Fixed Income and Risk Management

Interest Rate Models

Fall 2003, Term 2

Agenda and key issues

- **Pricing with binomial trees**
  - Replication
  - Risk-neutral pricing
- **Interest rate models**
  - Definitions
  - Uses
  - Features
  - Implementation
- **Binomial tree example**
- **Embedded options**
  - Callable bond
  - Putable bond
- **Factor models**
  - Spot rate process
  - Drift and volatility functions
  - Calibration

1-period binomial model

- Stock with price S = \$60 and one-period risk-free rate of r = 20%
- Over next period stock price either falls to \$30 or rises to \$90

$$\begin{tikzcd} & S_u = \$90 \\ S = \$60 \arrow[ur, dash] \arrow[dr, dash] & \\ & S_d = \$30 \end{tikzcd}$$

- Call option with strike price K = \$60 pays either \$0 or \$30

$$\begin{tikzcd} & C_u = \$30 \\ C = ? \arrow[ur, dash] \arrow[dr, dash] & \\ & C_d = 0 \end{tikzcd}$$

- Buy Δ = ½ share of stock and borrow L = \$12.50

$$\begin{tikzcd} & S_u/2 - 1.2 \times \$12.5 = \$45 - \$15 = \$30 \\ \$60/2 - \$12.50 = \$17.50 \arrow[ur, dash] \arrow[dr, dash] & \\ & S_d/2 - 1.2 \times \$12.5 = \$15 - \$15 = \$0 \end{tikzcd}$$

1-period binomial model (cont)

- **Portfolio replicates option payoff Þ *C* = $17.50**
- **Solving for replicating portfolio**
  - – Buy Δ shares of stock and borrow L
  - – If stock price rises to $90, we want the portfolio to be worth

$$\$90 \times \Delta - 1.2 \times L = \$30$$

  - – If stock price drops to $30, we want the portfolio to be worth

$$\$30 \times \Delta - 1.2 \times L = \$0$$

  - – Δ = 0.5 and *L* = $12.50 solve these two equations

1-period binomial model (cont)

- **"Delta"**
  - Δ is chosen so that the value of the replicating portfolio ($\Delta \times S - L$) has the same sensitivity to S as the option price C

$$\Delta = \frac{dC}{dS} = \frac{\$30 - \$0}{\$90 - \$30} = \frac{1}{2}$$

  - Δ is called the <u>hedge ratio</u> of "<u>delta</u>" of the option
  - Delta-hedging an option is analogous to duration-hedging a bond

1-period binomial model (cont)

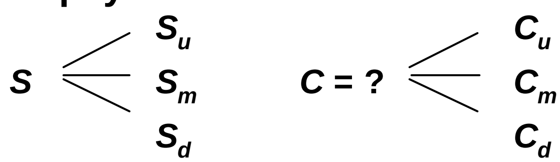
- **Very important result**

  The option price does <u>not</u> depend on the probabilities of a stock price up-move or down-move

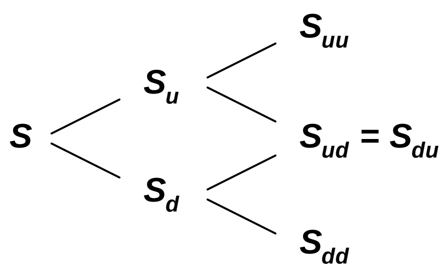
- **Intuition**
  - If $C \neq \Delta \times S - L$, there exist an arbitrage opportunity
  - Arbitrage opportunities deliver riskless profits
  - Riskless profits cannot depend on probabilities
  - Therefore, the option price cannot depend on probabilities

1-period binomial model (cont)

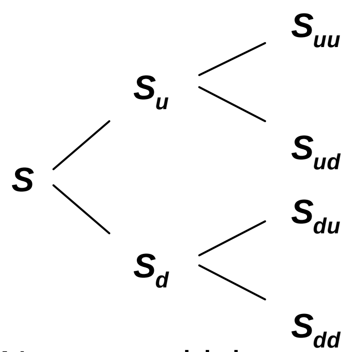
- **Unfortunately, this simple replication argument does not work with 3 or more payoff states**



- **Rather than increase the number of payoff states per period, increase the number of binomial periods Þ <u>binomial tree</u>**



<u>Recombining</u>



<u>Non-recombining</u>

1-period binomial model (cont)

- **Define**
  - u = 1 + return if stock price goes up
  - d = 1 + return if stock price goes down
  - r = per-period riskless rate (constant for now)
  - p = probability of stock price up-move
- **No arbitrage requires $d \le 1 + r \le u$**
- **Stock and option payoffs**

$$\begin{tikzcd}[column sep=small, row sep=small] & S \times u \\ S \arrow[ur, dash] \arrow[dr, dash] & \\ & S \times d \end{tikzcd} \qquad \begin{tikzcd}[column sep=small, row sep=small] & C_u = f(S \times u) \\ C = ? \arrow[ur, dash] \arrow[dr, dash] & \\ & C_d = f(S \times d) \end{tikzcd}$$

1-period binomial model (cont)

- **Payoff of portfolio of Δ shares and *L* dollars of borrowing**

$$\Delta \times S \times u - L \times (1+r)$$

$$\Delta S - L$$

$$\Delta \times S \times d - L \times (1+r)$$

- **Replication requires**

$$\Delta \times S \times u - L \times (1+r) = C_u$$
$$\Delta \times S \times d - L \times (1+r) = C_d$$

- **Two equations in two unknowns (Δ and *L*) with solution**

$$\Delta = \frac{C_u - C_d}{S \times (u-d)} \qquad L = \frac{d \times C_u - u \times C_d}{(1+r) \times (r-d)}$$

- **Option price**

$$C = \Delta \times S - L$$

Risk-neutral pricing (cont)

- **Define**

$$q = \frac{(1+r)-d}{u-d} \qquad (1-q) = \frac{u-(1+r)}{u-d}$$

- **No-arbitrage condition *d* £ 1 + *r* £ *u* implied 0 £ *q* £ 1**

- **Rearrange option price**

$$\begin{aligned} C &= \Delta \times S - L \\ &= \frac{C_u - C_d}{S \times (u-d)} \times S - \frac{d \times C_u - u \times C_d}{(1+r) \times (u-d)} \\ &\ldots \\ &= \frac{q \times C_u + (1-q) \times C_d}{1+r} \end{aligned}$$

