Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: http://www.tandfonline.com/loi/rquf20

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Quantitative Finance, 2015
Vol. 15, No. 7, 1191–1204, http://dx.doi.org/10.1080/14697688.2014.926020

Computing optimal rebalance frequency for log-optimal portfolios in linear time

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(*Received 9 February 2014; accepted 8 May 2014*)

The pure form of log-optimal investment strategies are often considered to be impractical due to the inherent need for continuous rebalancing. It is however possible to improve investor log utility by adopting a discrete-time periodic rebalancing strategy. Under the assumptions of geometric Brownian motion for assets and approximate log-normality for a sum of log-normal random variables, we find that the optimum rebalance frequency is a piecewise continuous function of investment horizon. One can construct this rebalance strategy function, called the *optimal rebalance frequency function*, up to a specified investment horizon given a limited trajectory of the *expected log of portfolio growth* when the initial portfolio is never rebalanced. We develop the analytical framework to compute the optimal rebalance strategy in linear time, a significant improvement from the previously proposed search-based quadratic time algorithm.

Keywords: Log-optimal portfolio; Log-normal; Portfolio optimization; Rebalancing frequency; Discrete Rebalancing; Portfolio growth rate; Instantaneous portfolio growth

JEL Classification: G11, C63, C65

1. Introduction

Algorithmic trading systems are becoming the foundation of today's commercial proprietary financial applications (Johnson 2010). With increasing computational power, advanced trading systems are built to deploy complex algorithms to make intelligent buy and sell decisions of tradable assets. Since the viable window for trading opportunity exists only for very short time period, all trading decisions need to be made extremely fast in the order of seconds or less. All such trading systems have to reliably decide when and how to rebalance an investment portfolio. Too much or too frequent rebalancing can be cost prohibitive due to the transaction costs involved. Procrastinated rebalancing decisions may also prove to be costly in terms of lost opportunity to respond to market signals.

In this research, we set out to gain insight into the question of how often an initial portfolio should be adjusted. We assume that the investor's has a log utility function and chooses the *log-optimal strategy* to maximize the expected value of the log of portfolio growth. A good survey for log-optimal portfolios with extensive references for related research in a continuous time framework can be found in Györfi *et al.* (2008), see also Luenberger (1998). The investor has to continuously rebalance the portfolio to the initial estimation of the weights in order to achieve maximum portfolio growth† in the long run. Both researchers and practitioners generally acknowledge the severe practical limitation of this strategy due to the continuous rebalancing condition.

Many researchers have studied the efficacy of discrete-time rebalancing in comparison to theoretical continuous rebalancing (Sun *et al.* 2006, Kritzman *et al.* 2009). They have found that the investor's utility loss when the investor switches from active continuous trading to infrequent discrete rebalancing may not be substantial. In a similar study, researchers have used Monte Carlo simulation to examine the investor utility loss if a discrete-time trading strategy is adopted (Branger *et al.* 2010). They conclude that in an incomplete market, where derivatives are not used to construct portfolios, the investor incurs very small loss. They have demonstrated that for a 10 year horizon a passive buy-and-hold strategy yields the same expected investor utility as continuous trading. Similarly, other researchers have developed a dynamic programming algorithm to compute the optimal rebalancing schedule (Sun *et al.* 2006). They show

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†In the context of this research *portfolio growth* means *expected log of portfolio growth* unless otherwise stated.

that using the schedule, the suboptimality cost for not using continuous rebalancing is very small, limited to only five basis points.

A more realistic investment proposition for the log-optimal investor is to maximize the log utility by rebalancing the portfolio periodically at discrete non-zero time intervals for a finite desired investment horizon. Das *et al.* (2014) investigated the possibility of discrete-time trading for log-optimal portfolios. The underlying log-normal approximation assumption for a sum of log-normal variables in Fenton-Wilkinson (Fenton 1960) approach made it possible to derive analytical expressions for passive portfolio mean and variance analogous to these for the log-optimal strategy. We outlined an analytical approach and an algorithm to compute the periodic optimal frequency.

In the proposed approach, the investor will rebalance periodically, in discrete time, to the optimal portfolio weights to maximize the *portfolio growth*. We showed that for a certain class of portfolio assets, such a periodic *optimal frequency* indeed exists. Simulation studies showed that there is considerable improvement in investor log loss when the investor uses the computed optimal frequency. In particular, for our portfolio example, for medium to long-term investors the log loss was found to be less than 2% compared to 6% or higher if the investor had used continuous rebalancing.

It turns out that the periodic optimal frequency is a function of the investment horizon. Often investment portfolio managers have to make rebalancing decisions for funds invested by multitude of diverse investors. These investors have different preference for the length of investment horizon. For example, a pension fund manager needs to worry about investors of all ages. Hence the fund manager has to make rebalance decision for a continuum of investment horizons. In such scenarios, there is a need to compute the value of optimal frequency function for a range of investment horizons. The search-based algorithms proposed by Das *et al.* (2014) for this purpose are inherently quadratic in time. The computing time rapidly explodes as the range of the investment horizon expands.

By using mathematical analysis, we reduce the complexity of the algorithm to linear time in two steps. First, we show that the optimal frequency can only be chosen from a finite set of numbers determined only by the intended investment horizon. Thus we obtain a substantial improvement in performance for finding the optimal frequency by limiting the search space to a discrete and countable set instead of a continuous range of numbers. Then we show that the entire investment horizon range can be divided into piecewise non-overlapping segments. The optimal frequency within each horizon segment is the ratio of the investment horizon to a fixed positive integer called the *rebalance divisor*. Therefore, we reduce the task of computing the optimal frequency function to merely finding the horizon segmentation boundaries, called *rebalance inflection points*, and the corresponding rebalance divisors.

The rest of the paper is organized in the following manner. In section 2, we review the mathematics of the discrete-time rebalancing methodology for log-optimal portfolios proposed in our earlier research (Das *et al.* 2014). In this research, we proposed two search based computations for optimal frequency. In section 3, we extend the prior research to eliminate the need to search and compute the optimal frequency analytically. Using this approach, the optimal frequency function can be computed in linear time. We then measure and compare the computational complexity of three increasingly sophisticated algorithms in section 4. Finally, we summarize the results in the section 5.

2. Review of rebalancing methodology

Suppose the investor has the choice of setting up an investment portfolio from a set of N *risky* financial assets and a *risk-free* asset.† Typical risky assets are stocks and funds, and often are correlated with other risky financial assets. These risky assets $i = 1, \ldots, N$ are provided with *a priori* expected returns and standard deviations. We assume that returns are stationary random variables and hence the expected return and standard deviations don't change over time. We consider risk-free asset $i = N + 1$ such as T-bills offering constant fixed rate of return. We will use the following symbols in our mathematical derivations and analysis for $\forall i, j = 1$ to $N + 1$.

T = investment horizon in years (periods)
μ_i = expected rate of return for asset i
σ_i = standard deviation for asset i
ρ_{ij} = correlation between returns of asset i and j
σ_{ij} = covariance of asset i and $j = \rho_{ij}\sigma_i\sigma_j$
w_i = proportion of investment in asset i in portfolio for log-optimal allocation
$\mu_p(t)$ = expected rate of return of portfolio of assets at time t
$\sigma_p(t)$ = standard deviation of portfolio of assets at time t
$V(t)$ = value (in dollars) of portfolio at time t

Without loss of generality, throughout our analysis, we will assume an initial value of $V(0) = \$1$. For asset $N+1$ which is risk-free, we will use $r_f = \mu_{N+1}$ alternatively. Since the asset is risk free, we also have $\sigma_{N+1} = 0$ and

$$\rho_{(N+1)j} = \rho_{j(N+1)} = 0 \quad \forall j = 1 \text{ to } N \tag{1}$$

2.1. Active log-optimal portfolio

We assume that asset price dynamics $S(t)$ follows Geometric Brownian motion. Geometric Brownian motion assumption is widely used in financial assets and derivative valuations (Neftci 2000, Hull 2011).

$$dS(t) = \mu S(t)dt + \sigma S(t)dz \tag{2}$$

where

μ = expected rate of return of the asset expressed in decimal form.
σ = volatility of the asset price.
Variable $dz = \epsilon\sqrt{dt}$ follows *Wiener process*, where $\epsilon \sim \phi(0, 1)$ is the standard normal variable.

In this asset dynamics framework, the *continuously compounded rate of return* per annum realized between time 0 and t denoted by x is characterized by the following normal

†The reader is strongly encouraged to refer Das *et al.* (2014) for in depth treatment of the topic covered in this section.

distribution:

$$x \sim \phi\left[\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{t}\right] \tag{3}$$

The asset price in terms of x is given by the following expression:

$$S(t) = S(0)e^{xt} \tag{4}$$

In the log-optimal investment strategy, portfolio weights are *continuously rebalanced* to maximize the long-term growth rate of log of portfolio return. The reader can find a good treatment of this strategy in Luenberger (1998). Log-optimal and semi-log optimal portfolios are also analysed in Györfi *et al.* (2007).

