CHAPTERS 3–6: REPETITION

DIT410/TIN174, Artificial Intelligence

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DIRECTED GRAPHS

A *graph* consists of a set N of *nodes* and a set A of ordered pairs of nodes, called *arcs*.

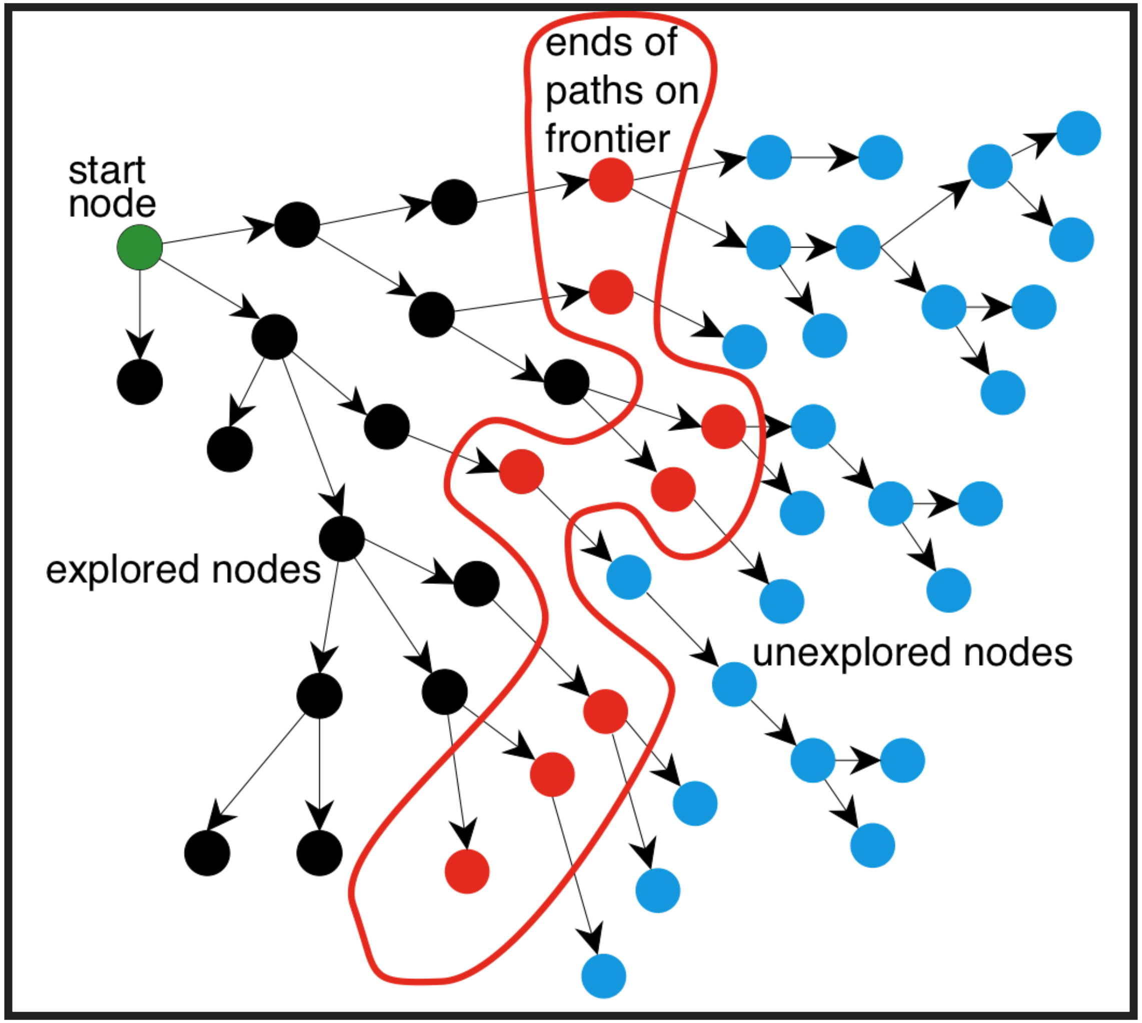
- Node n_2 is a *neighbor* of n_1 if there is an arc from n_1 to n_2. That is, if $(n_1, n_2) \in A$.
- A *path* is a sequence of nodes $(n_0, n_1, \ldots, n_k)$ such that $(n_{i-1}, n_i) \in A$.
- The *length* of path $(n_0, n_1, \ldots, n_k)$ is k.
- A *solution* is a path from a start node to a goal node, given a set of *start nodes* and *goal nodes*.
- (Russel & Norvig sometimes call the graph nodes *states*).

HOW DO WE SEARCH IN A GRAPH?

A generic search algorithm:

- Given a graph, start nodes, and a goal description, incrementally explore paths from the start nodes.
- Maintain a *frontier* of nodes that are to be explored.
- As search proceeds, the frontier expands into the unexplored nodes until a goal node is encountered.
- The way in which the frontier is expanded defines the search strategy.

ILLUSTRATION OF SEARCHING IN A GRAPH



THE GENERIC TREE SEARCH ALGORITHM

Tree search: Don't check if nodes are visited multiple times

function Search(*graph*, *initialState*, *goalState*):
    initialise *frontier* using the *initialState*
    while *frontier* is not empty:
        select and **remove** *node* from *frontier*
        if *node*.state is a *goalState* **then return** *node*
        for each *child* **in** ExpandChildNodes(*node*, *graph*):
            add *child* to *frontier*
    return failure

DEPTH-FIRST AND BREADTH-FIRST SEARCH

THESE ARE THE TWO BASIC SEARCH ALGORITHMS

Depth-first search (DFS)

- implement the frontier as a Stack
- space complexity: $O(bm)$
- incomplete: might fall into an infinite loop, doesn't return optimal solution

Breadth-first search (BFS)

- implement the frontier as a Queue
- space complexity: $O(b^m)$
- complete: always finds a solution, if there is one
- (when edge costs are constant, BFS is also optimal)

ITERATIVE DEEPENING

BFS is guaranteed to halt but uses exponential space.
DFS uses linear space, but is not guaranteed to halt.

Idea: take the best from BFS and DFS — recompute elements of the frontier rather than saving them.

- Look for paths of depth 0, then 1, then 2, then 3, etc.
- Depth-bounded DFS can do this in linear space.

Iterative deepening search calls depth-bounded DFS with increasing bounds:

- If a path cannot be found at *depth-bound*, look for a path at *depth-bound* + 1.
- Increase *depth-bound* when the search fails unnaturally
  (i.e., if *depth-bound* was reached).

ITERATIVE DEEPENING COMPLEXITY

Complexity with solution at depth k and branching factor b:

level	breadth-first	iterative deepening	# nodes
1	1	k	b
2	1	$k-1$	b^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	1	2	b^{k-1}
k	1	1	b^k
total	$\geq b^k$	$\leq b^k\left(\frac{b}{b-1}\right)^2$	

Numerical comparison for $k = 5$ and $b = 10$:

BFS = 10 + 100 + 1,000 + 10,000 + 100,000 = 111,110
IDS = 50 + 400 + 3,000 + 20,000 + 100,000 = 123,450

Note: IDS recalculates shallow nodes several times, but this doesn't have a big effect compared to BFS!

BIDIRECTIONAL SEARCH

Idea: search backward from the goal and forward from the start simultaneously.

- This can result in an exponential saving, because $2b^{k/2} \ll b^k$.
- The main problem is making sure the frontiers meet.

One possible implementation:

- Use BFS to gradually search backwards from the goal, building a set of locations that will lead to the goal.
  - this can be done using *dynamic programming*
- Interleave this with forward heuristic search (e.g., A*) that tries to find a path to these interesting locations.

COST-BASED SEARCH

IMPLEMENT THE FRONTIER AS A PRIORITY QUEUE, ORDERED BY $\mathbf{f(n)}$

Uniform-cost search (this is not a heuristic algorithm)

- expand the node with the lowest path cost
- $f(n) = g(n)$
- complete and optimal

Greedy best-first search

- expand the node which is closest to the goal (according to some heuristics)
- $f(n) = h(n)$
- incomplete: might fall into an infinite loop, doesn't return optimal solution

A* search

- expand the node which has the lowest estimated cost from start to goal
- $f(n) = g(n) + h(n)$ = estimated cost of the cheapest solution through n
- complete and optimal (if $h(n)$ is admissible/consistent)

A* TREE SEARCH IS OPTIMAL!

A* always finds an optimal solution first, provided that:

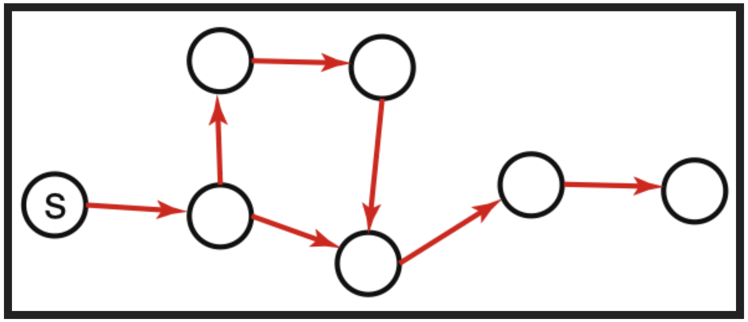
- the branching factor is finite,
- arc costs are *bounded above zero* (i.e., there is some $\epsilon > 0$ such that all of the arc costs are greater than ϵ), and
- $h(n)$ is ***admissible***
  - i.e., $h(n)$ is *nonnegative* and an *underestimate* of the cost of the shortest path from n to a goal node.

