Homework 5: Pareto and Kuznets on the Grand Tour

```r
library(ggplot2)
```

We continue working with the World Top Incomes Database [https://wid.world], and the Pareto distribution, as in the lab. We also continue to practice working with data frames, manipulating data from one format to another, and writing functions to automate repetitive tasks.

We saw in the lab that if the upper tail of the income distribution followed a perfect Pareto distribution, then

$$\left(\frac{P99}{P99.9}\right)^{-a+1} = 10 \tag{1}$$
$$\left(\frac{P99.5}{P99.9}\right)^{-a+1} = 5 \tag{2}$$
$$\left(\frac{P99}{P99.5}\right)^{-a+1} = 2 \tag{3}$$

We could estimate the Pareto exponent by solving any one of these equations for a; in lab we used

$$a = 1 - \frac{\log 10}{\log\left(P99/P99.9\right)}\ , \tag{4}$$

Because of measurement error and sampling noise, we can't find find one value of a which will work for all three equations (1)–(3). Generally, trying to make all three equations come close to balancing gives a better estimate of a than just solving one of them. (This is analogous to finding the slope and intercept of a regression line by trying to come close to all the points in a scatterplot, and not just running a line through two of them.)

1. We estimate a by minimizing

$$\left(\left(\frac{P99}{P99.9}\right)^{-a+1} - 10\right)^2 + \left(\left(\frac{P99.5}{P99.9}\right)^{-a+1} - 5\right)^2 + \left(\left(\frac{P99}{P99.5}\right)^{-a+1} - 2\right)^2$$

   Write a function, `percentile_ratio_discrepancies`, which takes as inputs `P99`, `P99.5`, `P99.9` and `a`, and returns the value of the expression above. Check that when `P99=1e6`, `P99.5=2e6`, `P99.9=1e7` and `a=2`, your function returns `0`.

```r
percentile_ratio_discrepancies <- function(P99, P99.5, P99.9, a) {
  term1 <- ((P99 / P99.9)^(-a + 1) - 10)^2
  term2 <- ((P99.5 / P99.9)^(-a + 1) - 5)^2
  term3 <- ((P99 / P99.5)^(-a + 1) - 2)^2

  return(term1 + term2 + term3)
}

test_result <- percentile_ratio_discrepancies(P99 = 1e6, P99.5 = 2e6, P99.9 = 1e7, a = 2)
test_result
```

```
## [1] 0
```

2. Write a function, `exponent.multi_ratios_est`, which takes as inputs `P99`, `P99.5`, `P99.9`, and estimates `a`. It should minimize your `percentile_ratio_discrepancies` function. The starting value for the minimization should come from (4). Check that when `P99=1e6`, `P99.5=2e6` and `P99.9=1e7`, your function returns an `a` of 2.

```
exponent.multi_ratios_est <- function(P99, P99.5, P99.9) {
  func <- function(a) {
    percentile_ratio_discrepancies(P99, P99.5, P99.9, a)
  }

  result <- optimize(func, interval = c(0.1, 10), maximum = FALSE)

  return(result$minimum)
}

test_a <- exponent.multi_ratios_est(P99 = 1e6, P99.5 = 2e6, P99.9 = 1e7)
test_a
```

```
## [1] 2.000014
```

3. Write a function which uses `exponent.multi_ratios_est` to estimate a for the US for every year from 1913 to 2012. (There are many ways you could do thi, including loops.) Plot the estimates; make sure the labels of the plot are appropriate.

```
wtid_data <- read.csv("./data/wtid-report.csv")

us_data <- wtid_data[wtid_data$Country == "United States", ]

years <- us_data$Year
P99 <- us_data$P99.income.threshold
P99.5 <- us_data$P99.5.income.threshold
P99.9 <- us_data$P99.9.income.threshold

head(us_data[, c("Year", "P99.income.threshold", "P99.5.income.threshold", "P99.9.income.threshold")])
```