Risk-neutral pricing (cont)

- **Interpretation of *q***
  - Expected return on the stock
    $$\mathrm{E}\left[\frac{S_1}{S_0}\right] = \frac{p \times S \times u + (1-p) \times S \times d}{S} = p \times u + (1-p) \times d$$
  - Suppose we were risk-neutral
    $$\mathrm{E}\left[\frac{S_1}{S_0}\right] = p \times u + (1-p) \times d = (1+r)$$
  - Solving for p
    $$p = \frac{(1+r)-d}{u-d} = q$$

- **Very, very important result**

  ***q* is the probability which sets the expected return on the stock equal to the riskfree rate Þ <u>risk-neutral probability</u>**

Risk-neutral pricing (cont)

- **Very, very, very important result**

  The option price equals its expected payoff discounted by the riskfree rate, where the expectation is formed using risk-neutral probabilities instead of real probabilities Þ <u>risk-neutral pricing</u>

- **Risk-neutral pricing extends to multiperiod binomial trees and applies to <u>all</u> derivatives which can be replicated**

$$\text{Derivatives price} = \mathrm{PV}_r\left[\mathrm{E}^q\left[\text{payoff}\right]\right]$$

Risk-neutral pricing intuition

- **Step 1**
  - Derivatives are priced by no-arbitrage
  - No-arbitrage does not depend on risk preferences or probabilities
- **Step 2**
  - Imagine a world in which all security prices are the same as in the real world but everyone is risk-neutral (a "risk-neutral world")
  - The expected return on any security equals the risk-free rate r
- **Step 3**
  - In the risk-neutral world, every security is priced as its expected payoff discounted by the risk-free rate, including derivatives
  - Expectations are taken wrt the risk-neutral probabilities q
- **Step 4**
  - Derivative prices must be the same in the risk-neutral and real worlds because there is only <u>one</u> no-arbitrage price

2-period binomial model

- **Stock and option payoffs**

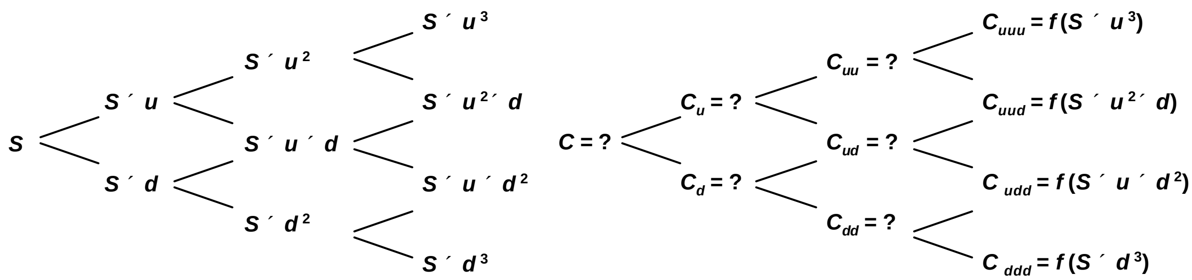
$$
\begin{tikzcd}[column sep=small, row sep=tiny]
 & & S \times u^2 \\
 & S \times u \arrow[ur, dash] \arrow[dr, dash] & \\
S \arrow[ur, dash] \arrow[dr, dash] & & S \times u \times d \\
 & S \times d \arrow[ur, dash] \arrow[dr, dash] & \\
 & & S \times d^2
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=tiny]
 & & C_{uu} = f(S \times u^2) \\
 & C_u = ? \arrow[ur, dash] \arrow[dr, dash] & \\
C = ? \arrow[ur, dash] \arrow[dr, dash] & & C_{ud} = f(S \times u \times d) \\
 & C_d = ? \arrow[ur, dash] \arrow[dr, dash] & \\
 & & C_{dd} = f(S \times d^2)
\end{tikzcd}
$$

- **By risk-neutral pricing**

$$C = \frac{q^2 \times C_{uu} + 2 \times q \times (1-q) \times C_{ud} + (1-q)^2 \times C_{dd}}{(1+r)^2}$$

3-period binomial model

- **Stock and option payoffs**



- **By risk-neutral pricing**

$$C = \frac{1}{(1+r)^3} \times \Big[q^3 \times C_{uuu} + 3 \times q^2 \times (1-q) \times C_{uud} + \ldots$$
$$3 \times q \times \times (1-q)^2 \times C_{udd} + (1-q)^3 \times C_{ddd} \Big]$$

Definitions

- **An <u>interest rate model</u> describes the dynamics of either**
  - 1-period spot rate
  - <u>Instantaneous</u> spot rate = t-year spot rate $r(t)$ as $t \to 0$
- **Variation in spot rates is generated by either**
  - One source of risk $\Rightarrow$ <u>single-factor models</u>
  - Two or more sources of risk $\Rightarrow$ <u>multifactor models</u>

Model uses

- **Characterize term structure of spot rates to price bonds**
- **Price interest rate and bond derivatives**
  - Exchange traded (e.g., Treasury bond or Eurodollar options)
  - OTC (e.g., caps, floors, collars, swaps, swaptions, exotics)
- **Price fixed income securities with embedded options**
  - Callable or putable bonds
- **Compute price sensitivities to underlying risk factor(s)**
- **Describe risk-reward trade-off**

Model features

- **Interest rate models should be**
  - Arbitrage free = model prices agree with current market prices
    - Spot rate curve
    - Coupon yield curve
    - Interest rate and bond derivatives
  - Time-consistent = model implied behavior of spot rates and bond prices agree with their observed behavior
    - Mean reversion
    - Conditional heteroskedasticity
    - Term structure of volatility and correlation structure
- **Developing an interest rate model which is both arbitrage free and time consistent is the holy grail of fixed income research**

Model implementation

- **In practice, two model implementations**
  - Cross-sectional calibration
    - Calibrate model to match exactly all market prices of liquid securities on a single day
    - Used for pricing less liquid securities and derivatives
    - Arbitrage free but probably not time-consistent
    - Usually one or two factors
  - Time-series estimation
    - Estimate model using a long time-series of spot rates
    - Used for hedging and asset allocation
    - Time-consistent but not arbitrage free
    - Usually two and more factors

