Reasoning under Uncertainty
Part I

Artificial Intelligence, 2015
TIN172/DIT410

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based on slides by

Poole, Mackworth and slides from 2015

Chalmers University of Technology

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Learning Objectives

At the end of the class you should be able to:

- justify the use and **semantics** of probability
- know how to compute **marginals** and apply **Bayes' theorem**
- build a **belief network** for a domain
- perform **inference** in a **belief network**
- explain the predictions of a **causal model**

Using Uncertain Knowledge

- Complete knowledge about the world not possible.
- Decisions are still needed!
- **Example:** wearing a seat belt.

Why Probability?

- Prediction approaches:
  - definitive: you will be run over tomorrow
  - point probabilities:
    $P(\text{you will be run over tomorrow}) = 0.002$
  - probability ranges:
    $P(\text{you will be run over tomorrow}) \in [0.001, 0.34]$
- Acting is gambling! Dutch books.

- Probabilities can be learned from data.
- Bayes rule specifies how to combine data and prior knowledge.

Probability

- Probability can model one's belief in some proposition — **subjective probability.**
- An agent's belief depends on its prior assumptions and on observations.

Numerical Measures of Belief

- a - a proposition
- $P(a)$ - **probability of** a, or the belief in a, is a number between 0 and 1
  - $P(a) = 0$ - a is believed to be definitely false.
  - $P(a) = 1$ - a is believed to be definitely true.
- Using 0 and 1 is purely a convention.

Random Variables

- A **random variable** is a variable that can take one of a number of different values.
- The **range** of a variable X, written $range(X)$, is the set of values X can take.
- Assignment $X = x$ means variable X has value x.
- Each assignment to a random variable is associated to a probability, $P(X = x)$.
- A **proposition** is a Boolean formula made from assignments of values to variables.

Axioms of Probability

Three axioms define what follows from a set of probabilities:

Axiom 1 $0 \leq P(a)$ for any proposition a.

Axiom 2 $P(true) = 1$

Axiom 3 $P(a \vee b) = P(a) + P(b)$ if a and b cannot both be true.

Probability Distributions

- A probability distribution $P(X)$ on a random variable X is a function $range(X) \to [0, 1]$.
- Joint distribution of several variables: $P(X, Y, Z)$.
- A (discrete) distribution always has to sum to one:

$$\sum_{x \in range(X)} P(X = x) = 1$$

.

- For continuous random variables, the distribution has a probability density function (PDF).

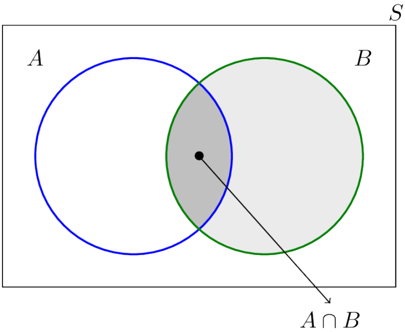
Probabilistic Conditioning

- How to revise beliefs based on new information.
- **Prior probability**: the belief before observing evidence
- Let e be the observed **evidence**, the **conditional probability** $P(h|e)$ of h given e is the **posterior probability** of h.

Conditional Probability

- The conditional probability of h given evidence e is

$$P(h|e) = \frac{P(h \wedge e)}{P(e)}$$



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability: Example

We toss a die.
Someone tells you that the outcome is an even number.
What is the probability that the outcome is 6?

