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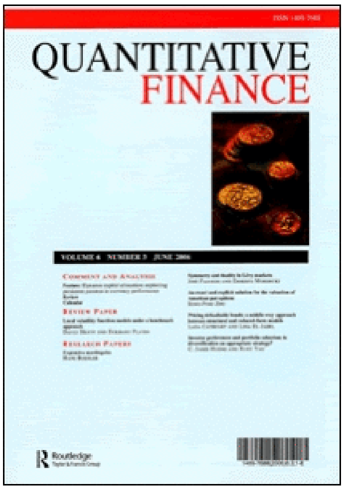

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Pseudospectral methods for pricing options

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Models with two or more risk sources have been widely applied in option pricing in order to capture volatility smiles and skews. However, the computational cost of implementing these models can be large—especially for American-style options. This paper illustrates how numerical techniques called 'pseudospectral' methods can be used to solve the partial differential and partial integro-differential equations that apply to these multifactor models. The method offers significant advantages over finite-difference and Monte Carlo simulation schemes in terms of accuracy and computational cost.

Keywords: American options; Options pricing; Partial differential equations; Stochastic volatility

1. Introduction

Most of the numerical techniques for valuing derivative assets fall into three broad categories: finite-difference methods (numerical solutions of PDEs), lattice methods (binomial and trinomial trees), and Monte Carlo techniques. Monte Carlo is useful in high-dimensional cases with many state variables, but it is relatively slow, even when implemented with acceleration schemes. Lattice and finite-difference methods (FDM) are fast in low dimensions but are impractical if there are more than a few state variables. Researchers have proposed a variety of multifactor models in recent years to account for volatility smiles and smirks in option prices. The stochastic volatility models of Heston (1993), Scott (1997) and others capture the persistence of volatility and improve the predictions for prices of options with moderate-to-long times to expiration. Bates (1996, 2000) and Bakshi *et al*. (1997) show that introducing discontinuous price processes subject to Poisson-driven jumps helps in pricing short-maturity options. Eraker *et al*. (2003) advocate allowing for jumps in volatility as well as price. Empirical tests of these multifactor models have been carried out mainly with European-style derivatives, where martingale methods yield computationally efficient pricing formulae. Applied to American options there is another complication besides the higher dimensionality: the presence of jump components requires solving partial integro-differential equations (PIDEs) instead of PDEs. This makes finite-difference methods much harder to implement.

This paper applies to option pricing an alternative technique for solving PDEs and PIDEs numerically. While these *pseudospectral methods* (PSM) are well known to physicists and engineers, they seem not to have been used in pricing derivatives. Pseudospectral methods approximate solution functions *globally* as high-order orthogonal polynomials, whereas finite-difference procedures use *local* linear approximations between finely spaced grid points. Global approximations are more efficient in many classes of problems. Boyd (2001) gives a comprehensive overview of pseudospectral methods and a comparison with conventional techniques. We find that they offer significant advantages over finite-difference and Monte Carlo simulation methods in pricing American options when underlying prices are subject to multiple sources of risk. Applications to other path-dependent claims are treated by Suh (2005).

It is noteworthy that there are several attempts to use basis functions for option pricing in different ways. Madan and Milne (1994) show that by modelling contingent claims as elements of a separable Hilbert space we may change measure to a reference measure under which asset prices are Gaussian and for which the family of Hermite polynomials serves as an orthonormal basis. We infer from observed option prices the implicit prices of basis elements and use them to construct the

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implied equivalent martingale measure density which is an input for pricing options. Linetsky (2004a) develops a spectral approach to the valuation of contingent claims when the underlying asset follows a one-dimensional diffusion process. Under the approach are developed many explicit valuation expressions in terms of an infinite number of eigenfunctions for quite a wide range of option models as well as option contracts: see, for example, Linetsky (2004b) for Asian options. Dempster *et al.* (2000) and Dempster and Eswaran (2001) illustrate applications of a wavelet method to several option pricing problems. Option prices are represented as a linear combination of wavelets which serve as basis functions and are substituted into PDEs for numerical solutions. Madan and Milne's (1994) approach is apt for pricing path-independent options non-parametrically but not for pricing American or other path-dependent options. The spectral expansion approach has not yet been developed for multi-factor models.

The following are the main contributions and findings of this paper.

- We illustrate the use of PSM for pricing European- and American-style options under several standard models for underlying price: geometric Brownian motion (BS), Heston's (1993) stochastic volatility (SV), and Bates' (1996) stochastic volatility with jumps (SVJ).
- A way of handling PIDEs by quadrature methods is proposed.
- The Broadie–Detemple (1996) technique is found to improve efficiency and extend PSM's advantage over FDM.
- Forward PDEs and PIDEs are developed for American puts under models SV and SVJ to allow options with different strikes to be priced all at once.
- An empirical application to exchange-traded equity options is provided.

2. Pricing under BS dynamics

To illustrate the pseudospectral approach in the simplest possible setting, we first apply the method to European and American vanilla options under Black–Scholes (1973) dynamics. This lets us compare accuracy and speed against exact formulae for European options and the many analytic approximations for Americans.

Let us begin with the usual geometric Brownian motion model for the price of the underlying, $\mathrm{d}S_t/S_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, and the fundamental PDE for the time-t value, $D(S_t, T-t)$, of a T-expiring derivative asset is given by

$$-D_{T-t} + rS_tD_S + \frac{\sigma^2}{2}S_t^2D_{SS} - rD = 0, \tag{1}$$

where r is the constant riskless interest rate, and subscripts indicate partial derivatives.

2.1. European options

The initial condition for the PDE depends on the option type, and for the European put struck at X it is

$$D(S_T, 0) = \max(0, X - S_T). \tag{2}$$

Supposing that the exact solution of (1) subject to (2) can be represented as an infinite linear combination of orthogonal basis functions, approximate the solution at time $t \in [0, T)$ by truncating after the first N_S terms, as

$$D(S,\tau) \doteq \sum_{n=0}^{N_S-1} b_{n,t}\phi_n(S),$$

where $\tau \equiv T - t$ and $\{\phi_n\}_{n=0}^{\infty}$ are the basis functions. Among the possible candidates for basis functions are Chebyshev, Legendre, Hermite, and Laguerre polynomials, the choice depending on the characteristics of the solution. As explained by Boyd (2001), Chebyshev polynomials are recommended for many applications. We choose Chebyshev polynomials of the first kind, which are defined for $x \in [-1, 1]$ as $P_n(x) = \cos[n \cdot \arccos(x)]$ or, equivalently, by the following recursion:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x), \quad \text{for } n \geq 1.$$

Derivatives are given by

$$P_n^{(1)}(x) \equiv \frac{\mathrm{d}P_n(x)}{\mathrm{d}x} = \frac{n\sin(ny)}{\sin(y)},$$
$$P_n^{(2)}(x) \equiv \frac{\mathrm{d}^2P_n(x)}{\mathrm{d}x^2} = -\frac{n^2\cos(ny)}{\sin^2(y)} + \frac{n\cos(y)\sin(ny)}{\sin^3(y)},$$

where $y = \arccos(x)$.

The first step is to transform from the $[-1, 1]$ Chebyshev domain to a different one, $S \in [\bar{S}, \underline{S}]$, that better suits the option at hand:

$$x = \frac{2S - (\bar{S} + \underline{S})}{\bar{S} - \underline{S}} \in [-1, 1].$$

Now writing $D(S,\tau) = \tilde{D}(x,\tau)$, the PDE becomes

$$-\tilde{D}_\tau + rS\frac{2}{\bar{S} - \underline{S}}\tilde{D}_x + \frac{\sigma^2}{2}S^2\left(\frac{2}{\bar{S} - \underline{S}}\right)^2\tilde{D}_{xx} - r\tilde{D} = 0,$$

and the trial solution at time t is

$$\tilde{D}(x,\tau) = \sum_{n=0}^{N_S-1} b_{n,t}P_n(x). \tag{3}$$

Coefficients $\{b_{n,t}\}$ are obtained by substituting (3) into the PDE, evaluating at N_S 'collocation' points $\{x_n\}$, and solving the resulting simultaneous linear equations system. Of the two kinds of collocation points for Chebyshev polynomials we choose the Gauss–Lobatto system,

$$x_n = \cos\left(\frac{\pi n}{N_S - 1}\right), \quad n \in \{0, \ldots, N_S - 1\}.$$