Spot rate tree

- **1-period spot rates (*m*-period compounded APR)**

$$
\begin{array}{ccccc}
 & & & & r_{2,2}(1) = 12\% \\
 & & r_{1,1}(1) = 11\% & \nearrow & \\
 & \nearrow & & \searrow & \\
r_{0,0}(1) = 10\% & & & & r_{2,1}(1) = 10\% \\
 & \searrow & & \nearrow & \\
 & & r_{1,0}(1) = 9\% & \searrow & \\
 & & & & r_{2,0}(1) = 8\%
\end{array}
$$

- **Notation**
  - $r_{i,j}(n)$ = *n*-period spot rate *i* periods in the future after *j* up-moves
  - Δt = length of a binomial step in units of years
- **Set $\Delta t = 1/m$ and $m = 2$**
- **Assume $q_{i,j}$ = 0.5 for all steps *i* and nodes *j***

Road-map

- **Calculate step-by-step**
  - Implied spot rate curve $r_{0,0}(1)$, $r_{0,0}(2)$, $r_{0,0,}(3)$
  - Implied changes in the spot rate curve

$$
\begin{array}{ccc}
 & \nearrow & r_{1,1}(1),\ \mathrm{r}_{1,1}(2) \\
r_{0,0}(1),\ \mathrm{r}_{0,0}(2) & & \\
 & \searrow & r_{1,0}(1),\ \mathrm{r}_{1,0}(2)
\end{array}
$$

  - Price 8% 1.5-yr coupon bond
  - Price 1-yr European call option on 8% 1.5-yr coupon bond
  - Price 1-yr American put option on 8% 1.5-yr coupon bond

1-period zero-coupon bond prices

- **At time 0**

$$P_{0,0}(1) = ? \quad \begin{cases} P_{1,1}(0) = \$100 \\ P_{1,0}(0) = \$100 \end{cases}$$

$$P_{0,0}(1) = \frac{q_{0,0} \times P_{1,1}(0) + (1 - q_{0,0}) \times P_{1,0}(0)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}$$

$$= \frac{\$100}{(1 + 0.1/2)^1} = \$95.24$$

1-period zero-coupon bond prices (cont)

- **At time 1**

$$P_{1,1}(1) = \frac{q_{1,1} \times P_{2,2}(0) + (1 - q_{1,1}) \times P_{2,1}(0)}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}}$$

$$= \frac{\$100}{(1 + 0.11/2)^1} = \$94.79$$

$$P_{1,0}(1) = \frac{q_{1,0} \times P_{2,1}(0) + (1 - q_{1,0}) \times P_{2,0}(0)}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}}$$

$$= \frac{\$100}{(1 + 0.09/2)^1} = \$95.69$$

1-period zero-coupon bond prices (cont)

- **At time 2**

$$P_{2,2}(1) = \frac{q_{2,2} \times P_{3,3}(0) + (1 - q_{2,2}) \times P_{3,2}(0)}{(1 + r_{2,2}(1)/m)^{1 \times \Delta t \times m}}$$

$$= \frac{\$100}{(1 + 0.12/2)^1} = \$94.34$$

$$P_{2,1}(1) = \frac{q_{2,1} \times P_{3,2}(0) + (1 - q_{2,1}) \times P_{3,1}(0)}{(1 + r_{2,1}(1)/m)^{1 \times \Delta t \times m}} = \$95.24$$

$$P_{2,0}(1) = \frac{q_{2,0} \times P_{3,1}(0) + (1 - q_{2,0}) \times P_{3,0}(0)}{(1 + r_{2,0}(1)/m)^{1 \times \Delta t \times m}}$$

$$= \frac{\$100}{(1 + 0.08/2)^1} = \$96.15$$

1-period zero-coupon bond prices (cont)

$P_{2,2}(1) = \$94.34$

$P_{1,1}(1) = \$94.79$

$P_{0,0}(1) = \$95.24$

$P_{2,1}(1) = \$95.25$

$P_{1,0}(1) = \$95.69$

$P_{2,0}(1) = \$96.15$

2-period zero-coupon bond prices

- **At time 0**

$$P_{0,0}(2) = ? \quad \begin{cases} P_{1,1}(1) = \$94.79 \\ P_{1,0}(1) = \$95.69 \end{cases}$$

$$\begin{aligned} P_{0,0}(2) &= \frac{q_{0,0} \times P_{1,1}(1) + (1 - q_{0,0}) \times P_{1,0}(1)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} \\ &= \frac{\frac{1}{2} \times \$94.79 + \frac{1}{2} \times \$95.69}{(1 + 0.1/2)^1} = \$90.71 \end{aligned}$$

- **Implied 2-period spot rate**

$$P_{0,0}(2) = \frac{\$100}{(1 + r_{0,0}(2)/m)^{2 \times \Delta t \times m}} \quad \Rightarrow r_{0,0}(2) = 9.9976\%$$

2-period zero-coupon bond prices (cont)

- **At time 1**

$$P_{1,1}(2) = \frac{q_{1,1} \times P_{2,2}(1) + (1 - q_{1,1}) \times P_{2,1}(1)}{(1 + r_{1,1}(1)/m)^{1\times\Delta t\times m}}$$

$$= \frac{\frac{1}{2} \times \$94.34 + \frac{1}{2} \times \$95.24}{(1 + 0.11/2)^1} = \$89.85$$

$$\Rightarrow \quad r_{1,1}(2) = 10.9976\%$$

$$P_{1,0}(2) = \frac{q_{1,0} \times P_{2,1}(1) + (1 - q_{1,0}) \times P_{2,0}(1)}{(1 + r_{1,0}(1)/m)^{1\times\Delta t\times m}}$$

$$= \frac{\frac{1}{2} \times \$95.24 + \frac{1}{2} \times \$96.15}{(1 + 0.09/2)^1} = \$91.58$$

$$\Rightarrow \quad r_{1,0}(2) = 8.9976\%$$

3-period zero-coupon bond price

- **At time 0**

$$P_{0,0}(3) = ? \quad \begin{cases} P_{1,1}(2) = \$89.85 \\ P_{1,0}(2) = \$91.58 \end{cases}$$

$$P_{0,0}(3) = \frac{q_{0,0} \times P_{1,1}(2) + (1 - q_{0,0}) \times P_{1,0}(2)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}$$
$$= \frac{\frac{1}{2} \times \$89.85 + \frac{1}{2} \times \$91.58}{(1 + 0.1/2)^1} = \$86.39$$

