Regret Analysis – Take 2

Source: Chap 20, Bandit Algorithms, LAS (textbook).

Setting: Actions $A_1, A_2, \dots, A_t \in \mathbb{R}^d$, with corresponding rewards $X_1, \dots, X_t \in \mathbb{R}$ s.t.

$$X_i = \langle \theta^*, A_i \rangle + \eta_i$$

$\eta_i \rightarrow$ 1-subG noise

and

$$\theta^* = \arg\min_{\theta} \sum_{s=1}^{t} \left(X_s - \langle \theta, A_s \rangle\right)^2 + \lambda \|\theta\|_2^2$$

$\lambda \rightarrow \geq 0$, a regularizer

Solution: $\hat{\theta}_t = V_t^{-1}(\lambda) \sum_{s=1}^{t} X_s A_s$, and

$$V_t(\lambda) = \lambda I + \sum_{s=1}^{t} A_s A_s^T \qquad \left(\text{if } \lambda = 0,\ V_t(0) = V_t\right)$$

Last time, we assumed that: $\exists\ \delta \in (0,1)$ s.t. with prob $\geq (1-\delta)$, $\forall\, t \in \{1, 2, \dots, n\}$, $\theta^* \in C_t$.

This round of analysis, we will justify the above assumption. <u>However</u>, we will assume:

1. $\lambda = 0$, V_t^{-1} exists: Note that this implies $t \geq d$, and that the actions $\{A_1, A_2, \cdots, A_t\}$ span $\mathbb{R}^d$. From now on (in Take 2):

$$V_t = \sum_{s=1}^{t} A_s A_s^T$$

Note that we need to actually use $\lambda > 0$ to ensure that the algo. works even with $\{A_s\}$ don't span $\mathbb{R}^d$ or we don't have enough samples yet $(t < d)$.

2. $\{\eta_s, s = 1, 2, \cdots, t\}$, are <u>independent</u>, $\eta_s \sim$ 1-subG.

Note that in reality, the noise is conditionally subG:

$$E\left[e^{\phi \eta_s} \,\middle|\, H_t\right] \leq e^{\phi^2/2} \quad \text{a.s.}$$

where $H_t = \{A_1, X_1, \cdots, A_{s-1}, X_{s-1}, A_s\}$.

i.e., the noise could depend on the current action, and all past action/rewards. (e.g. Bernoulli noise).

③ $\{A_1, \dots, A_t\}$ are chosen as a block at $s=0$, i.e., the actions do not depend on the rewards.

Clearly, this is not true in a bandit setting. However, below analysis provides the key ideas.

Analysis:

We will show that under the above assumptions,

$$P\left(\|\hat{\theta}_t - \theta^*\|_{V_t} \geq 2\sqrt{2\left(d\ln(6) + \ln(1/\delta)\right)}\right) \leq \delta.$$

where $d =$ dimension, $\delta \in (0,1)$, $V_t = \sum_{s=1}^{t} A_s A_s^T$, and

$$\hat{\theta}_t = V_t^{-1} \sum_{s=1}^{t} A_s X_s$$

Step 1: Concentration along a fixed direction:

Choose any direction $x \in \mathbb{R}^d$. We have the following chain:

$$\langle \hat{\theta}_t - \theta^*, x \rangle = \left\langle x, V_t^{-1} \sum_{s=1}^{t} A_s X_s - \theta^* \right\rangle$$

$$= \left\langle x, V_t^{-1} \sum_{s=1}^{t} A_s \left(A_s^T \theta^* + \eta_s \right) - \theta^* \right\rangle$$

$$= \left\langle x, V_t^{-1} V_t \theta^* + V_t^{-1} \sum_{s=1}^{t} A_s \eta_s - \theta^* \right\rangle$$

$$= \left\langle x, V_t^{-1} \sum_{s=1}^{t} A_s \eta_s \right\rangle = \sum_{s=1}^{t} \langle x, V_t^{-1} A_s \rangle \eta_s .$$

