Stochastic Finance (FIN 519)
Homework Solutions

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1. **HW 1-1 SCFA** Exercise 1.1

Solution: Let $T_{i,j}$ denote the expected time to go from level i to j. We are going to compute the answer as

$$T_{25,18} = T_{25,20} + T_{20,19} + T_{19,18}.$$

First, $T_{25,20} = 15$ from Eq. (1.15):

$$E(\tau \,|\, S_0 = 0) = \frac{B}{q-p} - \frac{A+B}{q-p}\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

with $p = 1/3$, $q = 2/3$, $A = \infty$, $B = 5$. We also know $T_{21,20} = 3$ from $B = 1$. Next, $T_{20,19} = 37$ is calculated from

$$T_{20,19} = \frac{1}{10}\cdot 1 + \frac{9}{10}(1 + T_{21,20} + T_{20,19}).$$

Finally $T_{19,18} = 77$ is obtained from

$$T_{19,18} = \frac{1}{3}\cdot 1 + \frac{2}{3}(1 + T_{20,19} + T_{19,18}).$$

Therefore, $T_{25,18} = 15 + 37 + 77 = 129$.

2. **HW 1-2 SCFA** Exercise 1.3

Solution: Let N_k be the number of visits to the level $k \neq 0$ before returning to 0 for the first time. First, we prove that $P(N_k \geq 1) = 1/(2k)$ for $k \geq 1$. In order for the event, $N_k > 0$, to happen, the first step should be $+1$, ($X_1 = +1$). If the first step is -1, the random walk has to hit 0 before it reaches $k \geq 1$. Given that the first step is $+1$, the probability to hit k ($A = k-1$ more steps up) before hitting 0 ($B = 1$ step down) is given as $1/k$ from Eq. (1.2). Combining the two results together, we have $P(N_k > 0) = 1/(2k)$ for $k \geq 1$.

Next, we prove that

$$P(N_k \geq j+1 \,|\, N_k \geq j) = \frac{1}{2} + \frac{k-1}{2k}.$$

Imagine that the random walk just hit the level k for the j-th time before hitting 0. If the next step is up, it is guaranteed that it will hit k at least one more time before returning to

0. If the next step is down, we know that the probability to hit k one more time ($A = 1$) before hitting 0 ($B = k - 1$) is $(k-1)/k$. Adding the two probabilities together, we obtain the result.

Therefore, we can say that, for $j \geq 0$,

$$\begin{aligned} P(N_k > j) &= P(N_k > 0)P(N_k > 1|N_k > 0) \cdots P(N_k > j|N_k > j-1) \\ &= \frac{1}{2k}\left(\frac{1}{2} + \frac{k-1}{2k}\right)^j = \frac{1}{2k}\left(\frac{2k-1}{2k}\right)^j. \end{aligned}$$

We can prove the final statement:

$$\begin{aligned} E(N_k) &= \sum_{j=1}^{\infty} j \cdot P(N_k = j) = \sum_{j=1}^{\infty} P(N_k \geq j) = \sum_{j=0}^{\infty} P(N_k > j) \\ &= \frac{1}{2k}\sum_{j=0}^{\infty}\left(\frac{2k-1}{2k}\right)^j = 1. \end{aligned}$$

3. **HW 2-1** In class, we derived the moments of the standard normal distribution:

$$E(Z^{2n}) = (2n-1)(2n-3)\cdots 3 \cdot 1 \quad \text{for} \quad Z \sim N(0,1).$$

We can derive the same result using the moment generating function. First, derive the moment generating function,

$$M_X(t) = E(\exp(tX)) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \quad \text{for} \quad X \sim N(\mu, \sigma^2).$$

Then, using the Taylor expansion of $M_X(t)$, derive the moment of Z. (After this problem, you can understand **SCFA Exercise 3.4** better. The solution is already provided.)

Solution: The PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Therefore,

$$\begin{aligned} M_X(t) = E(\exp(tX)) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\ &= \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)\int_{-\infty}^{\infty} f_X(x - \sigma^2 t)dx = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right). \end{aligned}$$

For $Z \sim N(0,1)$, the moment generating function is $M_Z(t) = \exp(t^2/2)$. Expanding $M_Z(t)$,

$$M_Z(t) = 1 + \frac{t^2}{2} + \cdots + \frac{1}{2^n \cdot n!}t^{2n} + \cdots$$

The $2n$-th moment is given as

$$E(Z^{2n}) = \frac{(2n)!}{2^n \cdot n!} = \frac{(2n)!}{(2n)(2n-2)\cdots 2} = (2n-1)(2n-3)\cdots 3 \cdot 1$$

4. **HW 2-2** Martingale page in (WIKIPEDIA) gives the following example of Martingale. Prove (or disprove) the statement.

