Valuation of American Options: Monte-Carlo Simulation and Mathematical Approximation Methods*

DAVID ANIMANTE

ABSTRACT

The use of American style equity options as hedging instrument has gained currency in recent times. This phenomenon devolves from the ever-expanding need by individuals, corporations and governments to hedge away their financial risks and the clarion call for derivative securities that give the holder increased flexibility in exercise. Nevertheless, pricing American options is complex and there exists no analytic solution to the problem except a profusion of approximation and finite difference techniques. Indeed, many researchers have shown that these methods cannot handle multifactor situations where the underlying asset follows a jump-diffusion process and where the derivative security depends on multiple sources of uncertainty such as stochastic volatility, stochastic interest rate among others. Monte-Carlo simulation techniques therefore developed out of the search for a pricing formula that has the capacity to accommodate all forms of uncertainty and at the same time able to produce speedy and accurate results. Some scholars at first rejected these techniques as yielding inaccurate results but in recent times, many researchers have demonstrated the efficacy of Monte-Carlo simulation in option pricing. The aim of this study is to assess the effectiveness of Monte-Carlo simulation methods in comparison with other option pricing techniques. To achieve this objective, the research builds an algorithm to compute Call and Put prices based on a wide range of input parameters. It also develops a model where volatility or interest rate is stochastic and a deterministic function of time. The results indicate that Monte-Carlo simulation techniques produce option values and exercise boundaries that are very similar to the Binomial, Barone-Adesi and Whaley as well as the Explicit Finite Difference methods. The results also show that the stochastic volatility and stochastic interest rate models yield slightly different but more accurate results. Consequently, the study recommends simulation techniques that incorporate multiple sources of uncertainty simultaneously for fast, efficient and more accurate option pricing.

JEL classification: D9, E3, E6, G11, G17

Keywords: Put option, call option, Monte-Carlo simulation, finite difference methods, mathematical approximation, antithetic variate, control variate, binomial trees, stochastic volatility, exercise boundary.

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CHAPTER ONE
INTRODUCTION

1.1. Background and Overview

Derivative securities have grown in importance in recent times as individuals, corporations and government agencies become more cognisant of their potential financial risks and seek ways of limiting, if not, eliminating their exposures. The task of hedging risk exposures becomes increasingly difficult when using American style options as hedging instrument, although these are now of superlative importance as financial market players seek to exercise their rights earlier than maturity. Valuing American options is difficult and there currently exists no closed-form analytical solution. The difficulty mainly stems from the early exercise feature and the need to ascertain the appropriate boundary conditions of the option. Valuation methods currently used comprise binomial and numerical methods. The numerical methods are either Approximating, Monte Carlo, Finite difference procedures among others. All these methods assume perfect market conditions of no interest rate asymmetry, no transaction costs and no income taxes. For many years, some held the view that Monte-Carlo simulation led to inaccurate results and therefore not a suitable tool for valuing American options while others believed the method was just as good as other numerical procedures.

In this research, I adopt different numerical procedures including Monte Carlo and non Monte Carlo methods to the valuation of American options under different exercise conditions. I analyse and compare the various results in order to reach a conclusion about whether other procedures produce better results than Monte Carlo simulation.

1.2 Justification of Research Topic

The use of financial derivatives including equity options has assumed greater importance as a means of hedging away financial risks. These hedging instruments continue to increase in variety and complexity with the passing of time. This phenomenon devolves from the ever expanding need to simplify and diversify the repertoire of hedging tools for financial market participants. The calls for increased flexibility give credence to

American style as well as many hybrid derivatives. American options enable the holder to exercise his right at any time before and including maturity. This contrasts with European options which can only be exercised on the maturity date. Thus, American options have assumed greater popularity. Nevertheless, while a closed-form analytical solution exists for the valuation of European options, the same cannot be said of American options. Consequently, Researchers and financial experts relentlessly continue to seek a solution to this problem which has led to a profuse of financial engineering methods all geared toward pricing American options and their hybrids. Current methods include Differential equations, Approximation and Monte Carlo simulation approaches. Of these three categories, the Monte Carlo methods are easier to understand and therefore used by many. However, there has been a longstanding debate regarding the appropriateness of their use, with some doubting their accuracy.

I do not take part in this debate but try to test whether or not the Monte Carlo methods actually do give option values that are accurate and comparable with other methods. Achieving this objective will help throw more light on a topical issue in Finance which has engendered unparalleled academic debate in current Mathematical Finance research.

1.3 Research Objectives

The research will enable me to value American options using different Monte Carlo methods and compare the results with solutions obtained from other methods such as Differential equations and Approximating methods. Specifically, it will enable me to examine the following important questions:

i) How to price American options using Monte Carlo simulation methods.

ii) How to price American options using Approximation and Finite difference methods.

iii) Do the different methods produce the same results?

iv) Do the Monte Carlo methods have any advantages or disadvantages relative to the other methods?

v) What improvements are necessary to make American option valuation easier and more accurate?

1.4 Organisation of the Study

This study is presented in five chapters. In this first chapter, the topic is introduced with the problems and objectives clearly and succinctly stated. In Chapter two, there is a review of some of the existing literature works on Options and other derivative securities in general and American options in particular. This review provides a frame of reference for the examination of the data. In Chapter three, the methodology used to achieve the set objectives is outlined and the procedures for the data collection and analysis are enumerated. Chapter four presents the results and significant findings of the study. Chapter five discusses the results, the implications and weaknesses of the study and ends with conclusions and recommendations arising out of the research.

The next section focuses on the literature review.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

In this chapter, I review some of the published literature on equity options. In particular, section 2.2 examines the Binomial Tree approaches and their underlying assumptions. Section 2.3 discusses stock price diffusion process and how the Black-Scholes method helps to value options in continuous time. In section 2.4 I discuss published information on American option pricing and Monte-Carlo simulation methods used in finance while in section 2.5 I consider other numerical methods. Section 2.6 focuses on the extension of the basic pricing model to include stochastic volatility while section 2.7 considers the case where the risk free interest rate is random.

2.2 Binomial Tree Approaches

The most popular method of option pricing is the binomial tree approach suggested by Sharpe (1978) and extended by Cox, Ross and Rubinstein (1979). This method is simplest when used for the pricing of Vanilla European options. European options comprise the class of derivative securities that can be exercised only at the maturity date of the contract. The alternative group of derivative securities or options is called American. The name has nothing to do with geography or origin but merely refers to options which unlike European ones can be exercised at any time before and including the maturity date of the contract. When the Binomial tree is used for American option valuation, it is necessary to work backwards on the Tree recursively in order to determine at which states it is more optimal to exercise the option earlier than maturity. Identifying these early exercise states enables one to determine what is technically called the early exercise boundary. This is so termed because it represents stock price level which produces the optimum payoff and hence the highest option price. This forms a payoff boundary beyond which it is no longer optimal to exercise the option before maturity. Although, the Binomial tree approaches are effective at pricing options, the final price has a diminished accuracy. This is because the approach assumes that the stock price moves in discrete steps and changes in an orderly fashion moving upwards or downwards.

However, it has long been shown by many early writers that the stock price process is far from orderly and hardly follows a discrete movement process. This means that prices do not jump in discrete steps from one value at a time to another. Rather they move randomly in a diffusion process in continuous time. Continuous price movement or diffusion process means that prices change continuously at every infinitesimal and imperceptible change in time to the extent that there is hardly any measurable difference between the current time and the immediately preceding one.

2.3 Stock Price Process and Continuous Time Valuation

Stock price diffusion or random process is akin to the kind of movement discovered by Brown (1828) which was later referred to as the Brownian motion. Although Einstein (1956) introduced this in his theory on "Relativity" and thus brought the process to the attention of Physicists, the process was already known to Financial Economists since Bachelier (1900) first introduced the idea in option pricing. Wiener (1923) observed that the stock price follows a standardised Brownian motion which is well known in finance as the geometric Brownian motion or Wiener process. The geometric Brownian motion is a standard Brownian motion from which drift has been eliminated in order to ensure that it is impossible for the stock price to drift to zero or negative as there is no such possibility in reality. Due to the randomness in the stock price process, it has always been difficult to predetermine the future price of a stock and hence difficult to value a derivative security whose value is contingent upon the future price of the underlying stock until the revolutionary discovery by Black and Scholes (1973). They found an effective way of valuing options in continuous time process in an intuitive and rational way which assumes risk neutrality. They envisaged a riskless world of no transaction costs, no taxes and no information asymmetry. In such circumstances the only applicable return on investment is the risk free rate which is also the interest rate for borrowing money. They used this simplification not to blur their view of reality but as a mathematical construct in order to make it possible to value derivative securities whose value depends on the future price of an asset that change in value randomly with time.

Merton (1973) updates Black and Scholes (1973) valuation process in order to take into account the existence of continuous dividend payments. Thus, Black-Scholes-Merton framework proved invaluable for pricing European options. Nevertheless, it was not possible to adopt their procedure to the valuation of options with early exercise features such as American options, but their article was an eye opener and set the pace for many others to try to search for different ways of valuing American style options. However, pricing American options proved to be an arduous task as the need to ascertain whether or not it was optimal to exercise the option earlier than maturity and to clearly delineate the so-called early exercise boundary introduced added complexity and presented additional challenge.

2.4 American Option Pricing and Monte Carlo Simulation

Boyle (1977) was the pioneering work in the quest for a numerical solution to the American option pricing problem. He develops a procedure that tries to simulate the price paths of the underlying stock as well as the respective option payoffs by the application of Monte-Carlo simulation. He shows that the approach can be used to obtain numerical solutions to option prices. But while he was able to apply his simulation to European options, he could not do same for American options although he mentioned that it was possible to do that in the case of dividend paying stocks.

Merton (1973) suggests that dividend paying stocks will have to be exercised just before the dividend date if the stock price exceeds a particular critical value and that as part of the valuation process it was necessary to determine this price at each date by calculating a series of option prices. This insight was however lost on many researchers until recently. This led scholars such as Hull and White (1977) to conclude that Monte-Carlo simulation could only be used for pricing European but not American options. According to them, this is because it is impossible to know at any time whether early exercise is optimal when a particular stock price is reached. This assertion has however been found to be untrue as Grant *et al* (1997) demonstrate that it is possible to apply simulation to the valuation of both American and Asian options by linking both the forward-moving simulation and the backward-moving recursion of dynamic programming through an iterative search process.

For their part Longstaff and Schwartz (2001) present a simple approach for finding an approximate value of an American option by simulation. They observe that linear regression can be used to estimate expected payoffs. At any exercise time the optimal exercise value depends on the higher of two possible outcomes; the payoff when the option is immediately exercised and the payoff of a later exercise. According to them, this conditional expectation can be estimated through a cross-sectional least squares regression and the fitted value from the regression can be used as a basis for accurate valuation of American option. The authors refer to this technique as the Least Squares Monte-Carlo (LSM) approach. An important feature of this Approach is that while it is impossible to use Finite difference techniques in multifactor situations where the underlying stock price follows a jump or jump-plus-diffusion process, their approach works well under all conditions including both path-dependent and multifactor situations. Again, unlike other methods, the LSM approach includes in the regression only paths for which the option is in-the-money (that is when it is beneficial to the option holder to exercise the option). This increases the efficiency of computational time.