- **Implied 3-period spot rate**

$$P_{0,0}(3) = \frac{\$100}{(1 + r_{0,0}(3)/m)^{3 \times \Delta t \times m}} \quad \Rightarrow r_{0,0}(3) = 9.9937\%$$

Implied spot rate curve

- **Current spot rate curve is slightly downward sloping**

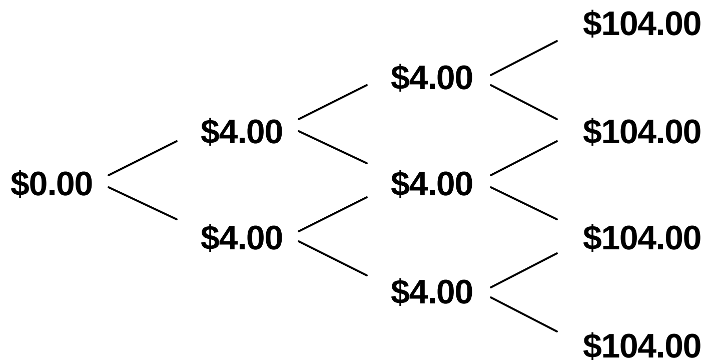
$$
\begin{aligned}
r_{0,0}(1) &= 10.0000\% \\
r_{0,0}(2) &= 9.9976\% \\
r_{0,0}(3) &= 9.9937\%
\end{aligned}
$$

- **From one period to the next, the spot rate curve shifts in parallel**

$$
\begin{array}{ccc}
 & & \begin{aligned} r_{1,1}(1) &= 11.0000\% \\ r_{1,1}(2) &= 10.9976\% \end{aligned} \\
\begin{aligned} r_{0,0}(1) &= 10.0000\% \\ r_{0,0}(2) &= 9.9976\% \end{aligned} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\
 & & \begin{aligned} r_{1,0}(1) &= 9.0000\% \\ r_{1,0}(2) &= 8.9976\% \end{aligned}
\end{array}
$$

Coupon bond price

- 8% 1.5-year (3-period) coupon bond with cashflow



Coupon bond price (cont)

- **Discounting terminal payoffs by 1 period**

$P_{3,3}$ = \$104.00

$P_{2,2}$ = \$104.00/1.06 = \$98.11

$P_{1,1}$ = ?

$P_{3,2}$ = \$104.00

$P_{0,0}$ = ?

$P_{2,1}$ = \$104.00/1.05 = \$99.05

$P_{1,0}$ = ?

$P_{3,1}$ = \$104.00

$P_{2,0}$ = \$104.00/1.04 = \$100

$P_{3,0}$ = \$104.00

Coupon bond price (cont)

- **By risk-neutral pricing**

$P_{3,3}$ = \$104.00

$P_{2,2}$ = \$104.00/1.06
= \$98.11

$P_{1,1}$ = \$97.23

$P_{3,2}$ = \$104.00

$P_{0,0}$ = ?

$P_{2,1}$ = \$104.00/1.05
= \$99.05

$P_{1,0}$ = ?

$P_{3,1}$ = \$104.00

$P_{2,0}$ = \$104.00/1.04
= \$100

$P_{3,0}$ = \$104.00

$$
\begin{aligned}
P_{1,1} &= \frac{c + q_{1,1} \times P_{2,2} + (1 - q_{1,1}) \times P_{2,1}}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}} \\
&= \frac{\$4.00 + \frac{1}{2} \times \$98.11 + \frac{1}{2} \times \$99.05}{1.055}
\end{aligned}
$$

Coupon bond price (cont)

- **By risk-neutral pricing (cont)**

$P_{3,3}$ = \$104.00

$P_{2,2}$ = \$104.00/1.06
= \$98.11

$P_{1,1}$ = \$97.23

$P_{3,2}$ = \$104.00

$P_{0,0}$ = ?

$P_{2,1}$ = \$104.00/1.05
= \$99.05

$P_{1,0}$ = \$99.07

$P_{3,1}$ = \$104.00

$P_{2,0}$ = \$104.00/1.04
= \$100

$P_{3,0}$ = \$104.00

$$
\begin{aligned}
P_{1,0} &= \frac{c + q_{1,0} \times P_{2,1} + (1 - q_{1,0}) \times P_{2,0}}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}} \\
&= \frac{\$4.00 + \frac{1}{2} \times \$99.05 + \frac{1}{2} \times \$100.00}{1.045}
\end{aligned}
$$

Coupon bond price (cont)

- By risk-neutral pricing (cont)

$P_{3,3}$ = \$104.00

$P_{2,2}$ = \$104.00/1.06 = \$98.11

$P_{1,1}$ = \$97.23

$P_{3,2}$ = \$104.00

$P_{0,0}$ = \$97.28

$P_{2,1}$ = \$104.00/1.05 = \$99.05

$P_{1,0}$ = \$99.07

$P_{3,1}$ = \$104.00

$P_{2,0}$ = \$104.00/1.04 = \$100

$P_{3,0}$ = \$104.00

$$P_{0,0} = \frac{c + q_{0,0} \times P_{1,1} + (1 - q_{0,0}) \times P_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}$$

$$= \frac{\$4.00 + \frac{1}{2} \times \$97.23 + \frac{1}{2} \times \$99.07}{1.05}$$

European call on coupon bond

- **1-yr <u>European style</u> <u>call option</u> on 8% 1.5-yr coupon bond with strike price K = \$99.00 pays max[0, $P_{2,?}$ – K]**

$V_{2,2}$ = max[0, \$98.11–\$99.00] = 0

$V_{1,1}$ = ?

$V_{0,0}$ = ?

$V_{2,1}$ = max[0, \$99.05–\$99.00] = \$0.05

$V_{1,0}$ = ?

$V_{2,0}$ = max[0,\$100.00–\$99.00] = \$1.00

European call on coupon bond

- **1-yr <u>European style</u> <u>call option</u> on 8% 1.5-yr coupon bond with strike price K = \$99.00 pays max[0, $P_{2,?} - K$]**

$V_{2,2}$ = max[0, \$98.11–\$99.00] = 0

$V_{1,1}$ = \$0.0237

$V_{0,0}$ = ?

$V_{2,1}$ = max[0, \$99.05–\$99.00] = \$0.05

$V_{1,0}$ = ?

