Stochastic Finance (FIN 519)
Final Exam

Instructor: Jaehyuk Choi

2017-18 Module 3 (2018. 4. 30)

BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. Assume that B_t is a standard **BM**.

1. ($1 \times 4 = 4$ points) Give **True or False**.
   (a) $(dB_t)^2 = dt$
   (b) $(dB_t)^4 = (dt)^2$
   (c) $d(B_t^2) = 2(B_t dB_t + dt)$
   (d) $d\sinh(B_t) = \cosh(B_t)dB_t + \frac{1}{2}\sinh(B_t)dt$ where $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$

Solution:

(a) True. $(dB_t)^2 = dt$.

(b) False. $(dB_t)^4 = 3(dt)^2$ from $E(B_t^4) = 3t^2$.

(c) False. $d(B_t^2) = 2B_t dB_t + dt$

(d) True. $d\sinh(B_t) = \cosh(B_t)dB_t + \frac{1}{2}\sinh(B_t)dt$ since $\sinh'(x) = \cosh(x)$ and $\sinh''(x) = \sinh(x)$.

2. ($2 \times 3 = 6$ points) **Black-Scholes and Importance sampling** Assume that $r = q = 0$ in this problem.
   (a) Suppose that a stock price follows $dS_t/S_t = \sigma dB_t$ and you want to price a call option with strike price K and maturity T. How do you get the option price by Monte-Carlo simulation? In other words, what should be the expression for $f(Z)$ in the following Monte-Carlo formula:

$$C = \frac{1}{N}\sum_{k=1}^{N} f(Z_k),$$

   where Z_k is a sequence of i.i.d. standard normal RVs.

   (b) Using Black-Scholes formula, find the call option price for $S_0 = 100, K = 2000, T = 1, \sigma = 0.5$. For computation, use approximation $20 \approx e^3$. You may also use the following CDF values for the standard normal distribution $N(z)$.

z	-6.5	-6.25	-6.0	-5.75	-5.5
$N(z)$	4.0×10^{-11}	2.1×10^{-10}	9.9×10^{-10}	4.5×10^{-9}	1.9×10^{-8}

   (c) As observed in (b), the call option value can be very small if K is extremely high compared to S_0 ($K \gg S_0$). Therefore, the Monte-Carlo price from (a) would be zero because no Z_k will make the stock price in-the-money. Apply important sampling and express the option price. (Hint: Select the shift amount to make the stock price at-the-money.)

Solution:

(a) The security price can be simulated as

$$S_T = S_0 \exp(\sigma\sqrt{T}Z - \sigma^2 T/2) \quad \text{for} \quad Z \sim N(0,1).$$

and the call option price is obtained as

$$C = \frac{1}{N}\sum_{k=1}^{N}\Big(S_0 \exp(\sigma\sqrt{T}Z_k - \sigma^2 T/2) - K\Big)^+$$

for i.i.d. samples $\{Z_k\}$ (assume $r = 0$) and $(x)^+ = \max(x, 0)$.

(b) From $d_{1,2} = \log(S_0/K)/\sigma\sqrt{T} \pm \sigma\sqrt{T}/2 = -6 \pm 0.25 = -5.75,\ -6.25$,

$$C = 100N(-5.75) - 2000N(-6.25) = 100 \times 4.5 \times 10^{-9} - 2000 \times 2.1 \times 10^{-10} = 3 \times 10^{-8}$$

(c) Remind that

$$E[f(Z)] = E\big[f(Z+\mu)\, e^{-\mu Z - \mu^2/2}\big].$$

Therefore,

$$C = \frac{1}{N}\sum_{k=1}^{N}\Big(S_0 \exp(\sigma\sqrt{T}(Z_k+\mu) - \sigma^2 T/2) - K\Big)^+ e^{-\mu Z_k - \mu^2/2}$$

where μ is determined as

$$S_0 \exp(\sigma\sqrt{T}\mu - \sigma^2 T/2) = K \quad \Rightarrow \quad \mu = \frac{\log(K/S_0)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}.$$

Notice that $\mu = -d_2$ in the Black-Scholes formula.

3. (4 points) **Solving SDE** Solve the modified "3/2" volatility process defined by

$$dV_t = \frac{3\sigma^2}{4} V_t^2\, dt + \sigma V_t^{3/2}\, dB_t.$$

The original "3/2" model has the drift term $a(b - V_t)V_t\, dt$ instead of $(3\sigma^2/4)V_t^2\, dt$ and was originally proposed for a stochastic model for volatility.