Now, by assumption, $\{\eta_s$ indep., 1-subG$\}$, and $\{A_s\}$, V_t deterministic. Thus,

$$\sum_{s=1}^{t} \langle x, V_t^{-1} A_s \rangle \eta_s \sim \left(\sum_{s=1}^{t} \langle x, V_t^{-1} A_s \rangle^2 \right)^{1/2} \text{- subG}$$

(why: compose $\eta_1 \sim \sigma_1$-subG and $\eta_2 \sim \sigma_2$-subG

$$\Longrightarrow \quad \eta_1 + \eta_2 \sim \sqrt{\sigma_1^2 + \sigma_2^2} \text{- subG} \,).$$

Note above works because we have also assumed $\{A_s, s = 1, 2, \ldots, t\}$ are determistic, and do not depend on the noise samples. Thus, we have

$$\langle \hat{\theta}_t - \theta^*, x \rangle \sim \left(\sum_{s=1}^{t} \langle x, V_t^{-1} A_s \rangle^2 \right)^{1/2} \text{- subGaussian}$$

Recall Sub Gaussian concentration: If $Z \sim \sigma$-subG, then,

$$P\left(Z \geq \sqrt{2\sigma^2 \ln(1/\delta)}\right) \leq \delta$$

for any $\delta \in (0,1)$. Thus,

$$P\left(\langle \hat{\theta}_t - \theta^*, x\rangle \geq \sqrt{2\left(\sum_{s=1}^{t} \langle x, V_t^{-1} A_s\rangle^2\right) \ln(1/\delta)}\right) \leq \delta$$

Now,

$$\sum_{s=1}^{t} \langle x, V_t^{-1} A_s\rangle^2 = \sum_{s=1}^{t} (x^T V_t^{-1} A_s)(A_s^T V_t^{-1} x)$$

$$= x^T V_t^{-1} \underbrace{\left(\sum_{s=1}^{t} A_s A_s^T\right)}_{V_t} V_t^{-1} x = \|x\|^2_{V_t^{-1}}$$

$$\therefore \boxed{P\left(\langle \hat{\theta}_t - \theta^*, x\rangle \geq \sqrt{2\|x\|^2_{V_t^{-1}} \ln(1/\delta)}\right) \leq \delta} \quad (*)$$

Step 2: Bounding concentrations for $\|\hat{\theta}_t - \theta^*\|_{V_t}$ by uniform concentrations for projections.

$$\|\hat{\theta}_t - \theta^*\|^2_{V_t} = (\hat{\theta}_t - \theta^*)^T V_t (\hat{\theta}_t - \theta^*)$$

$$= \langle \hat{\theta}_t - \theta^*, V_t^{1/2} \cdot V_t^{1/2} (\hat{\theta}_t - \theta^*)\rangle .$$

$\therefore$

$$\|\hat{\theta}_t - \theta^*\|_{V_t} = \left\langle \hat{\theta}_t - \theta^*, V_t^{1/2} \cdot \underbrace{\left(\frac{V_t^{1/2}(\hat{\theta}_t - \theta^*)}{\|\hat{\theta}_t - \theta^*\|_{V_t}} \right)}_{Y} \right\rangle$$

Observe that $Y \in \mathbb{R}^d$ is a random direction (that depends on noise samples), but

$$\|Y\|_2^2 = Y^T Y = \frac{(\hat{\theta}_t - \theta^*)^T V_t (\hat{\theta}_t - \theta^*)}{\|\hat{\theta}_t - \theta^*\|_{V_t}^2} = 1$$

$\therefore$ If we have a concentration for the projection $\langle \hat{\theta}_t - \theta^*, V_t^{1/2} \cdot z \rangle$, uniformly for any $z \in S^{d-1}$, where $S^{d-1} = \{ z \in \mathbb{R}^d : \|z\|_2^2 = 1 \}$,

unit ball surface in $\mathbb{R}^d$

then we can get a conc. for $\|\hat{\theta}_t - \theta^*\|_{V_t}$.

