The University of Texas at Austin
Department of Electrical and Computer Engineering

ECE 381V: Large-Scale Optimization II — Spring 2022

LECTURE 25

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Goal: In this lecture, we study the convergence properties of Cubic Regularization of Newton's method for convex and strongly convex problems.

1 From last lecture

Suppose we aim to solve the following unconstrained problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$$

Consider an open convex set $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathbb{R}^n$.
We first formally define the only assumption that we need to define the CRN method.

Assumption 1. *f is twice differentiable and its Hessian is L_2-Lipschitz continuous on $\mathcal{F}$, i.e.,*

$$\|\nabla^2 f(\mathbf{x}) - \nabla^2 f(\hat{\mathbf{x}})\|_2 \leq L_2\|\mathbf{x}-\hat{\mathbf{x}}\|, \qquad \text{for all} \quad \mathbf{x},\hat{\mathbf{x}} \in \mathcal{F}$$

Further, suppose we are given an arbitrary initial point $\mathbf{x}_0 \in \mathcal{F}$ with function value $f(\mathbf{x}_0)$. We define it corresponding sublevel set as

$$\mathcal{S}(f(\mathbf{x}_0)) := \{\mathbf{x}\in\mathbb{R}^n \mid f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$$

All we need that the convex set $\mathcal{F}$ is large enough that contains the sublevel set $\mathcal{S}(f(\mathbf{x}_0))$.

2 Cubic update

The main update of the CRN method requires solving the following cubic problem

$$\min_{\mathbf{y}} \phi_M(\mathbf{y};\mathbf{x}_k) = \min_{\mathbf{y}} \ \nabla f(\mathbf{x}_k)^\top(\mathbf{y}-\mathbf{x}_k) + \frac{1}{2}(\mathbf{y}-\mathbf{x}_k)^\top\nabla^2 f(\mathbf{x}_k)(\mathbf{y}-\mathbf{x}_k) + \frac{M}{6}\|\mathbf{y}-\mathbf{x}_k\|^3$$

Denote $T_M(\mathbf{x}_k)$ as an optimal solution of this problem for the case that cubic parameter is M.

$$f_M(\mathbf{x}) := \min_{\mathbf{y}} \ \left\{ f(\mathbf{x}) + \nabla f(\mathbf{x})^\top(\mathbf{y}-\mathbf{x}) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^\top\nabla^2 f(\mathbf{x})(\mathbf{y}-\mathbf{x}) + \frac{M}{6}\|\mathbf{y}-\mathbf{x}\|^3 \right\}$$

Initialization step: Select $\mathbf{x}_0$ and L_0 that is smaller than L_2 and set $M_0 = L_0$

At step k:

- Set $M = M_k$
- Solve the cubic problem with M and find $T_M(\mathbf{x}_k)$
- If $f(T_M(\mathbf{x}_k)) \leq f_M(\mathbf{x}_k)$ then set $\mathbf{x}_{k+1} = T_M(\mathbf{x}_k)$ and go to step $k+1$
- If $f(T_M(\mathbf{x}_k)) > f_M(\mathbf{x}_k)$ then set $M_k = 2M_k$ and go back to the second step

3 Convergence for Star-Convex functions

We state that a function is *star-convex* if f its set of global minimums is nonempty and for any optimal solution $\mathbf{x}^*$ we have

$$f(\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}) \leq \lambda f(\mathbf{x}^*) + (1-\lambda)f(\mathbf{x})$$

for any $\mathbf{x} \in \mathcal{F}$ and any $\lambda \in [0,1]$.
A particular example of a star-convex function is a usual convex function, but there are some nonconvex functions that are star-convex such as $f(\mathbf{x}) = f(x_1, x_2) = x_1^2 + x_2^2 + x_1^2 x_2^2$.
To prove the main result we first prove the following intermediate lemma.

Lemma 1. *For the iterates of CRN when Assumption 1 holds we have*

$$f_M(\mathbf{x}) \leq \min_{\mathbf{y}\in\mathcal{F}} \left\{ f(\mathbf{y}) + \frac{L_2 + M}{6}\|\mathbf{y}-\mathbf{x}\|^3 \right\}$$

Proof. Note that for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{F}$ we have

$$f(\mathbf{x}) + \nabla f(\mathbf{x})^\top(\mathbf{y}-\mathbf{x}) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^\top\nabla^2 f(\mathbf{x})(\mathbf{y}-\mathbf{x}) \leq f(\mathbf{y}) + \frac{L_2}{6}\|\mathbf{y}-\mathbf{x}\|^3$$

By adding $\frac{M}{6}\|\mathbf{y}-\mathbf{x}\|^3$ to both sides and computing the minimum with respect to $\mathbf{y}$ the claim follows. $\square$

The iterates generated by CRN are monotonically decreasing the objective hence they all belong to the set $\mathcal{S}(f(\mathbf{x}_0)) := \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$. For ease of notation we use D as the diameter of this set.