Solution: First we remove $\sqrt{V_t}$ from the coefficient of dB_t

$$\frac{dV_t}{V_t^{3/2}} = \frac{3\sigma^2}{4}\sqrt{V_t}\, dt + \sigma\, dB_t.$$

It turns out that the first candidate $\dfrac{dx}{x^{3/2}}$ works. Since $\displaystyle\int \frac{dx}{x^{3/2}} = -\frac{2}{\sqrt{x}}$,

$$-d\left(\frac{2}{\sqrt{V_t}}\right) = \frac{dV_t}{V_t^{3/2}} - \frac{3(dV_t)^2}{4V_t^{5/2}} = \frac{3\sigma^2}{4}\sqrt{V_t}\, dt + \sigma\, dB_t - \frac{3\sigma^2}{4}\sqrt{V_t}dt = \sigma dB_t$$

$$\frac{2}{\sqrt{V_0}} - \frac{2}{\sqrt{V_t}} = \sigma B_t \quad \Rightarrow \quad V_t = \left(\frac{1}{\sqrt{V_0}} - \frac{\sigma B_t}{2}\right)^{-2}$$

4. ($2 \times 4 = 8$ points) **Interest rate and bond price SDE**

(a) Find the mean and variance of the following expression:

$$I = \int_0^T B_t\, dt.$$

You may find that $TB_T = \int_0^T TdB_t$ is a helpful trick. Is I normally distribution?

(b) Suppose now you live in a real world where the risk-free rate, r_t, is stochastic rather than constant. Therefore the saving account β_T exponentially grows as $\beta_T = \exp\left(\int_0^T r_t dt\right)$. Show that, in risk-neutral measure Q, the zero bond price $P(t,T)$ is given as

$$P(0,T) = E^Q\left[\exp\left(-\int_0^T r_t dt\right)\right]$$

(c) Suppose that the SDE for r_t is given as

$$r_t = r_0 + \alpha t + \beta\, B_t,$$

where B_t is a standard BM under Q ($B_t = B_t^Q$). Calculate $P(0,T)$ using the results from (a) and (b).

(d) Suppose that the SDE for the bond price $P(t,T)$ in Q is given as

$$\frac{d\,P(t,T)}{P(t,T)} = \mu_P dt + \sigma_P\, dB_t.$$

From the result of (c), find $P(t,T)$ by replacing r_0 with r_t and T with $T-t$. Then, apply Itô's lemma to $\log P(t,T)$. By matching drift and volatility, find the expressions for μ_P and σ_P. (Hint: The answers are quite simple, so check your calculation again if the expressions are complicated. Especially μ_P is almost obvious in the risk-neutral measure. Even if you cannot get the derivation right, you still earn partial credit by correctly guessing μ_P. When applying Itô's lemma to $\log P(t,T)$, t is the time variable and T should be considered as a constant.)

Solution:

(a) (See the Exercise 6.2 of **SCFA**) Using Itô's lemma applied to tB_t, $d(tB_t) = tdB_t + B_t dt$, we can express I as

$$I = TB_T - \int_0^T tdB_t = T\int_0^t dB_t - \int_0^T tdB_t = \int_0^T (T-t)dB_t.$$

We get

$$E(I) = 0 \quad \text{and} \quad \mathrm{Var}(I) = \int_0^T (T-t)^2 dt = \frac{1}{3}T^3.$$

and I is normally distributed.

(b) The bond price $P(0,T)$ is the $t=0$ value of \$ 1 at $t=T$. Under the risk-neutral measure Q, any security value measured in the unit of β_t is a martingale,

$$\frac{P(0,T)}{\beta_0 = 1} = E_Q\left[\frac{P(T,T)=1}{\beta_T}\right].$$

Therefore,

$$P(0,T) = E_Q\left[\exp\left(-\int_0^T r_t dt\right)\right]$$

or, more generally,

$$P(t,T) = E_Q\left[\exp\left(-\int_t^T r_s dt\right)\middle|\mathcal{F}_t\right]$$

(c) From

$$-\int_0^T r_t dt = -Tr_0 - \frac{1}{2}\alpha T^2 - \beta I,$$

we get

$$\begin{aligned} P(0,T) &= \exp\left(-Tr_0 - \frac{1}{2}\alpha T^2\right) E_Q(e^{-\beta I}) \\ &= \exp\left(-Tr_0 - \frac{1}{2}\alpha T^2 + \frac{1}{6}\beta^2 T^3\right) \end{aligned}$$

Note that this is a simplified version of the Ho-Lee model (WIKIPEDIA). In the original model, the drift α is time-dependent:

$$dr_t = \alpha_t\,dt + \sigma dB_t.$$

(d) From

$$\log P(t,T) = -(T-t)r_t - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\beta^2(T-t)^3$$

we get the SDE for $\log P(t,T)$:

$$\begin{aligned} d\log P(t,T) &= -(T-t)dr_t + \left(r_t + \alpha(T-t) - \frac{1}{2}\beta^2(T-t)^2\right)dt \\ &= \left(r_t - \frac{1}{2}\beta^2(T-t)^2\right)dt - \beta(T-t)dB_t. \end{aligned}$$

and this should be equal to

$$d\log P(t,T) = (\mu_P - \sigma_P^2/2)dt + \sigma_P dB_t.$$

Therefore we get

$$\mu_P = r_t \quad \text{and} \quad \sigma_P = -\beta(T-t).$$

Note that (i) $\mu_P = r_t$ is obvious in the risk-neutral measure, (ii) the bond volatility σ_P is negative because the bond price moves in an opposite way to the interest rate, and (iii) $\sigma_P = 0$ at maturity $t = T$ because the bond price $P(t,T)$ converges to 1.

