Stochastic Finance (FIN 519)
Final Exam

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BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. Assume that B_t is a standard **BM**. The PDF and CDF for standard normal distribution is denoted by $n(z)$ and $N(z)$.

1. (6 points) Assume that a stochastic process X_t follows

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

for some functions, μ and σ. Find the stochastic differential equation (SDE) for the process $Y_t = \exp(X_t)$.

Solution: Applying, Itô's lemma,

$$\begin{aligned} dY_t &= \exp(X_t)dX_t + \frac{1}{2}\exp(X_t)(dX_t)^2 \\ &= Y_t\left(\mu(t, X_t)dt + \sigma(t, X_t)dB_t\right) + \frac{1}{2}Y_t\sigma^2(t, X_t)(dB_t)^2. \end{aligned}$$

Finally, we obtain the SDE as

$$\frac{dY_t}{Y_t} = \left(\mu(t, X_t) + \frac{1}{2}\sigma^2(t, X_t)\right)dt + \sigma(t, X_t)dB_t.$$

2. (6 points) **Option vega under the BSM model**. By direct computation, show that the vega (i.e., the derivative of the price with respect to the volatility σ) of both call and put option is

$$V = \frac{\partial C}{\partial \sigma} = S_0\, n(d_1)\sqrt{T} \quad \text{with} \quad d_1 = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

Since the terms, d_1 and d_2, in the Black-Scholes formula are defined via σ, you should also differentiate d_1 and d_2 rather than treating them as constants.

Solution: Using the properties

$$\frac{\partial d_1}{\partial \sigma} = -\frac{\log(S_0\, e^{rT}/K)}{\sigma^2\sqrt{T}} + \frac{1}{2}\sqrt{T} = -\frac{d_2}{\sigma}$$
$$\frac{\partial d_2}{\partial \sigma} = -\frac{\log(S_0\, e^{rT}/K)}{\sigma^2\sqrt{T}} - \frac{1}{2}\sqrt{T} = -\frac{d_1}{\sigma}$$

$$\frac{d_1^2 - d_2^2}{2} = \frac{(A+B)^2 - (A-B)^2}{2} = 2AB = \log(S_0\, e^{rT}/K),$$

where

$$A = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \quad \text{and} \quad B = \frac{\sigma\sqrt{T}}{2}.$$

We compute the vega as

$$\begin{aligned} V = \frac{\partial}{\partial\sigma}\left(S_0\, N(d_1) - e^{-rT} K N(d_2)\right) &= S_0\, n(d_1)\frac{\partial d_1}{\partial \sigma} - e^{-rT} K n(d_2)\frac{\partial d_2}{\partial \sigma} \\ = -S_0\, n(d_1)\frac{d_2}{\sigma} + e^{-rT} K n(d_2)\frac{d_1}{\sigma} &= S_0\,\frac{n(d_1)}{\sigma}\left(-d_2 + d_1 e^{(d_1^2 - d_2^2)/2}\frac{K}{S_0 e^{rT}}\right) \\ = S_0\,\frac{n(d_1)}{\sigma}(-d_2 + d_1) = S_0\,\frac{n(d_1)}{\sigma}\sigma\sqrt{T} &= S_0\, n(d_1)\sqrt{T}. \end{aligned}$$

3. (9 points) Assume that the stock price S_t follows a BM with the stochastic volatility σ_t following a GBM. The price and volatility are driven by the same standard BM B_t.

$$dS_t = \sigma_t dB_t \quad \text{and} \quad \frac{d\sigma_t}{\sigma_t} = -\nu dB_t \quad (\nu > 0).$$

(a) (3 points) Express the final price S_T in terms of S_0, σ_0, ν, T, and B_T by solving this SDE.

(b) (3 points) What is the call option price with strike price K and time-to-maturity T?

(c) (3 points) If $\nu = 0$, then $\sigma_t = \sigma_0$ for all t, that is, the volatility is no longer stochastic. Prove that, in the limit of $\nu \to 0$, the call option price from (b) converges to that of the normal model with normal volatility σ_0. You may need Taylor's expansion: $\log(1+\varepsilon) \approx \varepsilon - \varepsilon^2/2$ when ε is very small.

