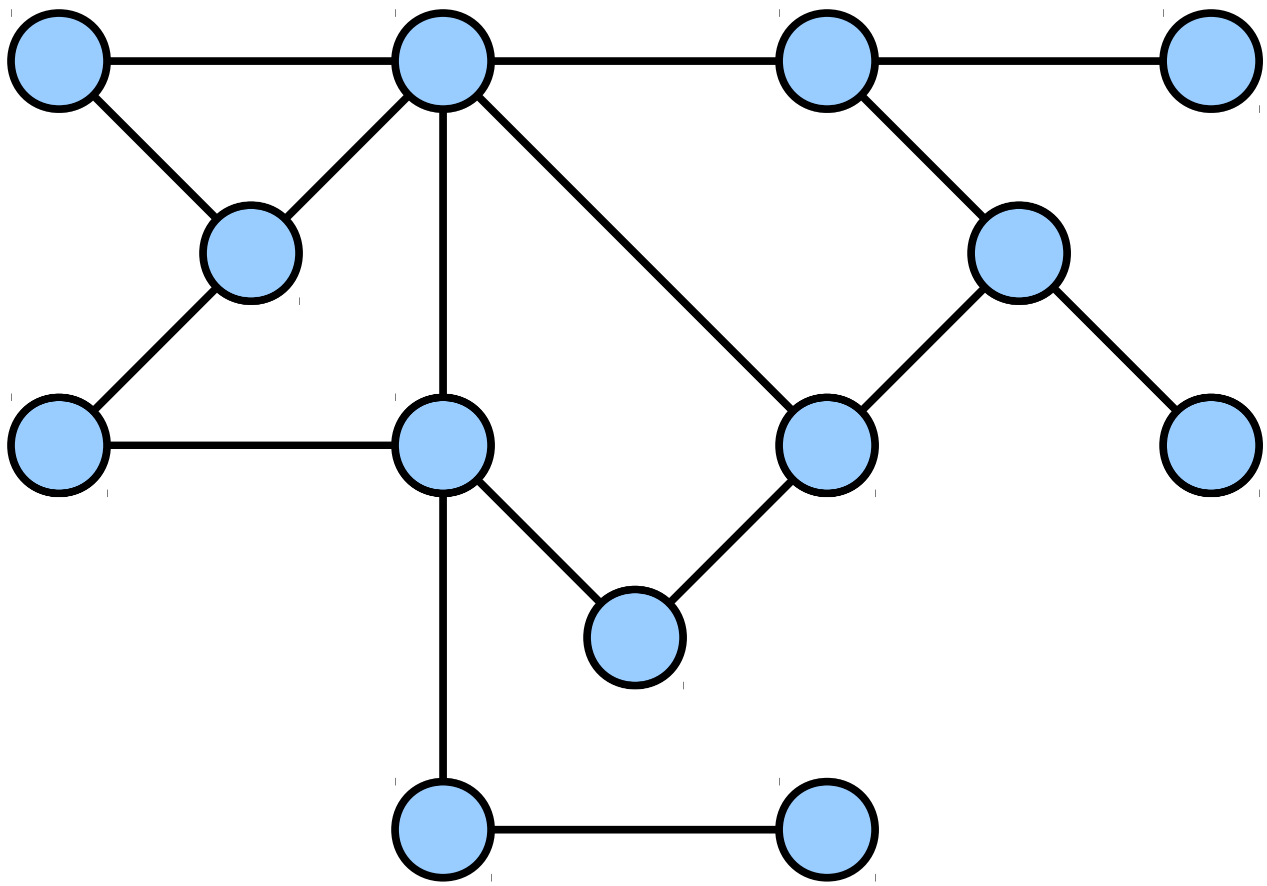