TURNING TREE SEARCH INTO GRAPH SEARCH

Tree search: Don’t check if nodes are visited multiple times
Graph search: Keep track of visited nodes

function Search(*graph*, *initialState*, *goalState*):
    initialise *frontier* using the *initialState*
    initialise *exploredSet* to the empty set
    while *frontier* is not empty:
        select and **remove** *node* from *frontier*
        if *node*.state is a *goalState* **then return** *node*
        add *node* to *exploredSet*
        for each *child* **in** ExpandChildNodes(*node*, *graph*):
            add *child* to *frontier* … **if** *child* is not in *frontier* or *exploredSet*
    return failure

GRAPH-SEARCH = MULTIPLE-PATH PRUNING



Graph search keeps track of visited nodes, so we don't visit the same node twice.

- Suppose that the first time we visit a node is not via the most optimal path

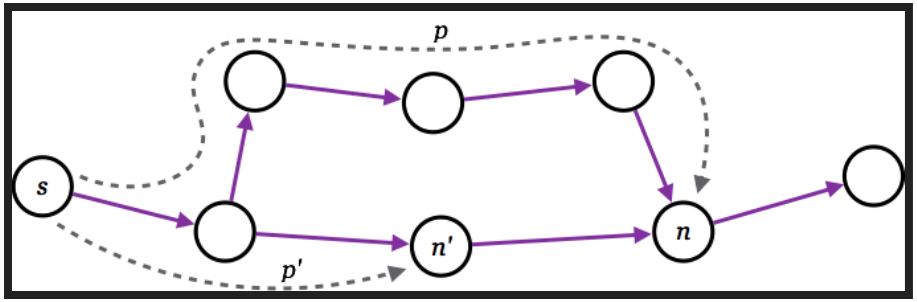
  $\Rightarrow$ then graph search will return a suboptimal path

- Under which circumstances can we guarantee that A* graph search is optimal?

WHEN IS A* GRAPH SEARCH OPTIMAL?

If h is *consistent*, then A* graph search is optimal:

- Consistency is defined as: $h(n') \leq cost(n', n) + h(n)$ for all arcs (n', n)
- **Lemma**: the f values along any path $[\ldots, n', n, \ldots]$ are nondecreasing:
    - **Proof**: $g(n) = g(n') + cost(n', n)$, therefore:
    - $f(n) = g(n) + h(n) = g(n') + cost(n', n) + h(n) \geq g(n') + h(n')$;
    - therefore: $f(n) \geq f(n')$, i.e., f is nondecreasing
- **Theorem**: whenever A* expands a node n, the optimal path to n has been found
    - **Proof**: Assume this is not true;
    - then there must be some n' still on the frontier, which is on the optimal path to n;
    - but $f(n') \leq f(n)$;
    - and then n' must already have been expanded $\Longrightarrow$ *contradiction*!



STATE-SPACE CONTOURS

The f values in A* are nondecreasing, therefore:

first	A* expands all nodes with $f(n) < C$
then	A* expands all nodes with $f(n) = C$
finally	A* expands all nodes with $f(n) > C$

A* will not expand any nodes with $f(n) > C*$, where $C*$ is the cost of an optimal solution.

SUMMARY OF OPTIMALITY OF A*

A* *tree search* is optimal if:

- the heuristic function $h(n)$ is **admissible**
- i.e., $h(n)$ is nonnegative and an underestimate of the actual cost
- i.e., $h(n) \leq cost(n, goal)$, for all nodes n

A* *graph search* is optimal if:

- the heuristic function $h(n)$ is **consistent** (or monotone)
- i.e., $|h(m) - h(n)| \leq cost(m, n)$, for all arcs (m, n)

SUMMARY OF TREE SEARCH STRATEGIES

Search strategy	Frontier selection	Halts if solution?	Halts if no solution?	Space usage
Depth first	Last node added	*No*	*No*	*Linear*
Breadth first	First node added	*Yes*	*No*	*Exp*
Greedy best first	Minimal $h(n)$	*No*	*No*	*Exp*
Uniform cost	Minimal $g(n)$	*Optimal*	*No*	*Exp*
A*	$f(n) = g(n) + h(n)$	*Optimal**	*No*	*Exp*

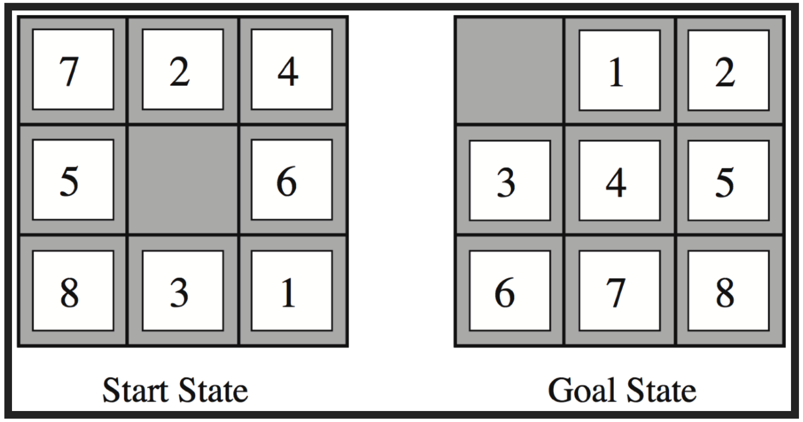
**Provided that $h(n)$ is admissible.*

Halts if: If there is a path to a goal, it can find one, even on infinite graphs.
Halts if no: Even if there is no solution, it will halt on a finite graph (with cycles).
Space: Space complexity as a function of the length of the current path.

RECAPITULATION: HEURISTICS FOR THE 8 PUZZLE

$h_1(n)$ = number of misplaced tiles

$h_2(n)$ = total Manhattan distance
(i.e., no. of squares from desired location of each tile)



$$h_1(StartState) = 8$$
$$h_2(StartState) = 3+1+2+2+2+3+3+2 = 18$$

DOMINATING HEURISTICS

If (admissible) $h_2(n) \geq h_1(n)$ for all n,
then h_2 **dominates** h_1 and is better for search.

Typical search costs (for 8-puzzle):

depth = 14	DFS ≈ 3,000,000 nodes
	A*(h_1) = 539 nodes
	A*(h_2) = 113 nodes
depth = 24	DFS ≈ 54,000,000,000 nodes
	A*(h_1) = 39,135 nodes
	A*(h_2) = 1,641 nodes

Given any admissible heuristics h_a, h_b, the **maximum** heuristics $h(n)$
is also admissible and dominates both:

$$h(n) = \max(h_a(n), h_b(n))$$

HEURISTICS FROM A RELAXED PROBLEM

Admissible heuristics can be derived from the exact solution cost of a relaxed problem:

- If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution
- If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution

Key point: the optimal solution cost of a relaxed problem is never greater than the optimal solution cost of the real problem

NON-ADMISSIBLE (NON-CONSISTENT) A* SEARCH

A* search with admissible (consistent) heuristics is optimal

But what happens if the heuristics is non-admissible?

- i.e., what if $h(n) > c(n, goal)$, for some n?
- the solution is not guaranteed to be optimal…
- …but it will find *some* solution!

Why would we want to use a non-admissible heuristics?

- sometimes it's easier to come up with a heuristics that is almost admissible
- and, often, the search terminates faster!

NON-CLASSICAL SEARCH

A problem is *nondeterministic* if there are several possible outcomes of an action

- deterministic — nondeterministic (chance)

It is *partially observable* if the agent cannot tell exactly which state it is in

- fully observable (perfect info.) — partially observable (imperfect info.)

A problem can be either nondeterministic, or partially observable, or both:

	deterministic	**chance**
perfect information	**chess, checkers, go, othello**	**backgammon monopoly**
imperfect information	**battleships, blind tictactoe**	**bridge, poker, scrabble nuclear war**

NONDETERMINISTIC SEARCH

We need a more general *result* function:

- instead of returning a single state, it returns a set of possible outcome states
- e.g., $\text{Results}(\text{Suck}, 1) = \{5, 7\}$ and $\text{Results}(\text{Suck}, 5) = \{1, 5\}$

We also need to generalise the notion of a *solution*:

- instead of a single sequence (path) from the start to the goal, we need a *strategy* (or a *contingency plan*)
- i.e., we need **if-then-else** constructs
- this is a possible solution from state 1:
    - [*Suck*, if *State*=5 then [*Right*, *Suck*] else []]

