Stochastic Finance (FIN 519)
Final Exam

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BM stands for Brownian motion.

1. (4 points) **Stochastic calculus.** Choose **all** surviving terms (i.e., non-zero terms) in stochastic calculus. Assume that $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$ for some functions μ and σ.
   (a) $dX_t \cdot dt$
   (b) $(dX_t)^2$
   (c) $dX_t \cdot dB_t$
   (d) $dB_t^1 \cdot dB_t^2$ for the two independent BMs, B_t^1 and B_t^2

2. (2×3 points) **Option price and delta under the BSM model.** You hold a call option with $K = 100$ maturing in 3 months. Assume that a stock's annual volatility is 32% of the current price. Also assume that $r = q = 0$. You may use the following CDF values for the standard normal distribution $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

   (a) The stock's current spot price is $S_0 = 100$. What is the price of the call option under the BSM model?
   (b) What is the delta (i.e., the sensitivity to S_0) of the option?
   (c) If the spot price changed to $S_0 = 100.5$, what is the new option price under the BSM model? Approximate the price with Taylor's expansion using the results from (a) and (b).

3. (2×3 points) **Itô's lemma.** The stochastic variance in Heston model is given by

$$dV_t = \alpha(V_\infty - V_t)dt + \sigma\sqrt{V_t}\, dB_t.$$

This process is also know as Cox-Ingersoll-Ross(CIR) model for stochastic interest rate. Additionally, the stochastic variance under the 3/2 model is also given as

$$dV_t = \lambda V_t(V_\infty - V_t)\, dt + \xi V_t\sqrt{V_t}\, dB_t.$$

   (a) Derive $E(V_t \,|\, \mathcal{F}_0)$ under the Heston model. (Hint: The transformation, $y_t = e^{\alpha t}(V_t - V_\infty)$, used in the Ornstein-Uhlenbeck process is also useful in this problem. Then, use the martingale property.) What is $E(V_t \,|\, \mathcal{F}_0)$ as $t \to \infty$.
   (b) To solve the 3/2 model, the inverse variance, $Y_t = 1/V_t$, is helpful. Find the SDE satisfied by Y_t.
   (c) What is $E(1/V_t \,|\, \mathcal{F}_0)$ under the 3/2 model? (Hint: You should recognize a similarity between the SDEs for V_t under Heston model and Y_t under the 3/2 model.)

4. (2×4 points) **SDE and martingale representation theorem.**
   (a) Apply Itô calculus to find the stochastic differentiation of $\cosh(B_t)$. Reminded that $\cosh x = (e^x + e^{-x})/2$.

(b) Find λ such that $X_t = e^{\lambda t}\cosh(B_t)$ is a martingale (i.e., dX_t has no dt term.)

(c) Using (b), find the martingale representation of $\cosh(B_T)$. In other words, find V_0 and ϕ_t satisfying

$$\cosh(B_T) = V_0 + \int_0^T \phi_t \, dB_t$$

(d) Assume a stock price S_t follows $dS_t = \sigma dB_t$ and $r = 0$. What is the price of a derivative that pays $\cosh(S_T - S_0)$ at time $t = T$?

5. (2×3 points) **T-forward measure.** In class, we learned that the forward price of buying a bond maturing at $T + \Delta$ at time $t = T$,

$$F_t = \frac{B(t, T+\Delta)}{B(t,T)},$$

is a martingale under the T-forward measure. We are going to prove this under an explicit setting. From a 2017 exam problem, we also know that

$$\frac{d\,B(t,T)}{B(t,T)} = r_t dt - \beta(T-t)\, dB_t^Q$$

when the risk-free rate changes according to $dr_t = \alpha dt + \beta\, dB_t^Q$. Here, B_t^Q is the standard BM under the risk-neutral measure.

(a) Derive the SDE for F_t. (Hint. First compute $d\log F_t$ and compute dF_t/F_t)

(b) If B_t^T is the standard BM under the T-forward measure, what is the relation between dB_t^Q and dB_t^T?

(c) From (a) and (b), finally derive the SDE for F_t under the T-forward measure. Is F_t a martingale? What is the volatility of dF_t/F_t?

(Not Exam) HW 3-4 Exponential Ornstein-Uhlenbeck process Solve the following SDE:

$$\frac{dP_t}{P_t} = \alpha(\mu - \log P_t)dt + \sigma dB_t.$$

What are $E(P_t)$ and $\mathrm{Var}(P_t)$ as $t \to \infty$? (Hint: use $X_t = \log P_t$.)

Solution. The SDE for X_t satisfies

$$dX_t = \frac{dP_t}{P_t} - \frac{(dP_t)^2}{2P_t^2} = \alpha\left(\mu - \frac{\sigma^2}{2\alpha} - X_t\right)dt + \sigma dB_t.$$

This is the Ornstein-Uhlenbeck with $X_\infty = \mu - \frac{\sigma^2}{2\alpha}$. Therefore,

$$X_t = X_\infty + e^{-\alpha t}(X_0 - X_\infty) + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B_{e^{2\alpha t}-1}$$

$$\log P_t = \mu - \frac{\sigma^2}{2\alpha} + e^{-\alpha t}\left(\log P_0 - \mu + \frac{\sigma^2}{2\alpha}\right) + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B_{e^{2\alpha t}-1}.$$

The mean and variance as $t \to \infty$ are given as

$$E(X_t) = X_\infty = \mu - \frac{\sigma^2}{2\alpha}, \quad \mathrm{Var}(X_t) = \frac{\sigma^2}{2\alpha}.$$

For P_t, we apply the properties of the lognoraml distributions:

$$E(P_t) = \exp\left(X_\infty + \frac{1}{2}\mathrm{Var}(X_t)\right) = \exp\left(\mu - \frac{\sigma^2}{4\alpha}\right).$$

$$\mathrm{Var}(P_t) = \left(\exp\left(\frac{\sigma^2}{2\alpha}\right) - 1\right)\exp\left(2\mu - \frac{\sigma^2}{2\alpha}\right) = \left(1 - \exp\left(-\frac{\sigma^2}{2\alpha}\right)\right)e^{2\mu}$$