Very recently, however, Choi and Song (2006) have proposed an improvement upon the Least Squares Monte-Carlo method. Firstly, their method improves upon the R-squares from the regressions by using either weighted regression with the same regressors or new regressors which are related to the discount factor from the current decision to exercise time. Secondly, they are able to achieve greater computational speed without sacrificing convergence by terminating early during the backward recursion and by decreasing the number of observations. However, any improvement achieved is proportionately far less than the complexity inherent in their approach. As a result, the LSM still remains the method of choice among practitioners.

2.5 Other Numerical Methods

Besides Monte Carlo simulation approaches, there is a profusion of other numerical methods currently adopted by researchers and practitioners for the valuation of equity options. Brennan and Schwartz (1977) propose two Finite difference methods based on a technique that values a derivative by solving the differential equation that the derivative satisfies. The Explicit finite difference method is usually easy to implement in a program but more delicate to execute. The Implicit finite difference method is more robust at execution but their use requires several systems of simultaneous equations to be solved in order to obtain a solution to the problem. As a result of this complexity, The Explicit finite difference method is more popular as it enables accurate and speedy valuation of derivative securities whiles at the same time avoiding the difficulty inherent in The Implicit finite difference procedure. Despite the effectiveness of Finite difference methods in option pricing, Fu *et al* (2001, p2) observe:

> *"In general (not just for American-style derivatives), the computational speed of these methods is significantly better than that of simulation methods for simple models and contracts. However, the major drawback of these methods is that they can often only handle one or two sources of uncertainty: for more state variables, they become computationally prohibitive, with computation times typically increasing exponentially with the number of state variables."*

Consequently, the use of simulation methods is still popular among option traders as it enables them to handle the pricing of complex derivative securities that depend on several sources of uncertainty namely asset prices, stochastic volatility and stochastic interest rates among others.

Barone-Adesi and Whaley (1987) as well as MacMillan (1986) on the other hand have proposed an efficient analytic approximation for American put options (that can also be adapted for call options) based upon a mathematical manipulation of the Black-Scholes-Merton equation. The Barone-Adesi and Whaley method has been a very useful tool, but the approximation works best only for short and very long maturities. The formula is less accurate for intermediate maturities of a couple of years, which nonetheless, are now very common for over-the-counter option contracts. Ju and Zhong (1999) propose what in their view is an improvement upon the Barone-Adesi and Whaley method. However, it remains to be seen the extent to which their method can withstand further scrutiny.

2.6 Stochastic Volatility

All the methods I have considered so far rest on the assumptions of constant volatility, a constant interest rate and no transaction costs. However, several researchers have tried extensions that relax the assumptions on which the theory has been based. Merton (1973) dropped the assumption of constant volatility and showed how the Black-Scholes formula can be extended to cover the situation in which the volatility is a deterministic function of time. This is thought to be more realistic as the strong assumption of constant volatility is known not to be true (Rubinstein, 1983). Cox and Ross (1976) and Rubinstein (1983) have solved the problem for the case when the volatility is a function of the underlying security price. While some scholars like Scott (1987) dispute this assumption and tries to show that the hypothesis that stock returns are distributed independently over time can be rejected, many including Bodurtha and Courtadon (1984), Heston (1993) and Hull and White (1987) support the hypothesis of stochastic volatility. Considering these results, it seems reasonable to model volatility as another stochastic variable.

2.7 Stochastic Interest Rate

Merton (1973) assumed that the price of a default-free discount bond, which matures and pays one dollar on the same date that the option expires, is a function of a stochastic interest rate. Subsequently, others such as Cox *et al* (1985) as well as Vasicek (1977) have derived closed-form valuation formulas for equity options under the assumption that the interest rate follows a mean-reverting Ornstein-Uhlenbeck process. Therefore, in this study, I have tried an extension of the basic option pricing model to include a situation where the risk free interest rate is assumed to be stochastic.

2.8 Conclusion

The review has shown that whiles the standard method for pricing European options is the Black-Scholes-Merton formula, the Binomial lattice tree approach can be used for the valuation of American options and that increased accuracy derives from setting the number of time steps to very large values. In addition, there are other Approximation and Differential equations methods that also give accurate option values. However, the Monte-Carlo simulation methods enable path dependent options to be priced in situations where there are several state variables such as stochastic volatility, stochastic interest rate and transaction costs among others.

In the next chapter I have described nine different option pricing methods comprising Monte Carlo simulations, Approximation as well as Differential equation approaches.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

This Chapter outlines the methodology to be employed in this research. The research design, data collection and analysis techniques as well as the issue of ethics will be discussed. This will agree with Guba and Lincoln's (1994) contention that the issue of paradigm is a fundamental starting point to guide research enquiry, and should come before the choice of methods. In section 3.2 nine quantitative approaches (including Black-Scholes) used for the investigation and upon which the analyses are based have been discussed. Sections 3.3 and 3.4 presents the models when there are stochastic volatility and stochastic interest rate respectively.

3.2 Option Valuation Methods

This study seeks to implement eight different American option valuation methods and compare the results. In particular, it seeks to determine whether the Monte-Carlo methods give results that are analytically close to the fair option price. The fair option price in this case is the results given by the Black-Scholes method (European options) and the Binomial model (for American options). The eight American methods (excluding Black-Scholes) implemented are as shown in Table 3.1.

Table 3.1: Classification of American Option Pricing Methods

Monte Carlo Simulation Methods	Approximation Methods		Differential Equation Methods
	Binomial Approximation	**Quadratic Approximation**	
Simple Monte Carlo	Binomial Lattice Tree	Barone-Adesi and Whaley	Explicit Finite Difference
Antithetic Variate Technique		Ju and Zhong	
Control Variate Technique			
Least Squares Monte Carlo (LSM)			

3.2.1 The Simple Monte-Carlo Method

In order to value American options using this method, I first computed the value of European options using Visual Basic for Application (VBA) programming and then extended the computations to determine optimum stopping times for American options. The computations were based on the theory that the stock price(S) follows the stochastic process $dS = \mu S dt + \sigma S dz$ where $\mu = r - d$, and where r = the risk free rate of interest and d = the continuous dividend yield on the stock, σ (sigma) = the volatility of the stock price and z = a stochastic process or standard Brownian motion. I simulated the path for the stock price using the discrete version of the formula $\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$ where ε is a random drawing from a normally distributed sample N (0, 1). This equation is only true in the limit as $\Delta t \to 0$. In that case the discrete version converges to the continuous model and because it is easier to manipulate the log transform, I used

$$\ln(S + \Delta S) - \ln(S) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t} .$$

I tried to simulate many paths for the stock price and obtained several different terminal values. I used these to estimate discounted payoffs and then took the average. The simulation procedure is as follows: For $i = 1$ to N , I computed an N (0, 1) sample ε_i and then estimated S_i and P_i as follows: $S_i = S_o e^{[(r-d-0.5*\sigma^2)T+\sigma\varepsilon_i\sqrt{T}]}$ and $P_i = \max(S_i - X, 0)$

where S_i represents the stock prices obtained for each ε_i , and P_i the payoffs. The average terminal payoff for N simulations is $M = \frac{1}{N}\sum_{i=1}^{N} P_i$ and the European call value C_0 is simply the payoff discounted to time 0 as follows:

$$C_0 = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} P_i$$

Assuming that an American option can be exercised at K (equally spaced) discrete times $0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_k = T$ then for each exercise time t , I estimated

$S_{ti} = S_0 e^{[(r-d-0.5*\sigma^2)t+\sigma\varepsilon_i\sqrt{t}]}$ and $P_{ti} = \max(S_{ti} - X, 0)$ for ε_i and for each t. The average payoff at each time is $M_t = \frac{1}{N}\sum_{i=1}^{N} P_{ti}$

Starting from the penultimate exercise time t_{k-1}, I worked back recursively to estimate the American call value, C_t at each time t by maximising the discounted terminal payoff M (assuming no early exercise) using the formula $C_t = \max(e^{-r(T-t)}M, M_t)$ until the value at time $t = 0$ is obtained.

In order for the estimated average discounted payoff to converge to the fair option price it is necessary to minimise the standard error, $SE = \frac{\sigma}{\sqrt{N}}$ of the estimate where

$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(C_{Ti} - \hat{C})^2}$ and where C_{Ti} is the i-th simulated payoff and $\hat{C}$ is the call price at time $t = 0$. SE is the standard deviation of the discounted payoff divided by the square root of the number of simulations. Thus SE reduces as N increases. Doubling N causes the SE to reduce to halve. $SE \to 0$ as $N \to \infty$ and this leads to improved results.

Before we consider the second Monte-Carlo Simulations method, it is important to discuss the Early Exercise Boundary and how this has been calculated in this study.

3.2.1.1 Early Exercise Boundary

To determine the early exercise boundary let the payoff function for European call option be $f(T) = C_T(S(T), T)$. The payoff for American option could then be at any optimally chosen stopping time, τ such that $f(\tau) = c\tau(S(\tau), \tau)$ where $t < \tau < T$ and $0 \le t < \tau$. It is then necessary to estimate the function $f(S(t)) = \max\left(e^{\tau}(-\tau(\tau - t))c\tau(S(\tau), \tau)\right) = S$. This is important because under that condition, we have $S = S^*(t)$ where $S^*(t)$ is the early exercise boundary and S the free boundary. In order to estimate $S^*(t)$ the following equations were used for American Call and Put Options respectively:

For Call Options:

$$S^*(t) = \begin{cases} X + c_1\sqrt{(T-t)\log\left[\frac{1}{T-t}\right]} & \text{If } \ d > r \\ X + c_2\sqrt{(T-t)\log\left[\frac{1}{T-t}\right]} & \text{If } \ d = r \\ \left(\frac{r}{d}\right)\left(X + c_3\sqrt{T-t}\right) & \text{If } \ 0 \le d < r \end{cases}$$

For Put Options:

$$S^*(t) = \begin{cases} X + c_1\sqrt{(T-t)\log\left[\frac{1}{T-t}\right]} & \text{If } \ 0 \le d < r \\ X + c_2\sqrt{(T-t)\log\left[\frac{1}{T-t}\right]} & \text{If } \ d = r \\ \left(\frac{r}{d}\right)\left(X + c_3\sqrt{T-t}\right) & \text{If } \ d > r \end{cases}$$

Where, c1, c2 and c3 are constants that depend on σ(sigma), d (dividend yield), and r (risk free rate) - (Caflisch and Chaudhary, 2004). The second Monte-Carlo method adopted in this study is the Antithetic Variate Technique which I consider next.

