Reasoning under Uncertainty Part II

Artificial Intelligence, 2015
TIN172/DIT410

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based on slides by
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Quick recap: Random Variables

- Upper case: X.
- Value is subject to chance.
  - Values: lower case.
  - Could represent the outcome of an experiment.
- A probability $\in [0, 1]$ is associated to each value that X can take.

Quick recap: Probability Distributions

- Describes the behaviour of a random variable.
- $P(X)$ is the probability measure of X.
- More than one variable:
  - Joint: $P(X, Y, Z)$
  - Marginal: $P(X) = \sum_Y P(X, Y)$
  - Conditional: $P(X|Y) = \frac{P(X,Y)}{P(Y)}$

Example: Probability Distributions

X - The outcome of a coin toss

X	$P(X)$
heads	0.5
tails	0.5

- This is called the **Binomial distribution**
- $P(X = heads)$ - the probability that coin comes up heads
- $P(X = tails) = 1 - P(X = heads)$

Chain Rule for Probabilities

$$P(X, Y, Z) = P(X|Y, Z)P(Y, Z)$$

Chain Rule for Probabilities

$$\begin{aligned} P(X, Y, Z) &= P(X|Y, Z)P(Y, Z) \\ &= P(X|Y, Z)P(Y|Z)P(Z) \end{aligned}$$

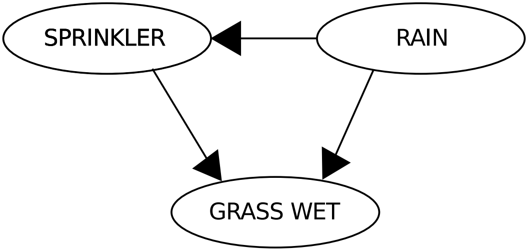
Conditional Independence

$$X \perp Y | Z \rightarrow P(X|Y,Z) = P(X|Z)$$

Probability Distributions ctd.

- Belief networks.
    - Nodes: random variables.
    - Arcs: causal dependence
    - Network encodes independence.
    - Flow of influence.

Example Belief Network



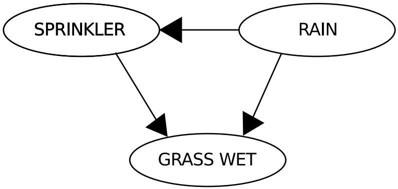
Chain Rule for Bayesian Networks

- Chain rule for Bayesian Networks:

$$P(X_1, X2, X3, ..., X_n) = \prod_i P(X_i | parents(X_i))$$

- P factorizes over the network.

Example Belief Network



- Using the chain rule: $P(Grass, Sprinkler, Rain) = P(Grass|Sprinkler, Rain)P(Sprinkler|Rain)P(Rain)$
- A factorization of P.

Querying Belief Networks

- Query variable(s): Q
- Observed evidence: $E_1 = e_1, E_2 = e_2, \ldots, E_n = e_n$
- How do you calculate $P(Q|e)$?

Inference in Belief Networks

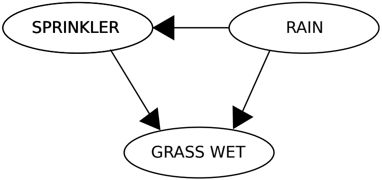
1. Set observed evidence: $E_1 = e_1, E_2 = e_2, \ldots, E_n = e_n$
2. Marginalize out non-query variables W .
3. $P(Q|e) \propto \sum_W P(Q, W, E = e)$
4. Renormalize.

Renormalization

- $\tilde{P}(X)$ - an unnormalized probability measure.
- $P(X) = \frac{\tilde{P}(X)}{\sum_X \tilde{P}(X)}$ - renormalized probability distribution over X.
- The denominator, $\sum_X \tilde{P}(X)$ is merely a constant.

Example Belief Network

	SPRINKLER	
RAIN	T	F
F	0.4	0.6
T	0.01	0.99



RAIN	
T	F
0.2	0.8

		GRASS WET	
SPRINKLER	RAIN	T	F
F	F	0.0	1.0
F	T	0.8	0.2
T	F	0.9	0.1
T	T	0.99	0.01

Factors in general

Function: $f(X_1, \ldots, X_j)$.

