

Stochastic Finance (FIN 519)

Midterm Exam

Instructor: Jaehyuk Choi

2016-17 Module 3 (2017. 3. 23)

BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively.

1. (4 points) **Standard BM**

If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Provide a brief reason for your answer.

- (a) $4B_{t/2}$
- (b) $tB_{1/t}$ with $B_0 = 1$.
- (c) $2(B_{1+t/4} - B_1)$
- (d) $\sqrt{t}Z$ for a standard normal RV Z

Solution:

- (a) No. $\text{Var}(4B_{t/2}) = 16 \times t/2 = 8t \neq t$.
- (b) No. B_0 should be 0.
- (c) Yes. $B_{1+t/4} - B_1$ is equivalent to $B_{t/4}$ and $2B_{t/4}$ is equivalent to B_t .
- (d) No. For any value of Z , $\sqrt{t}Z$ is not a stochastic process. For example, $\sqrt{s}Z$ and $(\sqrt{t} - \sqrt{s})Z$ for $s < t$ are correlated.

2. (2 points) **Martingale related to BM**

If B_t is a standard BM, find the value of the coefficient λ in order for each of the following expressions to be a martingale.

- (a) $B_{at}^2 - \lambda t$
- (b) $\exp(-B_{at} + \lambda t)$

Solution:

- (a) $\lambda t = E(B_{at}^2) = at$. Therefore, $\lambda = a$.
- (b) $\sqrt{a}B_t$ is a BM equivalent to $-B_{at}$. Therefore, $\lambda = -a/2$.

3. (3 points) **Average of BM path**

If B_t for $0 \leq t \leq 1$ is a standard BM, what is the distribution of the average of the BM values observed at three different times, $T = 1/3, 2/3$ and 1,

$$A = \frac{1}{3} \left(B_{1/3} + B_{2/3} + B_1 \right)?$$

Please make sure to provide the mean and the standard deviation of the distribution.

Solution:

$$\begin{aligned} \text{Var} \left(\frac{1}{3} (B_{1/3} + B_{2/3} + B_1) \right) &= \frac{1}{9} E \left((B_{1/3} + B_{2/3} + B_1)^2 \right) \\ &= \frac{1}{9} E \left(B_{1/3}^2 + B_{2/3}^2 + B_1^2 + 2B_{1/3}(B_{2/3} + B_1) + 2B_{2/3}B_1 \right) \\ &= \frac{1}{9} \left(\frac{1}{3} + \frac{2}{3} + 1 + 2 \cdot \frac{1}{3} \cdot 2 + 2 \cdot \frac{2}{3} \right) = \frac{1}{9} \frac{14}{3} = \frac{14}{27}. \end{aligned}$$

4. (8 points) **Generating RNs for correlated BMs**

Throughout this problem, assume that X_t and Y_t are two independent standard BMs.

- (a) Other than the examples we covered in the class, there are many ways to create standard BMs. A linear combination of the two BMs with the coefficients a and b ,

$$W_t = aX_t + bY_t$$

is also a BM. (No need to prove it.) What is the condition for a and b under which W_t is a **standard** BM.

- (b) What is the correlation between X_t and W_t ? We have not defined the correlation of two BMs yet, so simply compute the correlation of the two distributions of the BMs at $t = 1$, i.e., X_1 and W_1 . (In fact, the correlation is same for any time t .) You do not have to use the answer of (a).
- (c) Assume that $\{z_k\}$ for $k = 1, 2, \dots$ is a sequence of standard normal RVs, i.e., $N(0, 1)$, which are generated from computer (e.g., using Box-Muller algorithm). Use $\{z_k\}$ to generate RNs for X_t for a fixed time t .
- (d) Assume that we have two standard BMs, X_t and W_t , which have correlation ρ . How can you generate the pairs of RNs for X_t and W_t for a fixed time t ?

Solution:

- (a) $\text{Var}(W_t) = a^2 \text{Var}(X_t) + b^2 \text{Var}(Y_t) = (a^2 + b^2)t$ should be t . Therefore, $a^2 + b^2 = 1$.

