Lecture 1: Overview of Probability

1. Definition of prob. space

   $(\Omega, \mathcal{F}, \mathbb{P})$

2. Random variables.

3. pmf, pdf, $E[\cdot]$, $Var(\cdot)$

4. Conditional probability and conditional expectation

5. Concentrations | Chernoff Bound |

   Sub-Gaussian rvs. | MGF.

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* Module focuses on multiarmed bandits

Textbook: Bandit Algorithms, T. Lattimore and C. Czepesvári, Cambridge Univ. Press, 2019.

① Problem setting: explore vs. exploit

→ Online advertising
→ Recommendation systems
→ Clinical trials
→ Online resource allocation
→ Online search (e.g. MCTS)

② Setup:

$A_1, X_1, A_2, X_2, \dots$ ⟶ time.

action

reward

Key assumptions:

① Reward observed only corresponding to the action taken. (bandit feedback)

② Action does not alter the environment

③ Taking an action does not restrict future actions.

Distinguishes bandits from RL

Probability Review

$\boxed{\text{Probability triplet:}}$ $(\Omega, \mathcal{F}, \mathbb{P})$

Ω → sample space

$\mathcal{F}$ → σ-algebra (aka collection of events)

$(\Omega, \mathcal{F}, \mathbb{P})$ → prob. space

(Bertsekas & Tsitsiklis, Introduction to Probability Athena Scientific Press)

e.g.: ① Roll a pair of dies:

$$\Omega = \{ (1,1), (1,2), \ldots, (1,6), (2,1), \ldots \ldots, (6,6) \}$$

$$\mathcal{F} = \{ \phi, \Omega, \{(1,1)\}, \{(1,1),(1,2)\}, \{(1,1),(2,2)\}, \ldots \}$$

events are subsets of Ω

$\mathbb{P}: \mathcal{F} \longmapsto [0,1]$ the probability "measures" the "size" of an event.

Fair dies: $\mathbb{P}(\{(1,1)\}) = \frac{1}{36}$

$$\mathbb{P}(\{(1,1),(2,2),(3,3)\}) = \frac{3}{36}$$

$\boxed{\text{Random Variable:}}$ A map from outcomes of the experiment to the real line.

ex: Continuing roll of dies:

X = Sum of the two rolls.

X can take values: $\{2, 3, 4, \ldots \ldots, 12\}$.

$\boxed{\text{pmf}}$: Probability mass function

$$P_X(x) = P(X = x)$$

example: $P_X(2) = P(X=2) = \frac{1}{36}$

$$P_X(3) = P(X=3) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}$$

$\vdots$

$$P_X(12) = P(X=12) = \frac{1}{36}$$

$\boxed{\text{pdf:}}$ Probability density function.

Used to model continuous random variables

$$f_X(x) : \quad P(a < X \leq b) = \int_a^b f_X(x)\,dx.$$

$\boxed{\text{Expectation:}}$ (aka mean)

$$E[X] = \begin{cases} \displaystyle\int_{-\infty}^{\infty} x\, f_X(x)\,dx & \text{continuous.} \\ \displaystyle\sum_{x=-\infty}^{\infty} x\, p_X(x) & \text{discrete} \end{cases}$$

$\boxed{\text{Variance:}}$ $Var(X) = \sigma^2 = E\left[(X - E[X])^2\right].$

$\boxed{\text{Conditioning}}$ A, B events; with $P(B) > 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

(re-normalizing w.r.t the universe restricted to B)

$\boxed{\text{Joint probability}}$:

$$P_{X,Y}(x,y) = P(X=x, Y=y).$$

Independence: $P_{X,Y}(x,y) = P_X(x) P_Y(y).$

Notation: $\{X_1, X_2, \ldots, X_n\}$ a sequence of iid r.vs.

↳ independent and identically distributed

i.e., $P_{X_1, X_2, \ldots, X_n}(x_1, \ldots, x_n)$ product form → independence.

$$= \underbrace{P_{X_1}(x_1) \cdot P_{X_1}(x_2) \cdot P_{X_1}(x_3) \cdots P_{X_1}(x_n)}_{\text{identically dist: all same pmf}}$$

$\boxed{\text{Conditional Expectation}}$

$$E[X \mid A] = \sum_x x \, p_{X|A}(x \mid A)$$

$$E[X \mid Y = y] = \sum_x x \, p_{X|Y}(x \mid y)$$

where $p_{X|Y}(x \mid y) = \dfrac{p_{X,Y}(x, y)}{p_Y(y)}$.