The University of Texas at Austin
Department of Electrical and Computer Engineering

ECE 381V: Large-Scale Optimization II — Spring 2022

Lecture 27

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Goal: In this lecture, we discuss the properties of the subproblem that needs to be solved in the cubic regularized Newton method.

1 The Main Subproblem

Recall that the cubic subproblem in cubic regularized Newton's method is given by

$$\phi_M(\mathbf{y};\mathbf{x}_k) = \nabla f(\mathbf{x}_k)^\top(\mathbf{y}-\mathbf{x}_k) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^\top \nabla^2 f(\mathbf{x}_k)(\mathbf{y}-\mathbf{x}_k) + \frac{M}{6}\|\mathbf{y}-\mathbf{x}_k\|^3,$$

and we define its optimal solution and optimal value as

$$T_M(\mathbf{x}_k) = \underset{\mathbf{y}\in\mathcal{Q}}{\operatorname{argmin}}\{\phi_M(\mathbf{y};\mathbf{x}_k)\},$$

and

$$f_M(\mathbf{x}_k) := \min_{\mathbf{y}\in\mathcal{Q}}\ \{\phi_M(\mathbf{y};\mathbf{x}_k)\},$$

respectively.

2 The subproblem in the nonconvex case without constraint

Note that in the case that the problem has no constraint, our subproblem is equivalent to solving the following problem

$$\min_{\mathbf{h}\in\mathbb{R}^n}\ \left\{\mathbf{g}^\top\mathbf{h} + \frac{1}{2}\mathbf{h}^\top\mathbf{H}\mathbf{h} + \frac{M}{6}\|\mathbf{h}\|^3\right\}$$

Note that in the case that the objective function is nonconvex and $\mathbf{H}$ is indefinite, the above subproblem is nonconvex. But, we will show that it is equivalent to a convex problem and its optimal solution can be found efficiently.
Now note that since

$$\|\mathbf{h}\|^3 = \min_{\tau\in\mathbb{R}}\left\{\ |\tau|^{3/2}, \qquad \text{subject to } \|\mathbf{h}\|^2 \le \tau\right\}$$

we can write our subproblem as

$$\min_{\mathbf{h}\in\mathbb{R}^n}\min_{\tau\in\mathbb{R}}\ \left\{\mathbf{g}^\top\mathbf{h} + \frac{1}{2}\mathbf{h}^\top\mathbf{H}\mathbf{h} + \frac{M}{6}|\tau|^{3/2}, \qquad \text{subject to } \frac{1}{2}\|\mathbf{h}\|^2 \le \frac{1}{2}\tau\right\}$$

Note that the Lagrangian of this constrained problem is given by

$$\mathcal{L}(\mathbf{h},\tau,\nu) = \mathbf{g}^\top\mathbf{h} + \frac{1}{2}\mathbf{h}^\top\mathbf{H}\mathbf{h} + \frac{M}{6}|\tau|^{3/2} + \nu\left(\frac{M}{4}\|\mathbf{h}\|^2 - \frac{M}{4}\tau\right)$$

It is also useful to derive its dual function which is

$$\psi(\nu) = \inf_{\mathbf{h}\in\mathbb{R}^n} \min_{\tau\in\mathbb{R}} \left\{ \mathbf{g}^\top \mathbf{h} + \frac{1}{2}\mathbf{h}^\top \mathbf{H}\mathbf{h} + \frac{M}{6}|\tau|^{3/2} + \nu\left(\frac{M}{4}\|\mathbf{h}\|^2 - \frac{M}{4}\tau\right)\right\}$$

It can be easily verified that the optimal value of τ satisfies

$$\frac{M}{4}\tau^{*1/2}\mathrm{sign}(\tau) = \frac{M}{4}\nu \qquad \Rightarrow \qquad \tau^*(\nu) = \nu|\nu|$$

We will use the above discussion to show that the subproblem can be solved efficiently

3 Main Argument

Let's define

$$v_u(\mathbf{h}) = \mathbf{g}^\top \mathbf{h} + \frac{1}{2}\mathbf{h}^\top \mathbf{H}\mathbf{h} + \frac{M}{6}\|\mathbf{h}\|^3$$

and

$$v_l(\nu) = -\frac{1}{2}\mathbf{g}^\top \left(\mathbf{H} + \nu\frac{M}{2}\mathbf{I}\right)^{-1}\mathbf{g} - \frac{M}{12}|\nu|^3$$

Now we proceed to show that the optimal value of these two functions are the same.