HOW TO FIND CONTINGENCY PLANS

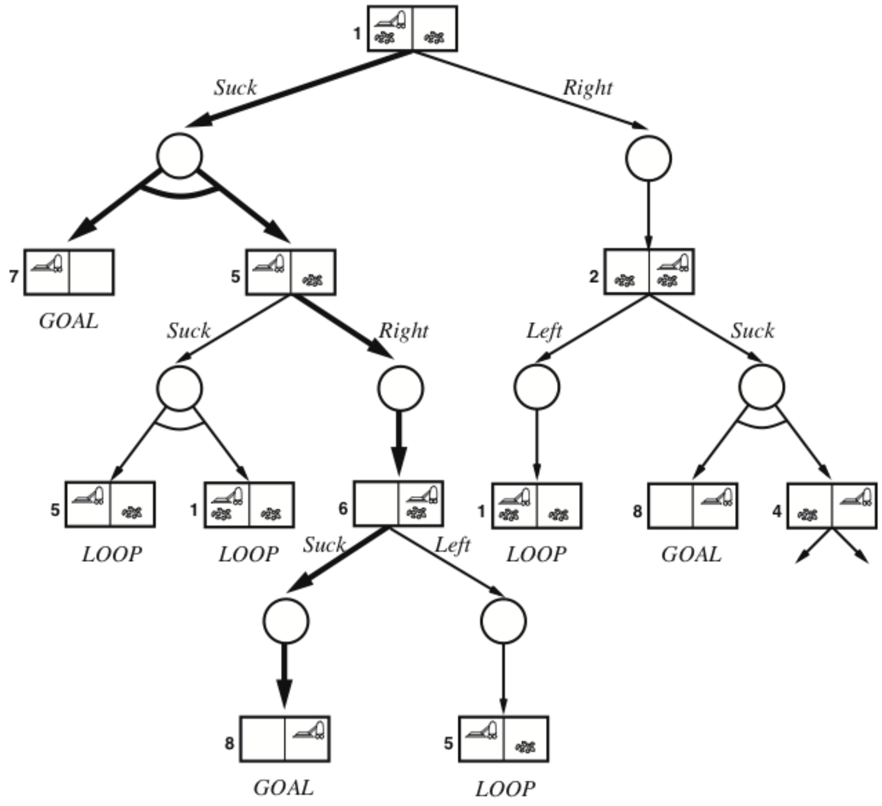
We need a new kind of nodes in the search tree:

- ***and** nodes*:
  these are used whenever an action is nondeterministic
- normal nodes are called ***or** nodes*:
  they are used when we have several possible actions in a state

A solution for an ***and-or*** search problem is a subtree that:

- has a goal node at every leaf
- specifies exactly one action at each of its ***or** node*
- includes every branch at each of its ***and** node*

A SOLUTION TO THE ERRATIC VACUUM CLEANER



The solution subtree is shown in bold, and corresponds to the plan:

[*Suck*, if *State*=5 then [*Right*, *Suck*] else []]

PARTIAL OBSERVATIONS: BELIEF STATES

Instead of searching in a graph of states, we use *belief states*

- A belief state is a *set of states*

In a sensor-less (or conformant) problem, the agent has *no information at all*

- The initial belief state is the set of all problem states
    - e.g., for the vacuum world the initial state is {1,2,3,4,5,6,7,8}

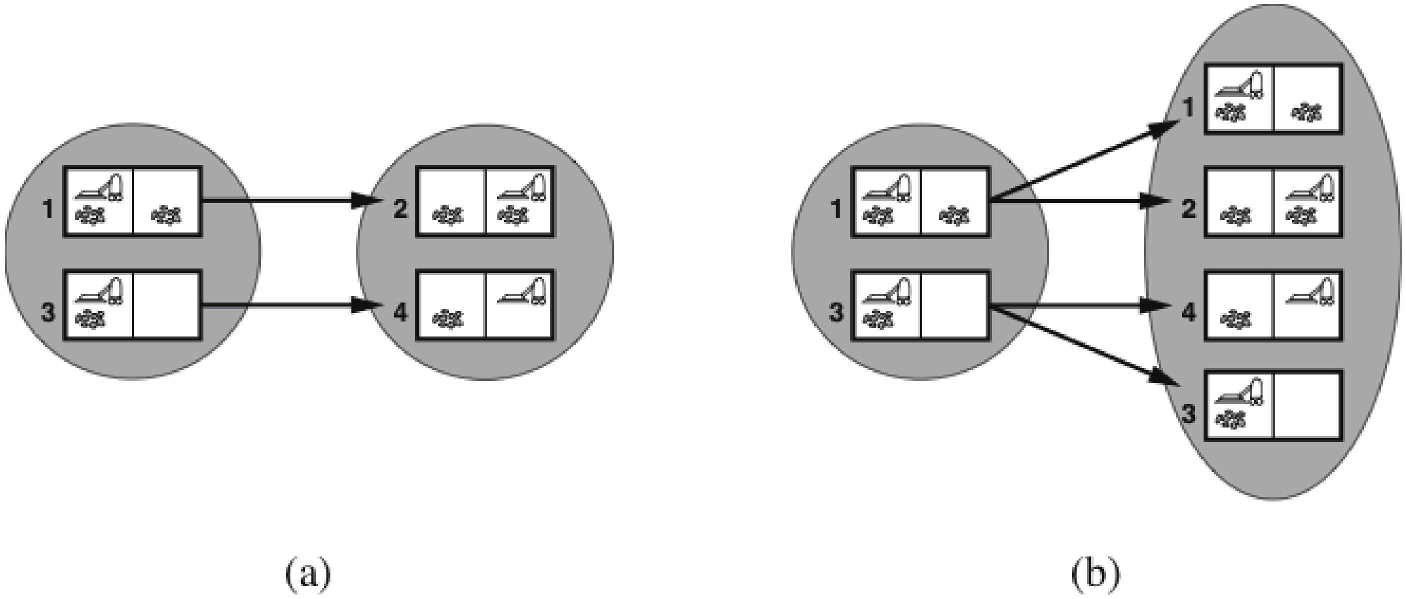
The goal test has to check that *all* members in the belief state is a goal

- e.g., for the vacuum world, the following are goal states: {7}, {8}, and {7,8}

The result of performing an action is the *union* of all possible results

- i.e., $\text{Predict}(b, a) = \{\text{Result}(s, a) \text{ for each } s \in b\}$
- if the problem is also nondeterministic:
    - $\text{Predict}(b, a) = \bigcup\{\text{Results}(s, a) \text{ for each } s \in b\}$

PREDICTING BELIEF STATES IN THE VACUUM WORLD



(a) Predicting the next belief state for the sensorless vacuum world with a deterministic action, *Right*.

(b) Prediction for the same belief state and action in the nondeterministic slippery version of the sensorless vacuum world.

ADVERSARIAL SEARCH (R&N 5.1–5.5)

TYPES OF GAMES

MINIMAX SEARCH

IMPERFECT DECISIONS

STOCHASTIC GAMES

GAMES AS SEARCH PROBLEMS

The main difference to chapters 3–4:
now we have more than one agent that have different goals.

- All possible game sequences are represented in a game tree.
- The nodes are states of the game, e.g. board positions in chess.
- Initial state (root) and terminal nodes (leaves).
- States are connected if there is a legal move/ply.
  (a ply is a move by one player, i.e., one layer in the game tree)
- Utility function (payoff function). Terminal nodes have utility values $+x$ (player 1 wins), $-x$ (player 2 wins) and 0 (draw).

PERFECT INFORMATION GAMES: ZERO-SUM GAMES

Perfect information games are solvable in a manner similar to fully observable single-agent systems, e.g., using forward search.

If two agents are competing so that a positive reward for one is a negative reward for the other agent, we have a two-agent *zero-sum game*.

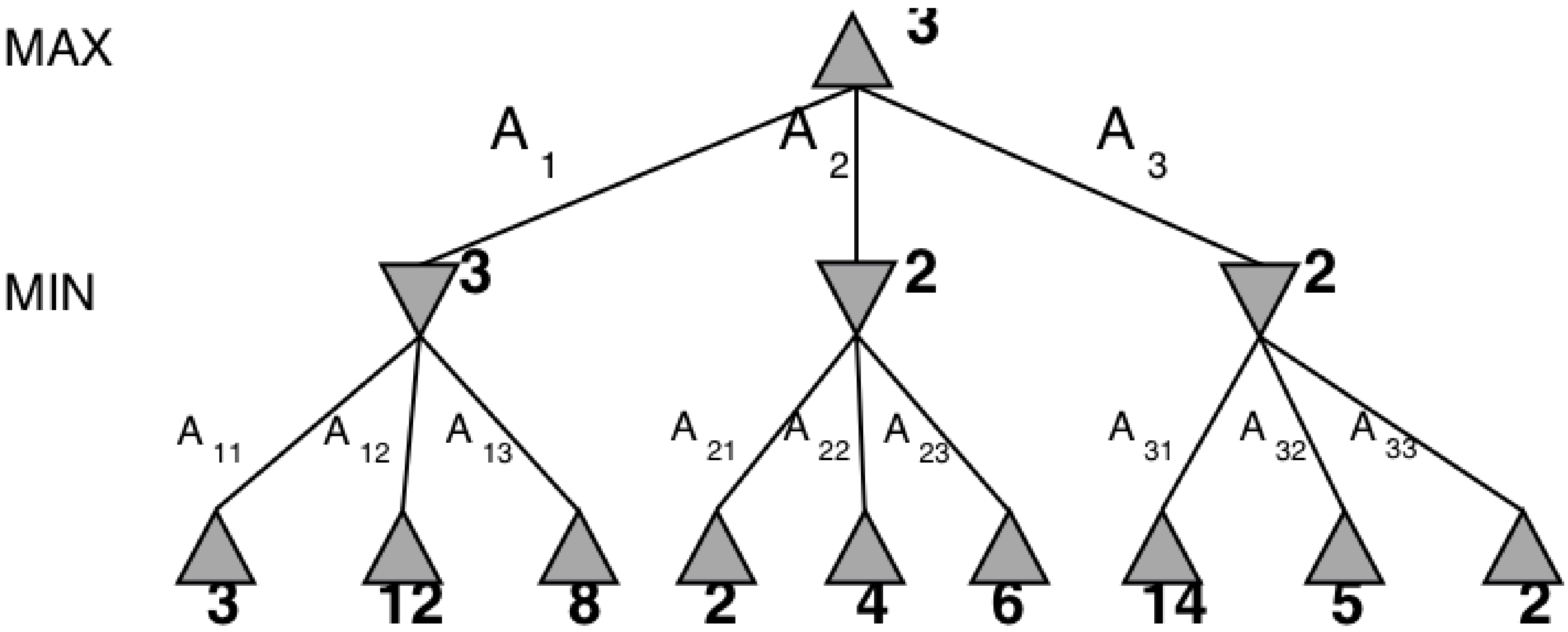
The value of a game zero-sum game can be characterized by a single number that one agent is trying to maximize and the other agent is trying to minimize.