Theorem 1. *Suppose the objective function f is star-convex and also the condition in Assumption 1 holds. Then the iterates of CRN satisfy the following condition:*

- *If $f(\mathbf{x}_0) - f^* \geq \frac{3}{2}L_2D^3$, then $f(\mathbf{x}_1) - f^* \leq \frac{1}{2}L_2D^3$*
- *If $f(\mathbf{x}_0) - f^* \leq \frac{3}{2}L_2D^3$, then the rate of convergence is*

$$f(\mathbf{x}_k) - f^* \leq \frac{3L_2D^3}{2(1+\frac{1}{3}k)^2}$$

Proof. Considering the fact that $f(\mathbf{x}_{k+1}) \leq f_M(\mathbf{x}_k)$ and the result in Lemma 1 we have

$$f(\mathbf{x}_{k+1}) \leq f_{M_k}(\mathbf{x}_k) \leq \min_{\mathbf{y}\in\mathcal{F}} \left\{ f(\mathbf{y}) + \frac{L_2 + M_k}{6}\|\mathbf{y}-\mathbf{x}_k\|^3 \right\}$$

Hence, using the fact that $M_k \leq 2L_2$ we have

$$\begin{aligned}
f(\mathbf{x}_{k+1}) - f^* &\leq \min_{\mathbf{y}\in\mathcal{F}} \left\{ f(\mathbf{y}) - f^* + \frac{L_2}{2}\|\mathbf{y}-\mathbf{x}_k\|^3 \right\} \\
&\leq \min_{\mathbf{y}} \left\{ f(\mathbf{y}) - f^* + \frac{L_2}{2}\|\mathbf{y}-\mathbf{x}_k\|^3; \quad \mathbf{y} = \alpha\mathbf{x}^* + (1-\alpha)\mathbf{x}_k,\ \alpha \in [0,1] \right\} \\
&\leq \min_{\alpha} \left\{ \alpha f^* + (1-\alpha)f(\mathbf{x}_k) - f^* + \frac{L_2}{2}\|\alpha\mathbf{x}^* + (1-\alpha)\mathbf{x}_k - \mathbf{x}_k\|^3 \right\} \\
&= \min_{\alpha} \left\{ f(\mathbf{x}_k) - f^* - \alpha(f(\mathbf{x}_k) - f^*) + \alpha^3\frac{L_2}{2}\|\mathbf{x}^* - \mathbf{x}_k\|^3 \right\} \\
&\leq \min_{\alpha} \left\{ f(\mathbf{x}_k) - f^* - \alpha(f(\mathbf{x}_k) - f^*) + \alpha^3\frac{L_2D^3}{2} \right\}
\end{aligned}$$

where the second inequality follows from the fact that $\{\mathbf{y} \in \mathcal{F}\} \supseteq \{\mathbf{y}|\mathbf{y} = \alpha\mathbf{x}^*+(1-\alpha)\mathbf{x}_k, \alpha \in [0,1]\}$, and the third inequality follows from the definition of star-convexity. It can be easily verified that the global minimizer of the right hand-side for $\alpha \geq 0$ is

$$\alpha_k = \sqrt{\frac{2(f(\mathbf{x}_k) - f^*)}{3L_2D^3}}$$

If $\alpha_k \geq 1$, then the optimal value corresponds to $\alpha = 1$. Hence, if $f(\mathbf{x}_k) - f^*) \geq \frac{3}{2}L_2D^3$, then

$$f(\mathbf{x}_{k+1}) - f^* \leq \frac{1}{2}L_2D^3$$

and the first claim follows. Note that the process is monotonically decreasing hence this only happens at the first step, if happens.

Now if $f(\mathbf{x}_k) - f^*) \leq \frac{3}{2}L_2D^3$, which implies that $\alpha_k \leq 1$, then we have

$$f(\mathbf{x}_{k+1}) - f^* \leq f(\mathbf{x}_k) - f^* - \left[\frac{2(f(\mathbf{x}_k) - f^*)}{3}\right]^{3/2} \frac{1}{\sqrt{L_2D^3}}$$

which leads to

$$\frac{2(f(\mathbf{x}_{k+1}) - f^*)}{3L_2D^3} \leq \frac{2f(\mathbf{x}_k) - f^*}{3L_2D^3} - \left[\frac{2(f(\mathbf{x}_k) - f^*)}{3}\right]^{3/2} \frac{2}{3(L_2D^3)^{3/2}}$$

which is equivalent to

$$\alpha_{k+1}^2 \leq \alpha_k^2 - \frac{2}{3}\alpha_k^3 \quad \Longleftrightarrow \quad \frac{\alpha_k^2 - \alpha_{k+1}^2}{2\alpha_k^3} \geq \frac{1}{3}$$

