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pantherfile.uwm.edu/mukul/public/
414-229-5001

Computing Optimal Rebalance Frequency For Log-Optimal Portfolio

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Technical Report UWM-CS-TR-1234 *

Abstract

We study the question of *how often* an investment portfolio be rebalanced to achieve the goal of maximizing the investor utility for any given investment horizon. We choose to use the log-optimal strategy that is appropriate when investors adhere to a logarithm utility function. This form of *active* investment strategy is cost-prohibitive and even impractical due to significant overhead of continuous rebalancing and trading cost. We develop an analytical framework to compute the expected value of portfolio growth of log-optimal investor when a given periodic rebalance frequency is used. We show that it is possible to improve investor log utility using this quasi-passive or *hybrid* rebalancing strategy. We present an algorithm to compute the optimal rebalance frequency for the given portfolio assets following Geometric Brownian motion. Simulation studies show that an investor shall gain significantly by using the optimal rebalance frequency in lieu of continuous rebalancing.

1 Introduction

Nearly half a century ago Markowitz investigated the problem of how an investor should distribute her wealth in a given set of financial securities like stocks or bonds with *a priori* estimation of return, risk or variance and the correlation parameters of each security in the portfolio with the goal of maximizing her return. He published his seminal paper ([Markowitz1952]) providing a simple and intuitive mathematical formulation of this portfolio selection problem. He characterized the relationship between an investor's return on investment and the variance or equivalently the risk of the return with the help of a two dimensional plot famously called the *Mean Variance Frontier* of investment. To optimize her return on investment an investor chooses her portfolio or the proportion of wealth distribution from a point on the frontier. The exact point on the frontier depends on the *utility* function she chooses to describe her risk bearing tolerance characteristics.

Once the portfolio is set up after determining the proper asset mix, the investor needs to address the issue of rebalancing the portfolio. [Calvet *et al.*2009] have studied the behavioral aspects of portfolio rebalancing for Swedish household investors. The changes in the household risky share is decomposed into two components, viz., *passive change* resulting from realized returns of risky assets in the absence of any trading and *active changes* due to asset trading and portfolio rebalancing. Their regression based analysis show strong household propensity to rebalance. Specifically, they show that wealthy and educated investors with better diversified portfolios rebalance more actively. There is strong evidence that households rebalance by adding more risky assets when they perform poorly and vice versa. This is known as *disposition effect* which has been examined in many papers including [Shefrin and Statman1985] and [Odean1999].

For the past few decades, rebalancing has been a thorny issue for researchers as well as practitioners. A sound investment philosophy needs to consider several aspects of rebalancing. Prominent among them are the questions of *when* one should rebalance and *what* the portfolio should be rebalanced to. In this context [Arnott and Lovell1993] have studied the historical stock and bond return and standard deviation data between 1968 to 1991 to conclude

*Submitted to Journal of Quantitative Finance.

that disciplined rebalancing can indeed boost returns. They have studied three types of alternate rebalancing techniques. In *calendar rebalancing* the portfolio mix is returned to initial asset mix in regular periodic intervals. In *rebalancing to allowed range*, the portfolio is always brought back within the allowed range of drift. In the third technique called *threshold rebalancing*, the portfolio is always rebalanced to the initial mix whenever it drifts beyond a predefined range. They conclude that more frequent rebalancing has been more beneficial before large deviation from the original mix occurs. The study also indicates that a monthly periodic rebalance frequency wins over other frequencies when both portfolio risk and return are considered. In a somewhat similar study [Thompson2002] uses historical data between 1997 to 2002 and finds that annual rebalancing frequency outperforms others such as monthly, quarterly and passive or no rebalancing. In yet another significant study, [Odean1999] finds that investors actually loose as they indulge in excessive trade.

This apparent inconsistent empirical conclusions based on historical data mining add to the the confusion in making sound rebalance decisions. Most of these exercises are at best qualitative in nature. Some even have attempted formulating ad-hoc intuitive rules ([Masters2002] and [Bernstein1996]) to determine when to rebalance. Although these rules do provide useful guidance to portfolio managers, they are at best heuristic judgements and are not based on solid mathematical or analytical foundation. There have been very little attempts to use quantitative and model-based approaches to corroborate the empirical observations.

[Tokat2007] uses Monte Carlo simulation technique to study effectiveness of various rebalancing frequencies, viz. monthly, quarterly and annually to control investment risk. His study considers three different types of market price behaviors and finds the appropriate rebalancing strategy for each type. In *trending markets*, where the prices show either a strong upward or downward trend, more frequent rebalancing produces both reduced expected portfolio return and risk. In *mean-reverting markets*, where the prices tend to reverse after following an upward or downward trend, faster rebalancing frequencies produced better risk control and slightly inferior portfolio returns. Lastly, when returns follow *random walks* ([Hull2008]), speedier rebalancing again produces better risk control and generally reduces portfolio return.

In this research, we set out to gain insight on the question of how often an initial portfolio be adjusted. For an investor, frequent rebalancing incurs cost in both time and money. An investor may not miss the opportunity to rebalance if she increases her chance to get a better return. On the other hand, an investor will benefit by knowing when to be passive. Hence we explore two questions: when and how often one needs to rebalance and, when she can afford not to rebalance and just be passive after initial investment decision. Informed passivity brings worry-free investment and saves paying undue trading fees. Our goal is to develop a computational system to provide analytical insight into these investment questions. For simplicity and computational tractability, we assume a static investment environment where the asset characteristics do not change over time.

We assume the investor has a log utility function and chooses the *log-optimal strategy* to maximize expected value of log of portfolio growth. [Luenberger1998] provides exhaustive analytical treatment to compute the optimal weights that the assets need to be divided to in a continuous time framework. The investor has to continuously rebalance the portfolio to the initial estimation of the weights in order to achieve maximal growth of log of portfolio in the long run. Both researchers and practitioners generally acknowledge the severe practical limitation of this strategy due to the continuous rebalancing condition.

A natural question to ask if the investor can benefit by remaining passive and delaying the rebalance decision. In other words, instead of rebalancing the portfolio continuously to initial set of weights, can she rebalance back to the initial portfolio weights less frequently? By doing so we must not, at any time during the investment horizon, sacrifice the investor goal of maximizing the log utility as achieved under active strategy. If such a rebalancing frequency exists, then the practical limitation set by the continuous rebalancing condition can be overcome. We show that, for certain class of portfolio assets, such a rebalance frequency indeed exists. In fact the investor can choose from a range of rebalance frequencies to rebalance her portfolio to the optimal weights. We can compute the single rebalance frequency in this range that will maximize the expected value of log of portfolio growth for a given investment horizon.

It is necessary to contrast our approach to that followed in [Kuhn and Luenberger2010]. In that paper, for a given discrete rebalance time the authors compute a *different* set of portfolio weights to maximize the expected log of portfolio growth. They demonstrate that there is no significant loss of portfolio performance in the long run if the investor chooses an annual frequency.

In our approach, we rebalance to the *same* portfolio weights that maximize the expected log of portfolio growth in the continuous rebalancing log-optimal approach. We merely want to know if the investor can afford to wait a certain time $\tau \neq 0$ to rebalance. This proposition obviates the need to continuously rebalance, yet achieves the same or higher level of expected log of portfolio growth. In order to answer this question, we first analyze the portfolio dynamics in a purely passive approach when the investor does not rebalance at all. This is an alternative extreme approach that follows a diametrically opposite investment philosophy about rebalancing compared to the purely active continuous rebalancing log-optimal approach.

A natural question to ask is if the investor can utilize Monte-Carlo simulation to compute the true underlying optimal rebalance frequency instead of using the analytical method presented in this report. The core issue in simulation has always been the tradeoff between speed and accuracy ([Glasserman2004]). The accuracy of simulation largely depends on the number of paths and size of discrete time step. We argue that the speed of simulation to determine the optimal rebalance frequency is not suitable for most modern-day investment scenarios. It is more so in today's era of algorithmic, micro-second and high-frequency trading environment that demands extremely fast determination of rebalance frequency in a dynamic changing market ([Hendershott *et al.*2011]). During our validation exercise with 5000 Monte-Carlo paths and 0.01 year time steps using a dual core 2.20 GHz, 4 GB Intel Pentium computer, the simulation takes a few days to complete compared to milliseconds latency of the analytical algorithms presented in this report.

The rest of the paper is organized in the following manner. After listing the basic notations in section 2, we review the basics of log-optimal portfolio in section 3 and present a generic algorithm to execute a rebalancing investment strategy. In section 4 we develop the analytical framework necessary to estimate the moments of log portfolio growth under passive strategy and introduce the notion of simple and stable rebalancing. Based on these two rebalancing concepts, in section 5 we estimate the expected log growth of portfolio when a periodic rebalancing is adopted in the so-called hybrid strategy. We prove that stable rebalancing outperforms simple and active continuous rebalancing. We also establish the relationship that enables us to compute the expected value of log hybrid portfolio from the expected value of log passive portfolio. With the help of these results we compute optimal rebalancing frequency that maximizes the expected value of log portfolio for any given horizon. We validate the analytical and computational results using Monte Carlo simulation in section 6. Finally, we summarize the results in the section 7.

2 Notations

Suppose the investor has the choice of setting up an investment portfolio from a set of N *risky* financial assets and a *risk-free* asset. Typical risky assets are stocks and funds, and often are correlated with other risky financial assets. These risky assets $i = 1, \dots, N$ are provided with *a priori* expected returns and standard deviations. We assume that returns are stationary random variables and hence the expected return and standard deviations don't change over time. We consider risk-free asset $i = N + 1$ such as T-bills offering constant fixed rate of return. We will use the following symbols in our mathematical derivations and analysis for $\forall i, j = 1$ to $N + 1$.

$T =$ investment horizon in years (periods)
$\mu_i =$ expected rate of return for asset i
$\sigma_i =$ standard deviation for asset i
$\rho_{ij} =$ correlation between returns of asset i and j
$\sigma_{ij} =$ covariance of asset i and $j = \rho_{ij}\sigma_i\sigma_j$
$w_i(t) =$ proportion of investment in asset i in portfolio at time t
$\mu_p(t) =$ expected rate of return of portfolio of assets at time t
$\sigma_p(t) =$ standard deviation of portfolio of assets at time t
$V(t) =$ value (in dollars) of portfolio at time t
$V(0) =$ initial value of portfolio wealth (in dollars)

For asset $N + 1$ which is risk-free, we will use $r_f = \mu_{N+1}$ alternatively. Since the asset is risk free, we also have $\sigma_{N+1} = 0$ and

$$\rho_{(N+1)j} = \rho_{j(N+1)} = 0 \; \forall j = 1 \text{ to } N \tag{1}$$

3 Active Portfolio

3.1 Asset Price Dynamics

In this section we briefly review the well known dynamics of asset prices. For more details and discussion of asset price modeling the reader may refer to [Hull2008] and [Luenberger1998]. We assume that asset price dynamics follows Geometric Brownian motion. Geometric Brownian motion assumption is widely used in financial assets and derivative valuations ([Neftci2000]).

$$dS(t) = \mu S(t)dt + \sigma S(t)dz \quad (2)$$

where
μ = expected rate of return of the asset expressed in decimal form.
σ = volatility of the asset price.
Variable $dz = \epsilon\sqrt{dt}$ follows *Wiener process*, where $\epsilon \sim \phi(0,1)$ is the standard normal variable. Rearranging equation 2, *instantaneous rate of return* of the asset will be,

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz \quad (3)$$

in this report we assume both the rate of return and volatility are constants for a given asset.
In this setting, asset price $S(t)$ has a lognormal distribution.

$$ln\ S(t) \sim \phi[ln\ S(0) + (\mu - \frac{\sigma^2}{2})t, \sigma^2 t] \quad (4)$$

The first and second terms in the above equation represent the mean and variance of the distribution respectively. Lognormality assumption precludes any negative price for assets. Expected and variance of asset prices are given by the following relationships:

$$E[S(t)] = S(0)e^{\mu t} \quad (5)$$

$$Var[S(t)] = S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1) \quad (6)$$

Lognormality of asset prices also lead to the following relationships of expected and variance of log of growth of asset price:

$$E[ln\{\frac{S(t)}{S(0)}\}] = \nu t \quad (7)$$

$$Var[ln\{\frac{S(t)}{S(0)}\}] = \sigma^2 t \quad (8)$$

where, asset *growth rate* ν is given by:

$$\nu = \mu - \frac{\sigma^2}{2} \quad (9)$$

In this asset dynamics framework the *continuously compounded rate of return* per annum realized between time 0 and t denoted by x is characterized by the following normal distribution ([Hull2008]):

$$x \sim \phi[\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{t}] \quad (10)$$

The asset price in terms of x is given by the following expression:

$$S(t) = S(0)e^{xt} \quad (11)$$

Risk-free asset dynamics is a special case of the risky asset dynamics described above. From equation 2:

$$dS_{N+1}(t) = r_f S_{N+1}(t)dt \quad (12)$$

The future price of risk-free asset will be deterministic and follows from equation 11:

$$S_{N+1}(t) = S_{N+1}(0)e^{r_f t} \quad (13)$$

3.2 Log-optimal Portfolio

In this investment strategy, portfolio weights are *continuously rebalanced* to maximize the long term growth rate of log of portfolio return. The reader can find a good treatment of this strategy in [Luenberger1998]. Log-optimal and semi-log optimal portfolios are also analyzed in [GyRfi *et al.*2007].

Since the portfolio is constructed using assets $i = 1$ through $N + 1$ with each asset taking up w_i proportion of the total investment outlay, we have:

$$\sum_{i=1}^{N+1} w_i = 1 \tag{14}$$

Note again that portfolio consists of $N + 1$ assets, one risk-free and N risky assets. If $V(t)$ is the value of the portfolio, then the *instantaneous rate of return of the portfolio* is equal to the weighted sum of the instantaneous rates of returns of the individual assets, i.e.

$$\frac{dV(t)}{V(t)} = \sum_{i=1}^{N+1} w_i \frac{dS_i(t)}{S_i(t)} \tag{15}$$

Plugging equation 3 in equation 15 we get,

$$\frac{dV(t)}{V(t)} = \sum_{i=1}^{N+1} (w_i \mu_i dt + w_i \sigma_i dz) \tag{16}$$

In the above equation, the first term is a fixed term with variance 0. The second term is a stochastic term with mean 0 and variance given by:

$$\begin{aligned} Var[\sum_{i=1}^{N+1} w_i\sigma_i dz] &= E(\sum_{i=1}^{N+1} w_i\sigma_i dz)^2 - (E(\sum_{i=1}^{N+1} w_i\sigma_i dz))^2 = E(\sum_{i=1}^{N+1} w_i\sigma_i dz)^2 \\ &= E(\sum_{i=1}^{N+1} w_i\sigma_i dz)(\sum_{j=1}^{N} w_j\sigma_j dz) = \sum_{i,j=1}^{N+1} w_i\sigma_{ij} w_j dt \end{aligned} \tag{17}$$

Note that in the above simplification, the second term goes away as it is the square of sum of expected values of multiples of standard normal variables. The expected value of a multiple of standard normal variable is zero ([Trivedi2001]). Now, we can write equation 15 in the following Geometric Brownian motion form analogous to the dynamics of asset price in equation 3:

$$dV(t) = \mu_p V(t)dt + \sigma_p V(t)dz \tag{18}$$

where the mean and variance of the portfolio are given by:

$$\mu_p = \sum_{i=1}^{N+1} w_i \mu_i \tag{19}$$

$${\sigma_p}^2 = \sum_{i,j=1}^{N+1} w_i \sigma_{ij} w_j \tag{20}$$

Analogous to asset price dynamics, applying *Itô's lemma* ([Neftci2000]) portfolio value $V(t)$ has a lognormal distribution.

$$ln\ V(t) \sim \phi[ln\ V(0) + (\mu_p - \frac{\sigma_p^2}{2})t, \sigma_p^2 t] \tag{21}$$

From above lognormality relationships, we can derive the following expected and variance for the growth of protfolio value and log of portfolio value in the following equations:

$$E[\frac{V(t)}{V(0)}] = e^{\mu_p t} \tag{22}$$

$$Var[\frac{V(t)}{V(0)}] = e^{2\mu_p t}(e^{\sigma_p^2 t} - 1) \tag{23}$$

$$E[ln\{\frac{V(t)}{V(0)}\}] = \nu_p t \tag{24}$$

$$Var[ln\{\frac{V(t)}{V(0)}\}] = \sigma_p^2 t \tag{25}$$

where, portfolio *growth rate* ν_p is given by:

$$\nu_p = \mu_p - \frac{\sigma_p^2}{2} \tag{26}$$

For notational simplicity we will use $\chi(t)$ to denote the expected log of portfolio growth at time t. Since $V(0) = 1$, we can rewrite equations 24 as,

$$\chi(t) = \nu_p t \tag{27}$$

In the lognormal portfolio, the growth rate ν_p is maximized by solving the following optimization problem:

$$\begin{aligned} &\underset{\mathbf{w}}{\text{maximize}} && \nu_p \\ &\text{subject to} && \sum_{i=1}^{N+1} w_i = 1 \end{aligned}$$

w defines the vector of asset weights. The solution to the above optimization problem is to select the weight of each risky asset i satisfying the following relationship ([Luenberger1998]):

$$\sum_{j=1}^{N} \sigma_{ij} w_j = \mu_i - r_f \tag{28}$$

There will be N linear equations corresponding to each risky asset with same number of unknown weight variables. We can then solve for the values of the portfolio weights for risky assets. Finally we can find out the portfolio weight w_{N+1} of the risk free asset using equation 14. We will extend the example used in [Luenberger1998] for demonstrating different investment strategies studied in this report. In this example, there are three risky assets, $i = 1, 2$ and 3. A portfolio manager or an investor needs to specify the asset mean, variance and correlation coefficients. She also specifies the risk free rate and investment horizon. The following is the set of input parameters specified for this example:

1. Initial portfolio value: $V(0) = 1$ (dollar)
2. Mean vector:
$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \end{bmatrix} = \begin{bmatrix} 0.24 & 0.20 & 0.15 \end{bmatrix}$$
3. Asset standard deviation vector:
$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 0.3000 & 0.2646 & 0.1732 \end{bmatrix}$$
4. Asset correlation coefficients:
$$\boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.2520 & 0.1925 \\ 0.2520 & 1.0000 & -0.2182 \\ 0.1925 & -0.2182 & 1.0000 \end{bmatrix}$$
5. Risk-free rate: $r_f = 0.1$
6. Investment horizon: $T = 30$ years.

We can derive the covariance matrix from the given asset variances and correlation coefficients using Matlab like syntax for matrix operations:

$$\mathbf{S} = \boldsymbol{\rho}. * (\boldsymbol{\Sigma}' * \boldsymbol{\Sigma}) \tag{29}$$

In the above syntax, $\boldsymbol{\Sigma}'$ is the compliment of $\boldsymbol{\Sigma}$ and $.*$ is element-wise multiplication of two matrices.

Using equation 29 we obtain:

$$\mathbf{S} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 0.09 & 0.02 & 0.01 \\ 0.02 & 0.07 & -0.01 \\ 0.01 & -0.01 & 0.03 \end{bmatrix}$$

Using the above matrix notations, system of linear equations in 28 can be written as:

$$\mathbf{S}\mathbf{w} = \boldsymbol{\mu} - r_f \tag{30}$$

We can solve the above set of linear equations easily by using a linear equation solver package. In Matlab, we can solve for $\mathbf{w}$ by using the backslash or matrix left division operator:

$$\mathbf{w} = \mathbf{S}/(\boldsymbol{\mu} - r_f) \tag{31}$$

For the example investment problem, we obtain:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1.0509 \\ 1.3818 \\ 1.7770 \end{bmatrix}$$

Using equation 14, we can derive the portfolio weight for risk free asset $w_4 = -3.2098$. The negative sign indicates that the risk free asset needs to be borrowed. A portfolio set up using the above weights will maximize the log growth of portfolio in the long run if the weights are always maintained to the original value by continuously rebalancing.

The mean μ_{opt} and variance σ^2_{opt} of the portfolio corresponding to the set of optimum weights can be computed using equation 19 and 20 respectively:

$$\mu_{opt} = 0.4742, \sigma^2_{opt} = 0.3742$$

Using equation 26 constant growth rate $\nu_p = 0.2871$.

We now summarize the above steps in the form of an computational algorithm. Algorithm 1 computes the optimal weight vector and the corresponding growth rate for the active investment strategy. The algorithm takes in the mean, variance and correlation vectors along with the constant risk free rate. It returns the portfolio growth rate, weight vector and mean vector to the calling procedure. Note that the output mean vector contains the risk free rate, i.e. the mean of the risk-free asset as well.