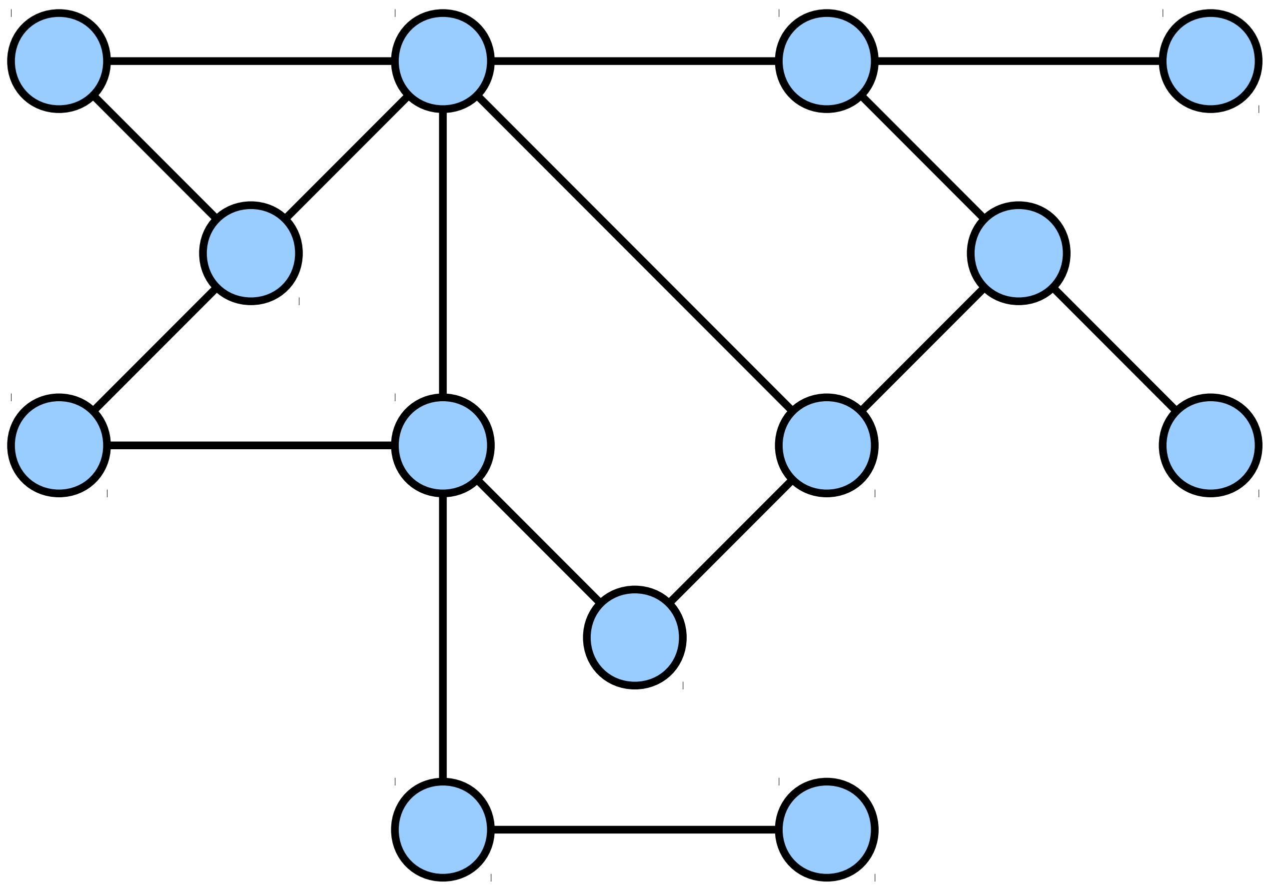