- (b)

$$\text{Corr}(W_t, X_t) = \frac{\text{Cov}(X_t, W_t)}{\sqrt{\text{Var}(X_t)\text{Var}(W_t)}} = \frac{at}{\sqrt{t \cdot (a^2 + b^2)t}} = \frac{a}{\sqrt{a^2 + b^2}}$$

(c) $\{\sqrt{t} z_k\}$ is the RNs for X_t .

(d) We can rewrite W_t as $W_t = \rho X_t + \sqrt{1 - \rho^2} Y_t$. Therefore, the random numbers for X_t and W_t can be generated as

$$\begin{aligned} &(\sqrt{t} z_1, \rho\sqrt{t} z_1 + \sqrt{1 - \rho^2}\sqrt{t} z_2) \\ &(\sqrt{t} z_3, \rho\sqrt{t} z_3 + \sqrt{1 - \rho^2}\sqrt{t} z_4) \\ &\quad \dots \\ &(\sqrt{t} z_{2k-1}, \rho\sqrt{t} z_{2k-1} + \sqrt{1 - \rho^2}\sqrt{t} z_{2k}) \end{aligned}$$

5. (3 points) **Wald's equation**

When $\{X_k\}$ are independent identically distributed random variable and N is a random variable taking positive integer values, Wald's equation says

$$E(X_1 + X_2 + \dots + X_N) = E(N) E(X_1)$$

if either (i) N is independent from $\{X_k\}$ or (ii) N is a stopping time with respect to $\{X_k\}$.

Consider an example where $X_k = 0$ or 1 with 50% and 50% probability and N is given as

$$N = X_2 + 1.$$

Obviously, $E(X_k) = 1/2$ and $E(N) = 1/2 + 1 = 3/2$. Find $E(X_1 + X_2 + \dots + X_N)$ and explain why Wald's equation does not hold in this example. If N is given instead as

$$N = X_1 + 1,$$

does Wald's equation hold? Is N a stopping time?

Solution: Branching on the value of X_2 :

$$N = \begin{cases} 1 & (X_2 = 0, \text{ Prob} = 1/2) \\ 2 & (X_2 = 1, \text{ Prob} = 1/2), \end{cases}$$

we compute

$$\begin{aligned} E(X_1 + X_2 + \dots + X_N) &= \frac{1}{2}E(X_1) + \frac{1}{2}E(X_1 + 1) \\ &= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} + 1\right) = 1 \neq E(X_k)E(N) = \frac{3}{4}. \end{aligned}$$

Wald's equation does not hold because $N = X_1 + X_2 + 1$ is

(i) not independent from $\{X_k\}$ as N is defined via X_2 and (ii) not a stopping time w.r.t. $\{X_k\}$ because we need future information X_2 in order to determine N (X_1 is not enough to tell $N = 1$ or not.)

If $N = X_1 + 1$, N is a stopping time. Based on X_1 we can tell whether $N = 1$ or not: $N = 1$ if $X_1 = 0$ and $N \neq 1$ (in fact, $N = 2$) if $X_1 = 1$. Therefore Wald's inequality should hold. We can directly verify that

$$E(X_1 + X_2 + \cdots + X_N) = \frac{1}{2}E(X_1) + \frac{1}{2}E(X_1 + X_2) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = \frac{3}{4}.$$

6. (5 points) Please circle the most appropriate item to you in each of the following questions.
- (a) The difficulty of this course is (i) easy (ii) appropriate (iii) difficult.
 - (b) Compared to the other required courses in the 1st year, the load of this course is (i) lower (ii) similar (iii) higher.
 - (c) If this course was **not** a required course, you would (i) still register (ii) not register.
 - (d) Professor's preparation for the course and communication with students are (i) satisfactory (ii) acceptable (iii) unsatisfactory.
 - (e) To your opinion (not professor's), this course is going to be (i) useful (ii) not useful for your future career path.