Algorithm 1 ComputeLogOptimalParams

Require: $\boldsymbol{\mu}$,$\boldsymbol{\Sigma}$,$\boldsymbol{\rho}$,r_f,N
$\mathbf{S} \leftarrow \boldsymbol{\rho}.*(\boldsymbol{\Sigma}' * \boldsymbol{\Sigma})$ # equation 29
$\mathbf{w} \leftarrow \mathbf{S}/\ (\boldsymbol{\mu} - r_f)$ # equation 31
$\mu[N+1] \leftarrow r_f$
$wSum \leftarrow 0$
for $i = 1$ to N **do**
    $wSum \leftarrow wSum + w[i]$
end for
$w[N+1] \leftarrow 1 - wSum$ # equation 14
$\mu_p \leftarrow 0, v_p \leftarrow 0$
for $i = 1$ to N+1 **do**
    $\mu_p \leftarrow \mu_p + w[i]\mu[i]$ # equation 19
    for $j = 1$ to N+1 **do**
        $v_p \leftarrow v_p + w[i]\sigma[i,j]w[j]$ # equation 20
    end for
end for
$\nu_p \leftarrow \mu_p - \frac{1}{2}v_p$ # equation 26
return $\nu_p, \mathbf{w}, \boldsymbol{\mu}$

Figure 3.2 plots the expected value of the portfolio in active strategy using equations 19 and 22. This seems to have rich growth potential to an investor. However, an investor also needs to look at the risk in this strategy. One

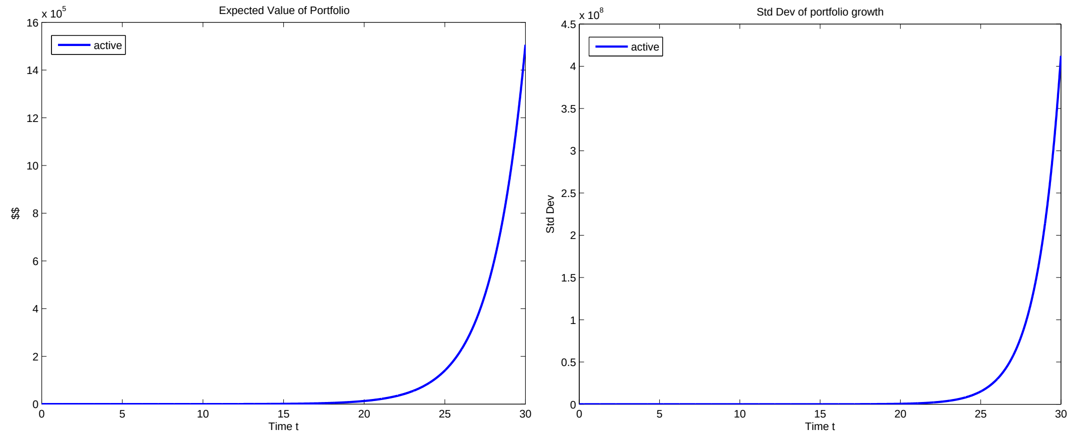

Figure 1: Moments of active portfolio value

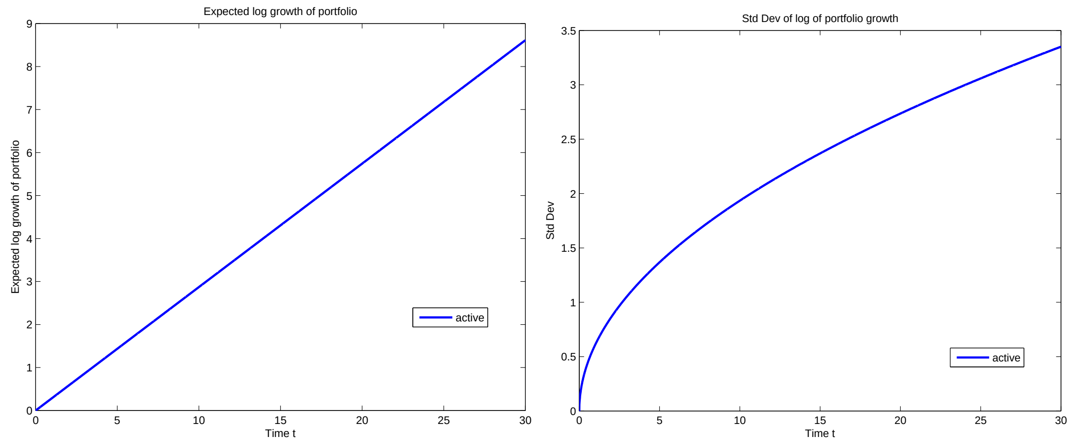

Figure 2: Moments of active log of portfolio value

measure of the risk is the variability or standard deviation of this portfolio value given by equation 23 and traced in figure 3.2. We can see that the upside potential of the portfolio growth comes at the expense of exponential increase in variability of standard deviation of the portfolio value.

The reader is reminded that the active log-optimal strategy maximizes the *log* of portfolio growth. For such log investor utility, we need to look at the expected value of the log of growth of portfolio over the investment horizon as given by equation 27 and plotted in figure 3.2. The uncertainty or risk in this estimation is given by equation 23 and plotted in figure 3.2. Notice that unlike exponential growth of standard deviation for the portfolio growth, the standard deviation of log of portfolio growth shows only quadratic growth.

Before we discuss alternative investment strategies, we will outline two important well-known properties of the log-optimal active strategy as stated in [Luenberger1998]. Suppose Z is an alternative investment strategy other than log-optimal active strategy.

1. Log-optimal strategy maximizes the expected portfolio ν_p growth rate in the long run, i.e. $\nu_p \geq \lim_{t\to\infty} \nu_p^Z(t)$.

2. Suppose $V^Z(t)$ is the value of portfolio at t under any investment strategy other than log-optimal strategy. Then, $E[\frac{V^Z(t)}{V(t)}] \leq 1$.

An obvious, yet important characteristic of active strategy is that it satisfies *reinvestment principle*. In other words, it produces identical portfolio value when the assets are liquidated in the middle and reinvested back in the same assets in the same proportion as before. From equation 22 it is easy to see how this is satisfied in active strategy. If $V(0)$, $V(t')$ and $V(t)$ are the portfolio values at time 0, t' and t such that $0 < t' < t$ then,

$$E[V(t)] = E[V(0)]e^{\mu_p t} = E[V(0)]e^{\mu_p t'}e^{\mu_p(t-t')} = E[V(t')]e^{\mu_p(t-t')} \tag{32}$$

Algorithm 2 outlines the generic steps for executing an investment strategy that uses a given rebalancing frequency τ. At every rebalancing time, it uses the market price for the assets to compute the total portfolio value (step 4 through 10). In steps 11 through 13, asset count is recomputed after rebalancing the portfolio to the initial optimal weights. A trader must buy and sell assets appropriately to arrive at the new asset counts. The algorithm returns the terminal portfolio log growth χ.

Algorithm 2 ExecuteRebalanceStrategy

Require: $\boldsymbol{\mu}$,$\boldsymbol{\Sigma}$,$\boldsymbol{\rho}$,r_f,N,T,τ

1: $V = 1$ # Initial investment of 1$

2: $[\nu_p, \mathbf{w}, \boldsymbol{\mu}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho}, r_f, N)$

3: **for** $t = 0$ to T by τ **do**

4:   **if** $t \geq \tau$ **then**

5:     $V \leftarrow 0$

6:     **for** $j = 1$ to $N + 1$ **do**

7:       Obtain $P_t[j]$

8:       $V \leftarrow V + P_t[j] * acnt[j]$ # total portfolio value

9:     **end for**

10:   **end if**

11:   **for** $j = 1$ to $N + 1$ **do**

12:     $acnt[j] \leftarrow \frac{w[j]V}{P_t[j]}$ # rebalance portfolio to **w**

13:   **end for**

14: **end for**

15: $V \leftarrow 0$

16: **for** $j = 1$ to $N + 1$ **do**

17:   $V \leftarrow V + P_T[j] * acnt[j]$ # liquidate the portfolio at horizon T

18: **end for**

19: $\chi = log(V)$

20: **return** χ

For a practical implementation, active strategy can employ daily rebalancing to emulate closely the effect of continuous rebalancing. Since in a typical year, there are 252 trading days, one can set $\tau = \frac{1}{252} = 0.004$ year. Thus, one will invoke the following command to execute active strategy for 30 year horizon:

$\chi = ExecuteRebalanceStrategy(30, 0.004)$

In the above statement, we have assumed that all other input parameters specific to the given set of portfolio assets have already been provided.

4 Passive Portfolio

In the prior section we elaborated the log-optimal strategy for portfolio growth where the portfolio is continuously rebalanced with $\tau = 0$. In this section, we will develop the framework to assess the nature of portfolio growth when the investor sets up the portfolio with the optimal weight vector **w** and never rebalances again. Consequently, we assume the rebalance frequency under such passive strategy to be $\tau = \infty$.

Throughout our analysis, we will use a superscript $.^P$ to denote the passive strategy relevant parameters. In the absence of any such superscript, the parameter pertains to active strategy. Note that the initial investment parameters enumerated under section 2 will be applicable to all strategies discussed in this report.

Lemma 1 *Consider an initial portfolio with value $V(0)$ constructed using N risky assets with weights $w_i(0)$, $i = 1, \ldots, N$ and a risk-free asset with weight $w_0(0)$. When left unadjusted, the portfolio will grow such that the value $V(t)$ at any time $t > 0$ will be given by:*

$$V^P(t) = V(0) \sum_{i=1}^{N+1} w_i(0)e^{x_i t} \tag{33}$$

where x_i is a random variable specified by equation 10.

Proof 1 *At $t = 0$ the value of the portfolio invested in asset i is $V(0)w_i(0)$. This translates to the number of shares n_i to be purchased and held for asset i at time $t = 0$:*

$$n_i = \frac{V(0)w_i(0)}{S_i(0)}$$

Since the portfolio remains unadjusted, the value of n_i shares of asset i at time $t > 0$ will be:

$$V_i^P(t) = \frac{V(0)w_i(0)}{S_i(0)}S_i(t) = \frac{V(0)w_i(0)}{S_i(0)}S_i(0)e^{x_i t} = V(0)w_i(0)e^{x_i t} \tag{34}$$

We have used equation 11 in simplifying the above. Now the result in equation 33 follows since the portfolio value is the sum of values of constituent assets.

The result in lemma 1 can be extended for any initial time $t_1 \geq 0$ and stated as the following corollary.

Corollary 1 *Consider an initial portfolio with value $V(t_1)$ constructed using N risky assets with weights $w_i(t_1)$, $i = 1, \ldots, N$ and a risk-free assets with weight $w_{N+1}(t_1)$. When left unadjusted, the portfolio will grow such that the value $V^P(t_2)$ at any time $0 < t_1 < t_2$ will be given by:*

$$V^P(t_2) = V(t_1)\sum_{i=1}^{N+1} w_i(t_1)e^{x_i(t_2-t_1)} \tag{35}$$

Hence the value of the passive portfolio is characterized by a sum of correlated random variables as per equation 33. In future, we will omit the time index from w_i when referring to the optimal weights calculated based on equation 31.

We now review some of the statistical properties of log normal random variable. The reader can find a very good overview in [Wikipedia2011a]. A comprehensive treatment of log-normal distribution will be found in [Crow and Shimizu1988].
Let Y be a normal random variable with mean m and standard deviation s.
Let X be another random variables such that:

$$X = e^Y$$

X is said to be a *log-normal* random variable since logarithm of the variable follows normal distribution. The first two moments of X are given as below:

$$E[X] = e^{m+\frac{s^2}{2}} \tag{36}$$

$$Var[X] = (e^{s^2} - 1)E[X]^2 = (e^{s^2} - 1)e^{2m+s^2} \tag{37}$$

When there are two correlated normal random variables Y_1 and Y_2 with correlation coefficient ρ_{12}, the covariance between the corresponding log-normal variables $X_1 = e^{Y_1}$ and $X_2 = e^{Y_2}$ is given by:

$$Cov[X_1X_2] = (e^{\rho_{12}s_1s_2} - 1)E[X_1]E[X_2] = (e^{\rho_{12}s_1s_2} - 1)e^{m_1+\frac{s_1^2}{2}}e^{m_2+\frac{s_2^2}{2}} \tag{38}$$

Given log-normal X, one can compute the mean m and variance s^2 of the underlying normal variable Y by using the following relationships:

$$s^2 = ln(1 + \frac{Var[X]}{E[X]^2}) \tag{39}$$

$$m = ln(E[X]) - \frac{1}{2}ln(1 + \frac{Var[X]}{E[X]^2}) = ln(E[X]) - \frac{1}{2}s^2 \tag{40}$$

Now we can proceed to compute the statistics for the passive portfolio evolution.

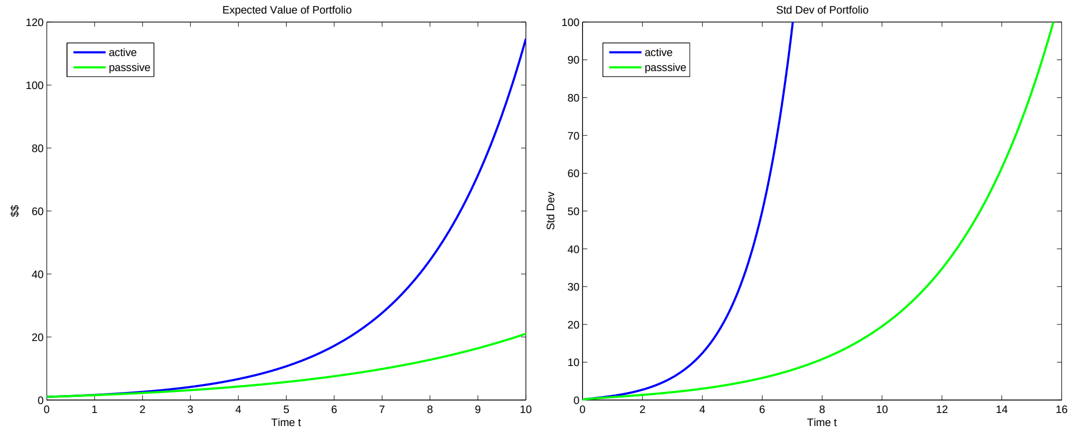

Figure 3: Expected Value and Std Dev of Portfolio Growth.

Lemma 2 *Under passive investment strategy, the expected value of portfolio growth at any time $t > 0$ is the weighted sum of the individual expected asset growths, i.e.,*

$$E[\frac{V^P(t)}{V(0)}] = \sum_{i=1}^{N+1} w_i e^{\mu_i t} \tag{41}$$

Proof 2 *From equation 33, we can compute the passive portfolio growth as:*

$$\begin{aligned} \frac{V^P(t)}{V(0)} &= \sum_{i=1}^{N+1} w_i e^{x_i t} = \sum_{i=1}^{N+1} e^{ln(w_i)+x_i t} \\ \Rightarrow \quad E[\frac{V^P(t)}{V(0)}] &= E[\sum_{i=1}^{N+1} e^{ln(w_i)+x_i t}] = \sum_{i=1}^{N+1} E[e^{ln(w_i)+x_i t}] \end{aligned} \tag{42}$$

We have made use of the fact that the expected value of a sum of random variables is same as the sum of expected values of the individual random variables ([Trivedi2001]). Now, given that x_i's are normal random variables as specified in equation 10, $ln(w_i) + x_i t$ will also be normal with the following moments:

$$ln(w_i) + x_i t \sim \phi[ln(w_i) + (\mu_i - \frac{\sigma_i^2}{2})t, \sigma_i^2 t] \tag{43}$$

Note that $Var(aX + b) = a^2 Var(X)$ for any random variable X and constants a and b.

We can now find out the first moment of $e^{ln(w_i)+x_i t}$ using log-normal properties of equation 36,

$$E[e^{ln(w_i)+x_i t}] \quad = \quad e^{ln(w_i)+(\mu_i - \frac{\sigma_i^2}{2})t + \frac{\sigma_i^2 t}{2}} = e^{ln(w_i)+\mu_i t} = w_i e^{\mu_i t} \tag{44}$$

Plugging the above in equation 42 we get the desired result.

Figure 4 shows the evolution of expected value of portfolio growth for our example investment scenario. In this case the passive strategy produces lower expected portfolio growth than the active strategy.

Lemma 3 *Under passive investment strategy, the variance of portfolio growth at any time $t > 0$ is given by:*

$$Var[\frac{V^P(t)}{V(0)}] = \sum_{i=1}^{N+1} w_i^2 e^{2\mu_i t}(e^{\sigma_i^2 t} - 1) + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}(e^{\rho_{ij}\sigma_i\sigma_j t} - 1) \tag{45}$$

Proof 3 *Similar to lemma 2, variance of passive portfolio growth is:*

$$Var[\frac{V^P(t)}{V(0)}] = Var[\sum_{i=1}^{N+1} e^{ln(w_i)+x_it}] = \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} Cov[e^{ln(w_i)+x_it}, e^{ln(w_j)+x_jt}] \tag{46}$$

The reader may refer [Wikipedia2011b] for the rule to obtain the sum of correlated random variables. We can now split the above equation into the variance (of the same variable) component and covariance (between two different variables) component:

$$Var[\frac{V^P(t)}{V(0)}] = \sum_{i=1}^{N+1} Var[e^{ln(w_i)+x_it}] + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N+1} Cov[e^{ln(w_i)+x_it}, e^{ln(w_j)+x_jt}] \tag{47}$$

We can use equation 37 to simplify the first variance component in the above equation:

$$\begin{aligned} Var[e^{ln(w_i)+x_it}] &= (e^{\sigma_i^2 t}-1)e^{2(ln(w_i)+(\mu_i-\frac{\sigma_i^2}{2})t)+\sigma_i^2 t} \\ = \quad & e^{2(ln(w_i)+\mu_i t)}(e^{\sigma_i^2 t}-1) = w_i^2 e^{2\mu_i t}(e^{\sigma_i^2 t}-1) \end{aligned} \tag{48}$$

Similarly we use equation 38 and 44 to simplify the covariance component in equation 47:

$$\begin{aligned} Cov[e^{ln(w_i)+x_it}, e^{ln(w_j)+x_jt}] &= (w_i e^{\mu_i t})(w_j e^{\mu_j t})(e^{\rho_{ij}\sigma_i\sqrt{t}\sigma_j\sqrt{t}}-1) \\ &= w_i w_j e^{(\mu_i+\mu_j)t}(e^{\rho_{ij}\sigma_i\sigma_j t}-1) \end{aligned} \tag{49}$$

Plugging the results in equations 48 and 49 in equation 47, we obtain the desired passive portfolio variance expression of equation 45.

Figure 4 shows the evolution of variance of portfolio growth for our example investment scenario. In this case the passive strategy has less variance compared to the active strategy. This alone indicates that passive strategy will be less risky which is good for risk-averse investors.

The reader is reminded that the active strategy is optimal only when the expected value of the log of portfolio growth is maximized for the investor. The corresponding expected value of the log of portfolio growth is given by equation 22. In order to have a fair portfolio performance comparison between active and passive strategy we need to analyze the expected log of the portfolio growth under passive strategy.

The problem here is to compute the first, and if possible, the second moment of the log of the portfolio growth under passive strategy. Using equation 33:

$$ln(\frac{V^P(t)}{V(0)}) = ln(\sum_{i=1}^{N+1} w_i(0)e^{x_it}) \tag{50}$$

The need to characterize the sum of lognormal variables arises in many domains. There have been many approximations to characterize the probability density function for sum of log normal. Two analytical methods to determine the moments of sum of correlated random variables widely used by researchers in many engineering disciplines. The first one proposed by Fenton and Wilkinson in 1960 is still being used because of its simplicity and analytical tractability ([Fenton1960]). More recently, the second method was proposed in [Schwartz and Yeh1982]. Both of these methods assume that the sum of lognormal is also lognormal. [Safak and Safak1994] compare the two approaches to formulate the outage probability in a mobile radio systems. Fenton's approach allows the use of closed form analytical expression for the moments of log of sum of lognormal random variables. Schwartz and Yeh method employs a recursive algorithm to obtain the moments. in this report, we will adapt Fenton's method because of its analytical tractability.

Lemma 4 *The variance of the log of portfolio growth under passive strategy is given by:*

$$\begin{aligned} Var[ln(\frac{V^P(t)}{V(0)})] &= ln(1+\frac{1}{[\sum_{i=1}^{N+1} w_i e^{\mu_i t}]^2}[\sum_{i=1}^{N+1} w_i^2 e^{2\mu_i t}(e^{\sigma_i^2 t}-1) \\ &+ 2\sum_{i=1}^{N}\sum_{j=i+1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}(e^{\rho_{ij}\sigma_i\sigma_j t}-1)]) \end{aligned} \tag{51}$$

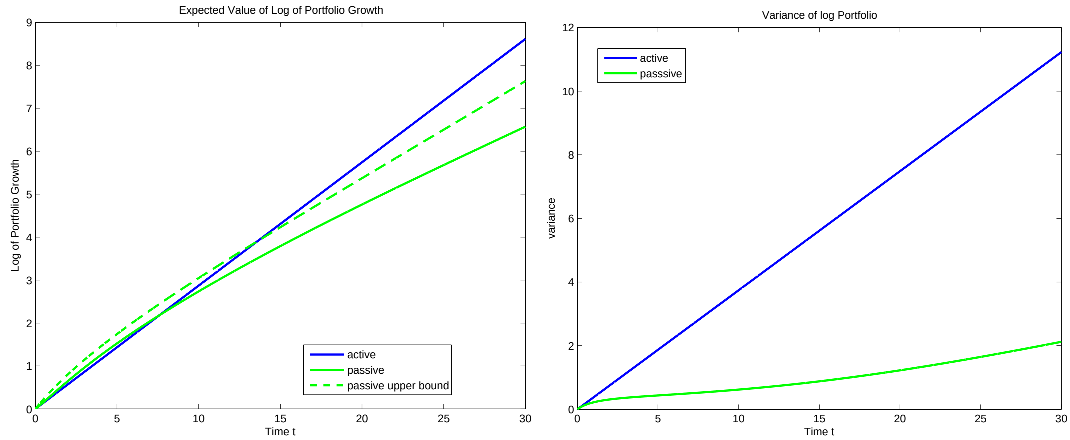

Figure 4: Expected Value and Variance of Log of Portfolio Growth.

Proof 4 *We assume that sum of lognormal random variables is also lognormal as is assumed in Fenton-Wilkinson approach. Thus the passive portfolio growth $\frac{V^P(t)}{V(0)}$ is lognormal. This implies that log of passive portfolio growth $ln(\frac{V^P(t)}{V(0)})$ is normal. Mathematically,*

$$e^{ln(\frac{V^P(t)}{V(0)})} = \frac{V^P(t)}{V(0)} \tag{52}$$

Using lognormal property given by equation 39, we obtain,

$$Var[ln(\frac{V^P(t)}{V(0)})] = ln(1 + \frac{Var[\frac{V^P(t)}{V(0)}]}{E[\frac{V^P(t)}{V(0)}]^2}) \tag{53}$$

Plugging in the values of expected value and variance of portfolio growth from equations 41 and 45 in the above equation we obtain the desired resulting equation of 51.

For our example investment scenario, the variance of log of portfolio growth in figure 4 is more favorable to the risk-averse investors.

Now we derive the expected value of log of portfolio growth which is the investor utility in log-optimal investment strategy.

Lemma 5 *The expected value of the log of portfolio growth under passive strategy is given by:*

$$\chi^P(t) = E[ln(\frac{V^P(t)}{V(0)})] = ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) - \frac{1}{2} Var[ln(\frac{V^P(t)}{V(0)})] \tag{54}$$

Proof 5 *The derivation is straightforward when we follow the lognormal assumption in lemma 4 and using lognormal property given by equation 40.*

The expected value thus obtained is an approximation due to the inherent log-normality assumption in Fenton-Wilkinson's approach. Using Jenson's inequality ([Wikipedia2011c]) we can derive a true upper bound.

Lemma 6 *The expected value of the portfolio growth under passive strategy will always be bounded, i.e.,*

$$\chi^P(t) \leq ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) \tag{55}$$

Proof 6 *Knowing that logarithm is a concave function and using Jensen's inequality:*

$$\chi^P(t) = E[ln(\frac{V^P(t)}{V(0)})] \leq ln(E[\frac{V^P(t)}{V(0)}]) \tag{56}$$

Plugging in expected value expression from equation 41 we obtain equation 55.

Notice that the estimation in equations 54 obtained using Fenton-Wilkinson approach will always meet the upper bound condition of equation 55. This is easy to see as the first term in equation 54 is the upper bound. The estimation is always going to be less than this bound as the positive variance term in the equation will always be subtracted from the first term.

Figure 4 shows the comparison of expected value of log of portfolio growth for our example investment scenario. We see that the passive strategy provides better performance for the initial few years. Since the investor wants to maximize the expected value of log of portfolio growth, he will choose passive strategy over active strategy for this initial period since the passive strategy offers higher expected value of portfolio growth. Passive strategy will be seen as more favorable if we had considered the transaction cost incurred in continuous rebalancing used in active strategy.

