Online Optimization of an Energy Storage System Profitability with Reserve Commitment

A Competitive Analysis Approach

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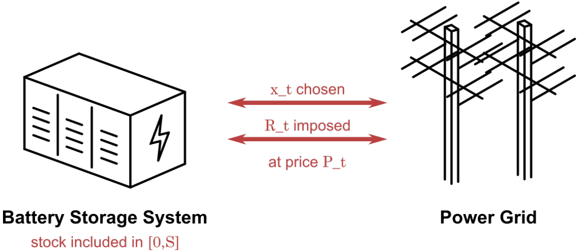
December 17th, 2024

EA Recherche MAP 511

Plan

1. Problem formulation and chosen approach
2. Contributions
3. Proofs

Industrial problem



The operating cost is $\sum_t P_t(x_t + R_t)$

Initial paper (Sossou Edou 2024), PGMO project.
Stochastic approach, tries to minimize $\mathbb{E}[\sum_t P_t x_t]$

Problem formulation

The problem in its general form is

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{t=1}^{T} P_t(x_t + R_t) \\
\text{s.t.} & \\
& \begin{array}{ll}
s_t = s_{t-1} + x_t + R_t & \forall t \in [T] \\
s_t \in [0, S] & \forall t \in [T] \\
x_t + R_t \in [-C^{\max}, +C^{\max}] & \forall t \in [T] \\
x_t \text{ depends only on } R_1, \ldots, R_{t-1}, P_1, \ldots, P_t & \forall t \in [T]
\end{array}
\end{array}
$$

The R_t and P_t are unknown beforehand but are respectively in the intervals $[-R^{\max}, +R^{\max}]$ and $[p^{\min}, p^{\max}]$ with $p^{\min} \geqslant 0$

Competitive Analysis

Given an online decision algorithm ALG and an instance I of the problem, we denote

- $ALG(I)$ the resulting cost when the algorithm ALG is applied on I.
- $OPT(I)$ the optimal value of the cost for the offline problem on the instance I (the whole instance is known beforehand).

Competitive Analysis

Given an online decision algorithm ALG and an instance I of the problem, we denote $ALG(I)$ the resulting cost when the algorithm ALG is applied on I and $OPT(I)$ the optimal value of the cost for the offline problem on the instance I.

Definition

Let $c \in \mathbb{R}$ and ALG an online decision algorithm. ALG is c-competitive if for all instances I, $ALG(I) \leqslant c\, OPT(I)$

Definition Competitive ratio

The competitive ratio of ALG is

$$\inf\{c \in \mathbb{R};\ ALG \text{ is } c\text{-competitive}\}$$

Acceptable algorithm

We call an algorithm *acceptable* if for every instance I, the inequalities $s_t \in [0, S]$ and $|x_t + R_t| \leq C^{\max}$ are checked at every time t.

Proposition

An algorithm is acceptable if and only if for every time t we have

$$x_t \leqslant \min(C^{\max} - R^{\max}, S - s_{t-1} - R^{\max})$$

$$x_t \geqslant \max(-C^{\max} + R^{\max}, -s_{t-1} + R^{\max})$$

Remark: An unacceptable algorithm has a competitive ratio of $+\infty$.

Plan

1. Problem formulation and chosen approach
2. Contributions
   - Constant prices
   - Known prices, unknown demand
   - No demand, unknown prices
   - Worst-case cost
3. Proofs

Plan

Constant prices

With constant positive prices, the problem can be written as

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=1}^{T} (x_t + R_t) \\
\text{s.t.} \quad & \\
& s_t = s_{t-1} + x_t + R_t && \forall t \in [T] \\
& s_t \in [0, S] && \forall t \in [T] \\
& x_t + R_t \in [-C^{\max}, +C^{\max}] && \forall t \in [T] \\
& x_t \text{ depends only on } R_1, \dots, R_{t-1} && \forall t \in [T]
\end{aligned}
$$

Intuition: $\sum_t (x_t + R_t) = s_T - s_0$ so we just want to minimize s_T, i.e sell as much as possible.