Step 3: Uniform concentrations over S^{d-1}.

Idea: Construct an ε-lattice over S^{d-1}. Show a conc. uniformly over all the lattice points. Then, Show that any other point is close to some lattice point, and extend the conc. to all points using Δ'le inequality.

$\boxed{\text{Lattice } D_\varepsilon}$: $D_\varepsilon \subseteq S^{d-1}$, with $|D_\varepsilon| \leq (3/\varepsilon)^d$, for any $\varepsilon \in (0,1)$.

finite covering set

Claim (Lemma 20.1 in text): $\exists\ D_\varepsilon \subseteq S^{d-1}$ with $|D_\varepsilon| \leq (3/\varepsilon)^d$ such that $\forall\ z \in S^{d-1}$, $\exists\ y \in D_\varepsilon$ with

$$\|z-y\|_2 \leq \varepsilon .$$

Pf: HW / Try yourself.

Lemma: For any $\varepsilon \in (0,1)$, $\delta \in (0,1)$, let

$$E = \left\{ \exists\ y \in D_\varepsilon \text{ s.t. } \left\langle \hat{\theta}_t - \theta^*, V_t^{1/2} y \right\rangle \geq \sqrt{2 \ln\left(\frac{|D_\varepsilon|}{\delta}\right)} \right\}.$$

Then, $P(E) \leq \delta$.

Proof: Recall from Step 1, ⊛, that for any $x \in \mathbb{R}^d$,

$$P\left(\langle \hat{\theta}_t - \theta^*, x\rangle \geq \sqrt{2\|x\|^2_{V_t^{-1}} \ln(1/\delta)}\right) \leq \delta$$

Here,

$$\|V_t^{1/2} y\|^2_{V_t^{-1}} = \left(y^T V_t^{1/2} V_t^{-1} V_t^{1/2} y\right) = y^T y = 1.$$

Thus, for any fixed $y \in D_\varepsilon$, we have

$$P\left(\langle \hat{\theta}_t - \theta^*, V_t^{1/2} y\rangle \geq \sqrt{2 \ln(|D_\varepsilon|/\delta)}\right) \leq \frac{\delta}{|D_\varepsilon|}$$

Using a union bound over all $y \in D_\varepsilon$, we have:

$$P\left(\exists\, y \in D_\varepsilon \text{ s.t. } \langle \hat{\theta}_t - \theta^*, V_t^{1/2} y\rangle \geq \sqrt{2 \ln(|D_\varepsilon|/\delta)}\right)$$

$$\leq \delta \qquad \blacksquare$$

Now, choose any $z \in S^{d-1}$. Then, we have:

$$\|\hat{\theta}_t - \theta^*\|_{V_t} \leq \max_{z \in S^{d-1}} \langle V_t^{1/2} z, \hat{\theta}_t - \theta^* \rangle$$

$$= \max_{z \in S^{d-1}} \left[\langle V_t^{1/2}(z-y), \hat{\theta}_t - \theta^* \rangle + \langle V_t^{1/2} y, \hat{\theta}_t - \theta^* \rangle \right]$$

for any $y \in D_\varepsilon$. (adding and subtracting $\langle V_t^{1/2} y, \hat{\theta}_t - \theta^* \rangle$)

Thus, for any chosen $z \in S^{d-1}$, choose the closest $y(z) \in D_\varepsilon$, and the above holds, i.e.,

$$\|\hat{\theta}_t - \theta^*\|_{V_t} \leq \max_{z \in S^{d-1}} \min_{y \in D_\varepsilon} \left[\langle V_t^{1/2}(z-y), \hat{\theta}_t - \theta^* \rangle + \langle V_t^{1/2} y, \hat{\theta}_t - \theta^* \rangle \right]$$

From Cauchy-Schwartz: ($\langle a, b \rangle \leq \|a\|_{Q^{-1}} \|b\|_Q$, for Q a symmetric, p.d. matrix).

