Concentration Review

(Ref: Chap 5, Bandit Algorithms)

$\{X_1, X_2, \ldots, X_n, \ldots\}$ sequence of

iid r.vs → random variables

↳ independent, identically distributed

Law of large numbers:

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow E[X] \quad \text{a.s.}$$

Notation: $E[X] = \mu$, $Var(X) = \sigma^2$.

Question: With n samples, $\{X_1, \ldots, X_n\}$, how "close" is the empirical estimate of the mean to the true mean?

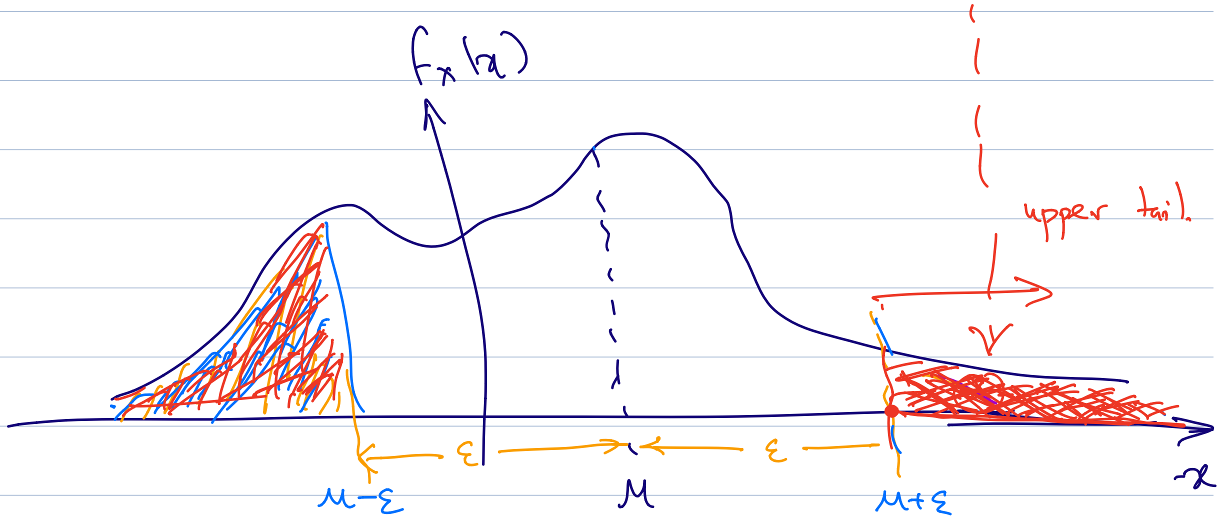
i.e., $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ a r.v. (empirical estimate)

$\mu = E[X]$ a fixed number.

$P(\hat{\mu}_n - \mu > \varepsilon) \leq \boxed{?}$ *

$P(|\hat{\mu}_n - \mu| > \varepsilon) \leq \boxed{?}$

tail probability



Why do we care: In bandit algorithms, past sample of rewards allows us to estimate "goodness" of actions. Concentrations permit us to quantify how much we trust these estimates.

Basic inequalities

Markov Inequality: $X \geq 0$ a non-negative r.v. and $\varepsilon > 0$. Then,

$$P(X \geq \varepsilon) \leq \frac{E[X]}{\varepsilon}$$

Chebyshev's Inequality: X a r.v. with $E[X] = \mu$, $Var(X) = \sigma^2$.

$$P\left(|X - \mu| > \varepsilon\right) \leq \frac{\sigma^2}{\varepsilon^2}.$$

Chernoff Bound: X a r.v. with $\underbrace{\text{MGF}}_{\text{Moment Generating Function}}$ $M_X(\theta)$,

$$M_X(\theta) = E\left[e^{\theta X}\right]$$

exists atleast for "small" θ

Then, $$P(X > \varepsilon) \leq e^{-I(\varepsilon)}$$ $\boxed{\varepsilon > E[X]}$

$$I(\varepsilon) = \max_{\theta} \left\{\varepsilon\theta - \ln M_X(\theta)\right\}.$$