Constant prices

Algorithm Minimizing x

---

for all $t \in [T]$ **do**

$x_t = \max(R^{\max} - C^{\max}, R^{\max} - s_{t-1})$

end for

Theorem 1, Optimal algorithm for constant prices

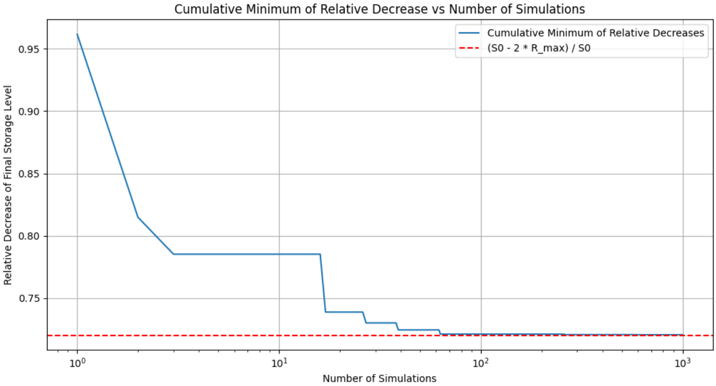
If $\min(C^{\max}, S) \geqslant 2R^{\max}$, then the previous algorithm is acceptable, and has a competitive ratio of

$$c = \frac{\max(2R^{\max} - s_0, -T(C^{\max} - 2R^{\max}))}{\max(-s_0, -TC^{\max})}$$

This is the best competitive ratio that can be achieved.

Constant Price : Numeric validation

- Simulate many instances
- Keep track of the minimum score (compared to optimal case)
- plot the results ; convergence to the true competitiveness ratio



Plan

Known prices Unknown Rt

With known prices beforehand, the problem can be written as

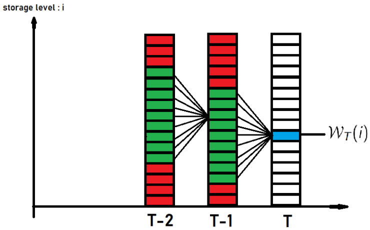
$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=1}^{T} P_t(x_t + R_t) \\
\text{s.t.} \quad & \\
& s_t = s_{t-1} + x_t + R_t && \forall t \in [T] \\
& s_t \in [0, S] && \forall t \in [T] \\
& x_t + R_t \in [-C^{\max}, +C^{\max}] && \forall t \in [T] \\
& x_t \text{ depends only on } R_1, \ldots, R_{t-1} && \forall t \in [T] \\
& P_t \in \mathbb{R}^T
\end{aligned}
$$

Known prices Unknown Rt

we define

$$\mathcal{W}_T(i) = \min_{x \in E} \left(\mathcal{W}_{T-1}(x) + gain(x, i) \right)$$

- E is the feasible domain
- $\mathcal{W}_T(i)$ is the best worst score at T for storage level i
- gain(x,i) is the worst gain when going from storage level x to i



Known prices Unknown Rt

now we can write the dynamic programming equation

Theorem , Best worst-case gain

To obtain such W_T it must obey the following equation:

$$\mathcal{W}_T(i) = \min_{x \in E} \left\{ \begin{aligned} \max \Big[& \mathcal{W}_{T-1}(x) + (x - R_m - i)P_T + P_{t-1} \cdot R_m, \\ & \mathcal{W}_{T-1}(x) + (x + R_m - i)P_T - P_{t-1} \cdot R_m \Big] \end{aligned} \right\}$$

.

