Lecture 2: Eigenvalues and Eigenvectors

Eigenvalues, Eigenvectors, and Solvability

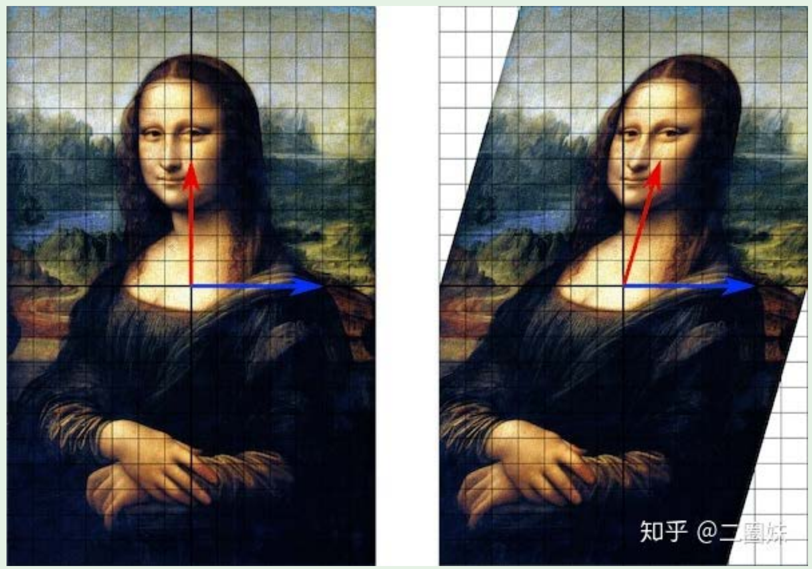
Another class of linear systems of equations which are of fundamental importance are known as **eigenvalue(特征值)** problems. Unlike the system $Ax = b$ which has the single unknown vector x, eigenvalue problems are of the form

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

which have the unknowns x and λ . The values of λ are known as the **eigenvalues** and the corresponding x are the **eigenvectors(特征向量).**

我们先来理解这个**为什么**叫特征值和特征向量： $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

矩阵A当然是一个变换，然后这个变换的特殊之处是当它作用在特征向量 x 上的时候，x 只发生了缩放变换，它的方向并没有改变，并没有旋转。



这幅图片在水平方向没有改变，$\begin{bmatrix}1\\0\end{bmatrix}$ 就是一个它的特征向量，对应的特征值是 $\lambda = 1$.

特征向量是经过了变换，这个向量可能会 scale， 但是仍旧保持其原有的方向，与特定的特征值对应。所以特征向量某种意义上展示了这个变换的‘特征’。

Eigenvalue problems often arise from **differential equations** (微分方程). Specifically, we consider the example of a linear set of **coupled differential equations** (线性耦合微分方程)

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}$$

By attempting a solution of the form(希望解的形式如下)

$$\mathbf{y} = \mathbf{x}\exp(\lambda t)$$

where all the time-dependence is captured in the exponent (指数), the resulting equation for *x* is

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

which is just the eigenvalue problem.

推导过程

$$\frac{dy}{dt} = Ay$$

$$y = xe^{\lambda t}$$

$$\frac{dy}{dt} = \frac{d\left(xe^{\lambda t}\right)}{dt} = xe^{\lambda t}\lambda = Ay = Axe^{\lambda t}$$

$$xe^{\lambda t}\lambda = Axe^{\lambda t}$$

$$Ax = \lambda x$$

Once the full set of **eigenvalues and eigenvectors** of this equation are found, the solution of the **differential equation** is written as

$$\vec{y} = c_1\mathbf{x}_1 \exp\left(\lambda_1 t\right) + c_2\mathbf{x}_2 \exp\left(\lambda_2 t\right) + \cdots + c_N\mathbf{x}_N \exp\left(\lambda_N t\right)$$

where N is the number of linearly independent solutions to the eigenvalue problem for the matrix A which is of size $N \times N$.

Thus solving a linear system of differential equations relies on the solution of an associated eigenvalue problem.

The questions remains: how are the eigenvalues and eigenvectors found? To consider this problem, we rewrite the eigenvalue problem as

$$\mathbf{Ax} = \lambda\mathbf{x} \quad \Rightarrow \quad \mathbf{Ax} = \lambda\mathbf{Ix}$$

where a multiplication by unity (单位矩阵) has been performed, i.e. ***Ix* = *x***.

Moving the right hand side to the left side of the equation gives

$$\mathbf{Ax} - \lambda\mathbf{Ix} = \mathbf{0}$$

Factoring out (提取) the vector *x* then gives the desired result

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

Two possibilities now exist.

Option I: The determinant (行列式) of the matrix $(\mathbf{A}-\lambda\mathbf{I})$ is not zero. If this is true, the matrix is nonsingular (非奇异的) and its inverse (逆矩阵), $(\mathbf{A}-\lambda\mathbf{I})^{-1}$, can be found. The solution to the eigenvalue problem is then

$$\mathbf{x}=(\mathbf{A}-\lambda\mathbf{I})^{-1}\mathbf{0}$$

which implies that

$$\mathrm{x}=0$$

This trivial (平凡的) solution could have been guessed. However, it is not relevant as we require nontrivial solutions for *x*.

Option II: The determinant of the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is zero.

If this is true, the matrix is singular (奇异) and its inverse, $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ cannot be found.