3.2.2 The Antithetic Variate Technique

This method is an extension of the Simple Monte-Carlo Method described above except that here it is necessary to calculate two values of the derivative. The first value is calculated in the usual way as described above while the second is done by setting the random variable to negative. Thus each ϵ_i has a corresponding -ϵ_i which gives two sets of payoffs for each simulation. For i=1 to N

I computed an N (0, 1) sample ε_i and then estimated S_i, $\overline{S_i}$, P_i, $\overline{P_i}$ and A_i as follows:

$$S_i = S_o \, e^{[(r-d-0.5*\sigma^2)T+\sigma\varepsilon_i\sqrt{T}]} \quad \text{and} \quad \overline{S_i} = S_o \, e^{[(r-d-0.5*\sigma^2)T-\sigma\varepsilon_i\sqrt{T}]}$$

$$P_i = \max(S_i - X, 0) \qquad \text{and} \qquad \overline{P_i} = \max(\overline{S_i} - X, 0)$$

Where S_i and $\overline{S_i}$ are the stock prices obtained when I use ε_i and $-\varepsilon_i$ respectively and P_i and $\overline{P_i}$ are the corresponding payoffs. The average payoff for each simulation is $A_i = \frac{1}{2}(P_i + \overline{P_i})$ and that for N simulations was computed as $M = \frac{1}{N}\sum_{i=1}^{N} A_i$

Assuming that an American option can be exercised at K (equally spaced) discrete times $0 < t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_k = T$ the optimal stopping strategy for all τ where $t < \tau \leq T$ is given as follows:

$$S_{ti} = S_o\, e^{[(r-d-0.5*\sigma^2)t+\sigma\varepsilon_i\sqrt{t}]} \qquad \text{and} \qquad \overline{S_{ti}} = S_o\, e^{[(r-d-0.5*\sigma^2)t-\sigma\varepsilon_i\sqrt{t}]}$$

$$P_{ti} = \max(S_{ti} - X, 0) \qquad \text{and} \qquad \overline{P_{ti}} = \max(\overline{S_{ti}} - X, 0)$$

for ε_i and $-\varepsilon_i$ respectively and for each τ. For the Antithetic algorithm, the average payoff is $M\tau = \frac{1}{N}\sum_{i=1}^{N} A_i\tau$ where $A_i\tau = \frac{1}{2}(P_i\tau + \overline{P_i}\tau)$. Starting from the penultimate exercise time t_k I worked back recursively to estimate the American call value just as was done under the Simple Monte-Carlo procedure. (Please see Appendix A for the VBA pricing algorithm).

The Antithetic Variate is expected to improve the distribution of the payoffs to symmetric and to make option pricing more efficient than the Simple Monte-Carlo as a result of significant reduction in variance as well as in the standard error. Another variance minimisation technique that can be used to improve upon the Simple Monte-Carlo is the Control Variate.

3.2.3 The Control Variate Technique

This method involves using the Black-Scholes figure that can be calculated analytically and then adjusting it to arrive at another figure that has no analytically tractable value.

The following is the procedure as adopted in this study. Let V_A be the numerical price of an American option calculated by N simulations. Using the same N simulations to calculate a European option value V_E that has analytically tractable Black-Scholes value, V_B, an algorithm based on VBA programming was used to estimate the Control Variate option price V_A^{CV} using the formula

$$V_A^{CV} = V_A + (V_B - V_E)$$

Furthermore, in this study two attempts were made to improve the efficiency of the Control Variate. First, the values V_A and V_E used were computed based on the Antithetic Variate technique instead of the Simple Monte-Carlo and secondly, a variance minimising parameter β was introduced following Tian and Burrage (2003) so that we have $V_A^{CV} = V_A + \beta\ (V_B - V_E)$ where $\beta = \dfrac{Cov(V_A, V_E)}{Var(V_E)}$.

The variance of the estimator $Var(V_A^{CV}) = Var(V_A) + \beta^2 Var(V_E) - 2\beta Cov(V_A, V_E)$ was therefore significantly lower and represents significant improvement upon the two preceding methods.

Having considered three Monte-Carlo methods let us turn our attention to a fourth technique that although also a Monte-Carlo procedure is significantly different from those that we have already discussed. This is the Least Squares Monte-Carlo (LSM) method.

3.2.4 The Least Squares Monte Carlo

This approach to American option valuation uses a Least Squares interpolation to obtain the conditional expectation function of continuing to hold the option at each exercise decision time. The procedure is described as follows:

Let i=1 to N where N is the number of random stock price paths generated, let n=1 to k where k is the number of equally spaced discrete times interval from time 0 to T and let Δt be each discrete time space. Then $t_n = n\Delta t$ is the exercise decision time for each price path. For i=1 to N

For n=k to 0 Step-1

$P_{i,k} = \max(S_{i,k} - X, 0)$ is the terminal payoff for path i at time k. At time $t_{i,k-1}$ if $S_{i,k-1} >$ X then estimate dependent variable $y_{i,k-1} = e^{-r*n\Delta t} P_{i,k}$,the independent variables $x_{i,k-1} = S_{i,k-1}$ and $x^2{}_{i,k-1} = S^2{}_{i,k-1}$ and regress $y_{i,k-1} = e^{-r*n\Delta t} P_{i,k}$ across section on the functions $x_{i,k-1} =$ $S_{i,k-1}$ and $x^2{}_{i,k-1} = S^2{}_{i,k-1}$ to estimate the conditional expectation function $y_{k-1} = \alpha + \beta x_{k-1} + \beta x^2{}_{k-1}$ and calculate fitted value $\hat{y}_{k-1}$. If $S_{i,k-1}$-X $> \hat{y}_{k-1}$ then early exercise for $P_{i,k-1} = S_{i,k-1} - X$ and set $P_{i,k} = 0$ else $P_{i,k-1} = 0$. At time $t_{i,k-2}$ if $S_{i,k-2} > X$ then regress $y_{i,k-2} = \max(e^{-r*2\Delta t} P_{i,k}, e^{-r\Delta t} P_{i,k-1})$ on $x_{i,k-2} = S_{i,k-2}$ and $x^2{}_{i,k-2} = S^2{}_{i,k-2}$ and fit the resulting function to obtain $\hat{y}_{k-2}$.

Next If $S_{i,k-2}$- X> $\hat{y}_{k-2}$ then early exercise for $P_{i,k-2} = S_{i,k-2} - X$ and set $P_{i,k-1} = P_{i,k} = 0$ Else $P_{i,k-2} = 0$. This process is repeated until $t_{i,0}$ is reached.

Generally, for each exercise time $t_{i,n}$ where $n < k$ (k is the option maturity and n=1 to k), if $S_{i,n}$>X (option is in the money), we estimate $y_{i,n} = \max(e^{-r*n\Delta t} P_{i,n})$ and regress it on the functions $x_{i,n} = S_{i,n}$ and $x^2{}_{i,n} = S^2{}_{i,n}$ to obtain the conditional expectation function $y_n = \alpha + \beta x_n + \beta x^2{}_n$. We then calculate the fitted value of the regression, $\hat{y}_n$ and finally, $P_{i,n} = S_{i,n} - X$ if it is greater than $\hat{y}_n$ else $P_{i,n} = 0$

3.2.5 Approximation Methods

The two approximation methods implemented in this study were the Quadratic Approximation and the Binomial Tree methods. The Quadratic approximations are discussed first followed by the Binomial.

3.2.5.1 Barone-Adesi and Whaley Quadratic Approximation

This is the second most popular method of pricing American options after the Binomial lattice tree. This is because it is much easier to program compared with other methods and it provides quick and fairly accurate results. To describe the approach;

Let S, X, r, d, σ be stock price, strike price, risk free interest rate, continuous dividend rate and stock price volatility respectively and $b = r - d$ be the stock holding cost. Then American call $C(S_0)$ is given by the expression:

$$C(S_0) = \begin{cases} c(S_0) + A\left(\dfrac{S_0}{S_0^*}\right)^q & \text{If } S_0 < S_0^* \\ S_0 - X & \text{If } S_0 \geq S_0^* \end{cases}$$

where the early exercise boundary S_0^* is the figure that solves the equation

$$S_0^* - X = c(S_0^*) + \frac{1 - e^{(b-r)T} N(b_1)}{q} S_0^*$$

$c(S_0)$ is the European call value given by the Black-Scholes-Merton formula; $A\left(\dfrac{S_0}{S_0^*}\right)^q > 0$ is the early exercise premium; $A = \dfrac{1 - e^{(b-r)T} N(b_1)}{q} S_0^*$

$$b_1 = \frac{\ln \dfrac{S_0^*}{X} + \left(b + \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad q = \frac{-\left(\dfrac{2b}{\sigma^2} - 1\right) + \sqrt{\left(\dfrac{2b}{\sigma^2} - 1\right)^2 + \dfrac{8r}{\sigma^2 h}}}{2}$$

and $\quad h = h(T) = 1 - e^{-rT}$

where $c(S_0^*)$ is the Black-Scholes-Merton European call value when S_0^* is the stock price input.

For American put option $P(S_0)$, the formula is similar as follows:

$$P(S_0) = \begin{cases} p(S_0) + A\left(\dfrac{S_0}{S_0^*}\right)^q & \text{If } S_0 > S_0^* \\ X - S_0 & \text{If } S_0 \leq S_0^* \end{cases}$$

where

$$A = \frac{1 - e^{(b-r)T} N(-b_1)}{q} S_0^* \qquad b_1 = \frac{\ln \frac{S_0^*}{X} + \left(b + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

$$q = \frac{-\left(\frac{2b}{\sigma^2} - 1 \right) - \sqrt{\left(\frac{2b}{\sigma^2} - 1 \right)^2 + \frac{8r}{\sigma^2 h}}}{2} \qquad \text{and} \qquad h = h(T) = 1 - e^{-rT}$$

The early exercise boundary S_0^* is the figure that solves the equation

$$S_0^* - X = c(S_0^*) + \frac{1 - e^{(b-r)T} N(-b_1)}{q} S_0^*$$

Where $p(S_0^*)$ is Black-Scholes-Merton European put value based on S_0^* as the stock price input. (Please see Appendix B for the VBA pricing algorithm).

3.2.5.2 Ju and Zhong Modified Quadratic Model

This analytical approximation is expected to improve upon the Barone-Adesi and Whaley model. According to Ju and Zhong (1999), it yields good results for intermediate maturity options for which the Barone-Adesi and Whaley sometimes performs poorly. To implement this model: Let S, X, r, d, σ be stock price, strike price, risk free interest rate, continuous dividend rate and volatility respectively and let $b = r - d$ be the cost of carry. The value of an American option V_A is given by:

$$V_A(S) = \begin{cases} V_E(S) + \dfrac{A(h) * \left(\dfrac{S}{S^*} \right)^{\lambda(h)}}{1-k} & \text{If } \phi(S^* - S) > 0 \\ \phi(S - X) & \text{If } \phi(S^* - S) \le 0 \end{cases}$$

Where $V_E(S)$ is the Black-Scholes-Merton European option formula ($\phi = 1$ for calls and $\phi = -1$ for puts), $A(h) = \phi(S^* - X) - V_E(S^*)$ and S^* is the figure that solves:

$$\phi = \phi e^{-d\tau} N(\phi d_1(S^*))^2 + \frac{\lambda(h) * (\phi(S^* - X) - V_E(S^*))}{S^*}$$

and k, b and c are given by

$$k = b(\log(S/S^*))^2 + c\log(S/S^*), \qquad b = \frac{(1-h)\alpha\lambda^{'}(h)}{2(2\lambda+\beta-1)},$$

$$c = -\frac{(1-h)\alpha}{2\lambda+\beta-1}\left(\frac{1}{A(h)}\frac{\partial V_E(S^*,h)}{\partial h} + \frac{1}{h} + \frac{\lambda^{'}(h)}{2\lambda+\beta-1}\right)$$

where

$$\tau = T - t, \qquad h(\tau) = 1 - e^{-r\tau}, \qquad \alpha = \frac{2r}{\sigma^2}, \qquad \beta = \frac{2(r-d)}{\sigma^2},$$

$$\lambda(h) = \frac{-(\beta-1) + \phi\sqrt{(\beta-1)^2 + 4\alpha/h}}{2}, \qquad \lambda^{'}(h) = -\frac{\phi\alpha}{h^2\sqrt{(\beta-1)^2 + 4\alpha/h}},$$

$$\frac{\partial V_E(S^*,h)}{\partial h} = \frac{S^* n(d_1(S^*))\sigma e^{(r-d)\tau}}{2r\sqrt{\tau}} - \phi d S^* N(\phi d_1(S^*))e^{(r-d)\tau}/r + \phi X N(\phi d_2(S^*)),$$

$$d_1(S^*) = \frac{\log(S^*/X) + (r-d-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$

