# Assignment 5

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## 1 Quadratic Forms 2.35

Find the area enclosed by the parabola $y = x^2$ and the line $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 4$.

## 2 SOLUTION

**Lemma 2.1.** *The points of intersection of **Line** $L$ :* $\mathbf{x} = \mathbf{q} + \mu\mathbf{m}$ *with **parabola***

$$y = ax^2 + bx + c \tag{2.0.1}$$

*or*

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{2.0.2}$$

*where*

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} \frac{b}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.4}$$

$$f = c \tag{2.0.5}$$

*are given by:*

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{2.0.6}$$

*where,*

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{Vm}} \left( -\mathbf{m}^T \left( \mathbf{Vq} + \mathbf{u} \right) \right.$$
$$\left. \pm \sqrt{\left[ \mathbf{m}^T \left( \mathbf{Vq} + \mathbf{u} \right) \right]^2 - \left( \mathbf{q}^T\mathbf{Vq} + 2\mathbf{u}^T\mathbf{q} + f \right) \left( \mathbf{m}^T\mathbf{Vm} \right)} \right) \tag{2.0.7}$$

*Proof.* The points of intersection must satisfy the line and parabola equation. Thus,

$$(\mathbf{q} + \mu\mathbf{m})^\mathbf{T}\mathbf{V}(\mathbf{q} + \mu\mathbf{m}) + 2\mathbf{u^T}(\mathbf{q} + \mu\mathbf{m}) + f = 0 \tag{2.0.8}$$

On expansion, we get

$$\mu^2\mathbf{m^T Vm} + \mu \left[ \mathbf{m^T Vq} + \mathbf{q^T Vm} + 2\mathbf{u^T m} \right]$$
$$+ \mathbf{q^T Vq} + 2\mathbf{u^T m} + f = 0 \tag{2.0.9}$$

Since, $\mathbf{q^T Vm}, 2\mathbf{u^T m}$ are scalars

$$\mathbf{q^T Vm} = \mathbf{m^T V^T q} \tag{2.0.10}$$

$$2\mathbf{u^T m} = 2\mathbf{m^T u} \tag{2.0.11}$$

Solving the above quadratic equation we get

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{Vm}} \left( -\mathbf{m}^T \left( \mathbf{Vq} + \mathbf{u} \right) \right.$$
$$\left. \pm \sqrt{\left[ \mathbf{m}^T \left( \mathbf{Vq} + \mathbf{u} \right) \right]^2 - \left( \mathbf{q}^T\mathbf{Vq} + 2\mathbf{u}^T\mathbf{q} + f \right) \left( \mathbf{m}^T\mathbf{Vm} \right)} \right) \tag{2.0.12}$$

$\square$

The matrix parameters of the parabola are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}, f = 0 \tag{2.0.13}$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.0.14}$$

$$\mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.15}$$

The vertex of the parabola can be expressed as

$$\begin{pmatrix} \mathbf{u^T} + \eta\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -\mathbf{f} \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.16}$$

where,

$$\eta = \mathbf{u^T p_1} = \frac{-1}{2} \tag{2.0.17}$$

$$\implies \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.18}$$

or

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

From (2.0.7),

$$\mu_1 = 2, \mu_2 = -2 \tag{2.0.20}$$

The given line is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2 \tag{2.0.21}$$

In parametric form, the given line can be written as:

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{2.0.22}$$

$$\implies \mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.23}$$

Substituting $\mu_1$ and $\mu_2$ in (2.0.23),the points of intersection

$$\mathbf{K} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \tag{2.0.24}$$

1) Thus, from Fig. 4 the area enclosed by parabola and line can be given as

$$A = \text{Area under line} - \text{Area under parabola} \tag{2.0.25}$$

$$A = Ar(KLNMK) - Ar(KCLNCMK) \tag{2.0.26}$$

$$A = A_1 - A_2 \tag{2.0.27}$$

2) Area under the line y=4 i.e, $A_1$-

$$A_1 = \int_{-2}^{2} y dx \tag{2.0.28}$$

$$= \int_{-2}^{2} 4 dx \tag{2.0.29}$$

$$= 16 \text{ units} \tag{2.0.30}$$

3) Area under the parabola that is $A_2$-

$$A_2 = \int_{-2}^{2} y dx \tag{2.0.31}$$

$$= \int_{-2}^{2} x^2 dx \tag{2.0.32}$$

$$= \frac{16}{3} \text{ units} \tag{2.0.33}$$

4) Putting (2.0.30) and (2.0.33) in (2.0.27) we get required area A as:

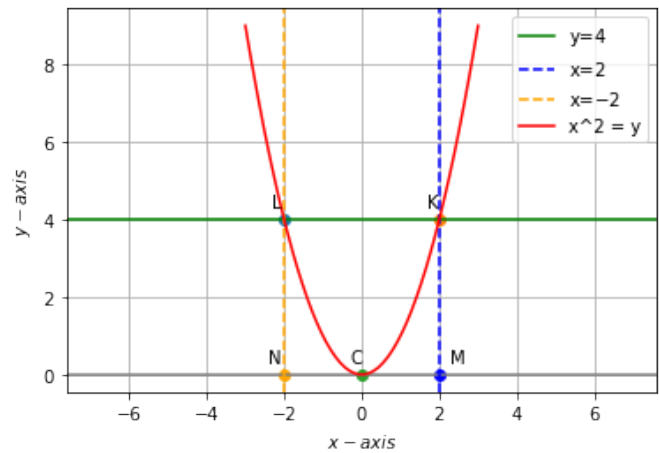$$A = A_1 - A_2 \tag{2.0.34}$$

$$A = \frac{32}{3} \text{ units} \tag{2.0.35}$$



Fig. 4: Plot of the parabola and line