Stochastic Finance (FIN 519)
Midterm Exam

Instructor: Jaehyuk Choi

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BM stands for Brownian motion. Assume that B_t is a standard **BM**. **RN** and **RV** stand for random number and random variable respectively. The PDF and CDF of the standard normal variable are denoted by $n(z)$ and $N(z)$ respectively. Assume interest rate and dividend rate are zero in option pricing.

1. (8 points) **Standard BM.** If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Explain briefly why it is a standard BM or not.
   (a) $(1/\sqrt{2})B_{2t}$
   (b) $\begin{cases} B_t & \text{if} \quad t \le \tau_a \\ 2a - B_t & \text{if} \quad t > \tau_a \end{cases}$, where τ_a is the first time B_t hitting the level a.
   (c) $\frac{1}{13}(5B_t + 12W_t)$ where W_t is another standard BM independent from B_t.
   (d) $B_{2t} - B_t$

Solution:

(a) Yes. The scaling property.

(b) Yes. The reflection principle.

(c) Yes.

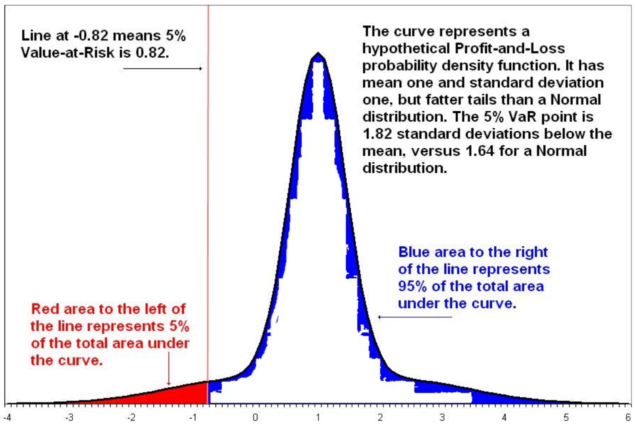
$$\frac{5^2 + 12^2}{13^2} = 1$$

(d) No. Let $X_t = B_{2t} - B_t$.

$$\begin{aligned} \mathrm{Cov}(X_s, X_t) &= E((B_{2s} - B_s)(B_{2t} - B_t) = E(B_{2s}B_{2t} - B_{2s}B_t - B_sB_{2t} + B_sB_t) \\ &= 3\min(s,t) - \min(2s,t) - \min(s,2t) \end{aligned}$$

If X_t is a standard BM, $\mathrm{Cov}(X_1, X_2) = \min(1,2) = 1$. However, $\mathrm{Cov}(X_1, X_2) = 3 - 2 - 1 = 0$.

2. (6 points) **Probability** (From Wikipedia) Value-at-risk (VaR) is a measure of the risk of loss for investments. It estimates how much a set of investments might lose (with a given probability p). VaR is typically used by firms and regulators in the financial industry to gauge the amount of assets needed to cover possible losses. For a given portfolio, time horizon, and probability p, the p-VaR is defined such that the probability of a loss greater than VaR is (at most) p while the probability of a loss less than VaR is (at least) $1 - p$. In other words, p-VaR is the loss <u>at</u> the worst p percentile.



(From Wikipedia. The graph is illustration only. **Ignore** the numbers in the graph.)

Assume that you invest in one share of stock today and that your profit & loss is distributed as $S_T - S_0 = X$ for some random variable X with the CDF, $F_X(x)$, and the PDF, $f_X(x)$.

(a) If $X \sim N(0, 10^2)$ (i.e., $\sigma = 10$), what is your 5%-VaR, $\text{VaR}(p = 0.05)$? You may use $N(-1.64) \approx 0.05$.

(b) Express the put option price with strike price K, $P(K)$, in terms of $f_X(x)$. (You may use integral in the answer.)

(c) Conditional VaR (CVaR or expected shortfall) is another risk measure to improve VaR. It is defined as the expected loss conditional on that the loss is within the worst p percentile. Find the expression for $\text{CVaR}(p)$. You can simplify the expression using $\text{VaR}(p)$ and the put option price, $P(K)$. Between $\text{VaR}(p)$ and $\text{CVaR}(p)$, which one assumes bigger loss?

Solution:

(a) $X \sim 10Z$ where $Z \sim N(0, 1)$.

$$\text{VaR}(0.05) = 10\, N^{-1}(0.05) = 10 \cdot (-1.64) = -16.4$$

(b)

$$\begin{aligned} P(K) &= \int_{x=-\infty}^{K-S_0} (K - S_0 - x) f_X(x) dx \\ \text{or} \quad &= \int_{x=-\infty}^{K} (K - x) f_X(x - S_0) dx \end{aligned}$$

(c)

$$\begin{aligned} \text{CVar}(p) &= \frac{1}{p} \int_{x=-\infty}^{K-S_0} x f_X(x) dx \quad \text{for} \quad K - S_0 = \text{VaR}(p) \\ &= \frac{K - S_0}{p} \int_{x=-\infty}^{K-S_0} f_X(x) dx + \frac{1}{p} \int_{x=-\infty}^{K} (x - K + S_0) f_X(x) dx \\ &= \frac{\text{VaR}(p)}{p} p - \frac{P(K)}{p} = \text{VaR}(p) - \frac{P(S_0 + \text{VaR}(p))}{p} \end{aligned}$$

CVaR assumes more severe loss.