Where $r = 0$ and the option was a call, the following limiting values were used,

$$\lambda(h) = \frac{-(\beta-1) + \phi\sqrt{(\beta-1)^2 + 8/\sigma^2\tau}}{2}, \qquad b = \frac{-2}{\sigma^4\tau^2((\beta-1)^2 + 8/\sigma^2\tau},$$

$$c = \frac{-\phi}{\sqrt{(\beta-1)^2 + 8/\sigma^2\tau}}\left(\frac{S^* n(d_1(S^*))e^{-d\tau}}{A(h)\sigma\sqrt{\tau}} - \frac{\phi 2dS^* N(\phi d_1(S^*))e^{-d\tau}}{A(h)\sigma^2} + \frac{2}{\sigma^2\tau} - \frac{4}{\sigma^4\tau^2((\beta-1)^2 + 8/\sigma^2\tau)}\right)$$

3.2.6 The Binomial Tree Method

Given the stock price (S), strike price (X), interest rate, (r), dividend yield (d), time to option's maturity (T), stock price volatility (σ), and number of price paths (N), I have built an algorithm to price American call and put options based on the assumption that the stock price moves in discrete steps up with risk neutral probability q and down with probability 1-q. Where, $q = \dfrac{e^{(r-d)\Delta t} - down}{up - down}$, $\Delta t = \dfrac{T}{N}$, $up = e^{\sigma\sqrt{\Delta t}}$ and $down = 1/up$. For j=0 to N, we have ${}^{N}C_j$ sample paths and

For $0 \le \text{i} \le \text{N-1}$ and for $0 \le \text{j} \le \text{i}$ the terminal payoff for a call, is given as

$f_{N,j} = \max(S*(up^j)*(down^{N-J}) - X, 0)$. The nodal values are given by

$f_{i,j} = \max(S*up^j*down^{i-j} - X, e^{-r\Delta t}(qf_{i+1,j} + (1-q)f_{i+1,j}))$.

Please see Appendix C for the VBA pricing algorithm for American call options.

Having discussed two approximation methods, it is necessary to consider differential equation methods.

3.2.7 The Explicit Finite Difference Method

According to Black-Scholes (1973), the partial differential equation satisfied by an option, f is:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} - rf = 0$$

The Explicit finite difference method proposed by Brennan and Schwartz (1978) requires the approximation of the derivatives as follows:

$$\frac{\partial f}{\partial t} \approx \frac{f_{i+1,j} - f_{i-1,j}}{\Delta t}, \quad \frac{\partial f}{\partial S} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}, \quad \text{and} \quad \frac{\partial^2 f}{\partial S^2} \approx \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta S)^2}$$

In order to set up an algorithm for the explicit finite difference method, let $f_{i,j}$ be the value of f at time $i\Delta t$ when the stock price is $j\Delta S$ and let

i=0 to N be the length of the time dimension

j=0 to M be the length of the price dimension

$S_{\max} = e^y$ be the maximum value that the stock could attain at $j = M$.

The following boundary conditions apply:

For $i = N$, the option is at maturity $T = N\Delta t$ and bounded by the payoff.

For $j = 0$, the stock price is $y = 0$ and $S = 1$ and the boundary condition is $f = 0$ for a put and $f = S_{\max}$.

For all the computations done under this method, I have set N=1000, M=500, $y=5$ and ensured that $\Delta y = 0.05$ satisfies the necessary condition $\Delta y \leq \dfrac{\sigma^2}{\left| r - \dfrac{\sigma^2}{2} \right|}$.

For $j = 1$ to M-1

For $i = 0$ to N-1 Step-1

American Call value is $C = \max(f_{i,j}, S - X)$ and the Put value is $P = \max(f_{i,j}, X - S)$ where

$$f_{i,j} = af_{i+1,j-1} + bf_{i+1,j} + cf_{i+1,j+1}, \text{ and}$$

$$a = \frac{1}{1 + r\Delta t} \frac{1}{2} \frac{\Delta t}{\Delta y} \left[\sigma^2 \left(\frac{1}{2} + \frac{1}{\Delta y} \right) - r \right]$$

$$b = \frac{1}{1 + r\Delta t} \left(1 - \frac{\sigma^2 \Delta t}{(\Delta y)^2} \right)$$

$$c = \frac{1}{1 + r\Delta t} \frac{1}{2} \frac{\Delta t}{\Delta y} \left[\sigma^2 \left(\frac{1}{\Delta y} - \frac{1}{2} \right) + r \right]$$

The variables a, b and c are non-negative constants such that $a + b + c = \dfrac{1}{1 + r\Delta t}$ is the discount factor for each Δt.

3.2.8 The Black-Scholes Method

This method was developed in the early 1970's and plays a vital role in the pricing and hedging of options. In order to price European option whose value is the starting point for the valuation of an American option, I have used the following formulae as they appear in Hull (2006). The European call (c) and European put (p) options on dividend paying stocks are given by

$$c = S_0 e^{-dT} N(d_1) - X e^{-rT} N(d_2)$$

$$p = X e^{-rT} N(-d_2) - S_0 e^{-dT} N(-d_1)$$

Where $d_1 = \dfrac{\ln\left(\dfrac{S_0}{X}\right) + \left(r - d + \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $\quad d_2 = \dfrac{\ln\left(\dfrac{S_0}{X}\right) + \left(r - d - \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$

And N(x) is the cumulative probability distribution function for a variable that is normally distributed with a mean of zero and standard deviation of one.

3.3 Stochastic Volatility Model

In order to extend the pricing model to accommodate a stochastic variance process, it is important to recall the stochastic process followed by the stock price

$$dS(t) \equiv \mu S dt + \sqrt{v(t)} S dz_1(t)$$

where $\mu = r - d$ is the instantaneous expected return, r is the risk free rate d is the continuous dividend rate and $v(t)$, the stock price variance. We assume that, $v(t)$ follows the mean reverting Ornstein-Uhlenbeck process $d\sqrt{v(t)} = -\beta\sqrt{v(t)}dt + \delta dz_2(t)$, where dz_1 and dz_2 are Wiener processes and according to Heston (1993), Ito's lemma shows that the variance $v(t)$ follows the process $dv(t) = \left[\delta^2 - 2\beta v(t)\right]dt + 2\delta\sqrt{v(t)}dz_2(t)$. Assuming that the correlation (between z_1 and z_2) ρ is equal to zero, this can be rewritten as a square-root process $dv(t) = m[\theta - v(t)]dt + \sigma\sqrt{v(t)}dz_2(t)$ where m is mean reversion speed, θ is long-term variance, and σ is the volatility of the variance parameter, $v(t)$. Using these new formulae, I simulated paths for the stock price following the procedure described in section 3.2.1 under the Simple Monte-Carlo method.

3.4 Stochastic Interest Rate Model

Given that the stock price follows the normal stochastic process

$$dS(t) \equiv \mu S(t)dt + \sigma S(t)dz_1(t)$$

The variables are as defined in section 3.4.1 above except that this time σ replaces $\sqrt{v(t)}$ and the interest rate r follows another stochastic process such that $dr = m(\theta - r)dt + \sigma\sqrt{r}dz_2$, where m is the mean reversion parameter, $m(\theta - r)$ is the instantaneous expected change in the short term rate, and θ is the long-term mean of the interest rate. Sigma, σ is the interest rate's volatility and dz_2 is proportional to $\varepsilon\sqrt{\Delta t}$ and is the standard Weiner process (Brownian motion) followed by r which is independent of dz_1 (the stochastic process for S). The expression $\sigma\sqrt{r}$ is the basis point volatility and in order to ensure non-zero and non-negativity in interest rates, I used the necessary boundary condition, $2m\theta \geq \sigma^2$ (Cox *et al*, 1985).

3.5 Conclusion

In this Chapter, eight American option pricing methods plus the Black-Scholes-Merton formula (for pricing standard European options) have been described. The American option pricing methods considered comprise four Monte-Carlo, one differential equations and three approximation methods including the Binomial Tree approach. I have also described the methodology for implementing two extensions to the basic model to include either stochastic volatility or random interest rates.

For each of the above-mentioned approaches, I developed an algorithm based on VBA and used them to price options. The results are presented in the next chapter.

CHAPTER FOUR
DATA ANALYSIS AND FINDINGS

4.1 Introduction

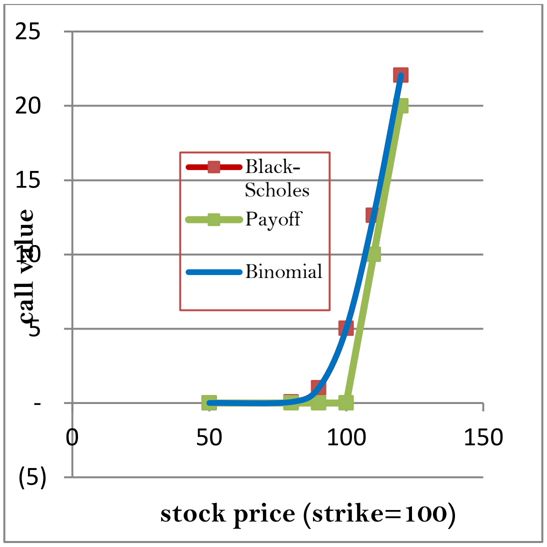
In this chapter I present the results of the quantitative investigation and reserve all comments and discussion of the findings to the next chapter. Section 4.2 contains results of American call options computed under the eight different methods based on a wide range of input parameters. Section 4.3 contains similar results for Put options. In section 4.4 I have presented my findings on the behaviour of the early exercise boundaries based on different ranges of input parameters while section 4.5 contains findings on extensions to the basic pricing model.

4.2 Call Options

For the nine methods described in this chapter (including the Black-Scholes for European options), an algorithm based on VBA was developed to enable flexible and easy data manipulation. For all the methods implemented in this study, N, the number of price movements within time T (that is Δt) was set to 1000. With respect to the Monte-Carlo Simulations methods I used 20,000 simulated payoffs with the objective of minimising the standard error. These, however, had the tendency to slow down the computation. Nevertheless, they significantly improved the results. In particular, the Binomial model produced the same European option prices as the Black-Scholes. As a result, I have used the Binomial model as the benchmark upon which the accuracy of other American models would be assessed. The following diagrams compare the call values of the Binomial and the Black-Scholes. Figures 4.1 and 4.2 provide the graphical representation of European and American call option values as well at the terminal payoff function.

Figure 4.1: American Call Options
(With no dividend)

Figure 4.2: American Call Options
(With continuous dividend)



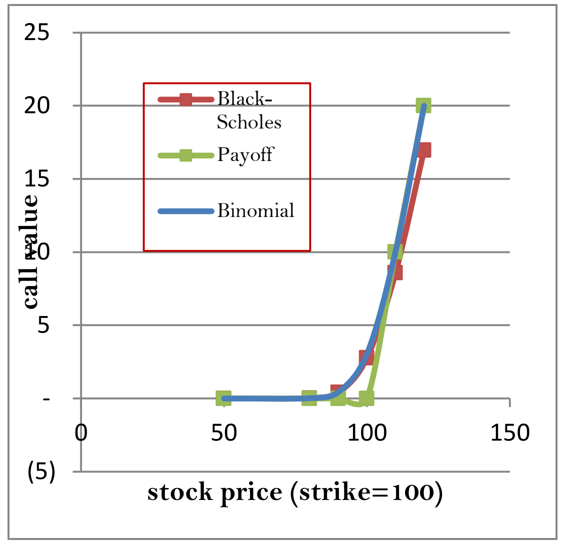


Figure 4.1 shows that when there are no dividends, the Black-Scholes (European) call values are the same as the Binomial (American) and do converge to the terminal payoff at maturity. Figure 4.2 on the other hand shows that with the present of continuous dividend payment the American call can be exercised at a certain time before maturity when it has values that exceed the European and do converge to the payoff function. This early exercise time is called the stoppage time and the stock price at that point is the early exercise boundary. Thus the above diagrams illustrate the fundamental difference between European and American call options.