For an initial portfolio value of $V(0) = 1\$$ the expected value and variance for the log of portfolio growth are given by the following equations:

$$\chi(t) = E[ln\{V(t)\}] = \nu_p t \tag{5}$$

$$\Upsilon(t) = Var[ln\{V(t)\}] = \sigma_p^2 t \tag{6}$$

where, portfolio mean μ_p, variance σ_p and *growth rate* ν_p are respectively given by:

$$\mu_p = \sum_{i=1}^{N+1} w_i \mu_i \tag{7}$$

$$\sigma_p^2 = \sum_{i,j=1}^{N+1} w_i \sigma_{ij} w_j \tag{8}$$

$$\nu_p = \mu_p - \frac{\sigma_p^2}{2} \tag{9}$$

For notational simplicity, we will use $\chi(t)$ to denote the portfolio growth at time t. Since $V(0) = 1$, we can rewrite equation (5) as,

$$\chi(t) = \nu_p t \tag{10}$$

In the log-optimal portfolio, the growth rate ν_p is maximized in the long run by solving the following optimization problem:

$$\begin{aligned} &\underset{\mathbf{w}}{\text{maximize}} && \nu_p \\ &\text{subject to} && \sum_{i=1}^{N+1} w_i = 1 \end{aligned}$$

$\mathbf{w}$ defines the vector of asset weights. The solution to the above-optimization problem is to select the weight of each risky asset i satisfying the following relationship (Luenberger 1998):

$$\sum_{j=1}^{N} \sigma_{ij} w_j = \mu_i - r_f \tag{11}$$

We will extend the example used in Luenberger (1998) for demonstrating different investment strategies studied in this paper. In this example, there are three risky assets, $i = 1, 2$ and 3. A portfolio manager or an investor needs to specify the asset mean, variance and correlation coefficients. She also specifies the risk-free rate and investment horizon. The following is the set of input parameters specified for this example:

(i) Initial portfolio value: $V(0) = 1\$$
(ii) Mean vector:

$$\boldsymbol{\mu} = \begin{bmatrix}\mu_1\ \mu_2\ \mu_3\end{bmatrix} = \begin{bmatrix}0.24\ 0.20\ 0.15\end{bmatrix}$$

(iii) Asset standard deviation vector:

$$\boldsymbol{\Sigma} = \begin{bmatrix}\sigma_1\ \sigma_2\ \sigma_3\end{bmatrix} = \begin{bmatrix}0.3000\ 0.2646\ 0.1732\end{bmatrix}$$

(iv) Asset correlation coefficients:

$$\boldsymbol{\rho} = \begin{bmatrix}\rho_{11} & \rho_{12} & \rho_{13}\\ \rho_{21} & \rho_{22} & \rho_{23}\\ \rho_{31} & \rho_{32} & \rho_{33}\end{bmatrix} = \begin{bmatrix}1.0000 & 0.2520 & 0.1925\\ 0.2520 & 1.0000 & -0.2182\\ 0.1925 & -0.2182 & 1.0000\end{bmatrix}$$

(v) Risk-free rate: $r_f = 0.1$
(vi) Investment horizon: $T = 30$ years.

For the example investment problem, we obtain:

$$\mathbf{w} = \begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix} = \begin{bmatrix}1.0509\\ 1.3818\\ 1.7770\\ -3.2098\end{bmatrix}$$

The negative sign indicates that the risk-free asset needs to be borrowed. A portfolio set-up using the above weights will maximize the portfolio growth in the long run if the weights are always maintained to $\mathbf{w}$ by continuously rebalancing.

Using $\mathbf{w}$ and the equations (7)–(9) we obtain:

$$\mu_p = 0.4742, \sigma_p^2 = 0.3742, \nu_p = 0.2871$$

2.2. *Passive strategy*

We use τ to denote the *periodic rebalance frequency* which measures the time interval between two consecutive act of portfolio rebalancing. In the prior section, we reviewed the log-optimal strategy to maximize portfolio growth where the portfolio is continuously rebalanced with $\tau = 0$ to optimal weight vector $\mathbf{w}$. In this section, we review the framework to assess the nature of portfolio growth when the investor sets up the portfolio with the optimal weight vector $\mathbf{w}$ and never rebalances again. We term such a strategy as *passive strategy* where the rebalance frequency $\tau = \infty$.

Throughout our analysis, we use the pertinent rebalance frequency as the superscript with parameters. All parameters for passive strategy will have a superscript of ∞. In the absence of any such superscript, the parameter pertains to active strategy. Note that the initial investment parameters enumerated under section 2.1 will be applicable to all strategies discussed in this paper.

Under passive investment strategy, the expected value and variance of portfolio growth at any time $t > 0$ are given by:

$$X(t) = E[V^{\infty}(t)] = \sum_{i=1}^{N+1} w_i e^{\mu_i t} \tag{12a}$$

$$Y(t) = Var[V^{\infty}(t)] = \sum_{i,j=1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}(e^{\sigma_{ij}t} - 1) \tag{12b}$$

The variance of log of portfolio value and portfolio growth under passive strategy are given by:

$$\Upsilon^\infty(t) = Var[ln(V^\infty(t))]$$
$$= ln(1 + \frac{\sum_{i,j=1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}(e^{\sigma_{ij}t} - 1)}{(\sum_{i=1}^{N+1} w_i e^{\mu_i t})^2}) \quad (13a)$$

$$\chi^\infty(t) = E[ln(V^\infty(t))] = ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) - \frac{1}{2}\Upsilon^\infty(t) \quad (13b)$$

The variance and expected values in equations (13a) and (13b) are obtained under the assumption of log-normality for sum of log-normal random variables.

2.3. Rebalancing frequency

In one possible rebalancing approach, investor may continue to use passive strategy until passive portfolio growth drops below that of active strategy. In the absence of transaction cost, duration of this rebalance time will be determined by the point of intersection of equations (10) and (13b). The investor will perform even better if the investor continues to use passive strategy as long as *instantaneous portfolio growth* offered by passive strategy is higher than or equal to that under active strategy. This period, denoted by τ_s, satisfies the following condition:

$$\exists\tau_s \text{ s.t. } \xi^\infty(t) > \nu_p, \forall t \in (0\tau_s) \quad (14)$$

The instantaneous growth, ξ is the derivative of portfolio growth with respect to time t. In this notation, using equation (10), we can write:

$$\xi = \frac{d\chi(t)}{dt} = \nu_p \quad (15)$$

We see that under active strategy, the instantaneous growth ξ equals ν_p, an invariant of time. Using equivalent notation for passive strategy, we can write,

$$\xi^\infty(t) = \frac{d\chi^\infty(t)}{dt} \quad (16)$$

We assume that the investor will only select assets with positive returns for portfolio composition. Under such scenario the passive expected log growth, $\chi^\infty(.)$ will be a monotonically increasing function. Without going through rigorous mathematical proof, for all our subsequent analysis, we assume that under passive strategy such monotonicity results in *unimodal* instantaneous growth evolution with time. We denote τ_m as the unique maxima of *instantaneous growth* with a value of $\xi^\infty(\tau_m)$. At τ_s, the instantaneous growth for passive strategy becomes equal to that for active strategy.

$$\xi^\infty(\tau_s) = \frac{1}{X(t)}\Big[X'(t) - \frac{1}{2}\frac{X(t)Y'(t) - 2X'(t)Y(t)}{X(t)^2 + Y(t)}\Big]\Big|_{t=\tau_s} = \nu_p \quad (17)$$

where,

$X(t) =$ expected portfolio growth at time t, given by equation (12a)

$Y(t) =$ variance of portfolio growth at time t, given by equation (12b)

$$X'(t) = \frac{dX}{dt} = \sum_{i=1}^{N+1} w_i \mu_i e^{\mu_i t} \quad (18a)$$

$$Y'(t) = \frac{dY}{dt}$$
$$= \sum_{i,j=1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}[(\mu_i + \mu_j)(e^{\sigma_{ij}t} - 1) + \sigma_{ij}e^{\sigma_{ij}t}] \quad (18b)$$

For our example portfolio, this rebalancing frequency τ_s = 3.7 years. We denote $\psi^\infty(t) = \chi^\infty(t) - \chi(t)$ as the *excess growth* relative to active strategy. It turns out that the excess passive growth $\psi^\infty(t)$ is a monotonously increasing function for $0 < t < \tau_s$ is maximized at τ_s.

2.4. Hybrid strategy

Our goal is to find the periodic frequency $\tau = \tau_o$ at which the investor can rebalance the portfolio to the initial optimal weights to maximize portfolio growth for the intended investment horizon. The frequency τ is the time interval measured in years. Under such a *hybrid strategy* the portfolio is rebalanced periodically every τ years till the end of investment horizon. We use superscript $\tau \neq \infty$ to denote a hybrid strategy that uses τ as the rebalance frequency.

It turns out that for longer horizons, the investor can maintain higher investor utility by periodically rebalancing the portfolio every τ_s. Moreover, one can compute the portfolio growth for such a hybrid strategy using the portfolio growths from passive strategy. The necessary passive to hybrid growth transformation rules are given by the theorem known as the *growth map theorem*. The theorem is applicable for any hybrid strategy that periodically rebalances the portfolio using any non-zero frequency τ.

The *law of additive growth* states that the passive portfolio growth is additive, i.e.

$$V^\tau(k\tau + t') = V^\tau(k\tau)\sum_{i=1}^{N+1} w_i e^{x_i(t')} \quad (19)$$

where t^r is the most recent time when the portfolio is rebalanced and t' is the time for which the portfolio grows passively after t^r. The *law of multiplicative growth* states that the portfolio growth multiplies with the number of times periodic rebalancing is executed, i.e.

$$\chi^\tau(k\tau) = k\chi^\infty(\tau), \forall k \in \mathbb{N}^+, \tau \in \mathbb{R}^+ \quad (20)$$

where τ is the periodic rebalance frequency. With the help of these two laws, we can arrive at the growth map theorem the proof of which can be found in Das *et al.* (2014).

THEOREM 2.1 *Assume that $\chi^\tau(t) = \chi^\infty(t)$, $\forall t \in (0\tau]$ is known following equation (13b). Then $\forall t > \tau > 0$,*

$$\chi^\tau(t) = \begin{cases} \nu_p t & \text{if } \tau = 0 \\ k\chi^\infty(\tau) + \chi^\infty(t') & \text{otherwise} \end{cases} \quad (21)$$

where $t = k\tau + t'$, $k = \lfloor t/\tau \rfloor$ and $t' = t$ mod τ.

The growth map theorem 2.1 establishes the relationship between passive and hybrid strategy portfolio growths. It states that under any hybrid strategy where rebalancing is done with periodicity of τ, the portfolio growth at subsequent rebalancing points can be obtained by multiplying the portfolio growth at the first rebalance point by the number of times the portfolio has been rebalanced. Note that in our approach, we always rebalance to the initial optimal weights, $\mathbf{w}$. Once we obtain the portfolio growth at the last rebalance point, portfolio growth for any additional time $t' < \tau$ will follow the passive trajectory identical to the initial rebalance period. This theorem holds true for any positive finite value of periodic rebalance frequency τ, not just simple or stable rebalance frequencies.