Known prices Unknown Rt

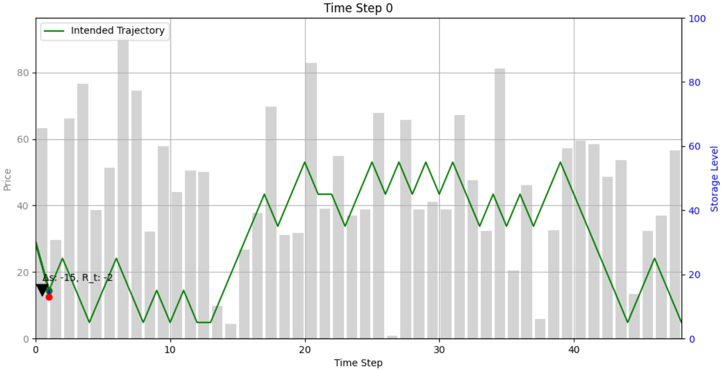


Figure: buy/sell strategy

Known prices Unknown Rt

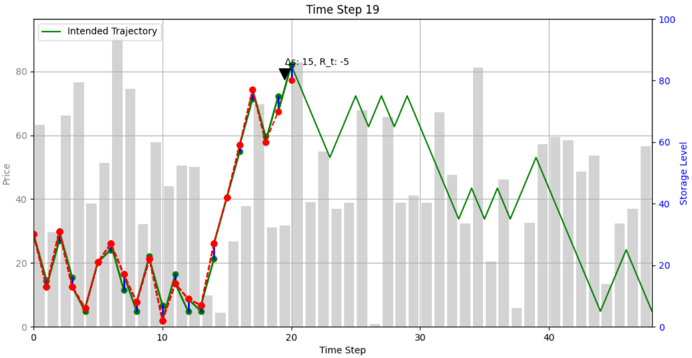


Figure: update as new Rt gets revealed

Known prices Unknown Rt

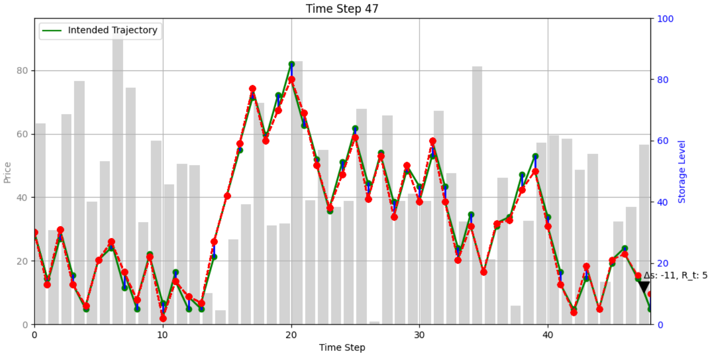


Figure: End of simulation

Plan

No demand, unknown prices

Theorem 2, Competitve ratio of OTA

Suppose that the prices are lower and upper bounded by L and U, respectively. Let $\theta := U/L$. In the case where no demand is observed, suppose that $S_0 + TC^{max} - S \leqslant 0$ and that $S_0 \leqslant C^{max}$.

- Algorithm 1 with ϕ^* as the threshold function achieves an optimal competitive ratio of $(1 + \ln(\theta))$.

Online Threshold Algorithm

Consider the following algorithm:

Algorithm 1 OTA with Threshold Function ϕ (OTA_ϕ) for DC

Input: Threshold function $\phi := \{\phi(\cdot)\}$, and $w^{(1)} = 0$;
While item i arrives and verify the constraints
For each $j \leqslant i$ (do one step in the solving process of SubPj)

$$y(j,i) = \begin{cases} \phi_j^{-1}(P_i) - w_j^{(i)} & P_i \geqslant \phi_j(w_j^{(i)}) \\ 0 & P_i < \phi_j(w_j^{(i)}) \end{cases}$$

update the utilization $w_j^{(i+1)} = w_j^{(i)} + y(j,i)$,
$y_i = \sum_{j=1}^{i} y(j,i)$
$x_i = C^{max} - y_i$

Algorithm

$$
\begin{array}{cccc|c}
y_{1,1} & & & & y_1 \\
y_{1,2} & y_{2,2} & & & y_2 \\
y_{1,3} & y_{2,3} & y_{3,3} & & y_3 \\
\vdots & & \ddots & & \\
y_{1,n} & \cdots & & y_{n,n} & y_n
\end{array}
$$