Owing to the difficulty of calculating American option prices, several methods have been developed some of which I have already discussed in the preceding chapter and under the literature review. In particular, it was noted that some think that the Monte-Carlo simulation methods do not give accurate values. In order to test the accuracy of the various pricing methods and to ascertain the veracity of that assertion, I have computed American call prices using eight different methods under wide ranges of input parameters. The results have been summarised in Table 4.1 below for easy comparison.

Table 4.1: American Call Option Values Computed under Various Pricing Methods (Strike Price=100)

	Stock Price	Black-Scholes (Europe)	Binomial Tree Method	Simple Monte Carlo	Antithetic Variate Technique	Control Variate Technique	Least Squares (LSM)	Quadratic Approx. (BAW)	Ju and Zhong's Method	Explicit Finite Difference
	80	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
r=0.08	90	0.57	0.58	0.57	0.58	0.59	0.58	0.59	0.57	0.58
σ=0.20	100	3.42	3.52	3.44	3.44	3.41	3.53	3.52	3.44	3.53
T=0.25	110	9.85	10.36	9.93	9.91	9.96	10.36	10.31	10.00	10.35
d=0.12	120	18.62	20.00	19.95	19.89	19.86	20.00	20.00	20.00	20.00
	80	1.05	1.06	1.05	1.05	1.08	1.06	1.07	1.05	1.06
r=0.08	90	3.23	3.28	3.26	3.21	3.21	3.27	3.28	3.25	3.24
σ=0.4	100	7.29	7.39	7.34	7.32	7.25	7.41	7.40	7.46	7.40
T=0.25	110	13.25	13.54	13.14	13.13	13.19	13.53	13.52	13.55	13.51
d=0.12	120	20.73	21.30	20.73	20.78	20.72	21.29	21.19	21.31	21.29
	80	0.21	0.21	0.23	0.23	0.22	0.22	0.22	0.21	0.20
r=0.08	90	1.31	1.37	1.33	1.32	1.30	1.36	1.39	1.31	1.35
σ=0.20	100	4.46	4.70	4.41	4.51	4.50	4.72	4.70	4.52	4.71
T=0.5	110	10.16	11.00	10.15	10.16	10.14	11.00	10.96	10.00	11.03
d=0.12	120	17.85	20.00	20.05	20.03	19.99	20.00	20.00	20.00	20.00
	80	1.29	1.29	1.35	1.27	1.28	1.29	1.29	1.29	1.29
r=0.08	90	3.82	3.84	3.86	3.77	3.77	3.82	3.80	3.84	3.81
σ=0.4	100	8.35	8.33	8.41	8.42	8.40	8.35	8.35	8.50	8.35
T=0.25	110	14.80	14.81	14.64	14.87	15.06	14.80	14.81	14.98	14.82
d=0.04	120	22.71	22.72	22.69	22.79	22.89	22.71	22.72	23.03	22.71
	80	0.41	0.41	0.40	0.41	0.39	0.41	0.41	0.41	0.41
r=0.08	90	2.18	2.18	2.19	2.18	2.15	2.18	2.18	2.18	2.18
σ=0.20	100	6.50	6.48	6.53	6.46	6.50	6.50	6.50	6.54	6.52
T=0.5	110	13.42	13.43	13.39	13.51	13.47	13.42	13.43	13.49	13.44
d=0.04	120	22.06	22.06	22.17	21.98	22.04	22.06	22.06	22.08	22.06
	80	1.93	2.34	1.91	1.95	1.96	2.34	2.52	1.93	2.34
r=0.08	90	3.75	4.77	3.76	3.72	3.71	4.75	4.97	3.94	4.76
σ=0.20	100	6.36	8.48	6.30	6.33	6.25	8.51	8.67	7.51	8.49
T=3	110	9.75	13.79	9.69	9.77	9.75	13.77	13.88	13.80	13.79
d=0.12	120	13.87	20.89	13.69	13.76	13.99	20.87	20.88	20.90	20.89
	80	3.79	3.98	3.69	3.80	3.76	3.98	4.20	3.79	3.98
r=0.08	90	6.81	7.27	6.73	6.92	6.84	7.25	7.54	7.17	7.25
σ=0.20	100	10.82	11.68	11.00	10.94	11.06	11.71	12.03	13.11	11.70
T=3	110	15.71	17.33	15.88	15.91	15.90	17.31	17.64	18.02	17.31
d=0.08	120	21.35	24.03	21.33	21.42	21.26	24.01	24.30	24.00	24.02
	80	6.88	6.88	13.31	6.96	7.15	1.64	6.97	6.86	6.88
r=0.08	90	11.49	11.51	11.68	11.45	11.51	4.58	11.62	11.97	11.48
σ=0.20	100	17.20	17.18	16.95	17.10	17.11	9.54	17.40	20.26	17.19
T=3	110	23.80	23.86	23.81	23.30	23.21	16.30	24.09	23.78	23.80
d=0.04	120	31.08	31.18	31.09	31.36	31.21	24.36	31.49	31.00	31.10

Using the Binomial call prices as the benchmark, it is easy to observe from the above table whether there are any marked differences between the call values produced by the Monte-Carlo methods and the benchmark or between the Monte-Carlo and the other methods. At various input ranges, I have plotted the call values to enable easy appraisal as to how the Monte-Carlo methods perform. For instance, Figure 4.3 shows how some Monte-Carlo methods compare with the Binomial and the Barone-Adesi & Whaley.

Figure 4.3: American Call Options
(Inputs: r=0.08, T=3, d=0.12, sigma=0.80)

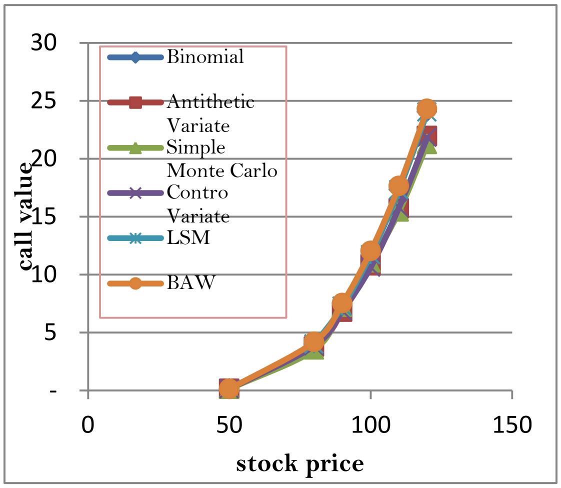
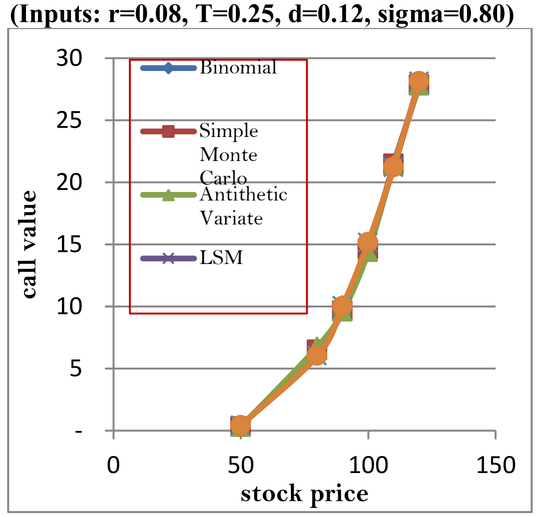


Figure 4.4: American Call Options
(Inputs: r=0.08, T=0.25, d=0.12, sigma=0.80)



In Figure 4.4 when we consider a situation where the time to maturity is reduced to $T = 0.25$, it is very difficult to distinguish the Monte Carlo prices from those of other methods. In fact, at very large T Monte Carlo simulations produce results which compare with the other methods as shown in Figures 4.5 and 4.6 below:

Figure 4.5: American Call Options
(Inputs: r=0.08, T=3, d=0.04, sigma=0.80)

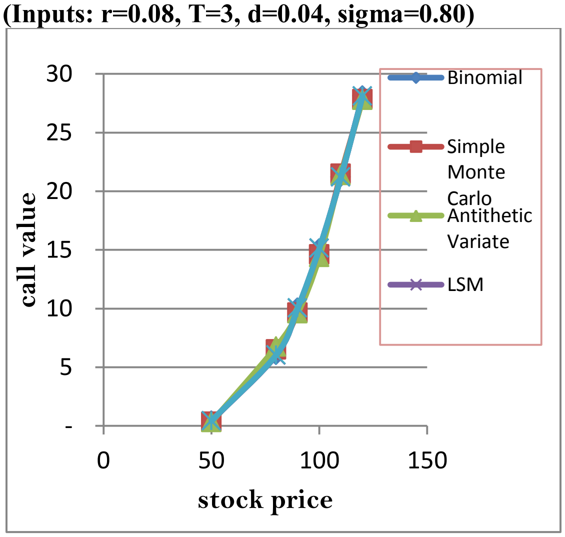


Figure 4.6: American Call Options
(Inputs: r=0.08, T=5, d=0.04, sigma=0.05)

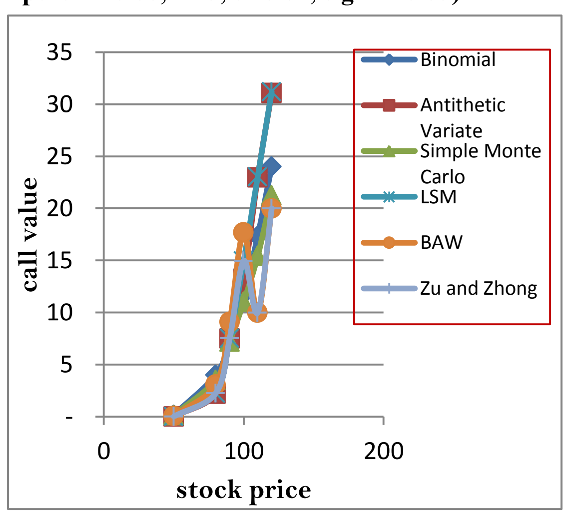


Figure 4.5 shows that at high values of T and σ, Monte-Carlo call prices are cannot be easily distinguished from the other methods. When T is adjusted upwards to say 5 and σ is reduced to say 0.05, Monte-Carlo methods present errors but we can judge whether their values are comparatively better or worse than those of Barone-Adesi & Whaley and Ju and Zhong by observing the shapes of the diagrams in Figure 4.6 above.

4.3 Put Options

In contrast with Call options, early exercise opportunity exists under Put options when there are no dividends. Using $r = 0.08, T = 0.25, \sigma = 0.18$ and strike price, $X = 100$ to compute Put prices, Figure 4.7 shows that European and American puts have the same prices when there is continuous dividend payment of $d = 0.18$ on the underlying stock. Figure 4.8 on the other hand reveals the effect of no dividend ($d = 0$) in the computations.