Solution:

(a)

$$\sigma_T = \sigma_0 \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right)$$

$$\sigma_T - \sigma_0 = -\nu \int_0^T \sigma_t dB_t$$

$$S_T - S_0 = \int_0^T \sigma_t dB_t = \frac{1}{\nu}(\sigma_0 - \sigma_T) = \frac{\sigma_0}{\nu}\left(1 - \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right)\right)$$

(b) The final stock price S_T is expressed by a standard normal variable z:

$$S_T - S_0 = \frac{\sigma_0}{\nu}\left(1 - \exp\left(-\nu z\sqrt{T} - \frac{1}{2}\nu^2 T\right)\right).$$

If $z = -d_1$ is the root of $S_T = K$, d_1 is obtained as

$$\log\left(1 - \frac{\nu}{\sigma_0}(K - S_0)\right) = \nu d_1\sqrt{T} - \frac{1}{2}\nu^2 T$$
$$d_1 = \frac{\log\left(1 - \frac{\nu}{\sigma_0}(K - S_0)\right)}{\nu\sqrt{T}} + \frac{1}{2}\nu\sqrt{T}.$$

The call option price can be obtained as

$$\begin{aligned}
C = E\Big((S_T - K)^+\Big) &= \int_{-d_1}^{\infty}\left[\frac{\sigma_0}{\nu}\left(1 - \exp\left(-\nu z\sqrt{T} - \frac{1}{2}\nu^2 T\right)\right) + S_0 - K\right] n(z)dz \\
&= \left(S_0 - K + \frac{\sigma_0}{\nu}\right)\int_{-d_1}^{\infty} n(z)dz - \frac{\sigma_0}{\nu}\int_{-d_1}^{\infty} n(z + \nu\sqrt{T})dz \\
&= \left(S_0 - K + \frac{\sigma_0}{\nu}\right)(1 - N(-d_1)) - \frac{\sigma_0}{\nu}(1 - N(-d_1 + \nu\sqrt{T})) \\
&= \left(S_0 - K + \frac{\sigma_0}{\nu}\right)N(d_1) - \frac{\sigma_0}{\nu}N(d_2) \\
\text{where}\quad d_{1,2} &= \frac{\log\left(1 - \frac{\nu}{\sigma_0}(K - S_0)\right)}{\nu\sqrt{T}} \pm \frac{1}{2}\nu\sqrt{T}
\end{aligned}$$

(c) Let

$$d = \frac{(S_0 - K)}{\sigma_0\sqrt{T}},$$

then d_1 and d_2 converge to d as $\nu \to 0$:

$$d_{1,2} = \frac{\log\left(1 - \frac{\nu}{\sigma_0}(K - S_0)\right)}{\nu\sqrt{T}} \pm \frac{1}{2}\nu\sqrt{T} = \frac{(S_0 - K)}{\sigma_0\sqrt{T}} + A\nu \pm \frac{1}{2}\nu\sqrt{T} \to d \quad \text{for some } A.$$

Therefore, the call option price converges to

$$\begin{aligned}
C &= (S_0 - K)N(d_1) + \frac{\sigma_0}{\nu}(N(d_1) - N(d_2)) \\
&= (S_0 - K)N(d) + \sigma_0\sqrt{T}\,n(d),
\end{aligned}$$

where we used the L'Hopital's rule,

$$\lim_{\nu\to 0}\frac{N(d_1) - N(d_2)}{\nu} = \lim_{\nu\to 0}\frac{N(d + A\nu + \nu\sqrt{T}/2) - N(d + A\nu - \nu\sqrt{T}/2)}{\nu} = n(d)\sqrt{T}$$

4. (9 points) **Black-Scholes and martingale representation theorem.** For this question, assume that the current time is $t = t$ (instead of $t = 0$) and the option expiry is $t = T$. Therefore, the time-to-maturity is $T - t$ (instead of T). Also assume that $r = 0$ to make the

problem simple. Then, the underlying stock price follows a geometric BM, $dS_t = \sigma S_t\, dB_t$, and the call option price at time t is given by the Black-Scholes formula,

$$C_t = S_t N(d_1) - K\, N(d_2), \quad \text{where} \quad d_{1,2} = \frac{\log(S_t/K)}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t}.$$

From the derivation of the BS formula in class, we know

$$dC_t = D_t\, dS_t,$$

where D_t is the delta of the option, i.e., the amount of the underlying stock to hold at time t to hedge the option:

$$D_t = \frac{\partial C_t}{\partial S_t} = N(d_1).$$

The martingale representation also tells us that the option premium, C_0, and the P&L from the hedge position from $t = 0$ to T will exactly add up to the payoff of the option, $C_T = (S_T - K)^+$:

$$C_T = (S_T - K)^+ = C_0 + \int_0^T N(d_1) dS_t.$$

In this question, we are going to show $dC_t = D_t\, dS_t$ by direct computation.