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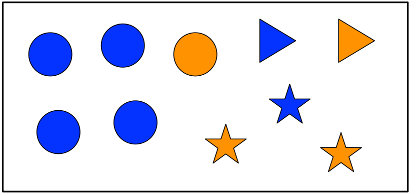
1/3

Conditioning

Possible values before evidence:



Conditioning

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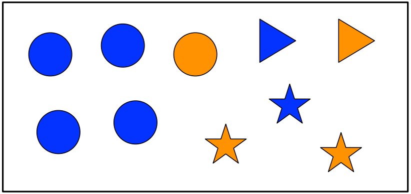
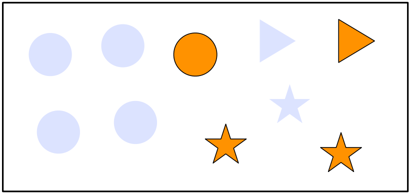


Observe $Color = orange$:



Marginals

If you have a joint probability distribution $P(X,Y)$ over some random variables X,Y, the marginal distribution $P(X)$ can be computed by summing over all values of Y:

- $P(X) = \sum_Y P(X,Y)$

Exercise

Flu	$Sneeze$	$Snore$	μ
true	true	true	8/125
true	true	false	12/125
true	false	true	2/125
true	false	false	3/125
false	true	true	12/125
false	true	false	18/125
false	false	true	28/125
false	false	false	42/125

What is:

(a) $P(flu \wedge sneeze)$

(b) $P(flu \wedge \neg sneeze)$

(c) $P(flu)$

(d) $P(sneeze \mid flu)$

(e) $P(\neg flu \wedge sneeze)$

(f) $P(flu \mid sneeze)$

(g) $P(sneeze \mid flu \wedge snore)$

(h) $P(flu \mid sneeze \wedge snore)$

Chain Rule

$$P(f_1 \wedge f_2 \wedge \ldots \wedge f_n)$$
$$=$$

Chain Rule

$$
\begin{aligned}
& P(f_1 \wedge f_2 \wedge \ldots \wedge f_n) \\
& \quad = \quad P(f_n | f_1 \wedge \cdots \wedge f_{n-1}) \times \\
& \quad P(f_1 \wedge \cdots \wedge f_{n-1}) \\
& \quad =
\end{aligned}
$$

Chain Rule

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& P(f_{n-1} | f_1 \wedge \cdots \wedge f_{n-2}) \times \\
& P(f_1 \wedge \cdots \wedge f_{n-2}) \\
= \quad & P(f_n | f_1 \wedge \cdots \wedge f_{n-1}) \times \\
& P(f_{n-1} | f_1 \wedge \cdots \wedge f_{n-2}) \\
& \times \cdots \times P(f_3 | f_1 \wedge f_2) \times P(f_2 | f_1) \times P(f_1) \\
= \quad & \prod_{i=1}^{n} P(f_i | f_1 \wedge \cdots \wedge f_{i-1})
\end{aligned}
$$

Bayes' theorem

The chain rule and commutativity of conjunction ($h \wedge e$ is equivalent to $e \wedge h$) gives us:

$$P(h \wedge e) \quad =$$

Bayes' theorem

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$$P(h \wedge e) \quad = \quad P(h|e) \times P(e)$$

Bayes' theorem

The chain rule and commutativity of conjunction ($h \wedge e$ is equivalent to $e \wedge h$) gives us:

$$\begin{aligned} P(h \wedge e) &= P(h|e) \times P(e) \\ &= P(e|h) \times P(h). \end{aligned}$$

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If $P(e) \neq 0$, divide the right hand sides by $P(e)$:

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Bayes' theorem

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If $P(e) \neq 0$, divide the right hand sides by $P(e)$:

$$P(h|e) = \frac{P(e|h) \times P(h)}{P(e)}.$$

This is **Bayes' theorem.**

Why is Bayes' theorem interesting?

- Often you have causal knowledge: $P(symptom \mid disease)$
- and want to do evidential reasoning: $P(disease \mid symptom)$

Exercise

A cab was involved in a hit-and-run accident at night. Two cab companies, the Green and the Blue, operate in the city. You are given:

- 85% of the cabs in the city are Green and 15% are Blue.
- A witness identified the cab as Blue.
- The witness reliability is 80%.

What is the probability that the cab involved in the accident was Blue?

