Stochastic Finance (FIN 519)
Homework Solutions

Instructor: Jaehyuk Choi

2017-18 Module 3 (Spring 2018)

HW 1-1 Using martingale property, re-drive that

$$E(\tau) = AB \quad \text{for} \quad \tau = \min\{n : S_n = A \text{ or } S_n = -B\}.$$

Answer You can find the answer in the textbook section 2.3.

HW 1-2. SCFA Exercise 2.4

Answer From the hint, let us define

$$A_{n+1} = A_n + E[(M_{n+1} - M_n)^2 | \mathcal{F}_n]$$

and prove the three required properties:

(i) N_n is a martingale.

$$\begin{aligned}
E[N_{n+1} \,|\, \mathcal{F}_n] &= E[M_{n+1}^2 - A_{n+1} \,|\, \mathcal{F}_n] \\
&= E[2M_{n+1}M_n - M_n^2 - A_n \,|\, \mathcal{F}_n] \\
&= 2E[M_{n+1} \,|\, \mathcal{F}_n]\, M_n - M_n^2 - A_n \\
&= M_n^2 - A_n = N_n.
\end{aligned}$$

(ii) $A_{n+1} \geq A_n$ is trivial.

(iii) A_n is non-anticipating because it is defined via the expectation under $\mathcal{F}_n$.

HW 2-1 SCFA Exercise 3.1 (b) Assume that $0 \leq s \leq t \leq 1$. Compute the self-covariance of Brownian bridge, $\mathrm{Cov}(U_s, U_t)$ where $U_t = B_t - tB_1$.

Answer

$$\begin{aligned}
\mathrm{Cov}(U_s, U_t) &= E\Big((B_s - sB_1)(B_t - tB_1)\Big) = E\Big(B_sB_t - sB_1B_t - tB_sB_1 + stB_1^2\Big) \\
&= \min(s,t) - s\min(1,t) - t\min(s,t) + st = s(1-t)
\end{aligned}$$

HW 2-2 Calculate $\mathrm{Var}(aB_t + bB_s)$ for constants a and b.

Answer

$$\begin{aligned}
\mathrm{Var}(aB_t + bB_s) &= E[a^2\, B_t^2 + 2abB_t\, B_s + b^2\, B_s^2] \\
&= a^2\, t + 2ab\, \min(s,t) + b^2\, s
\end{aligned}$$

HW 2-3 Derive the price of down-and-out call option with knock-out strike K_D and option strike K. (Obviously, $K_D < F$ and $K_D < K$) See the derivation for up-and-out call option and down-and-out digital option from the previous HW and exams.

Answer Assume $B_T^M = \max_{0\le t\le T} B_t$ and $B_T^m = \min_{0\le t\le T} B_t$. From textbook and class, we know

$$P(\sigma B_T^M < v,\ \sigma B_T < u) = P(B_T^M < v/\sigma,\ B_T < u/\sigma) = N\Big(\frac{u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{u-2v}{\sigma\sqrt{T}}\Big),$$

where $N(\cdot)$ is the normal distribution CDF. Using the reflection, $B_t \to -B_t$, we get

$$(-B)_T^m = \min_{0\le t\le T}(-B_t) = -\max_{0\le t\le T} B_t = -B_T^M$$

and

$$P(\sigma B_T^m > v,\ \sigma B_T > u) = N\Big(\frac{-u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{2v-u}{\sigma\sqrt{T}}\Big).$$

As the stock price is given as $S_T = F + \sigma B_T$,

$$P(S_T^m > v,\ S_T > u) = N\Big(\frac{F-u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{2v-u-F}{\sigma\sqrt{T}}\Big).$$

The probability density function on u with the joint condition, $\sigma B_T^m > v$ is obtained from the partial derivative w.r.t. u (with negative sign),

$$f(u) = \frac{1}{\sigma\sqrt{T}}\left(n\Big(\frac{F-u}{\sigma\sqrt{T}}\Big) - n\Big(\frac{2v-u-F}{\sigma\sqrt{T}}\Big)\right) \quad \text{for} \quad -\infty < v \le u.$$

Let $z = (u-F)/\sigma\sqrt{T}$, $d = (F-K)/\sigma\sqrt{T}$ and $d^* = (F-K_D)/\sigma\sqrt{T}$. Then, the down-and-out call option price is given as

$$\begin{aligned} C(K, K_D) =& \int_{u=K}^{\infty} (u-K)\, f(u) du = \int_{z=-d}^{\infty} (F-K+\sigma\sqrt{T}\,z)(n(z) - n(z+2d^*))dz \\ =&(F-K)N(d) + \sigma\sqrt{T}\,n(d) - (F-K-2d^*\sigma\sqrt{T})N(d-2d^*) - \sigma\sqrt{T}\,n(d-2d^*). \end{aligned}$$

The first two terms are exactly the regular call option price, $C(K) = (F-K)N(d) + \sigma\sqrt{T}\,n(d)$. Therefore, the down-and-out option is cheaper than the regular option by $(F-K-2d^*\sigma\sqrt{T})N(d-2d^*) + \sigma\sqrt{T}\,n(d-2d^*)$.