Assignments:

- $f(X_1 = x_1, X_2, \ldots, X_j)$, is a factor on $X_2, \ldots, X_j$.
- $f(X_1 = x_1, X_2 = x_2, \ldots, X_j = x_j)$

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	
f	t	
f	f	

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	
f	t	
f	f	

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	
f	

$r(X{=}t, Y{=}f, Z{=}f) =$

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	
f	t	
f	f	

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

$$r(X{=}t, Y{=}f, Z{=}f) = 0.8$$

Multiplying factors

The **product** of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z}) \quad = \quad f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 \times f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	
t	f	t	
t	f	f	
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 \times f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

Variable Elimination Algorithm

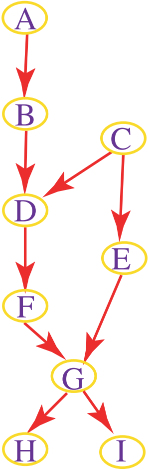
Compute the distribution of some query variable X_q

1. Use chain rule to get factorization.
2. Set observed variables.
3. Elimination: Marginalize all variables except X_q:
   - "Push in" the summations:

$$\sum_{Y} P(X)P(Y) = P(X)\sum_{Y} P(Y)$$

4. Multiply the remaining factors.
5. Renormalize.

Variable elimination example



$$\left.\begin{array}{l} P(A) \\ P(B|A) \end{array}\right\} \xrightarrow{\text{elim } A} f_1(B)$$

$$\left.\begin{array}{l} P(C) \\ P(D|B,C) \\ P(E|C) \end{array}\right\} \xrightarrow{\text{elim } C} f_2(BDE)$$

$$P(F|D)$$

$$P(G|F,E)$$

$$P(H|G) \;\} \xrightarrow{\text{obs } H} f_3(G)$$

$$P(I|G) \;\} \xrightarrow{\text{elim } I} f_4(G)$$

Variable Elimination example

$$P(D, H = h) = \frac{\sum_{A,B,C,E,F,G,I} P(A, B, C, D, E, F, G, H = h, I)}{Z}$$

$$= \frac{\sum_{A,B,C,E,F,G,I} P(I|G)P(H{=}h|G)P(G|F,E)P(F|D)P(E|C)P(D|B,C)P(C)P(B|A)P(A)}{Z}$$

$$= \frac{\sum_{I,G} P(I|G)P(H = h|G) \sum_{E,F} P(G|F, E)P(F|D) \sum_{C} P(E|C) \sum_{B} P(D|B, C)P(C) \sum_{A} P(B|A)P(A)}{Z}$$

Z is the (re)normalizing constant.

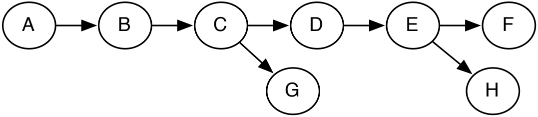
Variable Elimination example, ctd.

$$\underbrace{\sum_{G} f_4(G) f_3(G) \underbrace{\sum_{E,F} P(G|F,E) P(F|D) \underbrace{\sum_{B} f_2(B,D,E) f_1(B)}_{f_5(D,E)}}_{f_6(D,G)}}_{f_7(D)}$$

$$P(D, H = h) = \frac{f_7(D)}{Z}$$

Z is the (re)normalizing constant. f_1, f_2, f_3, see previous slide.