D. Kahneman, Thinking Fast and Slow, 2011, p. 166.

Exercise: Solution

$P(\text{cab is blue}|\text{witness says cab is blue}) =$

$\frac{P(\text{witness says blue}|\text{cab is blue}) \times P(\text{cab is blue})}{P(\text{witness says blue})} =$

$\frac{0.8 \times 0.15}{0.29} \approx$

0.41

(The normalizing constant ($P(\text{witness says blue})$) can be computed by marginalizing (summing over hypotheses): $0.8 * 0.15 + 0.2 * 0.85 = 0.29$.)

Conditional independence

Random variable X is **independent** of random variable Y **given** random variable Z if,

$$P(X|Y, Z) = P(X|Z)$$

Conditional independence

Random variable X is **independent** of random variable Y **given** random variable Z if,

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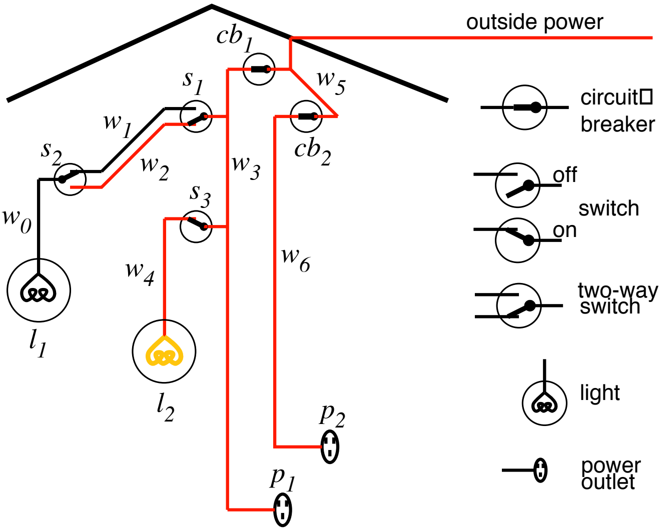
i.e. for all $x_i \in dom(X)$, $y_j \in dom(Y)$, $y_k \in dom(Y)$ and $z_m \in dom(Z)$,

$$\begin{aligned} & P(X = x_i | Y = y_j, Z = z_m) \\ = & \quad P(X = x_i | Y = y_k, Z = z_m) \\ = & \quad P(X = x_i | Z = z_m). \end{aligned}$$

That is, knowledge of Y's value doesn't affect the belief in the value of X, given a value of Z.

Example domain (diagnostic assistant)



Examples of conditional independence?

- The identity of the queen of Canada is dependent or independent of whether light $l1$ is lit given whether there is outside power?
- Whether there is someone in a room is independent of whether a light $l2$ is lit given what?
- Whether light $l1$ is lit is independent of the position of light switch $s2$ given what?
- Every other variable may be independent of whether light $l1$ is lit given

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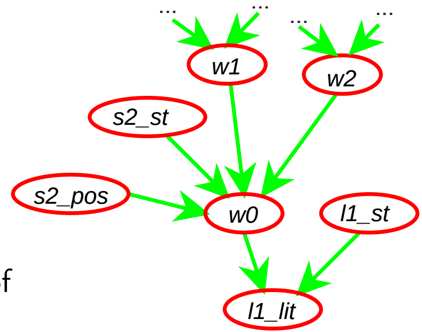
- Whether there is someone in a room is independent of whether a light $l2$ is lit given what?
- Whether light $l1$ is lit is independent of the position of light switch $s2$ given what?
- Every other variable may be independent of whether light $l1$ is lit given whether there is power in wire w_0 and the status of light $l1$ (if it's ok, or if not, how it's broken).