For any hybrid strategy with periodic rebalance frequency τ, once the passive trajectory $\chi^{\infty}(.)$ is calculated for the initial duration up to the first rebalance time, i.e. $[0\ \tau]$, we can completely construct the hybrid trajectory $\chi^{\tau}(.)$ for any future investment horizon. In real life investment, the performance of hybrid strategy will even be better once we factor in the cost of rebalancing. Figure 1(a) shows the evolution of expected instantaneous growth under hybrid strategy with a periodic rebalance frequency of τ_s. The growth is never allowed to slip below the corresponding value ν_p under active strategy. Under such a strategy, the instantaneous growth during the entire investment horizon always remains higher or equal to that under active strategy.

As shown in figure 1(b), hybrid strategy with τ_s as the periodic rebalancing frequency yields higher portfolio growth. It turns out that such hybrid strategy will always outperform a hybrid strategy with higher rebalancing frequency, i.e. $\chi^{\tau_s}(T) > \chi^{\tau_x}(T)$, where $T > \tau_x > \tau_s$.

2.5. *Optimal rebalance frequency function*

The obvious question now is if there exists a rebalance frequency at which the portfolio growth is maximum for a given investment horizon. The key to find the answer is to study the results of growth map theorem 2.1. The theorem provides the portfolio growth attained for a given horizon T when a particular rebalance frequency τ is used.

For a given investment horizon, one can use equation (21) to compute the portfolio growth for any non-zero value of rebalance frequency τ. We need to obtain the partial derivative of equation (21) with respect to τ to search for a maxima. In its current form, equation (21) is expressed in terms of floor and mod functions which are non-continuous piecewise linear functions. It turns out, it is difficult to differentiate this equation. Thus, our first attempt is to follow a numerical approach to search for the maxima of the equation for a given T when τ is varied.

$$\tau_o(T) = \tau,\ \text{s.t.} \max_{0\le\tau\le min(\tau_s,T)} \chi^{\tau}(T) \tag{22}$$

Figure 2 illustrates the variation of $\tau_o(T)$ when horizon T is varied from 0 to 30 years for our example portfolio. Observe that the optimal frequency is a function of length of investment horizon. Observe that optimal frequency function is piecewise linear and shows a sawtooth-like pattern of fluctuation. The amplitude optimal frequency fluctuation is larger for smaller values of T. As T increases, the amplitude decreases. For very large horizon, the optimal frequency converges to a single value. We call this converged frequency as *asymptotic* optimal frequency denoted by τ_{ao}. The following theorem establishes the convergence condition, the proof of which can be found in Das *et al.* (2014).

Theorem 2.2 *For large values of investment horizon T, the optimal frequency will asymptotically converge to τ_{ao}, the time at which instantaneous growth becomes equal to expected portfolio growth rate under a passive strategy, i.e.*

$$\xi^{\infty}(\tau_{ao}) = \nu_p^{\infty}(\tau_{ao}) \tag{23}$$

For the example portfolio τ_{ao} is found to be 1.65 years.

For any given horizon T, equation (22) examines a set of candidate rebalance frequencies before selecting the optimal choice of τ_o. We define this set as *rebalance frequency domain* $\Im_s(T)$. For this algorithm $\Im_s(T)$ contains the following elements:

$$\Im_s(T) = \left\{ m\delta t : \forall m \in \mathbb{N}^+ \text{ s.t. } m \le \left\lfloor \frac{min(\tau_s, T)}{\delta t} \right\rfloor \right\} \tag{24}$$

For our example portfolio, we have seen the value of $\tau_s = 3.7$. For a reasonable value of $\delta t = 0.001$, the cardinality of $\Im_s(30)$ shall be $\lfloor \frac{min(3.7,30)}{0.001} \rfloor = 3700$. Thus, one has to search

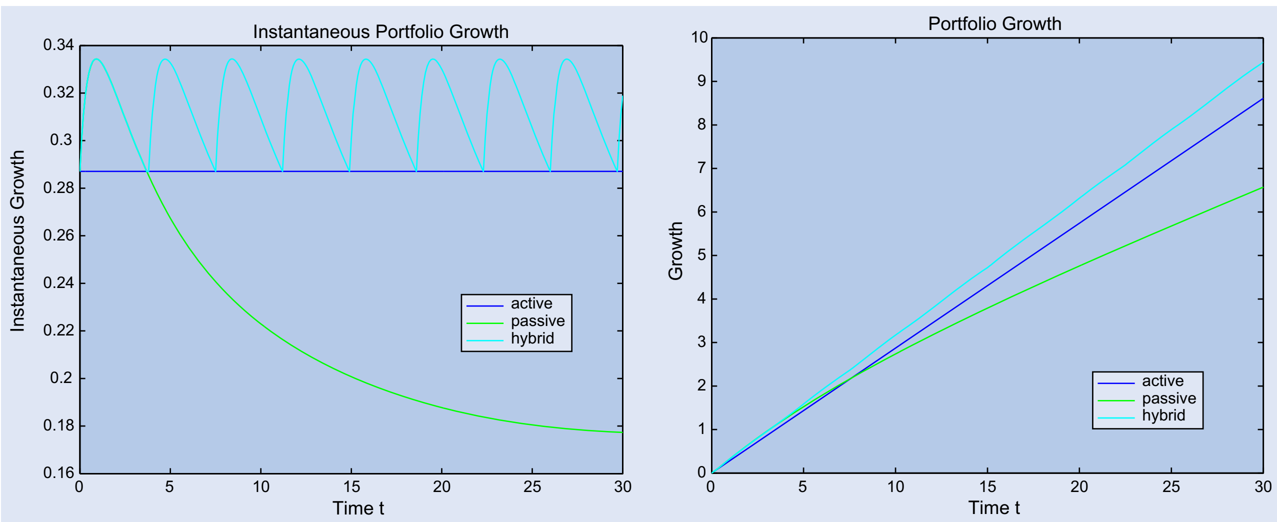

Figure 1. Expected portfolio value and instantaneous growth in stable hybrid strategy.

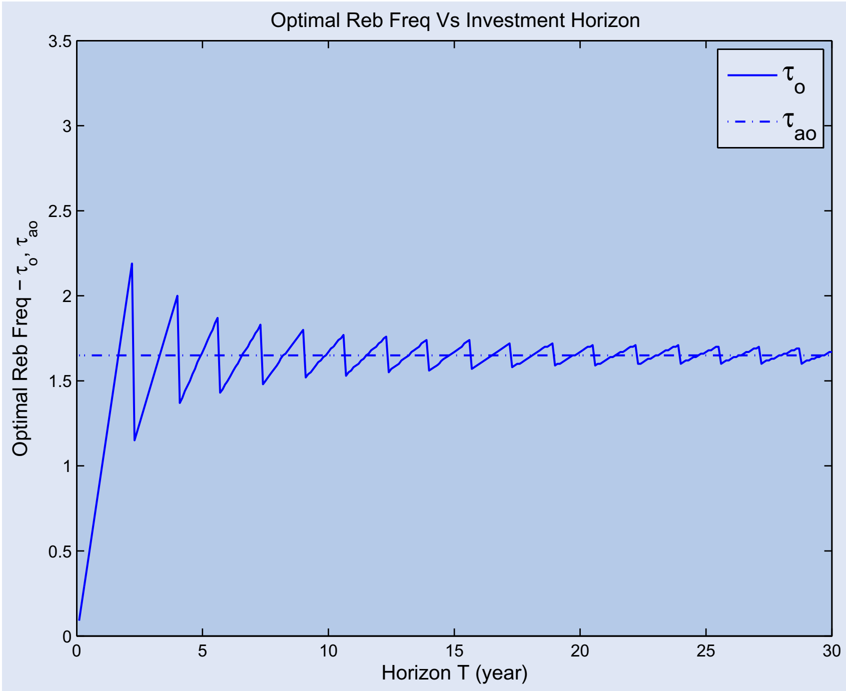

Figure 2. Fluctuation of optimal rebalance frequency.

for 3700 possible candidates to find the optimal frequency $\tau_o(30)$. The search space increases as the horizon value is increased from 0 to $\tau_s = 3.7$. It remains the same for any investment horizon longer than $\tau_s = 3.7$.

2.6. *Discrete rebalance divisor*

It turns out that, not all time values between 0 and τ_s are candidates for rebalance frequency domain. The domain of any log-optimal portfolio is restricted to only the factors of horizon T with positive integer divisors, i.e.

$$\Im_k(T) = \left\{ \frac{T}{k} : \forall k \in \mathbb{N}^+ \right\} \tag{25}$$

The portfolio growth at $\tau_o = \left(\frac{T}{k}\right)$ is given by:

$$\chi^{\tau_o}(T) = k\chi^{\infty}\left(\frac{T}{k}\right) \tag{26}$$

The rebalance frequency domain is restricted to only a infinite set of rational numbers. Henceforth, we describe the positive integer k as the *rebalance divisor* of the portfolio. A rebalance divisor divides the prescribed horizon into k equal segments. The portfolio has to be rebalanced after each segment to attain the terminal portfolio growth. In this way, the wealth grows following passive dynamics for k equal time periods. The portfolio growth at the end of the passive period $\frac{T}{k}$ multiplies k fold at the end of the horizon T.