$V_{2,0}$ = max[0,\$100.00–\$99.00] = \$1.00

$$V_{1,1} = \frac{q_{1,1} \times V_{2,2} + (1 - q_{1,1}) \times V_{2,1}}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}}$$

$$= \frac{0.5 \times \$0.00 + 0.5 \times \$0.05}{1.055}$$

European call on coupon bond

- **1-yr European style call option on 8% 1.5-yr coupon bond with strike price K = \$99.00 pays max[0, $P_{2,?}$ – K]**

$V_{2,2}$ = max[0, \$98.11–\$99.00] = 0

$V_{1,1}$ = \$0.0237

$V_{2,1}$ = max[0, \$99.05–\$99.00] = \$0.05

$V_{0,0}$ = ?

$V_{1,0}$ = \$0.5024

$V_{2,0}$ = max[0,\$100.00–\$99.00] = \$1.00

$$V_{1,0} = \frac{q_{1,0} \times V_{2,1} + (1 - q_{1,0}) \times V_{2,0}}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}}$$

$$= \frac{0.5 \times \$0.05 + 0.5 \times \$1.00}{1.045}$$

European call on coupon bond

- **1-yr European style call option on 8% 1.5-yr coupon bond with strike price K = \$99.00 pays max[0, $P_{2,?}$ – K]**

$V_{2,2}$ = max[0, \$98.11–\$99.00] = 0

$V_{1,1}$ = \$0.0237

$V_{0,0}$ = \$0.2505

$V_{2,1}$ = max[0, \$99.05–\$99.00] = \$0.05

$V_{1,0}$ = \$0.5024

$V_{2,0}$ = max[0,\$100.00–\$99.00] = \$1.00

$$
\begin{aligned}
V_{0,0} &= \frac{q_{0,0} \times V_{1,1} + (1 - q_{0,0}) \times V_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} \\
&= \frac{0.5 \times \$0.0237 + 0.5 \times \$0.5024}{1.05}
\end{aligned}
$$

American put on coupon bond

- **1-yr <u>American style</u> <u>put option</u> on 8% 1.5-yr coupon bond with strike price K = $99.00 pays max[0, K – $P_{i,?}$]**

$V_{2,2}$ = max[0,$99.00– $98.11]
= $0.89

$V_{1,1}$ = ?

$V_{2,1}$ = max[0,$99.00– $99.05]
= $0.00

$V_{0,0}$ = ?

$V_{1,0}$ = ?

$V_{2,0}$ = max[0,$99.00–$100.00]
= $0.00

American put on coupon bond

- **1-yr American style put option on 8% 1.5-yr coupon bond with strike price K = \$99.00 pays max[0, $K - P_{i,?}$]**

$V_{0,0}$ = ?

$V_{1,1}$ = \$1.7674

$V_{1,0}$ = ?

$V_{2,2}$ = max[0,\$99.00– \$98.11] = \$0.89

$V_{2,1}$ = max[0,\$99.00– \$99.05] = \$0.00

$V_{2,0}$ = max[0,\$99.00–\$100.00] = \$0.00

$$V_{1,1} = \max\left[\frac{q_{1,1} \times V_{2,2} + (1 - q_{1,1}) \times V_{2,1}}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}}, \underbrace{K - P_{1,1}}_{\text{exercise}}\right]$$

$$= \max\left[\frac{0.5 \times \$0.89 + 0.5 \times \$0.00}{1.055}, \$99.00 - \$97.23\right]$$

American put on coupon bond

- **1-yr American style put option on 8% 1.5-yr coupon bond with strike price K = \$99.00 pays max[0, $K - P_{i,?}$]**

$V_{0,0}$ = ?

$V_{1,1}$ = \$1.7674

$V_{1,0}$ = \$0.00

$V_{2,2}$ = max[0,\$99.00– \$98.11] = \$0.89

$V_{2,1}$ = max[0,\$99.00– \$99.05] = \$0.00

$V_{2,0}$ = max[0,\$99.00–\$100.00] = \$0.00

$$V_{1,0} = \max\left[\frac{q_{1,0} \times V_{2,1} + (1 - q_{1,0}) \times V_{2,0}}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}}, \underbrace{K - P_{1,0}}_{\text{exercise}}\right]$$

$$= \max\left[\frac{0.5 \times \$0.00 + 0.5 \times \$0.00}{1.045}, \$99.00 - \$99.07\right]$$

American put on coupon bond

- **1-yr <u>American style</u> <u>put option</u> on 8% 1.5-yr coupon bond with strike price K = \$99.00 pays max[0, $K - P_{i,?}$]**

$V_{2,2}$ = max[0,\$99.00– \$98.11] = \$0.89

$V_{1,1}$ = \$1.7674

$V_{0,0}$ = \$1.7150

$V_{2,1}$ = max[0,\$99.00– \$99.05] = \$0.00

$V_{1,0}$ = \$0.00

$V_{2,0}$ = max[0,\$99.00–\$100.00] = \$0.00

$$V_{0,0} = \max\left[\frac{q_{0,0} \times V_{1,1} + (1 - q_{0,0}) \times V_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}, \underbrace{K - P_{0,0}}_{\text{exercise}}\right]$$

$$= \max\left[\frac{0.5 \times \$1.7674 + 0.5 \times \$0.00}{1.05}, \$99.00 - \$97.28\right]$$

Callable Bond

- **Suppose we want to price a 10% 5-yr coupon bond callable (by the issuer) at the end of year 3 at par**
  - – Step 1: Determine the price of the non-callable bond, P_{NCB}
  - – Step 2: Determine the price of the call option on the non-callable bond with expiration after 3 years and strike price at par, O_{NCB}
  - – Step 3: The price of the callable bond is

$$P_{\text{CB}} = P_{\text{NCB}} - O_{\text{NCB}}$$

- **Intuition**
  - – The bondholder grants the issuer an option to buy back the bond
  - – The value of this option must be subtracted from the price the bondholder pays the issuer for the non-callable bond

Putable Bond

- **Suppose we want to price a 10% 5-yr coupon bond putable (by the bondholder to the issuer) at the end of year 3 at par**
  - Step 1: Determine the price of the non-putable bond, P_{NPB}
  - Step 2: Determine the price of the put option on the non-putable bond with expiration after 3 years and strike price at par, O_{NPB}
  - Step 3: The price of the putable bond is

$$P_{\text{PB}} = P_{\text{NPB}} + O_{\text{NPB}}$$

- **Intuition**
  - The bond issuer grants the holder an option to sell back the bond
  - The value of this option must be added to the price the bondholder pays the issuer for the non-putable bond