Idea of belief networks

- $l1$ is lit ($L1\_lit$) depends only on the status of the light ($L1\_st$) and whether there is power in wire $w0$.
- In a belief network, $W0$ and $L1\_st$ are **parents** of $L1\_lit$.
- $W0$ depends only on



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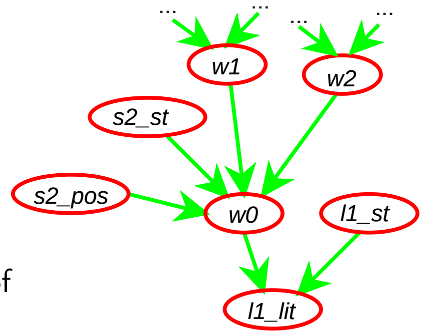

- $W0$ depends only on whether there is power in $w1$, whether there is power in $w2$, the position of switch $s2$ ($S2\_pos$), and the status of switch $s2$ ($S2\_st$).

Belief networks

- Totally order the variables of interest: $X_1, \ldots, X_n$
- Theorem of probability theory (chain rule):
  $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | X_1, \ldots, X_{i-1})$
- The **parents** $parents(X_i)$ of X_i are those predecessors of X_i that render X_i independent of the other predecessors. That is,

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- The **parents** $parents(X_i)$ of X_i are those predecessors of X_i that render X_i independent of the other predecessors. That is, $parents(X_i) \subseteq X_1, \ldots, X_{i-1}$ and $P(X_i | parents(X_i)) = P(X_i | X_1, \ldots, X_{i-1})$
- So $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | parents(X_i))$
- A **belief network** is a graph: the nodes are random variables; there is an arc from the parents of each node into that node.

Example: fire alarm belief network

Variables:

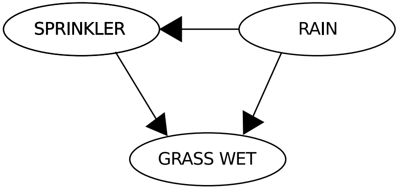
- Fire: there is a fire in the building
- Tampering: someone has been tampering with the fire alarm
- Smoke: what appears to be smoke is coming from an upstairs window
- Alarm: the fire alarm goes off
- Leaving: people are leaving the building *en masse*.
- Report: a colleague says that people are leaving the building *en masse*. (A noisy sensor for leaving.)

Components of a belief network

A belief network consists of:

- a directed acyclic graph with nodes labeled with random variables
- a domain for each random variable
- a set of conditional probability tables for each variable given its parents (including prior probabilities for nodes with no parents).

Example belief network



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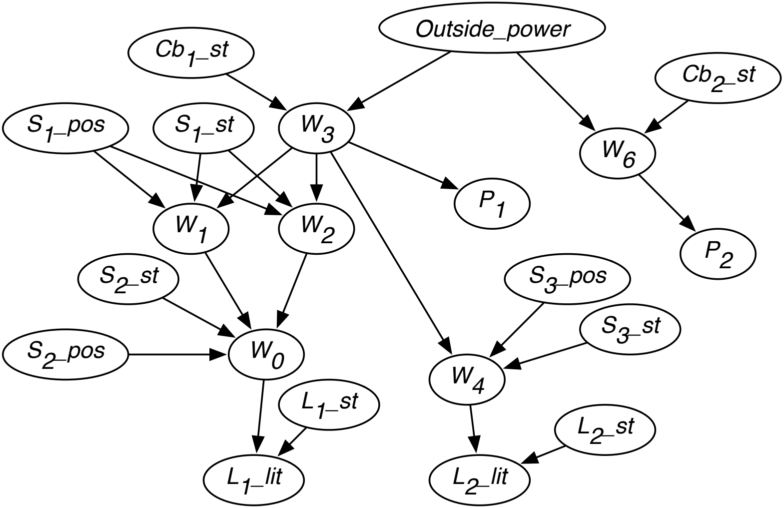
Example belief network



Example belief network (continued)

The belief network also specifies:

- The domain of the variables:
  $W_0, \ldots, W_6$ have domain $\{live, dead\}$
  S_1_pos, S_2_pos, and S_3_pos have domain $\{up, down\}$
  S_1_st has
  $\{ok, upside\_down, short, intermittent, broken\}$.
- Conditional probabilities, such as:
  $P(W_1 = live | s_1\_pos = up \wedge S_1\_st = ok \wedge W_3 = live)$

Belief network summary

- A belief network is a directed acyclic graph (DAG).
- Its nodes are random variables.
- The **parents** of X are those that X directly depends on.
- Acyclic by construction.
- A representation of **dependence** and **independence**:
  - X is independent of its non-descendants given its parents.