Figure 4.7: American Put Options (With continuous dividend)

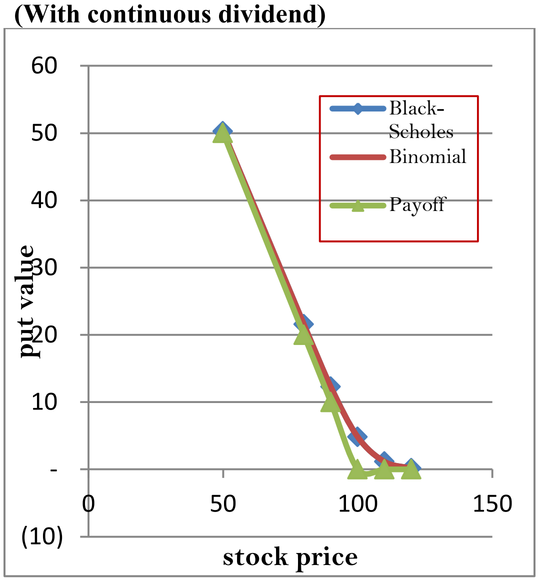
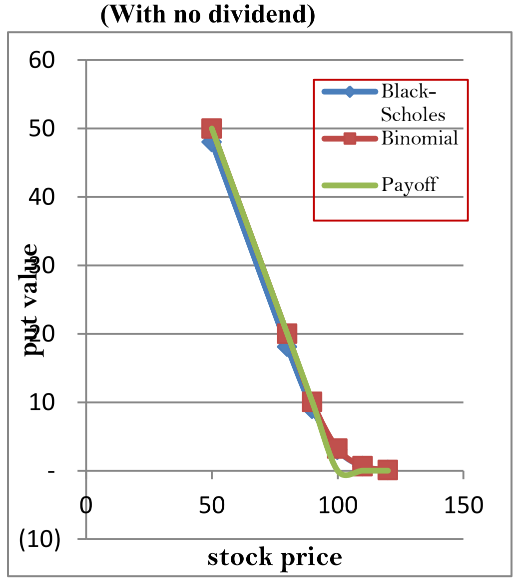
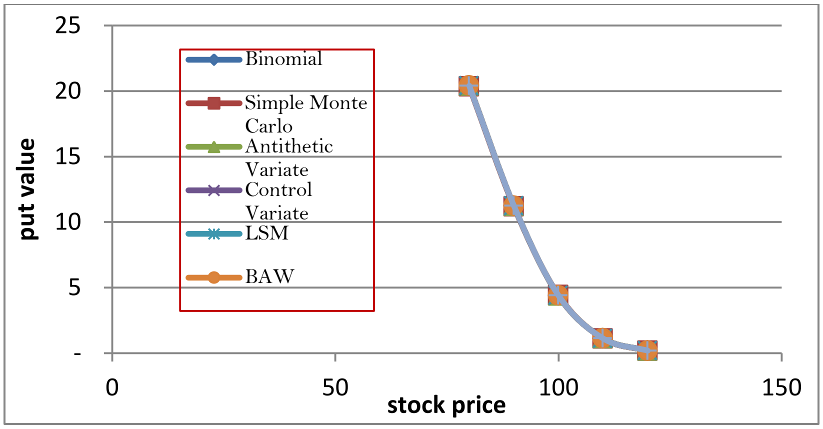


Figure 4.8: American Put Options (With no dividend)



In order to ascertain the accuracy of the Monte-Carlo put prices it is necessary to compare them to the benchmark and also to other non Monte-Carlo methods as has been presented in Figure 4.9

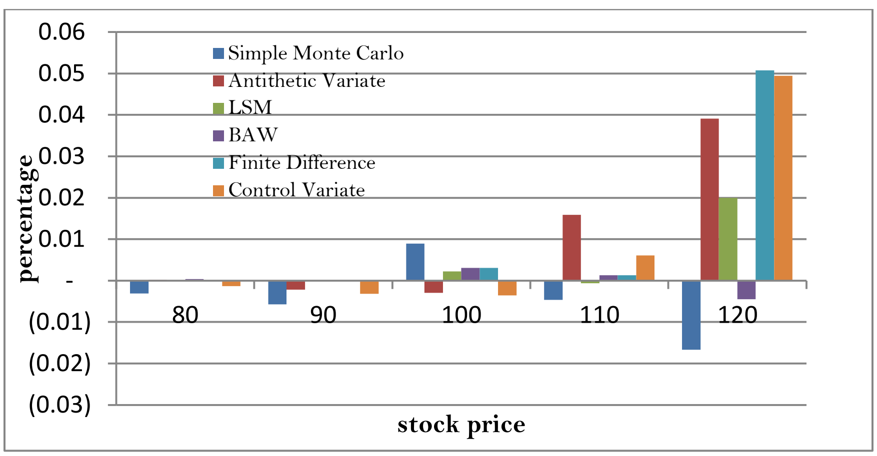
Figure 4.9: Comparison of Put Prices



Inputs: $r = 0.08,\ T = 0.25,\ d = 0.12,\ \sigma = 0.20,\ strike = 100$

From the above diagram no difference is apparent and it seems that all the different methods produce the same results. To be certain, I have computed the percentage deviation of the various prices from the benchmark (Binomial prices) and presented the results in the form of a bar graph as can be observed in Figure 4.10.

Figure 4.10: Percentage Deviation of Put Price from Benchmark



Inputs: $r = 0.08,\ T = 0.25,\ d = 0.12,\ \sigma = 0.20,\ strike = 100$

The above graph provides additional evidence that enables a more detailed comparison of the results. Furthermore, I have computed several Put prices using a range of input parameters and summarised the results in Table 4.2 below to enable further evaluation of the various pricing methods.

Table 4.2: American Put Option Values Computed under Various Pricing Methods (Strike Price=100)

	Stock Price	Black-Scholes (Europe)	Binomial Tree Method	Simple Monte Carlo	Antithetic Variate Technique	Control Variate Technique	Least Squares (LSM)	Quadratic Approx. (BAW)	Ju and Zhong's Method	Explicit Finite Difference
	80	20.41	20.41	20.35	20.42	20.39	20.41	20.42	20.41	20.41
r=0.08	90	11.25	11.25	11.19	11.23	11.22	11.25	11.25	11.25	11.25
σ=0.20	100	4.40	4.39	4.43	4.37	4.37	4.40	4.40	4.40	4.40
T=0.25	110	1.12	1.12	1.11	1.14	1.13	1.12	1.12	1.12	1.12
d=0.12	120	0.18	0.18	0.18	0.19	0.19	0.18	0.18	0.18	0.19
	80	21.43	21.45	21.51	21.39	21.55	21.44	21.46	21.43	21.44
r=0.08	90	13.91	13.93	13.95	13.83	13.75	13.92	13.93	13.91	13.91
σ=0.4	100	8.27	8.25	8.22	8.35	8.42	8.27	8.27	7.84	8.26
T=0.25	110	4.52	4.53	4.50	4.51	4.50	4.52	4.52	4.48	4.52
d=0.12	120	2.29	2.30	2.31	2.30	2.30	2.29	2.30	2.29	2.29
	80	20.95	20.95	20.97	20.99	20.94	20.96	20.96	20.95	20.98
r=0.08	90	12.63	12.64	12.66	12.63	12.55	12.63	12.63	12.63	12.64
σ=0.20	100	6.37	6.35	6.37	6.28	6.25	6.37	6.37	6.28	6.37
T=0.5	110	2.65	2.66	2.70	2.71	2.67	2.65	2.65	2.64	2.65
d=0.12	120	0.92	0.92	0.94	0.89	0.90	0.92	0.92	0.92	0.92
	80	20.11	20.59	20.12	20.15	20.10	20.59	20.59	21.43	20.53
r=0.08	90	12.74	12.97	12.63	12.80	12.70	12.96	12.95	13.91	12.93
σ=0.4	100	7.36	7.45	7.32	7.30	7.30	7.47	7.46	7.84	7.46
T=0.25	110	3.91	3.97	3.96	3.96	3.95	3.95	3.95	4.48	3.96
d=0.04	120	1.93	1.95	1.92	1.92	1.96	1.94	1.94	2.29	1.95
	80	18.08	20.00	19.96	19.98	20.00	20.01	20.00	20.00	20.00
r=0.08	90	10.04	10.76	10.16	9.96	9.91	10.76	10.75	10.00	10.71
σ=0.20	100	4.55	4.76	4.54	4.54	4.57	4.77	4.77	4.46	4.77
T=0.5	110	1.68	1.74	1.67	1.71	1.72	1.74	1.74	1.68	1.76
d=0.04	120	0.51	0.53	0.54	0.51	0.50	0.53	0.53	0.51	0.55
	80	24.78	25.65	24.67	24.73	24.70	25.65	25.66	24.78	26.25
r=0.08	90	19.62	20.09	14.49	15.51	15.45	20.08	20.08	19.62	20.64
σ=0.20	100	15.25	15.47	15.23	15.09	15.15	15.49	15.50	15.25	15.99
T=3	110	11.67	11.82	11.65	11.61	11.70	11.80	11.80	10.72	12.22
d=0.12	120	8.81	8.90	8.83	8.75	8.75	8.88	8.81	8.48	9.23
	80	19.52	22.21	22.17	23.31	23.22	22.20	22.20	20.00	22.40
r=0.08	90	14.68	16.22	14.64	14.80	14.71	16.20	16.21	14.68	16.50
σ=0.20	100	10.82	11.68	10.97	10.86	11.01	11.71	11.70	8.32	12.03
T=3	110	7.85	8.38	8.02	7.84	7.92	8.36	8.37	7.33	8.69
d=0.08	120	5.62	5.95	5.78	5.64	5.69	5.93	5.93	5.43	6.22
	80	14.59	20.33	14.71	21.21	21.16	20.33	20.35	20.00	20.33
r=0.08	90	10.33	13.50	10.25	10.37	10.39	13.48	13.50	10.33	13.56
σ=0.20	100	7.17	8.93	6.00	7.14	7.16	8.96	8.94	5.93	9.11
T=3	110	4.90	5.92	4.75	4.86	4.81	5.90	5.91	4.63	6.12
d=0.04	120	3.32	3.91	3.27	3.24	3.22	3.89	3.90	3.22	4.12

The results show how option prices under each model compare with the benchmark as well as with other pricing models. This enables proper assessment to be made with respect to the accuracy of the model results.

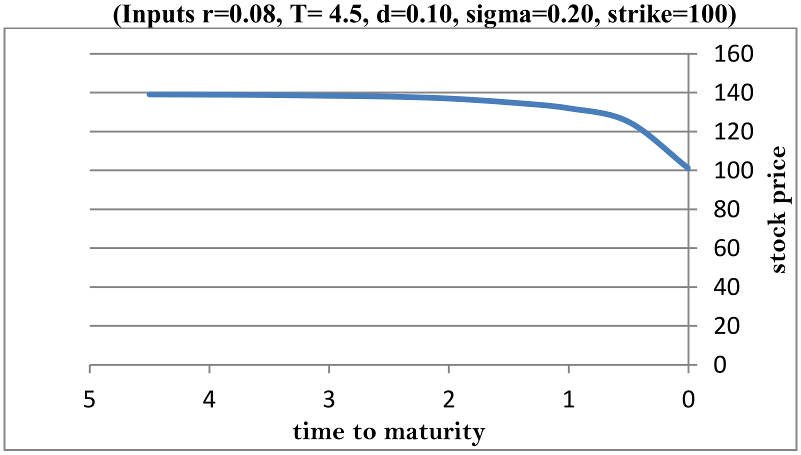
4.4 Exercise Boundaries

After observing how Call and Put prices computed under various methods compare with one another and how these prices change depending on the input parameters used, it is also important to examine in detail the early exercise feature of American options, the shape of the boundaries under each method and how they change under different conditions.

