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Computing optimal rebalance frequency for log-optimal portfolios

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Log-optimal investment portfolio is deemed to be impractical and cost-prohibitive due to inherent need for continuous rebalancing and significant overhead of trading cost. We study the question of *how often* a log-optimal portfolio should be rebalanced for any given finite investment horizon. We develop an analytical framework to compute the *expected log of portfolio growth* when a given discrete-time periodic rebalance frequency is used. For a certain class of portfolio assets, we compute the *optimal rebalance frequency*. We show that it is possible to improve investor log utility using this quasi-passive or *hybrid* rebalancing strategy. Simulation studies show that an investor shall gain significantly by rebalancing periodically in discrete time, overcoming the limitations of continuous rebalancing.

Keywords: Log-optimal portfolio; Log-normal; Portfolio optimization; Rebalancing frequency; Discrete rebalancing; Portfolio growth rate; Instantaneous portfolio growth

JEL Classification: C63, C61, G11

1. Introduction

In this research, we set out to gain insight on the question of how often an initial portfolio should be adjusted. For an investor, frequent rebalancing incurs cost in both time and money. The investor may not want to miss the opportunity to rebalance if there is a higher chance to get a better return. On the other hand, the investor will benefit by knowing when to be passive. Informed passivity brings worry-free investment and saves paying undue trading fees. Hence we explore two questions: when and how often the investor needs to rebalance and, when it is worthwhile to be passive after an initial investment decision.

Once the portfolio is set up, after determining the proper asset mix the investor needs to address the issue of rebalancing the portfolio. Conventional rebalancing strategies have been studied extensively by both researchers and practitioners (Mulvey and Simsek 2002, Masters 2003, Guastaroba *et al.* 2009). Collins and Stampfli (2005) is an excellent survey of modern principles and practice of rebalancing. There are three types of conventional rebalancing techniques discussed in literature (Arnott and Lovell 1993). In *calendar rebalancing* the portfolio mix is returned to the initial asset mix in regular periodic intervals. In *rebalancing to allowed range*, the portfolio is always brought back within the allowed range of drift. In the third technique called *threshold rebalancing*, the portfolio is always rebalanced to the initial mix whenever it drifts beyond a predefined range.

In this paper we assume the investor has a log utility function and chooses the *log-optimal strategy* to maximize *expected log of portfolio growth* given a stationary and time-independent return process for correlated assets. Luenberger (1998) provides exhaustive analytical treatment to compute the optimal weights that the portfolio needs in order to be rebalanced to the constituent assets in a continuous time framework. The investor has to continuously rebalance the portfolio to the initial estimation of the weights in order to achieve maximize growth rate in the long run when the investment horizon approaches infinity. Both researchers and practitioners generally acknowledge the severe practical limitation of this strategy due to the continuous rebalancing condition.

The log-optimal investment strategy, also known as Kelly's criterion, has long been of interest to researchers in the investment community. MacLean *et al.* (2011) provides an extensive treatment on the topic. The strategy has also several limitations. The strategy is very risky in short term. The strategy can also fare poorly with potential huge losses as a result of a sequence of bad scenarios no matter how long the finite investment horizon is. The asset means need to be carefully and conservatively estimated since portfolio log growth is very sensitive to these values. Despite the log-optimal strategy's

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established superiority over other similar investment strategies in the long run, it can take a very long time to show this.

Many researchers have studied the efficacy of discrete-time rebalancing (Sun *et al.* 2006, Kritzman and Page 2009, Branger *et al.* 2010). It is found that the investor loss when the investor abandons active continuous trading for discrete-time rebalancing may not be substantial. Using Monte Carlo simulation, Branger *et al.* (2010) conclude that in an incomplete market where derivatives are not used to construct portfolios, the utility loss due to discrete-time trading is very small. For a 10 year horizon, a passive buy-and-hold strategy will yield the same expected investor utility as continuous trading while needing merely about 10 basis points higher implied initial capital.

A natural question to ask if the log-optimal investor can benefit by adopting discrete-time rebalancing. In other words, instead of rebalancing the portfolio continuously to the initial set of weights, can the investor rebalance back to these weights less frequently? By doing so we must not, at any time during the investment horizon, sacrifice the investor goal of maximizing the portfolio log growth that could be achieved under the log-optimal active strategy. If such a rebalancing frequency exists, then the practical limitation set by the continuous rebalancing condition can be overcome. We show that, for certain class of portfolio assets and a finite investment horizon, such a rebalance frequency indeed exists and can be computed.

It is however necessary to contrast our approach to that followed in Kuhn and Luenberger (2010). In that paper, the authors formulate and solve the problem of maximizing the log-optimal portfolio's expected log growth rate when a periodic discrete-time rebalancing is used. The resulting portfolio weights may differ from those in optimal continuous-time rebalancing. They demonstrate that for long-term investors continuous rebalancing only slightly outperforms discrete-rebalancing if the investor chooses a rebalancing interval slightly shorter than a year. In our proposed approach, the investor maximizes the expected log growth for a more realistic short term horizon while rebalancing periodically to the same optimal weights used in optimal continuous-time rebalancing.

We merely want to know if the investor can afford to wait a certain finite time $\tau \neq 0$ to rebalance. This proposition obviates the need to continuously rebalance, yet achieves the same or higher level of portfolio growth.† In order to answer this question, we first analyze the portfolio dynamics in a purely passive approach when the investor does not rebalance at all. This is an alternative extreme approach that follows a diametrically opposite investment philosophy about rebalancing compared to the purely active continuous rebalancing log-optimal approach.

The rest of the paper is organized in the following manner. After listing the notation in section 2, we review the basics of log-optimal portfolio theory in section 3. In section 4 we develop the analytical framework necessary to estimate the moments of log of portfolio growth under a passive strategy. In section 5 we estimate the portfolio growth when a discrete-time periodic rebalancing is adopted in the so-called hybrid strategy. In section 6, we compute the *optimal rebalance frequency* that maximizes the portfolio growth for any given horizon. We validate the analytical and computational results using Monte Carlo simulation in section 7. Finally, we summarize the results in the section 8.

†In the context of this research *portfolio growth* means *expected log of portfolio growth* unless otherwise stated.

2. Notations

Suppose the investor has the choice of setting up an investment portfolio from a set of N *risky* financial assets and a *risk-free* asset. Typical risky assets are stocks and funds, and often are correlated with other risky financial assets. These risky assets $i = 1, \ldots, N$ are provided with *a priori* expected returns and standard deviations. We assume that returns are stationary random variables and hence the expected return and standard deviations don't change over time. We consider risk-free asset $i = N + 1$ such as T-bills offering constant fixed rate of return. We will use the following symbols in our mathematical derivations and analysis for $\forall i, j = 1$ to $N + 1$.

T	investment horizon in years (periods)
μ_i	expected rate of return for asset i
σ_i	standard deviation for asset i
ρ_{ij}	correlation between returns of asset i and j
σ_{ij}	covariance of asset i and $j = \rho_{ij}\sigma_i\sigma_j$
w_i	proportion of investment in asset i in portfolio for log-optimal allocation
$\mu_p(t)$	expected rate of return of portfolio of assets at time t
$\sigma_p(t)$	standard deviation of portfolio of assets at time t
$V(t)$	value (in dollars) of portfolio at time t

Without loss of generality, throughout our analysis we will assume an initial value of $V(0) = 1\$$.

3. Active log-optimal portfolio

We assume that asset price dynamics $S(t)$ follows Geometric Brownian motion. Geometric Brownian motion assumption is widely used in financial assets and derivative valuations (Neftci 2000, Hull 2011).

$$dS(t) = \mu S(t)dt + \sigma S(t)dz \tag{1}$$

where

μ	expected rate of return of the asset expressed in decimal form.
σ	volatility of the asset price.
dz	$\epsilon\sqrt{dt}$, the *Wiener process*, where $\epsilon \sim \phi(0, 1)$ is the standard normal variable.

In this asset dynamics framework the *continuously compounded rate of return* per annum realized between time 0 and t denoted by x is characterized by the following normal distribution:

$$x \sim \phi\left[\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{t}\right] \tag{2}$$

The asset price in terms of x is given by the following expression:

$$S(t) = S(0)e^{xt} \tag{3}$$

In the log-optimal investment strategy, portfolio weights are *continuously rebalanced* to maximize the long term growth rate of log of portfolio return. The reader can find a good treatment of this strategy in Luenberger (1998). Log-optimal and semi-log optimal portfolios are also analyzed in Györfi *et al.* (2007).

For an initial portfolio value of $V(0) = 1\$$ the expected value and variance for the log of portfolio growth are given by the following equations:

$$\chi(t) = E[ln\{V(t)\}] = \nu_p t \tag{4}$$

$$\Upsilon(t) = Var[ln\{V(t)\}] = \sigma_p^2 t \tag{5}$$

where, portfolio mean μ_p, variance σ_p and *growth rate* ν_p are respectively given by:

$$\mu_p = \sum_{i=1}^{N+1} w_i \mu_i \tag{6}$$

$$\sigma_p^2 = \sum_{i,j=1}^{N+1} w_i \sigma_{ij} w_j \tag{7}$$

$$\nu_p = \mu_p - \frac{\sigma_p^2}{2} \tag{8}$$

In the log-optimal portfolio, the growth rate ν_p is maximized in the long run by solving the following optimization problem:

$$\begin{aligned} &\underset{\mathbf{w}}{\text{maximize}} \quad \nu_p \\ &\text{subject to} \quad \sum_{i=1}^{N+1} w_i = 1 \end{aligned}$$

w defines the vector of asset weights. The solution to the above optimization problem is to select the weight of each risky asset i satisfying the following relationship (Luenberger 1998):

$$\sum_{j=1}^{N} \sigma_{ij} w_j = \mu_i - r_f \tag{9}$$

We will extend the example used in Luenberger (1998) for demonstrating different investment strategies studied in this paper. In this example, there are three risky assets, $i = 1$, 2 and 3. A portfolio manager or an investor needs to specify the asset mean, variance and correlation coefficients. She also specifies the risk free rate and investment horizon. The following is the set of input parameters specified for this example:

(i) Initial portfolio value: $V(0) = 1\$$
(ii) Mean vector:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \end{bmatrix} = \begin{bmatrix} 0.24 & 0.20 & 0.15 \end{bmatrix}$$

(iii) Asset standard deviation vector:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 0.3000 & 0.2646 & 0.1732 \end{bmatrix}$$

(iv) Asset correlation coefficients:

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.2520 & 0.1925 \\ 0.2520 & 1.0000 & -0.2182 \\ 0.1925 & -0.2182 & 1.0000 \end{bmatrix}$$

(v) Risk-free rate: $r_f = 0.1$
(vi) Investment horizon: $T = 30$ years.