$\hookrightarrow$ rate function. = Legendre-Fenchel Transform of "the log-MGF

Most often, we will simplify and assume noise terms are subgaussian

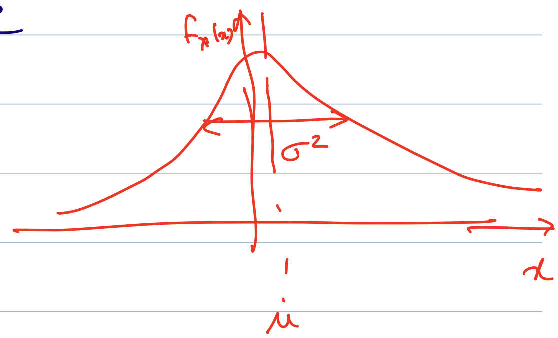
Recall: $X \sim N(\mu, \sigma^2) \longrightarrow$ Gaussian r.v. aka Normal r.v

$\hookrightarrow$ "distributed as"

"has the pdf of"

if

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$



For $Z \sim N(0, \sigma^2)$

$$\underbrace{E\left[e^{\theta Z}\right]} = e^{\theta^2\sigma^2/2}$$

$$\boxed{P(X > \mu + \varepsilon) \leq e^{-\varepsilon^2/2\sigma^2}}$$

MGF = Moment Generating Function

$Y \sim$ SubGaussian(σ) (also σ-subgaussian) if

ⓐ $E[Y] = 0$

ⓑ $E\left[e^{\theta Y}\right] \leq e^{\theta^2 \sigma^2 / 2}$

Properties $\quad Y \sim \text{subgaussian}(\sigma)$

① $P(Y \geq \varepsilon) \leq e^{-\varepsilon^2 / 2\sigma^2}$

② $P\left(Y \geq \sqrt{2\sigma^2 \ln(1/\delta)}\right) \leq \delta$, for any $\delta > 0$

③ $P\left(|Y| \geq \sqrt{2\sigma^2 \ln(2/\delta)}\right) \leq \delta$ for any $\delta > 0$.

④ $\text{Var}(Y) \leq \sigma^2$

⑤ $Z = cY \sim \text{subgaussian}(\overbrace{|c|\sigma}^{\text{std-dev}})$

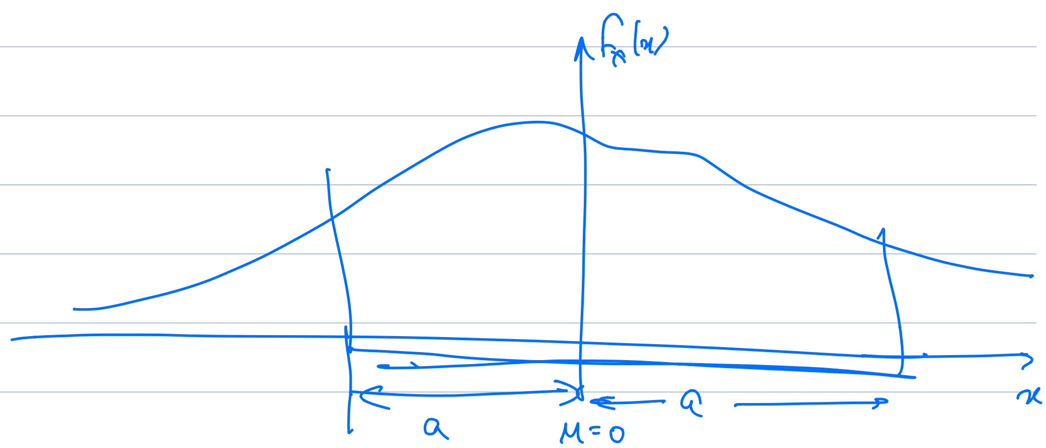
(Proof: See chapter 5, Bandit Algorithms).

⑥ $Z_1 \sim \sigma_1$ - subgaussian, $Z_2 \sim \sigma_2$ - subgaussian

$$(Z_1 + Z_2) \sim \sqrt{\sigma_1^2 + \sigma_2^2} \text{ - subgaussian.}$$

<u>Corollary:</u> $\{X_i\} \sim 1$ - subgaussian, iid

$$\frac{1}{m}\sum_{i=1}^{m} x_i \sim \frac{1}{\sqrt{m}} \text{ - subgaussian}$$



$$P\left(\underbrace{\frac{1}{m}\sum_{i=1}^{m} x_i}_{} > \varepsilon\right) \leq e^{\varepsilon^2 / 2\left(\frac{1}{\sqrt{m}}\right)^2}$$

$$= e^{-\varepsilon^2 / 2/m}$$

$$= \underline{e^{-(m\varepsilon^2/2)}}.$$