Threshold function

ϕ^*

Let ϕ^* defined as :

$$\phi^*(w) = \begin{cases} L & w \in [0, \beta^*) \\ Le^{(1+\ln\theta)w/b-1} & \text{if } w \in [\beta^*, b] \\ \infty & \text{if } w > b \end{cases}, \qquad (1)$$

where $\beta^* = \frac{b}{\alpha_{\phi^*}}$ is the utilization threshold, where $b = C^{max}$

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Worst-case cost

Definition Worst-case cost

For a minimization problem and a given algorithm ALG, the worst-case cost is

$$\sup\{ALG(I);\ I \text{ an instance of the online problem}\}$$

The algorithm is guaranteed to do better (i.e lower) than its worst-case cost.
The better the algorithm, the lower the worst-case cost is.
Concept close to competitive ratio but different and easier to work with.

Dynamic programming formulation for best worst-case cost

Define by induction the functions $W_1, \ldots, W_{T+1}$:

$$W_t(s) = \sup_{p \in [p^{\min}, p^{\max}]} \inf_{|x| \leqslant C^{\max} - R^{\max}} \sup_{|r| \leqslant R^{\max}} \big(p(x+r) + W_{t+1}(s+x+r) \big)$$

$$W_t(s \notin [0, S]) = +\infty \quad \forall t; \quad W_{T+1}(s \in [0, S]) = 0$$

Theorem 3, Best worst-case cost

An algorithm choosing at time t

$$x_t \in \operatorname*{arg\,min}_{|x| \leqslant C^{\max} - R^{\max}} \sup_{|r| \leqslant R^{\max}} \big(P_t(x + r) + W_{t+1}(s_{t-1} + x + r) \big)$$

has a worst-case cost of $W_1(s_0)$. This is the lowest worst-case cost that can be achieved.

Propositions on recursive functions W

We can have the explicit expression of W_T (after that it is getting difficult)

$$W_T(s) = \begin{cases} p^{\max}(2R^{\max} - s) & \text{if } s \leqslant 2R^{\max} \\ p^{\min}(2R^{\max} - s) & \text{if } s \in [2R^{\max}, C^{\max}] \\ p^{\min}(2R^{\max} - C^{\max}) & \text{if } s \geqslant C^{\max} \end{cases}$$

Proposition

For all $t \leqslant T$ and $s \in [0, S]$,

$$W_t(s) \geqslant p^{\min}(2R^{\max} - s)$$

Propositions on recursive functions W

Proposition Convexity of W

For all $t \leqslant T + 1$, W_t is convex.

Several remarks can be made from this:

- $r \mapsto P_t(x + r) + W_{t+1}(s_{t-1} + x + r)$ is convex, so the worst case demand R_t is always either $-R^{\max}$ or $R^{\max}$
- $x \mapsto \sup_{|r| \leqslant R^{\max}} \big(P_t(x + r) + W_{t+1}(s_{t-1} + x + r)\big)$ is convex too, so choosing x_t at time t amounts to solving a convex optimization problem (provided that W_{t+1} is known).

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Elements of Theorem 1's proof

Theorem 1, Optimal algorithm for constant prices

If $\min(C^{\max}, S) \geqslant 2R^{\max}$, then the algorithm choosing at time t

$$x_t = \max(R^{\max} - C^{\max}, R^{\max} - s_{t-1})$$

is acceptable, and has a competitive ratio of

$$c = \frac{\max(2R^{\max} - s_0, -T(C^{\max} - 2R^{\max}))}{\max(-s_0, -TC^{\max})}$$

This is the best competitive ratio that can be achieved.