$$\begin{aligned}
v_u^* &= \min_{\mathbf{h}\in\mathbb{R}^n} \left[\mathbf{g}^\top \mathbf{h} + \frac{1}{2}\mathbf{h}^\top \mathbf{H}\mathbf{h} + \frac{M}{6}\|\mathbf{h}\|^3\right] \\
&= \min_{\mathbf{h}\in\mathbb{R}^n} \min_{\tau\in\mathbb{R}} \max_{\nu\geq 0} \left\{ \mathbf{g}^\top \mathbf{h} + \frac{1}{2}\mathbf{h}^\top \mathbf{H}\mathbf{h} + \frac{M}{6}|\tau|^{3/2} + \nu\left(\frac{M}{4}\|\mathbf{h}\|^2 - \frac{M}{4}\tau\right)\right\} \\
&\geq \max_{\nu\geq 0} \min_{\mathbf{h}\in\mathbb{R}^n} \min_{\tau\in\mathbb{R}} \left\{ \mathbf{g}^\top \mathbf{h} + \frac{1}{2}\mathbf{h}^\top \mathbf{H}\mathbf{h} + \frac{M}{6}|\tau|^{3/2} + \nu\left(\frac{M}{4}\|\mathbf{h}\|^2 - \frac{M}{4}\tau\right)\right\} \\
&\geq \max_{\nu\geq 0, \mathbf{H}+\frac{M}{2}\nu\mathbf{I}\succ\mathbf{0}} \min_{\mathbf{h}\in\mathbb{R}^n} \min_{\tau\in\mathbb{R}} \left\{ \mathbf{g}^\top \mathbf{h} + \frac{1}{2}\mathbf{h}^\top \mathbf{H}\mathbf{h} + \frac{M}{6}|\tau|^{3/2} + \nu\left(\frac{M}{4}\|\mathbf{h}\|^2 - \frac{M}{4}\tau\right)\right\} \\
&= \max_{\nu\geq 0, \mathbf{H}+\frac{M}{2}\nu\mathbf{I}\succ\mathbf{0}} \min_{\mathbf{h}\in\mathbb{R}^n} \min_{\tau\in\mathbb{R}} \left\{ \mathbf{g}^\top \mathbf{h} + \frac{1}{2}\mathbf{h}^\top \mathbf{H}\mathbf{h} + \frac{M}{6}|\nu|^3 + \nu\frac{M}{4}\|\mathbf{h}\|^2 - |\nu|^3\frac{M}{4}\right\} \\
&= \max_{\nu\geq 0, \mathbf{H}+\frac{M}{2}\nu\mathbf{I}\succ\mathbf{0}} \min_{\mathbf{h}\in\mathbb{R}^n} \left\{ \mathbf{g}^\top \mathbf{h} + \frac{1}{2}\mathbf{h}^\top \left(\mathbf{H} + \nu\frac{M}{2}\mathbf{I}\right)\mathbf{h} - \frac{M}{12}|\nu|^3\right\} \\
&= \max_{\nu\geq 0, \mathbf{H}+\frac{M}{2}\nu\mathbf{I}\succ\mathbf{0}} \left\{ -\frac{1}{2}\mathbf{g}^\top \left(\mathbf{H} + \nu\frac{M}{2}\mathbf{I}\right)^{-1}\mathbf{g} - \frac{M}{12}|\nu|^3\right\} \\
&= v_l^*
\end{aligned}$$

Now consider direction $\mathbf{h}(\nu) = -\left(\mathbf{H} + \nu\frac{M}{2}\mathbf{I}\right)^{-1}\mathbf{g}$ and note that for any $\nu \in \mathcal{D}$ we have

$$\mathbf{g} = -\mathbf{H}\mathbf{h}(\nu) - \frac{M}{2}\nu\mathbf{h}(\nu)$$

Therefore, we have

$$
\begin{aligned}
v_u(\mathbf{h}(\nu)) &= \mathbf{g}^\top \mathbf{h}(\nu) + \frac{1}{2}\mathbf{h}(\nu)^\top \mathbf{H}\mathbf{h}(\nu) + \frac{M}{6}\|\mathbf{h}(\nu)\|^3 \\
&= -\frac{1}{2}\mathbf{h}(\nu)^\top \mathbf{H}\mathbf{h}(\nu) - \frac{M}{2}\nu\|\mathbf{h}(\nu)\|^2 + \frac{M}{6}\|\mathbf{h}(\nu)\|^3 \\
&= -\frac{1}{2}\mathbf{h}(\nu)^\top \left(\mathbf{H} + \frac{M}{2}\nu\right)\mathbf{h}(\nu) - \frac{M}{4}\nu\|\mathbf{h}(\nu)\|^2 + \frac{M}{6}\|\mathbf{h}(\nu)\|^3 \\
&= v_l(\nu) + \frac{M}{12}|\nu|^3 - \frac{M}{4}\nu\|\mathbf{h}(\nu)\|^2 + \frac{M}{6}\|\mathbf{h}(\nu)\|^3 \\
&= v_l(\nu) + \frac{M}{12}(\nu + 2\|\mathbf{h}(\nu)\|)(\|\mathbf{h}(\nu)\| - \nu)^2
\end{aligned}
$$