Spot rate process

- **Binomial trees are based on spot rate values $r_{i,j}(1)$ and risk-neutral probabilities $q_{i,j}$**
- **In single-factor models, these values are determined by a <u>risk-neutral</u> spot rate process of the form**

$$r_{t+\Delta t}(1) - r_t(1) = \underbrace{\mu\Big[r_t(1), t\Big]}_{\text{drift fct}} \times \Delta t + \underbrace{\sigma\Big[r_t(1), t\Big]}_{\text{volatility fct}} \times \sqrt{\Delta t} \times \epsilon_t$$

with

$$\text{Mean}[\epsilon_t] = 0 \quad \text{Var}[\epsilon_t] = 1$$

such that

$$\text{Mean}[r_{t+\Delta t}(1) - r_t(1)] = \mu\Big[r_t(1), t\Big] \times \Delta t$$

$$\text{Var}[r_{t+\Delta t}(1) - r_t(1)] = \sigma\Big[r_t(1), t\Big]^2 \times \Delta t$$

Spot rate process (cont)

- **In an *N*-factor models, these values are determined by a <u>risk-neutral</u> spot rate process of the form**

$$r_t(1) = z_{1,t} + z_{2,t} + \cdots z_{N,t}$$

with

$$z_{i,t+\Delta t} - z_{i,t} = \underbrace{\mu_i\Big[z_{1,t}, z_{2,t}, \cdots, z_{N,t}, t\Big]}_{\text{drift fct}} \times \Delta t \; + \underbrace{\sigma_i\Big[z_{1,t}, z_{2,t}, \cdots, z_{N,t}, t\Big]}_{\text{volatility fct}} \times \sqrt{\Delta t} \times \epsilon_{1,t}$$

and

$$\text{Mean}[\epsilon_{i,t}] = 0 \quad \text{Var}[\epsilon_{i,t}] = 1$$

Drift function

- **Case 1: Constant drift**

$$r_{t+\Delta t} - r_t = \underbrace{\lambda}_{\text{drift fct}} \times \Delta t + \sigma \times \sqrt{t} \times \epsilon_t$$

with

$$\epsilon_t \sim \mathrm{N}[0, 1]$$

- **Implied distribution of 1-period spot rates**

$$r_{t+\Delta t} \sim \mathrm{N}\Big[\underbrace{r_t + \lambda \times \Delta t}_{\text{mean}}, \underbrace{\sigma^2 \times \Delta t}_{\text{var}}\Big]$$

Drift function (cont)

- **Binomial tree representation**

$$
\begin{array}{lll}
 & & q = 1/2 \quad r_{0,0} + 2 \times \lambda \times \Delta t + 2 \times \sigma \times \mathrm{v}\ \Delta t \\
 & q = 1/2 \quad r_{0,0} + \lambda \times \Delta \mathrm{t} + \sigma \times \mathrm{v}\ \Delta t & \\
r_{0,0} & & q = 1/2 \quad r_{0,0} + 2 \times \lambda \times \Delta t \\
 & r_{0,0} + \lambda \times \Delta t - \sigma \times \mathrm{v}\ \Delta t & \\
 & & r_{0,0} + 2 \times \lambda \times \Delta t - 2 \times \sigma \times \mathrm{v}\ \Delta t
\end{array}
$$

- **Properties**
  - No mean reversion
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model ln[*r*(1)]
    $\Rightarrow$ “Rendleman-Bartter model”
  - 2 parameters
  - $\Rightarrow$ fit only 2 spot rates

Drift function (cont)

- **Example**
  - $r_{0,0} = 5\%$
  - $\lambda = 1\%$
  - $\sigma = 2.5\%$
  - $\Delta\, t = 1/m$ with $m = 2$

$$
\begin{array}{ccccc}
 & & & q = 1/2 & r_{2,2} = 9.54\% \\
 & q = 1/2 & r_{1,1} = 7.27\% & & \\
r_{0,0} = 5\% & & & q = 1/2 & r_{2,1} = 6.00\% \\
 & & r_{1,0} = 3.73\% & & \\
 & & & & r_{2,0} = 2.47\%
\end{array}
$$

Drift function (cont)

- **Case 2: Time-dependent drift**

$$r_{t+\Delta t} - r_t = \underbrace{\lambda(t)}_{\text{drift fct}} \times \Delta t + \sigma \times \sqrt{t} \times \epsilon_t$$

  with

$$\epsilon_t \sim \mathrm{N}[0,1]$$

- **Implied distribution of 1-period spot rates**

$$r_{t+\Delta t} \sim \mathrm{N}\Big[\underbrace{r_t + \lambda(t) \times \Delta t}_{\text{mean}}, \underbrace{\sigma^2 \times \Delta t}_{\text{var}}\Big]$$

- **Ho and Lee (1986, *J. of Finance*) Þ "Ho-Lee model"**

Drift function (cont)

- **Binomial tree representation**

$$
\begin{array}{ccccc}
 & & & q = 1/2 & r_{0,0} + [\lambda(1) + \lambda(2)] \times \Delta t + 2 \times \sigma \times \sqrt{\Delta t} \\
 & q = 1/2 & r_{0,0} + \lambda(1) \times \Delta \mathrm{t} + \sigma \times \sqrt{\Delta t} & & \\
r_{0,0} & & & & r_{0,0} + [\lambda(1) + \lambda(2)] \times \Delta t \\
 & & r_{0,0} + \lambda(1) \times \Delta t - \sigma \times \sqrt{\Delta t} & q = 1/2 & \\
 & & & & r_{0,0} + [\lambda(1) + \lambda(2)] \times \Delta t - 2 \times \sigma \times \sqrt{\Delta t}
\end{array}
$$

- **Properties**
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model $\ln[r(1)]$
    - $\Rightarrow$ "Salomon Brothers model"
  - Arbitrarily many parameters
    - $\Rightarrow$ fit term structure of spot rates but not necessarily spot rate volatilities (i.e., derivative prices)

Drift function (cont)