Notice that the rebalance frequency domain is entirely determined only by the length of investment T independent of any other portfolio characteristics. Portfolio parameters only help determine the optimal frequency from the domain. There are infinite choices of rebalance divisors. Any search algorithm has to search for infinite possible alternative divisors to find the optimal τ_o. It is, however, possible to restrict the domain $\Im_k(T)$ to only a finite and countable set by finding upper and lower bounds for the rebalance divisor. While τ_s determines the lower bound of rebalance divisor as per equation (27), τ_m determines the upper bound k_{mx} as per equation (28) below:

$$k_{mn} = max\left(1, \left\lceil \frac{T}{\tau_s} \right\rceil\right) \tag{27}$$

$$k_{mx} = max\left(1, \left\lfloor \frac{T}{\tau_m} \right\rfloor\right) \tag{28}$$

For sufficiently large horizon τ_o converges to τ_{ao} Das *et al.* (2014). We refine the rebalance frequency domain further to a finite countable set as follows:

$$\Im_k(T) = \left\{ \frac{T}{k} : \forall k \in \mathbb{N}^+ \text{ and } k_{mn} \le k \le k_{mx} \right\} \tag{29}$$

Contrast the above rebalance frequency domain $\Im_k(T)$ to the prior domain $\Im_s(T)$ defined in equation (24). For our example portfolio, the values for $\tau_m = 0.91$ and $\tau_s = 3.7$ lead to $k_{mn} = 9$ and $k_{mx} = 32$ for horizon $T = 30$ years. This restricts the rebalance frequency domain to $\Im_k(30) = \{3.3, 3, 2.7, 2.5, \ldots, 1.0, 0.97, 0.94\}$. Thus, one has to search for only 24 possible candidates to find the optimal frequency $\tau_o(30)$. Similarly, an investment horizon of 12 years leads to $k_{mn} = 4$ and $k_{mx} = 13$. One has to search the domain $\Im_k(12) = \{3, 2.4, 2, 1.7, 1.5, 1.33, 1.2, 1.09, 1, 0.92\}$ consisting of only 10 discrete values to find $\tau_o(12)$. There is significant reduction in search space compared to earlier search-based rebalance domain. For any horizon value $\forall T \ge 3.7$, $\Im_s(T)$ has a cardinality of 370 and 3700 for δt values of 0.01 and 0.001 respectively.

Finally, one can define optimal rebalance frequency as:

$$\tau_o(T) = \tau, \text{ s.t. } \max_{k_{mn} \le k \le k_{mx}} k\chi^{\infty}\left(\frac{T}{k}\right) \tag{30}$$

3. Rebalance divisor optimality

Comparing equations (22) and (30), we see that we have significantly reduced the search space. In this section, we will explore the possibility to completely avoid searching for the optimal frequency. The first step in this direction is to analyse the nature of the equation (30).

3.1. Log utility rebalance contour

The hybrid portfolio evolution is governed by the function $k\chi^{\infty}(\frac{t}{k})$. The investor has a finite choice of such evolution paths or contours, one for each possible value of k between k_{mn} and k_{mx}. For any horizon T, τ_o is determined by the rebalance divisor k defining the contour that yields the maximum portfolio growth for $t = T$†. We describe each such contour as a *rebalance contour*. The function Ł: $(\mathbb{N}^+, \mathbb{R}^+) \mapsto \mathbb{R}^+$ defines time evolution of the kth rebalance contour as follows:

$$Ł(k,t) = k\chi^{\infty}\left(\frac{t}{k}\right), \forall k \in \mathbb{N}^+, \forall t \in \mathbb{R}^+ \tag{31}$$

The rebalance contour function Ł(k, t) defines evolution of portfolio growth if the investor adopts a hybrid strategy rebalancing after every $\frac{t}{k}$ time interval. Using equation (29), for a given horizon, the investor needs to adopt a hybrid strategy that corresponds to one of the finite possible rebalance contours to optimize the log utility.

The functional notation in general describes all the three strategies we have discussed so far. Ł$(0, t)$ and Ł$(1, t)$ describe the portfolio growth under active and passive strategies, respectively. The following are some generic equivalent notations:

$$Ł(0,t) = \chi^{\infty}(t) = \chi(t) \tag{32a}$$

$$Ł(1,t) = \chi^{t}(t) = \chi^{\infty}(t) \tag{32b}$$

$$Ł(k,t) = \chi^{\frac{t}{k}}(t) = k\chi^{\infty}\left(\frac{t}{k}\right) \tag{32c}$$

Given a horizon t there are only finite such *rebalance contours* we need to consider corresponding to all possible rebalance divisors k used in defining $\Im_k(t)$. Figure 3(a) illustrates conceptual $rebalancecontours$ for three values of rebalance divisors k, $k+1$ and $k+2$. Before we proceed, we will derive a few important properties of rebalance contour.

Lemma 3.1 *The instantaneous growth of the kth rebalance contour is given by:*

$$\frac{\partial Ł(k,t)}{\partial t} = \xi^{\infty}\left(\frac{t}{k}\right) \tag{33}$$

Proof The proof is straight forward. Differentiating equation (31) with respect t gives us the following:

$$\frac{\partial Ł(k,t)}{\partial t} = k\frac{\partial \chi^{\infty}(\frac{t}{k})}{\partial(\frac{t}{k})}\frac{\partial(\frac{t}{k})}{\partial t} = \xi^{\infty}\left(\frac{t}{k}\right) \tag{34}$$

□

Let's look at the nature of instantaneous growth evolution for various rebalance contours. The first rebalance contour with $k = 1$ always evolves following the passive portfolio growth pattern. As we increase the value of k, instantaneous growth of rebalance contour becomes increasingly flatter as illustrated in figure 3(b). Instantaneous growth of all rebalance contours have the same maximum value of $\xi^{\infty}(\tau_m)$ although occurring at different time points. The instantaneous growth of kth rebalance contour maximizes at $k\tau_m$. Note that for a given time interval, the area under the instantaneous growth of kth rebalance contour calculates the change in portfolio growth during the interval. Because of the instantaneous growth asymmetry, two different rebalance contours will have different levels of performance in generating portfolio growth for different lengths of horizon. As an example, referring to the figure 3(a), after horizon $T_{k,k+1}$, $(k+1)$th rebalance contour surpasses kth rebalance contour in performance to generate higher portfolio growth.

3.2. Inflection point

We describe the horizon at which two rebalance contours intersect as an *inflection point*. At any inflection point one rebalance contour's performance surpasses the performance of another rebalance contour. We will denote the inflection point of two different rebalance contours for kth and k'th as $T_{k,k'}$. As an illustration figure 3(a) shows three different inflection points generated by kth, $(k+1)$th and $(k+2)$th rebalance contours. We describe $T_{k,k+a}, a \in \mathbb{N}^+$ as the ath inflection point for the kth rebalance contour. For example $T_{k,k+1}$, $T_{k,k+2}$ and $T_{k,k+3}$ are the first, second and third inflection points, respectively, for the kth rebalance contour. Note that by definition k'th inflection point of kth rebalance contour is same as kth inflection point of k'th rebalance contour. Using these notations, $T_{k,k'} = T_{k',k}, \forall k \neq k', \{k,k\}' \in \mathbb{N}^+$. As a convention, we prefer to use $T_{k,k'}$ where $k < k'$ to denote the inflection point of kth and k'th rebalance contours.

Lemma 3.2 *An inflection point shall have a lower bound given by:*

$$T_{k,k+a} > (k+a)\tau_m, \quad \forall a >\in \mathbb{N}^+ \tag{35}$$

Proof For simplicity of notation, we will use $T_{ka} = T_{k,k+a}$. We start with the definition of $T_{k,k+a}$.

$$Ł(k, T_{ka}) = Ł(k+a, T_{ka})$$

$$\Rightarrow k\chi^{\infty}\left(\frac{T_{ka}}{k}\right) = (k+a)\chi^{\infty}\left(\frac{T_{ka}}{k+a}\right) \tag{36}$$

Using lemma 3.1 we know that instantaneous growth for Ł(k, t) is increasing and peaks at $k\tau_m$. Thus, Ł(k, t) has higher instantaneous growth than that of Ł$(k+a, t)$ when $t \leq k\tau_m$. Hence $T_{ka} > k\tau_m$ since Ł(k, t) will not intersect Ł$(k+a, t)$ otherwise. Therefore,

$$T_{ka} > k\tau_m \Rightarrow \frac{T_{ka}}{k} > \tau_m$$

$$\Rightarrow \frac{T_{ka}}{k} = \tau_m + \Delta t_1, \text{ for some } \Delta t_1 > 0 \tag{37a}$$

$$\Rightarrow T_{ka} = k(\tau_m + \Delta t_1) \tag{37b}$$

We will prove the lemma's proposition by contradiction. Suppose the proposition is not true, i.e.

†In subsequent discussions we use both t and T interchangeably. We prefer to use t when the horizon value is used in the context of a variable of a function.

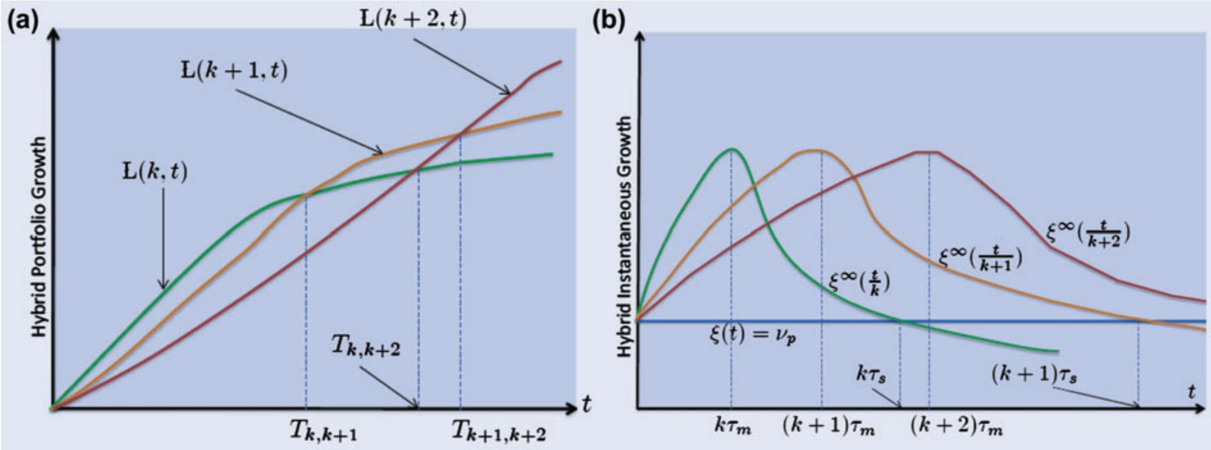

Figure 3. Illustrations of rebalance contour, inflection point and instantaneous growth of rebalance contour.

$$T_{ka} \leq (k+a)\tau_m \Rightarrow \frac{T_{ka}}{k+a} \leq \tau_m$$

$$\Rightarrow \frac{T_{ka}}{k+a} = \tau_m - \Delta t_2, \text{ for some } \Delta t_2 \geq 0 \tag{38a}$$

$$\Rightarrow T_{ka} = (k+a)(\tau_m - \Delta t_2) \tag{38b}$$

Substituting equations (37a) and (38a) in equation (36), we obtain:

$$k\chi^\infty(\tau_m + \Delta t_1) = (k+a)\chi^\infty(\tau_m - \Delta t_2) \tag{39}$$