   > Suppose each amoeba either splits into two amoebas, with probability p, or eventually dies, with probability $q = 1 - p$. Let X_n be the number of amoebas surviving in the n-th generation (in particular $X_n = 0$ if the population has become extinct by that time). Let r be the probability of eventual extinction. (Finding r as a function of p is an instructive exercise. Hint: The probability that the descendants of an amoeba eventually die out is equal to the probability that either of its immediate offspring dies out, given that the original amoeba has split.) Then
   >
   > $$\{ r^{X_n} : n = 1, 2, 3, \dots \}$$
   >
   > is a martingale with respect to $\{X_n : n = 1, 2, 3, \dots\}$.

   See **2017-18 Midterm Exam Problem 1** to derive the extinction probability r as a function of p (and q).

   Solution: First, we find r in terms of p and q. Since the survival of amoeba's are independent, the probability of n amoebas' extinction is r^n. Branching on the 2nd generation (i.e., multiplication vs death of the first amoeba), we get the following recurrence relation,

   $$r = p \cdot r^2 + q,$$

   and we can solve $r = q/p$. In fact, the relation above indicate the first step proof of the Martingale:

   $$E(r^{X_2} | X_1 = 1) = E(r^{X_1}) = r.$$

   For the rest, we do not need $r = q/p$, but just the relation, $1 = pr + q/r$.

   At each step, each of X_n amoebas either becomes 2 or 0 with probability of p and q respectively. We can write

   $$X_{n+1} = X_n + \sum_{k=1}^{X_n} I_k, \quad r^{X_{n+1}} = r^{X_n} \cdot r^{I_1 + \cdots + I_{X_n}}$$

   where I_k takes value of $+1$ or -1 with probability p and q respectively and $\{I_k\}$ are independent events. The probability of j amoebas multiplying by 2 ($X_n - j$ amoebas die) is given by the binomial distribution, $\binom{n}{j} p^j \, q^{X_n - j}$. Therefore,

   $$E(r^{I_1 + \cdots + I_{X_n}}) = \sum_{j=0}^{X_n} \binom{X_n}{j} p^j \, q^{X_n - j} \cdot \frac{r^j}{r^{X_n - j}} = \sum_{j=0}^{X_n} \binom{X_n}{j} (pr)^j \, \left(\frac{q}{r}\right)^{X_n - j} = \left(pr + \frac{q}{r}\right)^{X_n} = 1.$$

   Therefore,

   $$E(r^{X_{n+1}} | X_n) = r^{X_n} \cdot E(r^{I_1 + \cdots + I_{X_n}}) = r^{X_n}.$$

5. **HW 3-1** SCFA Exercise Problem 6.1

6. **HW 3-2** SCFA Exercise Problem 6.2

7. **HW 3-3** 2017-18 Final Exam Problem 4 (Interest rate and bond price SDE)

8. **HW 3-4 Exponential Ornstein-Uhlenbeck process** Solve the following SDE:

$$\frac{dP_t}{P_t} = \alpha(\mu - \log P_t)dt + \sigma dB_t.$$

What are $E(P_t)$ and $\mathrm{Var}(P_t)$ as $t \to \infty$? (Hint: use $X_t = \log P_t$.)

Solution: The SDE for X_t satisfies

$$dX_t = \frac{dP_t}{P_t} - \frac{(dP_t)^2}{2P_t^2} = \alpha\left(\mu - \frac{\sigma^2}{2\alpha} - X_t\right)dt + \sigma dB_t.$$

This is the Ornstein-Uhlenbeck with $X_\infty = \mu - \frac{\sigma^2}{2\alpha}$. Therefore,

$$X_t = X_\infty + e^{-\alpha t}(X_0 - X_\infty) + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B_{e^{2\alpha t}-1}$$

$$\log P_t = \mu - \frac{\sigma^2}{2\alpha} + e^{-\alpha t}\left(\log P_0 - \mu + \frac{\sigma^2}{2\alpha}\right) + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B_{e^{2\alpha t}-1}.$$

The mean and variance as $t \to \infty$ are given as

$$E(X_t) = X_\infty = \mu - \frac{\sigma^2}{2\alpha}, \quad \mathrm{Var}(X_t) = \frac{\sigma^2}{2\alpha}.$$

For P_t, we apply the properties of the lognoraml distributions:

$$E(P_t) = \exp\left(X_\infty + \frac{1}{2}\mathrm{Var}(X_t)\right) = \exp\left(\mu - \frac{\sigma^2}{4\alpha}\right).$$

$$\mathrm{Var}(P_t) = \left(\exp\left(\frac{\sigma^2}{2\alpha}\right) - 1\right)\exp\left(2\mu - \frac{\sigma^2}{2\alpha}\right) = \left(1 - \exp\left(-\frac{\sigma^2}{2\alpha}\right)\right)e^{2\mu}$$