Stochastic Finance (FIN 519)
Midterm Exam (Online / Take-home)

Instructor: Jaehyuk Choi

2019-20 Module 3 (2020. 3. 20.)

BM stands for Brownian motion. Assume that B_t is a standard **BM**. **RN** and **RV** stand for random number and random variable respectively. The PDF and CDF of the standard normal variable are denoted by $n(z)$ and $N(z)$ respectively.

1. (10 points) **(Equity Linked Note)** An equity-linked note (ELN) is a debt instrument, usually a bond, that differs from a standard fixed-income security in that the final payout is based on the return of the underlying equity, which can be a single stock, basket of stocks, or an equity index. Equity-linked notes are a type of structured products (WIKIPEDIA). We (as a security firm) want to design and sell an ELN based on a stock following a BM:

$$S_t = S_0 + \sigma B_t.$$

This note has coupon N periods, $t = k\Delta t$ for $k = 1, 2, \cdots, N$. At $t = 0$, investors buy this note at the price of P for the notional value of \$1. At the end of the k-th period, $t = k\Delta t$, it pays coupon μ if the stock price did not fall more than δ (i.e., if $S_{k\Delta t} - S_{(k-1)\Delta t} \geq -\delta$) and continues to the next period. At the maturity $t = N\Delta t$, it pays $1 + \mu$ (1 is the redemption of the notional value). If the price falls more than δ at $t = k\Delta t$, (i.e., $S_{k\Delta t} - S_{(k-1)\Delta t} < -\delta$), the note terminates immediately by redeeming $(1 - L)$ (at the loss of L).

Assume that the discounting rate for one period, Δt, is r. So, the present value of \$1 paid after time Δt is $1/(1 + r)$. To simplify notation, you can use $D = 1/(1 + r)$.

The basic design of this ELN is that investors receive coupon μ higher than the risk-free rate r if stock market does not crash. However, they take a risk of heavy loss L if market crashes.

   (a) (4 points) Obtain the price P by calculating the expected value of the payout of this ELN.

   (b) (3 points) Assume the following specific parameters:

$$N = 8,\ \Delta t = 0.25,\ S_0 = 100,\ \sigma = 10,\ \delta = 10,\ D = 0.97\ (r \approx 3\%),\ L = 0.5.$$

   That is, this ELN observes the price every 3 months and the maturity is 2 years. Given the parameters, determine the return μ to make the price of this ELN par (i.e., $P = 1$). You may use spreadsheet. How does it compares to the risk-free rate $r \approx 3\%$?

   (c) (3 points) Right after clients buy this ELN at the price $P = 1$, the volatility of the underlying stock suddenly increased to $\sigma = 20$ due to the spread of the Corvid-19 virus. (Assume that $S_0 = 100$ is unchanged.) What is client's loss?

Solution:

(a) From the independent increment of BM, the probability p to pay the coupon μ is same at every period. The probability, p, and $q = 1 - p$ is given by

$$\begin{aligned} p &= \mathrm{Prob}(S_{k\Delta t} - S_{(k-1)\Delta t} \geq -\delta) = \mathrm{Prob}(\sigma B_{\Delta t} \geq -\delta) \\ &= 1 - N\left(-\frac{\delta}{\sigma\sqrt{\Delta t}}\right) = N\left(\frac{\delta}{\sigma\sqrt{\Delta t}}\right). \end{aligned}$$

The present value of the ELN is decomposed into three components:

- The coupon μ paid at the k-th period ($k = 1, \cdots, n$):

  $$\text{Present Value} = \mu D^k, \quad \text{Probability} = p^k$$

  The expectation is given by

  $$\sum_{k=1}^{n} \mu D^k \cdot p^k = \mu(pD)\frac{1-(pD)^n}{1-pD}$$

- The early terminated redemption with loss, $(1 - L)$, at the k-th period ($k = 1, \cdots, n$):

  $$\text{Present Value} = (1-L)D^k, \quad \text{Probability} = p^{k-1}q.$$

  The expectation is give by

  $$\sum_{k=1}^{n}(1-L)D^k \cdot p^{k-1}q = (1-L)\frac{q}{p}\sum_{k=1}^{n}(pD)^k = (1-L)(qD)\frac{1-(pD)^n}{1-pD}$$

- The redemption of the notional \$1 at the n-th period (maturity):

  $$\text{Present Value} = D^n, \quad \text{Probability} = p^n.$$

  The expectation is $(pD)^n$.