By discretizing the time domain, the PDE can be solved at each step from $T - \Delta t$ back to $t = 0$. This reduces the number of dimensions of the approximating polynomials, speeds up computation, and simplifies programming. For discretization we use the backwards Euler algorithm. Dividing $[0, T]$ into N_T subperiods of equal length $\Delta t = T/N_T$, the option price at time to maturity $m\Delta t$ ($m \in \{0, 1, \ldots, N_T - 1\}$) satisfies

$$0 = -\frac{\tilde{D}[x,(m+1)\Delta t] - \tilde{D}(x, m\Delta t)}{\Delta t} + rS\frac{2}{\Delta S}\tilde{D}_x[x,(m+1)\Delta t] + \sigma^2 S^2 \frac{2}{(\Delta S)^2}\tilde{D}_{xx}[x,(m+1)\Delta t] - r\tilde{D}[x,(m+1)\Delta t],$$

where $\Delta S \equiv \bar{S} - \underline{S}$. Last, we impose conditions that determine D at boundaries of $[\underline{S}, \bar{S}]$. For European puts under model BS they are $D(\underline{S}, \tau) = X\exp(-r\tau) - \underline{S}$ and $D(\bar{S}, \tau) = 0$ for all $t \in [0, T]$. To develop the equation in matrix form, define $(N_S \times N_S)$ matrices $\mathbf{P}$, $\mathbf{P}^{(1)}$, $\mathbf{P}^{(2)}$ as Chebyshev polynomials and their derivatives evaluated at collocation points, i.e.

$$\begin{aligned}
\mathbf{P}_{ij} &= P_{j-1}(x_{i-1}), \quad i, j \in \{1, 2, \ldots, N_S\}, \\
\mathbf{P}^{(1)}_{ij} &= P^{(1)}_{j-1}(x_{i-1}), \quad i, j \in \{1, 2, \ldots, N_S\}, \\
\mathbf{P}^{(2)}_{ij} &= P^{(2)}_{j-1}(x_{i-1}), \quad i, j \in \{1, 2, \ldots, N_S\}.
\end{aligned}$$

Also, define diagonal matrices $\mathbf{S}$ and $\mathbf{S}^2$ having elements $\{S_0, \ldots, S_{N_S-1}\}$ and their squares, respectively, on the principal diagonals, where

$$S_i = \frac{(\bar{S} + \underline{S}) + (\bar{S} - \underline{S})x_i}{2}, \quad i \in \{0, 1, \ldots, N_S - 1\}.$$

The approximate solution at stage $t \in \{0, \Delta t, 2\Delta t, \ldots, (N_T - 1)\,\Delta t\}$ for all N_S values of S then takes the form

$$\left\{ D(S_i, \tau) = \sum_{n=0}^{N_S-1} b_{n,t} P_n(x_i) \right\}_{i=0}^{N_S-1} \equiv \mathbf{P}\mathbf{b}_t.$$

To develop an efficient solution, express the PDE in matrix form as

$$\mathbf{R}_{t+\Delta t} = \mathbf{Q}\mathbf{b}_t, \tag{4}$$

where

$$\begin{aligned}
\mathbf{R}_{t+\Delta t} &\equiv -(\Delta t)^{-1} \begin{bmatrix} \tilde{D}(x_0, \tau - \Delta t) \\ \tilde{D}(x_1, \tau - \Delta t) \\ \cdots \\ \tilde{D}(x_{N_S-1}, \tau - \Delta t) \end{bmatrix}, \\
\mathbf{Q} &\equiv -\left(r + \frac{1}{\Delta t}\right)\mathbf{P} + \frac{2r}{\Delta S}\mathbf{S}\mathbf{P}^{(1)} + \frac{2\sigma^2}{(\Delta S)^2}\mathbf{S}^2\mathbf{P}^{(2)}, \\
\mathbf{b}_t &= (b_{0,t}, b_{1,t}, \ldots, b_{N_S-1,t})', \\
\Delta S &\equiv \bar{S} - \underline{S}.
\end{aligned}$$

Boundary conditions are imposed by setting the top and bottom elements of vector $\mathbf{R}_{t+\Delta t}$ (corresponding to extreme prices $\bar{S}$ and $\underline{S}$) to $\tilde{D}(x_0, \tau) = 0$ and $\tilde{D}(x_{N_S-1}, \tau) = X\exp(-r\tau) - \underline{S}$, respectively, for European puts.

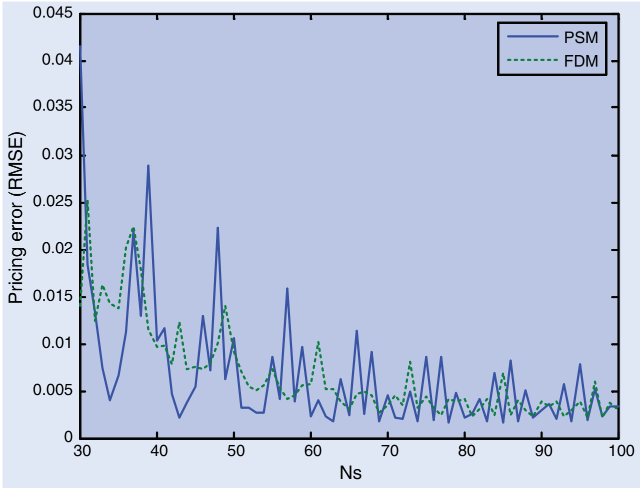

Figure 1. Pricing errors of PSM and FDM for European puts: BS model.

The top and bottom rows of matrix $\mathbf{Q}$ should simply be the top and bottom rows of $\mathbf{P}$. Starting at $t = (N_T - 1)\Delta t$ (one step before expiration) vector $\mathbf{R}_{N_T\Delta t} \equiv \mathbf{R}_T$ is known, and the above equations can be solved for $\mathbf{b}_{(N_T-1)\Delta t}$. Moving back to $t = (N_T - 2)\Delta t$, set $\mathbf{R}_{(N_T-1)\Delta t} = -\mathbf{P}\mathbf{b}_{(N_T-1)\Delta t}/\Delta t$, replace the top and bottom elements as above, then solve $\mathbf{R}_{(N_T-1)\Delta t} = \mathbf{Q}\mathbf{b}_{(N_T-2)\Delta t}$ for $\mathbf{b}_{(N_T-2)\Delta t}$. The process is continued until $\mathbf{b}_0$ is found, whence $\mathbf{P}\mathbf{b}_0$ gives the approximate solution for the vector of initial values of the derivative at underlying prices $\{S_i\}_{i=0}^{N_S-1}$. Note that matrix $\mathbf{Q}$ remains the same at each time step and that (for puts) only the bottom element of $\mathbf{R}_t$ needs to be altered (to reflect the changing discount factor).

To compare the rates of convergence of PSM and FDM, root-mean-squared pricing errors (RMSE) of European puts with moneyness $S/X \in \{0.8, 0.9, 1.0, 1.1, 1.2\}$ are calculated for values of N_S ranging from 30 to 100. The same values of N_S are used for the FDM price grid. Model parameters are $X = 100$, $r = 0.05$, $T - t = 0.5$, $\sigma = 0.3$, and $N_T = 100$. Figure 1 shows the patterns of pricing errors. Error levels for both methods tend to decrease with N_S, but PSM errors are typically larger and show pronounced fluctuations. This is contrary to expectation, because the approximation through PSM should converge exponentially fast, vs. rate N_S^{-2} for FDM.†

The explanation can be seen in figure 2, which depicts the weights $\{w_i\}$ with which a given period's prices are calculated from those of the next period, as $D(S_0, t) = \sum_i w_i D(S_i, t + \Delta t)$.‡ (The figures are based on $N_S = 60$ and $S_0 = 110$ for FDM, and $S_0 = 111.6$ for PSM) The weights of PSM are more dispersed than those of FDM and their signs alternate in the tails, as shown in the closer views depicted in the right-hand panels. This feature, combined with the kinked terminal payoff structure of standard options, seems to account

†See, for example, Wilmott *et al.* (1993) or Tavella and Randall (2000).
‡For this, aside from the boundary conditions, we can rewrite equation (4) as $\mathbf{b}_t = \mathbf{Q}^{-1}\,\mathbf{R}_{t+\Delta t} = -(\Delta t)^{-1}\,\mathbf{Q}^{-1}\,\mathbf{D}_{t+\Delta t}$ and multiply both sides by $\mathbf{P}$ to get $\mathbf{D}_t = -(\Delta t)^{-1}\,\mathbf{P}\mathbf{Q}^{-1}\,\mathbf{D}_{t+\Delta t}$, where $\mathbf{D}_t$ is a vector of option prices at t.