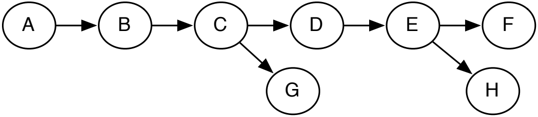
Variable Elimination example



Query: $P(G|f)$; elimination ordering: A, H, E, D, B, C

$$P(G|f) \quad \propto$$

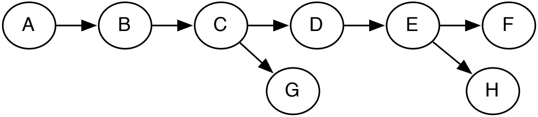
Variable Elimination example



Query: $P(G|f)$; elimination ordering: A, H, E, D, B, C

$$P(G|f) \propto \sum_C \sum_B \sum_D \sum_E \sum_H \sum_A P(A)P(B|A)P(C|B) \\ P(D|C)P(E|D)P(f|E)P(G|C)P(H|E)$$

Variable Elimination example



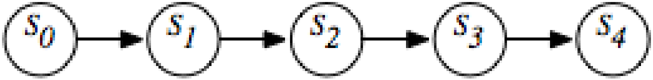
Query: $P(G|f)$; elimination ordering: A, H, E, D, B, C

$$P(G|f) \propto \sum_C \sum_B \sum_D \sum_E \sum_H \sum_A P(A)P(B|A)P(C|B) P(D|C)P(E|D)P(f|E)P(G|C)P(H|E)$$

$$= \sum_C \left(\sum_B \left(\sum_A P(A)P(B|A) \right) P(C|B) \right) P(G|C) \left(\sum_D P(D|C) \left(\sum_E P(E|D)P(f|E) \sum_H P(H|E) \right) \right)$$

Markov chain

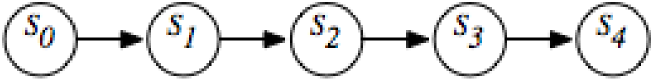
- A Markov chain is a special case of belief network:



What probabilities need to be specified? What Independence assumptions are made?

Markov chain

- A Markov chain is a special case of belief network:



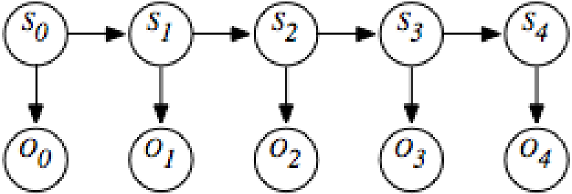
- $P(S_0)$ specifies initial conditions
- $P(S_{t+1}|S_t)$ specifies the dynamics
- $P(S_{t+1}|S_0, \ldots, S_t) = P(S_{t+1}|S_t)$.
- Often S_t represents the **state** at time t. Intuitively S_t conveys all of the information about the history that can affect the future states.
- "The future is independent of the past given the present."

Stationary Markov chain

- A **stationary Markov chain** is when for all $t > 0$, $t' > 0$, $P(S_{t+1}|S_t) = P(S_{t'+1}|S_{t'})$.
- We specify $P(S_0)$ and $P(S_{t+1}|S_t)$.
  - Simple model, easy to specify
  - Often the natural model
  - The network can extend indefinitely

Hidden Markov Model

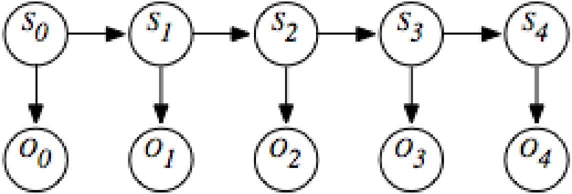
- A **Hidden Markov Model (HMM)** is a belief network:



The probabilities that need to be specified:

Hidden Markov Model

- A **Hidden Markov Model (HMM)** is a belief network:

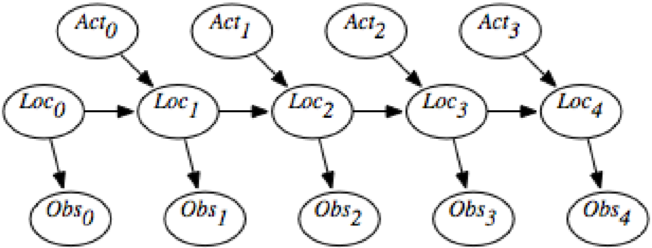


The probabilities that need to be specified:

- $P(S_0)$ specifies initial conditions
- $P(S_{t+1}|S_t)$ specifies the dynamics
- $P(O_t|S_t)$ specifies the sensor model

