# Bretagnolle-Huber Inequality and KL Divergence.

Sources: 1. Chap 14, Bandit Algorithms, textbook
2. Intro. to non-parametric estimation, A. L. Tsybakov, Springer 2008

**Defn:** (KL Divergence | Relative Entropy):

$P, Q$ prob. measures over a sample space $(\Omega, \mathcal{F})$. Suppose $\exists$ $\sigma$-finite measure $\lambda$ s.t. $P, Q$ are both absolutely continuous wrt $\lambda$, i.e., for any $B \in \mathcal{F}$, $\lambda(B) = 0 \implies P(B) = Q(B) = 0$.

Let $p = \frac{dP}{d\lambda}$, $q = \frac{dQ}{d\lambda}$

Radon-Nikodym derivative

Then, $D(P,Q) = KL(P,Q) = \int p \ln\left(\frac{p}{q}\right) d\lambda.$

Special Cases:

(1) $P, Q$ over common finite alphabet $\{x_1, x_2, \cdots, x_N\}$.

$$D(P,Q) = \sum_i p_i \ln\left(\frac{p_i}{q_i}\right),$$

where $p_i = P(Z = x_i)$, $q_i = Q(Z = x_i)$, $i = 1, 2, \cdots, N$.

(2) $P$ abs. cont. wrt $Q$:

$$D(P,Q) = \int \ln\left(\frac{dP}{dQ}(\omega)\right) dP(\omega)$$

(3) $D\left(N(\mu_1, \sigma_1^2), N(\mu_2, \sigma_2^2)\right)$

$$= \frac{\sigma_1^2 + (\mu_1 - \mu_2)^2}{2\sigma_2^2} + \ln\left(\frac{\sigma_2}{\sigma_1}\right) - \frac{1}{2}$$

(4) $D\left(\text{Bernoulli}(p), \text{Bernoulli}(q)\right)$, $p, q \in [0,1]$.

$$= p \ln\left(\frac{p}{q}\right) + (1-p) \ln\left(\frac{1-p}{1-q}\right)$$

**Thm (14.2 in text) Bretagnolle - Huber Inequality:**

$P, Q$ prob. measures over $(\Omega, \mathcal{F})$. Further, let $A \in \mathcal{F}$ be an event. Then:

$$P(A) + Q(A^c) \geq \frac{1}{2} e^{-\min\{D(P,Q), D(Q,P)\}}.$$

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**Note:** We will prove $P(A) + Q(A^c) \geq \frac{1}{2} e^{-D(P,Q)}$, for any $P, Q$, $A \in \mathcal{F}$. Thus,

$$Q(B) + P(B^c) \geq \frac{1}{2} e^{-D(Q,P)}.$$

Choose $B = A^c$.

$$\Rightarrow \quad P(A) + Q(A^c) \geq \frac{1}{2} e^{-D(Q,P)}$$

$$\Rightarrow \quad P(A) + Q(A^c) \geq \frac{1}{2} e^{-\min\{D(P,Q), D(Q,P)\}}.$$

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**Proof:** Recall $D(P,Q) = \int p \ln\left(\frac{p}{q}\right) d\lambda$, where

$$p = \frac{dP}{d\lambda}, \quad q = \frac{dQ}{d\lambda}$$

e.g. density functions if $\lambda$ = Borel measure.

notation: $a \wedge b = \min\{a,b\}$

$a \vee b = \max\{a,b\}$

(A): $P(A) + Q(A^c) \geq \int_\Omega p \wedge q$

Pf:

$$\int p \wedge q = \int_A p \wedge q + \int_{A^c} p \wedge q$$

$\int_\Omega p \wedge q \, d\lambda$

$$\leq \int_A p + \int_{A^c} q = P(A) + Q(A^c)$$

$\square$

(B): Le Cam's inequality: $\int_\Omega p \wedge q \geq \frac{1}{2}\left(\int_\Omega \sqrt{pq}\right)^2$

Pf:

$$\left(\int \sqrt{pq}\right)^2 = \left(\int \sqrt{(p \wedge q)(p \vee q)}\right)^2$$