Although there is no longer a guarantee that there is a solution, it is the only scenario which allows for the possibility of $\mathrm{x} \neq 0$.

It is this condition which allows for the construction (构造) of eigenvalues and eigenvectors.

Indeed, we choose the eigenvalues so that this condition holds and the matrix is singular.

To illustrate how the eigenvalues and eigenvectors are computed, an example is shown. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$$

怎么计算特征值
和特征向量

This gives the eigenvalue problem

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} \mathbf{x} = \lambda \mathbf{x}$$

which when manipulated (操作) to the form $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$ gives

$$\left[\begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \mathrm{x} = \begin{pmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{pmatrix} \mathrm{x} = 0$$

We now require that the determinant(行列式) is zero

$$\det\begin{vmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda)+3 = \lambda^2-6\lambda+8 = (\lambda-2)(\lambda-4) = 0$$

which gives the two eigenvalues

$$\lambda = 2, 4$$

The eigenvectors are then found as follows:

$$\lambda = 2: \quad \begin{pmatrix} 1-2 & 3 \\ -1 & 5-2 \end{pmatrix} \mathrm{x} = \begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \mathrm{x} = 0$$

Given that $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$, this leads to the single equation

$$-x_1 + 3x_2 = 0$$

This is an underdetermined system of equations (不定方程). Thus we have freedom in choosing one of the values. Choosing $x_2 = 1$ gives $x_1 = 3$ and

$$\mathbf{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The second eigenvector comes as follows:

$$\lambda = 4: \quad \begin{pmatrix} 1-4 & 3 \\ -1 & 5-4 \end{pmatrix} \mathrm{x} = \begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix} \mathrm{x} = 0$$

Given that $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$, this leads to the single equation

$$-x_1 + x_2 = 0$$

This is an underdetermined system of equations. Thus we have freedom in choosing one of the values. Choosing $x_2 = 1$ gives $x_1 = 1$ and

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

These results can be found from MATLAB by using the ***eig*** command. Specifically, the command structure

$$[V \quad \mathrm{D}] = \mathrm{eig}(A)$$

gives the **matrix *V* containing the eigenvectors as columns** and the **matrix *D* whose diagonal elements (对角线) are the corresponding eigenvalues**.

Matlab代码

$$A = [\ 1 \quad 3; -1 \quad 5\ ]$$
$$[V \quad \mathrm{D}] = \mathrm{eig}(A)$$

$$\mathrm{A} = \begin{matrix} 1 & 3 \\ -1 & 5 \end{matrix}$$

$$\mathrm{V} = \begin{matrix} -0.9487 & -0.7071 \\ -0.3162 & -0.7071 \end{matrix}$$

$$D = \begin{matrix} 2 & 0 \\ 0 & 4 \end{matrix}$$

矩阵乘方

Another important operation which can be performed with eigenvalue and eigenvectors is the evaluation of

$$A^M$$

where M is a large integer. For large matrices A, this operation is computationally expensive.

However, knowing the eigenvalues and eigenvectors of A allows for a significant ease in computational expense. Assuming we have all the eigenvalues and eigenvectors of A, then

$$\begin{aligned}
\mathbf{A}\mathbf{x}_1 &= \lambda_1 \mathbf{x}_1 \\
\mathbf{A}\mathbf{x}_2 &= \lambda_2 \mathbf{x}_2 \\
&\vdots \\
\mathbf{A}\mathbf{x}_n &= \lambda_n x_n
\end{aligned}$$

如果矩阵很大，运算成本很高，怎么简化？？？

This collection of eigenvalues and eigenvectors gives the matrix system

$$AS = S\Lambda$$

where the columns of the matrix S are the eigenvectors of A,

$$\mathbf{S} = \begin{pmatrix} \mathrm{x}_1 & \mathrm{x}_2 & \cdots & \mathrm{x}_n \end{pmatrix}$$

and Λ is a matrix whose diagonals are the corresponding eigenvalues

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & 0 & \lambda_n \end{pmatrix}$$

By multiplying on the right by S^{-1}, the matrix A can then be rewritten as

$$\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^{-1}$$

The final observation comes from

$$\mathbf{A}^2 = \left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right)\left(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right) = \mathbf{S}\mathbf{\Lambda}^2\mathbf{S}^{-1}$$

This then generalizes to

$$\mathbf{A}^M = \mathbf{S}\mathbf{\Lambda}^M\mathbf{S}^{-1}$$

where the matrix Λ^M is easily calculated as

$$\Lambda^M = \begin{pmatrix} \lambda_1^M & 0 & \cdots & & 0 \\ 0 & \lambda_2^M & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & 0 & \lambda_n^M \end{pmatrix}$$

Since raising the diagonal terms to the M^{th} power is easily accomplished, the matrix A can then be easily calculated by multiplying the three matrices.

同样的，如果这个方程变成了微分方程，又该怎么办？？？
（后面会陆续讲解）

一阶微分方程

$$\begin{cases} \frac{dy(t)}{dt} = f(y(t), t) \\ y(0) = y_0 \end{cases}$$

二阶微分方程

$$\begin{cases} \frac{d^2y(t)}{dt^2} = f\left(y(t), t, y'(t)\right) \\ y(0) = y_0 \\ \frac{dy(0)}{dt} = v_0 \end{cases}$$