The price P is the sum of the three expected values:

$$\begin{aligned} P &= \mu(pD)\frac{1-(pD)^n}{1-pD} + (1-L)(qD)\frac{1-(pD)^n}{1-pD} + (pD)^n \\ &= \Big(\mu p + (1-L)q\Big)\, D\, \frac{1-(pD)^n}{1-pD} + (pD)^n. \end{aligned}$$

(b) The coupon μ to satisfy $P = 1$ is obtained as

$$\begin{aligned} \Big(\mu p + (1-L)q\Big)\, D\, \frac{1-(pD)^n}{1-pD} + (pD)^n &= 1 \\ \mu p + (1-L)q &= \frac{1-pD}{D} \\ \mu &= \frac{1}{p}\left(\frac{1-pD}{D} - (1-L)q\right) \end{aligned}$$

Using the parameter values, the coupon should be $\mu = 4.234\%$. The intermediate values are

$$p = N(2) = 97.725\% \quad \text{and} \quad q = 1 - p = 2.275\%.$$

(c) When σ is suddenly jump to $\sigma = 20$, the price drops down to $P = 0.6867$. The loss is about 31%. The intermediate values are

$$p = N(1) = 84.134\% \quad \text{and} \quad q = 1 - p = 15.866\%.$$

2. (14 points) **(First hitting time of a BM with drift)** From Chapter 5, we derived that the probability density of the first-hitting time τ such that $B_\tau = \delta > 0$ is given by

$$f(t) = \frac{\delta}{\sqrt{2\pi t^3}} e^{-\delta^2/2t}. \tag{1}$$

We also know from **Hitting time of a level** in Chapter 4.5 that the Laplace transform of τ is given by

$$E(e^{-r\tau}) = e^{-\delta\sqrt{2r}}$$

and that the result is interpreted as the present value (discounted with the interest rate r) of a derivative paying \$1 at the event. We are going to generalize the result to the first hitting time of a drifted BM, $B_t + \gamma t$ ($\gamma > 0$). The probability density for τ (i.e., $B_\tau + \gamma\tau = \delta > 0$) is given by

$$f(t) = \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{(\gamma t - \delta)^2}{2t}\right), \tag{2}$$

and you can use this result without proof.

(a) (4 points) What is the price of the derivative, $E(e^{-r\tau})$?
Hint: Consider the Laplace transform of Eq. (1). new

(b) (3 points) Obtain the mean and variance of τ, $E(\tau)$ and $\mathrm{Var}(\tau)$.
Hint: if $L(r) = E(e^{-r\tau})$ is the Laplace transform of τ, $L(-r)$ is the moment generating function because $L(-r) = E(e^{r\tau})$.

(c) (3 points) We further generalize the results to derivative pricing. Assume that a stock follows the process, $S_t = S_0 + \mu t + \sigma B_t$ and the derivative pays \$1 when S_t hits K ($> S_0$) for the first time, i.e., $S_\tau = K$. What is the price of the derivative at $t = 0$? (Modify the result from (a).) Assume that you sold the derivative to clients and that you need to hedge the position using the underlying stock. How many shares of the underlying stock do you need to long or short?

(d) (4 points) From class we know that the CDF for Eq. (1) is

$$P(\tau \le t) = 2 - 2N(\delta/\sqrt{t}),$$

where $N(\cdot)$ is the normal cumulative distribution function. Can you derive the CDF for the generalized density function, Eq. (2)? (This question might be challenging. Try it only when you have extra time.)

Solution: The distribution of τ, Eq. (2), is known as the inverse Gaussian distribution (Wikipedia). The inverse Gaussian distribution is typically parameterized by μ and λ:

$$f(t) = \sqrt{\frac{\lambda}{2\pi t^3}} \exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2\, t}\right).$$

The parameters (μ, λ) are related to our parameters (γ, δ) by

$$\mu = \delta/\gamma, \quad \lambda = \delta^2.$$

Any established results for the inverse Gaussian distribution can be expressed in terms of (γ, δ) using the above formula. The distribution is called *inverse* Gaussian because it describes the (first-hitting) time of Brownian motion at a fixed location, whereas the Gaussian distribution describes the location at a fixed time. The answers to this problem is well-known properties of the inverse Gaussian distribution. But we can derive the solution from the knowledge obtained in class except (d).