Elements of Theorem 1's proof

- The algorithm is acceptable.
- The offline problem has an optimal value of

$$OPT(R) = \max(-TC^{\max}, -s_0)$$

  regardless of the values taken by R.
- The algorithm described previously can always do better than

$$ALG(R) \leqslant \max(2R^{\max} - s_0, -T(C^{\max} - 2R^{\max}))$$

- For every acceptable algorithm ALG', the instance $R^* = (R^{\max})_{1 \leq t \leq T}$ satisfies

$$ALG'(R^*) \geqslant \max(2R^{\max} - s_0, -T(C^{\max} - 2R^{\max}))$$

Plan

Transformed Problem

After a variable transformation $y_t = C^{max} - x_t$, we obtain an equivalent formulation for the initial problem :

$$
\begin{aligned}
\text{maximize} \quad & \sum_{t=1}^{T} P_t y_t \\
\text{subject to} \quad & S_0 + tC^{max} - S \leqslant \sum_{i=1}^{t} y_i \leqslant S_0 + tC^{max}, \quad \forall t \quad \text{(SPeq)} \\
& 0 \leqslant y_t \leqslant 2C^{max}, \quad \forall t \\
& y_t \text{ depends only on past}, \ \forall t \in [T].
\end{aligned}
$$

Simplified Problem Under Assumptions

Under the assumptions $S_0 + TC^{max} - S \leqslant 0$ and $S_0 \leqslant C^{max}$, the problem reduces to:

$$\begin{aligned} \text{maximize} \quad & \sum_{t=1}^{T} P_t y_t \\ \text{subject to} \quad & \sum_{i=1}^{t} y_i \leqslant S_0 + tC^{max} =: b(t), \ \forall t \\ & 0 \leqslant y_t, \ \forall t \\ & y_t \text{ depends only on past, } \forall t \in [T]. \end{aligned} \tag{SP'}$$

Decoupling the Problem

We consider the following decoupled problem (DC):

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} \sum_{i=j}^{n} p(i) y(j,i) \\
\text{subject to} \quad & \sum_{i=j}^{n} y(j,i) \leqslant b(j) - b(j-1), \quad j = 1, \ldots, n \\
\text{variables} \quad & y(j,i) \geqslant 0, \; i \geqslant j.
\end{aligned}
\tag{DC}
$$

Equivalence

Lemma

Any optimal solution to (DC) yields an optimal solution to (SP')
by the construction:

$$y^*(i) = \sum_{j=1}^{i} y^*(j, i).$$

Moreover, both problems share the same optimal objective value.

Resulting Subproblems

Each subproblem for a fixed j is:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=j}^{n} p(i) y(j,i) \\ \text{subject to} \quad & \sum_{i=j}^{n} y(j,i) \leqslant b(j) - b(j-1), \\ & y(j,i) \geqslant 0, \ i \geqslant j. \end{aligned} \tag{SubPj}$$

Each of these subproblems corresponds to a "Generalized One-Way Trading" (GOT) problem, which we know how to solve with an online threshold algorithm with an optimal competitve ratio of $(1 + \ln(\theta))$ [2].

From the subproblems to the original

(GOT)

$\max_{y_n} \quad \sum_{n\in\mathcal{N}} g_n(y_n), \quad \text{s.t. } \sum_{n\in\mathcal{N}} y_n \le C, \quad 0 \le y_n \le D_n$

Algorithm Online Threshold-Based Algorithm

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input: threshold function $\phi := \{\phi_m(\cdot)\}_{m\in\mathcal{M}}$, and $w_m^{(1)} = 0$

While item n arrives determine $\boldsymbol{y}_n^*$ by solving the pseudo-utility maximization problem

$$\boldsymbol{y}_n^* = \arg\max_{\boldsymbol{y}_n\in\mathcal{Y}_n} \quad g_n(\boldsymbol{y}_n) - \int_{w^{(n)}}^{w^{(n)}+y_n} \phi(u)du; \tag{2}$$

update the utilization $w^{(n+1)} = w^{(n)} + y_n^*$.

