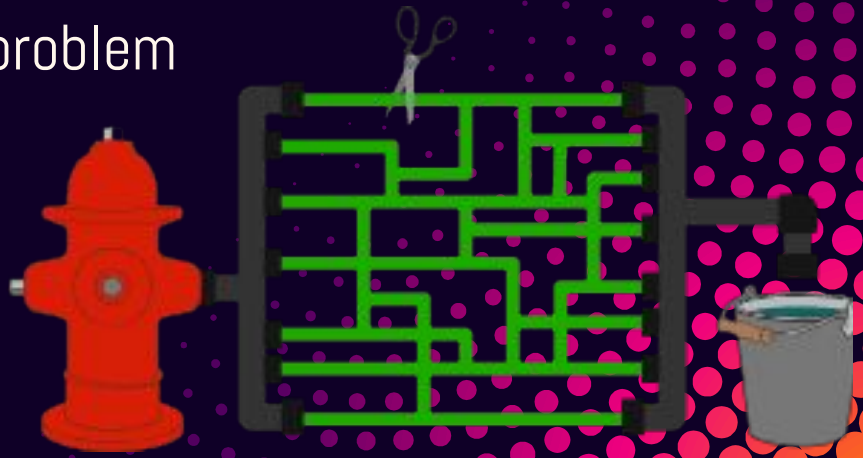
Maxflows and Mincuts

Week 4 AI Inspire Winter 2020

Real world problems

- Water pipes over a city network ⇒ each pipe has diff max capacities ⇒ want to get max flow across source to sink
- Any type of bipartite matching problem
  - People to tasks
  - Donating blood types
  - Stable marriage prob
  - Many other bipartite probs



Defining the maxflow problem

- Directed graph with weights on edges ⇒ edge-weighted digraph
  - Edge weights = capacities
- Source s and Sink t
- Flow on edges
  - Can't be greater than capacity
  - Flow in node **u** = Flow out of node **u**
    - s & t exceptions
  - Flow value = outflow of s = inflow at t
- Want to find MAXIMUM FLOW!!