For the example investment problem, we obtain:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1.0509 \\ 1.3818 \\ 1.7770 \\ -3.2098 \end{bmatrix}$$

The negative sign indicates that the risk free asset needs to be borrowed. A portfolio set up using the above weights will maximize the portfolio growth in the long run if the weights are always maintained to **w** by continuously rebalancing.

Using **w** and the equations (6)–(8) we obtain:

$$\mu_p = 0.4742, \quad \sigma_p^2 = 0.3742, \quad \nu_p = 0.2871$$

4. Passive portfolio

In the prior section we elaborated the log-optimal strategy for portfolio growth where the portfolio is continuously rebalanced with a periodicity of $\tau = 0$. Upon close scrutiny of the growth rate of the portfolio ν_p specified by equation (8), one finds that by not rebalancing, the portfolio effective mean μ_p deteriorates simultaneously decreasing the variance σ_p^2. For a short time if the second effect dominates the first, it will result in a net increase in growth rate. During this time the investor will benefit by avoiding continuous rebalancing. In this section, we will develop the framework to assess the nature of portfolio growth when the investor sets up the portfolio with the optimal weight vector **w** and never rebalances throughout the investment horizon T. Consequently, we assume the rebalance frequency under such passive strategy to be $\tau = \infty$.

Throughout our analysis, we use the pertinent rebalance frequency as the superscript with parameters. All parameters for passive strategy will have a superscript of ∞. In the absence of any such superscript, the parameter pertains to active strategy. Note that the initial investment parameters enumerated under section 2 will be applicable to all strategies discussed in this paper.

LEMMA 4.1 *Consider an initial portfolio with value $V(0) = 1\$$ constructed using N risky assets with weights w_{N+1}, $i = 1, \ldots, N$ and a risk-free asset with weight w_0. When left unadjusted, the portfolio will grow such that the value $V(t)$ at any time $t > 0$ will be given by:*

$$V^\infty(t) = \sum_{i=1}^{N+1} w_i e^{x_i t} \tag{10}$$

where x_i is a random normal variable specified by equation (2).

Proof At $t = 0$ the value of the portfolio invested in asset i is w_i. This translates to the number of shares n_i to be purchased and held for asset i at time $t = 0$:

$$n_i = \frac{w_i}{S_i(0)}$$

Since the portfolio remains unadjusted, the value of n_i shares of asset i at time $t > 0$ will be:

$$V_i^\infty(t) = \frac{w_i}{S_i(0)} S_i(t) = \frac{w_i}{S_i(0)} S_i(0) e^{x_i t} = w_i e^{x_i t} \tag{11}$$

We have used equation (3) in simplifying the above. Now the result in equation (10) follows since the portfolio value is the sum of values of constituent assets. □

Hence the value of the passive portfolio is characterized by a sum of correlated random variables as per equation (10). We now review some of the statistical properties of log normal random variable. A comprehensive treatment of log-normal distribution will be found in Crow and Shimizu (1988). Let Y be a normal random variable with mean m and standard deviation s. Let X be log-normal random variables such that:

$$X = e^Y$$

The first two moments of X are given as below:

$$E[X] = e^{m+\frac{s^2}{2}} \tag{12}$$

$$Var[X] = (e^{s^2} - 1)E[X]^2 = (e^{s^2} - 1)e^{2m+s^2} \tag{13}$$

When there are two correlated random normal variables Y_i with mean m_i, standard deviation s_i and correlation coefficient ρ_{12}, the covariance between the corresponding log-normal variables $X_i = e^{Y_i}$ for $i = 1, 2$ are given by:

$$\begin{aligned} Cov[X_1, X_2] &= (e^{\rho_{12}s_1s_2} - 1)E[X_1]E[X_2] \\ &= (e^{\rho_{12}s_1s_2} - 1)e^{m_1+\frac{s_1^2}{2}}e^{m_2+\frac{s_2^2}{2}} \end{aligned} \tag{14}$$

Given log-normal X, one can compute the variance s^2 and the mean m of the underlying normal variable Y by using the following relationships:

$$s^2 = ln\left(1 + \frac{Var[X]}{E[X]^2}\right) \tag{15}$$

$$m = ln(E[X]) - \frac{1}{2}ln\left(1 + \frac{Var[X]}{E[X]^2}\right) = ln(E[X]) - \frac{1}{2}s^2 \tag{16}$$

Now we can proceed to compute the statistics for the passive portfolio evolution.

Lemma 4.2 *Under passive investment strategy, the expected portfolio growth at any time $t > 0$ is the weighted sum of the individual expected asset growths, i.e.*

$$E[V^\infty(t)] = \sum_{i=1}^{N+1} w_i e^{\mu_i t} \tag{17}$$

Proof From equation (10), we can compute the passive portfolio growth as:

$$\begin{aligned} V^\infty(t) &= \sum_{i=1}^{N+1} w_i e^{x_i t} = \sum_{i=1}^{N+1} e^{ln(w_i)+x_i t} \\ \Rightarrow E[V^\infty(t)] &= E\left[\sum_{i=1}^{N+1} e^{ln(w_i)+x_i t}\right] = \sum_{i=1}^{N+1} E[e^{ln(w_i)+x_i t}] \end{aligned} \tag{18}$$

We have made use of the fact that the expected value of a sum of random variables is same as the sum of expected values of the individual random variables (Trivedi 2001). Now, given that x_i's are normal random variables as specified in equation (2), $ln(w_i) + x_i t$ will also be normal with the following moments:

$$ln(w_i) + x_i t \sim \phi\left[ln(w_i) + \left(\mu_i - \frac{\sigma_i^2}{2}\right)t, \sigma_i^2 t\right] \tag{19}$$

Note that $Var(aX+b) = a^2 Var(X)$ for any random variable X and constants a and b.

We can now find out the first moment of $e^{ln(w_i)+x_i t}$ using log-normal properties of equation (12),

$$E[e^{ln(w_i)+x_i t}] = e^{ln(w_i)+(\mu_i - \frac{\sigma_i^2}{2})t + \frac{\sigma_i^2 t}{2}} = e^{ln(w_i)+\mu_i t} = w_i e^{\mu_i t} \tag{20}$$

Substituting the above in equation (18) we get the desired result. □

Lemma 4.3 *Under passive investment strategy, the variance of portfolio growth at any time $t > 0$ is given by:*

$$Var[V^\infty(t)] = \sum_{i,j=1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}(e^{\sigma_{ij}t} - 1) \tag{21}$$

Proof Similar to lemma 4.2, variance of passive portfolio growth is:

$$Var[V^\infty(t)] = \sum_{i,j=1}^{N+1} Cov[e^{ln(w_i)+x_i t}, e^{ln(w_j)+x_j t}] \tag{22}$$

The reader may refer Wikipedia (2011) for the rule to obtain the sum of correlated random variables.

We use equations (14) and (20) to simplify equation (22):

$$\begin{aligned} &Cov[e^{ln(w_i)+x_i t}, e^{ln(w_j)+x_j t}] \\ &\quad = (w_i e^{\mu_i t})(w_j e^{\mu_j t})(e^{\rho_{ij}\sigma_i\sqrt{t}\sigma_j\sqrt{t}} - 1) \\ &\quad = w_i w_j e^{(\mu_i+\mu_j)t}(e^{\rho_{ij}\sigma_i\sigma_j t} - 1) \\ &\quad = w_i w_j e^{(\mu_i+\mu_j)t}(e^{\sigma_{ij}t} - 1) \end{aligned} \tag{23}$$

Substituting equation (23) in equation (22), we obtain the desired passive portfolio variance expression of equation (21). □

The reader is reminded that the active strategy is optimal only when the portfolio growth given in equation (4) is maximized for the investor. In order to have a fair portfolio performance comparison between active and passive strategy we need to analyze the portfolio growth under passive strategy.

The problem here is to compute the first and if possible, the second moment of the log of the portfolio growth under passive strategy. Using equation (10):

$$ln(V^\infty(t)) = ln\left(\sum_{i=1}^{N+1} w_i e^{x_i t}\right) \tag{24}$$

The need to characterize the sum of lognormal variables arises in many domains. There have been many approximations to characterize the probability density function for sum of log normal. Two analytical methods to determine the moments of sum of correlated random variables widely used by researchers in many engineering disciplines. The first one proposed by Fenton and Wilkinson in 1960 is still being used because of its simplicity and analytical tractability (Fenton 1960). More recently, the second method was proposed in Schwartz and Yeh (1982). Fenton's approach allows the use of closed form analytical expression for the moments of log of sum of log-normal random variables. Schwartz and Yeh method employs a recursive algorithm to obtain the moments. In this paper, we will use Fenton's method because of its analytical tractability.

LEMMA 4.4 *The variance of the log of portfolio growth under passive strategy is given by:*

$$\Upsilon^{\infty}(t) = Var[ln(V^{\infty}(t))]$$
$$= ln\left(1 + \frac{\sum_{i,j=1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}(e^{\sigma_{ij}t} - 1)}{\left(\sum_{i=1}^{N+1} w_i e^{\mu_i t}\right)^2}\right) \quad (25)$$

Proof We assume that sum of lognormal random variables is also lognormal as is assumed in Fenton–Wilkinson approach. Thus as per equation (10) the passive portfolio growth $V^{\infty}(t)$ is lognormal. This implies that log of passive portfolio growth $ln(V^{\infty}(t))$ is normal.