We can verify two cases:

1. If $K_D \to -\infty$ ($d^* \to \infty$), $C(K, K_D) = C(K)$ because the probably of being knocked out is zero. It is indeed the case because $N(d-2d^*) = n(d-2d^*) = 0$.
2. If $K_D \to F$ from below ($d^* \to 0$) on the other hand, the knock-out probability approaches to 100%, so the price should be zero. This is also the case from the formula.

HW 3-1 SCFA Exercise 8.2 The notation $h \in C^1(\mathbb{R}^+)$ means that the function $h(s)$ is differentiable for $s > 0$.

Answer From the SDE of $h(t)B_t$,

$$d(h(t)B_t) = h(t)dB_t + h'(t)B_t dt$$

we get

$$\int_0^t h(s)dB_s = h(t)B_t - \int_0^t h'(s)B_s\,ds.$$

Using that $B_T = \int_0^T dB_t$, we also get another useful result:

$$\int_0^T h'(s)B_s\,ds = h(T)B_T - \int_0^T h(t)dB_t = \int_0^T (h(T) - h(t))dB_t$$

HW 3-2 For a standard BM B_t, let

$$N_t = B_t^3 - 3t\,B_t.$$

(i) Prove that N_t is a martingale. (Hint: use Proposition 8.1) (ii) By applying Itô's lemma, express N_t as a stochastic integration. (iii) Calculate the variance of N_t.

Answer We set $N_t = f(t, B_t)$ where $f(t,x) = x^3 - 3tx$. Applying Itô's lemma,

$$dN_t = 3(B_t^2 - t)dB_t + 3B_t(dB_t)^2 - 3B_t dt = 3(B_t^2 - t)dB_t$$

As there is no drift term, N_t is a martingale and is represented as a stochastic integral:

$$N_t = \int_0^t 3(B_s^2 - s)dB_s.$$

The variance is calculated as

$$\begin{aligned}\mathrm{Var}(N_t) &= \int_0^t E[3^2(B_s^2 - s)^2]ds = 9\int_0^t (E(B_s^4) - 2sE(B_s^2) + s^2)ds \\ &= 9\int_0^t (3s^2 - 2s^2 + s^2)ds = 6t^2\end{aligned}$$

HW 3-3 SCFA Exercise 8.4 The sub-problem (b) is understood better after **HW 3-2** is solved.

Answer (a) If $f(t,x) = \phi(t)\psi(x)$, the condition $f_t = -\frac{1}{2}f_{xx}$ yields to

$$2\frac{\phi_t(t)}{\phi(t)} = -\frac{\psi_{xx}(x)}{\psi(x)} = \lambda,$$

where λ is a constant. If $\lambda > 0$ ($\lambda = \alpha^2$ for some α), we get the GBM solution,

$$\phi(t) = \phi(0)e^{-\alpha^2 t/2} \quad \text{and} \quad \psi(x) = Ae^{\alpha x} + Be^{-\alpha x} \text{ for constants } A, B$$

$$M_t = (Ae^{\alpha B_t} + Be^{-\alpha B_t})e^{-\alpha^2 t/2}$$

If $\lambda < 0$ ($\lambda = -\alpha^2$ for some α), we get

$$M_t = (A\cos(\alpha B_t) + B\sin(-\alpha B_t))e^{\alpha^2 t/2}$$

If $\lambda = 0$, we get $\phi(t) = \phi(0)$ and $\psi(x) = Ax + B$, therefore we have

$$M_t = AB_t + B.$$

(b)

$$\begin{aligned}M_t &= 1 + (\alpha B_t - \alpha^2 t/2) + \frac{1}{2}(\alpha B_t - \alpha^2 t/2)^2 + \frac{1}{6}(\alpha B_t - \alpha^2 t/2)^3 + \frac{1}{24}(\alpha B_t - \alpha^2 t/2)^4 + \cdots \\ &= 1 + (B_t)\alpha + \frac{1}{2}(B_t^2 - t)\alpha^2 + \frac{1}{6}(B_t^3 - 3tB_t)\alpha^3 + \frac{1}{24}(B_t^4 - 6tB_t^2 + 3t^2)\alpha^4 + \cdots\end{aligned}$$

