Stochastic Finance (FIN 519)
Final Exam

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2018-19 Module 3 (2019. 4. 19.)

BM stands for Brownian motion.

1. (4 points) **Stochastic calculus.** Choose **all** surviving terms (i.e., non-zero terms) in stochastic calculus. Assume that $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$ for some functions μ and σ.
   (a) $dX_t \cdot dt$
   (b) $(dX_t)^2$
   (c) $dX_t \cdot dB_t$
   (d) $dB_t^1 \cdot dB_t^2$ for the two independent BMs, B_t^1 and B_t^2

Solution: (b) $(dX_t)^2 = \sigma(t, X_t)^2\, dt$ and (c) $dX_t \cdot dB_t = \sigma(t, X_t)\, dt$

2. (2×3 points) **Option price and delta under the BSM model.** You hold a call option with $K = 100$ maturing in 3 months. Assume that a stock's annual volatility is 32% of the current price. Also assume that $r = q = 0$. You may use the following CDF values for the standard normal distribution $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

   (a) The stock's current spot price is $S_0 = 100$. What is the price of the call option under the BSM model?
   (b) What is the delta (i.e., the sensitivity to S_0) of the option?
   (c) If the spot price changed to $S_0 = 100.5$, what is the new option price under the BSM model? Approximate the price with Taylor's expansion using the results from (a) and (b).

Solution:

(a) Under the BSM model:

$$d_1 = \frac{\sigma_{\text{BS}}\sqrt{1/4}}{2} = 0.08, \quad d_2 = -0.08$$
$$C_0 = S_0 N(d_1) - K N(d_2) = 100 N(0.08) - 100(1 - N(0.08)) = 6.4$$

(b)

$$\frac{\partial C}{\partial S_0} = N(d_1) = 0.532$$

(c)

$$C_0' = C_0 + (S_0' - S_0) N(d_1) = 6.4 + 0.5 \times 0.532 = 6.666$$

3. (2×3 points) **Itô's lemma.** The stochastic variance in Heston model is given by

$$dV_t = \alpha(V_\infty - V_t)dt + \sigma\sqrt{V_t}\,dB_t.$$

This process is also know as Cox-Ingersoll-Ross(CIR) model for stochastic interest rate. Additionally, the stochastic variance under the 3/2 model is also given as

$$dV_t = \lambda V_t(V_\infty - V_t)\,dt + \xi V_t\sqrt{V_t}\,dB_t.$$

(a) Derive $E(V_t \,|\, \mathcal{F}_0)$ under the Heston model. (Hint: The transformation, $y_t = e^{\alpha\, t}(V_t - V_\infty)$, used in the Ornstein-Uhlenbeck process is also useful in this problem. Then, use the martingale property.) What is $E(V_t \,|\, \mathcal{F}_0)$ as $t \to \infty$.

(b) To solve the 3/2 model, the inverse variance, $Y_t = 1/V_t$, is helpful. Find the SDE satisfied by Y_t.

(c) What is $E(1/V_t \,|\, \mathcal{F}_0)$ under the 3/2 model? (Hint: You should recognize a similarity between the SDEs for V_t under Heston model and Y_t under the 3/2 model.)

Solution: See Cox-Ingersoll-Ross model (WIKIPEDIA).

(a) The transformation y_t is a martingale because

$$dy_t = \alpha e^{\alpha\, t}(V_t - V_\infty) + e^{\alpha\, t} dV_t = e^{\alpha\, t}\sigma\sqrt{V_t} dB_t.$$

Therefore, $y_0 = E(y_t|\mathcal{F}_0)$:

$$\begin{aligned} V_0 - V_\infty &= E(e^{\alpha\, t}(V_t - V_\infty) \,|\, \mathcal{F}_0) \\ E(V_t|\mathcal{F}_0) &= V_\infty + e^{-\alpha\, t}(V_0 - V_\infty) = (1 - e^{-\alpha\, t})V_\infty + e^{-\alpha\, t}V_0. \end{aligned}$$

$E(V_t|\mathcal{F}_0)$ goes to V_∞ as $t \to \infty$.

(b) Applying Itô's lemma:

$$\begin{aligned} dY_t &= -(1/V_t^2)dV_t + (1/V_t^3)(dV_t)^2 \\ &= -Y_t(\lambda(V_\infty - V_t)dt + \xi\sqrt{V_t}\,dB_t) - Y_t^3\xi^2 V_t^3 dt \\ &= (\lambda + \xi^2 - \lambda V_\infty Y_t)dt - \xi\sqrt{Y_t} dB_t. \\ &= \lambda V_\infty\left(\frac{\lambda + \xi^2}{\lambda V_\infty} - Y_t\right) dt - \xi\sqrt{Y_t} dB_t. \end{aligned}$$

(c) The process Y_t follows an CIR model with the parameters,

$$\alpha = \lambda V_\infty, \quad Y_\infty = \frac{\lambda + \xi^2}{\lambda V_\infty}, \quad \sigma = -\xi$$

Therefore,

$$\begin{aligned} E(1/V_t \,|\, \mathcal{F}_0) = E(Y_t \,|\, \mathcal{F}_0) &= (1 - e^{-\alpha\, t})Y_\infty + e^{-\alpha\, t}Y_0 \\ &= \left(1 - e^{-\lambda V_\infty\, t}\right)\frac{\lambda + \xi^2}{\lambda V_\infty} + e^{-\lambda V_\infty\, t}\frac{1}{V_0} \end{aligned}$$

4. (2×4 points) **SDE and martingale representation theorem.**

(a) Apply Itô calculus to find the stochastic differentiation of $\cosh(B_t)$. Reminded that $\cosh x = (e^x + e^{-x})/2$.