Using lognormal property given by equation (15), we obtain,

$$\Upsilon^{\infty} = Var[ln(V^{\infty}(t))] = ln\left(1 + \frac{Var[V^{\infty}(t)]}{E[V^{\infty}(t)]^2}\right) \quad (26)$$

Substituting the values of expected value and variance of portfolio growth from equations (17) and (21) in the above equation we obtain the desired resulting equation of (25). □

Now we derive the portfolio growth which is the investor utility in log-optimal investment strategy.

LEMMA 4.5 *The expected log of portfolio growth under passive strategy is given by:*

$$\chi^{\infty}(t) = E[ln(V^{\infty}(t))] = ln\left(\sum_{i=1}^{N+1} w_i e^{\mu_i t}\right) - \frac{1}{2}\Upsilon^{\infty}(t) \quad (27)$$

Proof The derivation is straightforward when we follow the lognormal assumption in lemma 4.4 and using lognormal property given by equation (16) and expected value equation (17). □

The expected value thus obtained is an approximation due to the inherent log-normality assumption in Fenton–Wilkinson's approach.

5. Periodic rebalancing and hybrid portfolio

There is little incentive for the investor to resort to continuous rebalancing if passive strategy yields equal or higher portfolio growth for a given finite horizon. As depicted in figure 1(a), for the example portfolio, passive strategy outperforms active strategy for the initial investment period of 7.61 years determined by the point of intersection of equations (4) and (27). This initial passive investment period will be longer if transaction costs are to be considered.

Our goal is to find the periodic frequency $\tau = \tau_o$ at which the investor can rebalance the portfolio to the initial optimal weights to maximize portfolio growth for the intended investment horizon. The frequency τ is the time interval measured in years. Under such a *hybrid strategy* the portfolio is rebalanced periodically every τ years till the end of investment horizon. We use superscript $\tau \neq \infty$ to denote a hybrid strategy that uses τ as the rebalance frequency.

Using the next theorem, we show that one can compute the portfolio growth for hybrid strategy using the portfolio growth from passive strategy. Hence, we name this theorem as the passive to hybrid *growth map theorem*. Before we state and prove the theorem, we will state and prove two hypothesis relevant to periodic rebalancing. The first one is called the *law of additive growth* whereas the second one is termed as *law of multiplicative growth*. First we state and prove the law of additive growth.

LEMMA 5.1 *Passive portfolio growth is additive, i.e.*

$$\chi^{\tau}(k\tau + t')$$
$$= \chi^{\tau}(k\tau) + \chi^{\infty}(t'), \quad \forall k \in \mathbb{N}^+, \tau \in \mathbb{R}^+, \text{and } 0 < t' < \tau \quad (28)$$

where $k\tau$ is the most recent time when the portfolio is rebalanced and t' is the time for which the portfolio grows passively after $k\tau$.

Proof Since $k\tau$ is the most recent rebalance time, the portfolio growth at $k\tau + t'$ is given by:

$$V^{\tau}(k\tau + t') = V^{\tau}(k\tau)\sum_{i=1}^{N+1} w_i e^{x_i(t')} \quad (29)$$

Taking first log and then expected value on both sides, we obtain,

$$\chi^{\tau}(k\tau + t') = \chi^{\tau}(k\tau) + E\left[ln\left(\sum_{i=1}^{N+1} w_i e^{x_i(t')}\right)\right]$$
$$= \chi^{\tau}(k\tau) + \chi^{\infty}(t') \quad (30)$$

□

LEMMA 5.2 *Portfolio growth multiplies with the number of times periodic rebalancing is executed, i.e.*

$$\chi^{\tau}(k\tau) = k\chi^{\infty}(\tau), \quad \forall k \in \mathbb{N}^+, \tau \in \mathbb{R}^+ \quad (31)$$

where τ is the periodic rebalance frequency.

Proof We prove this lemma by method of induction. For the base case $k = 1$, equation (31) is trivially true. We then assume equation (31) holds for k and prove below that it also holds for $k + 1$. For $k + 1$, we need to prove,

$$\chi^{\tau}(\overline{k+1}\tau) = (k+1)\chi^{\infty}(\tau) \quad (32)$$

We start with RHS of above equation (32):

$$(k+1)\chi^{\infty}(\tau) = k\chi^{\infty}(\tau) + \chi^{\infty}(\tau)$$
$$= \chi^{\tau}(k\tau) + \chi^{\infty}(\tau), \text{ as equation (31) holds for } k.$$
$$= \chi^{\tau}(k\tau + \tau), \text{ applying law of additive growth, lemma 5.1}$$
$$= \chi^{\tau}(\overline{k+1}\tau) = LHS \quad (33)$$

That completes the proof of equation (31) by induction. □

THEOREM 5.3 *Assume that $\chi^{\tau}(t) = \chi^{\infty}(t), \forall t \in (0\ \tau]$ is known following equation (27). Then $\forall t > \tau > 0$,*

$$\chi^{\tau}(t) = \begin{cases} \nu_p t & \text{if } \tau = 0 \\ k\chi^{\infty}(\tau) + \chi^{\infty}(t') & \text{otherwise} \end{cases} \quad (34)$$

where $t = k\tau + t'$, $k = \lfloor t/\tau \rfloor$ and $t' = t \mod \tau$.

Proof At the very outset, note that we consciously treat $\tau = 0$ case to be same as the active strategy for consistency of results between different strategies. Additionally while computing k and t', we avoid divide-by-zero scenarios. We only need to prove:

$$\chi^{\tau}(k\tau + t') = k\chi^{\infty}(\tau) + \chi^{\infty}(t') \quad (35)$$

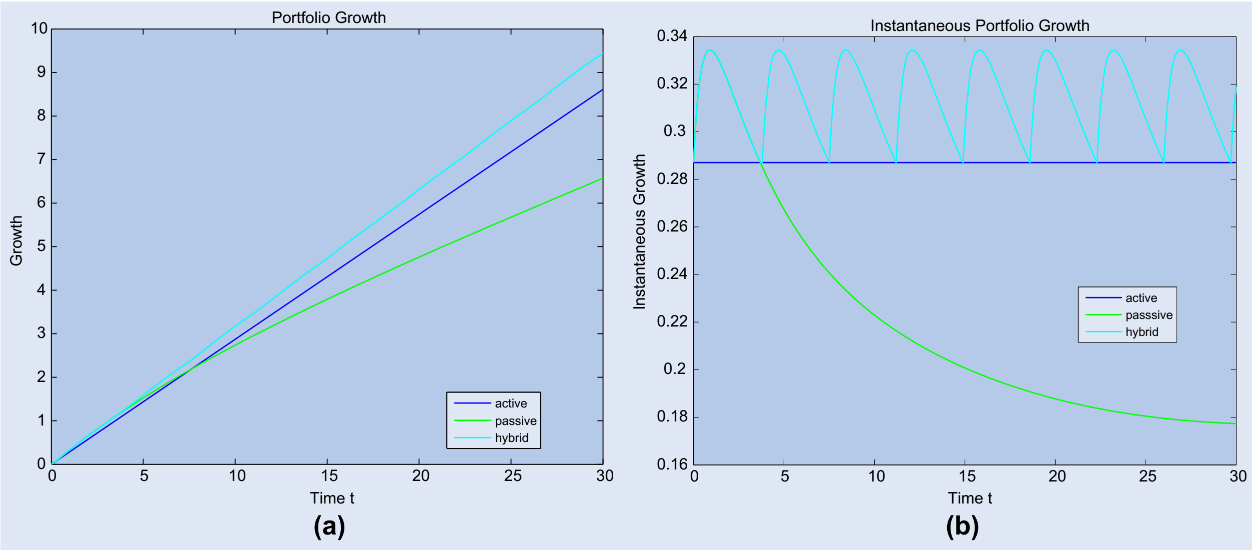

Figure 1. Comparison of absolute and instantaneous growth under different investment strategies.

We start with LHS of above equation (35).

$$\begin{aligned}\chi^{\tau}(k\tau+t') &= \chi^{\tau}(k\tau)+\chi^{\infty}(t'),\ \text{applying law of additive growth, lemma 5.1}\\ &= k\chi^{\infty}(\tau)+\chi^{\infty}(t'),\ \text{applying law of multiplicative growth, lemma 5.2} = LHS \end{aligned} \tag{36}$$

□

The growth map theorem 5.3 establishes the relationship between passive and periodic hybrid strategy portfolio growth. It states that under any hybrid strategy where rebalancing is done with periodicity of τ, the portfolio growth at subsequent rebalancing points can be obtained by multiplying the portfolio growth at the first rebalance point by the number of times the portfolio has been rebalanced to the initial optimal weights. Once we obtain the portfolio growth at the last rebalance point, growth for any additional time $t' < \tau$ will occur following the passive trajectory identical to the initial rebalance period. For any hybrid strategy with periodic rebalance frequency τ, once the passive χ^{∞} trajectory is calculated for the initial duration up to the first rebalance time, i.e. $[0\ \tau]$, we can completely construct the χ^{τ} trajectory for any future investment horizon.

5.1. Stable rebalance frequency

What rebalance frequency should the investor use in periodic hybrid strategy? An intuitive answer is to use the length of period when the passive strategy offers equal or higher portfolio growth than active strategy as the rebalance frequency τ. Due to growth map theorem, the portfolio growth under such periodic hybrid strategy will always be equal to or higher than the portfolio growth under active strategy.