$pq = (p \wedge q)(p \vee q)$

$$\leq \left(\int (p \wedge q)\right)\left(\int (p \vee q)\right)$$

Cauchy-Schwartz ineq.

$$= \left(\int (p\wedge q)\right)\left(\int p+q-(p\wedge q)\right)$$

$(p\wedge q)+(p\vee q)$
$= p+q$

$$= \left(\int (p\wedge q)\right)\left(2-\underbrace{\int (p\wedge q)}_{\geq 0}\right)$$

$\int p = 1$
$\int q = 1$

$$\leq 2\int (p\wedge q). \qquad \blacksquare$$

(C) $$\left(\int\sqrt{pq}\right)^2 \geq e^{-D(P,Q)}$$

Pf:

$$\left(\int\sqrt{pq}\right)^2 = e^{2\ln\int\sqrt{pq}}$$

$$= e^{2\ln\int p\sqrt{\frac{q}{p}}}$$

Jensen's inequality.

$$\geq e^{2\int p\ln\left(\sqrt{q/p}\right)}$$

use abs. cont. (read text, sec. 14.2)

$$= e^{-\int p\ln(p/q)}$$

$$= e^{-D(P,Q)} \qquad \blacksquare$$

Why is this useful? (see Chap 15 in textbook for formal argument)

Recall our running example: $\boxed{\Delta = \frac{1}{\sqrt{n}}}$

| System $1 = \nu$ | | System $2 = \nu'$ | |
|---|---|---|---|
| arm 1 | arm 2 | arm 1 | arm 2 |
| $N(\Delta, 1)$ | $N(0,1)$ | $N(\Delta, 1)$ | $N(2\Delta, 1)$ |

Fix some policy $\pi$, and let $\mathbb{P}$ be the probability measure induced by $(\pi, \nu)$ on the sequence $(A_1, X_1, A_2, X_2, \ldots, A_n, X_n)$.

Similarly, let $\mathbb{Q}$ be the prob. measure induced by $(\pi, \nu')$ on $(A_1, X_1, \ldots, A_n, X_n)$.

Since $\nu \approx \nu'$, it is plausible that $\mathbb{P} \approx \mathbb{Q}$, specifically, we will see later that:

$$D(\mathbb{P}, \mathbb{Q}) = \underbrace{E_\nu[T_2(n)]}_{\leq n} \left(\frac{\Delta^2}{2}\right) \cdot \leq \frac{n\Delta^2}{2}$$

Here: $E_\nu[T_2(n)]$ = Expected number of times arm 2 is played under system 1, i.e., under $(\pi, \nu)$.

Let $A = \chi_{\{T_1(n) \leq \frac{n}{2}\}}$

# of arm 1 plays.

A is a bad event under System 1 (good arm not played enough number of times)

$A^c = \chi_{\{T_1(n) > \frac{n}{2}\}}$

bad event under System 2 because arm 1 (which is bad in system 2) is played too many times.

B-H Inequality: $P(A) + Q(A^c) \geq e^{-D(P,Q)}$

$$\Rightarrow \quad \max\{P(A), Q(A^c)\} \geq \frac{e^{-D(P,Q)}}{2}.$$

$\Rightarrow$ Under at least one of the systems, we play the wrong arm for at least half the time.

$\Rightarrow$ Regret that scales $\geq \dfrac{\Delta \cdot n}{2} = \Theta(\sqrt{n})$

occurs for at least of the systems with prob. $\geq e^{-D(P,Q)}/2$.

But $e^{-D(P,Q)} \geq e^{-\frac{\Delta^2}{2} \underbrace{E_\nu[T_2(n)]}_{\leq n}}$.

$$\geq e^{-n\Delta^2/2} = e^{-1/2} = \Theta(1)$$

$$\Rightarrow \max\{R_n(\pi,\nu), R_n(\pi,\nu')\} \geq \Theta(\sqrt{n}) \cdot \Theta(1)$$

$$= \Theta(\sqrt{n}) \qquad \blacksquare$$

Formal Argument in Chap 15 in textbook; next lecture.