4.4.1 Exercise Boundaries- Call Options

The analysis showed that the shape of the early exercise boundary for call options depend on the range of input parameters used for the valuation. In Figure 4.11 when the continuous dividend rate is 10% the exercise boundary is low (below 140) and gently sloped at longer time to maturity. As maturity is approached, the boundary falls with greater velocity and tends to the strike price at time zero.

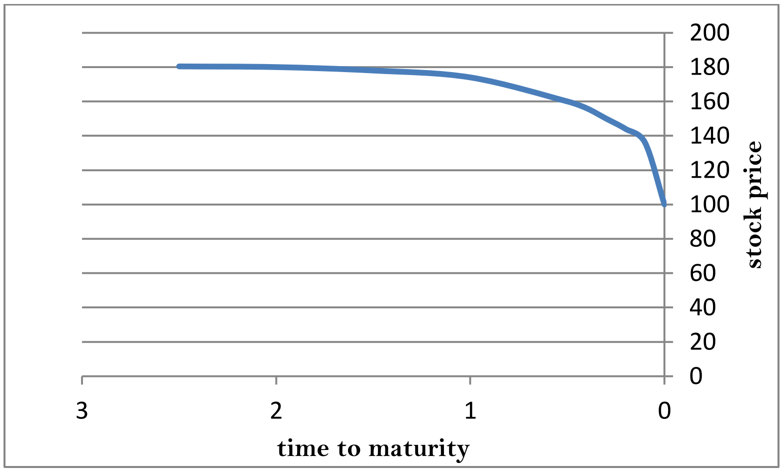
Figure 4.11: Exercise Boundary-Binomial Call Option (Inputs r=0.08, T= 4.5, d=0.10, sigma=0.20, strike=100)



When the dividend rate is reduced to 5% the exercise boundary shifts upwards (above 180) and has a steeper slope than the previous example as can be observed in Figure 4.12 below:

Figure 4.12: Exercise Boundary-Binomial Call Option

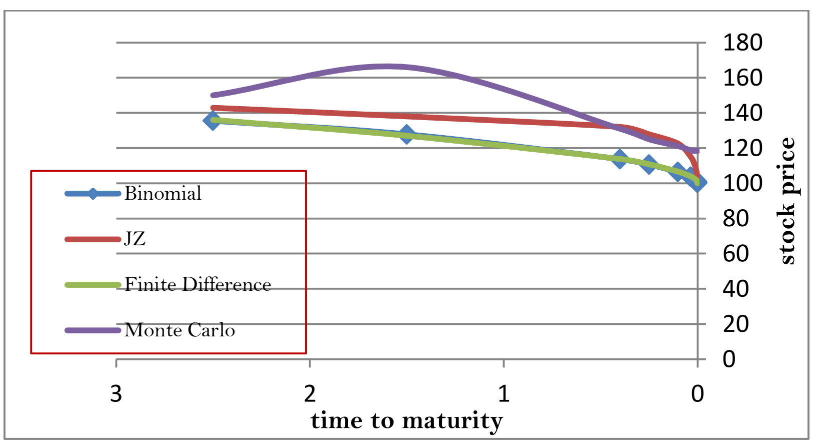
(Inputs r=0.08, T=4.5, d=0.05, sigma=0.20, strike=100)



Apart from this difference, the behaviour of the boundary tends to be similar with respect to time. This is because in both Figures (4.11 and 4.12) the exercise boundary approaches the strike price from above and assumes increased velocity at times near maturity.

It is however, noteworthy that the various option pricing methods produce slightly different exercise boundaries. Figure 4.13 is illustrative of such differences.

Figure 4.13: Comparison of Exercise Boundaries – Call Options



Inputs: $r = 0.08,\ T = 4.5,\ d = 0.10,\ \sigma = 0.20,\ strike = 100$

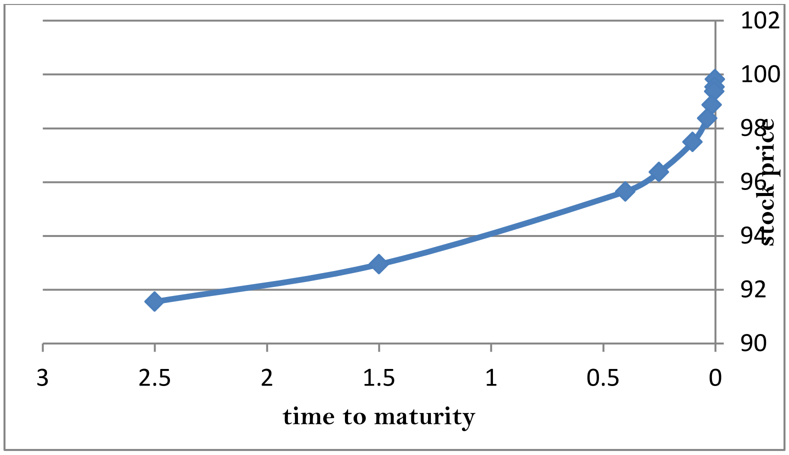
Having observed the behaviour of the exercise boundary with respect to American call, it is also necessary to analyse that of put options.

4.4.2 Exercise Boundaries- Put Options

The exercise boundary for American puts has a somewhat upward sloping shape and tends to the strike price at time zero. When maturity is far away, the slope is much gentle but velocity increases significantly as maturity is approached. This can be observed in Figure 4.14.

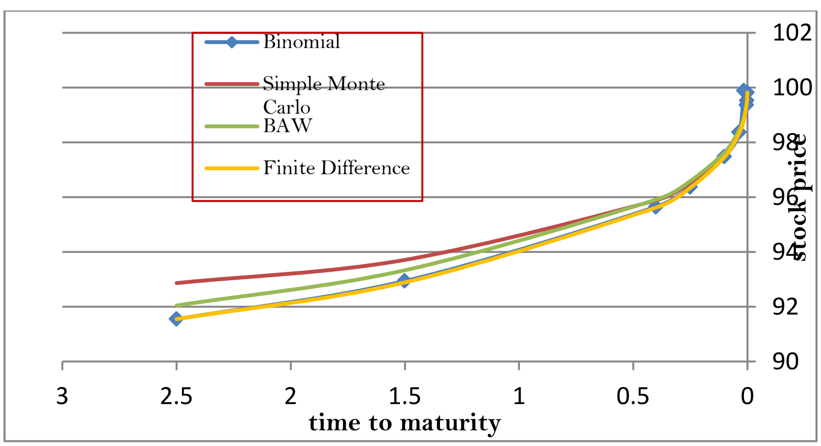
Figure 4.14: Exercise Boundary-Binomial Put Option

(Inputs r=0.08, T=2.5, d=0.05, sigma=0.20, strike=100)



Like call options, different pricing methods produced slightly different shapes of the exercise boundary. Figure 4.15 compares the shapes for four different methods.

Figure 4.15: Comparison of Exercise Boundaries – Put Options



Inputs: $r = 0.08,\ T = 2.5,\ d = 0.05,\ \sigma = 0.20,\ strike = 100$

After observing exercise boundaries, it must be remembered that all the analysis up to this point has been based on the assumptions that volatility as well as risk free interest rate is constant and that there is no transaction costs. In the following sections I present the results of extending the model to include situations where either volatility or interest rate is stochastic.

4.5 Stochastic Volatility

Modelling the variance to be random and applying it for option valuation in the Simple Monte-Carlo approach yielded option values that were slightly different from the case where volatility is assumed to be constant. For example, using correlation parameter $\rho = 0$, mean reversion parameter $m = 1.5$ Put prices computed under the random volatility model as well as their corresponding values under the basic model are as summarised in Table 4.3 below for easy comparison.

Table 4.3: Put Option Values When Volatility is Stochastic

Inputs	Stock Price	Basic Monte Carlo Model (A)	Stochastic Volatility Model (B)	Difference (A-B)	% Diff
	80	20.349	20.333	0.017	0.08
r=0.08	90	11.187	11.159	0.027	0.24
σ=0.20	100	4.426	4.401	0.025	0.55
T=0.25	110	1.113	1.080	0.033	2.96
d=0.12	120	0.178	0.148	0.030	16.78
	80	21.512	21.538	(0.027)	(0.12)
r=0.08	90	13.947	13.977	(0.030)	(0.22)
σ=0.40	100	8.225	8.247	(0.022)	(0.27)
T=0.25	110	4.498	4.515	(0.017)	(0.38)
d=0.12	120	2.311	2.331	(0.020)	(0.88)

4.6 Stochastic Interest Rate

The second extension to the model was the case where interest rate is modelled to be random. Using correlation parameter $\rho = 0$, mean reversion parameter $m = 1.5$ Call prices computed under the basic model as well as the case where interest rate is random together with the differences between the two models are as summarised in Table 4.4

Table 4.4: Put Option Values When Interest Rate is Stochastic

Inputs	Stock Price	Basic Monte Carlo Model (A)	Stochastic Interest Rate Model (B)	Difference (A-B)	% Diff
	80	20.349	20.679	(0.330)	(1.62)
r=0.08	90	11.187	11.327	(0.140)	(1.25)
σ=0.20	100	4.426	4.356	0.070	1.58
T=0.25	110	1.113	0.983	0.130	11.68
d=0.12	120	0.178	0.008	0.170	95.61
	80	21.512	21.282	0.230	1.07
r=0.08	90	13.947	14.637	(0.690)	(4.95)
σ=0.4	100	8.225	8.224	0.001	0.01
T=0.25	110	4.498	4.478	0.020	0.44
d=0.12	120	2.311	1.681	0.630	27.26

4.7 Conclusion

In this chapter, extensive results have been presented for several computations of call and put prices under a wide range of input parameters and how the values under different pricing methods compare with one another. Easy analysis of the results was achieved through a combination of tables and graphs. The analysis was also expanded to include an investigation of the shapes of the exercise boundaries for American calls and puts computed under various pricing models. Furthermore, I have analysed the results of two extensions to the basic model to include cases of stochastic volatility or stochastic interest rates and summarised the resulting option prices for comparison with the basic model's results.

In this chapter, I have simply presented the results of the quantitative analysis with very little if any comments. The next chapter contains the full and detailed discussions of the results.

CHAPTER FIVE

DISCUSSION AND CONCLUSION

5.1 Introduction

In this final chapter, I present in-depth analysis and discussion of the findings presented in the preceding chapter as well as the key issues arising from the literature review in section 5.2. In section 5.3 conclusions from the research findings are presented and recommendations are made as to how Monte-Carlo methods could be improved to make American option pricing more efficient whiles section 5.5 concludes the chapter and the whole report with the identification of the areas that this study could not address and which require future research.

5.2 Discussion of Results and Findings

Analysis has shown that Monte-Carlo simulations can be applied to the valuations of American options and that the accuracy of the Call and Put prices can be improved by adopting Variance reduction techniques such as the Control and the Antithetic Variates. Again, I observed that the Least Squares Monte-Carlo is another efficient method that uses simulation and Least squares regression to price options. Besides, the above-mentioned, Differential equations and Approximation methods also exist and I have demonstrated that these methods can also be used to accurately price American options.