This leads to a *minimax strategy*:

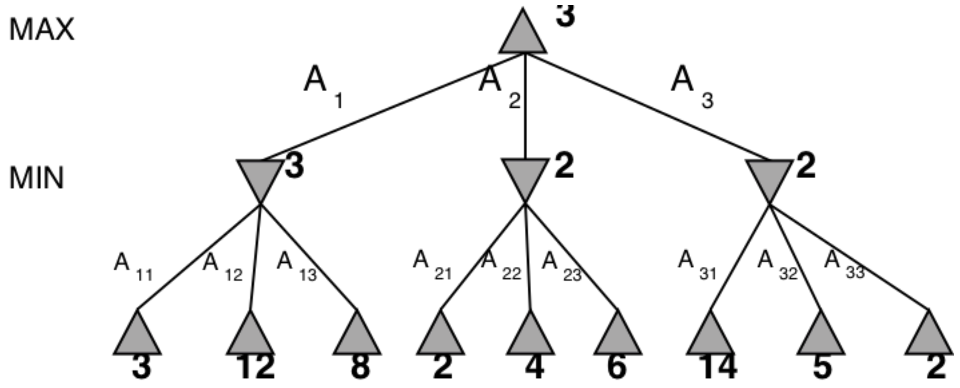
- A node is either a MAX node (if it is controlled by the maximising agent),
- or is a MIN node (if it is controlled by the minimising agent).

MINIMAX EXAMPLE

The Minimax algorithm gives perfect play for deterministic, perfect-information games.



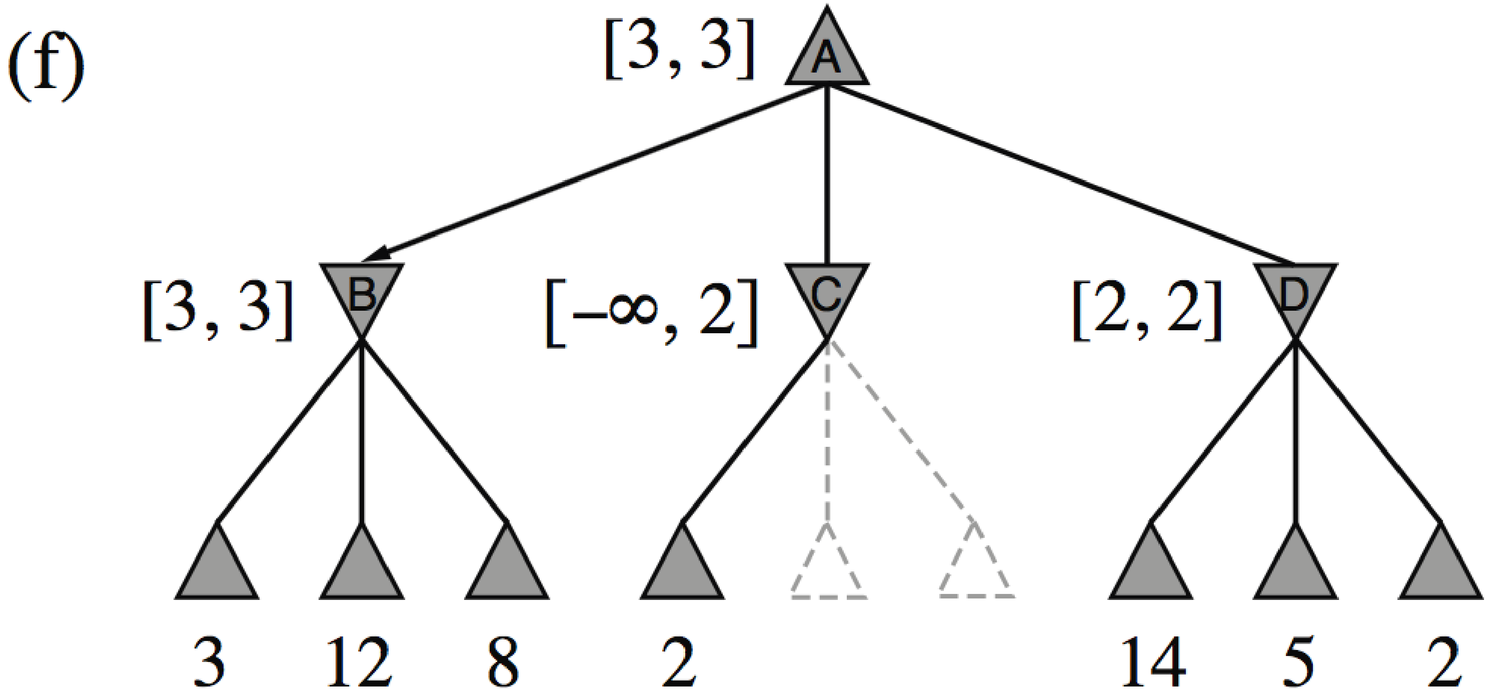
α–β PRUNING



$$
\begin{aligned}
\text{Minimax}(\textit{root}) &= \max(\min(3, 12, 8), \min(2, x, y), \min(14, 5, 2)) \\
&= \max(3, \min(2, x, y), 2) \\
&= \max(3, z, 2) \quad \text{where } z \leq 2 \\
&= 3
\end{aligned}
$$

I.e., we don't need to know the values of x and y!

MINIMAX EXAMPLE, WITH $\alpha-\beta$ PRUNING



HOW EFFICIENT IS $\alpha - \beta$ PRUNING?

The amount of pruning provided by the α-β algorithm depends on the ordering of the children of each node.

- It works best if a highest-valued child of a MAX node is selected first and if a lowest-valued child of a MIN node is returned first.
- In real games, much of the effort is made to optimise the search order.
- With a “perfect ordering”, the time complexity becomes $O(b^{m/2})$
    - this doubles the solvable search depth
    - however, $35^{80/2}$ (for chess) or $250^{160/2}$ (for go) is still impossible…

MINIMAX AND REAL GAMES

Most real games are too big to carry out minimax search, even with α-β pruning.

- For these games, instead of stopping at leaf nodes, we have to use a cutoff test to decide when to stop.
- The value returned at the node where the algorithm stops is an estimate of the value for this node.
- The function used to estimate the value is an evaluation function.
- Much work goes into finding good evaluation functions.
- There is a trade-off between the amount of computation required to compute the evaluation function and the size of the search space that can be explored in any given time.

MINIMAX VS H-MINIMAX

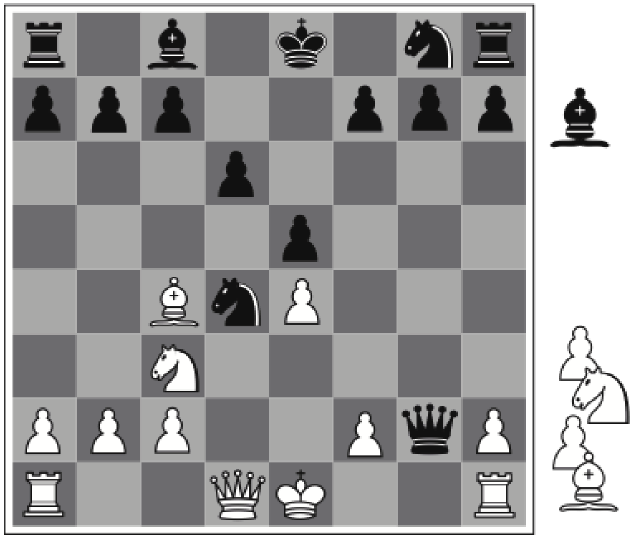
function Minimax(*state*):
  if TerminalTest(*state*) **then return** Utility(*state*)
  A := Actions(*state*)
  if *state* is a MAX node **then return** $\max_{a \in A}$ Minimax(Result(*state*, *a*))
  if *state* is a MIN node **then return** $\min_{a \in A}$ Minimax(Result(*state*, *a*))

The *Heuristic* Minimax algorithm is similar to normal Minimax

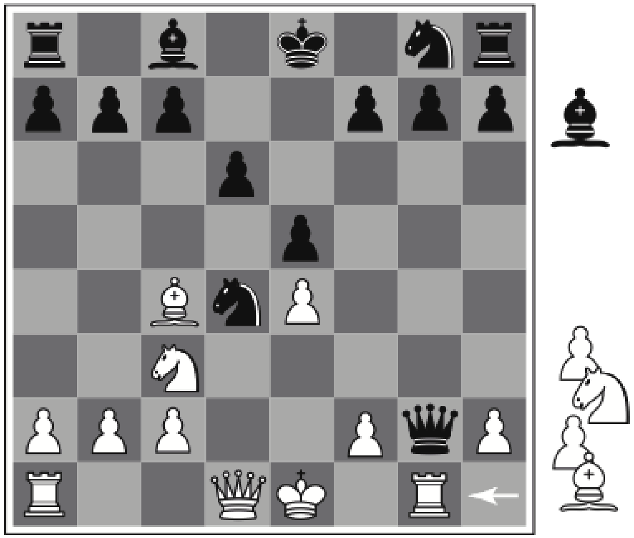
- it replaces **TerminalTest** and **Utility** with **CutoffTest** and **Eval**

function H-Minimax(*state*, *depth*):
  if CutoffTest(*state*, *depth*) **then return** Eval(*state*)
  A := Actions(*state*)
  if *state* is a MAX node **then return** $\max_{a \in A}$ H-Minimax(Result(*state*, *a*), *depth*+1)
  if *state* is a MIN node **then return** $\min_{a \in A}$ H-Minimax(Result(*state*, *a*), *depth*+1)

EVALUATION FUNCTIONS



(a) White to move



(b) White to move

A naive evaluation function will not see the difference between these two states.