As a result of the log-normality assumption inherent in Fenton-Wilkinson approach, we notice the analogous nature of the passive portfolio growth in equation 54 and the corresponding equation under active strategy in equation 27. Comparing both these equations portfolio growth rate under passive strategy ν_p^P will be given by,

$$\nu_p^P(t) = \frac{\chi^P(t)}{t} = \frac{1}{t}ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) - \frac{1}{2}(\frac{1}{t}Var[ln(\frac{V^P(t)}{V(0)})]) = \mu_p^P - \frac{\sigma_p^{P2}}{2} \tag{57}$$

where, mean μ_p^P and standard deviation σ_p^P of passive portfolio are given respectively by,

$$\mu_p^P(t) = \frac{1}{t}ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) \tag{58}$$

$$\sigma_p^{P2}(t) = \frac{1}{t}Var[ln(\frac{V^P(t)}{V(0)})] \tag{59}$$

One can compare equations 57, 58 and 59 with their counterpart equations 26, 19 and 20 for active strategy. Notice that the passive portfolio mean, standard deviation and growth rate are all time varying unlike the corresponding active strategy parameters. This is observed in the figures 4 and 4 for our example investment portfolio. Notice that the portfolio mean is lower than the corresponding mean in active strategy. However due to reduced portfolio standard deviation, we can still obtain higher portfolio growth rate under passive strategy for the initial period.

Similar to the expected value of portfolio growth, the growth rate in figure 4 demonstrates why the investor should choose to remain passive and not exercise her continuous rebalancing option to maximize his investment potential. It is prudent to only rebalance when the growth rate starts to fall below the active strategy.

Analogous to equations 22 and 23, we can now alternatively express the mean and variance of passive portfolio growth as following:

$$E[\frac{V^P(t)}{V(0)}] = e^{\mu_p^P(t)t} \tag{60}$$

$$Var[\frac{V^P(t)}{V(0)}] = e^{2\mu_p^P(t)t}(e^{\sigma_p^{P2}(t)t} - 1) \tag{61}$$

It is easy to show the equivalence of equations 41 and 60. Similarly, equations 45 and 61 are also equivalent.

We now end the analysis of passive strategy by looking at the mean-variance plot comparison of log portfolio as shown in figure 4. Notice that for our example portfolio, the plot for passive strategy lies above the plot for active strategy for the entire investment period. In other words, for a given standard deviation, passive portfolio will have higher expected value of log of portfolio growth. In this sense, the investor will find the passive strategy more favorable if she is willing to take a given level of risk quantified by the standard deviation of log of portfolio growth.

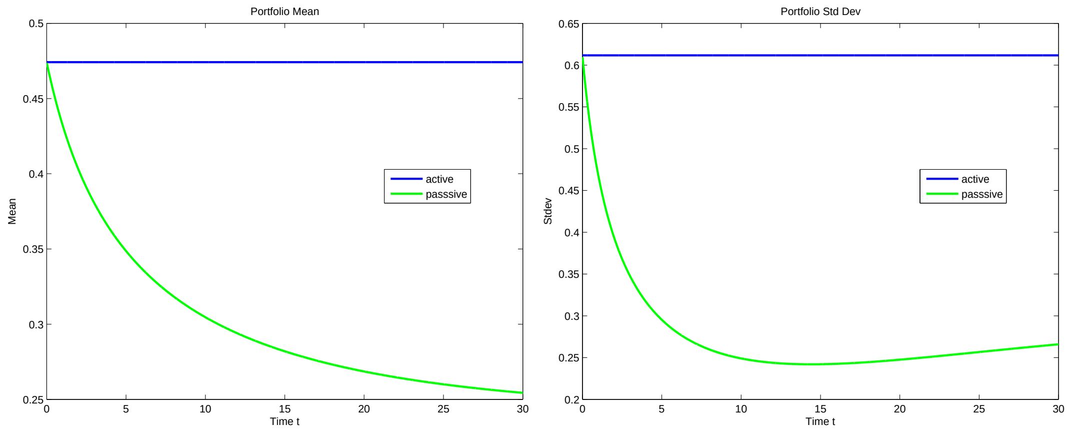

Figure 5: Portfolio mean and standard deviation evolution.

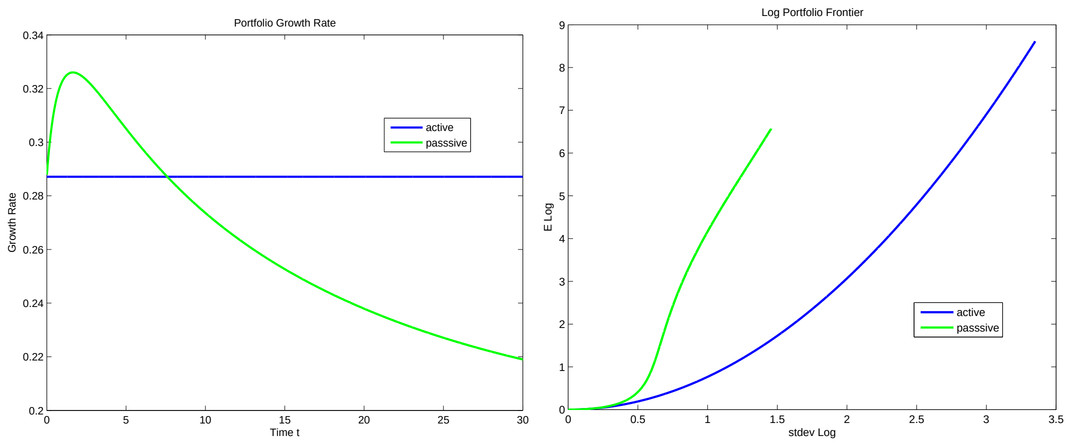

Figure 6: Growth rate and mean variance of log of portfolio.

4.1 Simple Rebalancing

As discussed previously, to attain the investor's log-optimality goal, the investor may not need to continuously rebalance. Figures 4 and 4 illustrate existence of opportunity to stay passive during the period when the expected log of portfolio growth and corresponding growth rate are higher than those in active strategy. We define this period $\tau_c > 0$ to be the *simple* rebalance time. During $(0\ \tau_c)$, passive strategy offers higher investor log utility as captured in the following condition:

$$\exists \tau_c \text{ s.t. } \chi^P(t) > \nu_p t, \forall t \in (0\ \tau_c) \tag{62}$$

The investor continues to use passive strategy until passive log utility drops and equals that of active strategy. In the absence of transaction cost, this first rebalance time τ_c will be determined by the point of intersection of equations 27 and 54 as in figure 4. We can express this mathematically as follows:

$$\chi^P(\tau_c) = \chi(\tau_c) = \nu_p \tau_c,\ \tau_c > 0 \tag{63}$$

Algorithm 3 ComputeSimpleRebalanceFrequency

Require: $\boldsymbol{\mu}$,$\boldsymbol{\Sigma}$,$\boldsymbol{\rho}$,r_f,T,δT

1: $\tau_c \leftarrow 0$ # default continuous rebalancing
2: $[\nu_p, \mathbf{w}, \boldsymbol{\mu}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho}, r_f, N)$
3: **if** $!IsPassiveStrategyPossible(\nu_p, \mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho})$ **then**
4:    **return** τ_c
5: **end if**
6: $k \leftarrow 0$
7: **for** $t = 0$ to T by δT **do**
8:    $k \leftarrow k + 1$
9:    $X[k] \leftarrow 0, Y_1[k] \leftarrow 0, Y_2[k] \leftarrow 0$
10:    **for** $i = 1$ to N+1 **do**
11:       $X[k] \leftarrow X[k] + w[i]e^{\mu[i]t}$ # equation 33
12:       $Y_1[k] \leftarrow Y_1[k] + w^2[i]e^{2\mu[i]t}(e^{\sigma^2[i]t} - 1)$ # first part of equation 45
13:       **for** $j = i + 1$ to N+1 **do** # second part of equation 45
14:          $Y_2[k] \leftarrow Y_2[k] + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\rho[i,j]\sigma[i]\sigma[j]t} - 1)$
15:       **end for**
16:    **end for**
17:    $\chi^P[k] \leftarrow ln(X[k]) - \frac{1}{2}ln(1 + \frac{Y_1[k]+2Y_2[k]}{X^2[k]})$ # equation 54
18:    $\chi[k] \leftarrow \nu_p t$ # equation 27
19:    **if** $k > 1$ **then**
20:       **if** $\chi^P[k] \leq \chi[k]$ and $\chi^P[k-1] \geq \chi[k-1]$ **then**
21:          **return** $\tau_c \leftarrow t$
22:       **end if**
23:    **end if**
24: **end for**
25: **return** τ_c

When τ_c exists, it is hard to obtain a closed loop solution for τ_c by solving equation 63 because of the non-linear nature of equation 54. However we can numerically solve the equation to obtain τ_c. Algorithm 3 outlines the computational steps required to compute the simple rebalance time τ_c for a given set of investment parameters. It records the time when the expected passive log portfolio growth transitions from being higher to a lower value relative to the value under active strategy. This is determined in lines 20 through 22. For our illustrative example we find $\tau_c = 7.61$ years. So, the investor's expected value of the log utility will be higher if the investment is unadjusted for 7.61 years than if it is to be rebalanced continuously to the optimum weights $\mathbf{w}$.

4.2 Stable Rebalancing

Is this rebalance time τ_c optimal? Can we do even better in maximizing expected log growth? In order to answer these questions we must investigate the robustness of τ_c. Note that the estimation of τ_c is based on the information available at time $t = 0$. Will our decision to rebalance change before the expected scheduled rebalance time τ_c expire?

We define a rebalance strategy to be *stable* if passive expected log growth exceeds active expected log growth throughout the passive investment period of τ_c. More formally, a stable rebalancing strategy satisfies the following condition:

$$E[ln(\frac{V^P(t, t+dt)}{V^P(t,t)})] \geq E[ln(\frac{V(t, t+dt)}{V^P(t,t)})], \forall t \in (0\ \tau_c) \text{ and } dt \to 0 \tag{64}$$

Here we have expanded our notation to denote the time when the expected log growth of value of portfolio value is measured. For example, $V(t, t')$ denotes the value of portfolio at time t' estimated at time t. The stability principle states that as the portfolio grows passively, at each time point before the rebalancing time, the investor should always expect to get higher or equal expected log growth using passive strategy. Should her expectation of log of portfolio growth using active strategy at any time be higher during passivity, she will opt to immediately switch to active strategy by rebalancing the portfolio to the set of initial optimal weights $\mathbf{w}$. A rebalancing interval τ_s is stable if the investor does not see the opportunity to switch to active strategy throughout the open interval $(0\ \tau_s)$.

Note that the denominator in the right hand side of equation 64 is $V^P(t,t)$, not $V(t,t)$. This is because up until time t portfolio follows passive strategy to attain the value of portfolio $V^P(t,t)$. At this time t, the investor examines the possibility to rebalance and switch to active strategy if needed.

The right hand side of inequality in equation 64 is $\nu_p dt$ as growth rate ν_p is always constant under active strategy. We now compute the left hand side of inequality in equation 64 in the following lemma.

Lemma 7 *The time t estimation of expected log growth under passive strategy for time $t + dt$ will be the difference of the initial growth estimation for the investment duration $t + dt$ and t, i.e.*

$$E[ln(\frac{V^P(t, t+dt)}{V^P(t,t)})] = E[ln(V^P(0, t+dt)] - E[ln(V^P(0,t)] \tag{65}$$

Proof 7 *Consider an initial investment amount of $V(0)$. From equation 33, under passive strategy*

$$V^P(0,t) = V(0) \sum_{i=1}^{N+1} w_i(0) e^{x_i t} \tag{66}$$

Similarly for investment duration $t + dt$,

$$V^P(0, t+dt) = V(0) \sum_{i=1}^{N+1} w_i(0) e^{x_i(t+dt)} \tag{67}$$

We also know that passive strategy will adhere to reinvestment principle. In other words, the growth archived for duration $t + dt$ will be same as summation of growth achieved for duration t and then growth for reinvesting $V^P(0,t)$ for duration dt. Mathematically,

$$V^P(0, t+dt) = V(0) \sum_{i=1}^{N+1} w_i(0) e^{x_i(t)} \sum_{i=1}^{N+1} w_i(t) e^{x_i(dt)} \tag{68}$$

Note that in equation 68, we have to use the weights at time t that have evolved and changed from their initial optimal values since no rebalancing to these original weights are done in passive strategy.

Equating equations 67 and 68, we get

$$\sum_{i=1}^{N+1} w_i(0) e^{x_i(t+dt)} = \sum_{i=1}^{N+1} w_i(0) e^{x_i(t)} \sum_{i=1}^{N+1} w_i(t) e^{x_i(dt)} \tag{69}$$

Readjusting the terms,

$$\sum_{i=1}^{N+1} w_i(t) e^{x_i(dt)} = \frac{\sum_{i=1}^{N+1} w_i(0) e^{x_i(t+dt)}}{\sum_{i=1}^{N+1} w_i(0) e^{x_i(t)}} \tag{70}$$

From corollary 1 and using equation 35 we can write the portfolio growth estimated at time t for the next dt as:

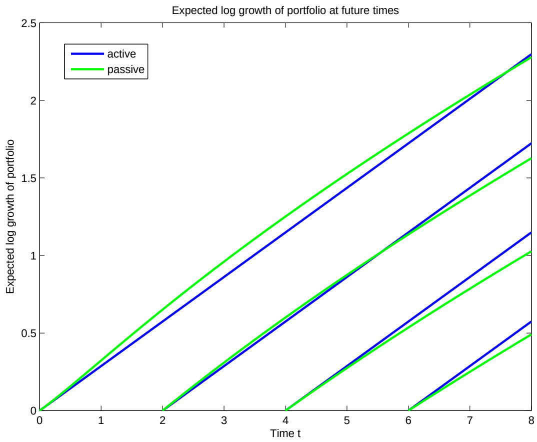

Figure 7: Expected log growth of portfolio at future times.

$$V^P(t, t+dt) = V^P(t,t) \sum_{i=1}^{N+1} w_i(t) e^{x_i dt} \tag{71}$$

First taking logarithm and then taking expectation on both sides, we obtain:

$$E[ln(\frac{V^P(t,t+dt)}{V^P(t,t)})] = E[ln(\sum_{i=1}^{N+1} w_i(t) e^{x_i dt})] \tag{72}$$

Plugging equation 70 in equation 72,

$$\begin{aligned} E[ln(\frac{V^P(t,t+dt)}{V^P(t,t)})] &= E[ln(\frac{\sum_{i=1}^{N+1} w_i(0) e^{x_i(t+dt)}}{\sum_{i=1}^{N+1} w_i(0) e^{x_i(t)}})] \\ &= E[ln(\sum_{i=1}^{N+1} w_i(0) e^{x_i(t+dt)})] - E[ln(\sum_{i=1}^{N+1} w_i(0) e^{x_i(t)})] \end{aligned} \tag{73}$$

Assuming $V(0) = 1$ *and using equations 66 and 67, we arrive at the desired result of equation 65.*

We can rewrite equation 65 in our familiar $\chi(.)$ notation as follows:

$$\chi^P(t, t+dt) = \chi^P(0, t+dt) - \chi^P(0,t) \tag{74}$$

Lemma 7 proves that for the same horizon, as time passes, our estimation of passive log growth undergoes parallel downward shift as depicted in the *log growth contours* in figure 7. The active log growth also parallel shifts downward at successive estimation time point t, but by a different magnitude equaling $\nu_p t$. As one moves along the estimation time line, the difference between passive and active log growth declines. After sometime passive log growth is no more advantageous over its active counterpart as depicted in figure 7. Thus, at that time, it is no more prudent to continue the portfolio passively and it needs to be rebalanced to the optimal set of weights. A fresh rebalance to these weights will reset the passive growth rate to its original value.

Lemma 8 *Portfolio rebalance time* τ_s *is stable if the initial estimation of rate of change of passive log growth is higher than optimal log growth rate* ν_p *during the passive investment period* $(0\ \tau_s)$*, i.e.*

$$\frac{d\chi^P(0,t)}{dt} \geq \nu_p, \forall t \in (0\ \tau_s) \tag{75}$$

Proof 8 *As noted earlier, the right hand side of equation 64 is* $\nu_p dt$. *Plugging in the equation 74 in the left hand side of equation 64, we get* $\forall t \in (0\ \tau_s)$*:*

$$\begin{aligned} &\chi^P(0, t+dt) - \chi^P(0,t) \geq \nu_p dt \\ \Rightarrow\ &\frac{\chi^P(0,t+dt) - \chi^P(0,t)}{dt} \geq \nu_p \end{aligned} \tag{76}$$

Letting $dt \to 0$, we get the desired equation 75.

Above lemma 8 states a very important result. It says that one can compute the rebalance time at time $t = 0$, by merely taking the derivative of expected passive log growth with respect to time t and equating it to optimal log growth rate of ν_p. This rebalance time τ_s shall be stable in the sense that at any time t' before τ_s, the passive investor will receive expected log growth higher than ν_p in the immediate future. Thus the investor has no incentive to shift to the continuous rebalance active strategy at any time before τ_s.

Intuitively the derivative of expected log growth with respect to time t is the *instantaneous portfolio growth* in an infinitely small time interval. We will use ξ to denote instantaneous portfolio growth. In this notation, we can write,

$$\xi = \frac{d\chi(t)}{dt} = \nu_p \tag{77}$$

Using equation 27, we see that under active strategy the instantaneous portfolio growth $\xi = \nu_p$, an invariant of time. Using equivalent notation for passive strategy, we can write,

$$\xi^P(t) = \frac{d\chi^P(t)}{dt} \tag{78}$$

One needs to distinguish between instantaneous portfolio growth and portfolio growth rate. Portfolio growth rate is the average portfolio growth for a specified duration of time. Instantaneous portfolio growth at any time is the incremental growth that is achieved for a infinitely small time interval. In the context of this paper, both are defined under log of portfolio value. From equation 77, under active strategy, these two measures are always equal and invariant of time.

The lemma 8 merely states that one needs to continue using passive strategy as long as the instantaneous portfolio growth offered by passive strategy is higher than or equal to that under active strategy. It also states that the stable rebalancing is possible only when the following condition is satisfied:

$$\exists \tau_s \text{ s.t. } \xi^P(t) > \nu_p, \forall t \in (0\ \tau_s) \tag{79}$$

Assuming that above condition is satisfied, the investor benefits by adopting passive strategy until τ_s, when the need to rebalance arises. At τ_s, the instantaneous portfolio growth for passive strategy becomes equal to that for active strategy, i.e. ν_p.

$$\xi^P(\tau_s) = \nu_p \tag{80}$$

We are now set to compute τ_s in terms of the given initial investment parameters.

Lemma 9 *The portfolio rebalance time τ_s is the solution of the following equation:*

$$\frac{1}{X(t)}[X'(t) - \frac{1}{2}\frac{X(t)Y'(t) - 2X'(t)Y(t)}{X(t)^2 + Y(t)}] = \nu_p \tag{81}$$

where,
$X(t)$ = expected portfolio growth at time t, given by equation 41
$Y(t)$ = variance of portfolio growth at time t, given by equation 45

$$X'(t) = \frac{dX}{dt} = \sum_{i=1}^{N+1} w_i \mu_i e^{\mu_i t} \tag{82}$$

$$\begin{aligned} Y'(t) = \frac{dY}{dt} \quad &= \quad \sum_{i=1}^{N+1} w_i^2 e^{2\mu_i t}[2\mu_i(e^{\sigma_i^2 t} - 1) + \sigma_i^2 e^{\sigma_i^2 t}] \\ &+2\sum_{i=1}^{N}\sum_{j=i+1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}[(\mu_i + \mu_j)(e^{\rho_{ij}\sigma_i\sigma_j t} - 1) + \rho_{ij}\sigma_i\sigma_j e^{\rho_{ij}\sigma_i\sigma_j t}] \end{aligned} \tag{83}$$

Proof 9 *We start with the resulting equation of lemma 8. At the stable rebalance time τ_s is given by the solution of the following equation when passive instantaneous portfolio growth becomes same as the instantaneous portfolio growth under active strategy which is ν_p:*

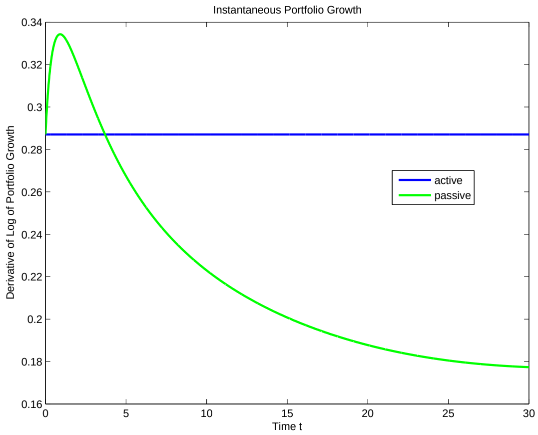

Figure 8: Instantaneous portfolio growth comparison with passive strategy.

$$\frac{d\chi^P(t)}{dt} = \nu_p \tag{84}$$

Note, for simplicity we have removed the first time index from above equation and assume initial time for these estimation. Using our notations, we can rewrite equation 51:

$$Var[ln(\frac{V^P(t)}{V(0)})] = ln(1 + \frac{Y(t)}{X^2(t)}) \tag{85}$$

Moreover, using our notations and equation 85, we can rewrite equation 54:

$$\chi^P(t) \quad = \quad E[ln(\frac{V^P(t)}{V(0)})] = ln(X(t)) - \frac{1}{2}Var[ln(\frac{V^P(t)}{V(0)})] = ln(X(t)) - \frac{1}{2}ln(1 + \frac{Y(t)}{X^2(t)}) \tag{86}$$

Taking the first derivative of equation 86, we get:

$$\begin{aligned}
\frac{d\chi^P(t)}{dt} \quad &= \quad \frac{X'(t)}{X(t)} - \frac{1}{2}\frac{X^2(t)}{X^2(t)+Y(t)}(1+\frac{Y(t)}{X^2(t)})' \\
&= \frac{X'(t)}{X(t)} - \frac{1}{2}\frac{X^2(t)}{X^2(t)+Y(t)}(\frac{Y(t)}{X^2(t)})' \\
&= \frac{X'(t)}{X(t)} - \frac{1}{2}\frac{X^2(t)}{X^2(t)+Y(t)}\frac{X^2(t)Y'(t) - 2X(t)X'(t)Y(t)}{X^4(t)} \\
&= \frac{X'(t)}{X(t)} - \frac{1}{2}\frac{1}{X^2(t)+Y(t)}\frac{X(t)Y'(t) - 2X'(t)Y(t)}{X(t)} \\
&= \frac{1}{X(t)}[X'(t) - \frac{1}{2}\frac{X(t)Y'(t) - 2X'(t)Y(t)}{X(t)^2+Y(t)}]
\end{aligned} \tag{87}$$

Figure 8 plots the instantaneous portfolio growth for passive strategy following equation 87 for our example investment scenario. As per lemma 9 non-zero intersection of the passive and the active instantaneous portfolio growth curves gives the stable rebalance time $\tau_s = 3.7$ for the portfolio. Now we summarize the computational steps in the form of algorithm 4 required to compute τ_s.