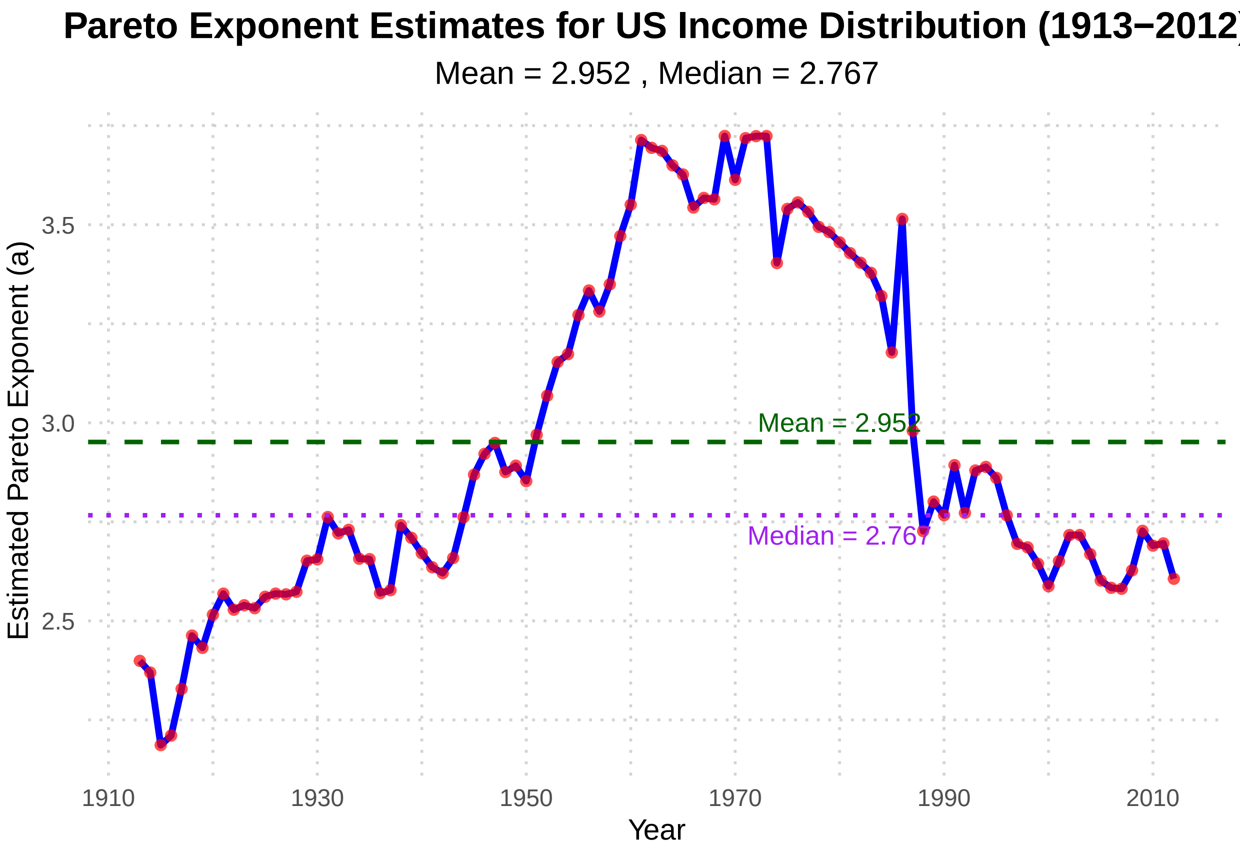
```
##   Year P99.income.threshold P99.5.income.threshold P99.9.income.threshold
## 1 1913             80087.90               131337.2               415206.4
## 2 1914             74012.72               122935.9               397671.6
## 3 1915             62392.24               118717.4               437522.8
## 4 1916             74869.18               133777.1               502094.2
## 5 1917             92341.21               149697.9               519558.7
## 6 1918             92221.06               143219.7               442731.1
```

```
estimate_pareto_exponents <- function(years, P99, P99.5, P99.9) {
  exponents <- numeric(length(years))

  for (i in 1:length(years)) {
    if (!is.na(P99[i]) && !is.na(P99.5[i]) && !is.na(P99.9[i])) {
      exponents[i] <- exponent.multi_ratios_est(P99[i], P99.5[i], P99.9[i])
    } else {
      exponents[i] <- NA
    }
  }

  return(exponents)
}
```

```r
# Estimate Pareto exponents for all years
pareto_exponents <- estimate_pareto_exponents(years, P99, P99.5, P99.9)

plot_data <- data.frame(
  Year = years,
  Pareto_Exponent = pareto_exponents
)

# Calculate summary statistics
mean_exponent <- mean(pareto_exponents, na.rm = TRUE)
median_exponent <- median(pareto_exponents, na.rm = TRUE)

# Create a comprehensive plot using ggplot2
ggplot(plot_data, aes(x = Year, y = Pareto_Exponent)) +
  geom_line(color = "blue", linewidth = 1.2) +
  geom_point(color = "red", size = 1.5, alpha = 0.7) +
  geom_hline(yintercept = mean_exponent, color = "darkgreen", linetype = "dashed", size = 0.8) +
  geom_hline(yintercept = median_exponent, color = "purple", linetype = "dotted", size = 0.8) +
  labs(
    title = "Pareto Exponent Estimates for US Income Distribution (1913-2012)",
    x = "Year",
    y = "Estimated Pareto Exponent (a)",
    subtitle = paste("Mean =", round(mean_exponent, 3), ", Median =", round(median_exponent, 3))
  ) +
  theme_minimal() +
  theme(
    plot.title = element_text(hjust = 0.5, size = 14, face = "bold"),
    plot.subtitle = element_text(hjust = 0.5, size = 12),
    panel.grid.major = element_line(color = "lightgray", linetype = "dotted"),
    panel.grid.minor = element_line(color = "lightgray", linetype = "dotted", size = 0.5)
  ) +
  scale_x_continuous(breaks = seq(1910, 2010, 20)) +
  annotate("text",
    x = 1980, y = mean_exponent + 0.05,
    label = paste("Mean =", round(mean_exponent, 3)),
    color = "darkgreen", size = 3.5
  ) +
  annotate("text",
    x = 1980, y = median_exponent - 0.05,
    label = paste("Median =", round(median_exponent, 3)),
    color = "purple", size = 3.5
  )
```

```
## Warning: Using `size` aesthetic for lines was deprecated in ggplot2 3.4.0.
## i Please use `linewidth` instead.
## This warning is displayed once every 8 hours.
## Call `lifecycle::last_lifecycle_warnings()` to see where this warning was
## generated.
```

```
## Warning: The `size` argument of `element_line()` is deprecated as of ggplot2 3.4.0.
## i Please use the `linewidth` argument instead.
## This warning is displayed once every 8 hours.
## Call `lifecycle::last_lifecycle_warnings()` to see where this warning was
## generated.
```