PROBLEMS WITH CUTOFF TESTS

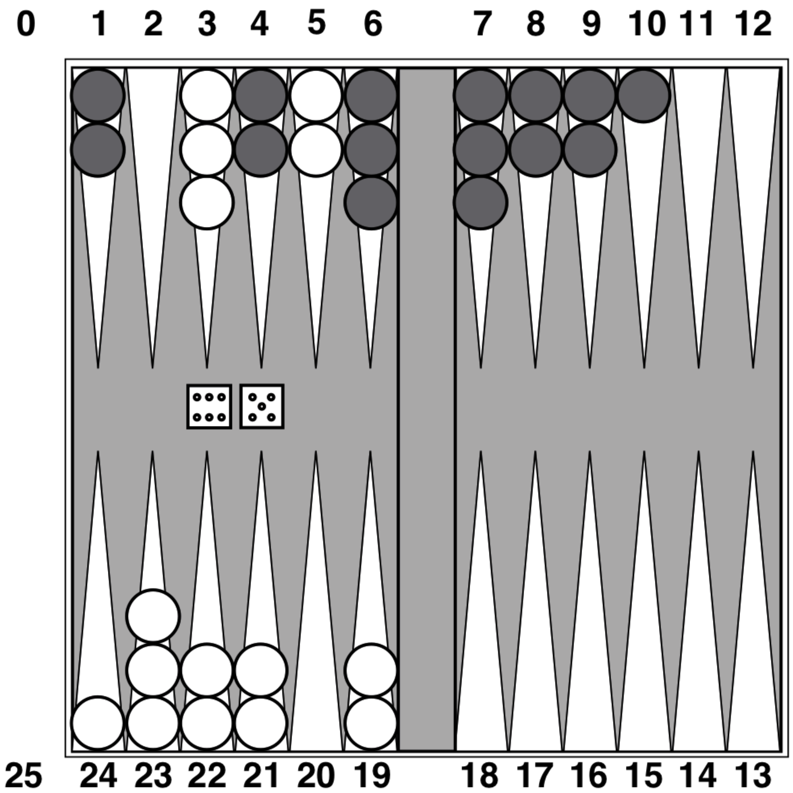
Too simplistic cutoff tests and evaluation functions can be problematic:

- e.g., if the cutoff is only based on the current depth
- then it might cut off the search in unfortunate positions (such as (b) on the previous slide)

We want more sophisticated cutoff tests:

- only cut off search in *quiescent* positions
- i.e., in positions that are “stable”, unlikely to exhibit wild swings in value
- non-quiescent positions should be expanded further

STOCHASTIC GAME EXAMPLE: BACKGAMMON

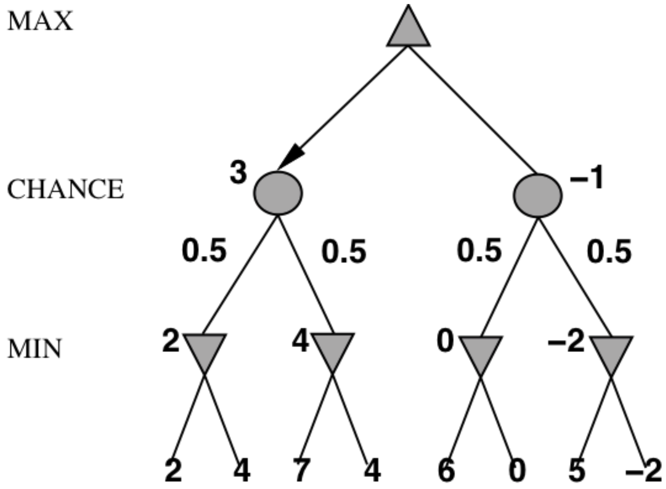


STOCHASTIC GAMES IN GENERAL

In stochastic games, chance is introduced by dice, card-shuffling, etc.

- We introduce *chance nodes* to the game tree.
- We can't calculate a definite minimax value,
  instead we calculate the *expected value* of a position.
- The expected value is the average of all possible outcomes.

A very simple example with coin-flipping and arbitrary values:



ALGORITHM FOR STOCHASTIC GAMES

The ExpectiMinimax algorithm gives perfect play;
it's just like Minimax, except we must also handle chance nodes:

function ExpectiMinimax(*state*):
  if TerminalTest(*state*) **then return** Utility(*state*)
  A := Actions(*state*)
  if *state* is a MAX node **then return** $\max_{a \in A}$ Minimax(*state*, *a*)
  if *state* is a MAX node **then return** $\min_{a \in A}$ Minimax(*state*, *a*)
  if *state* is a chance node **then return** $\sum_{a \in A} P(a)$ Minimax(*state*, *a*)

where $P(a)$ is the probability that action a occurs.

CONSTRAINT SATISFACTION PROBLEMS (R&N 4.1, 6.1–6.5)

CSP AS A SEARCH PROBLEM

CONSTRAINT PROGAGATION

PROBLEM STRUCTURE

LOCAL SEARCH FOR CSP

CSP: CONSTRAINT SATISFACTION PROBLEMS

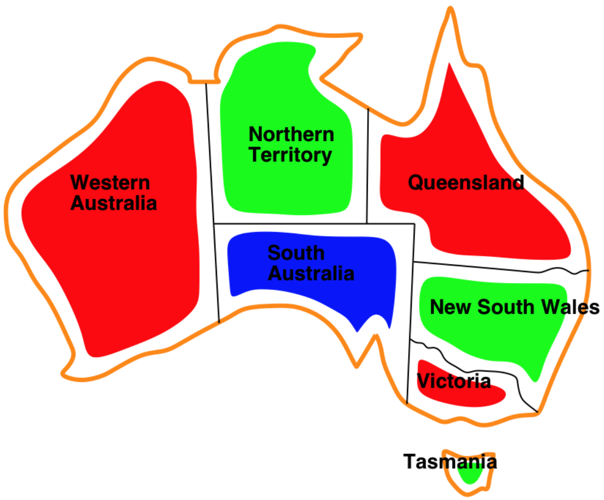
CSP is a specific kind of search problem:

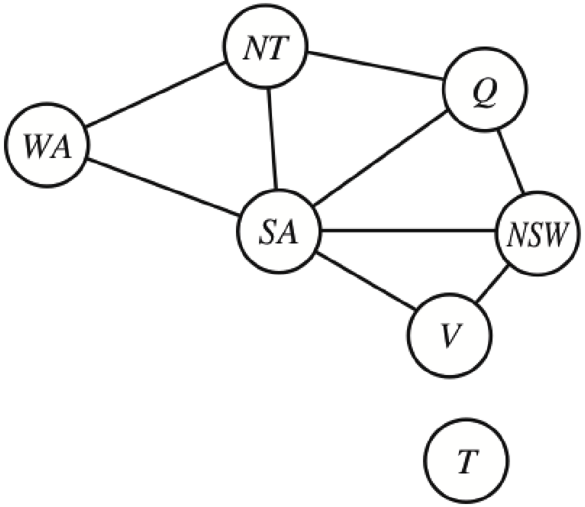
- the *state* is defined by *variables* X_i, each taking values from the domain D_i
- the *goal test* is a set of *constraints*:
  - each constraint specifies allowed values for a subset of variables
  - all constraints must be satisfied

Differences to general search problems:

- the path to a goal isn't important, only the solution is.
- there are no predefined starting state
- often these problems are huge, with thousands of variables,
  so systematically searching the space is infeasible

EXAMPLE: MAP COLOURING (BINARY CSP)

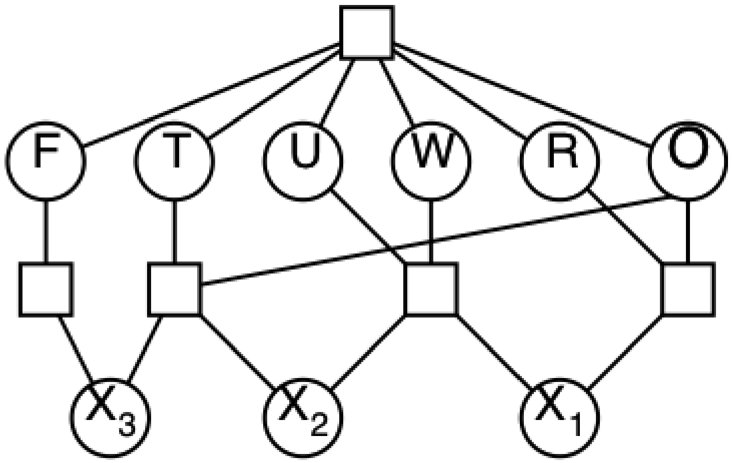




Variables:	*WA*, *NT*, *Q*, *NSW*, *V*, *SA*, *T*
Domains:	D_i = {red, green, blue}
Constraints:	*SA≠WA, SA≠NT, SA≠Q, SA≠NSW, SA≠V,* *WA≠NT, NT≠Q, Q≠NSW, NSW≠V*
Constraint graph:	Every variable is a node, every binary constraint is an arc.