Furthermore, from equations (37b) and (38b), we obtain:

$$k(\tau_m + \Delta t_1) = (k+a)(\tau_m - \Delta t_2) \tag{40}$$

Both equations (39) and (40) can only be true when passive portfolio evolution function, $\chi^\infty(t)$ is linear and takes the form of $\chi^\infty(t) = ct$ for any constant c. However, we know from equation (13b) that $\chi^\infty(t)$ is not linear. This contradiction shows that the supposition is false and so the given proposition of this lemma is true. □

LEMMA 3.3 *Let $T_{k,k+a}$ be the ath inflection point for the kth rebalance contour where $\{k, a\} \in \mathbb{N}^+$. The relative performance of the kth and $(k+a)$th rebalance contours shall meet the following constraints:*

$$Ł(k,t) > Ł(k+a,t), \forall 0 < t < T_{k,k+a} \tag{41a}$$

$$Ł(k,t) = Ł(k+a,t), \forall t = T_{k,k+a} \tag{41b}$$

$$Ł(k,t) < Ł(k+a,t), \forall t > T_{k,k+a} \tag{41c}$$

Proof We know that for initial investment of one dollar, all rebalance contours start with zero portfolio growth. From lemma 3.1, for any horizon t, the instantaneous growth for kth and $(k+a)$th rebalance contours shall be given by $\xi^\infty(\frac{t}{k})$ and $\xi^\infty(\frac{t}{k+a})$, respectively. From the unimodality assumption of instantaneous growth, $\xi^\infty(\frac{t}{k})$ will be increasing till $t = k\tau_m$. Observe that during the interval of $(0k\tau_m]$, $\xi^\infty(\frac{t}{k+a})$ is also increasing, albeit at a slower rate. During $(0k\tau_m]$, due to higher instantaneous growth, kth rebalance contour will have higher portfolio growth than $(k+a)$th rebalance contour. Since a rebalance contour is monotonically increasing kth rebalance contour will eventually catch up with $(k+a)$th rebalance contour at $T_{k,k+a}$. This proves that equation (41a) holds.

Equation (41b) holds from the definition of inflection point $T_{k,k+a}$.

Using the results of lemma 3.2, the following relationship shall always be satisfied:

$$\frac{T_{k,k+a}}{k} > \frac{T_{k,k+a}}{k+a} > \tau_m$$

$$\Rightarrow \xi^\infty\left(\frac{T_{k,k+a}}{k}\right) < \xi^\infty\left(\frac{T_{k,k+a}}{k+a}\right) \tag{42}$$

Equation (42) holds since both $\frac{T_{k,k+a}}{k}$ and $\frac{T_{k,k+a}}{k+a}$ fall on the decreasing part of passive instantaneous growth curve ξ^∞. Hence at $T_{k,k+a}$, $(k+a)$th rebalance contour will have higher instantaneous growth than kth rebalance contour. Note that at $T_{k,k+a}$ both the rebalance contours yield equal portfolio growth. But due to higher instantaneous growth at $T_{k,k+a}$, for $t > T_{k,k+a}$, $(k+a)$th rebalance contour will remain higher than kth rebalance contour satisfying equation (41c). □

LEMMA 3.4 *Let $T_{k,k+b}$ be the bth inflection point for the kth rebalance contour. Any rebalance contour with rebalance divisor higher than k shall have lower portfolio growth for investment horizon of $T_{k,k+b}$. Mathematically,*

$$Ł(k, T_{k,k+b}) > Ł(k+a, T_{k,k+b}), \forall a > b, \{k,a,b\} \in \mathbb{N}^+ \tag{43}$$

Proof For notational simplicity, we will use T_{kb} to denote $T_{k,k+b}$. By the definition of inflection point we can write:

$$k\chi^\infty\left(\frac{T_{kb}}{k}\right) = (k+b)\chi^\infty\left(\frac{T_{kb}}{k+b}\right)$$

$$\Rightarrow \chi^\infty\left(\frac{T_{kb}}{k}\right) = \frac{(k+b)}{b}\left[\chi^\infty\left(\frac{T_{kb}}{k}\right) - \chi^\infty\left(\frac{T_{kb}}{k+b}\right)\right]$$

$$= \frac{(k+b)}{b}\Delta\chi_k^{k+b} \tag{44}$$

where using the illustration in figure 4, $\Delta\chi_k^{k+b}$ is given by:

$$\Delta\chi_k^{k+b} = \chi^\infty\left(\frac{T_{kb}}{k}\right) - \chi^\infty\left(\frac{T_{kb}}{k+b}\right)$$

$$= \int_0^{\frac{T_{kb}}{k}} \xi^\infty(t)\,\mathrm{d}t - \int_0^{\frac{T_{kb}}{k+b}} \xi^\infty(t)\,\mathrm{d}t \tag{45}$$

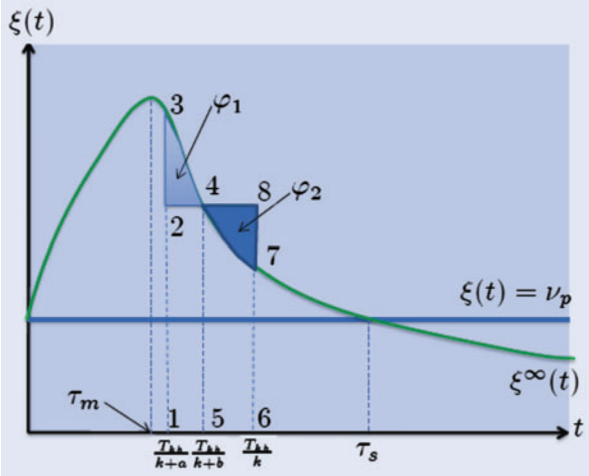

Figure 4. Illustration for proof of lemma 3.4.

Figure 4 illustrates the positions of different horizon points, viz. $\frac{T_{kb}}{k}$, $\frac{T_{kb}}{k+b}$ and $\frac{T_{kb}}{k+a}$. Note that due to the lower and upper bounds set by lemmas 3.2 and 3.10, the following relationship between the horizon points holds:

$$\tau_s > \frac{T_{kb}}{k} > \frac{T_{kb}}{k+b} > \tau_m \tag{46}$$

The horizon point $\frac{T_{kb}}{k+a}$ may lie anywhere within $(0\frac{T_{kb}}{k+b}]$. This figure places these time points on the right hand side of τ_m representing the worst case scenario that one needs to prove.

Expanding equation (43), we need to prove the following:

$$\begin{aligned}
& k\chi^{\infty}\left(\frac{T_{kb}}{k}\right) > (k+a)\chi^{\infty}\left(\frac{T_{kb}}{k+a}\right) \\
\Rightarrow & k\left[\chi^{\infty}\left(\frac{T_{kb}}{k}\right) - \chi^{\infty}\left(\frac{T_{kb}}{k+a}\right)\right] > a\chi^{\infty}\left(\frac{T_{kb}}{k+a}\right) \\
\Rightarrow & k\Delta\chi_k^{k+a} > a\left[\chi^{\infty}\left(\frac{T_{kb}}{k}\right) - \Delta\chi_k^{k+a}\right] \\
\Rightarrow & (k+a)\Delta\chi_k^{k+a} > a\chi^{\infty}\left(\frac{T_{kb}}{k}\right)
\end{aligned} \tag{47}$$

where, using the illustration in figure 4, $\Delta\chi_k^{k+a}$ is given by:

$$\begin{aligned}
\Delta\chi_k^{k+a} &= \chi^{\infty}\left(\frac{T_{kb}}{k}\right) - \chi^{\infty}\left(\frac{T_{kb}}{k+a}\right) \\
&= \int_0^{\frac{T_{kb}}{k}} \xi^{\infty}(t)\,\mathrm{d}t - \int_0^{\frac{T_{kb}}{k+a}} \xi^{\infty}(t)\,\mathrm{d}t
\end{aligned} \tag{48}$$

Substituting the value of $\chi^{\infty}\left(\frac{T_{kb}}{k}\right)$ from equation (44) in equation (47), we need to prove the following:

$$\begin{aligned}
& (k+a)\Delta\chi_k^{k+a} > \frac{a(k+b)}{b}\Delta\chi_k^{k+b} \\
\Rightarrow & (k+a)[\Delta\chi_k^{k+a} - \Delta\chi_k^{k+b}] > \frac{k(a-b)}{b}\Delta\chi_k^{k+b} \\
\Rightarrow & (k+a)\left[\int_0^{\frac{T_{kb}}{k}} \xi^{\infty}(t)\,\mathrm{d}t - \int_0^{\frac{T_{kb}}{k+a}} \xi^{\infty}(t)\,\mathrm{d}t \right. \\
& \left. - \int_0^{\frac{T_{kb}}{k}} \xi^{\infty}(t)\,\mathrm{d}t + \int_0^{\frac{T_{kb}}{k+b}} \xi^{\infty}(t)\,\mathrm{d}t\right] > \frac{k(a-b)}{b}\Delta\chi_k^{k+b},
\end{aligned}$$

using equations (45) and (48)

$$\begin{aligned}
\Rightarrow & (k+a)\left[\int_0^{\frac{T_{kb}}{k+b}} \xi^{\infty}(t)\,\mathrm{d}t - \int_0^{\frac{T_{kb}}{k+a}} \xi^{\infty}(t)\,\mathrm{d}t\right] \\
& > \frac{k(a-b)}{b}\Delta\chi_k^{k+b} \\
\Rightarrow & (k+a)\Delta\chi_{k+b}^{k+a} > \frac{k(a-b)}{b}\Delta\chi_k^{k+b}
\end{aligned} \tag{49}$$