Notice that the simple rebalance time $\tau_c = 7.61$ determined by algorithm 3 is much longer. Even though the investor can attain the same expected log growth as active strategy by remaining passive for $\tau_c = 7.61$ years, after $\tau_s = 3.7$ years, her incremental expected log growth shall be smaller compared to that offered by active strategy. We will soon see that by rebalancing earlier after $\tau_s = 3.7$ years, she can increase the potential gain measured in terms of expected log of portfolio growth.

We now define $\psi^P(t) = \chi^P(t) - \chi(t)$ which is the excess growth relative to active strategy. We show that the excess passive growth $\psi^P(t)$ is a monotonously increasing function for $0 < t < \tau_s$.

Lemma 10 *$\psi^P(t)$, the excess growth produced by passive strategy is increasing in the range $t \in (0\ \tau_s)$.*

Algorithm 4 ComputeStableRebalanceFrequency

Require: $\boldsymbol{\mu}$,$\boldsymbol{\Sigma}$,$\boldsymbol{\rho}$,r_f,N,T,δT

$\tau_s \leftarrow 0$ # default continuous rebalancing
$[\nu_p, \mathbf{w}, \boldsymbol{\mu}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho}, r_f, N)$
if $!IsPassiveStrategyPossible(\nu_p, \mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho})$ **then**
  return τ_s
end if
$k \leftarrow 0$
for $t = 0$ to T by δT **do**
  $k \leftarrow k + 1$
  $X[k] \leftarrow 0, Y_1[k] \leftarrow 0, Y_2[k] \leftarrow 0, Y_1'[k] \leftarrow 0, Y_2'[k] \leftarrow 0$
  for $i = 1$ to N+1 **do**
    $X[k] \leftarrow X[k] + w[i]e^{\mu[i]t}$ # equation 33
    $X'[k] \leftarrow X'[k] + w[i]\mu[i]e^{\mu[i]t}$ # equation 82
    $Y_1[k] \leftarrow Y_1[k] + w^2[i]e^{2\mu[i]t}(e^{\sigma^2[i]t} - 1)$ # 1^{st} half of equation 45
    # 1^{st} half of equation 83
    $Y_1'[k] \leftarrow Y_1'[k] + w[i]^2 e^{2\mu[i]t}[2\mu[i](e^{\sigma[i]^2 t} - 1) + \sigma[i]^2 e^{\sigma[i]^2 t}]$
    for $j = i + 1$ to N+1 **do**
      # 2^{nd} half of equation 45
      $Y_2[k] \leftarrow Y_2[k] + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\rho(i,j)\sigma[i]\sigma[j]t} - 1)$
      # 2^{nd} half of equation 83
      $Y_2'[k] \leftarrow Y_2'[k] + w[i]w[j]e^{(\mu[i]+\mu[j])t}[(\mu[i] + \mu[j])(e^{\rho[i,j]\sigma[i]\sigma[j]t} - 1) + \rho[i,j]\sigma[i]\sigma[j]e^{\rho[i,j]\sigma[i]\sigma[j]t}]$
    end for
  end for
  $Y[k] \leftarrow Y_1[k] + 2Y_2[k]$ # equation 45
  $Y'[k] \leftarrow Y_1'[k] + 2Y_2'[k]$ # equation 83
  $\xi^P[k] \leftarrow \frac{1}{X[k]}[X'[k] - \frac{1}{2}\frac{X[k]Y'[k] - 2X'[k]Y[k]}{X[k]^2 + Y[k]}]$ # equation 87
  if $k > 1$ **then**
    if $\xi^P[k] \leq \nu_p$ and $\xi^P[k-1] \geq \nu_p$ **then**
      return $\tau_s = t$
    end if
  end if
end for
return τ_s

Proof 10 *We need to prove that $\psi'^P(t) > 0$, $\forall t \in (0\ \tau_s)$. Let's start with the derivative of $\psi^P(t)$.*

$$\psi'^P(t) = \frac{d(\chi^P(t) - \nu_p t)}{dt} = \frac{d(\chi^P(t))}{dt} - \nu_p = \xi^P(t) - \nu_p \tag{88}$$

By definition of passive strategy, one needs to continue without rebalancing till $\xi^P(t) > \nu_p$. Since rebalancing occurs at τ_s, using equation 79, for all $t \in (0\ \tau_s)$, $\xi^P(t) > \nu_p$ implying $\psi'^P(t) > 0$.

Lemma 11 *$\psi^P(t)$, the excess growth produced by passive strategy is maximized at τ_s.*

Proof 11 *In order to prove that τ_s is a relative maxima, we need to prove the following two:*

$$\psi'^P(\tau_s) = 0 \tag{89}$$

$$\psi''^P(\tau_s) < 0 \tag{90}$$

Proof for equation 89 is straightforward:

$$\psi'^P(\tau_s) = \xi^P(\tau_s) - \xi(\tau_s) = \xi^P(\tau_s) - \nu_p = 0 \tag{91}$$

We have used the results of lemma 9 above. Hence we proved equation 89.

To prove equation 90, we will use fundamental definition of differentiation.

$$\begin{aligned}\psi''^P(\tau_s) &= \lim_{d\tau \to 0} \frac{\psi'^P(\tau_s + d\tau) - \psi'^P(\tau_s)}{d\tau} = \lim_{d\tau \to 0} \frac{(\chi'^P(\tau_s + d\tau) - \nu_p) - (\chi'^P(\tau_s) - \nu_p)}{d\tau} \\ &= \lim_{d\tau \to 0} \frac{\chi'^P(\tau_s + d\tau) - \chi'^P(\tau_s)}{d\tau} = \lim_{d\tau \to 0} \frac{\xi^P(\tau_s + d\tau) - \xi^P(\tau_s)}{d\tau}\end{aligned} \tag{92}$$

By definition of stable strategy $\xi^P(\tau_s) = \nu_p$ and $\xi^P(\tau_s + d\tau) < 0$. Therefore $\psi''^P(\tau_s) < 0$, proving equation 90.

Thus far we have not shed any light on conditions for the existence of rebalance time when an investor can take advantage of passive investment strategy. First, without going through formal mathematical proof, we will discuss the existence of initial rebalance time τ_c. When τ_c does not exists, then the opportunity to remain passive during certain duration of investment horizon will not be possible. In this case, the investor has to continuously rebalance in order to maximize her log utility for the given horizon. The initial rebalance time τ_c exists only when the following two conditions hold true:

$$\chi^P(\tau_c - d\tau_c) > \chi(\tau_c - d\tau_c) \text{ where, } 0 < d\tau_c < \tau_c \text{ and } d\tau_c \to 0 \tag{93a}$$

$$\chi^P(\tau_c + d\tau_c) > \chi(\tau_c + d\tau_c) \text{ where, } 0 < d\tau_c < \tau_c \text{ and } d\tau_c \to 0 \tag{93b}$$

From equation 27, $\chi(t)$ is a monotonically increasing function for $t \geq 0$ since its first derivative, the portfolio growth rate ν_p is a constant. This assumes a typical profit seeking investor chooses the assets for positive growth rate only. We also know that at $t = 0$, $\chi(t) = \chi^P(t) = ln[V(0)] = 0$. Hence, if equation 54 is a monotonically decreasing function for the given set of input investment parameters, then condition specified in equation 93a will never be satisfied for any $t > 0$. If, however, equation 54 is monotonically increasing with its first derivative or slope higher than than ν_p, then opportunity to remain passive exists. In other words, opportunity for passive strategy exists only if the slope or derivative of $E[ln(V^P(t))]$ at time $t = 0$ is greater than or equal to the corresponding slope ν_p for the active strategy. This is true for both simple as well as stable rebalancing strategy. Now we will derive the condition for existence of passive strategy.

Lemma 12 *Passive strategy is feasible only when:*

$$\sum_{i=1}^{N+1} w_i \mu_i - \frac{1}{2}[\sum_{i=1}^{N+1} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N} \sum_{j=i+1}^{N+1} w_i w_j \rho_{ij} \sigma_i \sigma_j] \geq \nu_p \tag{94}$$

Proof 12 *By definition, at time $t = 0$ the initial expected log of portfolio growth is the same under both active and passive strategy. One will consider using passive strategy only if the instantaneous portfolio growth is higher under this strategy at $t = 0$. Mathematically,*

$$\frac{d\chi^P(t)}{dt}|_{t=0} \geq \nu_p \tag{95}$$

Plugging in equation 87, the required condition becomes,

$$\frac{1}{X(0)}[X'(0) - \frac{1}{2}\frac{X(0)Y'(0) - 2X'(0)Y(0)}{X(0)^2 + Y(0)}] \geq \nu_p \tag{96}$$

Plugging in $t = 0$ in equations 41, 45, 82 and 83 respectively:

$$X(0) = \sum_{i=1}^{N+1} w_i = 1 \tag{97}$$

$$Y(0) = 0 \tag{98}$$

$$X'(0) = \sum_{i=1}^{N+1} w_i\mu_i \tag{99}$$

$$Y'(0) = \sum_{i=1}^{N+1} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N+1} w_iw_j\rho_{ij}\sigma_i\sigma_j \tag{100}$$

Finally, we obtain the condition of equation 94 by plugging in the above values in equation 96.

Hence, according lemma 12, the opportunity to take advantage of intermittent rebalancing and adherence to passive strategy is entirely determined by the set of asset mean and covariance characteristics. We now present this result in the form of the following algorithm 5.

Algorithm 5 IsPassiveStrategyPossible

Require: ν_p,$\mathbf{w}$,$\boldsymbol{\mu}$,$\boldsymbol{\Sigma}$,$\boldsymbol{\rho}$,N

$X' \leftarrow 0, Y_1' \leftarrow 0, Y_2' \leftarrow 0$
for $i = 1$ to N+1 **do**
  $X' \leftarrow X' + w[i]\mu[i]$ # equation 99
  $Y_1' \leftarrow Y_1' + w[i]^2\sigma[i]^2$ # 1^{st} half of equation 100
  for $j = i + 1$ to N+1 **do**
    $Y_2' \leftarrow Y_2' + w[i]w[j]\rho[i,j]\sigma[i]\sigma(j)$ # 2^{nd} half of equation 100
  end for
end for
$Y' \leftarrow Y_1' + 2Y_2'$ # equation 100
if $X' - \frac{Y'}{2} > \nu_p$ **then** # equation 94
  return true
else
  return false
end if

5 Hybrid Strategy

Thus far, we have analyzed the nature of log of portfolio growth when the investor opts to remain passive without performing any rebalancing. We have shown that for certain assets characteristics the investor can get higher expected log of portfolio growth during the initial investment period. During this investment period, which we term as rebalance time, she should not opt to rebalance the portfolio. First, we have derived the initial rebalance time τ_c when the expected log of portfolio growth is higher under passive strategy. Then, we designed a stable rebalance time τ_s when the expected instantaneous log growth of portfolio is higher under passive strategy. One can show that for a given set of assets $\tau_s \leq \tau_c$.

We now explore the nature of the portfolio growth after the first rebalance to determine the subsequent rebalance times. While determining the set of rebalance points or times, we must ensure that the investor utility, i.e. the expected log of portfolio growth, must not fall below the baseline value obtained using active rebalancing strategy. We define such investment strategy as a *hybrid* strategy wherein, the investor uses intermittent non-continuous rebalancing to maintain equal or higher expected log of portfolio growth throughout the investment horizon. In the simple approach, she will wait to rebalance as long as the expected log of portfolio value remains higher than the corresponding active strategy value. In the stable rebalancing approach, she will only rebalance when the expected instantaneous log growth dips below the corresponding value of ν_p under active strategy.

We will use the superscript H for hybrid strategy specific variables. We also use superscript H_τ to denote a hybrid strategy that uses τ as the rebalance frequency. Note that during the initial simple rebalance time period $(0\ \tau_c]$, hybrid strategy growth is identical to that of passive strategy. Thus we must satisfy equations 62 and 63. Using the expanded notation, the time 0 estimation of the expected log of portfolio growth satisfies the following two conditions:

$$\chi^{H_{\tau_c}}(0, \delta t) > \nu_p \delta t, \text{ where } 0 < \delta t < \tau_c \tag{101}$$

$$\chi^{H_{\tau_c}}(0, \tau_c) = \nu_p \tau_c \tag{102}$$

From fundamental definition,

$$\begin{aligned} \chi^{H_{\tau_c}}(0, \delta t) &= E[ln(V^P(0, \delta t))] = E[ln(V^P(0,0) \sum_{i=1}^{N+1} w_i^{x_i \delta t})] \\ &= E[ln(V^P(0,0))] + E[ln(\sum_{i=1}^{N+1} w_i^{x_i \delta t})] \\ &= E[ln(\sum_{i=1}^{N+1} w_i^{x_i \delta t})], \text{ since } V^P(0,0) = 1 \end{aligned} \tag{103}$$

Hence, equation 101 implies,

$$E[ln(\sum_{i=1}^{N+1} w_i^{x_i \delta t})] > \nu_p \delta t \tag{104}$$

Similarly, from fundamental definition we can derive:

$$\chi^{H_{\tau_c}}(0, \tau_c) = E[ln(\sum_{i=1}^{N+1} w_i^{x_i \tau_c})] \tag{105}$$

Hence, combining equations 102 and 105 we obtain:

$$E[ln(\sum_{i=1}^{N+1} w_i^{x_i \tau_c})] = \nu_p \tau_c \tag{106}$$

5.1 Simple Hybrid Strategy

Theorem 1 *Let τ_c, the initial simple rebalance time satisfying equation 62 and 63 exist (and which can be computed using algorithm 3). Then $i\tau_c$ will also be a rebalance time for simple hybrid strategy, where $i \in \mathbb{N}$.*

Proof 13 *We need to prove that for all $i \in \mathbb{N}$, i.e. for initial and all subsequent rebalance periods, $(i\tau_c\ (i+1)\tau_c]$, the following two conditions analogous to equations 101 and 102 must also hold.*

$$\chi^{H_{\tau_c}}(i\tau_c, i\tau_c + \delta t) > \nu_p(i\tau_c + \delta t), \ \forall i \in \mathbb{N}, \delta t < \tau_c \tag{107}$$

$$\chi^{H_{\tau_c}}(i\tau_c, (i+1)\tau_c) = \nu_p((i+1)\tau_c), \ \forall i \in \mathbb{N} \tag{108}$$

We will prove both of these equations by the method of induction. For initial step when $i = 0$, both equations 107 and 108 becomes equations 101 and 102 respectively. By definition of τ_c these will be true. For the inductive step, assume equation 107 and 108 hold for $i = k$ and hence $k\tau_c$ is also a rebalance time. That is,

$$\chi^{H_{\tau_c}}(k\tau_c, k\tau_c + \delta t) > \nu_p(k\tau_c + \delta t) \tag{109}$$

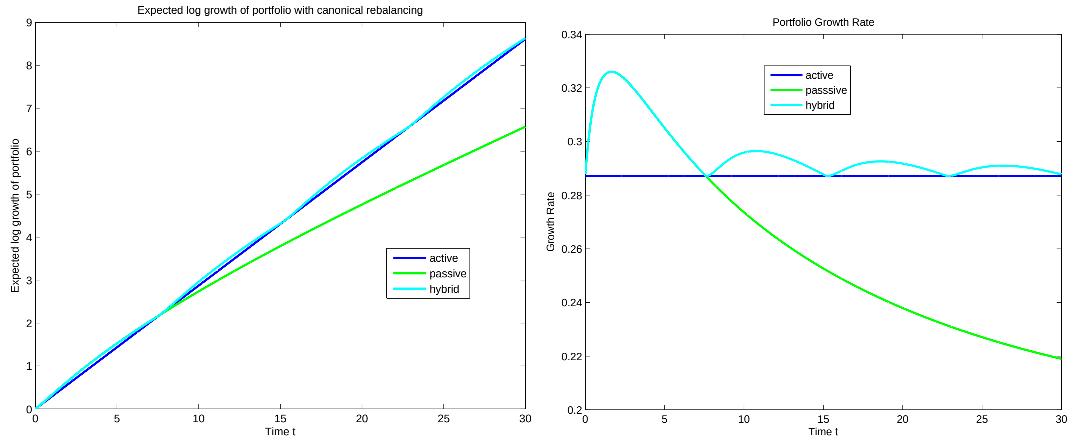

Figure 9: Expected value and growth rate of log portfolio.

$$\chi^{H_{\tau_c}}(k\tau_c,(k+1)\tau_c) = \nu_p((k+1)\tau_c) \tag{110}$$

Equation 110 indicates that $(k+1)\tau_c$ is a rebalance point. At rebalance points the expected log of portfolio value will always be equal to the corresponding value under active strategy. This value will not be driven by the time of estimation. Hence equation 110 can be written as:

$$\chi^{H_{\tau_c}}(.,(k+1)\tau_c) = E[ln(V^P(.,(k+1)\tau_c))] = \nu_p((k+1)\tau_c) \tag{111}$$

We must show that equations 107 and 108 also hold for $i = k+1$, i.e.

$$\chi^{H_{\tau_c}}((k+1)\tau_c,(k+1)\tau_c+\delta t) > \nu_p((k+1)\tau_c+\delta t) \tag{112}$$

$$\chi^{H_{\tau_c}}((k+1)\tau_c,(k+2)\tau_c) = \nu_p((k+2)\tau_c) \tag{113}$$

Following similar steps as of the derivation of equation 103,

$$\begin{aligned}\chi^{H_{\tau_c}}((k+1)\tau_c,(k+1)\tau_c+\delta t) &= E[ln(V^P((k+1)\tau_c,(k+1)\tau_c+\delta t))] \\ &= E[ln(V^P((k+1)\tau_c,(k+1)\tau_c)\sum_{i=1}^{N+1} w_i^{x_i\delta t})] \\ &= E[ln(V^P((k+1)\tau_c,(k+1)\tau_c))] + E[ln(\sum_{i=1}^{N+1} w_i^{x_i\delta t})]\end{aligned} \tag{114}$$

We have made use of the fact that $(k+1)\tau_c$ is a rebalance time and hence the initial asset weights are used. Now let's look at the two terms in the above equation. The first term is given by equation 111. The value of the second term is given by equation 104. Thus we establish the required relationship given by equation 112. To prove equation 113, we start with the LHS:

$$\begin{aligned}\chi^{H_{\tau_c}}((k+1)\tau_c,(k+2)\tau_c) &= E[ln(V^P((k+1)\tau_c,(k+2)\tau_c))] \\ &= E[ln(V^P((k+1)\tau_c,(k+1)\tau_c)\sum_{i=1}^{N+1} w_i^{x_i\tau_c})] \\ &= E[ln(V^P((k+1)\tau_c,(k+1)\tau_c))] + E[ln(\sum_{i=1}^{N+1} w_i^{x_i\tau_c})]\end{aligned} \tag{115}$$

Once again, we have made use of the fact that $(k+1)\tau_c$ is a rebalance time and hence the initial asset weights are used. As before, the first term is given by equation 111. The value of the second term is given by equation 106. Thus we establish the required relationship given by equation 113 and hence the equation 108.

This completes the proof of the theorem establishing the need to rebalance the assets to the initial optimal weights $\boldsymbol{w}$ at a periodic interval of τ_c.

As per theorem 1 under simple hybrid strategy the portfolio needs to be rebalanced at $\tau_c, 2\tau_c, 3\tau_c, \ldots$ regular time intervals in order to attain or exceed investor log utility for a given finite investment horizon. This is illustrated in figure 5.1 for our example investment portfolio. The portfolio only needs to be rebalanced successively at $7.61, 15.22$ and 22.83 years during the 30 year investment period.

We now determine the expected log of portfolio growth under hybrid strategy when the portfolio is rebalanced periodically. Using the next theorem, we show that one can compute the expected log of portfolio value for hybrid strategy using the expected log of portfolio values from passive strategy. Hence, we name this theorem as the passive to hybrid *growth map theorem*. The theorem is applicable for all rebalancing scenarios including simple and stable rebalancing. Before we state and prove the theorem, we will state and prove two hypothesis concerning periodic rebalancing. The first one is called the *law of additive growth* whereas the second one is termed as *law of multiplicative growth*. First we state and prove the law of additive growth.

Lemma 13 *Passive expected log of portfolio growth is additive, i.e.*

$$\chi^H(t^r + t') = \chi^H(t^r) + \chi^P(t') \tag{116}$$

where t^r is the most recent time when the portfolio is rebalanced and t' is the time for which the portfolio grows passively after t^r.

Proof 14 *Since t^r is the most recent rebalance time, the portfolio value at $t^r + t'$ is given by:*

$$V^H(t^r + t') = V^H(t^r) \sum_{i=1}^{N+1} w_i e^{x_i(t')} \tag{117}$$

Taking first log and then expected value on both sides, we obtain,

$$\chi^H(t^r + t') = \chi^H(t^r) + E[ln(\sum_{i=1}^{N+1} w_i e^{x_i(t')})] = \chi^H(t^r) + \chi^P(t') \tag{118}$$

Now we proceed to state and prove the law of multiplicative growth.

Lemma 14 *Expected log of portfolio value multiplies with the number of times periodic rebalancing is executed, i.e*

$$\chi^H(k\tau) = k\chi^P(\tau),\ \forall k \in \mathbb{N}^+ \tag{119}$$

where τ is the periodic rebalance frequency.

Proof 15 *We prove this lemma by method of induction. For the base case $k = 1$, equation 119 is trivially true. We then assume equation 119 holds for k and prove below that it also holds for $k + 1$. For $k + 1$, we need to prove,*

$$\chi^H(\overline{k+1}\tau) = (k+1)\chi^P(\tau) \tag{120}$$

We start with RHS of above equation 120:

$$\begin{aligned} (k+1)\chi^P(\tau) &= k\chi^P(\tau) + \chi^P(\tau) \\ &= \chi^H(k\tau) + \chi^P(\tau), \textit{ as equation 119 holds for k.} \\ &= \chi^H(k\tau + \tau), \textit{ applying law of additive growth, lemma 13} \\ &= \chi^H(\overline{k+1}\tau) = LHS \end{aligned} \tag{121}$$

That completes the proof of equation 119 by induction.

