Option Pricing under ‘Normal’ Model
Stochastic Finance (FIN 519)

Instructor: Jaehyuk Choi

Peking University HSBC Business School, Shenzhen, China

2019-20 Module 3 (Spring 2020)

Bachelier vs Black-Scholes-Merton model

- Let F_t be the forward price of stock price S_t:

$$F_t = e^{(r-q)(T-t)} S_t \quad (F_T = S_T),$$

where r is interest rate, q is dividend rate and T is the expiry of the forward contract.

- Then, F_t is a martingale. (However, let us safely assume $r = q = 0$, so $F_t = S_t$ for now.)
- Under Bachelier model, stock price follows an arithmetic Brownian motion (BM) with volatility σ_N:

$$S_t = S_0 + \sigma_\mathrm{N} B_t \quad (\text{SDE:} \quad dS_t = \sigma_\mathrm{N} dB_t)\,.$$

- Under Black-Scholes-Merton (BSM) model, stock follows an geometric BM:

$$S_t = S_0 \exp\left(-\frac{1}{2}\sigma_\mathrm{BSM}^2\, t + \sigma_\mathrm{BSM} B_t\right) \quad \left(\text{SDE:} \quad \frac{dS_t}{S_t} = \sigma_\mathrm{BSM} dB_t\right).$$

- The two models are approximately same if the two volatilities are related by

$$\sigma_\mathrm{N} = S_0\, \sigma_\mathrm{BSM}.$$

Normal model

Different names

- Normal process (vs Log-normal process)
- Arithmetic BM (vs Geometric BM)
- Bachelier model (vs Black-Scholes-Merton model)

Why normal model?

- Better dynamics for some underlying assets: interest rate
  - Price can be negative,
  - Daily changes are independent of the level of the price level
- More intuitive than Black-Scholes-Merton

Call Option Price

Underlying asset price at maturity T:

$$S_T = S_0 + \sigma B_T = S_0 + \sigma\sqrt{T}z, \quad \text{where} \quad z \sim N(0,1)$$

Payoff:

$$\max(S_T - K, 0) = (S_T - K)^+ = (S_0 - K + \sigma\sqrt{T}z)^+$$

$$S_T = K \quad \Rightarrow \quad z = -d = \frac{K - S_0}{\sigma\sqrt{T}} \quad \left(d = \frac{S_0 - K}{\sigma\sqrt{T}}\right)$$

Forward option value (undiscounted):

$$\begin{aligned} C(K) &= \int_{-d}^{\infty} (S_0 - K + \sigma\sqrt{T}z)\, n(z) dz \\ &= (S_0 - K)(1 - N(-d)) + \sigma\sqrt{T}\, n(-d) \\ &= (S_0 - K)N(d) + \sigma\sqrt{T}\, n(d) \end{aligned}$$

Here we used

$$\int z\, n(z) dz = \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = -n(z) + C.$$

Present option value (discounted):

$$C_0(K) = e^{-rT} C(K)$$

Put Option Price

Payoff:

$$(K - S_T)^+ = (K - S_0 - \sigma\sqrt{T}z)^+$$

$$\text{The root of } S_T = K \quad \Rightarrow \quad z = -d = \frac{K - S_0}{\sigma\sqrt{T}} \quad \left(d = \frac{S_0 - K}{\sigma\sqrt{T}}\right)$$

Forward option value (undiscounted):

$$\begin{aligned} P(K) &= \int_{-\infty}^{-d} (K - S_0 - \sigma\sqrt{T}z)\, n(z) dz \\ &= (K - S_0)N(-d) - \sigma\sqrt{T}\, n(-d) \\ &= (K - S_0)N(-d) + \sigma\sqrt{T}\, n(d) \end{aligned}$$

Present option value (discounted):

$$P_0(K) = e^{-rT} P(K)$$

Put-Call parity holds!

$$C(K) - P(K) = (S_0 - K)N(d) - (K - S_0)N(-d) = (S_0 - K)\big(N(d) + N(-d)\big) = S_0 - K$$

Option Price (At-The-Money)

If $K = S_0$ (at-the-money), $d = 0$ and the option prices are

$$C(K=S_0) = P(K=S_0) = \sigma\sqrt{T}n(0) = \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.4\,\sigma\sqrt{T}$$

$$\text{Straddle} = C + P \approx 0.8\,\sigma\sqrt{T}$$

$$C_0(K=S_0) = P_0(K=S_0) = \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}} \approx e^{-rT}\,0.4\,\sigma\sqrt{T}$$

Therefore the option price is proportional to the *width* (or stdev) of the distribution of the future price, $\sigma\sqrt{T}$, which is consistent with the intuition. Before we derive Black-Scholes formula, let's keep this relation between the volatility and the option price in mind. Even without the Black-Scholes formula (which is somewhat complicated), this relation should give you a very good intuition.