We can simplify LHS of equation (49) below:

$$\begin{aligned}
& (k+a)\Delta\chi_{k+b}^{k+a} \\
= & (k+a)(\text{Area of region 1-2-3-4-5}) \\
= & (k+a)[(\text{Area of region 1-2-4-5}) + (\text{Area of region 2-3-4})] \\
= & (k+a)\left[\left(\frac{T_{kb}}{k+b} - \frac{T_{kb}}{k+a}\right)\xi^{\infty}\left(\frac{T_{kb}}{k+b}\right) + \varphi_1\right] \\
= & (k+a)\left[\frac{(a-b)T_{kb}}{(k+b)(k+a)}\xi^{\infty}\left(\frac{T_{kb}}{k+b}\right) + \varphi_1\right] \\
= & \frac{(a-b)T_{kb}}{(k+b)}\xi^{\infty}\left(\frac{T_{kb}}{k+b}\right) + (k+a)\varphi_1
\end{aligned} \tag{50}$$

We can simplify RHS of equation (49) below:

$$\begin{aligned}
& \frac{k(a-b)}{b}\Delta\chi_k^{k+b} \\
= & \frac{k(a-b)}{b}(\text{Area of region 4-5-6-7}) \\
= & \frac{k(a-b)}{b}[(\text{Area of region 4-5-6-7-8}) - (\text{Area of region 4-7-8})] \\
= & \frac{k(a-b)}{b}\left[\left(\frac{T_{kb}}{k} - \frac{T_{kb}}{k+b}\right)\xi^{\infty}\left(\frac{T_{kb}}{k+b}\right) - \varphi_2\right] \\
= & \frac{k(a-b)}{b}\left[\frac{bT_{kb}}{k(k+b)}\xi^{\infty}\left(\frac{T_{kb}}{k+b}\right) - \varphi_2\right] \\
= & \frac{(a-b)T_{kb}}{(k+b)}\xi^{\infty}\left(\frac{T_{kb}}{k+b}\right) - \frac{k(a-b)}{b}\varphi_2
\end{aligned} \tag{51}$$

Since $\varphi_1, \varphi_2 > 0$, $a > b$ and $k \geq 1$, we see that the terms $(k+a)\varphi_1$ and $\frac{k(a-b)}{b}\varphi_2$ are both positive. Comparing equations (50) and (51), we conclude that equation (49) holds true. □

LEMMA 3.5 *For any given rebalance contour higher order inflection points are always longer. Equivalently if $T_{k,k+a}$ and $T_{k,k+b}$ be the ath and bth reflection points respectively for the kth rebalance contour, then the following must be true:*

$$T_{k,k+a} > T_{k,k+b}, \forall a > b, \{k, a, b\} \in \mathbb{N}^+ \tag{52}$$

Proof We will prove this proposition by contradiction. Suppose the proposition of this lemma is not true. Then either $T_{k,k+a} < T_{k,k+b}$ or $T_{k,k+a} = T_{k,k+b}$. Let's first suppose $T_{k,k+a} < T_{k,k+b}$.

Since $a > b$, from lemma 3.4 we can write:

$$Ł(k, T_{k,k+b}) > Ł(k+a, T_{k,k+b}) \tag{53}$$

Using the results of lemma 3.3, from equation (41c) we obtain:

$$Ł(k, t) < Ł(k+a, t), \forall t > T_{k,k+a} \tag{54}$$

Then under the assumption that $T_{k,k+b} > T_{k,k+a}$, the following must be true:

$$Ł(k, T_{k,k+b}) < Ł(k+a, T_{k,k+b}) \tag{55}$$

This contradiction in equations (53) and (55) shows that the supposition $T_{k,k+a} < T_{k,k+b}$ is false.

Let's suppose $T_{k,k+a} = T_{k,k+b}$. Then by fundamental definition of inflection point, at $T_{k,k+b}$ all three rebalance contours, viz. kth, $(k+a)$th and $(k+b)$th shall have identical portfolio growth. Mathematically,

$$Ł(k, T_{k,k+b}) = Ł(k+a, T_{k,k+b}) = Ł(k+b, T_{k,k+b}) \quad (56)$$

Once again we arrive at contradictory results between equations (53) and (56). Therefore the supposition $T_{k,k+a} = T_{k,k+b}$ is also false. Thus the given proposition of this lemma is true. □

LEMMA 3.6 *Rebalance contours with higher rebalance divisor shall have longer inflection point of the same order. Equivalently if $T_{k-a,k}$ and $T_{k-b,k}$ be the kth order reflection points for $(k-a)$th and $(k-b)$th rebalance contours respectively, then the following must be true:*

$$T_{k-a,k} < T_{k-b,k}, \forall k > a > b, \{k, a, b\} \in \mathbb{N}^+ \quad (57)$$

Proof Using the results of lemma 3.5 we know that the following must be true:

$$T_{k-a,k-b} < T_{k-a,k} \quad (58)$$

Using the results of lemma 3.3 and equation (41c) we also know that the following must be true:

$$Ł(k-a, t) < Ł(k-b, t), \forall t > T_{k-a,k-b} \quad (59)$$

From equations (58) and (59) we obtain:

$$Ł(k-a, T_{k-a,k}) < Ł(k-b, T_{k-a,k}) \quad (60)$$

By definition $Ł(k-a, T_{k-a,k}) = Ł(k, T_{k-a,k})$, i.e. at $T_{k-a,k}$ the portfolio growth for the kth and $(k-a)$th rebalance contours are identical. Therefore,

$$Ł(k, T_{k-a,k}) < Ł(k-b, T_{k-a,k}) \quad (61)$$

Again using the results of lemma 3.3 and equation (41a) we know that the following must be true:

$$Ł(k-b, t) > Ł(k, t), \forall t < T_{k-b,k} \quad (62)$$

Comparing equations (61) and (62), we conclude that the following relationship must hold true:

$$T_{k-a,k} < T_{k-b,k} \quad (63)$$

□

We now state and prove *inflection points seriality* theorem. Theorem 3.7 states that a given rebalance contour's inflection points increase as the intersecting rebalance contour's rebalance divisor increases. For example, the inflection points of the rebalance contour with $k = 4$ shall maintain an increasing sequence of $T_{1,4} < T_{2,4} < T_{3,4} < T_{4,5} < T_{4,6} < \dots$.

THEOREM 3.7 *A rebalance contour has distinct and increasing inflection points, i.e.*

$$T_{k,k_1} > T_{k,k_2}, \forall k_1 > k_2, k \neq k_1 \neq k_2, \{k, k_1, k_2\} \in \mathbb{N}^+ \quad (64)$$

Proof Note that $T_{k,k}$ is not defined since an inflection point has to involve two different rebalance contours. There are two cases we need to consider. First case is when $k > k_1 > k_2$. For such cases, according to lemma 3.6 $T_{k_1,k} > T_{k_2,k}$ which is equivalent to $T_{k,k_1} > T_{k,k_2}$.

The second case is when $k_1 > k_2 > k$. For such cases, according to lemma 3.5 $T_{k,k_1} > T_{k,k_2}$. To complete the proof, we need to show that the maximum inflection point for first case is less than the minimum inflection point for the second case. In other words, we need to prove the following:

$$T_{k-1,k} < T_{k,k+1} \quad (65)$$

We will prove the proposition in equation (65) by contradiction. Suppose the proposition is not true, i.e. one of the following two equations must hold:

$$T_{k-1,k} > T_{k,k+1} \quad (66a)$$

$$T_{k-1,k} = T_{k,k+1} \quad (66b)$$

According to lemma 3.4, $(k-1)$th and kth rebalance contours shall have higher portfolio growth than $(k+1)th$ rebalance contour for investment horizon of $T_{k-1,k}$. Mathematically,

$$Ł(k, T_{k-1,k}) > Ł(k+1, T_{k-1,k}) \quad (67)$$

From equation (41c) of lemma 3.3, the following must hold:

$$Ł(k, t) < Ł(k+1, t), \forall t > T_{k,k+1} \quad (68)$$

From equations (66a) and (68), the following relationship must hold:

$$Ł(k, T_{k-1,k}) < Ł(k+1, T_{k-1,k}) \quad (69)$$

We observe that equation (69) contradicts equation (67). Therefore the proposition in equation (66a) must be false.

Suppose equation (66b) is true. Then substituting equation (66b) in inequality (67) we obtain:

$$Ł(k, T_{k,k+1}) > Ł(k+1, T_{k,k+1}) \quad (70)$$

Moreover from equation (41b) of lemma 3.3, the following must hold:

$$Ł(k, T_{k,k+1}) = Ł(k+1, T_{k,k+1}) \quad (71)$$

Once again we arrive at contradiction in inequalities (70) and (71). Therefore equation (66b) must also be false. Thus, the relationship in equation (65) holds and hence we establish the proposition of this theorem. □

LEMMA 3.8 *The kth rebalance contour shall have higher expected log utility than all other rebalance contours with lower rebalance divisor $k' < k$ for any time horizon longer than inflection point $T_{k-1,k}$. Equivalently, the following must hold true:*

$$Ł(k, t) > Ł(k', t), \forall t > T_{k-1,k}, k > k', \{k, k'\} \in \mathbb{N}^+ \quad (72)$$

Proof We will use induction to prove this lemma. The base case is when $k = 2$ with only permissible value of $k' = 1$. We must prove that:

$$Ł(2, t) > Ł(1, t), \forall t > T_{1,2} \quad (73)$$

This is true due to the results of lemma 3.3 and equation (41c) for $k = a = 1$. Assume that the hypothesis in equation (72) holds for k. We also know from theorem 3.7 that the following relationship holds for inflection points:

$$T_{k,k+1} > T_{k-1,k} \quad (74)$$

Using the above relationship in equation (74) we can rewrite slightly less restrictive form of equation (72) as below:

$$Ł(k, t) > Ł(k', t), \forall t > T_{k,k+1} \quad (75)$$

We must now prove that equation (72) holds for $k+1$, i.e. the following must also be true:

$$Ł(k+1,t) > Ł(k',t), \forall t > T_{k,k+1}, k+1 > k' \quad (76)$$

Once again using equation (41c) of lemma 3.3, for $a = 1$ we obtain:

$$Ł(k+1,t) > Ł(k,t), \quad \forall t > T_{k,k+1} \quad (77)$$

Equations (75) and (77) jointly imply that equation (76) is true. Thus, we establish lemma 3.8. □

Thus far we have explored important properties of rebalance contours, rebalance divisors and inflection points. These set of properties will enable us to derive the maximum achievable portfolio growth with one of the permissible values of rebalance divisor. It turns out that the value of this optimum rebalance divisor depends on the desired investment horizon. We can divide the horizon into linear segments separated by predetermined inflection points. For each of the horizon segments an optimum rebalance divisor can be assigned that maximizes the portfolio growth for the horizon. Thus, each non-overlapping horizon segment can be associated with a *distinct* optimum rebalance divisor.