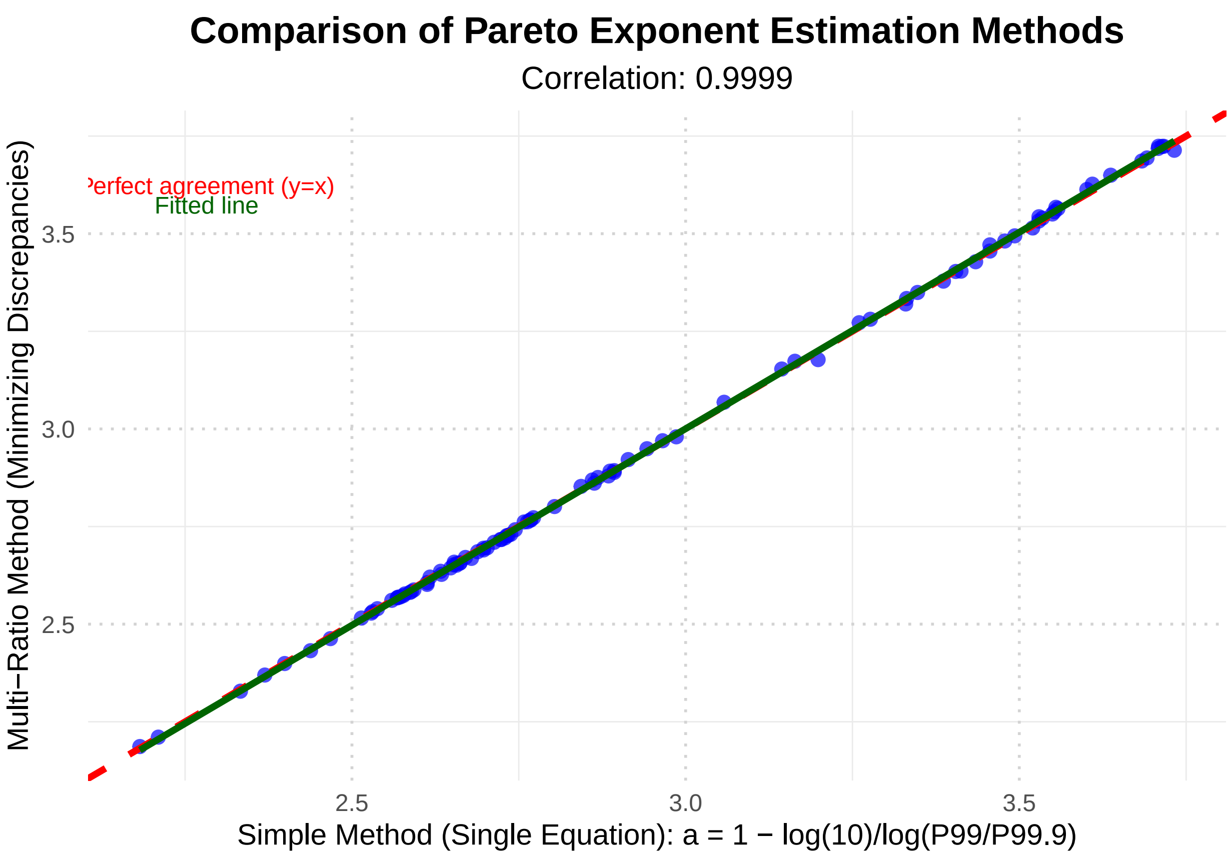


4. Use (4) to estimate a for the US for every year. Make a scatter-plot of these estimates against those from problem 3. If they are identical or completely independent, something is wrong with at least one part of your code. Otherwise, can you say anything about how the two estimates compare?

```r
# a = 1 - log(10) / log(P99/P99.9)
simple_pareto_estimate <- function(P99, P99.9) {
  return(1 - log(10) / log(P99 / P99.9))
}

simple_exponents <- numeric(length(years))
for (i in 1:length(years)) {
  if (!is.na(P99[i]) && !is.na(P99.9[i])) {
    simple_exponents[i] <- simple_pareto_estimate(P99[i], P99.9[i])
  } else {
    simple_exponents[i] <- NA
  }
}

# Create data frame for ggplot
comparison_data <- data.frame(
  Simple = simple_exponents,
  Multi_Ratio = pareto_exponents
)

# Calculate correlation and fit
correlation <- cor(simple_exponents, pareto_exponents, use = "complete.obs")
fit <- lm(pareto_exponents ~ simple_exponents)
```

```r
# Create the plot
ggplot(comparison_data, aes(x = Simple, y = Multi_Ratio)) +
  geom_point(color = "blue", size = 2, alpha = 0.7) +
  geom_abline(slope = 1, intercept = 0, color = "red", linetype = "dashed", size = 1.2) +
  geom_smooth(method = "lm", color = "darkgreen", size = 1.2, se = FALSE) +
  labs(
    title = "Comparison of Pareto Exponent Estimation Methods",
    x = "Simple Method (Single Equation): a = 1 - log(10)/log(P99/P99.9)",
    y = "Multi-Ratio Method (Minimizing Discrepancies)",
    subtitle = paste("Correlation:", round(correlation, 4))
  ) +
  theme_minimal() +
  theme(