EXAMPLE: CRYPTARITHMETIC PUZZLE (HIGHER-ORDER CSP)

```
  T W O
+ T W O
-------
F O U R
```



Variables:	*F, T, U, W, R, O,* X_1, X_2, X_3
Domains:	D_i = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
Constraints:	*Alldiff*(*F,T,U,W,R,O*), $O+O=R+10\cdot X_1$, etc.
Constraint graph:	This is not a binary CSP! The graph is a *constraint hypergraph*.

ALGORITHM FOR BACKTRACKING SEARCH

At each depth level, decide on one single variable to assign:

- this gives branching factor $b = d$, so there are d^n leaves

Depth-first search with single-variable assignments is called *backtracking search*:

```
function BacktrackingSearch(csp):
    return Backtrack(csp, { })

function Backtrack(csp, assignment):
    if assignment is complete then return assignment
    var := SelectUnassignedVariable(csp, assignment)
    for each value in OrderDomainValues(csp, var, assignment):
        if value is consistent with assignment:
            inferences := Inference(csp, var, value)
            if inferences ≠ failure:
                result := Backtrack(csp, assignment ∪ {var=value} ∪ inferences)
                if result ≠ failure then return result
    return failure
```

IMPROVING BACKTRACKING EFFICIENCY

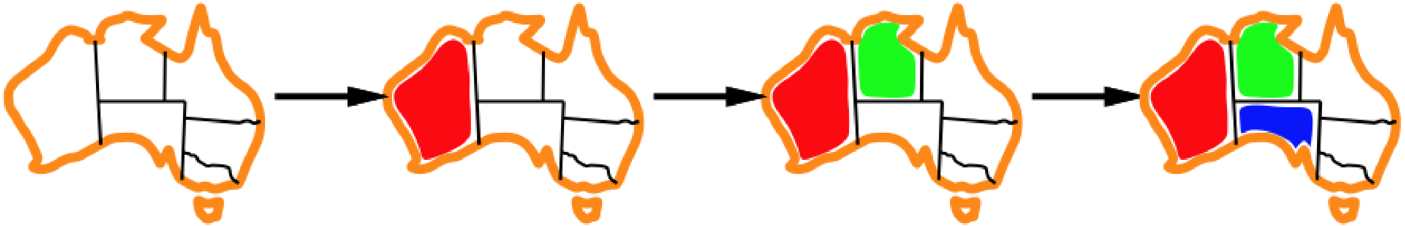
The general-purpose algorithm gives rise to several questions:

- Which variable should be assigned next?
  - SelectUnassignedVariable(*csp*, *assignment*)
- In what order should its values be tried?
  - OrderDomainValues(*csp*, *var*, *assignment*)
- What inferences should be performed at each step?
  - Inference(*csp*, *var*, *value*)
- Can the search avoid repeating failures?
  - Conflict-directed backjumping, constraint learning, no-good sets (R&N 6.3.3, not covered in this course)

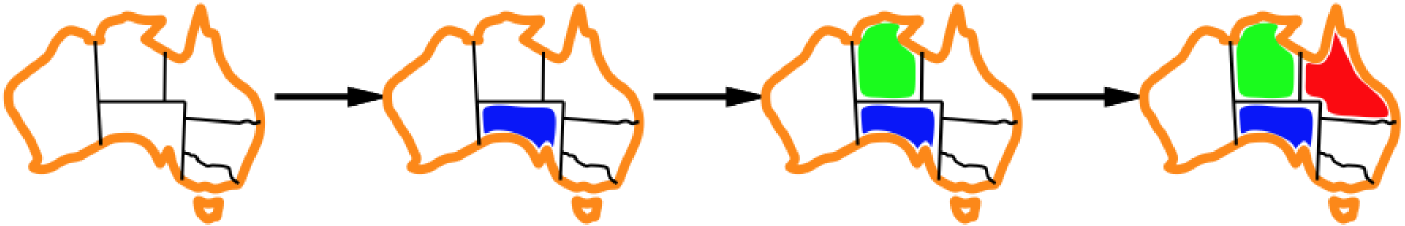
SELECTING UNASSIGNED VARIABLES

Heuristics for selecting the next unassigned variable:

- Minimum remaining values (MRV):
  $\Longrightarrow$ choose the variable with the fewest legal values



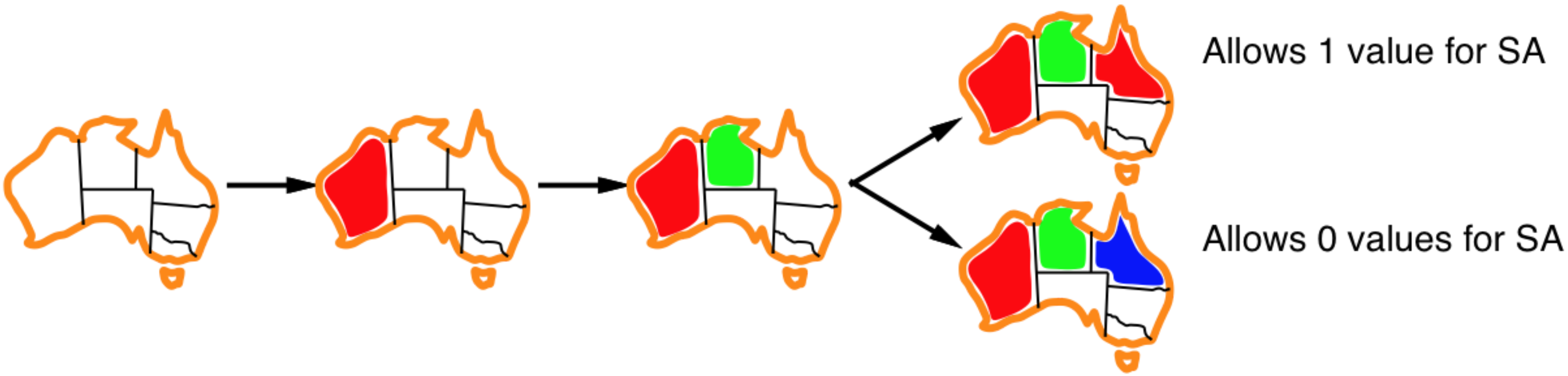
- Degree heuristic (if there are several MRV variables):
  $\Longrightarrow$ choose the variable with most constraints on remaining variables



ORDERING DOMAIN VALUES

Heuristics for ordering the values of a selected variable:

- Least constraining value:
  $\Longrightarrow$ prefer the value that rules out the fewest choices
  for the neighboring variables in the constraint graph



INFERENCE: ARC CONSISTENCY, AC-3

Keep a set of arcs to be considered: pick one arc (X, Y) at the time and make it consistent (i.e., make X arc consistent to Y).

- Start with the set of all arcs $\{(X, Y), (Y, X), (X, Z), (Z, X), \ldots\}$.

When an arc has been made arc consistent, does it ever need to be checked again?

- An arc (X, Y) needs to be revisited if the domain of Y is revised.

```
function AC-3(inout csp):
    initialise queue to all arcs in csp
    while queue is not empty:
        (X, Y) := RemoveOne(queue)
        if Revise(csp, X, Y):
            if  D_X = Ø  then return failure
            for each Z in X.neighbors–{Y} do add (Z, X) to queue

function Revise(inout csp, X, Y):
    delete every x from D_X such that there is no value y in D_Y satisfying the constraint C_XY
```

COMBINING BACKTRACKING WITH AC-3

What if some domains have more than one element after AC?

We can resort to backtracking search:

- Select a variable and a value using some heuristics
  (e.g., minimum-remaining-values, degree-heuristic, least-constraining-value)
- Make the graph arc-consistent again
- Backtrack and try new values/variables, if AC fails
- Select a new variable/value, perform arc-consistency, etc.

Do we need to restart AC from scratch?