Before we discuss alternative rebalancing frequency, we will introduce the concept of *incremental* portfolio growth. Mathematically portfolio growth in an infinitely small time interval can be obtained by taking the derivative of portfolio growth with respect to time t. Henceforth we use ξ to denote this denote this *instantaneous portfolio growth* or, in short, *instantaneous growth*. Thus the instantaneous growth for active and passive strategy are given respectively as follows:

$$\xi = \frac{d\chi(t)}{dt} = \nu_p \tag{37}$$

$$\xi^{\infty}(t) = \frac{d\chi^{\infty}(t)}{dt} \tag{38}$$

In an alternative approach, the investor may not rebalance till passive strategy yields equal or higher instantaneous growth compared to active strategy. We call this rebalancing interval τ_s as *stable* rebalance time. τ_s satisfies the following condition:

$$\exists\tau_s \text{ s.t. } \xi^{\infty}(t) > \nu_p, \quad \forall t \in (0\ \tau_s) \tag{39}$$

At τ_s, the instantaneous growth for both active and passive strategies become equal, i.e.

$$\xi^{\infty}(t) = \nu_p \tag{40}$$

Figure 1(b) shows the evolution of instantaneous growth under stable hybrid strategy. The instantaneous growth is never allowed to slip below the log-optimal growth rate ν_p. Using growth map theorem 5.3, we can obtain the portfolio growth under stable hybrid strategy as follows:

$$\chi^{\tau_s}(t) = k_s\chi^{\infty}(\tau_s) + \chi^{\infty}(t'_s) \tag{41}$$

where $t = k\tau_s + t'_s$, $k_s = \lfloor \frac{t}{\tau_s} \rfloor$ and $t'_s = t \mod \tau_s$. As shown in figure 1, stable hybrid strategy yields higher portfolio growth. For our running investment example, with a stable rebalance frequency of $\tau_s = 3.7$ years, the investor will obtain portfolio growth of 9.447 under stable hybrid strategy as compared to 8.613 obtained under active strategy. Thus stable hybrid strategy yields about 9.7% higher portfolio growth compared to baseline active strategy. It turns out that stable rebalancing time τ_s yields more portfolio growth compared to all higher rebalancing frequencies. We will formalize this property of τ_s in the form of theorem 5.6.

First we define $\psi^{\infty}(t) = \chi^{\infty}(t) - \chi(t)$ which is the *excess growth* relative to active strategy. We show that the excess passive growth $\psi^{\infty}(t)$ is a monotonously increasing function for $0 < t < \tau_s$.

LEMMA 5.4 *$\psi^{\infty}(t)$, the excess growth produced by passive strategy is increasing in the range $t \in (0\ \tau_s)$.*

Proof We need to prove that $\psi'^{\infty}(t) > 0,\ \forall t \in (0\ \tau_s)$. Let's start with the derivative of $\psi^{\infty}(t)$.

$$\psi'^{\infty}(t) = \frac{d(\chi^{\infty}(t) - \nu_p t)}{dt} = \frac{d(\chi^{\infty}(t))}{dt} - \nu_p = \xi^{\infty}(t) - \nu_p \tag{42}$$

Using equation (39), $\xi^{\infty}(t) > \nu_p, \forall t \in (0\ \tau_s)$ implying $\psi'^{\infty}(t) > 0$. □

LEMMA 5.5 *$\psi^{\infty}(t)$, the excess growth produced by passive strategy is maximized at τ_s.*

Proof In order to prove that τ_s is a relative maxima, we need to prove the following two:

$$\psi'^{\infty}(\tau_s) = 0 \tag{43a}$$

$$\psi''^{\infty}(\tau_s) < 0 \tag{43b}$$

Using equations (42) and (40), we obtain:

$$\psi'^{\infty}(\tau_s) = \xi^{\infty}(\tau_s) - \nu_p = 0 \tag{44}$$

Hence we proved equation (43a).

To prove equation (43b), we will use fundamental definition of differentiation.

$$\begin{aligned}\psi''^{\infty}(\tau_s) &= \lim_{d\tau \to 0} \frac{\psi'^{\infty}(\tau_s + d\tau) - \psi'^{\infty}(\tau_s)}{d\tau} \\ &= \lim_{d\tau \to 0} \frac{(\chi'^{\infty}(\tau_s + d\tau) - \nu_p) - (\chi'^{\infty}(\tau_s) - \nu_p)}{d\tau} \\ &= \lim_{d\tau \to 0} \frac{\chi'^{\infty}(\tau_s + d\tau) - \chi'^{\infty}(\tau_s)}{d\tau} \\ &= \lim_{d\tau \to 0} \frac{\xi^{\infty}(\tau_s + d\tau) - \xi^{\infty}(\tau_s)}{d\tau}\end{aligned} \tag{45}$$

By definition of τ_s, $\xi^{\infty}(\tau_s + d\tau) < \nu_p$. Therefore $\psi''^{\infty}(\tau_s) < 0$, proving equation (43b). □

THEOREM 5.6 *A periodic hybrid strategy with τ_s will always outperform another with higher rebalancing frequency, i.e. for any investment horizon $t > \tau_x$, $\chi^{\tau_s}(t) > \chi^{\tau_x}(t)$, $\forall \tau_x > \tau_s$.*

Proof Using the results of theorem 5.3:

$$\chi^{\tau_s}(t) = k_s \chi^{\infty}(\tau_s) + \chi^{\infty}(t'_s) \tag{46}$$

where $t = k_s \tau_s + t'_s$, $k_s = \lfloor \frac{t}{\tau_s} \rfloor$ and $t'_s = t \mod \tau_s$. Similarly,

$$\chi^{\tau_x}(t) = k_x \chi^{\infty}(\tau_x) + \chi^{\infty}(t'_x) \tag{47}$$

where $t = k_x \tau_x + t'_x$, $k_x = \lfloor \frac{t}{\tau_x} \rfloor$ and $t'_x = t \mod \tau_x$. Figure 2 depicts the two different rebalance frequencies under consideration. We need to prove the following inequality:

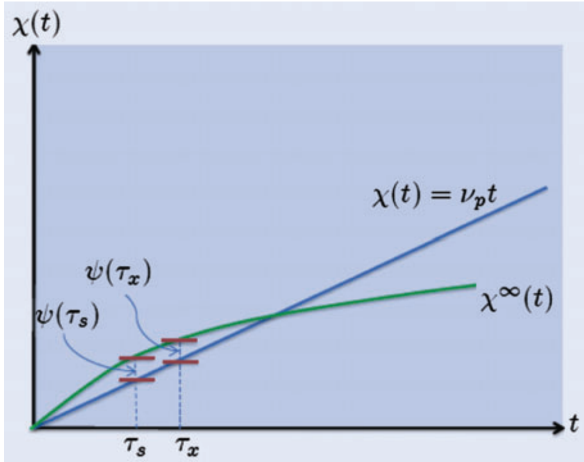

Figure 2. Illustration of excess growth at rebalance frequency τ_s and τ_x.

$$\begin{aligned}&\chi^{\tau_s}(t) > \chi^{\tau_x}(t) \\ &\Rightarrow k_s \chi^{\infty}(\tau_s) + \chi^{\infty}(t'_s) > k_x \chi^{\infty}(\tau_x) + \chi^{\infty}(t'_x) \\ &\Rightarrow k_s[\chi(\tau_s) + \psi^{\infty}(\tau_s)] + [\chi(t'_s) + \psi^{\infty}(t'_s)] \\ &\quad > k_x[\chi(\tau_x) + \psi^{\infty}(\tau_x)] + [\chi(t'_x) + \psi^{\infty}(t'_x)] \\ &\Rightarrow k_s[\nu_p \tau_s + \psi^{\infty}(\tau_s)] + [\nu_p t'_s + \psi^{\infty}(t'_s)] \\ &\quad > k_x[\nu_p \tau_x + \psi^{\infty}(\tau_x)] + [\nu_p t'_x + \psi^{\infty}(t'_x)] \\ &\Rightarrow \nu_p[k_s \tau_s - k_x \tau_x] + [k_s \psi^{\infty}(\tau_s) - k_x \psi^{\infty}(\tau_x)] \\ &\quad > \nu_p[t'_x - t'_s] + [\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)] \\ &\Rightarrow \nu_p[t - t'_s - t + t'_x] + [k_s \psi^{\infty}(\tau_s) - k_x \psi^{\infty}(\tau_x)] \\ &\quad > \nu_p[t'_x - t'_s] + [\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)] \\ &\Rightarrow k_s \psi^{\infty}(\tau_s) - k_x \psi^{\infty}(\tau_x) > \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)\end{aligned} \tag{48}$$

Since, $\tau_s < \tau_x$, we know that $k_s \geq k_x$. Let's define $\triangle k = k_s - k_x$ and substitute in the above inequality.

$$\begin{aligned}&(k_x + \triangle k)\psi^{\infty}(\tau_s) - k_x \psi^{\infty}(\tau_x) > \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s) \\ &\quad \Rightarrow k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] + \triangle k \psi^{\infty}(\tau_s) \\ &\quad > \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)\end{aligned} \tag{49}$$

We will now separately consider two possible cases for the value of $\triangle k$.

Case 1 – $\triangle k \geq 1$ The worst case scenario for equation (49) is when we consider the maximum possible value for the RHS expression. This will occur when $\psi^{\infty}(t'_s) \to 0$ and $\psi^{\infty}(t'_x) \to \psi^{\infty}(\tau_s)$(using lemma 5.5). Hence it is sufficient to prove:

$$\begin{aligned}&k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] + \triangle k \psi^{\infty}(\tau_s) \\ &\quad > max[\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)] \\ &\quad \Rightarrow k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] + \triangle k \psi^{\infty}(\tau_s) > \psi^{\infty}(\tau_s) \\ &\quad \Rightarrow k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] + [\triangle k - 1]\psi^{\infty}(\tau_s) > 0\end{aligned} \tag{50}$$

Again using lemma 5.5, we know $\psi^{\infty}(\tau_s) > \psi^{\infty}(\tau_x)$. We are considering investment horizons $t > \tau_x$. Hence $k_x \geq 1$. For this case, $[\triangle k - 1] \geq 0$. Lastly for valid passive strategy we need to have positive excess growth, i.e. $\psi^{\infty}(\tau_s) > 0$. With these conditions, inequality (50) will always hold.