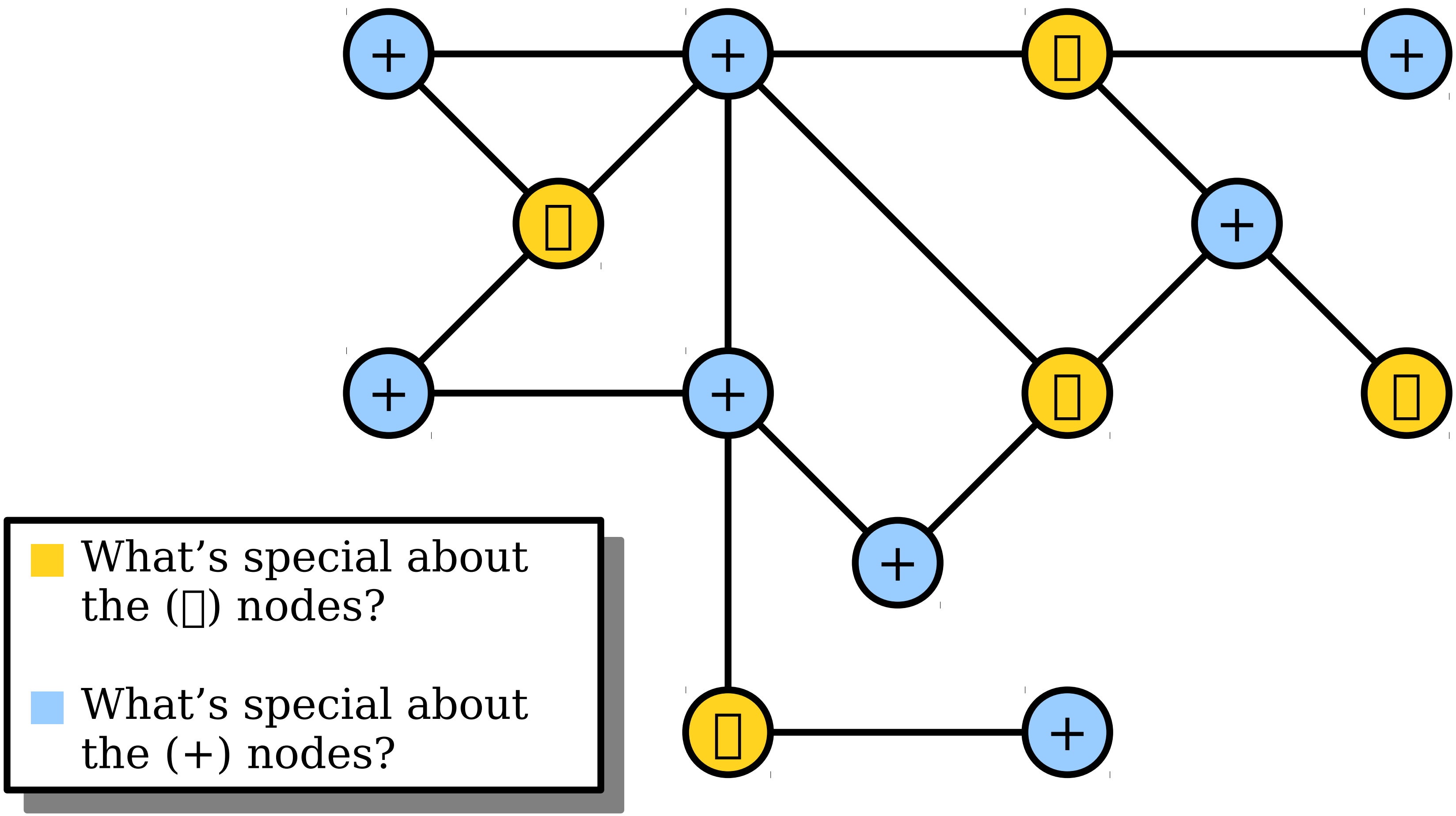