Competitive Ratio

Theorem

Suppose that $S_0 + TC^{max} - S \leqslant 0$ and that $S_0 \leqslant C^{max}$. When the threshold function of OTA_ϕ for SubPj is

$$\phi_j^*(w) = \begin{cases} L & w \in [0, \beta_j^*) \\ Le^{(1+\ln\theta)w/b_j - 1} & \text{if } w \in [\beta_j^*, b_j] \\ \infty & \text{if } w > b_j \end{cases}, \tag{3}$$

where $\beta_j^* = \frac{b_j}{\alpha_{\phi^*}}$ is the utilization threshold, where $b_j = b(j) - b(j-1)$, the competitive ratio of OTA_{ϕ^*} is $\alpha_{\phi^*} = 1 + \ln\theta$.

Online Threshold Algorithm

We thus construct the algorithm 1 for DC

Extensions and Conjectures

If the assumptions do not hold, the problem can still be decoupled but will involve bounds from both sides. In such a scenario, we conjecture a competitive ratio under $2(1 + \ln \theta)$.

Paths of progress:

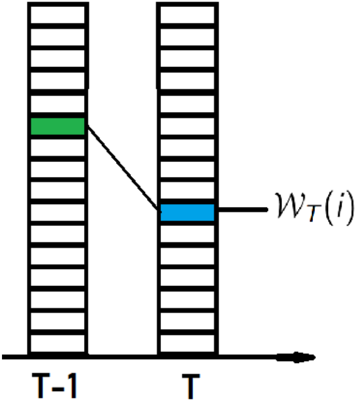
- Improve bound estimates step-by-step if price structure is can be learned.
- Use online algorithms for fractional knapsack in **random order model** to achieve competitive ratios under certain conditions.

Plan

Numeric implementation of Known prices, Unknown Rt

Let's specify that $E = A_i \cap F$:

- F : acceptable ending the feasible storage levels adjusted for random perturbations: $[R_{\max}, S - R_{\max}]$
- A_i : valid x positions where $\{\,|x - i| \leqslant C_{\max} - 2R_{\max}\}$



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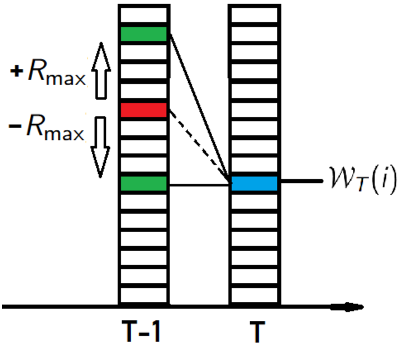

Numeric implementation of Known prices, Unknown Rt

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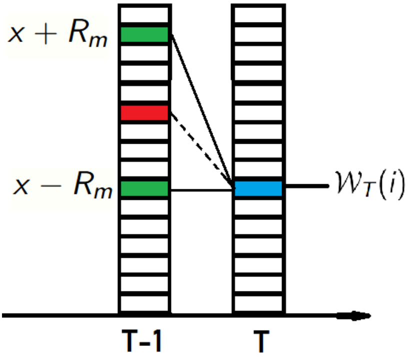
- A_i : valid x positions where $\{\, |x - i| \leqslant C_{\max} - 2R_{\max} \}$



Implementation Details

- A table was used to store the maximum worst-case gain $W_T(i)$ for each storage level i and time step T.
- Working forward simplifies the implementation:
  - At each time step t, all values for W_{t+1} are already computed.
  - we save the previous optimal state to recursively search for the optimal decision path later.
- Transitions between states were validated against E
- The table-filling process proceeds iteratively until all time steps are covered.

Implementation Details

Algorithm:

1. For each time step $t = 0$ to $T - 1$:
   1. For each storage level s:
      1. For each possible next storage level i such that $|i - s| \leqslant C_{\max} - 2R_{\max}$:
      2. Compute the two possible gains:

$$\text{gain1} = \text{gain\_matrix}[t][s] + (s - R_{\max} - i)P_t + P_{t-1} \cdot R_{\max}$$

$$\text{gain2} = \text{gain\_matrix}[t][s] + (s + R_{\max} - i)P_t - P_{t-1} \cdot R_{\max}$$

      3. Determine the worst-case gain:

$$\text{worse\_gain} = \max(\text{gain1}, \text{gain2})$$

      4. If $\text{worse\_gain} < \text{gain\_matrix}[t+1][i]$:
      5. Update $\text{gain\_matrix}[t+1][i] = \text{worse\_gain}$
      6. Update $\text{prev\_state}[t+1][i] = s$
2. Trace back the optimal path from $t = T$ to $t = 0$ using *prev_state*.