Note that it also implies that $\alpha_{k+1}^2 < \alpha_k^2$ Moreover, we have

$$\frac{1}{\alpha_{k+1}} - \frac{1}{\alpha_k} = \frac{\alpha_k - \alpha_{k+1}}{\alpha_k\alpha_{k+1}} = \frac{\alpha_k^2 - \alpha_{k+1}^2}{\alpha_k\alpha_{k+1}(\alpha_k + \alpha_{k+1})} \geq \frac{\alpha_k^2 - \alpha_{k+1}^2}{2\alpha_k^3} \geq \frac{1}{3}$$

Hence,

$$\frac{1}{\alpha_k} \geq \frac{1}{\alpha_0} + \frac{k}{3} \geq 1 + \frac{k}{3} \quad \Rightarrow \quad \alpha_k \leq \frac{1}{1+\frac{k}{3}} \quad \Rightarrow \quad \sqrt{\frac{2(f(\mathbf{x}_k) - f^*)}{3L_2D^3}} \leq \frac{1}{1+\frac{k}{3}}$$

and the second claim follows. $\square$

4 Convergence for functions with non-degenerate global minimum

Next we introduce the notion of non-degenerate global minimum for star-convex functions.

Definition 1. *The optimal solution of the star-convex function f is globally non-degenerate, if for any $\mathbf{x}^*$ in the optimal solution set we have*

$$f(\mathbf{x}) - f^* \geq \frac{\mu}{2}\|\mathbf{x} - \mathbf{x}^*\|^2$$

Indeed, strongly convex functions are a special case of the above definition, but the above definition also includes nonconvex functions such as $f(\mathbf{x}) = (\|\mathbf{x}\|^2 - 1)^2$

Theorem 2. *Suppose the objective function f is star-convex and its optimal solution set is μ-globally non-degenerate. Further, suppose Assumption 1 holds and consider the definition*

$$\omega = \frac{\mu^3}{8L_2^2}$$

Then the iterates of CRN satisfy the following condition:

- *If $f(\mathbf{x}_0) - f^* \geq \frac{4}{9}\omega$, then at the first phase of the process we have*

$$f(\mathbf{x}_k) - f^* \leq \left[(f(\mathbf{x}_0) - f^*)^{1/4} - \frac{k}{6}\sqrt{\frac{2}{3}}\ \omega^{1/4} \right]^4$$

  This phase is terminated as soon as $f(\mathbf{x}_k) - f^* \leq \frac{4}{9}\omega$, for some $k_0 \geq 0$.

- *For $k \geq k_0$, the sequence converges superlinearly:*

$$f(\mathbf{x}_{k+1}) - f^* \leq \frac{1}{2}(f(\mathbf{x}_k) - f^*)\sqrt{\frac{f(\mathbf{x}_k) - f^*}{\omega}}$$

Proof. If we assume that $\mathbf{x}_k^*$ is the projection of $\mathbf{x}_k$ onto the set of optimal solutions then we have

$$\begin{aligned} f(\mathbf{x}_{k+1}) - f^* &\leq \min_{\alpha \in [0,1]} \left\{ f(\mathbf{x}_k) - f^* - \alpha(f(\mathbf{x}_k) - f^*) + \alpha^3 \frac{L_2}{2} \|\mathbf{x}^* - \mathbf{x}_k\|^3 \right\} \\ &\leq \min_{\alpha \in [0,1]} \left\{ f(\mathbf{x}_k) - f^* - \alpha(f(\mathbf{x}_k) - f^*) + \alpha^3 \frac{L_2}{2} \left(\frac{2}{\mu}(f(\mathbf{x}_k) - f^*) \right)^{3/2} \right\} \end{aligned}$$

Now define $\Delta_k := \frac{f(\mathbf{x}_k) - f^*}{\omega}$ to obtain

$$\Delta_{k+1} \leq \min_{\alpha \in [0,1]} \left\{ \Delta_k - \alpha\Delta_k + \frac{1}{2}\alpha^3 \Delta_k^{3/2} \right\}$$