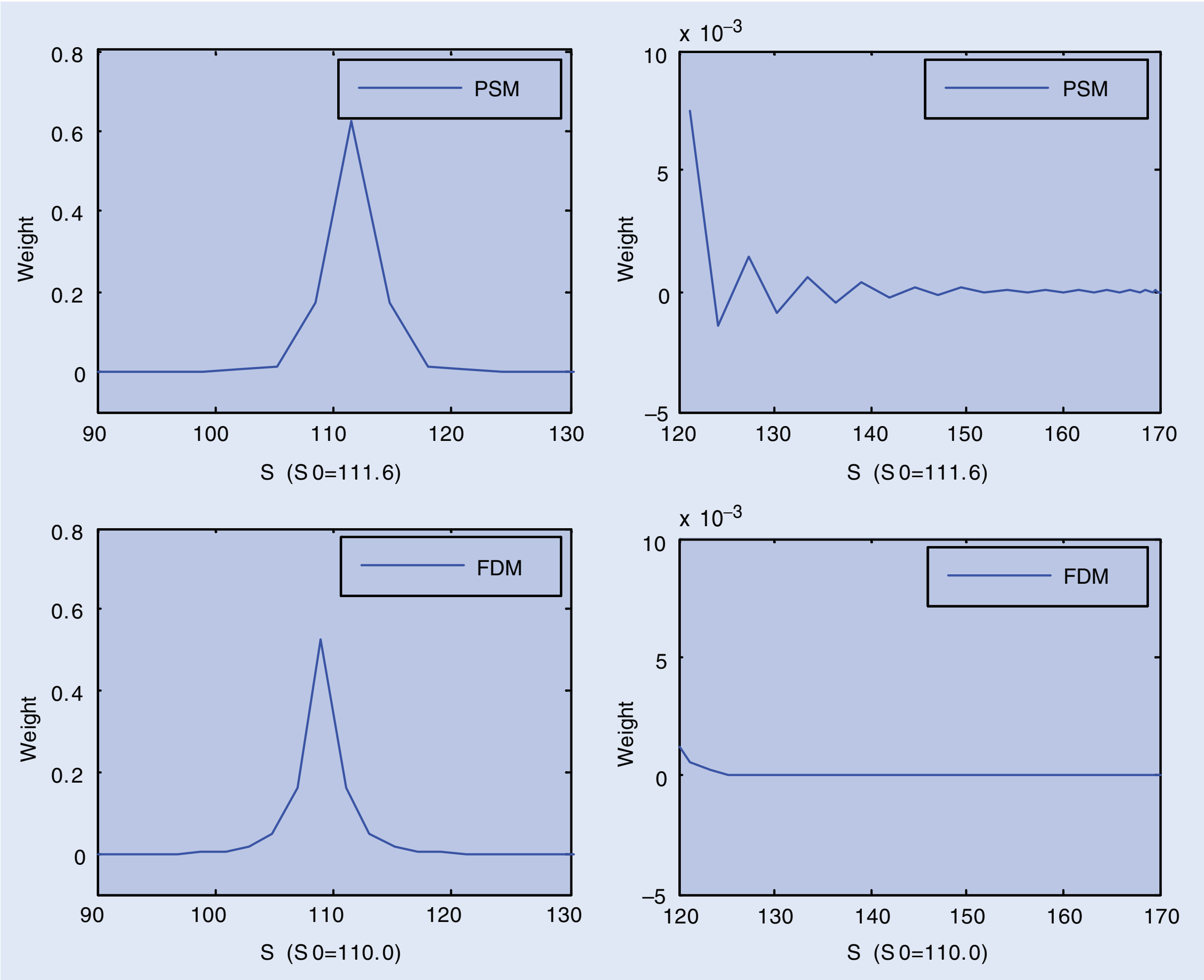

Figure 2. Pricing weights of PSM and FDM: BS model.

for the pronounced fluctuations in estimated prices. The weights $\{w_i\}$ and their positions change as N_S changes, causing oscillations in $D(S_i)$ as they move within the sharply curved region around X. The same effect is present in binomial pricing. As a remedy, Broadie and Detemple (1996) suggest replacing the kinked terminal value function with the smooth function corresponding to Black–Scholes prices at one time step before expiration. The technique also works for FDM—and especially well for PSM. Figure 3 depicts, as functions of N_S, the *maximum* pricing errors for all numbers of price steps $N'_S \geq N_S$. It is apparent that the Broadie–Detemple technique does allow PSM to achieve the faster convergence that theory leads us to expect.

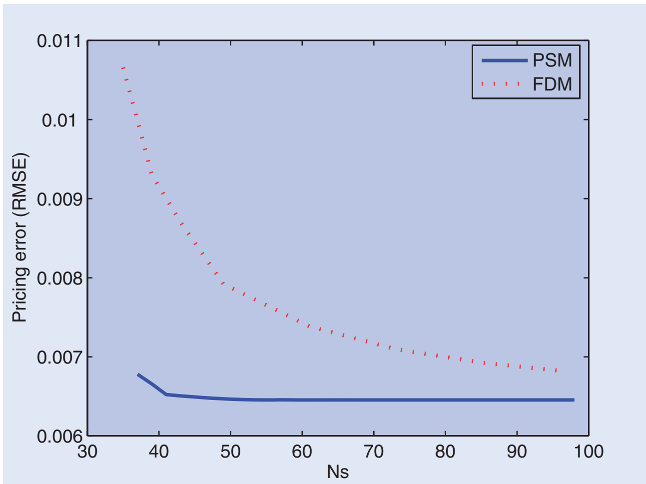

Figure 3. Pricing errors of PSM and FDM for European puts with the Broadie–Detemple scheme: BS model.

2.2. *American options*

There is as yet no exact solution of the fundamental PDE for American-style options, even under simple BS dynamics; however, many approximate solutions and numerical approaches have been developed; e.g., Geske and Johnson (1984), MacMillan (1986), Broadie and Detemple (1996), and Ju (1998). One simple approach to addressing the early exercise feature of American-style options is to advance the discrete solution over a time step ignoring the constraint, and then to apply the constraint explicitly. At each time step before expiration the option

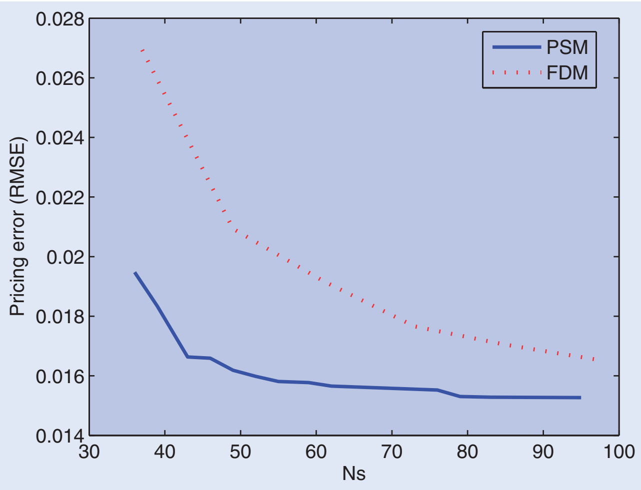

Figure 4. Pricing errors of PSM and FDM for American puts: BS model.

price is simply set to the greater of the intrinsic value and the value in solution vector $\mathbf{Pb}_t$ for the case of PSM. Figure 4 plots as functions of N_S the absolute differences of PSM and FDM prices from binomial solutions with 1000 time steps. (The options and parameters are as in figure 3.)

A comparison between figures 3 and 4 reveals higher pricing errors for American options than European options in both PSM and FDM. This may result from the slow convergence of this explicit application of the constraint.† Restoring a higher temporal convergence rate requires sophisticated methods. One nice approach probably applicable to PSM is to formalize the American option pricing problem as a linear complementarity problem (LCP) and to solve it using a penalty method. LCP formulations combined with the penalty method have been applied to American option pricing problems with FDM; e.g., Forsyth and Vetzal (2002), Zvan *et al.* (1998), and d'Halluin *et al.* (2004) for pricing American options with BS, SV, and jump diffusion dynamics, respectively. Applications of the LCP penalty or other methods to obtain faster temporal convergence with PSM would be beneficial but will be left as a future research topic so as to focus on the application of PSM to European option pricing problems in this paper.