Constructing belief networks

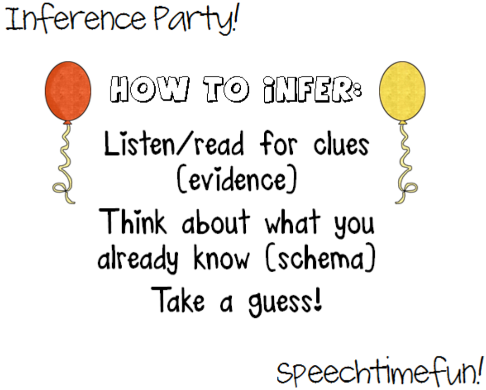
- What are the relevant variables?
  - Observed?
  - Query variables?
  - Variables that make the model simpler?
- What values should these variables take?
- What is the relationship between them?
- How does the value of each variable depend on its parents?

Using belief networks

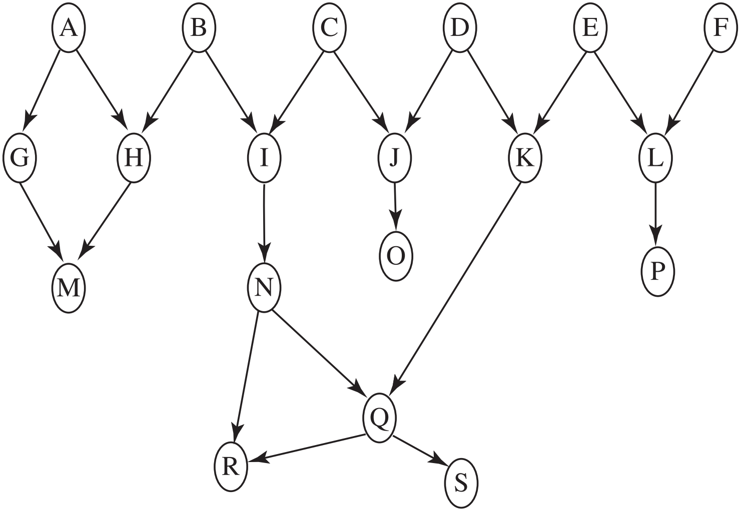
An example of how the power network can be used:

- Given values for:
  - switches,
  - outside power,
  - whether the lights are lit,
- you can determine the posterior probability that each switch or circuit breaker is ok or not.

This is called **inference**.



Understanding independence: example



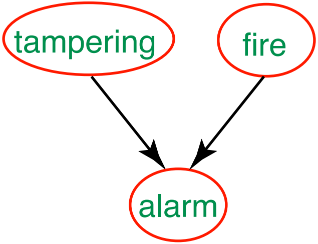
Understanding independence: questions

- On which given probabilities does $P(N)$ depend?
- If you were to observe a value for B, which variables' probabilities will change?
- If you were to observe a value for N, which variables' probabilities will change?
- Suppose you had observed a value for M; if you were to then observe a value for N, which variables' probabilities will change?
- Suppose you had observed B and Q; which variables' probabilities will change when you observe N?

What variables are affected by observing?