What is the *largest* Independent Set for this graph?
What is the *smallest* Vertex Cover for this graph?

A Connection



Independent sets and vertex covers are related.



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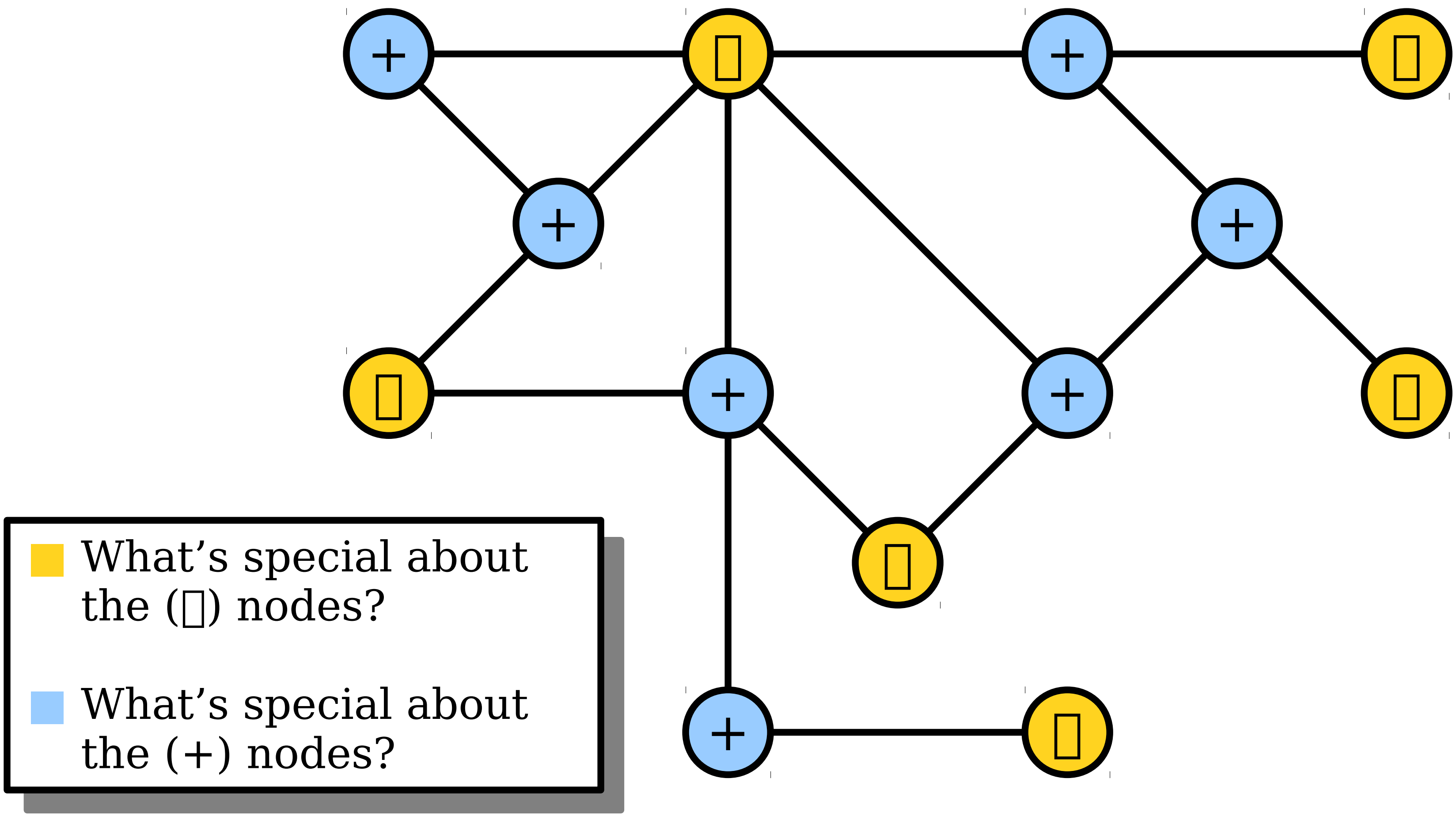