We now state and prove the *rebalance divisor optimality* theorem.

THEOREM 3.9 *The rebalance divisor k maximizes the portfolio growth for any investment horizon between $T_{k-1,k}$ and $T_{k,k+1}$. Mathematically,*

$$Ł(k,t) \geq Ł(k',t), \quad \forall t \in (T_{k-1,k}T_{k,k+1}], \\ k' \neq k, \{k,k'\} \in \mathbb{N}^+ \quad (78)$$

Proof Combining equations (41a) and (41b) of lemma 3.3, we know that the following is true:

$$Ł(k,t) \geq Ł(k_h,t), \quad \forall t \in (0T_{k,k_h}], k_h > k, \{k,k_h\} \in \mathbb{N}^+ \quad (79)$$

Furthermore using lemma 3.8, we obtain:

$$Ł(k,t) > Ł(k_l,t), \quad \forall t > T_{k-1,k}, k > k_l, \{k,k_l\} \in \mathbb{N}^+ \quad (80)$$

However from inflection point seriality theorem 3.7, we know that the following order of inflection points holds:

$$T_{k,k_h} \geq T_{k-1,k}, \quad \forall k_h > k \quad (81a)$$
$$T_{k-1,k} > T_{k,k+1}, \quad \forall k_h > k \quad (81b)$$

Using the relationship of inequality (81a), a less restrictive form of inequality (79) is as follows:

$$Ł(k,t) \geq Ł(k_h,t), \quad \forall t \in (0T_{k,k+1}] \quad (82)$$

Similarly, using the relationship of inequality (81b), a less restrictive form of inequality (80) is as follows:

$$Ł(k,t) > Ł(k_l,t), \quad \forall t \in (T_{k-1,k}T_{k,k+1}] \quad (83)$$

Equations (82) and (83) jointly imply the validity of the hypothesis in equation (78). □

3.3. Rebalance inflection point

Figure 5 illustrates a subset of possible rebalance contours with rebalance divisors $k-1$ through $k+2$. These rebalance contours participate in determining three distinct inflection points, viz. $T_{k-1,k}$, $T_{k,k+1}$ and $T_{k+1,k+2}$. As per theorem 3.9, the optimal rebalance divisor for any investment horizon between $T_{k-1,k}$ and $T_{k,k+1}$ shall be k. Therefore, the maximum possible portfolio growth shall be determined by kth rebalance contour as depicted by the bold uppermost segment during this horizon interval. Similarly, the optimal rebalance divisor for any investment horizon between $T_{k,k+1}$ and $T_{k+1,k+2}$ shall be $k+1$. In this case, $(k+1)$th rebalance contour shall determine the maximum possible portfolio growth traced in bold.

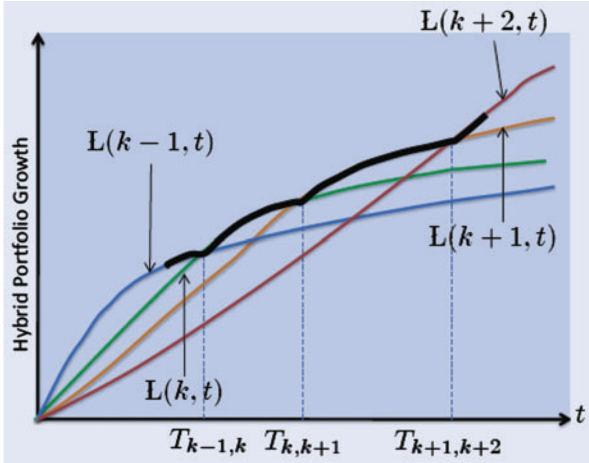

Figure 5. Rebalance inflection point and optimal log utility frontier (in dark bold line).

The inflection points of interest here are the ones which are generated by two adjacent rebalance contours. We term such a special inflection point $T_{k,k+1}, \forall k \in \mathbb{N}^+$ as *rebalance inflection point*. For completeness, we assume $T_{0,1}$ as the zeroth inflection point with a value of 0. For brevity of notation, henceforth we will drop the second subscript for specifying a inflection point. Thus, T_k denotes the kth inflection point equivalent to the expanded notation of $T_{k,k+1}$. In this parlance, 0 is the zeroth inflection point, T_1 is the first inflection point and so on.

By virtue of theorem 3.9, the entire investment horizon axis can be divided into piecewise intervals by the series of inflection points, $\{T_1, T_2, T_3, T_4, T_5 \dots\}$ with associated optimum rebalance divisors as $\{1, 2, 3, 4, 5, \dots\}$.

From the results of lemma 3.2, we have already seen that the kth inflection point has a lower bound as follows:

$$(k+1)\tau_m < T_k, \quad \forall k \in \mathbb{N}^+ \quad (84)$$

We now show that the inflection points also have an upper bound derived in lemma 3.10.

LEMMA 3.10 *The kth inflection point shall have an upper bound of $k\tau_s$, i.e.*

$$T_k < k\tau_s, \forall k \in \mathbb{N}^+ \quad (85)$$

Proof From the results of lemma 3.3, we know that for all values of $t > T_k$, the value of $Ł(k+1,t)$ exceeds $Ł(k,t)$ and vice versa. Then to prove that $T_k < k\tau_s$ it is suffice to show the following:

$$\begin{aligned}
&Ł(k+1, k\tau_s) > Ł(k, k\tau_s) \\
&\Rightarrow (k+1)\chi^{\infty}(\frac{k\tau_s}{k+1}) > k\chi^{\infty}(\frac{k\tau_s}{k}) \\
&\Rightarrow (k+1)\chi^{\infty}(\tau_s') > k\chi^{\infty}(\tau_s) \qquad (86)
\end{aligned}$$

We have substituted $\tau_s' = \frac{k}{k+1}$ above. Note that $\tau_s' < \tau_s$. As mentioned in section 2.3 $\psi^{\infty}(t)$, the excess growth produced by passive strategy, is maximized at τ_s Das *et al.* (2014). Therefore, the following must hold true:

$$\begin{aligned}
&\psi^{\infty}(\tau_s) > \psi^{\infty}(\tau_s') \\
&\Rightarrow \chi^{\infty}(\tau_s) - \chi(\tau_s) > \chi^{\infty}(\tau_s') - \chi(\tau_s') \\
&\Rightarrow \chi^{\infty}(\tau_s) > \chi^{\infty}(\tau_s') + \chi(\tau_s) - \chi(\tau_s') \\
&\Rightarrow \chi^{\infty}(\tau_s) > \chi^{\infty}(\tau_s') + \nu_p\tau_s - \nu_p\tau_s', \text{ using (16)} \\
&\Rightarrow \chi^{\infty}(\tau_s) > \chi^{\infty}(\tau_s') + \nu_p(\tau_s - \tau_s') \\
&\Rightarrow \chi^{\infty}(\tau_s) > \chi^{\infty}(\tau_s') + \nu_p(\tau_s - \frac{k}{k+1}\tau_s) \\
&\Rightarrow \chi^{\infty}(\tau_s) > \chi^{\infty}(\tau_s') + \nu_p\frac{\tau_s}{k+1} \qquad (87)
\end{aligned}$$

Substituting equation (87) in equation (86), it is suffice to show that:

$$\begin{aligned}
&(k+1)\chi^{\infty}(\tau_s') > k\Big(\chi^{\infty}(\tau_s') + \nu_p\frac{\tau_s}{k+1}\Big) \\
&\Rightarrow \chi^{\infty}(\tau_s') > \nu_p\frac{k}{k+1}\tau_s \\
&\Rightarrow \chi^{\infty}(\tau_s') > \nu_p\tau_s' \\
&\Rightarrow \chi^{\infty}(\tau_s') > \chi(\tau_s') \qquad (88)
\end{aligned}$$

By definition of stable rebalancing for all $\tau_s' < \tau_s$, portfolio growth will always be higher under passive strategy compared to active strategy. Hence the above equation (88) will always hold true. □

In prior section 2.6, we showed that the optimal rebalance divisor for a given horizon must be bounded by k_{mn} and k_{mx} given by equations (27) and (28), respectively. We must now examine that the choice of k as specified in rebalance divisor optimality theorem 3.9 conforms to these upper and lower bounds as well.

Lemma 3.11 *For any horizon T between inflection points T_{k-1} and T_k following must hold true:*

$$k \geq \left\lceil \frac{T}{\tau_s} \right\rceil \qquad (89a)$$

$$k \leq \left\lfloor \frac{T}{\tau_m} \right\rfloor \qquad (89b)$$

Proof By definition of horizon T the following is true:

$$T \leq T_k \Rightarrow \frac{T}{\tau_s} \leq \frac{T_k}{\tau_s} \qquad (90)$$

We can rewrite the inequality (85) as follows:

$$k > \frac{T_k}{\tau_s} \qquad (91)$$

Inequalities (90) and (91) together imply the following:

$$k > \frac{T}{\tau_s} \qquad (92)$$

Since k takes only positive integer values, inequality (92) implies inequality (89a). Again by definition of horizon T the following is true:

$$T > T_{k-1} \Rightarrow \frac{T}{\tau_m} > \frac{T_{k-1}}{\tau_m} \qquad (93)$$

We can rewrite the inequality (84) as follows:

$$k < \frac{T_{k-1}}{\tau_m} \qquad (94)$$

Inequalities (93) and (94) together imply the following:

$$k < \frac{T}{\tau_m} \qquad (95)$$

Since k takes only positive integer values, inequality (95) implies inequality (89b). □

3.4. Optimal frequency function

The uppermost rebalance contour between the two adjacent inflection points determines the maximum achievable portfolio growth following hybrid strategy. By combining these optimum contours for all the non-overlapping horizon segments, we obtain the *optimal log utility frontier* traced in bold in figure 5. We can completely specify the optimal frontier, $Ł_o(t)$ representing the maximum possible portfolio growth for all investment horizon $t \in \mathbb{R}$ as follows:

$$Ł_o(t) = \begin{cases} 0 & \text{if } t = 0 \\ Ł(k, t) & \text{if } t \in (T_{k-1,k} T_{k+1,k}], \quad \forall k \in \mathbb{N}^+ \end{cases} \qquad (96)$$

Given the optimal frontier specification, we can compute the optimal frequency function to determine the optimal frequency for any given horizon t as follows:

$$\tau_o(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{t}{k} & \text{if } t \in (T_{k-1,k} T_{k+1,k}], \forall k \in \mathbb{N}^+ \end{cases} \qquad (97)$$

3.5. An example

We use our familiar four-asset portfolio example to illustrate the concepts discussed so far. Table 1 presents the values of rebalance divisor, inflection point, optimal frequency at inflection point and the error, i.e. the deviation of rebalance frequency from the previous iteration. Note how the error diminish as we increase k. This is due to the rebalance frequency convergence theorem 2.2. Given an error tolerance, we can stop computing the inflection point further since frequency $\frac{T_k}{k}$ can be approximated to the last computed value when the error threshold tolerance ϵ is reached.