5. ($2 \times 4 = 8$ points) **Siegel's paradox** In foreign exchange market, each currency has a symbol of three characters. For example, `USD` is for US Dollar, `CNY` is for Chinese Yuan, and `EUR` is for Euro. Also, foreign exchange rate is quoted in one fixed direction such that the exchange rate is bigger than one. For example, the exchange between `USD` and `CNY` is always quoted in `USD/CNY`, not in `CNY/USD`. `USD/CNY` means the price of 1 `USD` in terms of `CNY`, which is around 6.3 as of today.

(a) Due to the recent trade war between US and China, suppose that the FX rate `USD/CNY` is going to be very volatile. Assuming `USD/CNY` = 6.0 today and it can change to either 4.0 or 8.0 after one year. Under the risk-neutral measure Q, what are the probabilities for the two scenarios? Assume that risk-free rate is zero in both US and China. (Hint: You can think of `USD/CNY` as the price of an asset (1 `USD`) in `CNY` unit.)

(b) Now consider the scenario from the perspective of an investor in US. Suppose a US investor invests in `CNY` by converting 1 `USD` to 6 `CNY` today and holding 6 `CNY` for one year. What is her expected return in terms of `USD` after one year? If the expected return is not zero, does it conflict with that the expected return under the risk-neutral measure Q should be same as the risk-less rate (which is zero in this problem)? (Hint: Consider the numeraire.)

(c) We can demonstrate this using stochastic calculus. If $X_t =$ `USD/CNY`, X_t is a martingale (no drift), so the SDE is given by

$$\frac{dX_t}{X_t} = \sigma dB_t^Q.$$

By applying Itô's lemma, find the SDE for X_0/X_t, which is the investment value in `USD`. Is the drift positive, negative or zero?

(d) In class, we show that, for a numeraire asset with volatility σ_N, the relation between the two standard BMs between the probability measures, Q^N and Q (risk-neutral measure), is given as

$$dB_t^{Q^N} + \sigma_N dt = dB_t^Q.$$

What is the numeraire to make X_0/X_t a martingale? From the relation above, show that X_0/X_t is indeed a martingale under the measure associated with the numeraire.

Solution: See (WIKIPEDIA) for **Siegel's paradox**.

(a)

$$4p + 8(1-p) = 6 \;\; \texttt{CNY} \quad \Rightarrow \quad p = 0.5$$

(b) After one year, the investor receives in

$$\frac{6}{4}\cdot\frac{1}{2} + \frac{6}{8}\cdot\frac{1}{2} = \frac{9}{8} \;\; \texttt{USD}$$

Therefore the return is 12.5%. This does not contract with the zero expected return of the risk-neutral measure because the numeraire of the risk-neutral measure is the `CNY` cash while the numeraire for the US investor is `USD` cash.

(c) In general, the SDE for the exchange rate is given as

$$\frac{dX_t}{X_t} = (r_d - r_f) + \sigma dB_t^Q,$$

where r_d is the risk-free rate of domestic currency (`CNY` in this case) and r_f is the foreign currency (`USD` in this case). We assume $r_d = r_f = 0$ for this problem, but we derive general result with r_d and r_f.

$$\begin{aligned} d\log X_t &= (r_f - r_d - \sigma^2/2)dt + \sigma dB_t^Q \\ d\log(X_0/X_t) &= (r_d - r_f + \sigma^2/2)dt - \sigma dB_t^Q \\ \frac{d(X_0/X_t)}{(X_0/X_t)} &= (r_d - r_f + \sigma^2)dt - \sigma dB_t^Q \end{aligned}$$

Assuming $r_d = r_f = 0$,

$$\frac{d(X_0/X_t)}{(X_0/X_t)} = \sigma^2\, dt - \sigma\, dB_t^Q.$$

(d) The `USD` cash should be the numeraire and the price of `USD` in terms of the risk-neutral numeraire `CNY` is nothing but X_t! Therefore the volatility is σ and we get

$$dB_t^{\texttt{USD}} + \sigma dt = dB_t^{\texttt{CNY}} (= dB_t^Q).$$

The SDE from (c) becomes

$$\frac{d(X_0/X_t)}{(X_0/X_t)} = \sigma\,(\sigma\, dt - dB_t^{\texttt{CNY}}) = \sigma dB_t^{\texttt{USD}}$$

and X_0/X_t is indeed a martingale under the `USD` numeraire.