3. Pricing under SV dynamics

Heston's (1993) stochastic volatility (SV) model is a two-factor model with state variables S_t and σ_t. The dynamics of the underlying price under the martingale measure $\hat{\mathbb{P}}$ are given by

$$\begin{aligned} \mathrm{d}S_t/S_t &= r\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_{1t}, \\ \mathrm{d}\sigma_t^2 &= (\alpha - \beta\sigma_t^2)\mathrm{d}t + \gamma\sigma_t(\rho\,\mathrm{d}W_{1t} + \bar{\rho}\,\mathrm{d}W_{2t}), \end{aligned}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$ and $\{W_{jt}\}_{j=1}^2$ are independent Brownian motions. A T-expiring derivative asset worth $D(S_t, \sigma_t^2, \tau)$ at time t satisfies

$$\begin{aligned} 0 = &-D_\tau + rS_tD_S + \frac{\sigma_t^2}{2}S_t^2D_{SS} + (\alpha - \beta\sigma_t^2)D_{\sigma^2} \\ &+ \gamma^2\frac{\sigma_t^2}{2}D_{\sigma^2\sigma^2} + \rho\gamma\sigma_t^2S_tD_{\sigma^2S} - rD, \end{aligned} \tag{5}$$

subject to initial and boundary conditions. Boundary conditions specify the behavior of D at extremes of the domains: $[\underline{S}, \bar{S}]$ for S_t and $[0, \bar{\sigma}^2]$ for σ_t^2. For S_t the conditions are as in the one-factor BS model; e.g., for the put option, $D(\underline{S}, \sigma_t^2, \tau) = X\exp(-r\tau) - \underline{S}$ and $D(\bar{S}, \sigma_t^2, \tau) = 0$ for $\sigma_t^2 \in [0, \bar{\sigma}^2], t \in [0, T]$. Appropriate boundary conditions for volatility are less apparent. Since the option value should be insensitive to further volatility change when σ_t^2 is very large, we set $D_{\sigma^2}(S, \bar{\sigma}^2, \tau) = 0$ for $S \in [\underline{S}, \bar{S}], t \in [0, T]$. Expressing the PDE at $\sigma_t^2 = 0$ gives as a lower boundary condition

$$\begin{aligned} 0 = &-D_\tau(S_t, 0, \tau) + rS_tD_S(S_t, 0, \tau) \\ &+ \alpha D_{\sigma^2}(S_t, 0, \tau) - rD(S_t, 0, \tau). \end{aligned}$$

Tensor products of one-dimensional basis functions serve as basis functions for two or more dimensions. The approximate solution for $D(S_t, \sigma_t^2, \tau) \equiv \tilde{D}(x, y, \tau)$ will take the form

$$\sum_{n_S=0}^{N_S-1}\sum_{n_\sigma=0}^{N_\sigma-1} b_{n_Sn_\sigma,t}P_{n_S}(x)P_{n_\sigma}(y) \equiv [\mathbf{P}(y) \otimes \mathbf{P}(x)]\mathbf{b}_t,$$

where

$$\begin{aligned} x &= [2S_t - (\bar{S} + \underline{S})]/(\bar{S} - \underline{S}), \\ y &= (2\sigma_t^2 - \bar{\sigma}^2)/\bar{\sigma}^2, \\ \mathbf{P}(x) &= \{P_0(x), \ldots, P_{N_S-1}(x)\}, \\ \mathbf{P}(y) &= \{P_0(y), \ldots, P_{N_\sigma-1}(y)\}, \end{aligned}$$

and $\mathbf{b}_t$ is now a vector with N_SN_σ elements.

Changing variables, expressing the PDE in matrix form, and imposing boundary conditions can be done in the same way as for model BS. The matrix form is $\mathbf{R}_{t+\Delta t} \equiv \mathbf{Qb}_t$, where

$$\mathbf{R}_{t+\Delta t} \equiv -(\Delta t)^{-1}\begin{bmatrix} \tilde{D}(x_0, y_0, \tau - \Delta t) \\ \cdots \\ \tilde{D}(x_{N_S-1}, y_0, \tau - \Delta t) \\ \cdots \\ \tilde{D}(x_0, y_{N_\sigma-1}, \tau - \Delta t) \\ \cdots \\ \tilde{D}(x_{N_S-1}, y_{N_\sigma-1}, \tau - \Delta t) \end{bmatrix},$$

$$\begin{aligned} \mathbf{Q} \equiv &-\left(r + \frac{1}{\Delta t}\right)\mathbf{P}_{XY} + \frac{2r}{\Delta S}(\mathbf{P}_Y \otimes \mathbf{SP}_X^{(1)}) \\ &+ \frac{2}{(\Delta S)^2}(\mathbf{\Sigma}P_Y \otimes \mathbf{S}^2\mathbf{P}_X^{(2)}) \end{aligned}$$

†For further discussion, refer to, for example, Forsyth and Vetzal (2002).

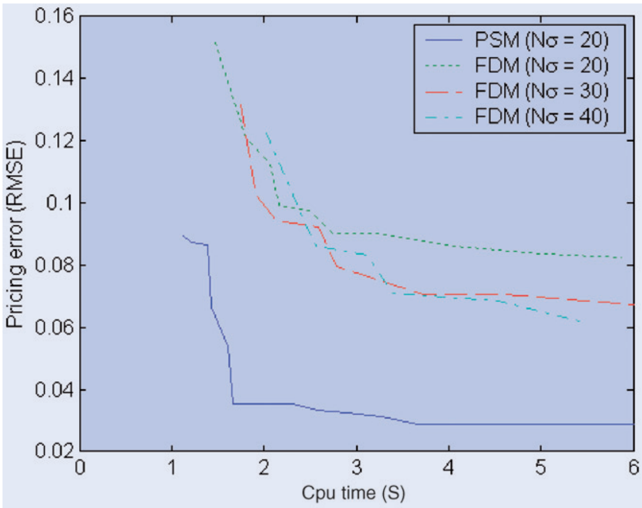

Figure 5. Pricing errors vs. CPU times of PSM and FDM for European puts: SV model.

$$+\frac{2}{\bar{\sigma}^2}[(\alpha \mathbf{I}_{N_\sigma} - \beta\boldsymbol{\Sigma})\mathbf{P}_Y^{(1)}]\otimes \mathbf{P}_X + \frac{2}{\bar{\sigma}^4}\gamma^2(\boldsymbol{\Sigma}\mathbf{P}_Y^{(2)}\otimes\mathbf{P}_X)$$
$$+\rho\gamma\frac{2}{\Delta S}\frac{2}{\bar{\sigma}^2}(\boldsymbol{\Sigma}\mathbf{P}_Y^{(1)}\otimes \mathbf{S}\mathbf{P}_X^{(1)}),$$

and

$$\mathbf{P}_{XY} = \mathbf{P}_Y \otimes \mathbf{P}_X, \quad \mathbf{P}_{XY}^{(k,l)} = \mathbf{P}_Y^{(l)} \otimes \mathbf{P}_X^{(k)}, \quad k,l \in \{0,1,2\},$$
$$\mathbf{P}_X^{(k)} = [P_{j-1}^{(k)}(x_{i-1})]_{ij}, \quad i,j \in \{1,\ldots,N_S\},$$
$$\mathbf{P}_Y^{(l)} = [P_{j-1}^{(l)}(y_{i-1})]_{ij}, \quad i,j \in \{1,\ldots,N_\sigma\},$$
$$\boldsymbol{\Sigma} = \mathrm{diag}\{\sigma_0^2,\ldots,\sigma_{N_\sigma-1}^2\},$$
$$\Delta S = \bar{S} - \underline{S}.$$

To evaluate the procedure, we compare estimates of prices of European options with analytical solutions for 15 contracts having initial moneyness $S_0/X \in \{0.8, 0.9, 1.0, 1.1, 1.2\}$, volatilities $\sigma_0 \in \{0.1, 0.2, 0.3\}$, $\tau = T - t = 0.5$, and $X = 100$. Model parameters are those reported by Eraker *et al.* (2003) for the S&P 500 index. Figure 5 compares accuracies and computation times of PSM and FDM, both with Broadie–Detemple implementation. Each plot shows the correspondence between the *maximum* RMSE value for price steps $N'_S \geq N_S$ and the required CPU time (in seconds);† i.e. $\mathrm{Max}_{N'_S \geq N_S}$ Error(N'_S) vs. Time(N_S). Plots for PSM and FDM are for $N_\sigma = 20$ and $N_\sigma \in \{20, 30, 40\}$ volatility steps, respectively. To make the contest fair, we tried several implementations of FDM and report just the best results.‡ For both methods, computation takes much longer than for the one-factor BS model, but PSM is again far more efficient. PSM yields roughly two or three times more accurate results than FDM at the same computational cost.