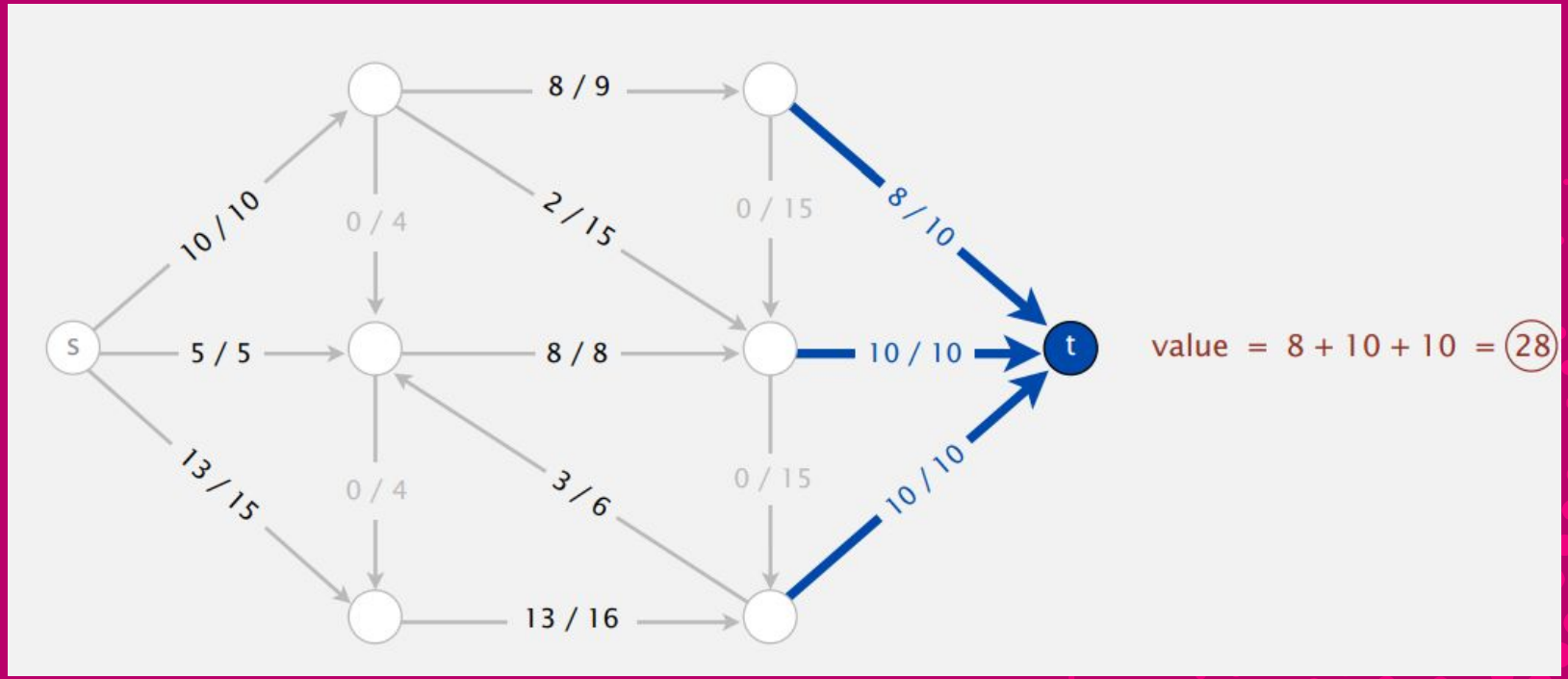
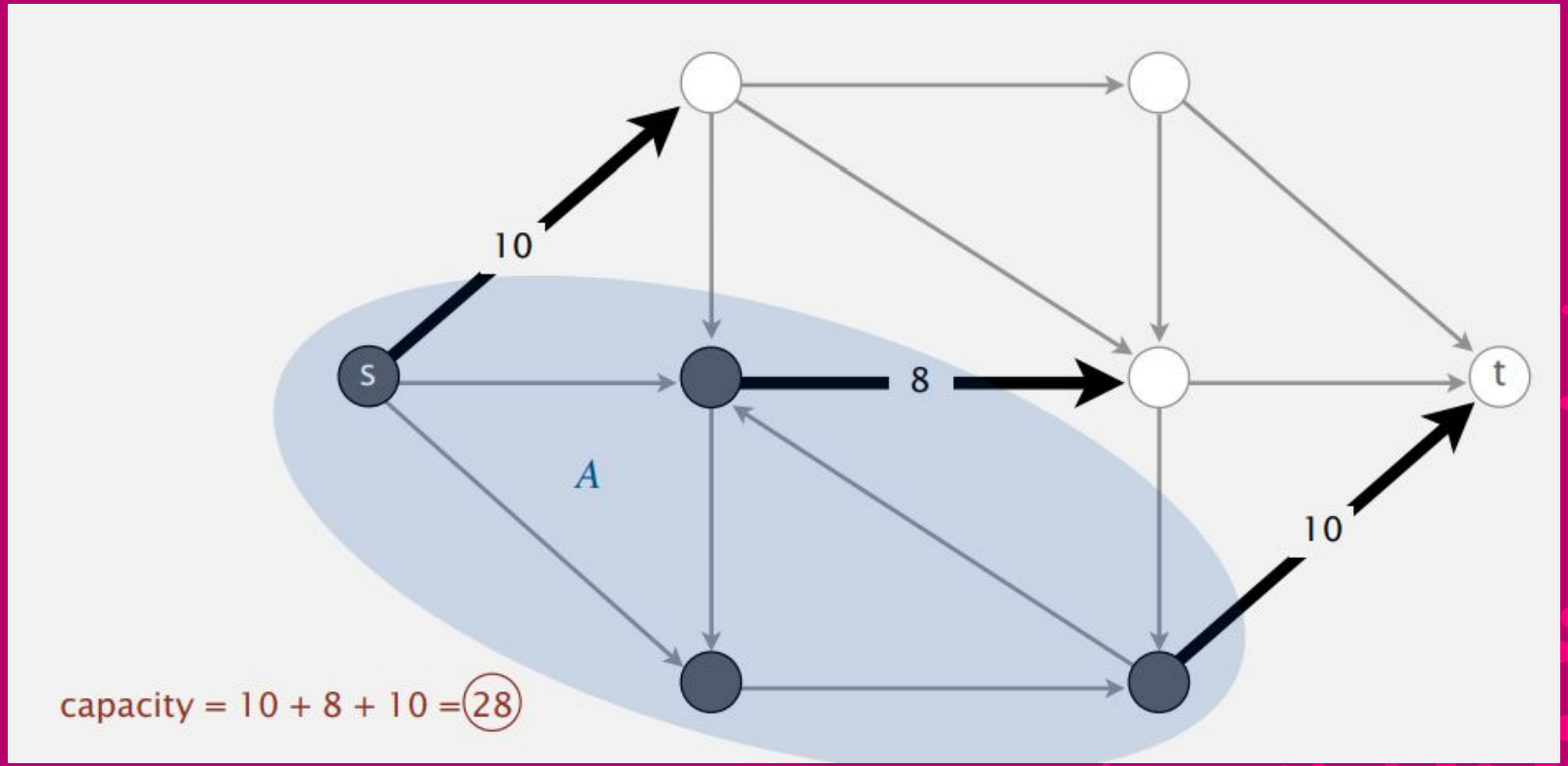