Greeks (risks of option)

Delta: sensitivity on the underlying price

$$\frac{\partial C}{\partial S_0} = N(d), \quad \frac{\partial P}{\partial S_0} = -N(-d) \quad \left(d = \frac{S_0 - K}{\sigma\sqrt{T}}\right)$$

$$\left(\frac{\partial C}{\partial S_0} - \frac{\partial P}{\partial S_0} = 1\right)$$

$N(d)$ measures how closely the call option price moves with the underlying stock, i.e., how much the option is in-the-money.

Gamma: convexity on the underlying price

$$\frac{\partial^2 C}{\partial S_0^2} = \frac{\partial^2 P}{\partial S_0^2} = \frac{n(d)}{\sigma\sqrt{T}}$$

Vega: sensitivity on the volatility

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = \sqrt{T}\, n(d)$$

Comparison of the two models

Model	Normal (Bachelier)	Lognormal (BSM)
Reference	Bachelier [1900]	Black-Scholes, Merton [1973]
SDE	Arithmetic BM: $dS_t = \sigma dW_t$	Geometric BM: $dS_t/S_t = \sigma dW_t$
Asset class	Interest rate, Inflation, Spread	Equity, FX
Call option price	$(S_0 - K)N(d) + \sigma\sqrt{T}\, n(d)$ $d = \frac{(S_0-K)}{\sigma\sqrt{T}}$	$S_0\, N(d_1) - K\, N(d_2)$ $d_{1,2} = \frac{\log(S_0/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$
Volatility conversion	$\sigma_{\text{N}} \approx S_0\, \sigma_{\text{BSM}}$	
Digital, $P(S_t > K)$	$N(d)$	$N(d_2)$
Delta $(\partial/\partial S_0)$	$N(d)$	$N(d_1)$
Gamma $(\partial^2/\partial S_0^2)$	$n(d)/\sigma\sqrt{T}$	$n(d_1)/S_0\sigma\sqrt{T}$
Vega $(\partial/\partial\sigma)$	$\sqrt{T}\, n(d)$	$S_0\sqrt{T}\, n(d_1)$

Generalization

The price at maturity T has normal distribution with variance V_T (stdev $\sqrt{V_T}$):

$$X_T = X_0 + \sqrt{V_T}\,z, \quad \text{where} \quad z \sim N(0,1)$$

Then, for the payoff $\max(\pm(X_T - K), 0)$, the option prices are given by

$$\begin{cases} C(K) = (X_0 - K)N(d) + \sqrt{V_T}\,n(d) \\ P(K) = (K - X_0)N(-d) + \sqrt{V_T}\,n(d), \\ C(K = X_0) = P(K = X_0) = 0.4\sqrt{V_T}, \end{cases} \qquad \text{where} \quad d = \frac{X_0 - K}{\sqrt{V_T}}$$

- Spread/Basket option

$$X_t = X_0 + aW_t + bZ_t \text{ with } E(W_t\,Z_t) = \rho t \quad \Rightarrow \quad V_T = (a^2 + 2\rho ab + b^2)T$$

- Asian option

$$X_t = X_0 + \frac{\sigma}{N}\sum_{k=1}^{N} W_{kT/N} \quad \Rightarrow \quad V_T = \sigma^2 T\sqrt{\frac{15}{32}} \quad (N = 4)$$

- Time-varying volatility

$$dS_t = f(t)dB_t \quad \Rightarrow \quad V_T = \int_0^T f^2(t)dt \;\; \text{(Itô's isometry)}$$

Previous Homework (solution available)

1. Derive the (forward) price of the digital(binary) call/put option struck at K at maturity T. The digital(binary) call/put option pays \$1 if S_T is above/below the strike K, i.e. $1_{S_T \ge K}/1_{S_T \le K}$.
2. The payoff of the call option, $\max(S_T - K, 0)$ can be decomposed into two parts,
$$S_T \cdot 1_{S_T \ge K} \quad - \quad K \cdot 1_{S_T \ge K}.$$
The first payout is the payout of the **asset-or-nothing** call option and the second payout if the binary call option multiplied with $-K$. What is the price of the asset-or-nothing call option?
3. Using the joint distribution of B_t and B_t^*, derive the price of the call option struck at K and knock-out at K_1 $(> K)$. First, generalize the joint CDF function $P(u < B_t, v < B_t^*)$ to σB_t. Next, derive the PDF on u by taking derivative on u. Then, integrate the payoff $(S_T - K)^+$ from K to K_1. (Assume that the risk-free rate is zero, $r = 0$, so that $S_0 = F$. Otherwise the problem is too complicated.)