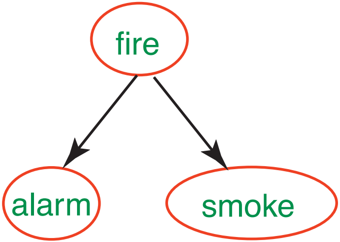
- If you observe variable $\overline{Y}$, the variables whose posterior probability is different from their prior are:
  - The ancestors of $\overline{Y}$ and
  - their descendants.
- Intuitively (if you have a causal belief network):
  - You do **abduction** to possible causes and
  - **prediction** from the causes.

Common descendants



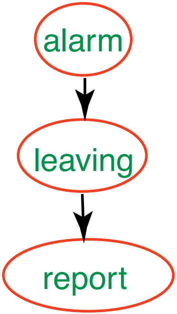
- $tampering$ and $fire$ are independent
- $tampering$ and $fire$ are dependent given $alarm$
- Intuitively, $tampering$ can **explain away** $fire$

Common ancestors



- $alarm$ and $smoke$ are dependent
- $alarm$ and $smoke$ are independent given $fire$
- Intuitively, $fire$ can **explain** $alarm$ and $smoke$; learning one can affect the other by changing your belief in $fire$.

Chain



- $alarm$ and $report$ are dependent
- $alarm$ and $report$ are independent given $leaving$
- Intuitively, the only way that the $alarm$ affects $report$ is by affecting $leaving$.

Belief network inference

- Variable Elimination: exploit the structure of the network to eliminate (sum out) the non-observed, non-query variables one at a time.
- Search-based approaches: enumerate some of the possible assignments, and estimate posterior probabilities.
- Stochastic simulation: generate random assignments according to the probability distributions.
- Variational methods: find the closest tractable distribution to the (posterior) distribution.

Factors

Function from a set of random variables to a number.
$f(X_1, \ldots, X_j)$.
Some or all of the variables of a factor can be assigned:

- $f(X_1 = v_1, X_2, \ldots, X_j)$, is a factor on $X_2, \ldots, X_j$.
- $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	
f	t	
f	f	

Example factors

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t	f	t	0.2
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f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

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Example factors

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f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	
f	t	
f	f	

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	
f	

$r(X{=}t, Y{=}f, Z{=}f) =$

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	
f	t	
f	f	

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

$$r(X{=}t, Y{=}f, Z{=}f) = 0.8$$

Multiplying factors

The **product** of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z}) \quad = \quad f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 \times f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	
t	f	t	
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 \times f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

Summing out variables

We can **sum out** a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$\begin{aligned}&(\sum_{X_1} f)(X_2, \ldots, X_j)\\ &= \quad f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)\end{aligned}$$

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	
f	t	
f	f	

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Exercise

Given factors:

s:

A	val
t	0.75
f	0.25

t:

A	B	val
t	t	0.6
t	f	0.4
f	t	0.2
f	f	0.8

o:

A	val
t	0.3
f	0.1

What is?

(a) $s \times t$

(b) $\sum_A s \times t$

(c) $\sum_B s \times t$

(d) $s \times t \times o$

(e) $\sum_A s \times t \times o$

(f) $\sum_b s \times t \times o$

Evidence

If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j$:

$$P(Z|Y_1 = v_1, \ldots, Y_j = v_j)$$

$$=$$

Evidence

If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \wedge \ \ldots \wedge Y_j = v_j$:

$$
\begin{aligned}
& P(Z|Y_1 = v_1, \ldots, Y_j = v_j) \\
= \quad & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{P(Y_1 = v_1, \ldots, Y_j = v_j)} \\
= \quad &
\end{aligned}
$$

Evidence

If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \wedge \ \ldots \wedge Y_j = v_j$:

$$\begin{aligned} & P(Z|Y_1 = v_1, \ldots, Y_j = v_j) \\ = & \quad \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{P(Y_1 = v_1, \ldots, Y_j = v_j)} \\ = & \quad \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{\sum_Z P(Z, Y_1 = v_1, \ldots, Y_j = v_j).} \end{aligned}$$

So the computation reduces to the probability of $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$.
We normalize at the end.