We emphasize also that PSM works just as well for pricing American options, where no fast analytical method—or even well-accepted benchmark—is yet available.

4. Pricing under SVJ dynamics

Merton (1976) first introduced jump processes in option pricing models, and Bates (1996) and Scott (1997) combined the jump process with SV. More recently, Duffie *et al.* (2000) proposed models with jumps in both price and volatility, and Eraker *et al.* (2003) have found empirical support for discontinuities in both. Allowing for jumps turns PDEs into partial integro-differential equations (PIDEs), which are harder to solve; however, pseudospectral methods are easily adapted to handle this complication. While we treat only Bates' SVJ model with discontinuities in price alone, also allowing for jumps in volatility would not be hard.

The underlying price under martingale measure $\hat{\mathbb{P}}$ now evolves as

$$\mathrm{d}S_t = (r - \theta\nu)S_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}W_t + S_{t-}U\,\mathrm{d}J_t,$$

where $\{W_t\}_{t\geq 0}$ and $\{J_t\}_{t\geq 0}$ are independent Wiener and Poisson(θ) processes. If $J_t > 0$, the price during $[0, t]$ will have undergone proportional jumps of sizes $\{U_j\}_{j=1}^{J_t}$—random variables independent of $\{W_t\}_{t\geq 0}$ and $\{J_t\}_{t\geq 0}$ and with $\log(1+U)$ i.i.d. as $N[\log(1+\nu) - \xi^2/2, \xi^2]$. Volatility behaves just as in model SV. The martingale property of $\{D(S_t, \sigma_t^2, \tau)/M_t\}_{0\leq t\leq T}$ under $\hat{\mathbb{P}}$ (where M_t is the $\mathcal{F}_0$-measurable value of a money fund) implies the following PIDE:

$$0 = -\tilde{D}_\tau + \left(r - \theta\nu - \frac{\sigma_t^2}{2}\right)\tilde{D}_s + \frac{\sigma_t^2}{2}\tilde{D}_{ss} + (\alpha - \beta\sigma_t^2)\tilde{D}_{\sigma^2}$$
$$+ \gamma^2\frac{\sigma_t^2}{2}\tilde{D}_{\sigma^2\sigma^2} + \rho\gamma\sigma_t^2\tilde{D}_{s\sigma^2} - r\tilde{D}$$
$$+ \theta\{E[\tilde{D}(s_{t-} + u, \sigma_t^2, \tau) \mid \mathcal{F}_{t-}] - \tilde{D}(s_{t-}, \sigma_t^2, \tau)\}, \qquad (6)$$

where $s_t \equiv \log(S_t)$, $u \equiv \log(1+U)$, and $\tilde{D}(s_t, \sigma_t^2, \tau) \equiv D(S_t, \sigma_t^2, T-t)$.

As for model SV, $\tilde{D}(s_t, \sigma_t^2, \tau)$ must be approximated with Chebyshev polynomials defined on $(s_t, \sigma_t^2) \in [\underline{s}, \bar{s}] \times [0, \bar{\sigma}^2]$; however, evaluating the expectation

†The required CPU times are not reported in figures 3 and 4 because they are typically short and of little importance in a simple model such as the BS model.

‡We used direct methods, such as LU decomposition with the sparse matrix specification in Matlab6.5 and Fortran95, and also iterative methods in Fortran95 with several packages: ITPACK (http://rene.ma.utexas.edu/CNA/ITPACK), NSPCG (http://rene.ma.utexas.edu/CNA/NSPCG) and SPARSKIT (ftp://ftp.cs.umn.edu/dept/sparse). Iterative methods with SPARSKIT perform best in this particular problem. Those are the results reported.

requires $\tilde{D}$ to be defined for all $s_t \in \Re$. We do it piecewise, as

$$\tilde{D}(s_t, \sigma_t^2, \tau) = \begin{cases} X e^{-r\tau} - e^{s_t}, & s_t < \underline{s}, \\ \sum_{n_S=0}^{N_S-1} \sum_{n_\sigma=0}^{N_\sigma-1} b_{n_S n_\sigma, t} P_{n_S}(x) P_{n_\sigma}(y), & s_t \in [\underline{s}, \bar{s}], \\ 0, & s_t > \bar{s}, \end{cases}$$

where $x \equiv (2s_t - \underline{s} - \bar{s})/(\bar{s} - \underline{s})$ and $y \equiv (2\sigma_t^2 - \bar{\sigma}^2)/\bar{\sigma}^2$. Gaussian quadrature can now be used to approximate $E[\tilde{D}(s_{t-} + u, \sigma_t^2, \tau) \mid \mathcal{F}_{t-})]$ as a weighted sum of values of $\tilde{D}$ at prescribed values of $s_{t-} + u$. Specifically, at some $s_{t-} = s_k$ and its corresponding collocation point x_k the approximation is

$$\begin{aligned} \sum_{i=1}^{I} w_i \tilde{D}(s_k + u_i, \sigma_t^2, \tau) &= \sum_{i \in F} w_i \sum_{n_S=0}^{N_S-1} \sum_{n_\sigma=0}^{N_\sigma-1} b_{n_S n_\sigma, t} P_{n_S}(x_k + \tilde{u}_i) \\ &\quad \times P_{n_\sigma}(y) \phi(u_i) \\ &\quad + \sum_{i \in G} w_i (X e^{-r\tau} - e^{s_k + u_i}) \phi(u_i) \\ &= \left(\sum_{i \in F} w_i \mathbf{P}_Y \otimes \mathbf{P}_X(x_k + \tilde{u}_i) \phi(u_i) \right) \mathbf{b}_t \\ &\quad + \sum_{i \in G} w_i (X e^{-r\tau} - e^{s_k + u_i}) \phi(u_i) \\ &\equiv \mathbf{Q}(x_k) \mathbf{b}_t + q_t(x_k), \end{aligned}$$

where (i) $\tilde{u} = 2u/(\bar{s} - \underline{s})$, (ii) $\phi(u_i)$ is the $N[\log(1+\nu) - \xi^2/2, \xi^2]$ pdf, and (iii) index sets F and G are defined as $F = \{i : s_k + u_i \in [\underline{s}, \bar{s}]\}$ (equivalently, $x_k + \tilde{u}_i \in [-1, 1]$) and $G = \{i : s_k + u_i < \underline{s}\}$ (equivalently, $x_k + \tilde{u}_i < -1$). Note that vector $\mathbf{Q}$ is fixed from one time step to the next while scalar q_t changes.

To assess the performance, we compare PSM and FDM against analytical solutions for prices of the same 15 European puts as for SV, again with the Eraker *et al.* (2003) parameters. Figure 6 shows the trade-offs between accuracy and CPU time. It appears in this case that PSM outperforms FDM only when high accuracy is required.† However, the gains in accuracy are not small. For example, PSM provides results that are two or three times more accurate than FDM in the same 10 s time period.

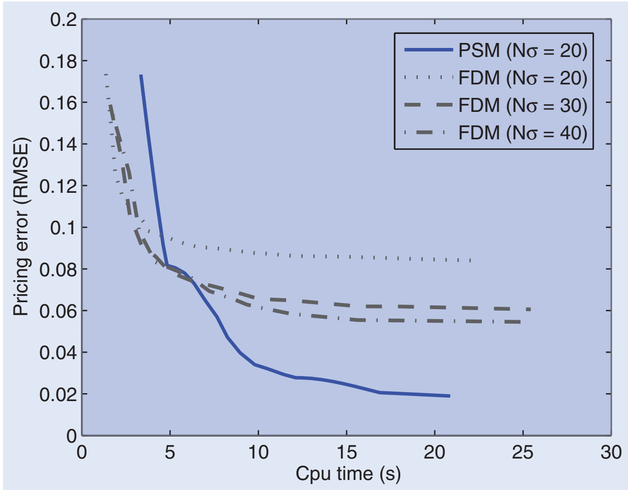

Figure 6. Pricing errors versus CPU times of PSM and FDM for European puts: SVJ model.

