Technical Report: Finite Difference Mode Solver Derivation

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Abstract

This report details the mathematical formulation and finite difference implementation for solving the scalar Helmholtz equation in optical waveguides. The derivation assumes a scalar field approximation for weakly guiding structures.

1. Governing Physical Equation

We start with the scalar Helmholtz equation for the electric field $E(x,y)$ propagating in the z-direction with propagation constant β:

$$\nabla_\perp^2 E(x,y) + k_0^2 n^2(x,y) E(x,y) = \beta^2 E(x,y)$$

Where $k_0 = 2\pi/\lambda$ is the vacuum wavenumber and $n(x,y)$ is the refractive index distribution.

2. Finite Difference Discretization

We discretize the domain into a grid with spacing Δx and Δy. The Laplacian operator is approximated using the central difference scheme (5-point stencil):

$$\frac{\partial^2 E}{\partial x^2} \approx \frac{E_{i+1,j} - 2E_{i,j} + E_{i-1,j}}{\Delta x^2}$$

$$\frac{\partial^2 E}{\partial y^2} \approx \frac{E_{i,j+1} - 2E_{i,j} + E_{i,j-1}}{\Delta y^2}$$

3. Matrix Formulation

Substituting the discrete approximations into the Helmholtz equation yields a linear algebraic eigenvalue problem:

$$(\mathbf{L} + \mathbf{V})\mathbf{\Phi} = \beta^2 \mathbf{\Phi}$$

Here, $\mathbf{L}$ is the sparse Laplacian matrix constructed from the Kronecker sum of 1D second-derivative operators:

$$\mathbf{L} = \mathbf{D}_{yy} \otimes \mathbf{I}_x + \mathbf{I}_y \otimes \mathbf{D}_{xx}$$

And $\mathbf{V}$ is the diagonal matrix representing the refractive index profile:

$$\mathbf{V} = \mathrm{diag}(k_0^2 n^2)$$

4. Computational Implementation

The solver is implemented in Python using scipy.sparse for efficient memory usage. The eigenvalue problem is solved using the implicitly restarted Arnoldi method (via scipy.sparse.linalg.eigs) to find the largest real eigenvalues corresponding to the guided modes.

5. Conclusion

This formulation provides a robust and computationally efficient method for analyzing optical modes in arbitrary refractive index distributions defined by Gerber files.