Case 2 – $\triangle k = 0$ Under this scenario, inequality 49 is simplified to:

$$k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] > \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s) \tag{51}$$

We now show that the above inequality always holds by establishing $k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] > 0$ and $[\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)] < 0$. Since $\triangle k = 0, k_s = k_x$. To prove that $[\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)] < 0$, we will start from the definition of horizon t:

$$\begin{aligned} t &= k_s\tau_s + t'_s = k_x\tau_x + t'_x \\ &\Rightarrow k_x\tau_s + t'_s = k_x\tau_x + t'_x,\ \text{since } k_s = k_x \\ &\Rightarrow k_x(\tau_x - \tau_s) = t'_s - t'_x \\ &\Rightarrow t'_s - t'_x > 0,\ \text{since } \tau_x > \tau_s, k_x \geq 1 \\ &\Rightarrow t'_s > t'_x \Rightarrow \tau_s > t'_s > t'_x,\ \text{since } \tau_s > t'_s \\ &\Rightarrow \psi(\tau_s) > \psi(t'_s) > \psi(t'_x),\ \text{using lemma 5.4} \\ &\Rightarrow \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s) < 0 \end{aligned} \tag{52}$$

It is easy to prove that $k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] > 0$. We know that $k_x \geq 1$ since we are concerned with investment horizon $t \geq \tau_x$ here. We also know by means of lemma 5.5 that $\psi^{\infty}(\tau_s) > \psi^{\infty}(\tau_x)$. Hence we showed that inequality 51 is always true since LHS is positive whereas RHS is negative. □

6. Optimal rebalance frequency

A pertinent question is, for a given investment horizon T, if there exists a rebalance frequency at which the portfolio growth of periodic hybrid portfolio is maximum. We denote this *optimal rebalance frequency* as τ_o. As per lemma 5.4, since excess growth produced by passive strategy is increasing for horizon values $T \leq \tau_s$, the investor will never rebalance in order to maximize portfolio growth. For such low range of horizon values, optimal frequency $\tau_o = T$ implying adherence to passive strategy.

For horizon values $T > \tau_s$, one can search for the maximum value for hybrid portfolio growth using growth map theorem 5.3. As per theorem 5.6, the search can be restricted to the continuous time interval $[0\ \tau_s]$. For this search we only need to compute the partial passive portfolio growth trajectory in the interval $[0\ \tau_s]$. In short, for any horizon T, one has to examine a set of candidate rebalance frequencies before selecting the optimal choice of τ_o. We define this set $\Im(T)$ as *rebalance frequency domain*. $\Im(T)$ is an infinite set containing all possible numerical values in $[0\ \tau_s]$. The following lemma restricts rebalance frequency domain to only a predefined set of numerical values.

Lemma 6.1 *For any given horizon T, the optimal frequency of any log-optimal portfolio is a function of T and is restricted to only the factors of T with positive integer divisors, i.e.*

$$\tau_o(T) = \left\{\frac{T}{k} : \forall k \in \mathbb{N}^+\right\} \tag{53}$$

Proof Using the results of growth map theorem 5.3:

$$\chi^{\tau}(T) = k'\chi^{\infty}(\tau) + \chi^{\infty}(T - k'\tau) \tag{54}$$

where τ is a periodic rebalance frequency and $k' = \lfloor T/\tau \rfloor \in \mathbb{N}$ is the set of positive natural numbers including 0. Taking the partial derivative with respect to τ, we obtain:

$$\begin{aligned} \frac{\partial \chi^{\tau}(T)}{\partial \tau} &= k'\frac{\partial \chi^{\infty}(\tau)}{\partial \tau} + \frac{\partial \chi^{\infty}(T - k'\tau)}{\partial \tau} \\ &= k'\xi^{\infty}(\tau) - k'\xi^{\infty}(T - k'\tau) \end{aligned} \tag{55}$$

To find the rebalance frequency $\tau = \tau_o$ at which $\chi^{\tau}(T)$ is maximized, we set the partial derivative in equation (55) to zero and solve for τ_o:

$$\begin{aligned} &k'\xi^{\infty}(\tau_o) - k'\xi^{\infty}(T - k'\tau_o) = 0 \Rightarrow \xi^{\infty}(\tau_o) \\ &\quad = \xi^{\infty}(T - k'\tau_o) \Rightarrow \tau_o = T - k'\tau_o \Rightarrow \tau_o = \frac{T}{k' + 1} \end{aligned} \tag{56}$$

Note that $k' > 0$ when $T > \tau_s \geq \tau_o$. Substituting $k = k' + 1$ such that $k \in \mathbb{N}^+$ in equation (56) we can state the value of τ_o as a function of T as follows:

$$\tau_o(T) = \frac{T}{k} \tag{57}$$

Hence, $\Im(T)$ can only have the factors for T as specified in equation (53). □

Corollary 6.2 *Portfolio growth at $\tau_o(T) = \frac{T}{k}$ is given by:*

$$\chi^{\tau_o}(T) = k\chi^{\infty}\left(\frac{T}{k}\right) \tag{58}$$

Proof The derivation is straight forward. Substituting equation (56) in equation (54), the maximum portfolio growth when τ_o is used as the periodic rebalance frequency:

$$\begin{aligned} \chi^{\tau_o}(T) &= k\chi^{\infty}\left(\frac{T}{k}\right) + \chi^{\infty}\left(T - \frac{kT}{k}\right) \\ &= k\chi^{\infty}\left(\frac{T}{k}\right) + \chi^{\infty}(0) = k\chi^{\infty}\left(\frac{T}{k}\right) \end{aligned} \tag{59}$$

□

Lemma 6.1 restricts the rebalance frequency domain to only an infinite set of rational numbers. Henceforth we describe the positive integer k as the *rebalance divisor* of the portfolio. A rebalance divisor divides the prescribed horizon into k equal segments. The portfolio has to be rebalanced after each segment to attain the terminal portfolio growth. In this way the wealth grows following passive dynamics for k equal time periods. The portfolio growth at the end of the passive period $\frac{T}{k}$ multiplies k-fold at the end of the horizon T.

Notice that the rebalance frequency domain is entirely determined by the length of investment T independent of any other portfolio characteristics. For example, any given portfolio that is eligible for rebalancing (see Appendix A) and has $T = 30$ year investment horizon, optimal frequency τ_o for a log-optimal investor must belong to $\Im(30) = \{30, 15, 10, 7.5, 6, 5, 4.3, 3.8, 3.3, 3, \ldots\}$. If the intended horizon is $T = 12$ years, then the investor must choose τ_o from $\Im(12) = \{12, 6, 4, 3, 2.4, 2, 1.7, 1.5, \ldots\}$. Portfolio parameters only help determine the unique optimal frequency from the rebalance frequency domain.

Yet there are infinite choices of rebalance divisors. It turns out we can do even better by finding upper and lower bounds for the rebalance divisor. Consequently we restrict the rebalance frequency domain $\Im(T)$ to only a finite and countable set. Theorem 5.6 prescribes portfolio specific upper bound for τ_o, i.e. $\tau_o \leq \tau_s$. That gives a lower bound k_{mn} for rebalance divisor:

$$k_{mn} = max\left(1, \left\lceil\frac{T}{\tau_s}\right\rceil\right) \tag{60}$$

The rebalance frequency domain is further restricted and yet has infinite cardinality as follows:

$$\Im(T) = \left\{\frac{T}{k} : \forall k \in \mathbb{N}^+ \text{ and } k \geq k_{mn}\right\} \tag{61}$$

For our example portfolio the rebalance frequency domain is reduced to $\{3.3, 3, 2.7, 2.5, \ldots\}$ and $\{3, 2.4, 2, 1.7, 1.5, \ldots\}$ for 30 and 12 year horizons respectively. It turns out that under the assumption that instantaneous growth is *unimodal*, the optimal frequency τ_o will also have a lower bound further restricting the domain to a countable finite set.

LEMMA 6.3 *For any given portfolio with horizon T, assume $\xi^\infty(t)$ is* unimodal *in $0 \le t \le T$ with the unique maxima at $t = \tau_m$. The portfolio growth $\chi^{\tau_o}(T)$ is maximized for optimal frequency $\tau_o \in \Im(T)$ such that $\tau_o \ge \tau_m$.*

Proof Let's choose two rebalance divisors $k+1$ and k. The corresponding rebalance frequencies are $\tau_1 = \frac{T}{k+1}$ and $\tau_2 = \frac{T}{k}$ belong to $\Im(T)$. By definition $\tau_1 < \tau_2$. We need to prove that if $\tau_1 < \tau_m$, τ_2 will always outperform τ_1 in generating higher portfolio growth for horizon T. τ_2 may take any value on either side of τ_m. Mathematically, it suffices to prove the following:

$$\begin{aligned}
&\chi^{\tau_1}(T) < \chi^{\tau_2}(T)\\
&\Rightarrow (k+1)\chi^\infty(\tau_1) < k\chi^\infty(\tau_2), \text{ using lemma 6.1}\\
&\Rightarrow \frac{1}{k}\chi^\infty(\tau_1) < \chi^\infty(\tau_2) - \chi^\infty(\tau_1)\\
&\Rightarrow \frac{1}{k}\int_0^{\tau_1}\xi^\infty(t)\,\mathrm{d}t < \int_0^{\tau_2}\xi^\infty(t)\,\mathrm{d}t - \int_0^{\tau_1}\xi^\infty(t)\,\mathrm{d}t \qquad (62)
\end{aligned}$$

We consider two possible scenarios for inequality (62) as illustrated in figure 3. The first scenario in figure 3(a) is applicable when $\tau_1 < \tau_2 \le \tau_m$. The LHS of inequality (62) is given by:

$$\begin{aligned}
&\frac{1}{k}\int_0^{\tau_1}\xi^\infty(t)\,\mathrm{d}t\\
&= \frac{1}{k}(\text{Area of region 1-2-4-5})\\
&= \frac{1}{k}(\text{Area of region 1-3-4-5}) - \frac{1}{k}(\text{Area of region 2-3-4})\\
&= \left(\frac{1}{k}\right)(\tau_1)\xi^\infty(\tau_1) - \frac{\varphi_1}{k} = \frac{T}{k(k+1)}\xi^\infty(\tau_1) - \frac{\varphi_1}{k} \qquad (63)
\end{aligned}$$

The RHS of inequality (62) is given by:

$$\begin{aligned}
&\int_0^{\tau_2}\xi^\infty(t)\,\mathrm{d}t - \int_0^{\tau_1}\xi^\infty(t)\,\mathrm{d}t\\
&= (\text{Area of region 4-5-6-7-8})\\
&= (\text{Area of region 4-5-6-7}) + (\text{Area of region 4-7-8})\\
&= (\tau_2 - \tau_1)\xi^\infty(\tau_1) + \varphi_2 = \left(\frac{T}{k} - \frac{T}{k+1}\right)\xi^\infty(\tau_1) + \varphi_2\\
&= \frac{T}{k(k+1)}\xi^\infty(\tau_1) + \varphi_2 \qquad (64)
\end{aligned}$$

Since $\varphi_1, \varphi_2 > 0$, comparing equations (63) and (64) we prove that inequality (62) holds true.

