Stochastic Finance (FIN 519)
Midterm Exam

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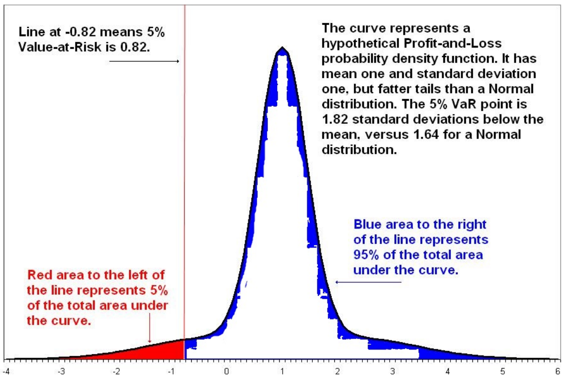
BM stands for Brownian motion. Assume that B_t is a standard **BM**. **RN** and **RV** stand for random number and random variable respectively. The PDF and CDF of the standard normal variable are denoted by $n(z)$ and $N(z)$ respectively. Assume interest rate and dividend rate are zero in option pricing.

1. (8 points) **Standard BM.** If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Explain briefly why it is a standard BM or not.
   (a) $(1/\sqrt{2})B_{2t}$
   (b) $\begin{cases} B_t & \text{if} \quad t \le \tau_a \\ 2a - B_t & \text{if} \quad t > \tau_a \end{cases}$, where τ_a is the first time B_t hitting the level a.
   (c) $\frac{1}{13}(5B_t + 12W_t)$ where W_t is another standard BM independent from B_t.
   (d) $B_{2t} - B_t$

Solution:
(a) Yes. The scaling property.
(b) Yes. The reflection principle.
(c) Yes.
$$\frac{5^2 + 12^2}{13^2} = 1$$
(d) No. Let $X_t = B_{2t} - B_t$.
$$\begin{aligned} \mathrm{Cov}(X_s, X_t) &= E((B_{2s} - B_s)(B_{2t} - B_t) = E(B_{2s}B_{2t} - B_{2s}B_t - B_sB_{2t} + B_sB_t) \\ &= 3\min(s,t) - \min(2s,t) - \min(s,2t) \end{aligned}$$
If X_t is a standard BM, $\mathrm{Cov}(X_1, X_2) = \min(1,2) = 1$. However, $\mathrm{Cov}(X_1, X_2) = 3 - 2 - 1 = 0$.

2. (6 points) **Probability** (From Wikipedia) Value-at-risk (VaR) is a measure of the risk of loss for investments. It estimates how much a set of investments might lose (with a given probability p). VaR is typically used by firms and regulators in the financial industry to gauge the amount of assets needed to cover possible losses. For a given portfolio, time horizon, and probability p, the p-VaR is defined such that the probability of a loss greater

than VaR is (at most) p while the probability of a loss less than VaR is (at least) $1-p$. In other words, p-VaR is the loss <u>at</u> the worst p percentile.



(From Wikipedia. The graph is illustration only. <u>**Ignore** the numbers in the graph.</u>)

Assume that you invest in one share of stock today and that your profit & loss is distributed as $S_T - S_0 = X$ for some random variable X with the CDF, $F_X(x)$, and the PDF, $f_X(x)$.

(a) If $X \sim N(0, 10^2)$ (i.e., $\sigma = 10$), what is your 5%-VaR, VaR$(p = 0.05)$? You may use $N(-1.64) \approx 0.05$.

(b) Express the put option price with strike price K, $P(K)$, in terms of $f_X(x)$. (You may use integral in the answer.)

(c) Conditional VaR (CVaR or expected shortfall) is another risk measure to improve VaR. It is defined as the expected loss <u>conditional on</u> that the loss is <u>within</u> the worst p percentile. Find the expression for CVaR(p). You can simplify the expression using VaR(p) and the put option price, $P(K)$. Between VaR(p) and CVaR(p), which one assumes bigger loss?

Solution:

(a) $X \sim 10Z$ where $Z \sim N(0,1)$.

$$\mathrm{VaR}(0.05) = 10\, N^{-1}(0.05) = 10 \cdot (-1.64) = -16.4$$

(b)

$$\begin{aligned} P(K) &= \int_{x=-\infty}^{K-S_0} (K - S_0 - x) f_X(x) dx \\ \text{or} \quad &= \int_{x=-\infty}^{K} (K - x) f_X(x - S_0) dx \end{aligned}$$

(c)

$$\begin{aligned} \mathrm{CVar}(p) &= \frac{1}{p} \int_{x=-\infty}^{K-S_0} x f_X(x) dx \quad \text{for} \quad K - S_0 = \mathrm{VaR}(p) \\ &= \frac{K - S_0}{p} \int_{x=-\infty}^{K-S_0} f_X(x) dx + \frac{1}{p} \int_{x=-\infty}^{K} (x - K + S_0) f_X(x) dx \\ &= \frac{\mathrm{VaR}(p)}{p} p - \frac{P(K)}{p} = \mathrm{VaR}(p) - \frac{P(S_0 + \mathrm{VaR}(p))}{p} \end{aligned}$$

CVaR assumes more severe loss.