- no, only some arcs risk becoming inconsistent after a new assignment
- restart AC with the queue $\{(Y_i, X) | X \rightarrow Y_i\}$,
  i.e., only the arcs (Y_i, X) where Y_i are the neighbors of X
- this algorithm is called *Maintaining Arc Consistency* (MAC)

CONSISTENCY PROPERTIES

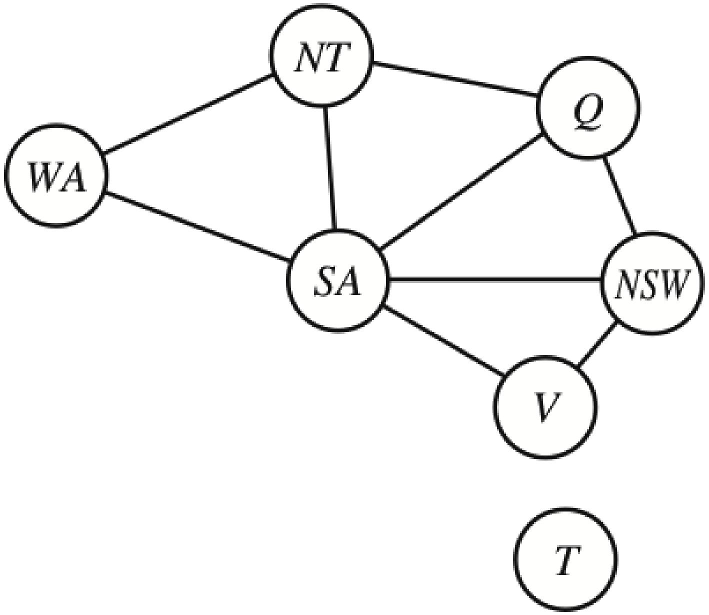
There are several kinds of consistency properties and algorithms:

- *Node consistency*: single variable, unary constraints (straightforward)
- *Arc consistency*: pairs of variables, binary constraints (AC-3 algorithm)
- *Path consistency*: triples of variables, binary constraints (PC-2 algorithm)
- k*-consistency*: k variables, k-ary constraints (algorithms exponential in k)
- Consistency for global constraints:
    - special-purpose algorithms for different constraints, e.g.:
    - *Alldiff*($X_1, \ldots, X_m$) is inconsistent if $m > |D_1 \cup \cdots \cup D_m|$
    - *Atmost*($n, X_1, \ldots, X_m$) is inconsistent if $n < \sum_i \min(D_i)$

PROBLEM STRUCTURE: INDEPENDENT SUBPROBLEMS

Tasmania is an *independent subproblem*:

- there are efficient algorithms for finding *connected components* in a graph



Suppose that each subproblem has c variables out of n total. The cost of the worst-case solution is $n/c \cdot d^c$, which is linear in n.

E.g., $n = 80, d = 2, c = 20$:

- 2^{80} = 4 billion years at 10 million nodes/sec

If we divide it into 4 equal-size subproblems:

- $4 \cdot 2^{20}$ =0.4 seconds at 10 million nodes/sec

Note: this only has a real effect if the subproblems are (roughly) equal size!

PROBLEM STRUCTURE: TREE-STRUCTURED CSP

A constraint graph is a tree when any two variables are connected by only one path.

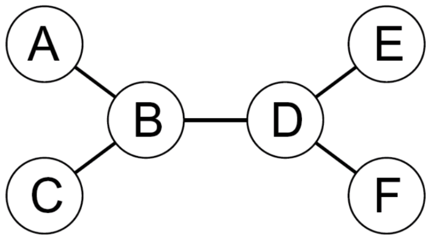
- then any variable can act as root in the tree
- tree-structured CSP can be solved in *linear time*, in the number of variables!

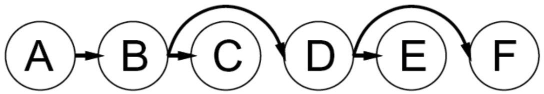
CSP is *directed arc-consistent* if:

- there is an orderning of variables $X_1, X_2, \ldots, X_n$ such that
- every X_i is arc-consistent with each X_j for all $j > i$

To solve a tree-structured CSP:

- first pick a variable to be the root of the tree
- then find a *topological sort* of the variables (with the root first)
- finally, make each arc consistent, in reverse topological order
- the total runtime is $O(nd^2)$, i.e., linear in n

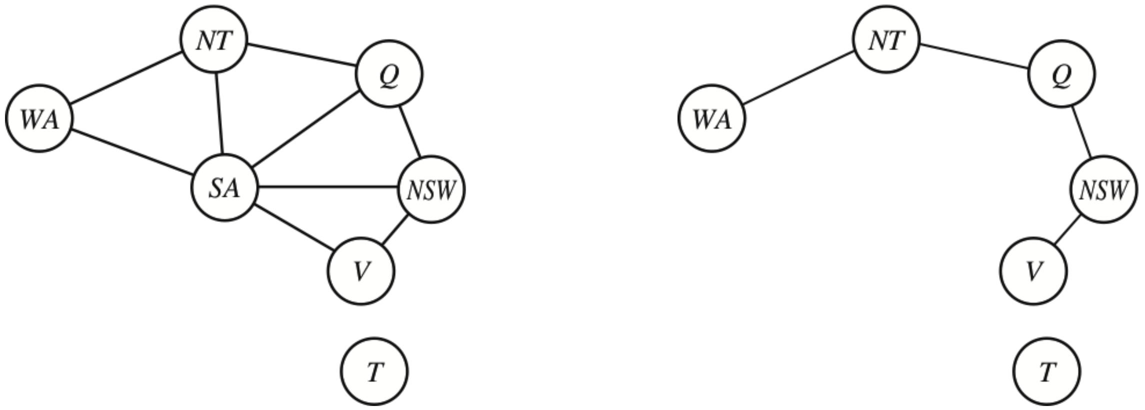




CONVERTING TO TREE-STRUCTURED CSP

Most CSPs are *not* tree-structured, but sometimes we can reduce a problem to a tree

- one approach is to assign values to some variables,
  so that the remaining variables form a tree



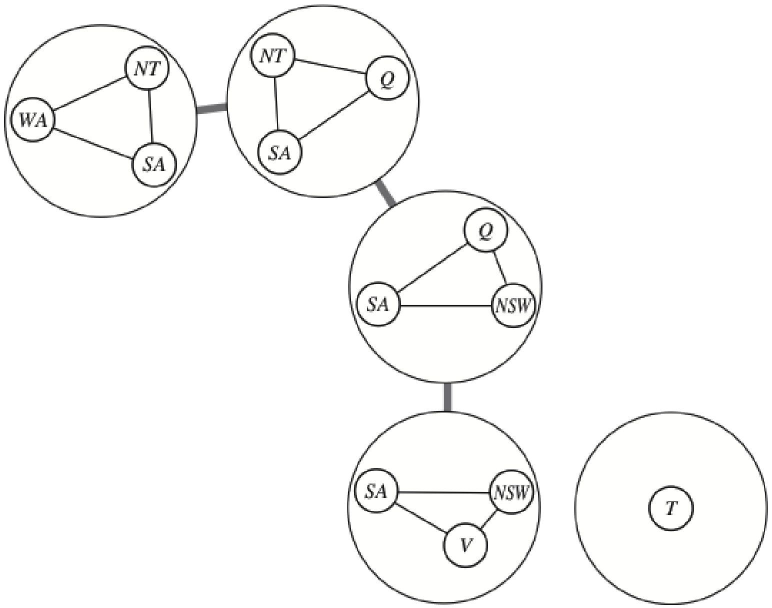
The set of variables that we have to assign is called a *cycle cutset*

- for Australia, {*SA*} is a cycle cutset and {*NT,Q,V*} is also a cycle cutset
- finding the smallest cycle cutset is NP-hard,
  but there are efficient approximation algorithms

TREE DECOMPOSITION

Another approach for reducing to a tree-CSP is *tree decomposition*:

- divide the original CSP into a set of connected subproblems, such that the connections form a *tree-structured graph*
- solve each subproblem independently
- since the decomposition is a tree, we can solve the main problem using directed arc consistency (the TreeCSPSolver algorithm)



LOCAL SEARCH FOR CSPS

Given an assignment of a value to each variable:

- A conflict is an unsatisfied constraint.
- The goal is an assignment with zero conflicts.
- Heuristic function to be minimized: the number of conflicts.
    - this is the *min-conflicts* heuristics

function MinConflicts(*csp*, *max_steps*)
    current := an initial complete assignment for *csp*
    repeat *max_steps* times:
        if *current* is a solution for csp **then return** *current*
        var := a randomly chosen conflicted variable from *csp*
        value := the value *v* for *var* that minimises Conflicts(*var*, *v*, *current*, *csp*)
        current[*var*] = *value*
    return failure

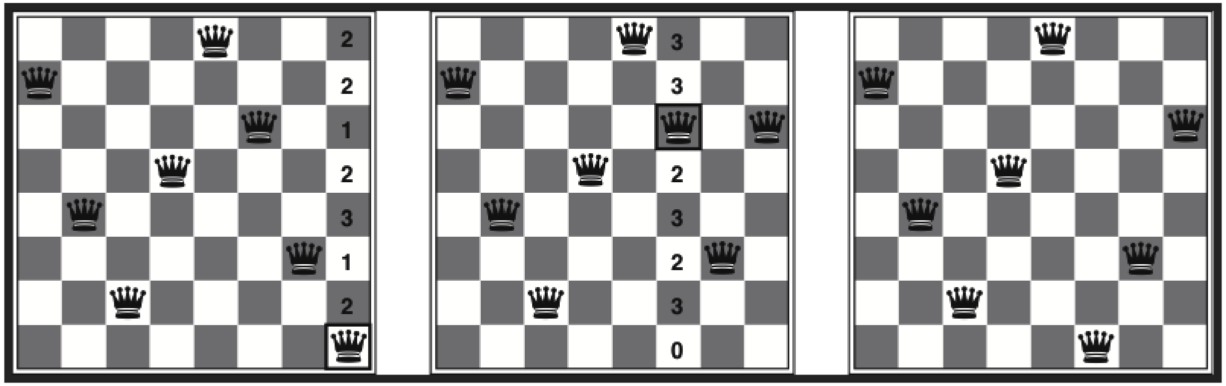
EXAMPLE: n-QUEENS (REVISITED)