With the help of scenario 1, we proved that the investor should prefer to use the *largest* value of rebalance frequency out of all possible rebalance frequencies in the interval of 0 to τ_m. This largest frequency shall have a corresponding rebalance divisor of $\lfloor \frac{T}{\tau_m} \rfloor$. We now show that the next higher periodic rebalance frequency shall always be a better choice for the investor to attain higher portfolio growth. All rebalance frequency candidates in the interval of 0 to τ_m shall be suboptimal for the investor. Therefore the optimal frequency τ_o shall always be higher than τ_m.

As illustrated in figure 3(b), τ_1 is the largest periodic rebalance frequency candidate less than τ_m with a rebalance divisor of $k+1$. τ_2 is the next higher rebalance frequency candidate with a rebalance divisor of k. We need to prove that the investor shall attain higher portfolio growth when rebalance frequency of τ_2 instead of τ_1 is used.

First we reckon that for sufficiently large value of horizon $T \gg \tau_m$, $k = \lfloor \frac{T}{\tau_m} \rfloor \gg 1$. For scenario 2, we can derive the values of τ_1 and τ_2 as follows:

$$\tau_1 = \frac{T}{k+1} = \frac{k\tau_m + t'}{k+1} = \tau_m - \frac{\tau_m - t'}{k+1}, \text{ where } 0 \le t' < \tau_m \qquad (65a)$$

$$\tau_2 = \frac{T}{k} = \frac{k\tau_m + t'}{k} = \tau_m + \frac{\tau_m}{k} \qquad (65b)$$

For $k \gg 1$, both $\frac{\tau_m - t'}{k+1}$ and $\frac{\tau_m}{k}$ shall be small compared to τ_m. Hence τ_1 and τ_2 will be very close to τ_m. This is illustrated in figure 3(b) where it is assumed $\xi^\infty(\tau_1) = \xi^\infty(\tau_2)$. Similar to scenario 1, the LHS of inequality (62) is given by equation (63). We now derive the RHS of inequality (62) as follows:

$$\begin{aligned}
&\int_0^{\tau_2}\xi^\infty(t)\,\mathrm{d}t - \int_0^{\tau_1}\xi^\infty(t)\,\mathrm{d}t\\
&= (\text{Area of region 4-5-6-7-8})\\
&= (\text{Area of region 4-5-6-7}) + (\text{Area of region 4-7-8})\\
&= (\tau_2 - \tau_1)\xi^\infty(\tau_1) + \varphi_2 = \left(\frac{T}{k} - \frac{T}{k+1}\right)\xi^\infty(\tau_1)\\
&\quad + \varphi_2 = \frac{T}{k(k+1)}\xi^\infty(\tau_1) + \varphi_2 \qquad (66)
\end{aligned}$$

Once again, since $\varphi_1, \varphi_2 > 0$, comparing equations (63) and (66) we prove that inequality (62) holds true for scenario 2. Therefore an investor will never choose any optimal frequency $\tau_o < \tau_m$. □

Thus we establish upper and lower bounds for the optimal frequency as τ_s and τ_m respectively. While τ_s determines the lower bound of rebalance divisor as per equation (60), τ_m determines the upper bound k_{mx} as per equation (67) below:

$$k_{mx} = max\left(1, \left\lfloor \frac{T}{\tau_m} \right\rfloor\right) \qquad (67)$$

Figure 4(a) illustrates the values of upper and lower bounds of rebalance divisors for various investment horizons. Notice that, for low values of horizon, optimal rebalance divisor $k_o = 1$ outperforms all other rebalance divisors. Hence the investor will follow passive strategy for such low value of investment horizon. As the horizon increases, we observe that k_o increases in steps of 1 resulting in faster optimal frequency τ_o for longer horizon. We refine the rebalance frequency domain further to a finite countable set as follows:

$$\Im(T) = \left\{\frac{T}{k} : \forall k \in \mathbb{N}^+ \text{ and } k_{mn} \le k \le k_{mx}\right\} \qquad (68)$$

For our example portfolio, the values for $\tau_m = 0.91$ and $\tau_s = 3.7$ lead to $k_{mn} = 9$ and $k_{mx} = 32$ for horizon $T = 30$ years. This restricts the rebalance frequency domain to $\Im(30) = \{3.3, 3, 2.7, 2.5, \ldots, 1.0, 0.97, 0.94\}$. Thus one has to search for only 24 possible candidates to find the optimal frequency $\tau_o(30)$. Similarly an investment horizon of 12 years leads to $k_{mn} = 4$ and $k_{mx} = 13$. One has to search the rebalance

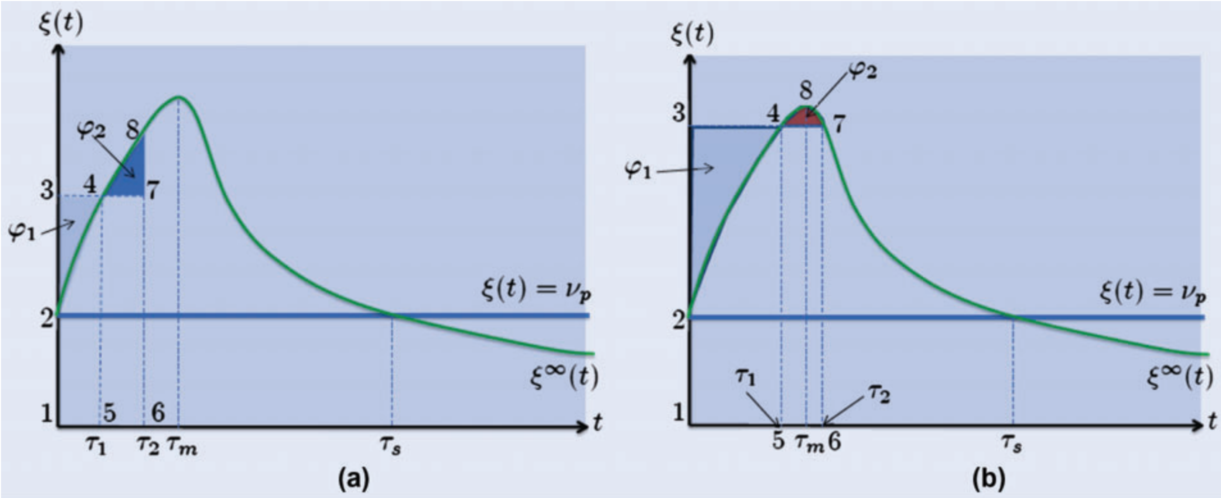

Figure 3. Two possible scenarios for deriving the lower bound for optimal frequency.

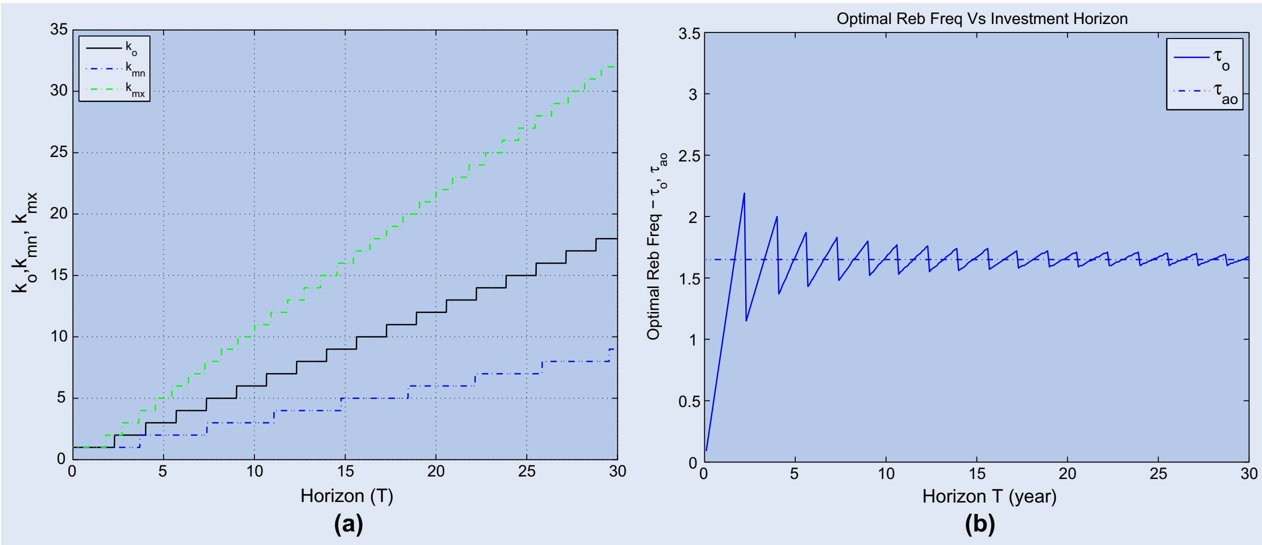

Figure 4. Optimal rebalance divisor and frequency change with investment horizon.

frequency domain $\Im(12)$={3, 2.4, 2, 1.7, 1.5, 1.33, 1.2, 1.09, 1, 0.92} consisting of only 10 discrete values to find $\tau_o(12)$.

Finally, one can define optimal rebalance frequency as:

$$\tau_o(T) = \tau, \quad \text{s.t.} \max_{k_{mn} \le k \le k_{mx}} k\chi^{\infty}\left(\frac{T}{k}\right) \tag{69}$$

6.1. Asymptotic optimal frequency

For our example portfolio, figure 4(b) illustrates the variation of $\tau_o(T)$ horizon T changes. It is interesting to observe the fluctuation pattern of $\tau_o(T)$. The fluctuation is vigorous for smaller values of T. As T increases, the amplitude of the fluctuation decreases. One would expect that for very large horizon, the optimal frequency will converge to a single value. We now prove that is indeed the case. Henceforth we will call this converged frequency as *asymptotic* optimal frequency and denote it by τ_{ao}.