Theorem 2 *Assume that $\chi^{H_\tau}(t) = \chi^P(t), \forall\, t \in (0\ \tau]$ is known following equation 54. Then $\forall\, t > \tau > 0$,*

$$\chi^{H_\tau}(t) = \begin{cases} \nu_p t & \textit{if } \tau = 0 \\ k\chi^P(\tau) + \chi^P(t') & \textit{otherwise} \end{cases} \tag{122}$$

where $t = k\tau + t'$, $k = \lfloor \frac{t}{\tau} \rfloor$ and $t' = t \bmod \tau$.

Proof 16 *At the very outset, note that we consciously treat $\tau = 0$ case to be same as the active strategy for consistency of results between different strategies. Additionally while computing k and t', we avoid divide-by-zero scenarios. We only need to prove:*

$$\chi^{H_\tau}(k\tau + t') = k\chi^P(\tau) + \chi^P(t') \tag{123}$$

We start with LHS of above equation 123.

$$\begin{aligned} \chi^{H_\tau}(k\tau + t') &= \chi^{H_\tau}(k\tau) + \chi^P(t'), \text{ applying law of additive growth, lemma 13} \\ &= k\chi^P(\tau) + \chi^P(t'), \text{ applying law of multiplicative growth, lemma 14} \\ &= LHS \end{aligned} \tag{124}$$

This completes the proof for the growth map theorem.

The growth map theorem 2 establishes the relationship between passive and hybrid strategy portfolio growths. It states that under any hybrid strategy where rebalancing is done with periodicity of τ, the expected log of portfolio growth at subsequent rebalancing points can be obtained by multiplying the expected log of portfolio growth at the first rebalance point by the number of times the portfolio has been rebalanced to the initial optimal weights. Once we obtain the expected log of portfolio growth at the last rebalance point, growth for any additional time $t' < \tau$ will occur following the passive trajectory identical to the initial rebalance period. An important aspect of this lemma is that the proposition is true for any positive finite value of period rebalance frequency τ, not just simple or stable rebalance frequencies. For any hybrid strategy with periodic rebalance frequency τ, once the passive $\chi)^P(.)$ trajectory is calculated for the initial duration up to the first rebalance time, i.e. $[0\ \tau]$, we can completely construct the $\chi^{H_\tau}(.)$ trajectory for any future investment horizon.

For our example portfolio, after 30 years, the expected log of portfolio growth will be 8.6125 and 8.632 under active and simple hybrid strategy respectively. In terms of the expected log of portfolio growth, throughout the investment period, we expect to outperform the continuous rebalance active strategy. We will now prove this assertion using the next lemma. In real life investment, the performance of hybrid strategy will even be better once we factor in the cost of rebalancing.

Lemma 15 *Simple hybrid strategy will always outperform active strategy, i.e. $\chi^{H_{\tau_c}}(t) \geq \chi(t)$.*

Proof 17 *Using the results of growth map theorem 2:*

$$\chi^{H_{\tau_c}}(t) = k\chi^P(\tau_c) + \chi^P(t') \tag{125}$$

where $t = k\tau_c + t'$, $k = floor(t/\tau_c)$ and $t' = t \bmod \tau_c$.

Using the same notations,

$$\chi(t) = \nu_p t = \nu_p(k\tau_c + t') = k\nu_p\tau_c + \nu_p t' = k\chi(\tau_c) + \chi(t') \tag{126}$$

By definition, during the initial rebalance period $[0\ \tau_c)$, passive strategy outperforms active strategy, i.e.

$$\chi^P(t') \geq \chi(t'), \forall t' \in [0\ \tau_c) \tag{127}$$

We know from equation 63,

$$\chi^P(\tau_c) = \chi(\tau_c) \tag{128}$$

Using the relations of equations 125 through 128, we obtain $\chi^{H_{\tau_c}}(t) \geq \chi(t)$.

5.2 Stable Hybrid Strategy

Similar to theorem 1, we will now derive subsequent stable rebalance times for stable hybrid strategy.

Theorem 3 *Let τ_s, the initial stable rebalance time satisfying equation 79 exists (and which can be computed using algorithm 4). Then $i\tau_s$ will also be a rebalance time for a stable hybrid strategy, where i is the set of natural numbers including 0, i.e. $i \in \mathbb{N}$.*

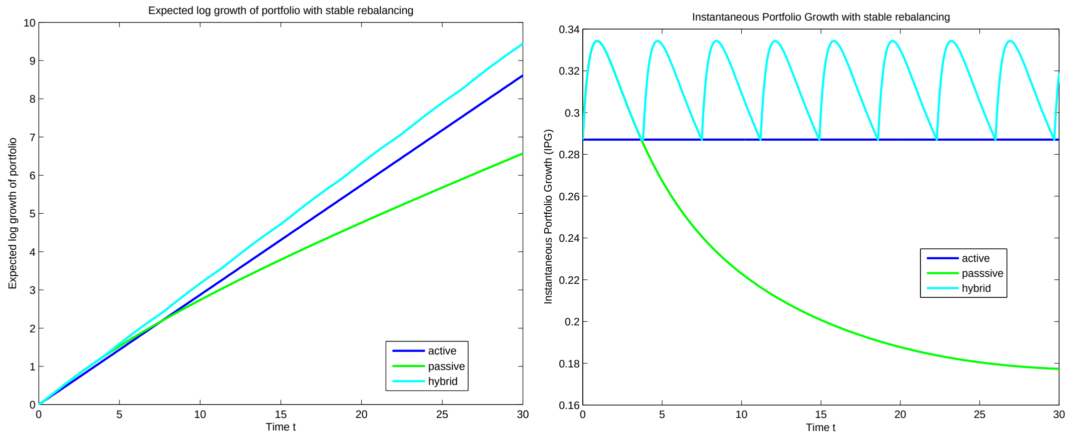

Figure 10: Expected portfolio value and instantaneous portfolio growth in stable hybrid strategy

The proof is similar to the proof of theorem 1. For brevity we provide the proof in appendix 9. Figure 5.2 shows the evolution of expected instantaneous portfolio growth under stable hybrid strategy. The growth is never allowed to slip below the corresponding value ν_p under active strategy. Under such a strategy the instantaneous growth expectation during the entire investment horizon always remains higher or equal to that under active strategy. Using lemma 2, we can obtain the expected log of portfolio value under stable hybrid strategy as follows:

$$\chi^{H_{\tau_s}}(t) = k_s\chi^P(\tau_s) + \chi^P(t'_s) \tag{129}$$

where $t = k\tau_s + t'_s$, $k_s = \lfloor \frac{t}{\tau_s} \rfloor$ and $t'_s = t \bmod \tau_s$. As shown in figure 5.2, stable hybrid strategy yields higher expected log of portfolio growth. We will formalize this property in the form of theorem 4 below.

Theorem 4 *Stable hybrid strategy will always outperform a hybrid strategy with higher rebalancing frequency, i.e. for any investment horizon* $t > \tau_x$, $\chi^{H_{\tau_s}}(t) > \chi^{H_{\tau_x}}(t)$, *where* $\tau_x > \tau_s$.

Proof 18 *Using the results of theorem 2:*

$$\chi^{H_{\tau_s}}(t) = k_s\chi^P(\tau_s) + \chi^P(t'_s) \tag{130}$$

where $t = k_s\tau_s + t'_s$, $k_s = \lfloor \frac{t}{\tau_s} \rfloor$ *and* $t'_s = t \bmod \tau_s$. *Similarly,*

$$\chi^{H_{\tau_x}}(t) = k_x\chi^P(\tau_x) + \chi^P(t'_x) \tag{131}$$

where $t = k_x\tau_x + t'_x$, $k_x = \lfloor \frac{t}{\tau_x} \rfloor$ *and* $t'_x = t \bmod \tau_x$. *Figure 11 depicts the two different rebalance frequencies under consideration relative to the simple rebalance point* τ_c. *We need to prove the following inequality:*

$$\begin{aligned}
& \chi^{H_{\tau_s}}(t) > \chi^{H_{\tau_x}}(t) \\
\Rightarrow\quad & k_s\chi^P(\tau_s) + \chi^P(t'_s) > k_x\chi^P(\tau_x) + \chi^P(t'_x) \\
\Rightarrow\quad & k_s[\chi(\tau_s) + \psi^P(\tau_s)] + [\chi(t'_s) + \psi^P(t'_s)] > k_x[\chi(\tau_x) + \psi^P(\tau_x)] + [\chi(t'_x) + \psi^P(t'_x)] \\
\Rightarrow\quad & k_s[\nu_p\tau_s + \psi^P(\tau_s)] + [\nu_p t'_s + \psi^P(t'_s)] > k_x[\nu_p\tau_x + \psi^P(\tau_x)] + [\nu_p t'_x + \psi^P(t'_x)] \\
\Rightarrow\quad & \nu_p[k_s\tau_s - k_x\tau_x] + [k_s\psi^P(\tau_s) - k_x\psi^P(\tau_x)] > \nu_p[t'_x - t'_s] + [\psi^P(t'_x) - \psi^P(t'_s)] \\
\Rightarrow\quad & \nu_p[t - t'_s - t + t'_x] + [k_s\psi^P(\tau_s) - k_x\psi^P(\tau_x)] > \nu_p[t'_x - t'_s] + [\psi^P(t'_x) - \psi^P(t'_s)] \\
\Rightarrow\quad & [k_s\psi^P(\tau_s) - k_x\psi^P(\tau_x)] > [\psi^P(t'_x) - \psi^P(t'_s)]
\end{aligned} \tag{132}$$

Since, $\tau_s < \tau_x$, *we know that* $k_s \geq k_x$. *Let's define* $\triangle k = k_s - k_x$ *and substitute in the above inequality.*

$$\begin{aligned}
& [(k_x + \triangle k)\psi^P(\tau_s) - k_x\psi^P(\tau_x)] > [\psi^P(t'_x) - \psi^P(t'_s)] \\
\Rightarrow\quad & k_x[\psi^P(\tau_s) - \psi^P(\tau_x)] + \triangle k\psi^P(\tau_s) > [\psi^P(t'_x) - \psi^P(t'_s)]
\end{aligned} \tag{133}$$

We will now separately consider two possible cases for the value of $\triangle k$. *Case 1:* $\triangle k \geq 1$
The worst case scenario for equation 133 is when we consider the maximum possible value for the RHS expression. This will

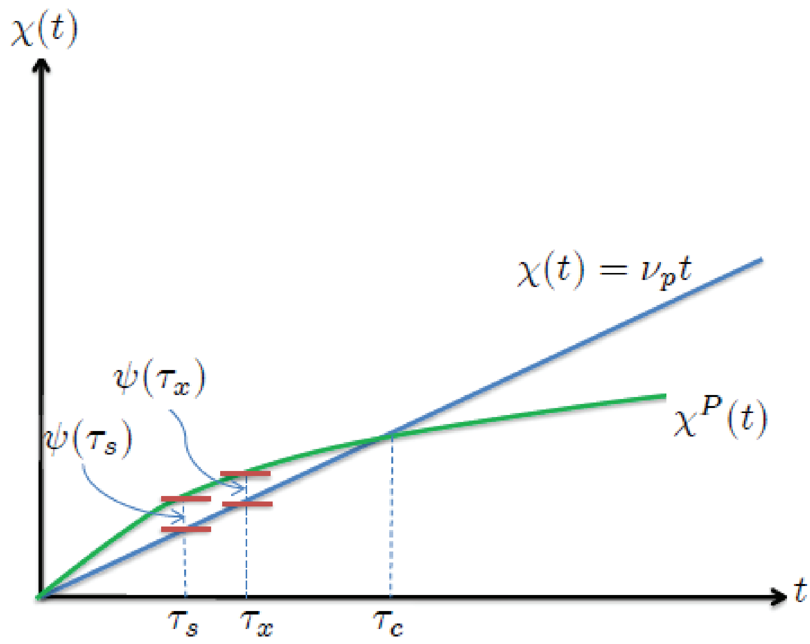

Figure 11: Illustration of excess growth at rebalance frequency τ_s and τ_x.

occur when $\psi^P(t'_s) \to 0$ and $\psi^P(t'_x) \to \psi^P(\tau_s)$(using lemma 11). Hence it is sufficient to prove:

$$\begin{aligned} & k_x[\psi^P(\tau_s) - \psi^P(\tau_x)] + \triangle k \psi^P(\tau_s) > max[\psi^P(t'_x) - \psi^P(t'_s)] \\ \Rightarrow \quad & k_x[\psi^P(\tau_s) - \psi^P(\tau_x)] + \triangle k \psi^P(\tau_s) > \psi^P(\tau_s) \\ \Rightarrow \quad & k_x[\psi^P(\tau_s) - \psi^P(\tau_x)] + [\triangle k - 1]\psi^P(\tau_s) > 0 \end{aligned} \tag{134}$$

Again using lemma 11, we know $\psi^P(\tau_s) > \psi^P(\tau_x)$. We are considering investment horizons $t > \tau_x$. Hence $k_x \geq 1$. For this case, $[\triangle k - 1] \geq 0$. Lastly for valid passive strategy we need to have positive excess growth, i.e. $\psi^P(\tau_s) > 0$. With these conditions, inequality 134 will always hold.

Case 2: $\triangle k = 0$
Under this scenario, inequality 133 is simplified to:

$$k_x[\psi^P(\tau_s) - \psi^P(\tau_x)] > [\psi^P(t'_x) - \psi^P(t'_s)] \tag{135}$$

We now show that the above inequality 135 always holds since $k_x[\psi^P(\tau_s) - \psi^P(\tau_x)] > 0$ and $[\psi^P(t'_x) - \psi^P(t'_s)] < 0$. Since $\triangle k = 0, k_s = k_x$. To prove that $[\psi^P(t'_x) - \psi^P(t'_s)] < 0$, we will start from the definition of horizon t:

$$\begin{aligned} t = \quad & k_s\tau_s + t'_s = k_x\tau_x + t'_x \\ \Rightarrow \quad & k_x\tau_s + t'_s = k_x\tau_x + t'_x, \text{ since } k_s = k_x \\ \Rightarrow \quad & k_x(\tau_x - \tau_s) = t'_s - t'_x \\ \Rightarrow \quad & t'_s - t'_x > 0, \text{ since } \tau_x > \tau_s, k_x \geq 1 \\ \Rightarrow \quad & t'_s > t'_x \\ \Rightarrow \quad & \tau_s > t'_s > t'_x, \text{ since } \tau_s > t'_s \\ \Rightarrow \quad & \psi(\tau_s) > \psi(t'_s) > \psi(t'_x), \text{ using lemma 10} \\ \Rightarrow \quad & \psi^P(t'_x) - \psi^P(t'_s) < 0 \end{aligned} \tag{136}$$

It is easy to prove that $k_x[\psi^P(\tau_s) - \psi^P(\tau_x)] > 0$. We know that $k_x \geq 1$ since we are concerned with investment horizon $t \geq \tau_x$ here. We also know by means of lemma 11 that $\psi^P(\tau_s) > \psi^P(\tau_x)$. Hence we showed that inequality 135 is always true since LHS is positive whereas RHS is negative.

Corollary 2 *Stable hybrid strategy will outperform simple hybrid strategy for any investment horizon exceeding τ_c, i.e. $\chi^{H_{\tau_s}}(t) > \chi^{H_{\tau_c}}(t), \forall t > \tau_c$.*

Proposition in corollary 2 is directly evident from theorem 4. For our running investment example, the investor is expected to obtain expected portfolio log growth of 9.447 under stable hybrid strategy as compared to the 8.632 and 8.6125 obtained under simple hybrid and active strategy respectively. Stable hybrid strategy yields about 9.7% higher expected log growth compared to baseline active strategy whereas simple hybrid strategy merely yields 0.23% higher expected log portfolio growth.

5.3 Optimal Hybrid Strategy

The obvious question now is if there exists a rebalance frequency at which the portfolio growth is maximum for a given investment horizon. The key to find the answer is to study the results of growth map theorem 2. The theorem provides the expected log of portfolio value attained for a given horizon T when a particular rebalance frequency τ is used. We will use a slightly different notation to rewrite equation 122 as a function of $T > 0$ and τ:

$$\chi(T, \tau) = \lfloor \frac{T}{\tau} \rfloor \chi(\tau, \infty) + \chi(T \bmod \tau, \infty) \tag{137}$$

Note we have used a more generic functional form above omitting use of any superscript to indicate the type of strategy. Instead, in order to indicate passive strategy we can merely set the rebalance frequency to ∞. A hybrid strategy will have a rebalance frequency τ, such that $0 < \tau < T$.

We need to obtain the partial derivative of equation 137 with respect to τ to search for a maxima. In its current form equation 137 is expressed in terms of floor and mod functions which are non-continuous piecewise linear functions. It turns out it is difficult to differentiate equations. Thus, our first attempt is to follow a numerical approach to search for the maxima of the equation.

For a given investment horizon, one can use equation 137 to compute the expected log of portfolio growth for any value of rebalance frequency τ. Figure 5.3 plots $\chi(30, \tau)$ for various values of τ. We notice that using a frequency of $\tau_o = 1.67$ year the expected log value of the portfolio is maximized in $T = 30$ years. Notice further that, the investor may use any rebalance frequency in $(0\ \tau_c]$ to obtain higher expected log of portfolio growth higher than if continuous rebalancing had been used. However, a rebalance frequency $\tau \in (0\ \tau_o)$ is not *efficient* since there is always a corresponding rebalance frequency $\tau' \in [\tau_o\ \tau_c)$ which will produce equal portfolio growth. More formally,

$$\forall \tau \in (0\ \tau_o), \exists \tau' \in [\tau_o\ \tau_c) \text{ s.t. } \chi(T, \tau) = \chi(T, \tau') \tag{138}$$

Since, by definition using all such $\tau < \tau'$, the investor will pay higher transaction cost for using τ instead of τ'. As a result investor will have no incentive to use τ when she can afford to remain passive longer without degrading her terminal expected log of portfolio value. In fact, she will improve her terminal expected log value when non-zero transaction costs are considered. Thus, investor will consider using a rebalance frequency τ only if it is on the *efficient rebalance frontier*, i.e. $\tau \in (\tau_o\ \tau_c)$. Figure 5.3 depicts the frontier as the shaded portion of the plot.

The following algorithm outlines the computational steps to search for the optimal rebalance frequency τ_o which maximizes the left hand side of the equation 137 for any given investment horizon T. The computational burden is greatly reduced as the search needs to be performed only in the range of $\tau \in (0\ \tau_s]$ as per theorem 4.

We now state the following corollary which is quite obvious from the exposition thus far.

Corollary 3 *If for any given portfolio, τ_c, τ_s and τ_o are the simple, stable and optimal rebalance frequencies respectively, then the following must hold true:*

$$\tau_o \leq \tau_s \leq \tau_c \tag{139}$$

The above relationship is also demonstrated in figure 5.3. Using algorithm 6 we can determine the optimal frequency for various values of investment horizon. Figure 5.3 illustrates the variation of τ_o against horizon T for our example portfolio. It is interesting to observe the fluctuation pattern of τ for different values of T. The fluctuation is vigorous for smaller values of T. As T increases, the amplitude of the fluctuation decreases. One would expect that for very large horizon, the optimal frequency will converge to a single value. We now prove that is indeed the case. Henceforth we will call this converged frequency as *asymptotic optimal* rebalance frequency and denote it by τ_{ao}.

Theorem 5 *For large values of investment horizon T, the optimal rebalance frequency will asymptotically converge to τ_{ao}, the time at which passive expected portfolio growth rate becomes equal to expected instantaneous growth rate, i.e.*

$$\xi^P(\tau_{ao}) = \nu_p^P(\tau_{ao}) \tag{140}$$

Algorithm 6 ComputeOptimalRebFreq

Require: $\boldsymbol{\mu}$,$\boldsymbol{\Sigma}$,$\boldsymbol{\rho}$,r_f,T,δT,N

$\chi^{H_{\tau_o}} \leftarrow \nu_p T$
$\tau_o \leftarrow 0$ # default continuous rebalancing
$[\nu_p, \mathbf{w}, \boldsymbol{\mu}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho}, r_f, N)$
$m \leftarrow 0$
for $t = 0$ to T by δT **do**
  $m \leftarrow m + 1$
  $X[m] \leftarrow 0, X'[m] \leftarrow 0, Y_1[m] \leftarrow 0, Y_2[m] \leftarrow 0, Y_1'[m] \leftarrow 0, Y_2'[m] \leftarrow 0$
  for $i = 1$ to N+1 **do**
    $X[m] \leftarrow X[m] + w[i]e^{\mu[i]t}$ # equation 33
    $X'[m] \leftarrow X'[m] + w[i]\mu[i]e^{\mu[i]t}$ # equation 82
    $Y_1[m] \leftarrow Y_1[m] + w^2[i]e^{2\mu[i]t}(e^{\sigma^2[i]t} - 1)$ # 1^{st} half of equation 45
    # 1^{st} half of equation 83
    $Y_1'[m] \leftarrow Y_1'[m] + w^2[i]e^{2\mu[i]t}[2\mu[i](e^{\sigma^2[i]t} - 1) + \sigma^2[i]e^{\sigma^2[i]t}]$
    for $j = i + 1$ to N+1 **do**
      # 2^{nd} half of equation 45
      $Y_2[m] \leftarrow Y_2[m] + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\rho(i,j)\sigma[i]\sigma[j]t} - 1)$
      # 2^{nd} half of equation 83
      $Y_2'[m] \leftarrow Y_2'[m] + w[i]w[j]e^{(\mu[i]+\mu[j])t}[(\mu[i] + \mu[j])(e^{\rho[i,j]\sigma[i]\sigma[j]t} - 1) + \rho[i,j]\sigma[i]\sigma[j]e^{\rho[i,j]\sigma[i]\sigma[j]t}]$
    end for
  end for
  $Y[m] \leftarrow Y_1[m] + 2Y_2[m]$ # equation 45
  $Y'[m] \leftarrow Y_1'[m] + 2Y_2'[m]$ # equation 83
  $\chi^P[m] \leftarrow ln(X[m]) - \frac{1}{2}ln(1 + \frac{Y[m]}{X^2[m]})$ # equations 53 and 54
  $\xi^P[m] \leftarrow \frac{1}{X[m]}[X'[m] - \frac{1}{2}\frac{X[m]Y'[m]-2X'[m]Y[m]}{X[m]^2+Y[m]}]$ # equation 87
  $k \leftarrow \lfloor \frac{T}{t} \rfloor$ # theorem 2
  $t' \leftarrow T \bmod t$ # theorem 2
  $\chi^{H_t}[m] \leftarrow k\chi^P[m] + \chi^P(t'/\delta t)$ # theorem 2
  if $\chi^{H_t}[m] > \chi^{H_{\tau_o}}$ **then** # look for maximum χ^H
    $\chi^{H_{\tau_o}} \leftarrow \chi^{H_t}[m]$
    $\tau_o \leftarrow t$
  end if
  if $(m > 1)$ and $(\xi^P(1) > \nu_p)$ **then**
    # found stable rebalance time?
    if $\xi^P[m] \leq \nu_p$ and $\xi^P[m-1] > \nu_p$ **then**
      return $(\tau_o, \chi^{H_{\tau_o}})$ # stop searching, theorem 4
    end if
  end if
end for
return $(\tau_o, \chi^{H_{\tau_o}})$ # stable reb is not found, keep searching till horizon

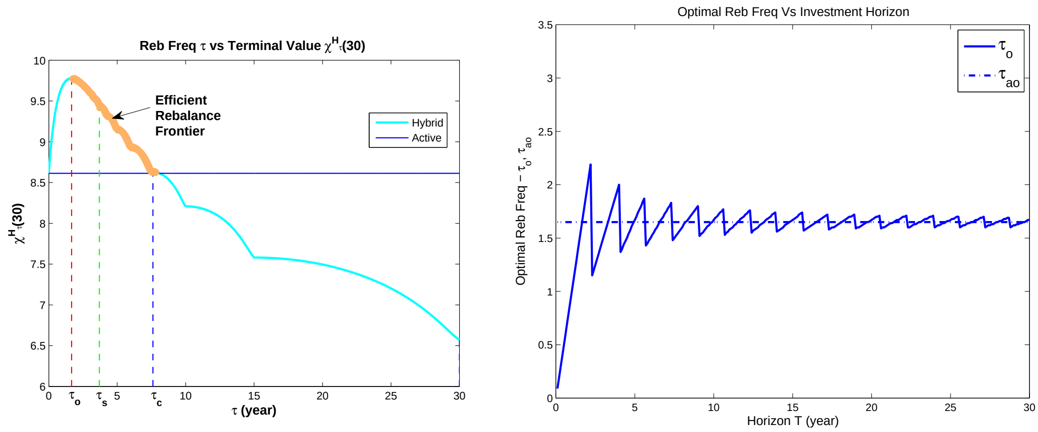

Figure 12: Optimal rebalancing frequency and its fluctuation with investment horizon

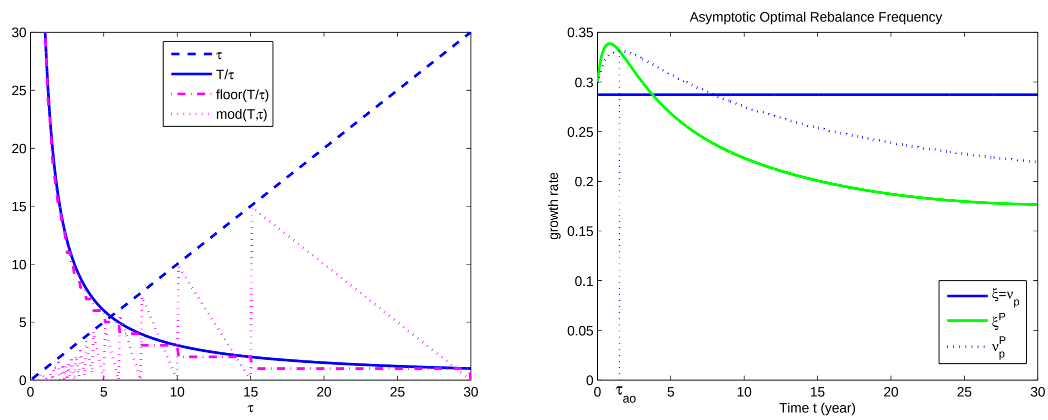

Figure 13: Illustration of derivation of asymptotic optimal rebalance frequency τ_{ao} for $T = 30$ years.