We get the first five martingales as below:

$$\begin{aligned}H_0(t, B_t) &= 1 \\ H_1(t, B_t) &= B_t \\ H_2(t, B_t) &= \frac{1}{2}(B_t^2 - t) \\ H_3(t, B_t) &= \frac{1}{6}(B_t^3 - 3tB_t) \\ H_4(t, B_t) &= \frac{1}{24}(B_t^4 - 6tB_t^2 + 3t^2).\end{aligned}$$

HW 3-4 SCFA Exercise 9.6 We first derive the mean and the variance of lognormal distribution, $Y \sim \exp(\mu + \sigma Z)$, where Z is a standard normal distribution (see the same result at (WIKIPEDIA)):

$$E(Y) = e^{\mu + \sigma^2/2}$$

and

$$\begin{aligned}\mathrm{Var}(Y) &= e^{2\mu} E((e^{\sigma Z} - e^{\sigma^2/2})^2) = e^{2\mu} E(e^{2\sigma Z} - 2e^{\sigma Z + \sigma^2/2} + e^{\sigma^2}) \\ &= e^{2\mu}(e^{2\sigma^2} - 2e^{\sigma^2} + e^{\sigma^2}) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1).\end{aligned}$$

Back to the problem, for $t = kh$ and $s = (k-1)h$,

$$R_k(h) + 1 = \frac{X_t}{X_s} = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) h + \sigma(B_t - B_s)\right) = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) h + \sigma\sqrt{h} Z\right),$$

where Z is standard normal distribution. Since $R_k(h) + 1$ is a lognormal distribution with $\mu := (\mu - \sigma^2/2)h$ and $\sigma := \sigma\sqrt{h}$, we obtain the mean and the variance of $R_k(h) + 1$ as

$$\begin{aligned}E(R_k(h) + 1) &= \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) h + \frac{\sigma^2}{2} h\right) = e^{\mu h} \\ \mathrm{Var}(R_k(h) + 1) &= e^{2\mu h}(e^{\sigma^2 h} - 1).\end{aligned}$$

Therefore,

$$E(R_k(h)) = e^{\mu h} - 1 \quad \text{and} \quad \mathrm{Var}(R_k(h)) = e^{2\mu h}(e^{\sigma^2 h} - 1)$$

and it follows that

$$\begin{aligned}\mathrm{Var}(R_k(h)) &= (1 + E(R_k(h))^2(e^{\sigma^2 h} - 1) \\ \sigma^2 &= \frac{1}{h}\log\left(1 + \frac{\mathrm{Var}(R_k(h))}{(1 + E(R_k(h))^2}\right).\end{aligned}$$

From sample data, we can estimate the mean and the variance as

$$E(R_k(h)) = \frac{1}{n}\sum_{k=1}^{n} R_k(h), \quad \mathrm{Var}(R_k(h)) = \frac{1}{n-1}\sum_{k=1}^{n}(R_k(h) - E(R_k(h))).$$

Below are the s and σ values for various values of $E(R_k(h))$ and $\mathrm{Stdev}(R_k(h))$:

(All numbers are in the unit of %. $h = 1/12$.)

Stdev$(R_k(h))$ $\to$		5	10	15	20	25
$E(R_k(h))$	s	17.3	34.6	52.0	69.3	86.6
-2		17.7	35.3	52.7	70.0	87.0
0		17.3	34.6	51.7	68.6	85.3
2	σ	17.0	33.9	50.7	67.3	83.7
4		16.6	33.2	49.7	66.0	82.1
6		16.3	32.6	48.8	64.8	80.6

The values of s and σ are not significantly different unless the average return $E(R_k(h))$ is high.

HW 3-5 (Martingale representation theory) For a standard BM B_t ($0 \le t \le T$), find the martingale representation of $X_t = E(B_T^3|\mathcal{F}_t)$. (In class, we did the same for $X_t = E(B_T^2|\mathcal{F}_t)$)

Answer Using the short notation $E_t(\cdot) = E(\cdot\,|\mathcal{F}_t)$,

$$\begin{aligned} X_t &= E_t((B_t + B_T - B_t)^3) = B_t^3 + 3B_t^2 E_t(B_T - B_t) + 3B_t\, E_t((B_T - B_t)^2) + E_t((B_T - B_t)^3) \\ &= B_t^3 + 0 + 3(T-t)B_t + 0 = B_t^3 + 3(T-t)B_t. \end{aligned}$$

In particular, $X_0 = 0$. From the SDE,

$$dX_t = 3B_t^2 dB_t + 3B_t(dB_t)^2 + 3(T-t)dB_t - 3B_t dt = 3(B_t^2 + T - t)dB_t,$$

the martingale representation is

$$X_T = \int_0^T 3(B_t^2 + T - t)\, dB_t$$