THEOREM 6.4 *For sufficiently large values of investment horizon T, the optimal frequency will asymptotically converge to τ_{ao}, the time at which instantaneous growth becomes equal to passive portfolio growth ν_p^{∞}, i.e.*

$$\xi^{\infty}(\tau_{ao}) = \nu_p^{\infty}(\tau_{ao}), \quad \text{where } \nu_p^{\infty}(t) = \frac{\chi^{\infty}(t)}{t} \tag{70}$$

Proof Using the growth map theorem 5.3 we can write:

$$\chi^{\tau}(T) = \lfloor T/\tau \rfloor \chi^{\infty}(\tau) + \chi^{\infty}(T \mod \tau) \tag{71}$$

From theorem 5.6 we know that $\tau_o \le \tau_s$. Note that for optimality of τ, our interest is only in $\forall \tau \le \tau_s$. We assume that horizon T is sufficiently large, such that $T \gg \tau_s > \tau_o$. Thus, $\lfloor \frac{T}{\tau} \rfloor \gg 1$. We also know that $\tau > (T \mod \tau)$ implying that $\chi^{\infty}(\tau) > \chi^{\infty}(T \mod \tau)$ since the passive portfolio growth will always be an increasing function of time for $t < \tau_s$ courtesy lemma 5.4. Combining these two, we get $\lfloor \frac{T}{\tau} \rfloor \chi^{\infty}(\tau) \gg \chi^{\infty}(T \mod \tau)$. In other words, the first term involving floor function shall dominate the second term. Hence, as a first order simplification we can ignore the second term:

$$\chi^{\tau}(T) \approx \lfloor \frac{T}{\tau} \rfloor \chi^{\infty}(\tau) \tag{72}$$

Furthermore, for $T \gg \tau$, $\lfloor \frac{T}{\tau} \rfloor \approx \frac{T}{\tau}$. Applying this second order of simplification, we obtain:

$$\chi^{\tau}(T) \approx \frac{T}{\tau}\chi^{\infty}(\tau) \tag{73}$$

In order to determine the value of τ at which the LHS of equation (73) is maximized, we take the partial derivative:

$$\begin{aligned}\frac{\partial \chi^{\tau}(T)}{\partial \tau} &\approx \frac{\partial\left(\frac{T}{\tau}\chi^{\infty}(\tau)\right)}{\partial \tau}\\ &\approx -\frac{T}{\tau^2}\chi^{\infty}(\tau) + \frac{T}{\tau}\frac{\partial \chi^{\infty}(\tau)}{\partial \tau}\\ &\approx \frac{T}{\tau}\left(\frac{\partial \chi^{\infty}(\tau)}{\partial \tau} - \frac{1}{\tau}\chi^{\infty}(\tau)\right)\end{aligned} \tag{74}$$

Setting (74) to zero, we obtain the value of τ_{ao} at which the hybrid portfolio growth value is maximized.

$$\begin{aligned}&\frac{T}{\tau_{ao}}\left(\left|\frac{\partial \chi^{\infty}(\tau)}{\partial \tau}\right|_{\tau=\tau_{ao}} - \frac{1}{\tau_{ao}}\chi^{\infty}(\tau_{ao})\right) = 0\\ &\Rightarrow \left|\frac{\partial \chi^{\infty}(\tau)}{\partial \tau}\right|_{\tau=\tau_{ao}} - \frac{1}{\tau_{ao}}\chi^{\infty}(\tau_{ao}) = 0,\\ &\qquad \text{since } T \neq 0 \text{ and } \tau_{ao} \neq \infty\\ &\Rightarrow \left|\frac{\partial \chi^{\infty}(\tau)}{\partial \tau}\right|_{\tau=\tau_{ao}} = \frac{1}{\tau_{ao}}\chi^{\infty}(\tau_{ao})\\ &\Rightarrow \xi^{\infty}(\tau_{ao}) = \nu_p^{\infty}(\tau_{ao})\end{aligned} \tag{75}$$

□

For our example portfolio the value of τ_{ao} is 1.65 years. As depicted in figure 4(b) τ_o fluctuates around τ_{ao} as the investment horizon changes.

7. Simulation results

As part of this study, we used Monte Carlo simulation to examine the accuracy of the analytical results presented in this paper. The simulation is run for the familiar portfolio example with four assets. The asset price equation used in generating Monte Carlo paths is given as follows (Hull 2011):

$$S(t+dt) = S(t)e^{(\mu-\frac{\sigma^2}{2})dt+\sigma\epsilon\sqrt{dt}} \tag{76}$$

To reduce variance in simulation, an antithetic variable is used (Hull 2011). For every asset price path generated using a set of random correlated standard normal variables ϵ, another path using $-\epsilon$ is generated. A total of 20 000 such Monte Carlo paths for correlated prices are generated using a discrete time step of 0.01 year for both T and τ.

An initial \$1 investment fund is distributed among the four assets as per the optimized proportion determined by $\mathbf{w}$. For each price path, the allocated funds are periodically rebalanced to the initial optimal weights $\mathbf{w}$ at the specified optimal frequency. Portfolio growth is computed as the average of the terminal portfolio values over all the price paths.

As plotted in figure 5(a), passive portfolio growth values from simulation closely track the values computed analytically using equation (27). Analytical approach slightly overestimates the portfolio growth values at the short-end of investment horizon while underestimating for longer horizons. For the entire horizon, except for the first two years, the analytical passive portfolio growth values are within +/−5% (figure 5(b)) of the true values obtained in simulation. Higher error percentages observed during the initial two year period is mostly because of the division by very small numbers.

To examine the accuracy of growth map theorem 5.3, hybrid portfolio growth is computed according to the theorem for every possible combination of T and τ using the passive trajectory of portfolio growth obtained in simulation as shown in figure 6(a). Except for small values of τ, there is very little deviation of the computed hybrid portfolio growth from the values obtained in simulation. Small but visible error for low values of τ is attributed to the inherent estimation error in simulation data.

Similar to the analytically computed optimal frequency, the true values obtained in simulation also exhibit saw-tooth pattern especially for lower values of horizon (figure 6(b)). The amplitude of fluctuation diminish for large horizons. The true optimal frequencies have a midpoint of 2.6 years compared to a more conservative analytical estimate of $\tau_{ao} = 1.65$ years. The true values suggest longer passivity with longer rebalancing intervals for investors than the recommendations obtained analytically.

Potential *loss to investor* is assessed for using a rebalancing frequency τ instead of the true underlying optimal frequency $\widehat{\tau_o}$ found in simulation. A wide hat $\widehat{(.)}$ is used to denote a parameter predicted by simulation. The loss is estimated by the fraction of the log wealth the investor gives up by using τ instead of $\widehat{\tau_o}$:

$$L(t,\tau) = \frac{\widehat{\chi}^{\widehat{\tau_o}(t)}(t) - \widehat{\chi}^{\tau(t)}(t)}{\widehat{\chi}^{\widehat{\tau_o}(t)}(t)} \tag{77}$$

Corroborating our hitherto claims, as depicted in figure 7(a), hybrid optimal strategy fares better than active strategy in terms of limiting investor loss. In spite of improved performance relative to active strategy, there is small albeit observable loss in using analytically predicted τ_o. The investor incurs higher loss in active continuously rebalanced strategy even without considering the adversarial effect of transaction cost. Following hybrid optimal strategy, the investor loss in the long run is limited to 1.6% compared to a much higher percentage of 6.2% for active strategy. As anticipated, passive strategy is far suboptimal with higher than 25% loss in the long run. The standard error estimate† of realized portfolio growth $\widehat{\chi}^{\tau_o(t)}(t)$ used in the loss calculation is small as shown in figure 7(b).

The simulation is also run for a more realistic portfolio comprising of four real risky assets and the risk-free asset. The representative risky assets are chosen from S&P 100 stock index representing four different industry sectors. Exxon Mobile Corp (ticker: XOM), Amgen Inc (ticker: AMGN) and Verizon Communications Inc (ticker: VZ) stocks are picked from oil, pharmaceutical and communication industries respectively. Gold Trust exchange traded fund (ticker: GLD) is the fourth risky asset representing the commodity market. The portfolio parameters are computed using the historical daily stock prices for six years recorded between 2007 and 2013. Analogous simulation results are obtained reinforcing the findings of earlier simulation with the fictitious portfolio scenario.

†The standard error estimate is square root of the ratio of variance of estimation to the number of simulation trials (Hull 2011).

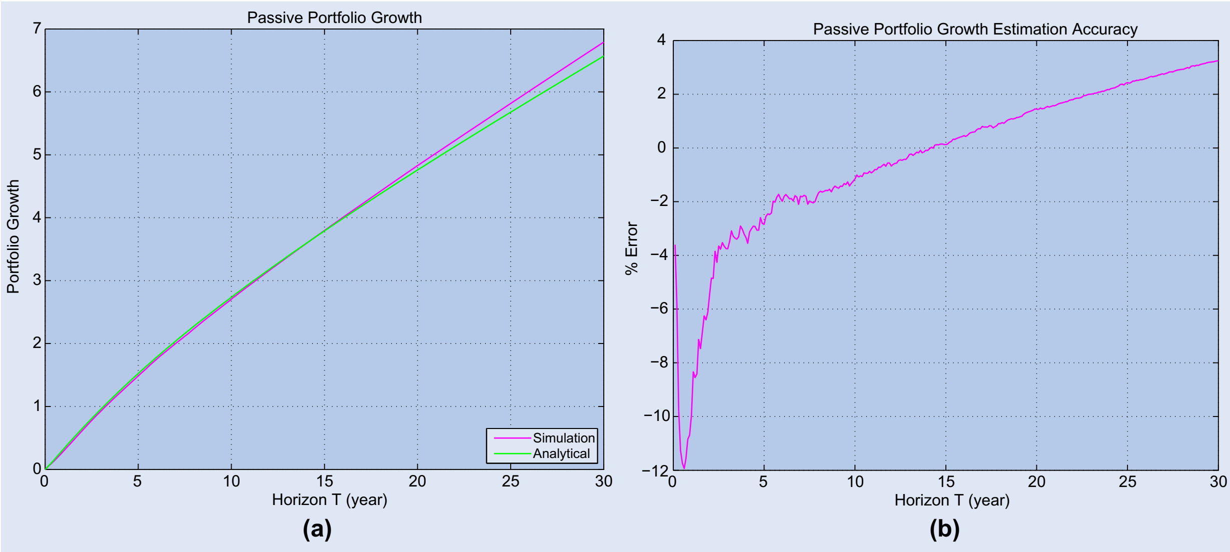

Figure 5. Accuracy of passive portfolio growth estimation.