3. (3 points) In the gambler's ruin problem,

$$S_n = X_1 + \cdots + X_n, \quad X_k = \pm 1 \quad \text{with probability} \quad p : q \; (p + q = 1),$$

what is the probability that S_n ever hits a level $A > 0$? How does this probability changes when p changes? (Hint: consider $B \to \infty$ from the results we know from class.)

Solution: The probability of hitting A before hitting $-B$ is given as

$$P(S_\tau = A) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1} \quad (p \neq q) \quad \text{or} \quad \frac{B}{A+B} \quad (p = q = 1/2)$$

If we let $B \to \infty$, the probability of S_n ever hitting A is

$$1 \quad \text{if} \quad p \geq 0.5 \quad \text{or} \quad (p/q)^A \quad \text{if} \quad p < 0.5$$

4. (2+3 points) **Merton's model.** From an accounting standpoint, a firm's equity (stock) value, S, is equal to $A - D$, where A is total asset value and D is total debt. When A goes below D, the firm defaults with equity value $S = 0$. In 1974, Merton proposed a model to price the current equity value S_0 as the expected asset value A_T above the constant debt value D, i.e., the call option on the asset A_T stuck at D, at some time T (expiry):

$$S_0 = E(\max(A_T - D, 0)) \quad \text{where} \quad A_0 > D.$$

If A_t follows a geometric BM, the resulting equity price is same as the call option price formula by Black and Scholes (1973). Thus, the formula is called Black-Scholes-Merton formula (and Merton was awarded Nobel prize with Scholes in 1997).

Instead, in this problem, assume that A_t follows an arithmetic BM with volatility σ:

$$A_t = A_0 + \sigma B_t \quad (\text{assume} \quad r = 0).$$

(a) What should be the current equity value S_0? You may use the result from class.

(b) Under this framework, we can also derive the corporate bond (issued by the firm) maturing at T. At the maturity $t = T$, the firm pays \$1 to the bond holder. If the firm defaults before T, however, the bond value becomes zero. Assume that the risk-free interest rate is zero. (Hint: use the result on the probability of the first hitting time, $P(\tau_a > T)$)

Solution:

(a) It is same as the call option value under normal model:

$$S_0 = (A_0 - D)N(d) + \sigma\sqrt{T}\, n(d) \quad \text{where} \quad d = \frac{A_0 - D}{\sigma\sqrt{T}}$$

(b) For a standard BM, the probability for the first time hitting the level a is given as

$$P(\tau_a < T) = 1 - P(\tau_a > T) = 2 - 2N\left(\frac{|a|}{\sigma\sqrt{T}}\right)$$

The bond price is same as the probability of A_t hitting D happening later than T. Therefore,

$$\text{Corporate bond price} = 2 - 2N\left(\frac{A_0 - D}{\sigma\sqrt{T}}\right)$$

5. (4 points) **Time-dependent volatility.** The at-the-money options on Meituan Dianping (IPO in September 2018 on Hong Kong stock exchange) with three-month ($T = 1/4$) and one-year ($T = 1$)

maturities are currently trading at the prices of 4.0 and 6.4 Hong Kong dollars, respectively. Assume that the stock price follows $dS_t = f(t)\,dB_t$ and that the option price can be approximated with 0.4 stdev(S_T). Find the piecewise constant instantaneous volatility $f(t)$ that satisfies the observed option prices.

Solution: We need to find

$$f(t) = \begin{cases} a & \text{if} \quad 0 \le t \le 0.25 \\ b & \text{if} \quad 0.25 \le t \end{cases}$$

For the two options,

$$\begin{aligned} 4 &= 0.4\sqrt{0.25\,a^2} \\ 6.4 &= 0.4\sqrt{0.25\,a^2 + 0.75\,b^2} \end{aligned}$$

We get $a = 20$ and $b = \sqrt{208} \approx 14.42$

6. (4 points) **Max option.** Assume that the stock price follows a BM, $S_t = S_0 + \sigma B_t$. As in the text book, assume that $S_t^* = \max_{0 \le s \le t} S_s$. Calculate the call option price whose payout at expiry $t = T$ is given by the maximum value on the path

$$\max(S_T^* - K, 0) \quad \text{where} \quad K > S_0$$

Intuitively, this option should be more expensive that the regular call option whose payout is given by the final price S_T. By how much more is it more expensive? (Hint: In class and textbook, we derived the PDF of B_T^*. Properly adjust σ.)

Solution: The PDF for the maximum of BM, B_t^*, is given as

$$f(x) = \frac{2}{\sqrt{t}}\, n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x > 0$$

This is equivalent to the normal distribution, $N(0, t)$, defined on the positive side only (that is why factor 2 is multiplied). Therefore, $S_T^* = S_0 + \sigma\sqrt{T}\,z$ where the PDF of z is $2n(z)$ and $z \ge 0$. The option price is twice as expensive as that of the regular call option:

$$C(K) = 2(S_0 - K)N(d) + 2\sigma\sqrt{T}\,n(d) \quad \text{where} \quad d = \frac{S_0 - K}{\sigma\sqrt{T}}$$