Proof 19 *From corollary 3 we know that $\tau_o \leq \tau_s$. Let's look at the two terms in the RHS of equation 137 with the help of the illustration in figure 19. Not that for optimality of τ, our interest is only in $\forall \tau \leq \tau_s$.*

We assume that horizon T is sufficiently large, such that $T \gg \tau_s > \tau_o$. Thus, $\lfloor \frac{T}{\tau} \rfloor \gg 1$. We also know that $\tau > (T \bmod \tau)$ implying that $\chi(\tau, \infty) > \chi(T \bmod \tau, \infty)$ since the passive portfolio growth will always be an increasing function of time for $t < \tau_s$. Now combining these two, we get $\lfloor \frac{T}{\tau} \rfloor \chi(\tau, \infty) \gg \chi(T \bmod \tau, \infty)$. In other words, the first term involving floor function shall dominate the second term. Hence, as a first order simplification we can ignore the second term :

$$\chi(T, \tau) \approx \lfloor \frac{T}{\tau} \rfloor \chi(\tau, \infty) \tag{141}$$

Furthermore, using the illustration in figure 19, for $T \gg \tau$, $\lfloor \frac{T}{\tau} \rfloor \approx \frac{T}{\tau}$. Applying this second order of approximation, we obtain:

$$\chi(T, \tau) \approx \frac{T}{\tau} \chi(\tau, \infty) \tag{142}$$

In order to determine the value of τ at which the LHS of equation 137 is maximized, let's take the partial derivative:

$$\frac{\partial \chi(T, \tau)}{\partial \tau} \approx \frac{\partial(\frac{T}{\tau}\chi(\tau, \infty))}{\partial \tau} \approx -\frac{T}{\tau^2}\chi(\tau, \infty) + \frac{T}{\tau}\frac{\partial \chi(\tau, \infty)}{\partial \tau} \approx \frac{T}{\tau}\left(\frac{\partial \chi(\tau, \infty)}{\partial \tau} - \frac{1}{\tau}\chi(\tau, \infty)\right) \tag{143}$$

Setting 143 to zero, we can obtain the value of τ_{ao} at which the expected log of hybrid portfolio value will be maximized.

$$\begin{aligned} & \frac{T}{\tau_{ao}}\left(\left|\frac{\partial \chi(\tau, \infty)}{\partial \tau}\right|_{\tau=\tau_{ao}} - \frac{1}{\tau_{ao}}\chi(\tau_{ao}, \infty)\right) = 0 \\ \Rightarrow \quad & \left|\frac{\partial \chi(\tau, \infty)}{\partial \tau}\right|_{\tau=\tau_{ao}} - \frac{1}{\tau_{ao}}\chi(\tau_{ao}, \infty) = 0, \textit{ since } T \neq 0 \textit{ and } \tau_{ao} \neq \infty \\ \Rightarrow \quad & \left|\frac{\partial \chi(\tau, \infty)}{\partial \tau}\right|_{\tau=\tau_{ao}} = \frac{1}{\tau_{ao}}\chi(\tau_{ao}, \infty) \\ \Rightarrow \quad & \xi^P(\tau_{ao}) = \nu_p^P(\tau_{ao}), \textit{ using equation 57} \end{aligned} \tag{144}$$

Figure 19 illustrates the application of the above theorem to compute τ_{ao} for our example portfolio. In this case τ_{ao} is found to be 1.5 years. Observe from figure 5.3 how τ_o fluctuates around τ_{a0} as the investment horizon changes. We now present our last algorithm 7 to compute τ_{ao}.

6 Simulation And Analysis

6.1 Methodology

We attempt to determine the accuracy of the portfolio strategies presented in this report. The strict notion of accuracy is to compare the analytical parameters such as the optimal rebalance frequency and the associated expected log growth with the corresponding *true* underlying parameters. The closest approximation of the true underlying parameters can be obtained via rigorous Monte-Carlo simulation. For our purpose we will assume that any parameter value predicted by simulation is indeed the best substitute for the true value exhibited by the underlying stochastic process. We validate our analytical results if the parameters predicted closely match with the corresponding simulation results.

As per notations, we use a wide hat $\widehat{(.)}$ to denote a parameter predicted by simulation which we have considered as the best available proxy for the corresponding true parameter. As an example τ_o and $\widehat{\tau}_o$ are the optimal frequency value computed by the analytical framework in this report and the corresponding true value best estimated using Monte-Carlo simulation.

We have two primary end goals here. First, we want to measure the error or equivalently the accuracy of the optimal rebalance frequency we compute in this report. Second, a larger goal, is to assess potential *loss to investor* if she would use this optimal rebalance frequency recommendation to execute the optimal hybrid strategy. This loss has to be estimated by the differential wealth creation using the true optimal rebalance frequency $\widehat{\tau}_o$ and the analytical optimal frequency τ_o. For our purpose, we will use the following two functions to estimate the loss and the percentage loss to investor respectively if τ_o is used as rebalance frequency:

$$L(t, \tau_o) = \widehat{\chi}^{H_{\widehat{\tau}_o(t)}}(t) - \widehat{\chi}^{H_{\tau_o(t)}}(t) \tag{145}$$

Algorithm 7 ComputeAsymptoticOptimalRebalanceFrequency

Require: $\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho}, r_f, N, T, \delta T$

$\tau_{ao} \leftarrow 0$ # default continuous rebalancing
$[\nu_p, \mathbf{w}, \boldsymbol{\mu}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho}, r_f, N)$
if $!IsPassiveStrategyPossible(\nu_p, \mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho})$ **then**
  return τ_{ao}
end if
$k \leftarrow 0, \nu_p^P[1] \leftarrow 0$
for $t = 0$ to T by δT **do**
  $k \leftarrow k + 1$
  $X[k] \leftarrow 0, Y_1[k] \leftarrow 0, Y_2[k] \leftarrow 0, Y_1'[k] \leftarrow 0, Y_2'[k] \leftarrow 0$
  for $i = 1$ to N+1 **do**
    $X[k] \leftarrow X[k] + w[i]e^{\mu[i]t}$ # equation 33
    $X'[k] \leftarrow X'[k] + w[i]\mu[i]e^{\mu[i]t}$ # equation 82
    $Y_1[k] \leftarrow Y_1[k] + w^2[i]e^{2\mu[i]t}(e^{\sigma^2[i]t} - 1)$ # 1^{st} half of equation 45
    # 1^{st} half of equation 83
    $Y_1'[k] \leftarrow Y_1'[k] + w[i]^2 e^{2\mu[i]t}[2\mu[i](e^{\sigma[i]^2 t} - 1) + \sigma[i]^2 e^{\sigma[i]^2 t}]$
    for $j = i + 1$ to N+1 **do**
      # 2^{nd} half of equation 45
      $Y_2[k] \leftarrow Y_2[k] + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\rho(i,j)\sigma[i]\sigma[j]t} - 1)$
      # 2^{nd} half of equation 83
      $Y_2'[k] \leftarrow Y_2'[k] + w[i]w[j]e^{(\mu[i]+\mu[j])t}[(\mu[i] + \mu[j])(e^{\rho[i,j]\sigma[i]\sigma[j]t} - 1) + \rho[i,j]\sigma[i]\sigma[j]e^{\rho[i,j]\sigma[i]\sigma[j]t}]$
    end for
  end for
  $Y[k] \leftarrow Y_1[k] + 2Y_2[k]$ # equation 45
  $Y'[k] \leftarrow Y_1'[k] + 2Y_2'[k]$ # equation 83
  $\xi^P[k] \leftarrow \frac{1}{X[k]}[X'[k] - \frac{1}{2}\frac{X[k]Y'[k] - 2X'[k]Y[k]}{X[k]^2 + Y[k]}]$ # equation 87
  if $t > 0$ **then**
    $\nu_p^P[k] \leftarrow \frac{1}{t}(ln(X[k]) - \frac{1}{2}ln(1 + \frac{Y[k]}{X^2[k]}))$ # equation 57
  end if
  if $k > 1$ **then**
    if $\xi^P[k-1] \geq \nu_p^P[k-1]$ and $\xi^P[k] \leq \nu_p^P[k]$ **then**
      return $\tau_{ao} = t$
    end if
  end if
end for
return τ_{ao}

$$\%L(t, \tau_o) = \frac{\widehat{\chi}^{H_{\widehat{\tau}_o(t)}}(t) - \widehat{\chi}^{H_{\tau_o(t)}}(t)}{\widehat{\chi}^{H_{\widehat{\tau}_o(t)}}(t)} \text{ x } 100 \tag{146}$$

It is important to understand the significance of equation 145. First and foremost, we are measuring the logarithmic loss. The first term in the numerator is the true expected log of portfolio value when the true optimal rebalance frequency $\widehat{\tau}_o$ is used. This is the best case expected log portfolio growth that is possible if the investor had known and used the true optimal $\widehat{\tau}_o$. The second term is the true expected log of portfolio value had the investor used the recommendation τ_o computed using the analytical framework. In a sense, this is the *realized* expected optimal log of portfolio value for the investor. Note that we consider true $\widehat{\chi}$ instead of analytical χ in the second term. Investor has only control over weather to use the rebalance frequency predicted by our analytical framework. Once used, she will get only the true underlying expected log of portfolio value. We assume both $\widehat{\tau}_o$ and τ_o change with horizon t. One can compute the investor loss $L(t, \tau_{ao})$ and $L(t, 0)$ for using τ_{ao} and continuous rebalancing respectively.

We will use the familiar four assets (one risk free and three risky) example portfolio for validation. Using commercial SaS/IML ([Wicklin2010]) software we generated 5000 correlated Monte Carlo asset price paths ([Fan2003]) for the given asset return and covariance vectors. The asset prices are log-normally distributed following the stochastic process of equation 4. The equivalent asset price equation used in generating Monte Carlo paths is given as follows ([Hull2008]).

$$S(t+dt) = S(t)e^{(\mu - \frac{\sigma^2}{2})dt + \sigma dz} \tag{147}$$

Alternatively one can also use equation 2 to generate Monte Carlo paths for correlated asset prices. As explained in [Hull2008], the advantage of using equation 147 is that it is valid for any value of dt whereas equation 2 is accurate only when dt is very small.

The value of dt is chosen to be 0.01 year to complete the simulation for 5000 Monte Carlo paths in a reasonable time. We start with an initial $1 investment fund and distribute it among the four assets as per the optimized proportion determined by equation 31. For each asset price path, we determine the path of the portfolio value at each time increment. We also rebalance the portfolio to the initial optimal weights for a specified rebalance frequency. We then determine the terminal log of portfolio value for the path. We finally compute the average of log of this terminal portfolio value over all 5000 paths.

6.2 Active Strategy Validation

We validated the correctness of active strategy by setting the rebalance frequency to a near zero value of 0.001 year. We recorded 8.5675429 as the average log of terminal portfolio value over all the paths for an investment horizon of 30 years. This is compared against the well known theoretical value of $\nu_p T = 0.2871 \text{ x } 30 = 8.613$. The simulated value is close to the theoretical value within 0.53% error.

6.3 Passive Strategy Validation

We simulated the effect of passive strategy by avoiding rebalancing for 30 years and recorded the evolution of average log of portfolio growth at each time increment over 5000 paths. This simulation output data series $\widehat{\chi}^P(t)$ is provided in table 10 of appendix. We also generated the equivalent series $\chi^P(t)$ by analytical method using equation 54. Both series are plotted in Figure 6.3. We observe that $\chi^P(t)$ very closely tracks $\widehat{\chi}^P(t)$. To be precise, $\chi^P(t)$ slightly overestimates at the short-end of investment horizon whereas it underestimates for longer investment horizons.

Figure 6.3 plots the estimates of error and error percentages. For the entire horizon, except the first two years, the analytical $\chi^P(t)$ series is within +/-5% of the true $\widehat{\chi}^P(t)$ series. Higher error percentages observed during the initial two year period is mostly because of the division by very small numbers.

6.4 Optimal Hybrid Strategy Validation

We first validated the results of the important growth map theorem 2. We generated $\widehat{\chi}^{H_\tau}(30)$ series by changing the rebalance frequency τ from 0.1 year to $T = 30$ years at an interval of 0.1 year. For each τ, we recorded the expected value for $\widehat{\chi}^{H_\tau}(30)$ as presented in table 10. We emphasize that $\widehat{\chi}^{H_\tau}(30)$ data series is generated purely from Monte-Carlo simulation for a 30 year horizon by applying different rebalancing frequency $\tau \in (0\ T]$. We then derived the expected value of log of hybrid portfolio using the growth map theorem from $\widehat{\chi}^P(t)$ data series. We denote this derived data series as $\tilde{\chi}^{H_\tau}(30)$. Figure 6.4 plots both the true $\widehat{\chi}^{H_\tau}(30)$ and the derived $\tilde{\chi}^{H_\tau}(30)$. We see that $\tilde{\chi}^{H_\tau}(30)$ faithfully tracks the true $\widehat{\chi}^{H_\tau}(30)$ series confirming the correctness of the growth map theorem.

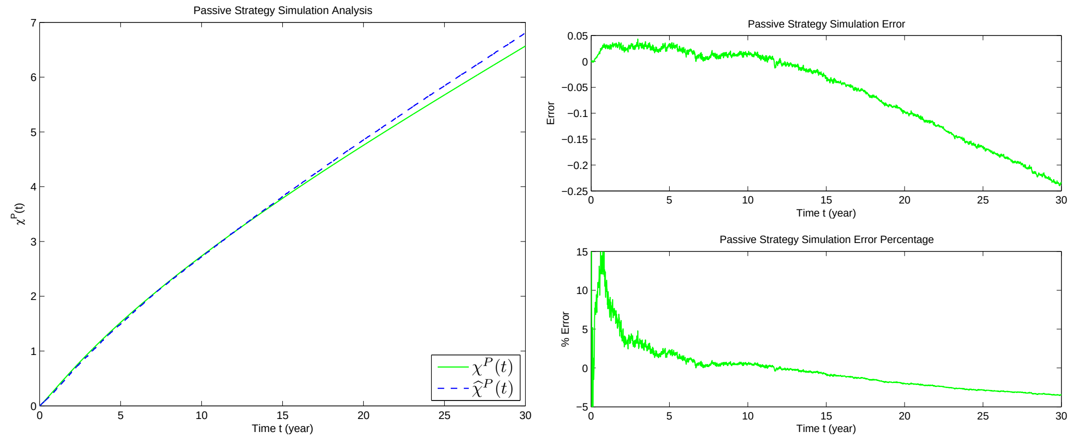

Figure 14: Analysis of simulation results for passive strategy.

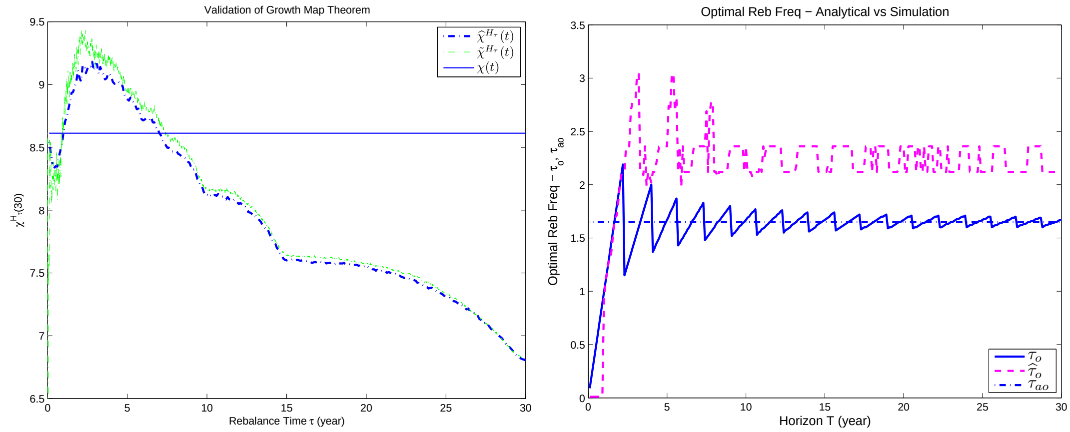

Figure 15: Analysis of simulation results for hybrid strategy.

We assume any visible difference between these two data series is entirely due to the inherent error caused by simulation parameters such as number of paths and width of time increment. This is an important validation step as we will use growth map theorem 2 along with $\widehat{\chi}^P(t)$ data series to project the true values for hybrid strategies when both τ and horizon T change.

Observation of $\tilde{\chi}^{H_\tau}(30)$ plot in figure 6.4 offers some interesting insights. For smaller rebalance frequency ($0 < \tau < 0.5$), the expected log of portfolio value decreases relative to continuous rebalancing case. However, as we increase the frequency beyond this range the performance of rebalancing continues to improve and peaks at $\tau = 2.18$ years. For higher rebalancing frequencies the performance continues to degrade. Rebalance frequencies in the range of $1 < \tau < 7.5$ years offer higher performance over continuous rebalancing case. However an investor will always benefit to use a rebalance frequency from the rebalance efficient frontier of $2.18 \le \tau < 7.5$ years. The reader should compare this frontier predicted by simulation with $1.67 \le \tau < 7.61$ years which is computed by our analytical framework and illustrated in figure 5.3. Our analytical framework's efficient rebalance frontier includes that predicted by simulation and slightly larger on both side of the interval.

Using $\widehat{\chi}^P(t)$ data series and growth map theorem we derived the optimal rebalance frequency $\widehat{\tau}_o$ for different values of horizon T. The data is plotted in figure 6.4. Observe the fluctuation pattern of true $\widehat{\tau}_o$. Like τ_o, the $\widehat{\tau}_o$ fluctuation is vigorous for small values of horizon T. For most part the true $\widehat{\tau}_o$ vary between 2.12 and 2.36 years. These values don't appear to converge for large T unlike their analytical counterpart. The true optimal frequencies have a midpoint $\widehat{\tau}_{ao}$ of 2.24 years. Note, that our estimation of $\tau_{ao} = 1.65$ years is a conservative one considering the higher values of optimal frequency exhibited in the simulation data.