Put n queens on an $n \times n$ board, in separate columns

Conflicts = unsatisfied constraints = n:o of threatened queens

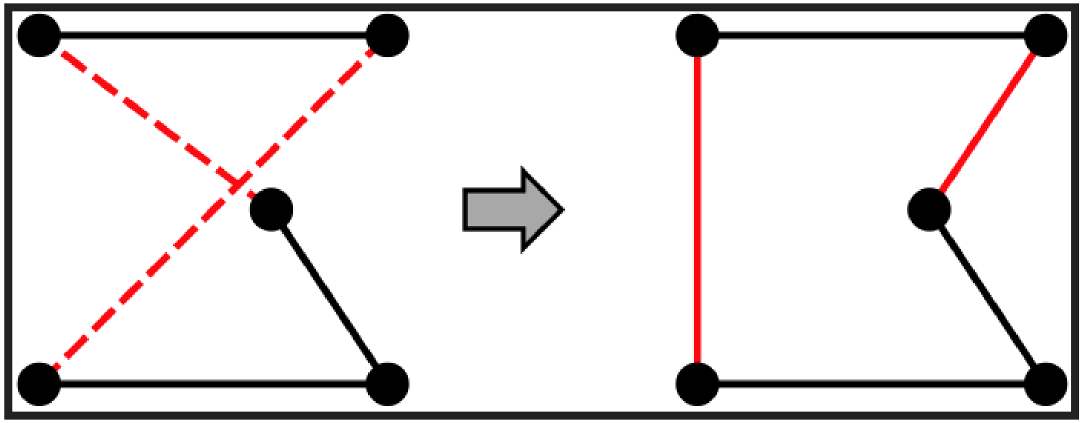
Move a queen to reduce the number of conflicts

- repeat until we cannot move any queen anymore
- then we are at a local maximum — hopefully it is global too



EXAMPLE: TRAVELLING SALESPERSON

Start with any complete tour, and perform pairwise exchanges



Variants of this approach get within 1% of optimal very quickly with thousands of cities

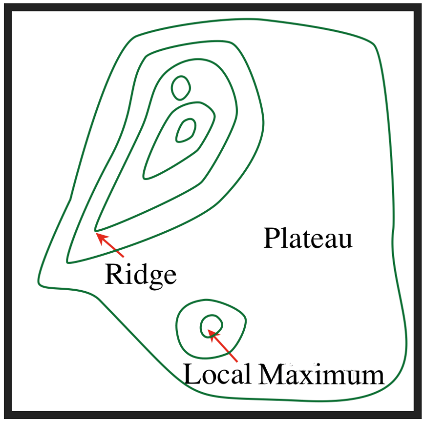
LOCAL SEARCH

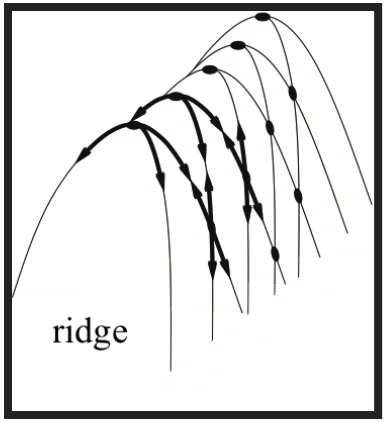
Hill climbing search is also called gradient/steepest ascent/descent, or greedy local search.

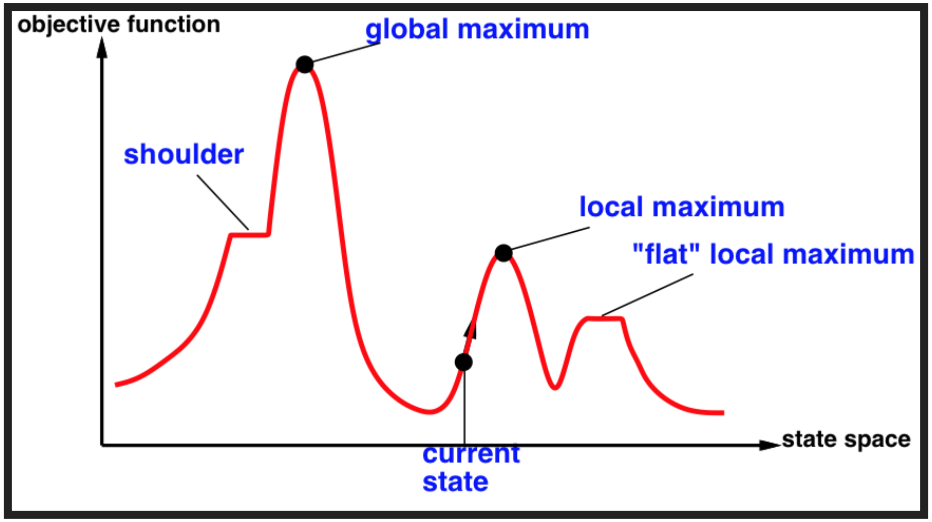
function HillClimbing(*graph*, *initialState*):
    current := *initialState*
    loop:
        neighbor := a highest-valued successor of *current*
        if *neighbor*.value ≤ *current*.value **then return** *current*
        current := *neighbor*

PROBLEMS WITH HILL CLIMBING

Local maxima — Ridges — Plateaux







RANDOMIZED HILL CLIMBING

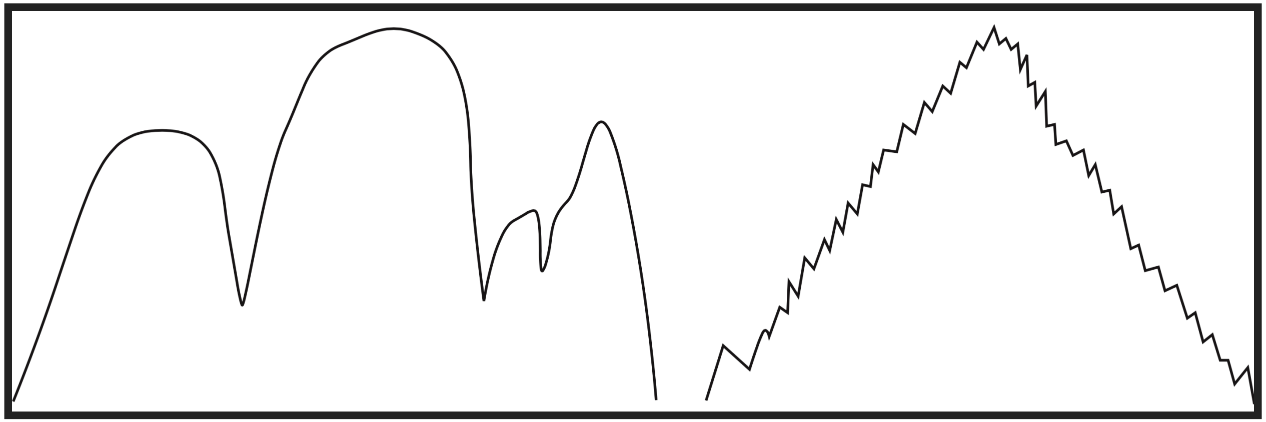
As well as upward steps we can allow for:

- *Random steps:* (sometimes) move to a random neighbor.
- *Random restart:* (sometimes) reassign random values to all variables.

Both variants can be combined!

1-DIMENSIONAL ILLUSTRATIVE EXAMPLE

Two 1-dimensional search spaces; you can step right or left:



Which method would most easily find the global maximum?

- random steps or random restarts?

What if we have hundreds or thousands of dimensions?

- …where different dimensions have different structure?

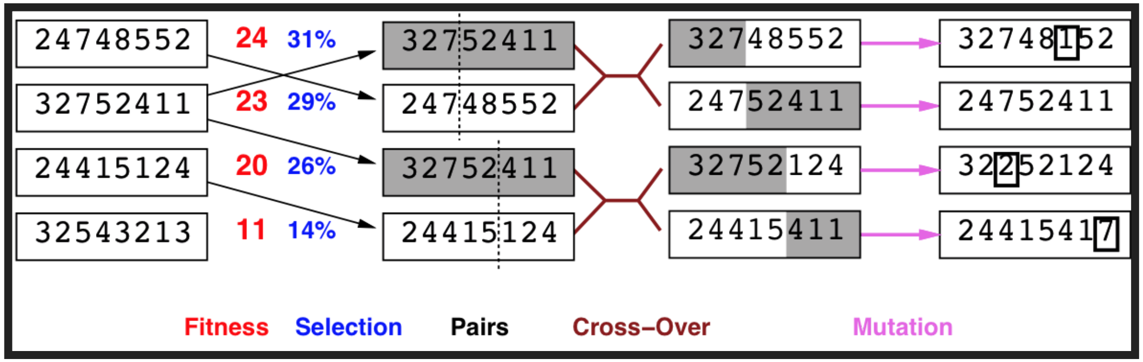
LOCAL BEAM SEARCH

Idea: maintain a population of k states in parallel, instead of one.

- At every stage, choose the k best out of all of the neighbors.
  - when $k = 1$, it is normal hill climbing search
  - when $k = \infty$, it is breadth-first search
- The value of k lets us limit space and parallelism.
- *Note*: this is not the same as k searches run in parallel!
- *Problem*: quite often, all k states end up on the same local hill.

GENETIC ALGORITHMS

Similar to stochastic beam search,
but *pairs* of individuals are combined to create the offspring.



For each generation:

- Randomly choose pairs of individuals where
  the fittest individuals are more likely to be chosen.
- For each pair, perform a cross-over:
  form two offspring each taking different parts of their parents:
- Mutate some values.

Stop when a solution is found.