    plot.title = element_text(hjust = 0.5, size = 14, face = "bold"),
    plot.subtitle = element_text(hjust = 0.5, size = 12),
    panel.grid.major = element_line(color = "lightgray", linetype = "dotted")
  ) +
  annotate("text",
    x = min(simple_exponents, na.rm = TRUE) + 0.1,
    y = max(pareto_exponents, na.rm = TRUE) - 0.1,
    label = "Perfect agreement (y=x)", color = "red", size = 3
  ) +
  annotate("text",
    x = min(simple_exponents, na.rm = TRUE) + 0.1,
    y = max(pareto_exponents, na.rm = TRUE) - 0.15,
    label = "Fitted line", color = "darkgreen", size = 3
  )
```

```
## `geom_smooth()` using formula = 'y ~ x'
```




```r
# Calculate correlation and other statistics
correlation <- cor(simple_exponents, pareto_exponents, use = "complete.obs")
print(paste("Correlation between methods:", round(correlation, 4)))
```

```
## [1] "Correlation between methods: 0.9999"
```

```r
# Print regression summary
print("Linear regression: Multi-ratio ~ Simple method")
```

```
## [1] "Linear regression: Multi-ratio ~ Simple method"
```

```r
print(summary(fit))
```

```
##
## Call:
## lm(formula = pareto_exponents ~ simple_exponents)
##
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0237225 -0.0038007 -0.0004056  0.0042395  0.0130011
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)      -0.018887   0.004637  -4.073 9.42e-05 ***
## simple_exponents  1.006435   0.001555 647.106  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.006568 on 98 degrees of freedom
```

```
## Multiple R-squared:  0.9998, Adjusted R-squared:  0.9998
## F-statistic: 4.187e+05 on 1 and 98 DF,  p-value: < 2.2e-16
```