We can trace the under estimation of τ_{ao} by about 0.59 years from $\widehat{\tau}_o$ to the slight over estimation of $\chi^P(t)$ in the short end as depicted in figure 6.4. For simplicity of exposition we assume that this estimation error in $\chi^P(t)$ is

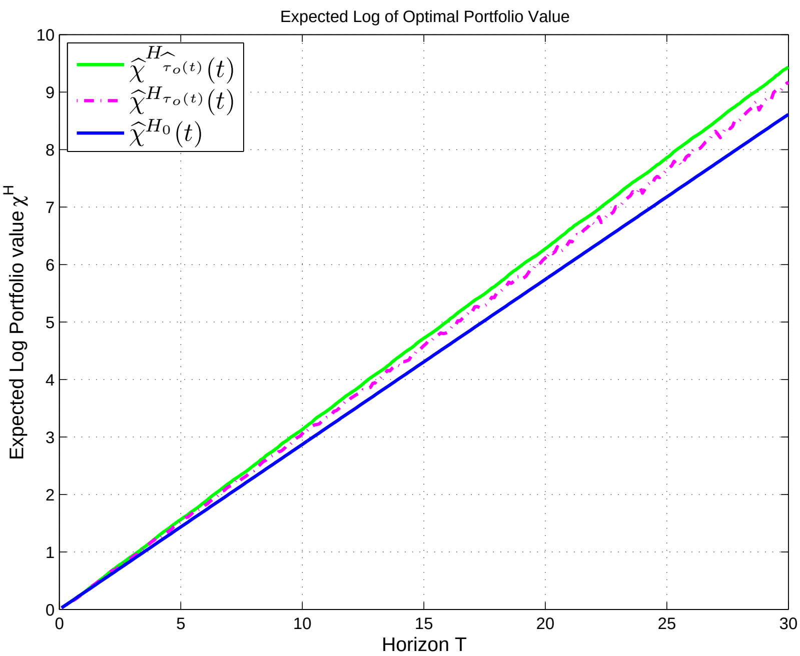

Figure 16: True and realized expected log of portfolio growth.

constant e in this short end. Using the notations used in theorem 5, we can write equation 142 for true expected log portfolio values for hybrid strategy as:

$$\begin{aligned} \widehat{\chi}(T,\tau) &\approx \frac{T}{\tau}\widehat{\chi}(\tau,\infty) \\ &\approx \frac{T}{\tau}(\chi(\tau,\infty)-e) \end{aligned} \tag{148}$$

Following derivation similar to theorem 5, equation 148 will be maximized when the following condition holds:

$$\begin{aligned} \left|\frac{\partial\chi(\tau,\infty)}{\partial\tau}\right|_{\tau=\widehat{\tau}_{ao}} &= \frac{1}{\widehat{\tau}_{ao}}(\chi(\widehat{\tau}_{ao},\infty)-e) \\ \Rightarrow \quad \xi^P(\widehat{\tau}_{ao}) = \nu_p^P(\widehat{\tau}_{ao}) - \frac{e}{\widehat{\tau}_{ao}} \end{aligned} \tag{149}$$

Hence, the value of $\widehat{\tau}_{ao}$ will be obtained by the intersection point of ξ^P curve and ν_p^P curve stretched downwards to adjust for the term $\frac{e}{\widehat{\tau}_{ao}}$. Referring to the illustration in figure 19, this intersection point $\widehat{\tau}_{ao}$ will occur at a higher value relative to the theoretical τ_{ao}.

Figure 16 plots the expected log of realized portfolio growth for optimal hybrid strategy using τ_o and active strategy using $\tau = 0$ relative to the true underlying portfolio growth. Corroborating our hitherto claims, hybrid optimal strategy fares better than active strategy for any horizon. In spite of improved performance relative to active strategy, there is some performance loss in real-term when we comare the output with that of underlying true expected portfolio growth.

Finally we look at the log loss the investor will incur for using τ_o and τ_{ao}. Figure 6.4 and 6.4 plot the investor loss following equations 145 and 146 respectively for three alternative rebalance frequencies we have studied in this report, viz. τ_o, τ_{ao} and $\tau = 0$. We observe that, investor loss grows with investment horizon irrespective of the employed rebalance frequency. This is anticipated based on error amplification lemma 16. Overall, τ_{ao} seems to be a very good substitute for τ_o for any horizon. The loss and loss percentages for τ_{ao} closely track those for time varying τ_o. Surprisingly τ_{ao} fares better in containing the percentage error to a relatively smaller band around 3.2%. Moreover, the investor incurs higher loss in active continuously rebalanced strategy even without considering the adversarial effect of transaction cost. Following hybrid optimal strategy, the investor loss in the long run is limited to 3.2% compared to a much higher percentage of slightly less than 9% for active strategy.

Please note that approximations of the true underlying values can be improved with increasingly smaller dt value. We can also improve the accuracy if we use higher number of Monte Carlo paths. Some of the deviations of

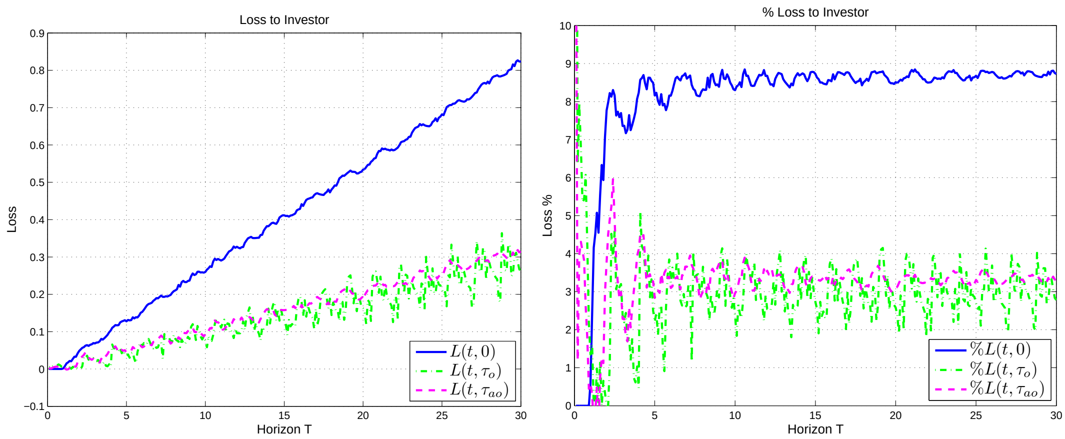

Figure 17: Investor loss in using optimal hybrid strategy.

analytical values reflected in the error percentage numbers can be attributed to these simulation constraints. More importantly, we are amenable to deviate from the true underlying values because of our central assumption of log-normality of sum of log-normal variables under Fenton-Wilkinson method.

7 Conclusion

We started with the assumption that the investor's utility is logarithmic implying that she desires to maximize expected log of her portfolio return for the investment horizon. For this utility, we considered the existing log-optimal framework, we termed as active strategy, wherein after the initial portfolio setup, the investor continuously rebalances to the initial optimal asset weights. In this setting, the change in asset prices are modeled as Geometric Brownian motion. We then developed an analytical framework to study the nature of the portfolio growth if it is left passive without any further rebalancing. For this analysis we assumed that the asset return mean and covariance characteristics are static parameters given as input. We used Fenton-Wilkinson log-normality assumption for sum of log-normal variables to determine the first and second moments of log of portfolio growth for the passive investment. The underlying log-normal assumption in Fenton-Wilkinson approach made it possible to derive analytical expression for passive portfolio mean, variance and growth rate analogous to active strategy.

We explored and proposed three different rebalancing approaches, viz. simple, stable and optimal rebalancing. In the simple rebalancing approach, the investor maintains higher expected portfolio log growth. In stable rebalancing she shall improve the expected terminal log growth by maintaining higher expected instantaneous log growth. In optimal rebalancing, the investor maximizes her expected log of terminal portfolio value by using the best suitable rebalancing frequency for a given horizon. We showed that in all three approaches the investor can use a periodic rebalancing strategy to maintain the respective rebalancing criterion intact until the horizon. We termed such periodic rebalancing as hybrid strategy.

We also derived the condition for existence of the rebalance possibility for a given set of input asset characteristics. The rebalance oppertunity exists if the time zero expected instantaneous portfolio growth under passive strategy is higher than corresponding value ν_p under active strategy. If this criterion is not satisfied, the investor will prefer to follow active strategy for some positive initial duration. We have left the topic of determining an appropriate rebalancing strategy when this criteria is violated for future research.

We established an important relationship, called growth map theorem. For any given investment horizon and rebalance frequency, with the help of the theorem, one can compute the expected value of portfolio growth under hybrid strategy by merely knowing the evolution of the same under passive strategy. We showed that using any other rebalancing frequency $\tau > \tau_s$ is suboptimal since τ_s shall always produce higher expected terminal portfolio growth. With the help of these two relationships we outlined a numerical algorithm to compute the optimal rebalancing frequency τ_o for any given investment horizon.

We defined efficient rebalance frontier as approximately the range of $[\tau_o\ \tau_c]$. If the investor uses a rebalance frequency from this range, she is guaranteed to obtain higher expected log of portfolio growth than if she had to use continuous rebalancing. The investor ought not use a rebalancing frequency from the range $[0\ \tau_o)$ which is demonstrated as inefficient. Finally, we showed that for large investment horizons, optimal rebalancing frequency converges to an asymptotic value τ_{ao}. At τ_{ao}, the expected portfolio growth rate becomes equal to the expected instantaneous growth when the portfolio is left to grow passively.

Simulation studies showed that our analytical framework predicts the expected growth of passive portfolio very accurately. It slightly underestimates at the short end while slightly overestimating at the long end of the horizon. This small discrepancy at the short end results in a smaller optimal rebalance frequency estimation. We showed that there is considerable improvement in investor log loss when the investor uses the estimated τ_o. In particular, for our portfolio example, the investor log loss was found to be only 3.2% compared to near 9% if the investor had used active continuous rebalancing strategy.

The above analytical framework is scalable to any number of risky assets to be considered for portfolio construction. Nevertheless, there are several future research topics. A critical one is to break the assumption that the asset return mean and variance are static values. In real tradable assets these characteristics may evolve dynamically, especially when the investment horizon is long ([Campbell and Viceira1999], [Ball and Kothari1989] and [Bodurtha and Mark1991]). There are several alternative models proposed in literature to make these parameters more dynamic. One needs to study and apply these models to modify our analytical framework suitably.

As we have noted before, we have ignored the transaction cost in our models. This simplification needs to be avoided by assuming appropriate transaction cost model suitable for the analytical framework. With reasonable transaction cost one should derive more accurate rebalance frequency than the conservative estimations presented in this report.

References

[Arnott and Lovell1993] Arnott, R.D. and Lovell, R.M., Rebalancing: Why? When? How Often?. *Journal of Investing*, 1993, pp. 5–10.

[Ball and Kothari1989] Ball, R. and Kothari, S., Nonstationary expected returns: Implications for tests of market efficiency and serial correlation in returns. *Journal of Financial Economics*, 1989, **25**, 51 – 74.

[Bernstein1996] Bernstein, W.J., The Rebalancing Bonus: Theory and Practice: http://www.efficientfrontier.com/ef/996/rebal.htm. , 1996.

[Bodurtha and Mark1991] Bodurtha, James N., J. and Mark, N.C., Testing the CAPM with Time-Varying Risks and Returns. *The Journal of Finance*, 1991, **46**, 1485–1505.

[Calvet *et al.*2009] Calvet, L.E., Campbell, J.Y. and Sodini, P., Fight of Flight? Portfolio Rebalancing by Individual Investors.. *Quarterly Journal of Economics*, 2009, **124**, 301 – 348.

[Campbell and Viceira1999] Campbell, J.Y. and Viceira, L.M., Consumption and Portfolio Decisions When Expected Returns Are Time Varying. *The Quarterly Journal of Economics*, 1999, **114**, 433–495.

[Crow and Shimizu1988] Crow, E. and Shimizu, K., *Lognormal Distributions: Theory and Applications*, Mathematics, Finance and Risk 1988 (Marcel Dekker, Inc.: New York and Basel).

[Fan2003] Fan, X., *SAS for Monte Carlo Studies: A Guide for Quantitative Researchers* 2003, SAS Publishing.

[Fenton1960] Fenton, L., The Sum of Log-Normal Probability Distributions in Scatter Transmission Systems. *Communications Systems, IRE Transactions on*, 1960, **8**, 57 –67.

[Glasserman2004] Glasserman, P., *Monte Carlo Methods in Financial Engineering*, Vol. 22, 2004, Springer.

[GyRfi *et al.*2007] GyRfi, L., UrbN, A. and Vajda, I., Kernel-Based Semi-Log-Optimal Empirical Portfolio Selection Strategies. *International Journal of Theoretical and Applied Finance (IJTAF)*, 2007, **10**, 505–516.

[Hendershott *et al.*2011] Hendershott, T., Jones, C.M. and Menkveld, A.J., Does Algorithmic Trading Improve Liquidity? 1. *The Journal of Finance*, 2011, **66**, 1–33.

[Hull2008] Hull, J.C., *Options, Futures, and other Derivatives*, 7 2008, Prentice Hall.

[Kuhn and Luenberger2010] Kuhn, D. and Luenberger, D., Analysis of the rebalancing frequency in log-optimal portfolio selection. *Quantitative Finance*, 2010, **10**, 221–234.

[Luenberger1998] Luenberger, D., *Investment science* 1998 (Oxford Univ. Press: New York [u.a.]).

[Markowitz1952] Markowitz, H., Portfolio selection. *Journal of Finance*, 1952, **7**, 77–91.

[Masters2002] Masters, S.J., Rules for Rebalancing. *Journal of Financial Planning*, 2002.

[Neftci2000] Neftci, S.N., *An Introduction to the Mathematics of Financial Derivatives*, Advanced Finance 2000, Academic Press.

[Odean1999] Odean, T., Do Investors Trade Too Much?. *American Economic Review*, 1999, **89**, 1279 – 1298.

[Safak and Safak1994] Safak, A. and Safak, M., Moments of the sum of correlated log-normal random variables. In *Proceedings of the Vehicular Technology Conference, 1994 IEEE 44th*, Jun., pp. 140 –144 vol.1, 1994.

[Schwartz and Yeh1982] Schwartz, S.C. and Yeh, Y.S., On the distribution function and moments of power sums with lognormal components. *Bell Syst. Tech. J.*, 1982, **61**, 14411462.

[Shefrin and Statman1985] Shefrin, H. and Statman, M., The Disposition to Sell Winners Too Early and Ride Losers Too Long: Theory and Evidence.. *Journal of Finance*, 1985, **40**, 777 – 790.

[Thompson2002] Thompson, J., What history tells us 'tis better to have rebalanced regularly than not at all. *Financial Planning*, 2002.

[Tokat2007] Tokat, Y., Portfolio Rebalancing in Theory and Practice. *Journal of Investing*, 2007, pp. 52–59.

[Trivedi2001] Trivedi, K.S., *Probability and Statistics with Reliability, Queueing, and Computer Science Applications, 2nd Edition* 2001, Wiley-Interscience.

[Wicklin2010] Wicklin, R., *Statistical Programming with SAS/IML Software* 2010, SAS Publishing.

[Wikipedia2011a] Wikipedia, Log-normal distribution: http://en.wikipedia.org/wiki/Log-normal_distribution. 2011a.

[Wikipedia2011b] Wikipedia, Variance: http://en.wikipedia.org/wiki/Variance. 2011b.

[Wikipedia2011c] Wikipedia, Wikipedia: Jensen's inequality: http://en.wikipedia.org/wiki/Jensen_Inequality. 2011c.

8 Error Amplification

Another consequence of growth map theorem 2 is that any error in the estimation of passive strategy portfolio growth projection will lead to amplified error in portfolio growth projection for hybrid strategy. We will quantify this error amplification in the following lemma.

Lemma 16 *Let the error in estimating the expected log of portfolio growth under passive and hybrid strategies are as follows:*

$$e^P(t) = \widehat{\chi}^P(t) - \chi^P(t) \tag{150}$$

$$e^{H_\tau}(t) = \widehat{\chi}^{H_\tau}(t) - \chi^{H_\tau}(t) \tag{151}$$

where $\widehat{\chi}^P(t)$ and $\widehat{\chi}^{H_\tau}(t)$ are the true *underlying expected log of portfolio growth values for a given rebalance frequency of τ. Then,*

$$e^{H_\tau}(t) = ke^P(\tau) + e^P(t') \tag{152}$$

where $t = k\tau + t'$, $k = \lfloor \frac{t}{\tau} \rfloor$ and $t' = t \bmod \tau$.

Proof 20 *We start with LHS of equation 152:*

$$\begin{aligned}
e^{H_\tau}(t) &= \widehat{\chi}^{H_\tau}(t) - \chi^{H_\tau}(t) \text{ (using equation 151)} \\
&= [k\widehat{\chi}^P(\tau) + \widehat{\chi}^P(t')] - [k\chi^P(\tau) + \chi^P(t')] \\
&\quad \text{(using theorem 2)} \\
&= k[\widehat{\chi}^P(\tau) - \chi^P(\tau)] + [\widehat{\chi}^P(t') - \chi^P(t')] \\
&= ke^P(\tau) + e^P(t') \text{ (using equation 150)}
\end{aligned} \tag{153}$$

As per lemma 150, effect of any error in passive strategy for shorter investment horizon will have noticeable error amplifying effect in hybrid strategy. As the illustration in figure 19 depicts, the value of rebalance frequency τ decreases and/or the value of investment horizon t increases, the value of $k(=\lfloor\frac{T}{\tau}\rfloor)$ becomes larger. This will have an adversarial effect on the hybrid strategy estimation of portfolio growth.

We anticipate some estimation error in passive log growth estimation using equation 54 since there is an underlying log-normality assumption in the Fenton-Wilkinson approach to obtain the moments of a sum of log-normal random variables. In our simulation section, we will observe and study the effect of this error for the hybrid strategy.

9 Proof of Theorem 3

Proof 21 *When $i = 0$, $i\tau_s = 0$ is trivially true as time 0 is the very first time when the portfolio is setup with the desired set of optimum asset weights. When $i = 1$, $i\tau_s = \tau_s$ is given as the first rebalance time after the initial setup. During this initial rebalancing period $(0\ \tau_s]$, we must satisfy equation 79 and 80. Using the expanded notation, the time 0 estimation of the passive instantaneous portfolio growth satisfies the following two conditions:*

$$\xi^{H_{\tau_s}}(0,\delta t) > \nu_p, \delta t < \tau_s \tag{154}$$

$$\xi^{H_{\tau_s}}(0,\tau_s) = \nu_p \tag{155}$$

When $i \geq 1$, i.e. for all subsequent rebalance periods, $(i\tau_s\ (i+1)\tau_s]$, the following two conditions analogous to equations 154 and 155 must also hold.

$$\xi^{H_{\tau_s}}(i\tau_s, i\tau_s+\delta t) > \nu_p,\ \forall i\ \in\ \mathbb{N}, \delta t < \tau_s \tag{156}$$

$$\xi^{H_{\tau_s}}(i\tau_s, (i+1)\tau_s) = \nu_p,\ \forall i\ \in\ \mathbb{N} \tag{157}$$

We will prove both of these equations 156 and 157 by the method of induction. Let's prove first equation 156. The base case is when $i = 0$. Then equation 156 simply becomes equation 154 which by definition is true. From fundamental definition,

$$\begin{aligned}
\xi^{H_{\tau_s}}(0,\delta t) &= \frac{dE[ln(V^P(0,\delta t))]}{dt} = \frac{dE[ln(V^P(0,0)\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt} \\
&= \frac{dE[ln(V^P(0,0))]}{dt} + \frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt} \\
&= \frac{dE[\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt}, \text{ since } V^P(0,0) = 1
\end{aligned} \tag{158}$$

Hence, equation 154 implies,

$$\frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt} > \nu_p \tag{159}$$

Now, assume equation 156 holds for $i = k$ and hence $k\tau_s$ is also a rebalance time. That is,

$$\xi^{H_{\tau_s}}(k\tau_s, k\tau_s+\delta t) > \nu_p \tag{160}$$

To complete the proof we must show that it also holds for $i = k+1$, i.e.

$$\xi^{H_{\tau_s}}((k+1)\tau_s, (k+1)\tau_s+\delta t) > \nu_p \tag{161}$$

Following similar steps as of the derivation of equation 158,

$$\begin{aligned}
&\xi^{H_{\tau_s}}((k+1)\tau_s, (k+1)\tau_s+\delta t) \\
=\ & \frac{dE[ln(V^P((k+1)\tau_s, (k+1)\tau_s+\delta t))]}{dt} \\
=\ & \frac{dE[ln(V^P((k+1)\tau_s, (k+1)\tau_s)\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt} \\
=\ & \frac{dE[ln(V^P((k+1)\tau_s, (k+1)\tau_s))]}{dt} + \frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt}
\end{aligned} \tag{162}$$

We have made use of the fact that $(k+1)\tau_s$ is a rebalance time and hence the initial asset weights are used. Now let's look at the two terms in the above equation. In the first term $V^P((k+1)\tau_s,(k+1)\tau_s)$ is a deterministic value as the estimation time is same as the time at which the portfolio value is being computed. It is same as asking for the current *portfolio value which is known at that instant and is invariant of time. Hence the derivative of a constant (i.e. log of the constant portfolio value) will be* 0*. The value of the second term is given by equation 159. Thus we establish the required relationship given by equation 161 and hence the equation 156.*

Now let's prove equation 157. The induction approach is similar to above with small differences. The base case is when $i=0$. Then equation 157 simply becomes equation 155 which by definition is true. Similar to the derivation of equation 158, we can show that,

$$\xi^{H_{\tau_s}}(0,\tau_s)=\frac{dE[\sum_{i=1}^{N+1} w_i^{x_i\tau_s})]}{dt} \tag{163}$$

Hence, equation 155 implies,

$$\frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i\tau_s})]}{dt}=\nu_p \tag{164}$$

Now, assume equation 157 holds for $i=k$ and hence $(k+1)\tau_s$ is also a rebalance time. That is,

$$\xi^{H_{\tau_s}}(k\tau_s,(k+1)\tau_s)=\nu_p \tag{165}$$

To complete the proof we must show that it also holds for $i=k+1$, i.e.

$$\xi^{H_{\tau_s}}((k+1)\tau_s,(k+2)\tau_s)=\nu_p \tag{166}$$

Following similar steps as of the derivation of equation 162,

$$\xi^{H_{\tau_s}}((k+1)\tau_s,(k+2)\tau_s)=\frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i\tau_s})]}{dt}=\nu_p,\ \textit{using equation 164} \tag{167}$$

Thus we establish the required relationship given by equation 166 and hence the equation 157. This completes the proof of the theorem stating that, in order to obtain stable rebalancing, the assets need to be rebalanced to the initial optimal weights at a periodic interval of τ_s.

