Time Series Models

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Chapter 11: Time Series Models

These notes are based on Chapter 11 of the book Mathematics & Statistics for Financial Risk Management by Michael Miller.

Time Series: an equation or set of equations describing how a random variable or variables evolve(s) over time.

NB: Could also refer to the observed data sample from such a random variable.

Random Walk

One of the conceptually simplest time series models is the so-called random walk.

$$
\begin{aligned}
x_t &= x_{t-1} + u_t \\
E(u_t) &= 0 \\
E(u_t^2) &= \sigma^2 \\
E(u_s u_t) &= 0 \quad \forall \quad s \neq t
\end{aligned}
$$

x is equal to its value in the previous period, plus a random disturbance, u_t

u_t is zero-mean with a constant variance

- It is implied that u's from different periods will be uncorrelated
- x_{t-1} is referred to as lag of x_t

We can also think in terms of changes in x

$$\Delta x_t = x_t - x_{t-1} = u_t$$

NB: Δx_t has all the properties of the disturbance

Evolution over time:

$$\begin{aligned} x_t &= x_{t-1} + u_t \\ x_{t-1} &= x_{t-2} + u_{t-1} \\ &\vdots \\ x_{t-i} &= x_{t-i-1} + u_{t-i} \end{aligned}$$

By recursive substitution, we can see that

$$x_t = x_{t-1} + u_t = x_{t-2} + u_{t-1} + u_t = x_0 + \sum_{i=1}^{t} u_i$$

NB: x_t is the initial value plus the accumulation of all the disturbances over time (random innovations)

We can now calculate the conditional mean and variance:

$$\begin{aligned} E(x_t|x_0) &= x_0 \\ Var(x_t|x_0) &= t\sigma^2 \end{aligned}$$

NB: the variance is proportional to time (it is explosive). Volatility is proportional to the square root of time.

For a random walk our best guess to predict the next observed value is simply the current value. But the probability of finding it near the current value becomes increasingly small.

For all practical purposes this makes a random walk unpredictable (unforecastable) or totally random.

NB: this is a good starting point for modeling prices that are informationally efficient.

See the paper Proof that Properly Anticipated Prices Fluctuate Randomly by Paul Samuelson.

Though there are some problems:

1. For equities we expect a positive return over time to compensate risk-averse investors for holding the investment.
2. Prices (typically) cannot be negative

The Drift-Diffusion Model

We can add a constant term as follows:

$$p_t = \alpha + p_{t-1} + u_t$$

The random variable (p) is now a function of:

1. The previous value p_{t-1}
2. The constant term α
3. The disturbance term u_t

Just as before:

- Variance of u_t is constant over time
- u's are uncorrelated

If pt is the log price, then

$$r_t = \Delta p_t = \alpha + u_t$$

- α often referred to as the drift term
- u_t often referred to as the diffusion term

NB: in physics this process is often used to describe the motion of particles (also known as Brownian motion)

When equity returns follow a drift-diffusion process we say that equity markets are perfectly efficient

- the limiting ase of Hayek's information aggregation process
- a simple way to parameterize the Efficient Markets Hypothesis (EMH) from received theory
- mathematically, expected conditional and unconditional returns are equal

$$E[r_t|r_{t-1}] = E[r_t] = \alpha$$

- If this were not true - if there were some information in the past that was helpful for forecasting tomorrow's return then speculators would step in and push prices towards this result!

We can still do recursive substitution

$$p_t = 2\alpha + p_{t-2} + u_t + u_{t-1} = t\alpha + p_0 + \sum_{i=1}^{t} u_i$$

And as before:

- $E[p_t | p_0] = p_0 + t\alpha$
- $Var[p_t | p_0] = t\sigma^2$

NB: variance is still proportional to time, however the mean is no longer constant but rather now moves around at rate α. Thus it is called the *drift* term.

Autoregression

Now modify the model by multiplying the lagged term by a constant:

$$r_t = \alpha + \rho r_{t-1} + u_t$$

- both α and ρ are constants
- depending on the value of ρ, the model's behavior can vary greatly
- when $|\rho| < 1$ it will produce a stable time series
- when $\rho = 1$ we get the random walk as a special case
- when $|\rho| > 1$ the system is explosive in a way that is not particularly interesting for financial time series

This is known as an *autoregressive* model ($AR(1)$ above)

We can write down the $AR(p)$ generalization:

$$r_t = \alpha + \rho_1 r_{t-1} + \rho_2 r_{t-2} + \ldots + \rho_p r_{t-p} + u_t$$

As before, do recursive substitution

$$r_t = \alpha \sum_{i=0}^{p} \rho^i + \rho^p r_{t-p} + \sum_{i=0}^{p-1} \rho^i$$

The condition mean and variance become:

$$\begin{aligned} E[r_t|r_{t-p}] &= \frac{1-\rho^p}{1-\rho}\alpha + \rho^p r_{t-p} \\ Var[r_t|r_{t-p}] &= \frac{1-\rho^{2p}}{1-\rho^2}\sigma^2 \end{aligned}$$

NB: for $|\rho| > 1$ the variance grows exponentially.

For values $|\rho| < 1$ the process is stable

If we extend back in time, and let p approach infinity, ρ^p becomes increasingly small causing $\rho^p r_{t-p}$ to approach zero

$$r_t = \frac{1}{1-\rho}\alpha + \sum_{i=0}^{\infty} \rho^i u_{t-i}$$

Using some results from geometric series, we can write the mean and variance

$$
\begin{aligned}
E[r_t] &= \frac{1}{1-\rho}\alpha \\
Var[r_t] &= \frac{1}{1-\rho}\sigma^2
\end{aligned}
$$

For values of $|\rho|$ less than one, as n approaches infinity, the initial state of the system ceases to matter. The mean and variance are only a function of the constants.

Stationarity

We say that a random variable X is stationary if for all t and n:

$$\begin{aligned} E[x_t] &= \mu \quad \text{and} \quad |\mu| < \infty \\ Var[x_t] &= \sigma^2 \quad \text{and} \quad \left|\sigma^2\right| < \infty \\ Cov[x_t, x_{t-n}] &= \sigma_{t,t-n} \end{aligned}$$

where μ, σ^2, and $\sigma_{t,t-n}$ are constants.

Moving Average

Besides $AR(p)$ models, the other major class of time series models is moving average models. An $MA(q)$ sereis takes the form:

$$x_t = u_t + \theta_1 u_{t-1} + \cdots + \theta_q u_{t-q}$$

We can combine the two to form the $ARMA(p,q)$ model as well:

$$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + \cdots + \rho_p x_{t-p} + u_t + \theta_1 u_{t-1} + \cdots + \theta_q u_{t-q}$$