(a) The density function, Eq. (1) and its Laplace transform are expressed as

$$E(e^{-r\tau}) = \int_{t=0}^{\infty} e^{-rt} f(t) dt = \int_{t=0}^{\infty} \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - rt\right) dt = e^{-\delta\sqrt{2r}}$$

Based on this, we can derive the Laplace transform of Eq. (2):

$$\begin{aligned}
E(e^{-r\tau}) &= \int_{t=0}^{\infty} \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{(\gamma t - \delta)^2}{2t} - rt\right) dt \\
&= \int_{t=0}^{\infty} \frac{\delta e^{\gamma\delta}}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - (r + \frac{\gamma^2}{2})t\right) dt \\
&= \int_{t=0}^{\infty} \frac{\delta e^{\gamma\delta}}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - r't\right) dt \quad \left(r' = r + \frac{\gamma^2}{2}\right) \\
&= \exp\left(\gamma\delta - \delta\sqrt{2r'}\right) \\
&= \exp\left(\gamma\delta\left(1 - \sqrt{1 + 2r/\gamma^2}\right)\right).
\end{aligned}$$

(b) From Taylor's expansion,

$$1 - \sqrt{1 + \frac{2r}{\gamma^2}} = 1 - \left(1 + \frac{r}{\gamma^2} - \frac{r^2}{2\gamma^4} + \cdots\right) = -\frac{r}{\gamma^2} + \frac{r^2}{2\gamma^4} + \cdots,$$

the Laplace transform is expanded into

$$E(e^{-r\tau}) = 1 + \gamma\delta\left(-\frac{r}{\gamma^2} + \frac{r^2}{2\gamma^4} + \cdots\right) + \frac{1}{2}\left(\frac{\delta^2}{\gamma^2} r^2 + \cdots\right) + \cdots.$$

Therefore, the first two moments and the variance are given by

$$M_1 = \frac{\delta}{\gamma}, \quad M_2 = \frac{\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2}, \quad \text{Var} = \frac{\delta}{\gamma^3}.$$

(c) From

$$S_\tau = S_0 + \mu\tau + \sigma B_\tau = K \quad \Longrightarrow \quad (\mu/\sigma)\tau + B_\tau = (K - S_0)/\sigma,$$

we can use

$$\delta = \frac{K - S_0}{\sigma}, \quad \gamma = \frac{\mu}{\sigma}.$$

From the result of (a),

$$P = E(e^{-r\tau}) = \exp\left(\frac{\mu(K-S_0)}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\right)$$

The amount of the underlying stock to hold for hedging the position is given by

$$\frac{\partial P}{\partial S_0} = -\frac{\mu}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\exp\left(\frac{\mu(K-S_0)}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\right)$$

(d) The CDF of the inverse Gaussian distribution is give by

$$F(t) = N\left(\gamma\sqrt{t}-\frac{\delta}{\sqrt{t}}\right)+e^{2\gamma\delta}N\left(-\gamma\sqrt{t}-\frac{\delta}{\sqrt{t}}\right).$$

This CDF was originally found by

> Shuster, J. (1968). On the inverse Gaussian distribution function. *Journal of the American Statistical Association*, 63(324), 1514–1516. https://doi.org/10.1080/01621459.1968.10480942

It can be also derived using Girsanov theorem in Chapter 13.

When $\gamma = 0$, the CDF $F(t)$ is reduced to the case we learned in class:

$$F(t) = P(\tau \le t) = N\left(-\frac{\delta}{\sqrt{t}}\right)+N\left(-\frac{\delta}{\sqrt{t}}\right) = 2-2N\left(\frac{\delta}{\sqrt{t}}\right)$$

3. (6 points) **Min put option** Assume that the stock price follows a BM, $S_t = S_0 + \sigma B_t$. Assume that $S_t^m = \min\{S_s : 0 \le s \le t\}$. Calculate the put option price whose payout at expiry $t = T$ is given by the minimum value along the path

$$\max(K - S_T^m, 0) \quad \text{where} \quad K < S_0$$

Intuitively, this option should be more expensive that the regular put option whose payout is given by the final price S_T. By how much is it more expensive?

Solution: Let $W_t = -B_t$, then W_t is also a standard BM. The minimum of B_t is the negative of the maximum of W_t:

$$B_t^m = \min\{B_s : 0 \le s \le t\} = -\max\{W_s : 0 \le s \le t\} = -W_t^M$$

Since the PDF for W_t^M, is given by

$$f(x) = \frac{2}{\sqrt{t}}n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \ge 0,$$

the PDF for B_t^m is same as

$$f(x) = \frac{2}{\sqrt{t}}n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \le 0.$$

because $n(x) = n(-x)$. This is two times the PDF for B_t defined on the negative side only. The minimum put option price is twice as expensive as that of the regular put option:

$$P(K) = 2(K - S_0)N(-d) + 2\sigma\sqrt{T}\,n(d) \quad \text{where} \quad d = \frac{S_0 - K}{\sigma\sqrt{T}}$$