My next objective in this study was to find out how the Monte-Carlo methods compare with the others and to determine whether they can practically be used for option valuation. This objective has been achieved by computing several Put and Call prices under various ranges of input parameters and comparing the results. The summarised Call prices in Table 4.1 show that the results of Monte-Carlo methods are very similar to those of the Binomial, Finite Difference and the Approximation methods. In particular, in Figure 4.4 when the time to maturity is 0.25 years (3 months), it is nearly impossible to distinguish the Monte-Carlo call option prices from the others. Even at large T, it was observed in Figure 4.3 that when the time to maturity is as long as 3 years the Monte Carlo methods produce very accurate results and that the values compare very well with the other methods. In fact, with the exception of the Simple Monte-Carlo, the other simulation methods produced exactly the same prices as the non simulation methods. An important finding from the results as presented in Chapter four is time to maturity and Volatility effects. The results show that differences in Call option values among the various methods magnify when there is simultaneous increase in T and decrease in sigma. This can be observed in Figure 4.6 when $T = 5$ and $\sigma = 0.05$. The Monte-Carlo methods produce errors but their results are more accurate than the Barone-Adesi & Whaley or the Ju and Zhong methods.

The results in Table 4.2 also show that the Monte-Carlo Put prices are not different from the other methods and therefore accurate especially when compared with the benchmark Binomial method.

A key issue arising out of the literature review was that some notable writers believe that Monte-Carlo simulation cannot be used for accurate valuation of options. However, the analysis has shown that assertion to be untrue and that Simulations not only provide an efficient method for pricing American options but also in certain situations (such as very long time to maturity) produce option values that are more accurate than some Approximation methods namely the Barone-Adesi & Whaley as well as the Ju and Zhong models as can be seen in Figure 4.6.

With respect to exercise boundaries, Figure 4.1 as well as Figure 4.5 shows that unlike European options, American options can be early exercised under certain conditions and therefore it is important to determine the early exercise boundary or the stopping time. It was observed that whenever $d > r$ the American call is higher than the European and hence there is the possibility of early exercise for the American option. However, the situation is different when $d < r$ since then, the European and American calls have the same values and hence no early exercise opportunity.

Figures 4.11 and 4.12 showed that for American calls, the exercise boundary converges to the Strike price from above as time to maturity tends to zero. Again, it was observed that all things being equal the exercise boundary shifts upwards as the continuous dividend rate on the underlying security is reduced and have seen the case when dividend was reduced from 10% to 5%. With respect to Put options, the exercise boundary is upward sloping and converges to the Strike price at maturity. Moreover, it was observed in Figure 4.14 as well as in Figure 4.15 that at times near option's maturity, the rate of convergence increases as the boundary approaches the Strike price with greater velocity just as can be found in Figure 4.11 and Figure 4.12 in the case of Calls. Furthermore, Figure 4.13 as well as Figure 4.15 shows that the various option pricing methods produced different exercise boundaries but such differences tend to diminish when nearing maturity. Thus, each method produces boundaries that have different concavities and convexities for Calls and Puts respectively. However, a critical investigation of the determinants of these curvatures was not one of my objectives in this study.

Where volatility was modelled to be stochastic the resultant option values were slightly different from the basic model when the correlation between the standard Wiener processes for the Stock price and the volatility were assumed to be zero. The results in Table 4.3 suggest that Put prices for the stochastic model are lower when initial volatility is low and larger for high initial volatility.

For the stochastic interest rate model, the results in Table 4.4 show that for Put options, differences between the two models generally increase with the Stock price. When the option is deep out-of-the-money, only small differences exist between the models. However, the differences in option values magnify when options are deep in-the-money.

Putting all the findings together and taking cognition of the themes emerging from the literature review, the research enables far reaching conclusions to be made which should be of particular relevance to the study of derivatives in general and American options in particular. I therefore summarise my conclusions in the next section.

5.3 Conclusions and Recommendation from the Research

One of the objectives of this study was to address the questions as to how Monte-Carlo simulations can be used to price American options. It was also my aim to show how Approximation and Differential equation methods can also be used for option pricing and whether there were any marked differences between the values produced under the Monte-Carlo approaches and other methods. These objectives have been achieved by implementing eight methods which include Monte-Carlo, Approximation as well as Differential equation procedures to value options with early exercise features. It has been observed that the Monte-Carlo methods produce accurate results and that there are no significant differences between their values and the other methods. If anything at all, the Monte-Carlo methods seem to produce better results at very long time to maturity compared with the Quadratic approximation models of Barone-Adesi and Whaley and its modified version suggested by Ju and Zhong (please. see Figure 4.6).

The findings from the investigation of the exercise boundaries also indicate that different methods produce boundaries with different curvatures; however, they are generally upward sloping for Puts and downward sloping for Calls and tend to converge to the Strike price at maturity.

Again, many writers have done researches, the results of which seem to suggest that neither volatility nor interest rates are constant in the real world. Modelling these parameters as random variables in the Simple Monte-Carlo model therefore helps to capture reality and my results show that this produces slightly different but more accurate results.

Consequently, it is recommended that in order to improve option valuation under Monte-Carlo simulation it is necessary to introduce randomness in volatility and interest rate.

5.4 Limitations and Areas for Future Research

A major limitation of this research was my inability to investigate further the curvatures of the exercise boundaries and why they differ under different methods. Again, the analyses of the effects of stochastic volatility and stochastic interest rates on option prices were based on limited data and hence it would be inappropriate to generalise the findings.

Consequently, areas suggested for further research are firstly, the investigation of the reasons why different pricing methods produce different concavities and convexities of exercise boundaries for Call and Put options respectively. Secondly, future research should focus on the effect on option prices of simultaneously modelling the volatility as well as the interest rate to be stochastic.

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APPENDICES

SAMPLE VBA PROGRAMMING CODES

A. ANTITHETIC VARIATE MONTE CARLO SIMULATIONS FOR AMERICAN CALL OPTIONS

```
Function MCAmCallAnti(S, X, r, d, T, sigma, N, Nsim As Integer)
Dim dt, rrT, sigmaT, RandNumber, payoff1, payoff2, Sum, S1, S2, m, up, down
Dim i As Integer
Dim j As Integer
Dim payoff As Variant
ReDim payoff(N)
rrT = (r - d - 0.5 * sigma ^ 2) * T
sigmaT = sigma * T ^ 0.5
discountFactor = Exp(-r * T)
Sum = 0
  For i = 1 To Nsim
    RandNumber = Randns()
    S1 = S * Exp(rrT + Randns * sigmaT)
    S2 = S * Exp(rrT - Randns * sigmaT)
    payoff1 = Application.WorksheetFunction.Max(S1 - X, 0)
    payoff2 = Application.WorksheetFunction.Max(S2 - X, 0)
    Sum = Sum + (payoff1 + payoff2) * 0.5
  Next i
    For m = 0.00001 To T
    For j = N - 1 To 0 Step -1
      dt = T / N * (N - j)
      payoff(j) = Sum / Nsim * Exp(-r * dt)
      If payoff(j) < S * Exp((r - d - 0.5 * sigma ^ 2) * m + sigma * Randns * m ^ 0.5) -
X Then payoff(j) = S * Exp((r - d - 0.5 * sigma ^ 2) * m + sigma * Randns * m ^ 0.5) - X
    Next j
  Next m
```

```
 MCAmCallAnti = payoff(0)
End Function
```

B. BARONE-ADESI AND WHALEY METHOD FOR AMERICAN CALL AND PUT

```
Function BSDOne(Snow, T, X, r, d, sigma)
Dim a, b, c
a = Log(Snow / X)
b = (r - d - 0.5 * sigma ^ 2) * T
c = sigma * Sqr(T)
BSDOne = (a + b) / c
End Function

Function BSDTwo(Snow, T, X, r, d, sigma)
Dim BSDOne
BSDTwo = BSDOne - sigma * Sqr(T)
End Function

Function BAWAmerPutValue(S, X, r, q, tyr, sigma)
Dim atol, Sstar, eqt, ht, alpha, beta, gam, Nd, Astar, va
atol = 0.0001
Sstar = MBWSstar(-1, S, X, r, q, tyr, sigma, atol)
eqt = Exp(-q * tyr)
ht = 1 - Exp(-r * tyr)
alpha = 2 * r / (sigma ^ 2)
beta = 2 * (r - q) / (sigma ^ 2)
gam = 0.5 * (-(beta - 1) - Sqr((beta - 1) ^ 2 + 4 * alpha / ht))
Nd = Application.NormSDist(-BSDOne(Sstar, X, r, q, tyr, sigma))
Astar = -(Sstar / gam) * (1 - eqt * Nd)
If S > Sstar Then
```

```
va = BSOptionValue(-1, S, X, r, q, tyr, sigma) + Astar * (S / Sstar) ^ (gam)
Else
va = X - S
End If
BAWAmerPutValue = va
End Function

Function BAWSstar(S, T, X, r, d, sigma, atol)
' Replicates Goal Seek or Excel Solver to find Sstar in BAW
' Uses BSOptionValue fn
' Uses BSDOne fn
Dim eqt, ht, alpha, beta, gam, Snow, cS, Nd, fS, Nd1, fdashS, Phi, BSDOne
Phi= 1  ‘ Phi=1 if option is a call and =2 if option is a put
eqt = Exp(-d * T)
ht = 1 - Exp(-r * T)
alpha = 2 * r / (sigma ^ 2)
beta = 2 * (r - d) / (sigma ^ 2)
gam = 0.5 * (-(beta - 1) + Phi * Sqr((beta - 1) ^ 2 + 4 * alpha / ht))
Snow = S
Do
   cS = BScall(S, T, X, r, d, sigma)
   Nd = Application.NormSDist(Phi * BSDOne(Snow, T, X, r, d, sigma))
   fS = iopt * (Snow - X) - cS - Phi * (1 - eqt * Nd) * Snow / gam
   Nd1 = Application.NormDist(Phi * BSDOne(Snow, T, X, r, d, sigma), 0, 1, False)
   fdashS = Phi * (1 - eqt * Nd) * (1 - 1 / gam) + (eqt * Nd1) / (gam * sigma * Sqr(T))
   Snow = Snow - (fS / fdashS)
Loop While Abs(fS) > atol
BAWSstar = Snow
End Function
```

C. PRICING AMERICAN CALL OPTIONS USING BINOMIAL TREES

```
Function BinomialAmCall(S, X, r, d, T, sigma, N As Integer) As Single
Dim i As Integer
Dim j As Integer
Dim up, down, q, dt, rr, sdd, discountFactor
Dim payoff As Variant
ReDim payoff(N)
dt = T / N
rr = Exp(r * dt)
sdd = sigma * dt ^ 0.5
up = Exp(sdd)
down = 1 / up
q = (Exp((r - d) * dt) - down) / (up - down)
discountFactor = rr
'Stock price movements on the binomial tree
 For j = 0 To N
  payoff(j) = Application.WorksheetFunction.Max((S * (up ^ j) * (down ^ (N - j)) - X), 0)
    'Work backwards recursively to determine the option price
   Next j
   For i = N - 1 To 0 Step -1
       For j = 0 To i
       payoff(j) = (q * payoff(j + 1) + (1 - q) * (payoff(j))) / discountFactor
       If payoff(j) < S * (up ^ j) * (down ^ (i - j)) - X Then payoff(j) = S * (up ^ j) * (down
^ (i - j)) - X
       Next j
   Next i
  BinomialAmCall = payoff(0)
  End Function
```