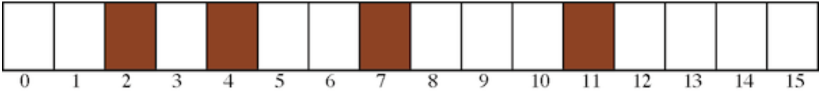
Example: localization

- Suppose a robot wants to determine its location based on its actions and its sensor readings: **Localization**
- This can be represented by the augmented HMM:



Example localization domain

- Circular corridor, with 16 locations:



- Doors at positions: 2, 4, 7, 11.
- Noisy Sensors
- Stochastic Dynamics
- Robot starts at an unknown location and must determine where it is.

Example Sensor Model

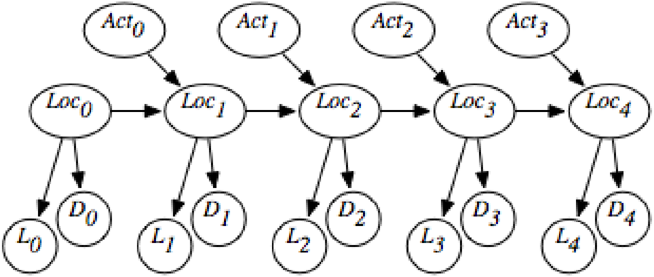
- $P(Observe\ Door \mid At\ Door) = 0.8$
- $P(Observe\ Door \mid Not\ At\ Door) = 0.1$

Example Dynamics Model

- $P(loc_{t+1} = L | action_t = goRight \wedge loc_t = L) = 0.1$
- $P(loc_{t+1} = L + 1 | action_t = goRight \wedge loc_t = L) = 0.8$
- $P(loc_{t+1} = L + 2 | action_t = goRight \wedge loc_t = L) = 0.074$
- $P(loc_{t+1} = L' | action_t = goRight \wedge loc_t = L) = 0.002$ for any other location L'.
    - All location arithmetic is modulo 16.
    - The action $goLeft$ works the same but to the left.

Combining sensor information

- Example: we can combine information from a light sensor and the door sensor **Sensor Fusion**

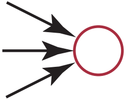


S_t robot location at time t
D_t door sensor value at time t
L_t light sensor value at time t

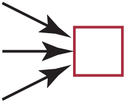
Decisions Networks

Decisions Networks

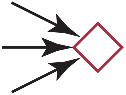
A **decision network** is a graphical representation of a finite sequential decision problem, with 3 types of nodes:



- A **random variable** is drawn as an ellipse. Arcs into the node represent probabilistic dependence.

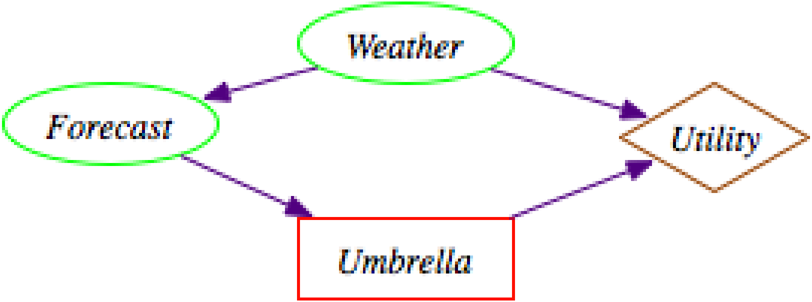


- A **decision variable** is drawn as an rectangle. Arcs into the node represent information available when the decision is make.



- A **utility** node is drawn as a diamond. Arcs into the node represent variables that the utility depends on.