Image from Princeton University COS 226 Fall 2019 Lecture Slides

Defining the mincut problem

- Edge-weighted digraph
- Source **s** and Sink **t**
- st cut = **partition** of nodes s and t into 2 disjoint sets A and B ⇒ s is in A and t is in B
- Capacity = sum of capacities from A to B NOT B to A
- GOAL ⇒ Find cut of MINIMUM CAPACITY



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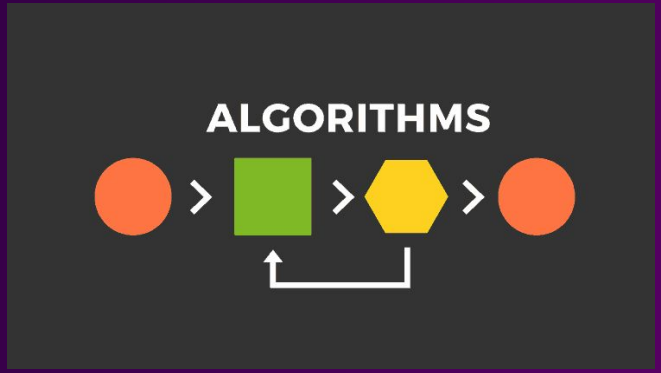
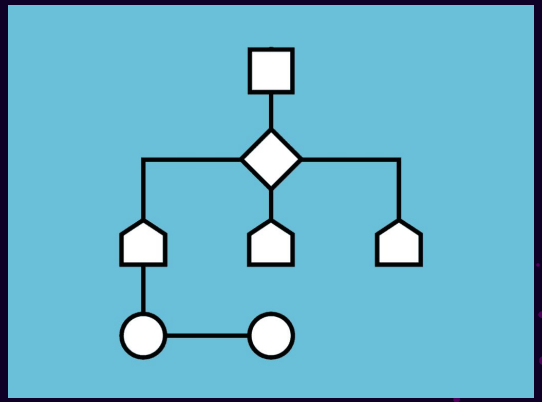
Ford-Fulkerson Algorithm



What is it?

- Greedy algorithm
- Finds maximum flow in network with constraints



What is an augmenting path?

- Path from source to sink
  - Can increase flow along non-full forward edges
  - Can decrease flow along non-empty backward edges
- Algorithm ends when no more augmenting paths (all paths will consist of full forward edge or empty backward edge)
- Goal ⇒ increment amt of flow through augmenting paths

How to use augmenting paths

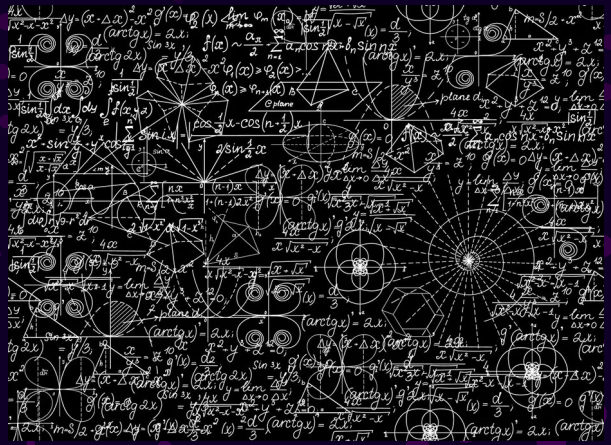
- Can increase flow along paths ⇒ closer towards max flow
- For each augmenting path ⇒ get bottleneck capacity and increase flow by bottleneck capacity
- Still many unanswered questions about how augmenting paths will work exactly in algo

Maxflow-Mincut Theorem

Most Important Theorems

- Augmenting path theorem
  - A flow f is a maxflow i.f.f. there are no augmenting paths
- Value of maxflow = capacity of mincut
- Proven lemma ⇒ netflow across cut = flow f of graph

Proving Theorems



How to get mincut from maxflow

1) Get sets A and B from partition
2) Set A ⇒ will contain source s with NO full forward edges & NO empty backward edges
    a) Make sure all nodes in A are connected to s in undirected graph through a path
    b) Similarly with set B
3) Capacity of cut = maxflow ⇒ mincut

Algorithm Analysis

- Bad ford fulkerson case demonstration
- Optimization methods
  - Find shortest augmenting path (fewest edges) using Queue (BFS used queue and found shortest path)
  - Find paths with maximum bottleneck capacity using Priority Queue

Applications

Bipartite Matching Application

- N people
- N tasks
- Assign people to tasks (Every person has 1 task and has a qualified person = edge goes in between)
- Construct flow network by adding source, sink, edges, and capacities
- Maxflow problem and use FF algo ⇒ get maxflow ⇒ yields in perfect matching