Assume that the inflection points have been computed as in table 1. We will now illustrate how one determines the optimal frequency for a specified investment horizon T. As an example consider the specified horizon values in table 2. From table 1, we notice that the optimal rebalance divisor for any investment horizon from 0 to 2.2916 is 1. Therefore for $T = 1$, the optimal frequency $\tau_o = \frac{T}{k} = 1$. Hence if the log-optimal investor desires to invest for 1 year, she should adhere to passive strategy without any rebalancing. With the passive strategy the investor shall have a portfolio growth of 0.3229. If instead the investor uses a lower rebalance frequency of 0.8, then the portfolio growth shall be lowered to 0.3161. Suppose the investor has a desire to invest till $T = 6$ years. From table 1, the optimum rebalance divisor shall be 4 since $5.6906 < T < 7.3548$ implying an optimal frequency of 1.5.

Table 1. Rebalance inflection points.

k	T_k	$\frac{T_k}{k}$	Error	k	T_k	$\frac{T_k}{k}$	Error
1	2.2916	2.2916	–	10	17.2725	1.7273	0.0086
2	4.0127	2.0063	0.2853	11	18.9221	1.7202	0.0071
3	5.6906	1.8969	0.1095	12	20.5713	1.7143	0.0059
4	7.3548	1.8387	0.0582	13	22.2202	1.7092	0.0050
5	9.0129	1.8026	0.0361	14	23.8690	1.7049	0.0043
6	10.6676	1.7779	0.0246	15	25.5175	1.7012	0.0038
7	12.3204	1.7601	0.0179	16	27.1659	1.6979	0.0033
8	13.9719	1.7465	0.0136	17	28.8142	1.6950	0.0029
9	15.6225	1.7358	0.0107	18	30.4624	1.6924	0.0026

Table 2. Investment horizon and optimal frequency.

T	k_o	τ_o	χ^{τ_o}	τ_l	χ^{τ_l}	τ_h	χ^{τ_h}
1	1	1.00	0.3229	0.80	0.3161	–	–
6	4	1.50	2.0862	1.20	1.9479	1.75	1.8447
30	18	1.67	9.7991	1.58	9.7854	1.76	9.7516
40	24	1.67	13.0654	1.60	13.0390	1.74	13.0442

This will generate 2.0862 as the portfolio growth. A lower (τ_l) or a higher (τ_h) rebalancing frequency shall generate lower portfolio growth for this horizon. Similarly for 30 year horizon, the optimum rebalance divisor and frequency are 18 and 1.67, respectively, resulting in a maximum portfolio growth of 9.7991.

The last example investment horizon we consider is $T = 40$. Let's assume that we will accept an optimal frequency error threshold of $\epsilon = 0.0026$. Using table 1, we will use the data for the highest inflection point in the very last row. We have assumed that for this inflection point the optimal frequency $\frac{T_k}{k}$ is very close to the asymptotic rebalance frequency τ_{ao}, i.e. $\tau_{ao} \approx \frac{T_k}{k}$. Instead of computing more higher order inflection points, we merely impute the optimum rebalance divisor by using $\lfloor \frac{T}{\tau_{ao}} \rfloor$. For $\epsilon = 0.0026$, we can apply this imputation for all $T > 30.4624$. For $T = 40$, the imputation results in an optimum rebalance divisor of 23. Note that we would have obtained the same optimum divisor had we continued computing higher order inflection point equal to or higher than $T = 40$. We would have to compute six additional inflection points, viz. 32.1105, 33.7586, 35.4065, 37.0545, 38.7023 and 40.3502 corresponding to rebalance divisors of 19 through 24. This would have resulted an optimum rebalance divisor of 24 instead of the imputed value of 23.

4. Computational efficiency

We analysed the computational complexity of each of the underlying algorithms presented in this paper. Pure search-based algorithm corresponding to equation (22) has a quadratic time complexity of $\mathrm{O}\left(\mathbb{T}^2\right)$. Let $\mathbb{K}$ be the domain of rebalance divisors that searches to find optimum k_o. Note that $\mathbb{K}$ is only limited to positive integer values between k_{mn} and k_{mx}. Hence the valid rebalance divisor domain $\mathbb{K}$ is much smaller compared to the time domain $\mathbb{T}$. Consequently searching the optimum rebalance divisor in equation (30) has significantly improved complexity of $\mathrm{O}\left(\mathbb{K}\mathbb{T}\right)$. Finally we designed a linear algorithm corresponding to equation (97) based on computing the inflection points. This algorithm has $\mathrm{O}\left(\mathbb{T}\right)$ complexity.

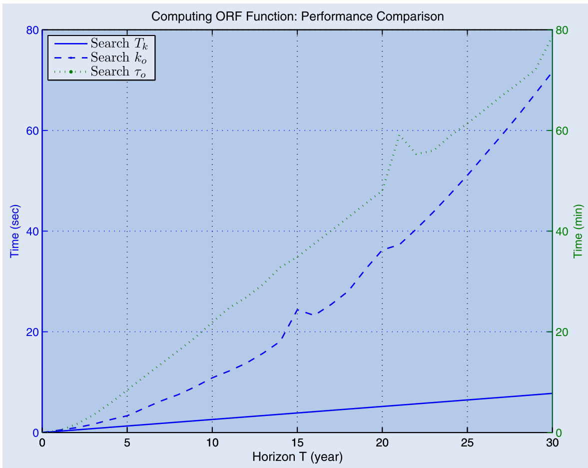

Figure 6. Algorithm performance comparison.

Figure 6 shows the execution timings of Matlab implementation of the three algorithms. The timings are generated for our example portfolio with four assets. Small values of discrete-time step $\delta t = 0.0001$ year and error threshold $\epsilon = 0.0001$ are used in order to achieve high accuracy of rebalance strategy. The measurements are taken in a 64-bit Intel 3 GHz computer with 32 GB of RAM.

To study the order of magnitude of performance improvements, we compare the times taken by the algorithms to compute optimal frequency function for 30 years of investment horizon. The pure search algorithm takes slightly more than an hour† to compute the strategy. In comparison, the algorithm

†The right hand side time axis in minutes is applicable to pure search algorithm.

that only searches optimum rebalance divisor k_o, reduces the computation time significantly to under one minute. The final algorithm that searches only the inflection points brings down the time to 6.5 s. Notice that for higher values of horizon, the performance difference between the algorithms widens rapidly.

5. Conclusion

The results presented in this paper are built on the foundation developed in Das *et al.* (2014) to compute the periodic rebalance frequency maximizing the utility of a log-optimal investor for a finite horizon. Mathematical insight into log-optimal portfolio rebalancing helped us simplify the computation of the optimal frequency function. Existing algorithms developed in our earlier research search for the optimal frequency in the continuous time range between 0 and τ_s. Therefore, the search speed is heavily dependent on the width of discrete-time interval used to break up this continuous range. Smaller time granularity increases the accuracy of the optimal frequency and simultaneously deteriorates the computational performance. Using the approach proposed in this paper, we reduced the complexity of the optimal frequency algorithms from quadratic to linear time.

First, we reduced the search space by showing that there is only a discrete set of finite possible candidates for the choice of optimal frequency. We introduced the concept of rebalance divisors, which are positive integer values. A rebalance divisor divides the investment horizon into equal intervals. At the end of each interval, the portfolio rebalancing is to be executed. For the first interval, the portfolio grows following a passive strategy. The terminal value of the portfolio growth is given by multiplying the portfolio growth at the end of the first interval with the rebalance divisor.

We then used mathematical analysis to determine the unique optimal rebalance divisor for any given investment horizon without resorting to search. We introduced the concepts of rebalance contour, inflection point and optimal frontier. The entire horizon time axis is mapped into unique non-overlapping intervals by the series of inflection points. A unique and ascending optimal rebalance divisor is assigned to each horizon interval. The optimal rebalance divisor is then assigned to the horizon interval of the specified horizon. The optimal frequency is the ratio of the horizon to the selected optimal rebalance divisor. This enabled us to specify the optimal frequency function as a piecewise continuous function of the investment horizon.

We assessed the relative computational performance of the three proposed algorithms by using the four-asset portfolio example. Using the inflection point-based algorithm gives an order of magnitude performance improvement over other algorithms. The execution time reduces from hours to seconds. We believe that these measurements can be improved further by using more sophisticated computing infrastructure such as grid computing.

It is important to highlight the key underlying assumptions we have made to arrive at the mathematically elegant solutions for computing optimal frequency. First, we assumed that the asset prices follow geometric Brownian motion and have static mean and standard deviations. Second, to derive mathematical expressions for passive evolution of portfolio we assumed log-normality for sum of log-normal random variables. We assumed unimodality for instantaneous growth function in order to simplify the mathematical analysis. We ignored the effect of trading cost for rebalancing as well. Our research can be extended to overcome the constraints put by one or more of these assumptions. Further research is also needed to explore mathematical framework to determine the existence of optimal frequency for portfolios other than log-normal.

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