Output: - Optimal storage levels and maximum worst-case gain.

Plan

Elements of Theorem 3's proof

Define by induction the functions $W_1, \ldots, W_{T+1}$:

$$W_t(s) = \sup_{p \in [p^{\min}, p^{\max}]} \inf_{|x| \leqslant C^{\max} - R^{\max}} \sup_{|r| \leqslant R^{\max}} \big(p(x+r) + W_{t+1}(s+x+r)\big)$$

$$W_t(s \notin [0, S]) = +\infty \quad \forall t; \quad W_{T+1}(s \in [0, S]) = 0$$

Theorem 3, Best worst-case cost

An algorithm choosing at time t

$$x_t \in \operatorname*{arg\,min}_{|x| \leqslant C^{\max} - R^{\max}} \sup_{|r| \leqslant R^{\max}} \big(P_t(x+r) + W_{t+1}(s_{t-1} + x + r)\big)$$

has a worst-case cost of $W_1(s_0)$. This is the lowest worst-case cost that can be achieved.

Elements of Theorem 3's proof

The key idea is to write W_t as

$$W_t(s) = \inf_{\substack{X_i \in \mathcal{X}_i^t \\ i \in [t,T]}} \sup_{\substack{R_t,\dots,R_T \in [\pm R^{\max}] \\ P_t,\dots,P_T \in [p^{\min}, p^{\max}]}} \left(\sum_{i=t}^{T} P_i(X_i(P_{[t,i]}, R_{[t,i-1]}, s) + R_i) \right)$$

where $P_{[t,i]} = (P_t, \dots, P_i)$ and
$\mathcal{X}_i^t \subset \{X : \mathbb{R}^{i-t+1} \times \mathbb{R}^{i-t} \times \mathbb{R} \to [\pm(C^{\max} - R^{\max})\}$
We use induction reasoning and the following lemma

Lemma Minimax

Let X, Y be sets and $F\colon X \times Y \to \mathbb{R}$. Then

$$\inf_{h\colon X \to Y} \sup_{x \in X} F(x, h(x)) = \sup_{x \in X} \inf_{y \in Y} F(x, y)$$

Proof of the convexity of W

We use two convexity lemmas

Lemma Danskin's lemma

Let X be a convex set, Y be a set and $F : X \times Y \to \mathbb{R}$. If for all $y \in Y$, $x \mapsto F(x, y)$ is convex, then $x \mapsto sup_{y \in Y} F(x, y)$ is convex.

Lemma Convexity of the optimal value function

Let X and Y be convex sets and $F : X \times Y \to \mathbb{R}$. If F is jointly convex, then $x \mapsto \inf_y F(x, y)$ is convex.

Proof of the convexity of W

Proof.

We proceed by reverse induction. Let t be such that W_{t+1} is convex. Then, for all p, r, $(s, x) \mapsto p(x + r) + W_{t+1}(s + x + r)$ is jointly convex.

By Danskin's lemma, for all p,

$(s, x) \mapsto \sup_r p(x + r) + W_{t+1}(s + x + r)$ is convex.

By convexity of the optimal value function lemma, for all p,

$s \mapsto \inf_x \sup_r p(x + r) + W_{t+1}(s + x + r)$ is convex.

Finally, by Danskin's lemma, $s \mapsto W_t(s)$ is convex. $\square$

References I

Yang, X. & Wang, S., "Online Linear Programming: Competitive Algorithms for Real-Time Decisions."

Sun, X., Qi, Q., Wierman, A., "Competitive Algorithms for Online Multiple-Choice Knapsack and One-Way Trading."