3. (3 points) In the gambler's ruin problem,

$$S_n = X_1 + \cdots + X_n, \quad X_k = \pm 1 \quad \text{with probability} \quad p : q \ (p + q = 1),$$

what is the probability that S_n ever hits a level $A > 0$? How does this probability changes when p changes? (Hint: consider $B \to \infty$ from the results we know from class.)

Solution: The probability of hitting A before hitting $-B$ is given as

$$P(S_\tau = A) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1} \quad (p \neq q) \quad \text{or} \quad \frac{B}{A+B} \quad (p = q = 1/2)$$

If we let $B \to \infty$, the probability of S_n ever hitting A is

$$1 \quad \text{if} \quad p \geq 0.5 \quad \text{or} \quad (p/q)^A \quad \text{if} \quad p < 0.5$$

4. (2+3 points) **Merton's model.** From an accounting standpoint, a firm's equity (stock) value, S, is equal to $A - D$, where A is total asset value and D is total debt. When A goes below D, the firm defaults with equity value $S = 0$. In 1974, Merton proposed a model to price the current equity value S_0 as the expected asset value A_T above the constant debt value D, i.e., the call option on the asset A_T stuck at D, at some time T (expiry):

$$S_0 = E(\max(A_T - D, 0)) \quad \text{where} \quad A_0 > D.$$

If A_t follows a geometric BM, the resulting equity price is same as the call option price formula by Black and Scholes (1973). Thus, the formula is called Black-Scholes-Merton formula (and Merton was awarded Nobel prize with Scholes in 1997).

Instead, in this problem, assume that A_t follows an arithmetic BM with volatility σ:

$$A_t = A_0 + \sigma B_t \quad (\text{assume} \quad r = 0).$$

(a) What should be the current equity value S_0? You may use the result from class.

(b) Under this framework, we can also derive the corporate bond (issued by the firm) maturing at T. At the maturity $t = T$, the firm pays \$1 to the bond holder. If the firm defaults before T, however, the bond value becomes zero. Assume that the risk-free interest rate is zero. (Hint: use the result on the probability of the first hitting time, $P(\tau_a > T)$)

Solution:

(a) It is same as the call option value under normal model:

$$S_0 = (A_0 - D)N(d) + \sigma\sqrt{T}\, n(d) \quad \text{where} \quad d = \frac{A_0 - D}{\sigma\sqrt{T}}$$

(b) For a standard BM, the probability for the first time hitting the level a is given as

$$P(\tau_a > T) = 2N\left(\frac{|a|}{\sqrt{T}}\right) - 1$$

The bond price is same as the probability of A_t hitting D happening later than T. Therefore,

$$\text{Corporate bond price} = 2N\left(\frac{A_0 - D}{\sigma\sqrt{T}}\right) - 1$$

5. (4 points) **Time-dependent volatility.** The at-the-money options on Meituan Dianping (IPO in September 2018 on Hong Kong stock exchange) with three-month ($T = 1/4$) and one-year ($T = 1$) maturities are currently trading at the prices of 4.0 and 6.4 Hong Kong dollars, respectively. Assume that the stock price follows $dS_t = f(t)\,dB_t$ and that the option price can be approximated with 0.4 stdev(S_T). Find the piecewise constant instantaneous volatility $f(t)$ that satisfies the observed option prices.

Solution: We need to find

$$f(t) = \begin{cases} a & \text{if} \quad 0 \le t \le 0.25 \\ b & \text{if} \quad 0.25 \le t \end{cases}$$

For the two options,

$$4 = 0.4\sqrt{0.25\,a^2}$$
$$6.4 = 0.4\sqrt{0.25\,a^2 + 0.75\,b^2}$$

We get $a = 20$ and $b = \sqrt{208} \approx 14.42$

6. (4 points) **Max option.** Assume that the stock price follows a BM, $S_t = S_0 + \sigma B_t$. As in the text book, assume that $S_t^* = \max_{0\le s\le t} S_s$. Calculate the call option price whose payout at expiry $t = T$ is given by the maximum value on the path

$$\max(S_T^* - K, 0) \quad \text{where} \quad K > S_0$$

Intuitively, this option should be more expensive that the regular call option whose payout is given by the final price S_T. By how much more is it more expensive? (Hint: In class and textbook, we derived the PDF of B_T^*. Properly adjust σ.)

Solution: The PDF for the maximum of BM, B_t^*, is given as

$$f(x) = \frac{2}{\sqrt{t}}\, n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x > 0$$

This is equivalent to the normal distribution, $N(0, t)$, defined on the positive side only (that is why factor 2 is multiplied). Therefore, $S_T^* = S_0 + \sigma\sqrt{T}\,z$ where the PDF of z

is $2n(z)$ and $z \geq 0$. The option price is twice as expensive as that of the regular call option:

$$C(K) = 2(S_0 - K)N(d) + 2\sigma\sqrt{T}\, n(d) \quad \text{where} \quad d = \frac{S_0 - K}{\sigma\sqrt{T}}$$