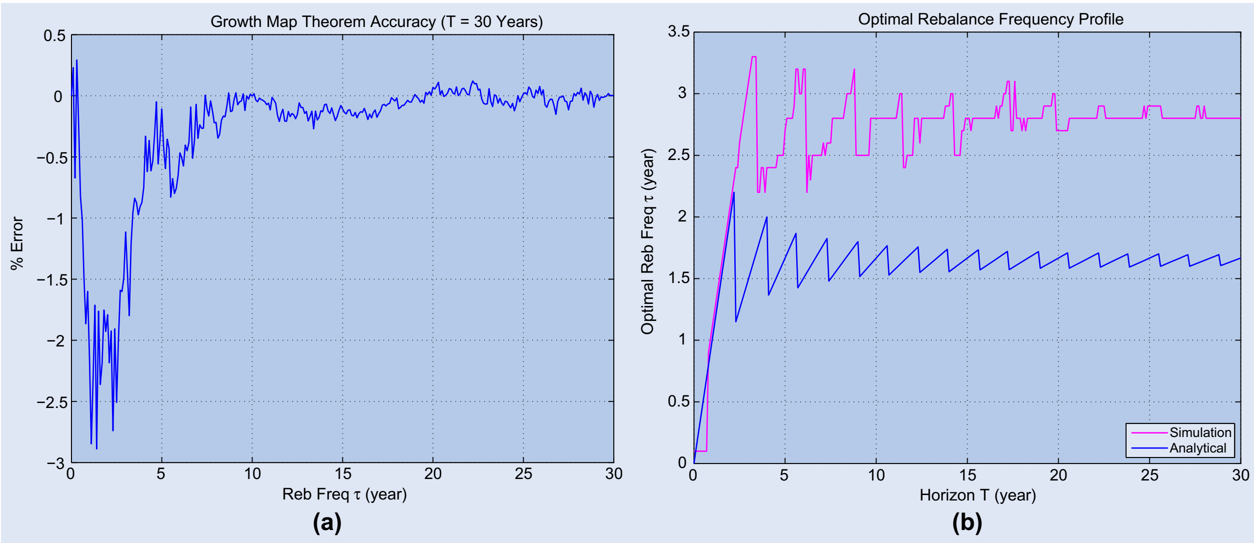

Figure 6. Accuracy of growth map theorem and optimal frequency estimation.

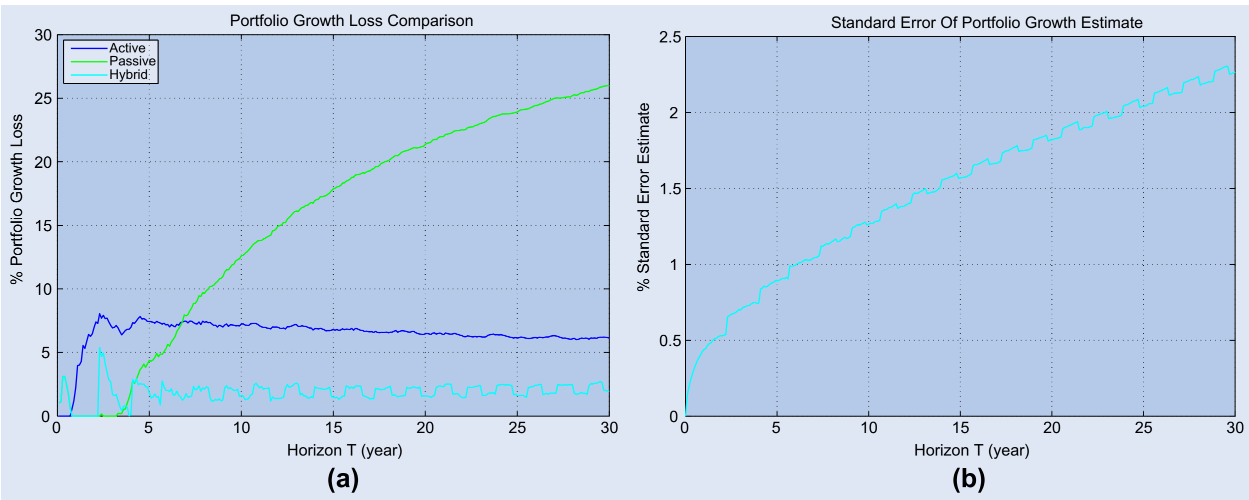

Figure 7. Loss and standard error of estimate of portfolio growth.

8. Conclusion

In the log-optimal investment strategy, to maximize the investor's log utility in the long run the investor continuously rebalances to the initial optimal asset weights. A more realistic investment proposition is to maximize the log-utility by rebalancing the portfolio periodically at discrete non-zero time intervals for a finite desired investment horizon. We investigated the existence of such a periodic optimal frequency by first developing an analytical framework to study the nature of the portfolio growth if it is left passive.

We used the Fenton–Wilkinson log-normality assumption for a sum of log-normal variables to determine the first and second moments of the log of portfolio growth for the passive investment. The underlying log-normal assumption in the Fenton–Wilkinson approach made it possible to derive an analytical expression for a passive portfolio mean and variance analogous to that of the active strategy.

We established an important relationship called the growth map theorem. For any given investment horizon and rebalance frequency, with the help of the theorem one can compute the portfolio growth under a hybrid strategy by merely knowing the evolution of the portfolio growth under the passive strategy. First we identified a special rebalancing frequency τ_s and showed that using a different rebalancing frequency $\tau > \tau_s$ is always suboptimal in the sense that it produces lower terminal portfolio growth. We then restricted the choice of optimal frequency τ_o to only those discrete factors of horizon within the interval of $[\tau_m \ \tau_s]$. Finally we showed that for sufficiently large investment horizons, the optimal frequency converges to an asymptotic value τ_{ao}. At τ_{ao}, the expected portfolio growth rate is equal to the instantaneous growth rate when the portfolio is left to grow passively.

Simulation studies showed that our analytical framework predicts the passive portfolio growth very accurately. It slightly underestimates at the short end, while slightly overestimating at the long end of the horizon. The growth map theorem also accurately transforms the passive portfolio values to hybrid values. The discrepancy in the passive portfolio value estimation results in a relatively smaller optimal frequency estimation. We showed that there is considerable improvement in investor log loss when the investor uses the estimated τ_o. In particular, for our portfolio example, for medium to long term investors the log loss was found to be less than 2% compared to 6% or higher if the investor had used active continuous rebalancing strategy.

We also derived the condition for existence of a rebalance possibility for a given set of input asset characteristics. The rebalance opportunity exists if the time zero instantaneous growth under the passive strategy is higher than the corresponding value ν_p under active strategy. If this criterion is not satisfied, the investor will prefer to follow the active strategy for some positive initial duration. Determination of an appropriate rebalancing strategy when this criterion is violated is a future research topic.

As we have noted before, we have ignored transaction costs in our models. This simplification needs to be avoided by assuming the appropriate transaction cost model suitable for the analytical framework. With reasonable transaction costs one should derive a more accurate optimal frequency than the conservative estimations presented in this paper.

An alternative to obtaining a higher growth rate is by reducing portfolio variance by diversification. For example, if we combine several stocks with the same mean and variance, the portfolio variance will reduce and the growth rate will increase. So a potential future research area is to investigate whether the diversification by itself will use up the potential for improvement that can be obtained by our proposed rebalancing method.

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Appendix A. Eligibility for discrete-time rebalancing

The investor should take advantage of discrete-time periodic rebalancing if the passive strategy can yield higher instantaneous growth relative to active strategy for a non-zero investment horizon. It turns out that the time zero instantaneous growth are equal under passive and active strategies, i.e.

$$\xi^{\infty}(0) = \xi = \nu_p \tag{A1}$$

We know two properties of instantaneous growth. First, as per equation (37) active strategy has constant instantaneous growth. Secondly both active and passive strategy start out with the same instantaneous growth at time zero. Consequently, to obtain higher passive portfolio growth for a non-zero initial time interval, the passive portfolio must have an increasing instantaneous growth at time zero. Mathematically this translates to the following conditions for eligibility for discrete-time periodic rebalancing for a given portfolio:

$$[P''(0) - P'(0)^2] - \frac{1}{2}\left[Q''(0) - Q'(0)^2\right] + 2P'(0)Q'(0) \geq 0 \quad \text{(A2)}$$

$P'(0)$, $P''(0)$, $Q'(0)$ and $Q''(0)$ are the time zero values of the first and second derivatives of $P(t)$ and $Q(t)$ respectively and are given as follows:

$$P'(0) = \sum_{i=1}^{N+1} w_i \mu_i \quad \text{and} \quad P''(0) = \sum_{i=1}^{N+1} w_i {\mu_i}^2 \quad \text{(A3a)}$$

$$Q'(0) = \sum_{i,j=1}^{N+1} w_i w_j \sigma_{ij} \quad \text{and}$$

$$Q''(0) = \sum_{i,j=1}^{N+1} w_i w_j \sigma_{ij} [2(\mu_i + \mu_j) + \sigma_{ij}] \quad \text{(A3b)}$$

where,

$P(t)$ expected portfolio growth at time t, given by equation (17)
$Q(t)$ variance of portfolio growth at time t, given by equation (21)

Hence, the opportunity to take advantage of discrete-time periodic rebalancing is entirely determined by the set of asset mean and covariance characteristics.