Note that the optimal value of α is obtained at

$$\alpha_k = \sqrt{\frac{2}{3}\Delta_k^{-1/2}}$$

Now if we assume that $f(\mathbf{x}_0) - f^* \geq \frac{4}{9}\omega$ which is equivalent to $\Delta_k \geq \frac{4}{9}$, we obtain that $\alpha_k \leq 1$ and hence we have

$$\Delta_{k+1} \leq \Delta_k - \left(\frac{2}{3}\right)^{3/2} \Delta_k^{3/4}$$

which leads to the following recursion

$$u_{k+1} \leq u_k - \frac{2}{3}u_k^{3/4}$$

where $u_k := \frac{9}{4}\Delta_k$.
Now note that in the regime where $\Delta_k \geq \frac{4}{9}$, we have $u_k \geq 1$. Note that the right hand side of the above inequality is increasing for $u_k \geq 1/16$ and using this fact we can show by induction that

$$u_k \leq \left(u_0^{1/4} - \frac{k}{6} \right)^4$$

Indeed, it holds for $k = 0$. Now suppose it holds for k. Then we have

$$u_{k+1} \le u_k - \frac{2}{3}u_k^{3/4} \le \left(u_0^{1/4} - \frac{k}{6}\right)^4 - \frac{2}{3}\left(\left(u_0^{1/4} - \frac{k}{6}\right)^4\right)^{3/4}$$

all we need to show is that

$$\left(u_0^{1/4} - \frac{k}{6}\right)^4 - \frac{2}{3}\left(u_0^{1/4} - \frac{k}{6}\right)^3 \le \left(u_0^{1/4} - \frac{k+1}{6}\right)^4$$

which is equivalent to

$$\begin{aligned}
&\frac{2}{3}\left(u_0^{1/4} - \frac{k}{6}\right)^3 \\
&\ge \left(u_0^{1/4} - \frac{k}{6}\right)^4 - \left(u_0^{1/4} - \frac{k+1}{6}\right)^4 \\
&= \frac{1}{6}\left[\left(u_0^{1/4} - \frac{k}{6}\right)^3 + \left(u_0^{1/4} - \frac{k}{6}\right)^2\left(u_0^{1/4} - \frac{k+1}{6}\right) + \left(u_0^{1/4} - \frac{k}{6}\right)\left(u_0^{1/4} - \frac{k+1}{6}\right)^2 + \left(u_0^{1/4} - \frac{k+1}{6}\right)^3\right]
\end{aligned}$$

which is indeed true. Hence, by induction we showed that for $u_k \ge 1$ we have

$$u_k \le \left(u_0^{1/4} - \frac{k}{6}\right)^4$$

which leads to the first claim.

Now we reach k_0 for which $\Delta_k \le \frac{4}{9}$, then the optimal value of α is $\alpha_k = 1$ for which we have

$$\Delta_{k+1} \le \frac{1}{2}\Delta_k^{3/2}$$

and the second claim follows. □

4.1 Overall complexity

Define

$$\zeta := \frac{L_2 D}{\mu}.$$

For the worst case analysis we assume that $\zeta \ge 1$. We divide our analysis into four phases

- After at most 1 iteration we can ensure that $f(\mathbf{x}) - f^* \le \frac{1}{2}L_2D^3$. Hence, $k_1 = 1$.
- The second phase corresponds to the case that the objective function error is smaller than $\frac{3}{2}L_2D^3$, but larger than $\frac{3}{2}\mu D^2$. In this case, we use the second case of Theorem 1 to show that the number of iterations is at most $k_2 = 3\sqrt{\zeta}$
- The third phase corresponds to the case that the objective function error is smaller than $\frac{3}{2}\mu D^2$, but larger than $\frac{1}{18}\mu^3 L_2^2$. In this case, we use the first case of Theorem 2 to show that the number of iterations is at most k_3 such that

$$\left(\frac{1}{18}\mu^3 L_2^2\right)^{1/4} \le \left(\frac{3}{2}\mu D^2\right)^{1/4} - \frac{k_3}{6}\sqrt{\frac{2}{3}}\left(\frac{\mu^3}{8L_2^2}\right)^{1/4} \iff k_3 \le 3.25\sqrt{\zeta}$$

- The fourth phase corresponds to the case that the objective function error is smaller than $\frac{1}{18}\mu^3 L_2^2 = \frac{4}{9}\omega$. In this case, we use the second case of Theorem 2 to show that the number of iterations is at most k_4 which satisfies

$$\delta_{k+1} \leq \delta_k^{3/2}$$

where $\delta_k := 9\frac{f(\mathbf{x}_k)-f^*}{16\omega}$. In this case $\delta_k \leq 1/4$, hence we have $k_4 \leq \log_{3/2} \log_4 \frac{2\mu^3}{9\epsilon L_2^2}$

Hence the overall number of iterations is at most

$$N \leq 1 + 6.25\sqrt{\frac{L_2 D}{\mu}} + \log_{3/2} \log_4 \frac{2\mu^3}{9L_2^2} + \log_{3/2} \log_4 \frac{1}{\epsilon}$$