5. Pricing with Monte Carlo methods

Thus far the pricing performances of PSM have been provided only with respect to FDM in terms of accuracy and computational cost. In addition to FDM, Monte Carlo (MC) methods can also be major contenders, especially for high-dimensional problems. This section provides some illustrative results of pricing the above three option models with MC simulations. For efficient implementations of MC methods, second-order refinements are employed in discretization of continuous asset price processes instead of the simple Euler approximation.‡ The regression-based approach proposed by Longstaff and Schwartz (2001) is adopted for pricing American as well as European options.§

An RMSE of 0.0233 over four European puts with the same parameters considered in section 2 is given by MC methods implemented with 50,000 simulations (25,000 plus 25,000 antithetic) and 25 time steps (table 1).¶ The standard error of RMSE is calculated as 0.0090 over 100 repetitions. The CPU time was more than six seconds for calculating *one* option contract, which is longer than the time taken by PSM. PSM typically require a CPU time of less than one second for *all* European (and also American) puts with different strikes under BS dynamics. Figure 3 confirms that PSM provides more accurate results with a shorter computation time for European puts under the BS model. Since the RMSE using MC methods increases to 0.2814 for American puts under BS dynamics while PSM yields a RMSE of less than 0.02, as shown in figure 4, PSM produces better results than MC methods also for American puts under BS dynamics.

†Appendix A describes how the SVJ model specified as equation (6) is implemented under FDM. Without the expectation terms, the PDE evaluated at a grid point will be approximated as a linear combination of only a small number of unknown solutions at other grid points. By approximating the PDE at all grid points and rearranging, we can express the simultaneous equation system in matrix form where the coefficient matrix becomes a sparse matrix having only a small number of non-zero elements. However, with the expectation terms arising from, for example, jump processes, the coefficients become non-zero, as shown in appendix A and the coefficient matrix under FDM becomes denser. Therefore, the computation times for FDM are substantially longer with SVJ than with SV because the denser coefficient matrix makes sparse-matrix solutions slower. If volatility jumps as well as price, as in models SVIJ and SVCJ of Eraker *et al.* (2003), then the matrix becomes denser still and computation slows still more.

‡See Glasserman (2004) for the second-order refinement in discretization. In particular, the discretization scheme for the SV model is provided on pp. 356–357.

§We gauge computational costs as if we use the regression-based approach even to price European options by MC simulations because of the necessity to emulate the situation of pricing American options.

¶Of the five contracts with $S/X = \{0.8, 0.9, 1.0, 1.1, 1.2\}$, the regression-based MC method fails to yield the price for $S/X = 1.2$.

Table 1. Pricing performances of MC methods.

	BS (European)	BS (American)	SV (European)	SVJ (European)
No. of simulations	50,000	50,000	50,000	5000
No. of time steps	25	25	25	25
No. of options	4	4	10	10
RMSE	0.0233	0.2814	0.0203	0.0931
SE	0.0090	0.0706	0.0058	0.0164
CPU time (s)	6.76×4	6.76×4	12.49×10	55.16×10

PSM also yields a RMSE of around 0.02 with a CPU time of three seconds, as shown in figure 5, while MC methods produce results with similar accuracy but with much higher computational costs when pricing 10 European puts under SV dynamics.† MC methods require much higher computational costs to price European puts under SVJ dynamics with larger pricing errors, suggestive of a better performance of PSM over MC methods. Taking these results, MC methods turn out to be inferior to PSM in all cases under consideration in this paper.

6. Empirical application

To demonstrate the feasibility of the pseudospectral technique in real applications, we fit models SV and SVJ to actual prices of CBOE-traded American equity puts. As is customary, we use nonlinear least squares to fit the models to a sample of options with different strikes and times to expiration. Whereas solving (5) and (6) values a single option at different levels of S_t, fitting the models would manifestly be easier were it possible to value many options all at once. Fortunately, this can be done by solving the *forward* versions of the PDE and PIDE, which will now be derived.

6.1. *Forward equations*

Noting the symmetric relation of option value to underlying price and strike, Dupire (1994) derived a *forward* PDE under model BS—the 'Dupire equation'—in which the roles of X and S_t are reversed. Carr and Hirsa (2003) developed a forward PIDE for American puts under the variance-gamma model. Here we derive the forward PDE and PIDE for European puts under models SV and SVJ.

The derivative value in both models, $D(S_t,\sigma_t^2,\tau;X)$, is linearly homogeneous in S_t and X, so Euler's theorem implies that $D(S_t,\sigma_t^2,\tau;X)=S_tD_S+XD_X$. Differentiating gives

$$D_{SS}(S_t,\sigma_t^2,\tau;X)=X^2D_{XX}/S_t^2,$$
$$D_{S\sigma^2}(S_t,\sigma_t^2,\tau;X)=(D_{\sigma^2}-XD_{X\sigma^2})/S_t,$$

and substituting these expressions for D, D_{SS}, and $D_{S\sigma^2}$ in backward PDE (5) gives as forward PDE for model SV

$$0=-D_\tau-rXD_X+\frac{\sigma_t^2}{2}X^2D_{XX}+(\alpha-\beta\sigma_t^2+\rho\gamma\sigma_t^2)D_{\sigma^2}$$
$$+\gamma^2\frac{\sigma_t^2}{2}D_{\sigma^2\sigma^2}-\rho\gamma\sigma_t^2XD_{X\sigma^2}.$$

For model SVJ, (i) shift to logs as $s_t=\log(S_t)$ and $x=\log(X)$, (ii) write

$$\tilde{D}(s,x,\sigma^2,\tau)=\tilde{D}_s+\tilde{D}_x,$$
$$\tilde{D}_{ss}(s,x,\sigma^2,\tau)-\tilde{D}_s(s,x,\sigma^2,\tau)=\tilde{D}_{xx}-\tilde{D}_x,$$
$$\tilde{D}_{s\sigma^2}(s,x,\sigma^2,\tau)=\tilde{D}_{\sigma^2}-\tilde{D}_{x\sigma^2},$$
$$\tilde{D}(s+\lambda,x+\lambda,\sigma^2,\tau)=e^{\lambda}\tilde{D}(s,x,\sigma^2,\tau),\lambda\in\Re,$$

and (iii) substitute into (6) to obtain as forward PIDE for model SVJ‡

$$0=-\tilde{D}_\tau-\left(r-\theta\nu-\frac{\sigma_t^2}{2}\right)\tilde{D}_x+\frac{\sigma_t^2}{2}\tilde{D}_{xx}+(\alpha-\beta\sigma_t^2+\rho\gamma\sigma_t^2)\tilde{D}_{\sigma^2}$$
$$+\gamma^2\frac{\sigma_t^2}{2}\tilde{D}_{\sigma^2\sigma^2}-\rho\gamma\sigma_t^2\tilde{D}_{x\sigma^2}-\theta(1+\nu)\tilde{D}$$
$$+\theta\widehat{E}[\tilde{D}(x-u,\sigma_t^2,\tau)\cdot e^u\mid\mathcal{F}_{t-})].$$

6.2. *Calibration*

We use the forward PDE and PIDE to fit models SV and SVJ to prices of American puts on Cisco Systems as of 09:30 on August 4, 2004, at which time the stock price itself was \$20.73. The data comprise averages of bid and ask prices of 67 puts with lifetimes of 11 to 609 days.

†The model parameters used in sections 3 and 4 are also employed in simulations except for γ. To avoid taking the square root of a negative number or dividing by zero, the simulated volatility is replaced by its absolute value at each time step. A large γ may induce a high possibility of replacement, yielding upward bias in pricing errors. Because of the large pricing errors with the original estimates of γ (2.27 for SV and 1.51 for SVJ), the parameter value of γ is taken to be 0.3 for both models. Of 15 option contracts, five are excluded: $(S_0/X,\sigma_0)=(1.2,0.1)$, $(1.2,0.2)$, $(1.2,0.3)$, $(1.1,0.1)$, $(1.1,0.2)$, whose prices the regression-based method fails to provide.