Umbrella Decision Network

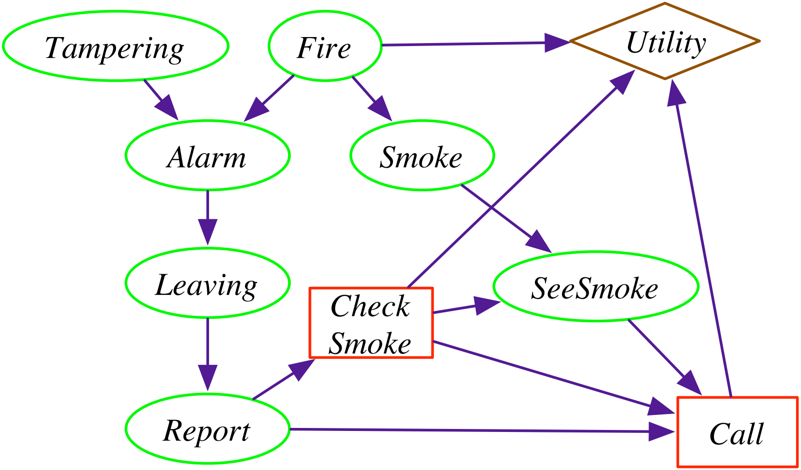


You don't get to observe the weather when you have to decide whether to take your umbrella. You do get to observe the forecast.

Expected Utility

- $EU[\mathcal{D}[a]] = \sum_x P(x|a)U(x,a)$
- We want to choose actions that maximizes this.
- $\hat{a} = argmax_a EU[\mathcal{D}[a]]$

Decision Network for the Alarm Problem



Example Initial Factors

Which Way	Accident	Value
long	true	0.01
long	false	0.99
short	true	0.2
short	false	0.8

Which Way	Accident	Wear Pads	Value
long	true	true	30
long	true	false	0
long	false	true	75
long	false	false	80
short	true	true	35
short	true	false	3
short	false	true	95
short	false	false	100

After summing out Accident

Which Way	Wear Pads	Value
long	true	74.55
long	false	79.2
short	true	83.0
short	false	80.6

Finding an optimal policy

- Create a factor for each conditional probability table and a factor for the utility.

Finding an optimal policy

- Create a factor for each conditional probability table and a factor for the utility.
- Repeat:
  - Sum out random variables that are not parents of a decision node.
  - Select a variable D that is only in a factor f with (some of) its parents.
  - Eliminate D by maximizing. This returns:
    - an optimal decision function for D: $\arg\max_D f$
    - a new factor: $\max_D f$
- until there are no more decision nodes.
- Sum out the remaining random variables. Multiply the factors: this is the expected utility of an optimal policy.

Initial factors for the Umbrella Decision

Weather	Value
norain	0.7
rain	0.3

Weather	Fcast	Value
norain	sunny	0.7
norain	cloudy	0.2
norain	rainy	0.1
rain	sunny	0.15
rain	cloudy	0.25
rain	rainy	0.6

Weather	Umb	Value
norain	take	20
norain	leave	100
rain	take	70
rain	leave	0

Eliminating By Maximizing

f:

Fcast	Umb	Val
sunny	take	12.95
sunny	leave	49.0
cloudy	take	8.05
cloudy	leave	14.0
rainy	take	14.0
rainy	leave	7.0

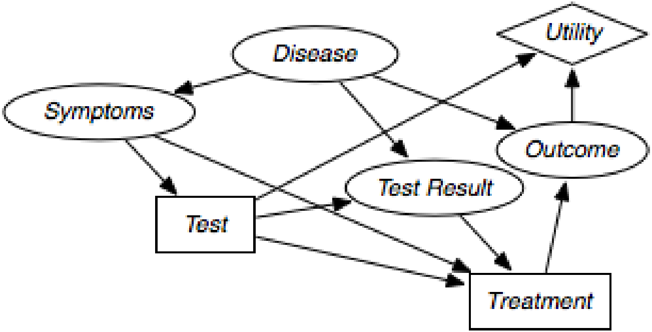
$\max_{Umb} f$:

Fcast	Val
sunny	49.0
cloudy	14.0
rainy	14.0

$\arg\max_{Umb} f$:

Fcast	Umb
sunny	leave
cloudy	leave
rainy	take

Exercise



What are the factors?
Which random variables get summed out first?
Which decision variable is eliminated? What factor is created?
Then what is eliminated (and how)?
What factors are created after maximization?