10 Monte Carlo Simulation Output

Passive Strategy Simulation Output.

t	$\chi^P(t)$	$\hat{\chi}^P(t)$	$e^P(t)$	t	$\chi^P(t)$	$\hat{\chi}^P(t)$	$e^P(t)$	t	$\chi^P(t)$	$\hat{\chi}^P(t)$	$e^P(t)$	t	$\chi^P(t)$	$\hat{\chi}^P(t)$	$e^P(t)$
0.1	0.0305	0.0251	0.0054	7.6	2.1978	2.1685	0.0293	15.1	3.8287	3.8418	-0.0131	22.6	5.2561	5.3718	-0.1157
0.2	0.0621	0.0567	0.0054	7.7	2.2218	2.1868	0.0350	15.2	3.8487	3.8657	-0.0170	22.7	5.2743	5.3914	-0.1171
0.3	0.0944	0.0827	0.0117	7.8	2.2458	2.2166	0.0292	15.3	3.8687	3.8794	-0.0107	22.8	5.2926	5.4104	-0.1178
0.4	0.1273	0.1099	0.0174	7.9	2.2696	2.2359	0.0337	15.4	3.8886	3.9047	-0.0161	22.9	5.3108	5.4316	-0.1208
0.5	0.1606	0.1377	0.0229	8.0	2.2934	2.2693	0.0241	15.5	3.9085	3.9287	-0.0202	23.0	5.3290	5.4514	-0.1224
0.6	0.1942	0.1657	0.0285	8.1	2.3171	2.2930	0.0241	15.6	3.9284	3.9483	-0.0199	23.1	5.3472	5.4721	-0.1249
0.7	0.2279	0.1957	0.0322	8.2	2.3407	2.3157	0.0250	15.7	3.9483	3.9676	-0.0193	23.2	5.3654	5.4962	-0.1308
0.8	0.2618	0.2255	0.0363	8.3	2.3642	2.3308	0.0334	15.8	3.9681	3.9887	-0.0206	23.3	5.3836	5.5126	-0.1290
0.9	0.2956	0.2572	0.0384	8.4	2.3877	2.3544	0.0333	15.9	3.9879	4.0065	-0.0186	23.4	5.4018	5.5307	-0.1289
1.0	0.3295	0.2871	0.0424	8.5	2.4111	2.3762	0.0349	16.0	4.0076	4.0324	-0.0248	23.5	5.4199	5.5486	-0.1287
1.1	0.3632	0.3226	0.0406	8.6	2.4344	2.4025	0.0319	16.1	4.0274	4.0505	-0.0231	23.6	5.4381	5.5700	-0.1319
1.2	0.3969	0.3612	0.0357	8.7	2.4576	2.4252	0.0324	16.2	4.0471	4.0698	-0.0227	23.7	5.4562	5.5905	-0.1343
1.3	0.4304	0.3917	0.0387	8.8	2.4808	2.4457	0.0351	16.3	4.0667	4.0916	-0.0249	23.8	5.4743	5.6104	-0.1361
1.4	0.4638	0.4214	0.0424	8.9	2.5039	2.4661	0.0378	16.4	4.0864	4.1121	-0.0257	23.9	5.4924	5.6309	-0.1385
1.5	0.4970	0.4537	0.0433	9.0	2.5269	2.4957	0.0312	16.5	4.1060	4.1345	-0.0285	24.0	5.5105	5.6490	-0.1385
1.6	0.5300	0.4827	0.0473	9.1	2.5499	2.5176	0.0323	16.6	4.1255	4.1536	-0.0281	24.1	5.5286	5.6725	-0.1439
1.7	0.5629	0.5188	0.0441	9.2	2.5727	2.5427	0.0300	16.7	4.1451	4.1742	-0.0291	24.2	5.5466	5.6866	-0.1400
1.8	0.5955	0.5481	0.0474	9.3	2.5956	2.5632	0.0324	16.8	4.1646	4.1990	-0.0344	24.3	5.5647	5.7105	-0.1458
1.9	0.6280	0.5876	0.0404	9.4	2.6183	2.5828	0.0355	16.9	4.1841	4.2162	-0.0321	24.4	5.5827	5.7246	-0.1419
2.0	0.6602	0.6228	0.0374	9.5	2.6410	2.6036	0.0374	17.0	4.2036	4.2373	-0.0337	24.5	5.6008	5.7466	-0.1458
2.1	0.6923	0.6573	0.0350	9.6	2.6636	2.6284	0.0352	17.1	4.2230	4.2595	-0.0365	24.6	5.6188	5.7685	-0.1497
2.2	0.7241	0.6859	0.0382	9.7	2.6862	2.6510	0.0352	17.2	4.2424	4.2790	-0.0366	24.7	5.6368	5.7823	-0.1455
2.3	0.7557	0.7186	0.0371	9.8	2.7087	2.6757	0.0330	17.3	4.2618	4.2976	-0.0358	24.8	5.6548	5.8017	-0.1469
2.4	0.7872	0.7469	0.0403	9.9	2.7311	2.6996	0.0315	17.4	4.2811	4.3191	-0.0380	24.9	5.6728	5.8202	-0.1474
2.5	0.8184	0.7762	0.0422	10.0	2.7535	2.7182	0.0353	17.5	4.3004	4.3394	-0.0390	25.0	5.6908	5.8414	-0.1506
2.6	0.8494	0.8022	0.0472	10.1	2.7758	2.7438	0.0320	17.6	4.3197	4.3603	-0.0406	25.1	5.7087	5.8612	-0.1525
2.7	0.8802	0.8404	0.0398	10.2	2.7981	2.7644	0.0337	17.7	4.3390	4.3851	-0.0461	25.2	5.7267	5.8809	-0.1542
2.8	0.9108	0.8730	0.0378	10.3	2.8203	2.7902	0.0301	17.8	4.3583	4.4066	-0.0483	25.3	5.7446	5.8993	-0.1547
2.9	0.9412	0.8936	0.0476	10.4	2.8424	2.8087	0.0337	17.9	4.3775	4.4234	-0.0459	25.4	5.7626	5.9190	-0.1564
3.0	0.9714	0.9251	0.0463	10.5	2.8645	2.8307	0.0338	18.0	4.3967	4.4446	-0.0479	25.5	5.7805	5.9409	-0.1604
3.1	1.0014	0.9519	0.0495	10.6	2.8865	2.8511	0.0354	18.1	4.4158	4.4655	-0.0497	25.6	5.7984	5.9612	-0.1628
3.2	1.0312	0.9826	0.0486	10.7	2.9085	2.8798	0.0287	18.2	4.4350	4.4833	-0.0483	25.7	5.8163	5.9756	-0.1593
3.3	1.0608	1.0167	0.0441	10.8	2.9304	2.9042	0.0262	18.3	4.4541	4.5068	-0.0527	25.8	5.8343	5.9970	-0.1627
3.4	1.0903	1.0467	0.0436	10.9	2.9523	2.9232	0.0291	18.4	4.4732	4.5278	-0.0546	25.9	5.8521	6.0148	-0.1627
3.5	1.1196	1.0738	0.0458	11.0	2.9741	2.9500	0.0241	18.5	4.4922	4.5480	-0.0558	26.0	5.8700	6.0364	-0.1664
3.6	1.1486	1.1027	0.0459	11.1	2.9958	2.9721	0.0237	18.6	4.5113	4.5720	-0.0607	26.1	5.8879	6.0552	-0.1673
3.7	1.1776	1.1352	0.0424	11.2	3.0175	2.9864	0.0311	18.7	4.5303	4.5921	-0.0618	26.2	5.9058	6.0728	-0.1670
3.8	1.2063	1.1640	0.0423	11.3	3.0392	3.0116	0.0276	18.8	4.5493	4.6108	-0.0615	26.3	5.9236	6.0896	-0.1660
3.9	1.2349	1.1968	0.0381	11.4	3.0608	3.0295	0.0313	18.9	4.5683	4.6326	-0.0643	26.4	5.9415	6.1115	-0.1700
4.0	1.2633	1.2245	0.0388	11.5	3.0823	3.0586	0.0237	19.0	4.5872	4.6532	-0.0660	26.5	5.9593	6.1308	-0.1715
4.1	1.2915	1.2530	0.0385	11.6	3.1038	3.0799	0.0239	19.1	4.6061	4.6717	-0.0656	26.6	5.9772	6.1517	-0.1745
4.2	1.3196	1.2842	0.0354	11.7	3.1253	3.1113	0.0140	19.2	4.6250	4.6929	-0.0679	26.7	5.9950	6.1691	-0.1741
4.3	1.3475	1.3145	0.0330	11.8	3.1467	3.1314	0.0153	19.3	4.6439	4.7119	-0.0680	26.8	6.0128	6.1900	-0.1772
4.4	1.3753	1.3391	0.0362	11.9	3.1680	3.1560	0.0120	19.4	4.6628	4.7311	-0.0683	26.9	6.0306	6.2110	-0.1804
4.5	1.4029	1.3632	0.0397	12.0	3.1893	3.1707	0.0186	19.5	4.6816	4.7519	-0.0703	27.0	6.0485	6.2272	-0.1787
4.6	1.4304	1.3819	0.0485	12.1	3.2106	3.1889	0.0217	19.6	4.7004	4.7717	-0.0713	27.1	6.0663	6.2479	-0.1816
4.7	1.4577	1.4102	0.0475	12.2	3.2318	3.2183	0.0135	19.7	4.7192	4.7896	-0.0704	27.2	6.0840	6.2660	-0.1820
4.8	1.4849	1.4441	0.0408	12.3	3.2530	3.2373	0.0157	19.8	4.7380	4.8151	-0.0771	27.3	6.1018	6.2823	-0.1805
4.9	1.5119	1.4697	0.0422	12.4	3.2741	3.2621	0.0120	19.9	4.7568	4.8386	-0.0818	27.4	6.1196	6.3021	-0.1825
5.0	1.5389	1.4907	0.0482	12.5	3.2952	3.2810	0.0142	20.0	4.7755	4.8552	-0.0797	27.5	6.1374	6.3207	-0.1833
5.1	1.5656	1.5229	0.0427	12.6	3.3162	3.3038	0.0124	20.1	4.7942	4.8749	-0.0807	27.6	6.1552	6.3436	-0.1884
5.2	1.5923	1.5456	0.0467	12.7	3.3372	3.3197	0.0175	20.2	4.8129	4.8937	-0.0808	27.7	6.1729	6.3619	-0.1890
5.3	1.6188	1.5738	0.0450	12.8	3.3581	3.3447	0.0134	20.3	4.8316	4.9138	-0.0822	27.8	6.1907	6.3842	-0.1935
5.4	1.6452	1.6008	0.0444	12.9	3.3790	3.3666	0.0124	20.4	4.8502	4.9325	-0.0823	27.9	6.2084	6.3989	-0.1905
5.5	1.6714	1.6270	0.0444	13.0	3.3999	3.3904	0.0095	20.5	4.8689	4.9504	-0.0815	28.0	6.2262	6.4195	-0.1933
5.6	1.6976	1.6570	0.0406	13.1	3.4207	3.4098	0.0109	20.6	4.8875	4.9730	-0.0855	28.1	6.2439	6.4384	-0.1945
5.7	1.7236	1.6831	0.0405	13.2	3.4414	3.4309	0.0105	20.7	4.9061	4.9916	-0.0855	28.2	6.2617	6.4646	-0.2029
5.8	1.7495	1.7142	0.0353	13.3	3.4622	3.4567	0.0055	20.8	4.9246	5.0086	-0.0840	28.3	6.2794	6.4791	-0.1997
5.9	1.7753	1.7371	0.0382	13.4	3.4829	3.4790	0.0039	20.9	4.9432	5.0328	-0.0896	28.4	6.2971	6.5012	-0.2041
6.0	1.8010	1.7597	0.0413	13.5	3.5035	3.4957	0.0078	21.0	4.9617	5.0495	-0.0878	28.5	6.3148	6.5218	-0.2070
6.1	1.8265	1.7945	0.0320	13.6	3.5241	3.5195	0.0046	21.1	4.9803	5.0722	-0.0919	28.6	6.3326	6.5373	-0.2047
6.2	1.8520	1.8214	0.0306	13.7	3.5447	3.5420	0.0027	21.2	4.9988	5.0921	-0.0933	28.7	6.3503	6.5575	-0.2072
6.3	1.8773	1.8408	0.0365	13.8	3.5652	3.5611	0.0041	21.3	5.0173	5.1135	-0.0962	28.8	6.3680	6.5776	-0.2096
6.4	1.9026	1.8628	0.0398	13.9	3.5857	3.5810	0.0047	21.4	5.0357	5.1336	-0.0979	28.9	6.3857	6.5923	-0.2066
6.5	1.9277	1.8965	0.0312	14.0	3.6061	3.6028	0.0033	21.5	5.0542	5.1508	-0.0966	29.0	6.4034	6.6123	-0.2089
6.6	1.9527	1.9166	0.0361	14.1	3.6266	3.6271	-0.0005	21.6	5.0726	5.1715	-0.0989	29.1	6.4211	6.6336	-0.2125
6.7	1.9777	1.9565	0.0212	14.2	3.6469	3.6439	0.0030	21.7	5.0910	5.1907	-0.0997	29.2	6.4388	6.6543	-0.2155
6.8	2.0025	1.9740	0.0285	14.3	3.6673	3.6682	-0.0009	21.8	5.1094	5.2086	-0.0992	29.3	6.4565	6.6713	-0.2148
6.9	2.0272	1.9991	0.0281	14.4	3.6876	3.6871	0.0005	21.9	5.1278	5.2279	-0.1001	29.4	6.4741	6.6956	-0.2215
7.0	2.0519	2.0230	0.0289	14.5	3.7078	3.7080	-0.0002	22.0	5.1462	5.2532	-0.1070	29.5	6.4918	6.7089	-0.2171
7.1	2.0764	2.0501	0.0263	14.6	3.7281	3.7305	-0.0024	22.1	5.1645	5.2699	-0.1054	29.6	6.5095	6.7329	-0.2234
7.2	2.1009	2.0807	0.0202	14.7	3.7483	3.7525	-0.0042	22.2	5.1829	5.2889	-0.1060	29.7	6.5272	6.7505	-0.2233
7.3	2.1253	2.1033	0.0220	14.8	3.7684	3.7738	-0.0054	22.3	5.2012	5.3103	-0.1091	29.8	6.5448	6.7672	-0.2224
7.4	2.1495	2.1276	0.0219	14.9	3.7885	3.7944	-0.0059	22.4	5.2195	5.3291	-0.1096	29.9	6.5625	6.7873	-0.2248
7.5	2.1737	2.1521	0.0216	15.0	3.8086	3.8196	-0.0110	22.5	5.2378	5.3503	-0.1125	30.0	6.5802	6.8025	-0.2223

Table 1: Rebalance Frequency Simulation Output - Hybrid Strategy

τ (yr)	$\widehat{\chi}^{H\tau}$ (30)	τ (yr)	$\widehat{\chi}^{H\tau}$ (30)	τ (yr)	$\widehat{\chi}^{H\tau}$ (30)	τ (yr)	$\widehat{\chi}^{H\tau}$ (30)
0.10	**8.542352**	**7.50**	**8.497746**	**15.10**	**7.605126**	**22.60**	**7.460833**
0.20	**8.455353**	**7.60**	**8.477886**	**15.20**	**7.605857**	**22.70**	**7.453264**
0.30	**8.368990**	**7.70**	**8.467232**	**15.30**	**7.605144**	**22.80**	**7.445571**
0.40	**8.334001**	**7.80**	**8.464937**	**15.40**	**7.609650**	**22.90**	**7.447380**
0.50	**8.350096**	**7.90**	**8.467263**	**15.50**	**7.600330**	**23.00**	**7.443484**
0.60	**8.346815**	**8.00**	**8.467839**	**15.60**	**7.603166**	**23.10**	**7.437876**
0.70	**8.419484**	**8.10**	**8.447780**	**15.70**	**7.603840**	**23.20**	**7.423039**
0.80	**8.473442**	**8.20**	**8.441657**	**15.80**	**7.599871**	**23.30**	**7.419542**
0.90	**8.521598**	**8.30**	**8.429594**	**15.90**	**7.597581**	**23.40**	**7.408796**
1.00	**8.626667**	**8.40**	**8.428899**	**16.00**	**7.602618**	**23.50**	**7.406879**
1.10	**8.709298**	**8.50**	**8.418490**	**16.10**	**7.596690**	**23.60**	**7.399510**
1.20	**8.739643**	**8.60**	**8.412785**	**16.20**	**7.593222**	**23.70**	**7.393206**
1.30	**8.803758**	**8.70**	**8.385438**	**16.30**	**7.595756**	**23.80**	**7.387769**
1.40	**8.859632**	**8.80**	**8.377513**	**16.40**	**7.591306**	**23.90**	**7.376424**
1.50	**8.933021**	**8.90**	**8.345923**	**16.50**	**7.585871**	**24.00**	**7.379830**
1.60	**8.940686**	**9.00**	**8.323310**	**16.60**	**7.580517**	**24.10**	**7.382556**
1.70	**8.983553**	**9.10**	**8.323823**	**16.70**	**7.584899**	**24.20**	**7.375637**
1.80	**9.024119**	**9.20**	**8.293214**	**16.80**	**7.587569**	**24.30**	**7.362645**
1.90	**9.113382**	**9.30**	**8.248139**	**16.90**	**7.580986**	**24.40**	**7.354784**
2.00	**9.052070**	**9.40**	**8.225336**	**17.00**	**7.584849**	**24.50**	**7.350422**
2.10	**9.024366**	**9.50**	**8.198339**	**17.10**	**7.592308**	**24.60**	**7.348662**
2.20	**9.151034**	**9.60**	**8.185290**	**17.20**	**7.584418**	**24.70**	**7.335963**
2.30	**9.150685**	**9.70**	**8.145177**	**17.30**	**7.575777**	**24.80**	**7.328365**
2.40	**9.090808**	**9.80**	**8.117516**	**17.40**	**7.580219**	**24.90**	**7.317851**
2.50	**9.126256**	**9.90**	**8.121853**	**17.50**	**7.576318**	**25.00**	**7.315764**
2.60	**9.150673**	**10.10**	**8.116371**	**17.60**	**7.579315**	**25.10**	**7.303295**
2.70	**9.144521**	**10.20**	**8.115589**	**17.70**	**7.579462**	**25.20**	**7.300355**
2.80	**9.181453**	**10.30**	**8.123384**	**17.80**	**7.580997**	**25.30**	**7.290861**
2.90	**9.111371**	**10.40**	**8.111238**	**17.90**	**7.581976**	**25.40**	**7.288126**
3.00	**9.162608**	**10.50**	**8.121841**	**18.00**	**7.571419**	**25.50**	**7.274491**
3.10	**9.163117**	**10.60**	**8.113584**	**18.10**	**7.579131**	**25.60**	**7.273108**
3.20	**9.085013**	**10.70**	**8.128323**	**18.20**	**7.573210**	**25.70**	**7.263738**
3.30	**9.140095**	**10.80**	**8.124641**	**18.30**	**7.575717**	**25.80**	**7.254038**
3.40	**9.106145**	**10.90**	**8.110909**	**18.40**	**7.576314**	**25.90**	**7.255741**
3.50	**9.081381**	**11.00**	**8.106902**	**18.50**	**7.575620**	**26.00**	**7.243761**
3.60	**9.041245**	**11.10**	**8.097604**	**18.60**	**7.579672**	**26.10**	**7.229281**
3.70	**9.048834**	**11.20**	**8.085180**	**18.70**	**7.570250**	**26.20**	**7.218271**
3.80	**9.089872**	**11.30**	**8.104539**	**18.80**	**7.570011**	**26.30**	**7.209375**
3.90	**9.078496**	**11.40**	**8.086047**	**18.90**	**7.570965**	**26.40**	**7.217678**
4.00	**9.056633**	**11.50**	**8.082976**	**19.00**	**7.567993**	**26.50**	**7.197334**
4.10	**9.020260**	**11.60**	**8.074162**	**19.10**	**7.560275**	**26.60**	**7.190700**
4.20	**9.005965**	**11.70**	**8.084117**	**19.20**	**7.565564**	**26.70**	**7.189377**
4.30	**9.020178**	**11.80**	**8.075804**	**19.30**	**7.565476**	**26.80**	**7.172624**
4.40	**9.024254**	**11.90**	**8.062688**	**19.40**	**7.565549**	**26.90**	**7.165610**
4.50	**9.014870**	**12.00**	**8.048993**	**19.50**	**7.558258**	**27.00**	**7.158003**
4.60	**8.990582**	**12.10**	**8.057178**	**19.60**	**7.558644**	**27.10**	**7.133600**
4.70	**8.990204**	**12.20**	**8.044947**	**19.70**	**7.550932**	**27.20**	**7.125669**
4.80	**8.927782**	**12.30**	**8.025242**	**19.80**	**7.550376**	**27.30**	**7.121147**
4.90	**8.898053**	**12.40**	**8.025183**	**19.90**	**7.554773**	**27.40**	**7.101257**
5.00	**8.884910**	**12.50**	**8.014069**	**20.00**	**7.549886**	**27.50**	**7.103818**
5.10	**8.871700**	**12.60**	**7.983777**	**20.10**	**7.550149**	**27.60**	**7.097771**
5.20	**8.895869**	**12.70**	**7.981845**	**20.20**	**7.541165**	**27.70**	**7.076035**
5.30	**8.884081**	**12.80**	**7.972173**	**20.30**	**7.538853**	**27.80**	**7.060809**
5.40	**8.845325**	**12.90**	**7.957285**	**20.40**	**7.529070**	**27.90**	**7.042808**
5.50	**8.835382**	**13.00**	**7.963722**	**20.50**	**7.528389**	**28.00**	**7.033106**
5.60	**8.803390**	**13.10**	**7.945089**	**20.60**	**7.518195**	**28.10**	**7.028621**
5.70	**8.807741**	**13.20**	**7.931469**	**20.70**	**7.525809**	**28.20**	**7.019760**
5.80	**8.760611**	**13.30**	**7.919481**	**20.80**	**7.514850**	**28.30**	**7.001764**
5.90	**8.722579**	**13.40**	**7.903196**	**20.90**	**7.514211**	**28.40**	**6.993319**
6.00	**8.714644**	**13.50**	**7.875091**	**21.00**	**7.521058**	**28.50**	**6.981541**
6.10	**8.722966**	**13.60**	**7.863023**	**21.10**	**7.512854**	**28.60**	**6.964666**
6.20	**8.728240**	**13.70**	**7.854180**	**21.20**	**7.513160**	**28.70**	**6.946877**
6.30	**8.716763**	**13.80**	**7.854901**	**21.30**	**7.500682**	**28.80**	**6.931088**
6.40	**8.718771**	**13.90**	**7.818607**	**21.40**	**7.507145**	**28.90**	**6.912075**
6.50	**8.740152**	**14.00**	**7.789904**	**21.50**	**7.502436**	**29.00**	**6.908937**
6.60	**8.698996**	**14.10**	**7.767849**	**21.60**	**7.495822**	**29.10**	**6.895214**
6.70	**8.698427**	**14.20**	**7.738175**	**21.70**	**7.487980**	**29.20**	**6.878146**
6.80	**8.653252**	**14.30**	**7.728568**	**21.80**	**7.491792**	**29.30**	**6.866248**
6.90	**8.654150**	**14.40**	**7.695856**	**21.90**	**7.487690**	**29.40**	**6.849868**
7.00	**8.615148**	**14.50**	**7.687919**	**22.00**	**7.477312**	**29.50**	**6.837249**
7.10	**8.597792**	**14.60**	**7.659117**	**22.10**	**7.478763**	**29.60**	**6.833582**
7.20	**8.555518**	**14.70**	**7.639103**	**22.20**	**7.469627**	**29.70**	**6.820091**
7.30	**8.534219**	**14.80**	**7.613359**	**22.30**	**7.465581**	**29.80**	**6.813326**
7.40	**8.520672**	**14.90**	**7.604649**	**22.40**	**7.459336**	**29.90**	**6.810321**
7.50	**8.497425**	**15.00**	**7.599636**	**22.50**	**7.458788**	**30.00**	**6.802497**