From this expression we obtain that

$$
0 \leq v_u(\mathbf{h}(\nu)) - v_l(\nu) = \frac{M}{12}(\nu + 2\|\mathbf{h}(\nu)\|)(\|\mathbf{h}(\nu)\| - \nu)^2 = \frac{4}{3M}\frac{\nu + 2\|\mathbf{h}(\nu)\|}{(\nu + \|\mathbf{h}(\nu)\|)^2} v_l'(\nu)
$$

where it follows from the fact that

$$
v_l'(\nu) = \frac{M}{4}(\|\mathbf{h}(\nu)\|^2 - \nu^2)
$$

Now if the optimal value v_l^* is attained at $\nu^* > 0$ form $\mathcal{D}$ then $v_l'(\nu^*) = 0$ and hence $v_r^* = v_l^*$. If $\nu^* = \frac{2}{M}(-\lambda_{min}(H))_+$, then using a continuity argument the main claim can be justified. We will discuss it further in the next section.

4 Implementation Details

The result in the previous section implies that in non-degenerate situation the solution of the subproblem can be found from one-dimensional equation

$$
\nu = \left\| \left(\mathbf{H} + \frac{M}{2}\nu\mathbf{I}\right)^{-1} \mathbf{g} \right\|, \qquad \nu \geq \frac{2}{M}(-\lambda_{min}(H))_+
$$

This problem can be solved using a trust region method. For now let's focus on the structure of the solution. For the sake of argument, let's focus on the case that the Hessian is diagonal.
In this case, if we define $\mathbf{g}_n$ is the coordinate that corresponds to the smallest eigenvalue of the Hessian then we have

$$
\nu^2 = \sum_{i=1}^{n} \frac{g_i^2}{(\lambda_i + \frac{M}{2}\nu)^2}, \qquad \nu \geq \frac{2}{M}(-\lambda_n)_+
$$

If $g_n^2 \neq 0$ then the solution belongs to the interior of the second condition and the optimal solution of the original problem can be obtained using

$$
\mathbf{h}^* = \left(\mathbf{H} + \frac{M}{2}\nu^*\mathbf{I}\right)^{-1} \mathbf{g}
$$

If $g_n^2 = 0$, then this argument does not work and we need to use a perturbation idea. Basically we need to solve the problem for $\hat{\mathbf{g}} = \mathbf{g} + \epsilon\mathbf{e}_n$.

5 Convex Constrained Case

Note that

$$\frac{1}{3}r^3 = \max_{\tau \geq 0}[r^2\tau - \frac{2}{3}\tau^{3/2}]$$

In the convex constrained setting we simply use the following argument

$$\begin{aligned}
&\min_{\mathbf{y}\in\mathcal{Q}} \left[\mathbf{g}_k^\top(\mathbf{y}-\mathbf{x}_k) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^\top \mathbf{H}_k(\mathbf{y}-\mathbf{x}_k) + \frac{M}{6}\|\mathbf{y}-\mathbf{x}_k\|^3\right] \\
&= \min_{\mathbf{y}\in\mathcal{Q}} \max_{\tau\geq 0} \left[\mathbf{g}_k^\top(\mathbf{y}-\mathbf{x}_k) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^\top \mathbf{H}_k(\mathbf{y}-\mathbf{x}_k) + \frac{M}{2}\left(\tau\|\mathbf{y}-\mathbf{x}_k\|^2 - \frac{2}{3}\tau^{3/2}\right)\right] \\
&= \max_{\tau\geq 0} \min_{\mathbf{y}\in\mathcal{Q}} \left[\mathbf{g}_k^\top(\mathbf{y}-\mathbf{x}_k) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^\top \mathbf{H}_k(\mathbf{y}-\mathbf{x}_k) + \frac{M}{2}\left(\tau\|\mathbf{y}-\mathbf{x}_k\|^2 - \frac{2}{3}\tau^{3/2}\right)\right] \\
&= \max_{\tau\geq 0} \left[-\frac{M}{3}\tau^{3/2} + \min_{\mathbf{y}\in\mathcal{Q}} \left[\mathbf{g}_k^\top(\mathbf{y}-\mathbf{x}_k) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^\top(\mathbf{H}_k + \tau M)(\mathbf{y}-\mathbf{x}_k)\right]\right]
\end{aligned}$$

Note that the minimization problem is a convex quadratic minimization problem, which very often can be solved efficiently. On the upper level, we have a problem of maximizing a concave univariate function which is also simple to solve.