- **Case 3: Mean reversion**

$$r_{t+\Delta t} - r_t = \underbrace{\kappa \times \left[\theta - r_t(1)\right]}_{\text{drift fct}} \times \Delta t + \sigma \times \sqrt{t} \times \epsilon_t$$

with

$$\epsilon_t \sim \mathrm{N}[0,1]$$

- **Implied distribution of 1-period spot rates**

$$r_{t+\Delta t} \sim \mathrm{N}\left[\underbrace{r_t + \kappa \times \left[\theta - r_t(1)\right] \times \Delta t}_{\text{mean}}, \underbrace{\sigma^2 \times \Delta t}_{\text{var}}\right]$$

- **Vasicek (1977, *J. of Financial Economics*) Þ "Vasicek model"**

Drift function (cont)

- **Binomial tree representation**

$$r_{0,0} \quad \xrightarrow{q=1/2} \quad \begin{cases} r_{0,0} + \kappa \times (\theta - r_{0,0}) \times \Delta t + \sigma \times \sqrt{\Delta t} & \xrightarrow{q=1/2} \begin{cases} r_{1,1} + \kappa \times (\theta - r_{1,1}) \times \Delta t + \sigma \times \sqrt{\Delta t} \\ r_{1,1} + \kappa \times (\theta - r_{1,1}) \times \Delta t - \sigma \times \sqrt{\Delta t} \end{cases} \\ r_{0,0} + \kappa \times (\theta - r_{0,0}) \times \Delta t - \sigma \times \sqrt{\Delta t} & \xrightarrow{q=1/2} \begin{cases} r_{1,0} + \kappa \times (\theta - r_{1,0}) \times \Delta t + \sigma \times \sqrt{\Delta t} \\ r_{1,0} + \kappa \times (\theta - r_{1,0}) \times \Delta t - \sigma \times \sqrt{\Delta t} \end{cases} \end{cases}$$

- **Properties**
  - – Non-recombining, but can be fixed
  - – No heteroskedasticity
  - – Spot rates can become negative, but not if we model ln[*r*(1)]
  - – 3 parameters
    - ⇒ fit only 3 spot rates

Drift function (cont)

- **Example**
  - – $r_{0,0} = 5\%$
  - – $\theta = 10\%$
  - – $\kappa = 0.25$
  - – $\sigma = 2.5\%$
  - – $\Delta\ t = 1/m$ with $m = 2$

				With $\kappa = 0$
		q = 1/2	$r_{2,2} = 9.49\%$	$r_{2,2} = 8.54\%$
	q = 1/2	$r_{1,1} = 7.39\%$		
$r_{0,0} = 5\%$			$r_{2,1} = 5.95\%$	$r_{2,1} = 5.00\%$
		q = 1/2	$r_{2,1} = 6.39\%$	$r_{2,1} = 5.00\%$
		$r_{1,0} = 3.86\%$		
			$r_{2,0} = 2.86\%$	$r_{2,0} = 1.46\%$

Drift function (cont)

- **Example**
  - $r_{0,0} = 15\%$
  - $\theta = 10\%$
  - $\kappa = 0.25$
  - $\sigma = 2.5\%$
  - $\Delta t = 1/m$ with $m = 2$

			With $\kappa = 0$
		q = 1/2 $r_{2,2} = 17.14\%$	$r_{2,2} = 18.54\%$
	q = 1/2 $r_{1,1} = 16.77\%$		
		$r_{2,1} = 13.61\%$	$r_{2,1} = 15.00\%$
$r_{0,0} = 15\%$		q = 1/2 $r_{2,1} = 14.05\%$	$r_{2,1} = 15.00\%$
	$r_{1,0} = 12.61\%$		
		$r_{2,0} = 10.51\%$	$r_{2,0} = 11.46\%$

Volatility function

- **Case 1: <u>Square-root volatility</u>**

$$r_{t+\Delta t} - r_t = \lambda \times \Delta t + \underbrace{\sigma \times \sqrt{r_t}}_{\text{vol fct}} \times \sqrt{t} \times \epsilon_t$$

with

$$\epsilon_t \sim \mathrm{N}[0,1]$$

- **Implied distribution of 1-period spot rates**

$$r_{t+\Delta t} \sim \mathrm{N}\Big[\underbrace{r_t + \lambda \times \Delta t}_{\text{mean}}, \underbrace{\sigma^2 \times r_t \times \Delta t}_{\text{var}}\Big]$$

- **Cox, Ingersoll, and Ross (1985, *Econometrics*) Þ "CIR model"**

Volatility function (cont)

- **Binomial tree representation**

$r_{0,0}$

$q = 1/2$: $r_{0,0} + \lambda \times \Delta t + \sigma \times \sqrt{r_{0,0}} \times \sqrt{\Delta t}$

- $q = 1/2$: $r_{1,1} + \lambda \times \Delta t + \sigma \times \sqrt{r_{1,1}} \times \sqrt{\Delta t}$
- $r_{1,1} + \lambda \times \Delta t - \sigma \times \sqrt{r_{1,1}} \times \sqrt{\Delta t}$

$r_{0,0} + \lambda \times \Delta t - \sigma \times \sqrt{r_{0,0}} \times \sqrt{\Delta t}$

- $q = 1/2$: $r_{1,0} + \lambda \times \Delta t + \sigma \times \sqrt{r_{1,0}} \times \sqrt{\Delta t}$
- $r_{1,0} + \lambda \times \Delta t - \sigma \times \sqrt{r_{1,0}} \times \sqrt{\Delta t}$

- **Properties**
  - – Non-recombining, but can be fixed
  - – No mean-reversion, but can be fixed by using different drift function
  - – Spot rates can become negative, but not as $\Delta t \to 0$
  - – 1 volatility parameter (and arbitrarily many drift parameters)
    $\Rightarrow$ fit term structures of spot rates but only 1 spot rate volatility

Volatility function (cont)

- **Example**
  - $r_{0,0} = 5\%$
  - $\lambda = 1\%$
  - $\sigma = 11.18\% \Rightarrow \sigma \times \sqrt{r_{0,0}} = 2.5\%$
  - $\Delta t = 1/m$ with $m = 2$

			With constant volatility
		q = 1/2 $r_{2,2} = 9.90\%$	$r_{2,2} = 9.54\%$
	q = 1/2 $r_{1,1} = 7.27\%$		
		$r_{2,1} = 5.64\%$	$r_{2,1} = 0.60\%$
$r_{0,0} = 5\%$		q = 1/2 $r_{2,1} = 5.56\%$	$r_{2,1} = 0.60\%$
	$r_{1,0} = 3.73\%$		
		$r_{2,0} = 2.70\%$	$r_{2,0} = 2.47\%$