(a) (3 points) Find the gamma, the second derivative with respect to the spot price S_t, and theta, the derivative with respect to time t:

$$G_t = \frac{\partial^2 C_t}{\partial S_t^2} \quad \text{and} \quad \Theta_t = \frac{\partial C_t}{\partial t}$$

(b) (3 points) Apply Itô's lemma to find the stochastic differential equation (SDE) for C_t. Use the result of (a).

(c) (3 points) Imagine that a situation where the volatility for pricing option is different from that for the underlying stock. You price and risk-manage option using volatility σ_i (i for *implied* volatility). That is, C_t and D_t is evaluated with

$$d_{1,2} = \frac{\log(S_t/K)}{\sigma_i\sqrt{T-t}} \pm \frac{1}{2}\sigma_i\sqrt{T-t}.$$

But the underlying stock has volatility σ_r (r for *realized* volatility),

$$dS_t = \sigma_r S_t\, dB_t.$$

Derive the SDE for C_t again under this new situation. In this situation, does the option premium and hedging P&L amount to the option payout? Compare the cases, $\sigma_i > \sigma_r$ and $\sigma_i > \sigma_r$.

Solution:

(a) Gamma and theta are obtained as

$$G_t = \frac{n(d_1)}{S_t\, \sigma\sqrt{T-t}} \quad \text{and} \quad \Theta_t = -\frac{\sigma\, S_t\, n(d_1)}{2\sqrt{T-t}}.$$

(b) The SDE for C_t is computed as

$$\begin{aligned}
dC_t &= \frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial C_t^2}{\partial^2 S_t}(dS_t)^2 \\
&= \left(\Theta_t + \frac{\sigma^2 S_t^2}{2}G_t\right)dt + D_t\, dS_t \\
&= \left(-\frac{\sigma\, S_t\, n(d_1)}{2\sqrt{T-t}} + \frac{\sigma^2 S_t^2}{2}\frac{n(d_1)}{S_t\,\sigma\sqrt{T-t}}\right)dt + N(d_1)\, dS_t \\
&= N(d_1)\, dS_t.
\end{aligned}$$

(c) Using $(dS_t)^2 = \sigma_r^2\, S_t^2\, dt$ instead, we obtain

$$\begin{aligned}
dC_t &= \left(-\frac{\sigma_i\, S_t\, n(d_1)}{2\sqrt{T-t}} + \frac{\sigma_r^2 S_t^2}{2}\frac{n(d_1)}{S_t\,\sigma_i\sqrt{T-t}}\right)dt + N(d_1)\, dS_t \\
&= \frac{\sigma_r^2 - \sigma_i^2}{2}\cdot\frac{S_t\, n(d_1)}{\sigma_i\sqrt{T-t}}dt + N(d_1)\, dS_t \\
&= \frac{S_t^2}{2}(\sigma_r^2 - \sigma_i^2)G_t dt + N(d_1)\, dS_t.
\end{aligned}$$

The sum of the premium and hedging P&L is

$$C_0 + \int_0^T N(d_1)dS_t = (S_T - K)^+ + \frac{\sigma_i^2 - \sigma_r^2}{2}\int_0^T S_t^2\, G_t\, dt.$$

Notice that $S_t^2 G_t$ is always positive. If $\sigma_i > \sigma_r$, the value you hold at $t = T$ is bigger than the option payout $(S_T - K)^+$. If $\sigma_r > \sigma_i$, the value is less than the option payout $(S_T - K)^+$. This is consistent with observation,

$$C_0 > C_0^r \quad \text{if} \quad \sigma_i > \sigma_r \quad \text{and} \quad C_0 < C_0^r \quad \text{if} \quad \sigma_i < \sigma_r,$$

where C_0^r is the *correct* option price evaluated with the realized volatility σ_r,

$$d_{1,2}^r = \frac{\log(S_t/K)}{\sigma_r\sqrt{T-t}} \pm \frac{1}{2}\sigma_r\sqrt{T-t}.$$

(Option price is a monotonically increasing function of volatility.)