‡To check the validity of these forward equations, we calculated American put prices by solving both forward and backward versions by pseudospectral methods. The solutions are virtually identical.

Table 2. Parameter estimates from Cisco puts.

Parameter	SV	SVJ
α	0.0094	0.0069
β	0.1016	0.0570
γ	0.1963	0.2889
ρ	−0.3668	−0.6420
σ_0^2	0.1121	0.1151
θ		0.2013
ν		0.0001
ξ		0.3230
RMSE	0.1337	0.0474

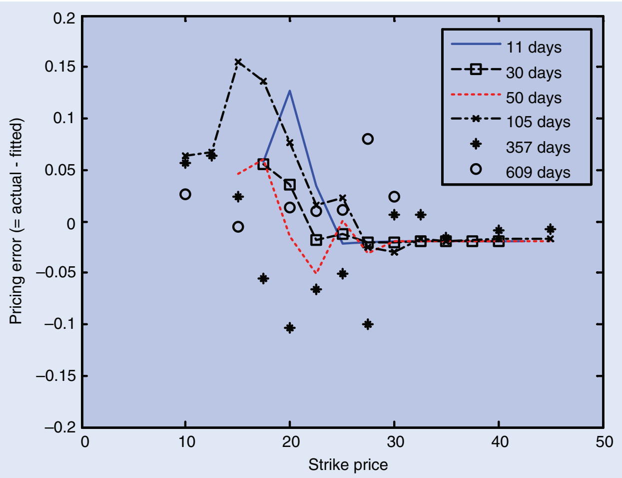

Figure 7. In-sample SVJ pricing errors for Cisco puts.

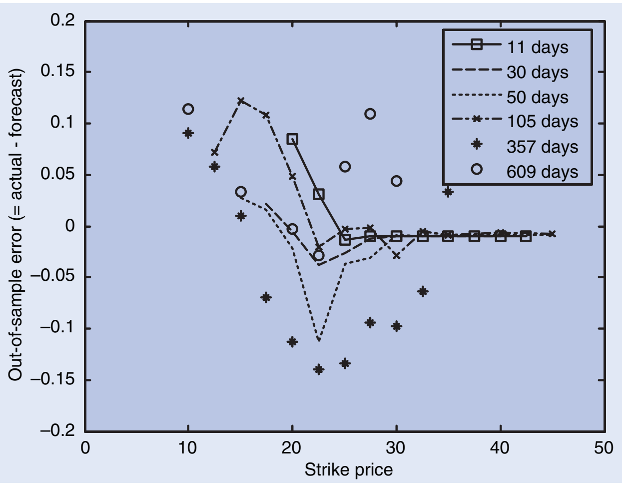

Figure 8. Out-of-sample SVJ pricing errors for Cisco puts.

Riskless interest rates were taken from the Treasury yield curve. Solving the forward equation a single time generates the prices of all options with different strikes but the same maturity. Calibration is by minimizing the sum of squared pricing errors. Table 2 reports estimates for the two models. Note that SVJ gives the better fit. Figures 7 and 8 show in-sample and out-of-sample SVJ pricing errors, respectively, the latter based on 15:30 prices on the same date.

7. Conclusion

Pseudospectral methods give approximate solutions to partial differential equations (PDEs) as linear combinations of orthogonal-polynomial basis functions. The methods are often applied in engineering and the physical sciences but are not widely used by researchers in finance. This paper has shown that pseudospectral methods are competitive with—and in some cases markedly superior to—conventional techniques for pricing options when analytical formulae are unavailable. We have developed specific ways to solve the PDEs or partial 'integro-differential' equations (PIDEs) that arise from no-arbitrage conditions when prices of underlying assets are stochastic-volatility diffusions or jump-diffusion processes. In particular, we have shown how to combine quadrature with pseudospectral methods to solve the PIDEs implied by jump dynamics. Like standard finite-difference techniques, pseudospectral methods are well adapted to pricing American-style and other path-dependent derivatives, even when underlying prices have multiple risk sources; however, we find that the pseudospectral time-accuracy frontier usually dominates that for finite-difference methods. Applying *forward* equations instead of the traditional backward PDEs or PIDEs, we can price options with different strikes all at once. We use the method to fit a stochastic-volatility/jump model to prices of American equity options, estimating risk-neutral parameters by nonlinear least squares.

Many factors affect the choice of numerical methods, but only the computational cost and accuracy have been discussed here. Canuto *et al.* (1988) and Boyd (2001) address the theoretical aspects of pseudospectral methods, including conditions for convergence and stability. For implementation, the programming difficulty is about the same as that for the finite-difference approach. Suh (2005) discusses the flexibility to handle a variety of model specifications and option types, including several well-known exotics.

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References

Bakshi, G., Cao, C. and Chen, Z., Empirical performance of alternative option pricing models. *J. Finan.*, 1997, **52**, 2003–2049.

Bates, D.S., Jumps and stochastic volatility: exchange processes implicit in Deutschemark options. *Rev. Finan. Stud.*, 1996, **9**, 69–108.

Bates, D.S., Post-'87 crash fears in S&P 500 futures options. *J. Econometr.*, 2000, **94**, 181–238.

Black, F. and Scholes, M., The pricing of options and corporate liabilities. *J. Polit. Econ.*, 1973, **81**, 637–654.
Boyd, J.P., *Chebyshev and Fourier Spectral Methods*, 2001 (Dover: New York).
Broadie, M. and Detemple, J., American option valuation: new bounds, approximations, and a comparison of existing methods. *Rev. Finan. Stud.*, 1996, **9**, 211–1250.
Canuto, C., Hussaini, M., Quarteroni, A. and Zang, T., *Spectral Methods in Fluid Dynamics*, 1988 (Springer: New York).
Carr, P. and Hirsa, A., Why be backward?: Forward equations for American options. *Risk*, 2003, **January**, 103–107.
Dempster, M.A.H. and Eswaran, A., Wavelet based PDE valuation of derivatives, In *Proceedings of the Third European Congress of Mathematics*, edited by C. Casacuberta *et al.*, pp. 347–365, 2001 (Birkhauser: Basel).
Dempster, M.A.H., Eswaran, A. and Richards, D.G., Wavelet methods in PDE valuation of financial derivatives, in *Data Mining, Financial Engineering, and Intelligent Agents: Intelligent Data Engineering and Automated Learning—IDEAL 2000 International Conference*, Hong Kong, China, 13–15 December 2000, pp. 215–238.
d'Halluin, Y., Forsyth, P. and Labahn, G., A penalty method for American options with jump diffusion processes. *Numerische Mathematik*, 2004, **97**, 321–352.
Duffie, D., Pan, J. and Singleton, K., Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 2000, **68**, 1343–1376.
Dupire, B., Pricing with a smile. *Risk*, 1994, **7**, 18–20.
Eraker, B., Johannes, M. and Polson, N., The impact of jumps in volatility and returns. *J. Finan.*, 2003, **58**, 1269–1300.
Forsyth, P. and Vetzal, K., Quadratic convergence of a penalty method for valuing American options. *SIAM J. Sci. Comput.*, 2002, **23**, 2095–2122.
Geske, R. and Johnson, H., The American put option valued analytically. *J. Finan.*, 1984, **39**, 1511–1554.
Glasserman, P., *Monte Carlo Methods in Financial Engineering*, 2004 (Springer: Berlin).
Heston, S., A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Finan. Stud.*, 1993, **6**, 327–343.
Ju, N., Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. *Rev. Finan. Stud.*, 1998, **11**, 627–646.
Linetsky, V., The spectral decomposition of the option value. *Int. J. Theoret. Appl. J.*, 2004a, **7**, 337–384.
Linetsky, V., Spectral expansions for Asian (average price) options. *Oper. Res.*, 2004b, **52**, 856–867.
Longstaff, F. and Schwartz, E.S., Valuing American options by simulation: a simple least-squares approach. *Rev. Finan. Stud.*, 2001, **14**, 113–147.
MacMillan, L.W., Analytic approximation for the American put option. *Adv. Fut. Opt. Res.*, 1986, **1**, 119–139.
Madan, D. and Milne, F., Contingent claims valued and hedged by pricing and investing in a basis. *Math. Finan.*, 1994, **4**, 223–245.
Merton, R.C., Option pricing when underlying stock returns are discontinuous. *J. Finan. Econ.*, 1976, **3**, 125–144.
Scott, L., Pricing stock options in a jump diffusion model with stochastic volatility and interest rates: applications of Fourier inversion methods. *Math. Finan.*, 1997, **7**, 413–424.
Suh, S., Orthogonal polynomials in option pricing by the p.d.e. and martingale approaches, Ph.D. Dissertation, 2005 (University of Virginia).
Tavella, D. and Randall, C., *Pricing Financial Instruments: The Finite Difference Method*, 2000 (Wiley: New York).
Wilmott, P., Dewynne, J. and Howison, S., *Option Pricing: Mathematical Models and Computation*, 1993 (Oxford Financial Press: Oxford).
Zvan, R., Forsyth, P. and Vetzal, K., Penalty methods for American options with stochastic volatility. *J. Comput. Appl. Math.*, 1998, **91**, 119–228.