Volatility function (cont)

- **Case 2: Time-Dependent volatility**

$$r_{t+\Delta t} - r_t = \lambda \times \Delta t + \underbrace{\sigma(t)}_{\text{vol fct}} \times \sqrt{t} \times \epsilon_t$$

with

$$\epsilon_t \sim \mathrm{N}[0,1]$$

- **Implied distribution of 1-period spot rates**

$$r_{t+\Delta t} \sim \mathrm{N}\Big[\underbrace{r_t + \lambda \times \Delta t}_{\text{mean}}, \underbrace{\sigma(t)^2 \times \Delta t}_{\text{var}}\Big]$$

- **Hull and White (1993, *J. of Financial and Quantitative Analysis*)**
  Þ **"Hull-White model"**

Volatility function (cont)

- **Binomial tree representation**

$q = 1/2$ $\quad r_{1,1} + \lambda \times \Delta\, \mathrm{t} + \sigma(2) \times \mathrm{v}\ \Delta\, t$

$q = 1/2$ $\quad r_{0,0} + \lambda \times \Delta\, t + \sigma(1) \times \mathrm{v}\ \Delta\, t$

$r_{1,1} + \lambda \times \Delta\, \mathrm{t} - \sigma(2) \times \mathrm{v}\ \Delta\, t$

$r_{0,0}$

$q = 1/2$ $\quad r_{1,0} + \lambda \times \Delta\, \mathrm{t} + \sigma(2) \times \mathrm{v}\ \Delta\, t$

$r_{0,0} + \lambda \times \Delta\, t - \sigma(1) \times \mathrm{v}\ \Delta\, t$

$r_{1,0} + \lambda \times \Delta\, \mathrm{t} - \sigma(2) \times \mathrm{v}\ \Delta\, t$

- **Properties**
  - – Non-recombining, but can be fixed
  - – No mean-reversion, but can be fixed by using different drift function
  - – Spot rates can become negative, but not if we model ln[*r*(1)]
    $\Rightarrow$ "Black-Karasinski model" and "Black-Derman-Toy model"
  - – Arbitrarily many volatility and drift parameters
    $\Rightarrow$ fit term structures of spot rates <u>and</u> volatilities

Calibration

- **To calibrate parameters of a factor model to bonds prices**
  - – Step 1: Pick arbitrary parameter values
  - – Step 2: Calculate implied 1-period spot rate tree
  - – Step 3: Calculate model prices for liquid securities
  - – Step 4: Calculate model pricing errors given market prices
  - – Step 5: Use solver to find parameter values which minimize the sum of squared pricing errors

Constant drift example

- **Step 1: Pick arbitrary parameter values**

Parameters

?	0.00%
s	0.10%

Observed 1-period spot rate

r(1)	6.21%

Constant drift example (cont)

- **Step 2: Calculate implied 1-period spot rate tree**

										6.92%
									6.85%	
								6.78%		6.78%
							6.70%		6.70%	
						6.63%		6.63%		6.63%
					6.56%		6.56%		6.56%	
				6.49%		6.49%		6.49%		6.49%
			6.42%		6.42%		6.42%		6.42%	
		6.35%		6.35%		6.35%		6.35%		6.35%
	6.28%		6.28%		6.28%		6.28%		6.28%	
6.21%		6.21%		6.21%		6.21%		6.21%		6.21%
	6.14%		6.14%		6.14%		6.14%		6.14%	
		6.07%		6.07%		6.07%		6.07%		6.07%
			6.00%		6.00%		6.00%		6.00%	
				5.93%		5.93%		5.93%		5.93%
					5.86%		5.86%		5.86%	
						5.79%		5.79%		5.79%
							5.72%		5.72%	
								5.64%		5.64%
									5.57%	
										5.50%

Constant drift example (cont)

- **Step 3: Calculate model prices for liquid securities**
  - E.g., for a 2.5-yr STRIPS

					$ 100.00
				$ 96.86	
			$ 93.87		$ 100.00
		$ 91.05		$ 96.92	
	$ 88.37		$ 94.00		$ 100.00
$ 85.82		$ 91.23		$ 96.99	
	$ 88.61		$ 94.13		$ 100.00
		$ 91.42		$ 97.06	
			$ 94.26		$ 100.00
				$ 97.12	
					$ 100.00

Constant drift example (cont)

- **Step 3: Calculate model prices for liquid securities (cont)**

Parameters		**Periods**	**Model implied spot rate**
?	0.00%	0.5	6.21%
s	0.10%	1.0	6.21%
		1.5	6.21%
		2.0	6.21%
Observed 1-period spot rate		2.5	6.21%
		3.0	6.21%
r(1)	6.21%	3.5	6.21%
		4.0	6.21%
		4.5	6.21%
		5.0	6.21%

Constant drift example (cont)

- **Step 4: Use solver**

Parameters

?	0.00%
s	0.10%

Observed 1-period spot rate

r(1)	6.21%

Periods	Model implied spot rate	Observed spot rate	Pricing error
0.5	6.21%	6.21%	0.00%
1.0	6.21%	6.41%	0.20%
1.5	6.21%	6.48%	0.27%
2.0	6.21%	6.56%	0.35%
2.5	6.21%	6.62%	0.41%
3.0	6.21%	6.71%	0.50%
3.5	6.21%	6.80%	0.59%
4.0	6.21%	6.87%	0.66%
4.5	6.21%	6.92%	0.71%
5.0	6.21%	6.97%	0.76%

Sum of squared errors 0.0002521

Minimize sum of squared errors by choice of parameters

Constant drift example (cont)

- **Solution**

Parameters		Periods	Model implied spot rate	Observed spot rate	Pricing error
?	0.57%	0.5	6.21%	6.21%	0.00%
s	3.63%	1.0	6.34%	6.41%	0.07%
		1.5	6.45%	6.48%	0.03%
		2.0	6.56%	6.56%	0.00%
Observed 1-period spot rate		2.5	6.65%	6.62%	-0.03%
		3.0	6.73%	6.71%	-0.02%
r(1)	6.21%	3.5	6.81%	6.80%	-0.01%
		4.0	6.87%	6.87%	0.00%
		4.5	6.92%	6.92%	0.00%
		5.0	6.96%	6.97%	0.01%

Sum of squared errors 7.882E-07