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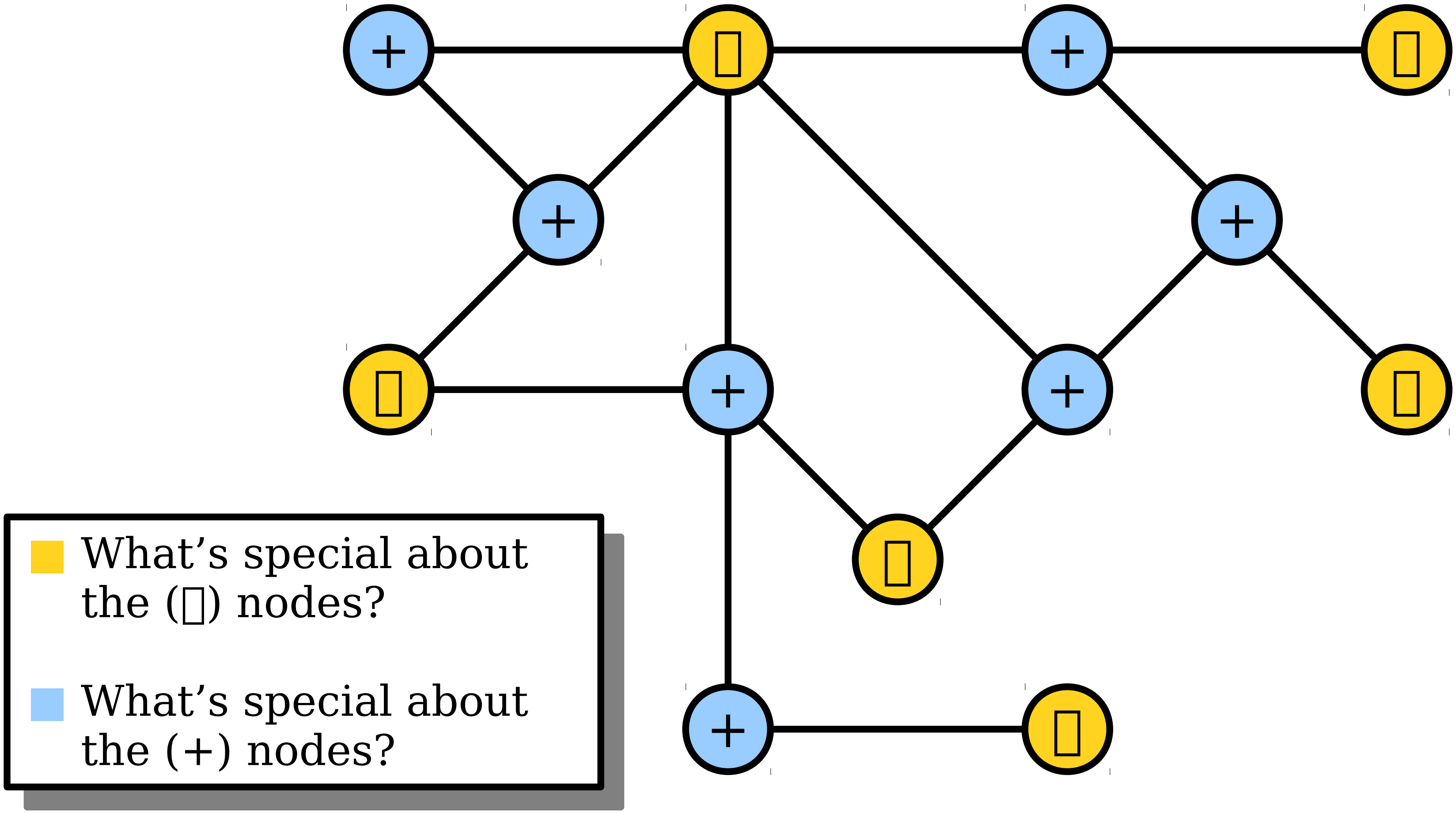

Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G if and only if $V - C$ is an independent set in G.

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Recap for Today

- A ***graph*** is a structure for representing items that may be linked together. ***Digraphs*** represent that same idea, but with a directionality on the links.

- ***Vertex covers*** and ***independent sets*** are useful tools for modeling problems with graphs.
- The complement of a vertex cover is an independent set, and vice-versa.

Minimum Vertex Cover and Maximum Independent Set are two complementary (and generally hard) problems.