$$\leq \max_{z \in S^{d-1}} \min_{y \in D^\varepsilon} \left[\|\hat{\theta}_t - \theta^*\|_{V_t} \cdot \left((z-y)^T V_t^{1/2} V_t^{-1} V_t^{1/2} (z-y) \right)^{1/2} + \langle V_t^{1/2} y, \hat{\theta}_t - \theta^* \rangle \right]$$

$$= \max_{z \in S^{d-1}} \min_{y \in D_\varepsilon} \left[\|z-y\|_2 \|\hat{\theta}_t - \theta^*\|_{V_t} + \langle V_t^{1/2} y, \hat{\theta}_t - \theta^* \rangle \right]$$

Now, $y \in D_\varepsilon$, and from Claim (Lemma 20.1), we have that $\exists\, y(z) \in D_\varepsilon$ s.t. $\|z - y(z)\|_2 \leq \varepsilon$.

$$\leq \max_{z \in S^{d-1}} \min_{y \in D_\varepsilon} \left(\varepsilon \|\hat{\theta} - \theta^*\|_{V_t} + \langle V_t^{1/2} y, \hat{\theta}_t - \theta^* \rangle \right)$$

Finally, recall that for $\underline{\underline{\text{any}}}$ $y \in D_\varepsilon$,

$$\langle V_t^{1/2} y, \hat{\theta}_t - \theta^* \rangle \leq \sqrt{2 \ln\left(\frac{|D_\varepsilon|}{\delta}\right)}$$

w.p. $\geq 1-\delta$.

$\therefore$

$$\|\hat{\theta}_t - \theta^*\|_{V_t} \leq \varepsilon \|\hat{\theta} - \theta^*\|_{V_t} + \sqrt{2 \ln\left(\frac{|D_\varepsilon|}{\delta}\right)}$$

w.p. $\geq (1-\delta)$.

i.e., $\|\hat{\theta}_t - \theta^*\|_{V_t} \leq \left(\frac{1}{1-\varepsilon}\right) \sqrt{2 \ln\left(\frac{|D_\varepsilon|}{\delta}\right)}$

w.p. $\geq (1-\delta)$.

Choose $\varepsilon = 1/2$. Then $|D_{1/2}| \leq \left(3/(1/2)\right)^d = 6^d$.

$$\therefore \; P\left(\|\hat{\theta}_t - \theta^*\|_{V_t} \geq 2\sqrt{2\left(d \ln(6) + \ln(1/\delta)\right)}\right) \leq \delta.$$

$\boxtimes$

General Result (Section 20.1 in textbook):

Setting:

① $\exists\ \theta^* \in \mathbb{R}^d$ s.t. $X_t = \langle \theta^*, A_t \rangle + \eta_t \quad \forall t \geq 1.$

② η_s is conditionally 1-subG, i.e.,

$$E\left[e^{\phi \eta_t} \,\middle|\, A_1, X_1, \ldots, A_{t-1}, X_{t-1}, A_t\right] \leq e^{\phi^2/2} \quad \text{a.s.}$$

③ Regularizer $\lambda > 0$.

Theorem (20.5 in book): Fix any $\delta \in (0,1)$. Then, w.p. $(1-\delta)$, $\forall\ t \geq 1$, we have:

$$\|\hat{\theta}_t - \theta^*\|_{V_t(\lambda)} < \sqrt{\lambda}\, \|\theta^*\|_2 + \sqrt{2 \ln(1/\delta) + \ln\left(\frac{\det V_t(\lambda)}{\lambda^d}\right)}$$

Further, if $\|\theta^*\|_2 \leq m_2 < \infty$, then

$$P\left(\exists\ t \geq 1 : \theta^* \notin C_t\right) \leq \delta, \quad \text{where}$$

$$C_t = \left\{ \theta \in \mathbb{R}^d : \|\hat{\theta}_{t-1} - \theta\|_{V_{t-1}(\lambda)} \leq m_2 \sqrt{\lambda} + \ln\left(\frac{\det(V_{t-1}(\lambda))}{\lambda^d}\right) \right\}$$

Please read 20.1 for proof details.