(b) Find λ such that $X_t = e^{\lambda t}\cosh(B_t)$ is a martingale (i.e., dX_t has no dt term.)

(c) Using (b), find the martingale representation of $\cosh(B_T)$. In other words, find V_0 and ϕ_t satisfying

$$\cosh(B_T) = V_0 + \int_0^T \phi_t \, dB_t$$

(d) Assume a stock price S_t follows $dS_t = \sigma dB_t$ and $r = 0$. What is the price of a derivative that pays $\cosh(S_T - S_0)$ at time $t = T$?

Solution:

(a)

$$d\cosh(B_t) = \sinh(B_t)dB_t + \frac{1}{2}\cosh(B_t)dt.$$

(b) The SDE for X_t is

$$dX_t = e^{\lambda t}\sinh(B_t)dB_t + e^{\lambda t}\left(\lambda + \frac{1}{2}\right)\cosh(B_t)dt.$$

Therefore X_t is a martingale when $\lambda = -1/2$.

(c)

$$X_T - X_0 = e^{-T/2}\cosh(B_T) - 1 = \int_0^T e^{-t/2}\sinh(B_t)\, dB_t$$
$$\cosh(B_T) = e^{T/2} + \int_0^T e^{(T-t)/2}\sinh(B_t)\, dB_t$$

Therefore, we obtain $V_0 = e^{T/2}$ and $\phi_t = e^{(T-t)/2}\sinh(B_t)$.

(d) With the presence of σ, the results above becomes

$$X_t = e^{\sigma^2 t/2}\cosh(\sigma B_t)$$

and

$$\cosh(S_T - S_0) = \cosh(\sigma B_T) = e^{\sigma^2 T/2} + \int_0^T e^{\sigma^2(T-t)/2}\sinh(\sigma B_t)\, dB_t$$

The price is $V_0 = e^{\sigma^2 T/2}$. It can also directly computed as

$$V_0 = E(\cosh(\sigma B_T)) = \frac{1}{2}E\left(e^{\sigma\sqrt{T}z} + e^{-\sigma\sqrt{T}z}\right) = \frac{1}{2}\left(e^{\sigma^2 T/2} + e^{\sigma^2 T/2}\right) = e^{\sigma^2 T/2}.$$

5. (2×3 points) T**-forward measure.** In class, we learned that the forward price of buying a bond maturing at $T + \Delta$ at time $t = T$,

$$F_t = \frac{B(t, T+\Delta)}{B(t,T)},$$

is a martingale under the T-forward measure. We are going to prove this under an explicit setting. From a 2017 exam problem, we also know that

$$\frac{d\,B(t,T)}{B(t,T)} = r_t dt - \beta(T-t)\, dB_t^Q$$

when the risk-free rate changes according to $dr_t = \alpha dt + \beta\, dB_t^Q$. Here, B_t^Q is the standard BM under the risk-neutral measure.

(a) Derive the SDE for F_t. (Hint. First compute $d \log F_t$ and compute dF_t/F_t)

(b) If B_t^T is the standard BM under the T-forward measure, what is the relation between dB_t^Q and dB_t^T?

(c) From (a) and (b), finally derive the SDE for F_t under the T-forward measure. Is F_t a martingale? What is the volatility of dF_t/F_t?

Solution: Assume $\sigma = -\beta(T-t)$ and $\sigma' = -\beta(T+\Delta-t)$ so that

$$\frac{d\,B(t,T)}{B(t,T)} = r_t dt + \sigma\, dB_t^Q, \quad \frac{d\,B(t,T+\Delta)}{B(t,T+\Delta)} = r_t dt + \sigma'\, dB_t^Q$$

(a) From,

$$d \log B(t,T) = (r_t - \sigma^2/2)dt + \sigma\, dB_t^Q, \quad d \log B(t,T+\Delta) = (r_t - \sigma'^2/2)dt + \sigma'\, dB_t^Q,$$

we get

$$\begin{aligned} d \log F_t &= -\frac{1}{2}(\sigma'^2 - \sigma^2)dt + (\sigma' - \sigma)dB_t^Q, \\ \frac{dF_t}{F_t} &= -\frac{1}{2}(\sigma'^2 - \sigma^2)dt + (\sigma' - \sigma)dB_t^Q \\ &= -\frac{1}{2}(\sigma'^2 - \sigma^2)dt + \frac{1}{2}(\sigma' - \sigma)^2 dt + (\sigma' - \sigma)dB_t^Q \\ &= -\sigma(\sigma' - \sigma)dt + (\sigma' - \sigma)dB_t^Q \\ &= -\Delta\beta^2(T-t)dt - \Delta\beta dB_t^Q \end{aligned}$$

(b) Because the volatility of the numeraire $B(t,T)$ is $\sigma = -\beta(T-t)$,

$$dB_t^Q = dB_t^T - \beta(T-t)dt.$$

(c)

$$\frac{dF_t}{F_t} = -\Delta\beta dB_t^T.$$

F_t is a martingale indeed and the volatility is $\sigma' - \sigma = -\Delta\beta$.