

美国数学月刊征解问题

(American Mathematical Monthly)

AMM2748 设 $f(x) = x^n \ln x$, 求极限 $\lim_{n \rightarrow \infty} \frac{1}{n!} f^{(n)}\left(\frac{1}{n}\right)$.

解法一 $f'(x) = nx^{n-1} \left(\ln x + \frac{1}{n}\right)$,

$$\begin{aligned}f''(x) &= \left[nx^{n-1} \left(\ln x + \frac{1}{n}\right) \right]' = n(n-1)x^{n-2} \left(\ln x + \frac{1}{n}\right) + nx^{n-1} \cdot \frac{1}{x} \\&= n(n-1)x^{n-2} \left(\ln x + \frac{1}{n} + \frac{1}{n-1}\right),\end{aligned}$$

对 $k = 1, 2, \dots, n$, 归纳出

$$f^{(k)}(x) = n(n-1)\cdots(n-k+1)x^{n-k} \left(\ln x + \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-k+1}\right).$$

特别地 $f^{(n)}(x) = n! \left(\ln x + \frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right)$. (记 $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$)

$$\frac{1}{n!} f^{(n)}\left(\frac{1}{n}\right) = H_n - \ln n$$

所以 $\lim_{n \rightarrow \infty} \frac{1}{n!} f^{(n)}\left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} (H_n - \ln n) = \gamma \approx 0.577$. (γ 为欧拉-马歇罗尼常数)

解法二 利用莱布尼兹法则有

$$\begin{aligned}f^{(n)}(x) &= \sum_{k=0}^n C_n^k (\ln x)^{(k)} (x^n)^{(n-k)} \\&= n! \ln x + \sum_{k=1}^n C_n^k (-1)^{k-1} (k-1)! \frac{1}{x^k} \cdot n(n-1)\cdots(k+1)x^k \\&= n! \ln x + n! \sum_{k=1}^n C_n^k (-1)^{k-1} \frac{1}{k}.\end{aligned}$$

下证等式 $\sum_{k=1}^n C_n^k (-1)^{k-1} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} (= H_n)$. (用一点积分知识)

$$\begin{aligned}\sum_{k=1}^n C_n^k (-1)^{k-1} \frac{1}{k} &= \sum_{k=1}^n C_n^k (-1)^{k-1} \int_0^1 x^{k-1} dx = \int_0^1 \sum_{k=1}^n C_n^k (-x)^{k-1} dx \\&= \int_0^1 \frac{1 - (1-x)^n}{x} dx \stackrel{x=1-t}{=} \int_0^1 (1+t+\cdots+t^{n-1}) dt = 1 + \frac{1}{2} + \cdots + \frac{1}{n},\end{aligned}$$

所以 $f^{(n)}\left(\frac{1}{n}\right) = n!(H_n - \ln n)$, $\lim_{n \rightarrow \infty} \frac{1}{n!} f^{(n)}\left(\frac{1}{n}\right) = \gamma$.