Appendix A

In this appendix we explain how the SVJ model, specifically equation (6), is implemented under FDM. The only difficulty lies in dealing with the expectation term, $E[D(s_t+u,\sigma_t^2,\tau)]$, where u follows normal distribution $N[\log(1+\nu)-\xi^2/2,\xi^2]$. Assume that s_t is discretized into M points $[s_0,\ldots,s_{M-1}]$, and denote $[D_0,\ldots,D_{M-1}]$ as the corresponding put prices. The expectation term will be expressed as

$$E[D(s+u,\sigma_t^2,\tau)] = c + \sum_{j=0}^{j=M-1} a_j D_j.$$

Assuming the solution functional form as

$$\begin{aligned}
&D(s) = X\exp(-r\tau) - \exp(s), \quad \text{for } s < s_0,\\
&D(s) = \frac{D_k - D_{k-1}}{\Delta s}(s - s_{k-1}) + D_{k-1}, \quad \text{for } s \in [s_{k-1}, s_k],\\
&D(s) = 0, \quad \text{for } s > s_{M-1},
\end{aligned}$$

we can express

$$\begin{aligned}
&E[D(s_k+u,\sigma_t^2,\tau)]\\
&\quad = \int_{-\infty}^{\infty} D(s_k+u)\phi(u)\mathrm{d}u\\
&\quad = \int_{-\infty}^{-s_k+s_0} [X\exp(-r\tau) - \exp(s_k+u)]\phi(u)\mathrm{d}u\\
&\qquad + \sum_{j=0}^{j=M-2} \int_{-s_k+s_j}^{-s_k+s_{j+1}} \left[\frac{D_{j+1}-D_j}{\Delta s}(u+s_k-s_j) + D_j\right]\phi(u)\mathrm{d}u,
\end{aligned}$$

where $\phi(u)$ is the pdf of u. From the facts that

$$\begin{aligned}
&\int_{-\infty}^{b} \mathrm{e}^u\phi(u)\mathrm{d}u = \exp\left(\mu + \frac{\xi^2}{2}\right)\cdot\Phi\left(\frac{b-(\mu+\xi^2)}{\xi}\right),\\
&\int_{-\infty}^{b} u\phi(u)\mathrm{d}u = -\xi^2\phi\left(\frac{b-\mu}{\xi}\right) + \mu\Phi\left(\frac{b-\mu}{\xi}\right),\\
&\int_{-\infty}^{b} \phi(u)\mathrm{d}u = \Phi\left(\frac{b-\mu}{\xi}\right),
\end{aligned}$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution and $\mu=\log(1+\nu)-\xi^2/2$, we can derive

$$\begin{aligned}
E[D(s_k+u,\sigma_t^2,\tau)] &= X\exp(-r\tau)\Phi\left(\frac{-s_k+s_0-\mu}{\xi}\right)\\
&\quad - \exp(s_k)\exp\left(\mu+\frac{\xi^2}{2}\right)\\
&\quad \cdot\Phi\left(\frac{-s_k+s_0-(\mu+\xi^2)}{\xi}\right)\\
&\quad + g_1 - g_2 + g_3 - g_4,
\end{aligned}$$

where

$$\begin{aligned}
g_1 = \sum_{j=0}^{j=M-2} \frac{D_{j+1}-D_j}{\Delta s}\Bigg[&-\xi^2\phi\left(\frac{-s_k+s_{j+1}-\mu}{\xi}\right)\\
&+ \mu\Phi\left(\frac{-s_k+s_{j+1}-\mu}{\xi}\right)\Bigg],
\end{aligned}$$

$$g_2 = \sum_{j=0}^{j=M-2} \frac{D_{j+1} - D_j}{\triangle s}\left[-\xi^2 \phi\left(\frac{-s_k + s_j - \mu}{\xi}\right) + \mu\Phi\left(\frac{-s_k + s_j - \mu}{\xi}\right)\right],$$

$$g_3 = \sum_{j=0}^{j=M-2}\left(\frac{D_{j+1} - D_j}{\triangle s}(s_k - s_j) + D_j\right)\Phi\left(\frac{-s_k + s_{j+1} - \mu}{\xi}\right),$$

$$g_4 = \sum_{j=0}^{j=M-2}\left(\frac{D_{j+1} - D_j}{\triangle s}(s_k - s_j) + D_j\right)\Phi\left(\frac{-s_k + s_j - \mu}{\xi}\right).$$

Rearranging terms gives the coefficients

$$c = X\exp(-r\tau)\Phi\left(\frac{-s_k + s_0 - \mu}{\xi}\right) - \exp(s_k)\exp\left(\mu + \frac{\xi^2}{2}\right)\cdot\Phi\left(\frac{-s_k + s_0 - (\mu + \xi^2)}{\xi}\right),$$

$$a_0 = \frac{1}{\triangle s}\left[\xi^2\phi\left(\frac{-s_k + s_1 - \mu}{\xi}\right) - \mu\Phi\left(\frac{-s_k + s_1 - \mu}{\xi}\right)\right] - \frac{1}{\triangle s}\left[\xi^2\phi\left(\frac{-s_k + s_0 - \mu}{\xi}\right) - \mu\Phi\left(\frac{-s_k + s_0 - \mu}{\xi}\right)\right] + \left(-\frac{s_k - s_0}{\triangle s} + 1\right)\left\{\Phi\left(\frac{-s_k + s_1 - \mu}{\xi}\right) - \Phi\left(\frac{-s_k + s_0 - \mu}{\xi}\right)\right\},$$

$$a_j = \frac{2}{\triangle s}\left[-\xi^2\phi\left(\frac{-s_k + s_j - \mu}{\xi}\right) + \mu\Phi\left(\frac{-s_k + s_j - \mu}{\xi}\right)\right] - \frac{1}{\triangle s}\left[-\xi^2\phi\left(\frac{-s_k + s_{j+1} - \mu}{\xi}\right) + \mu\Phi\left(\frac{-s_k + s_{j+1} - \mu}{\xi}\right)\right] - \frac{1}{\triangle s}\left[-\xi^2\phi\left(\frac{-s_k + s_{j-1} - \mu}{\xi}\right) + \mu\Phi\left(\frac{-s_k + s_{j-1} - \mu}{\xi}\right)\right] + \left(\frac{s_k - s_{j-1}}{\triangle s}\right)\left\{\Phi\left(\frac{-s_k + s_j - \mu}{\xi}\right) - \Phi\left(\frac{-s_k + s_{j-1} - \mu}{\xi}\right)\right\} + \left(-\frac{s_k - s_j}{\triangle s} + 1\right)\left\{\Phi\left(\frac{-s_k + s_{j+1} - \mu}{\xi}\right) - \Phi\left(\frac{-s_k + s_j - \mu}{\xi}\right)\right\},$$

where $j \in \{1, \ldots, M-2\}$ and

$$a_{M-1} = \frac{1}{\triangle s}\left[-\xi^2\phi\left(\frac{-s_k + s_{M-1} - \mu}{\xi}\right) + \mu\Phi\left(\frac{-s_k + s_{M-1} - \mu}{\xi}\right)\right] - \frac{1}{\triangle s}\left[-\xi^2\phi\left(\frac{-s_k + s_{M-2} - \mu}{\xi}\right) + \mu\Phi\left(\frac{-s_k + s_{M-2} - \mu}{\xi}\right)\right] + \left(\frac{s_k - s_{M-2}}{\triangle s}\right)\left\{\Phi\left(\frac{-s_k + s_{M-1} - \mu}{\xi}\right) - \Phi\left(\frac{-s_k + s_{M-2} - \mu}{\xi}\right)\right\}.$$