

# 美国数学月刊征解问题

(American Mathematical Monthly)

AMM2748 设  $f(x) = x^n \ln x$ , 求极限  $\lim_{n \rightarrow \infty} \frac{1}{n!} f^{(n)}\left(\frac{1}{n}\right)$ .

解法一  $f'(x) = nx^{n-1}(\ln x + \frac{1}{n})$ ,

$$\begin{aligned} f''(x) &= \left[ nx^{n-1}(\ln x + \frac{1}{n}) \right]' = n(n-1)x^{n-2}(\ln x + \frac{1}{n}) + nx^{n-1} \cdot \frac{1}{x} \\ &= n(n-1)x^{n-2}(\ln x + \frac{1}{n} + \frac{1}{n-1}), \end{aligned}$$

对  $k=1, 2, \dots, n$ , 归纳出

$$f^{(k)}(x) = n(n-1)\cdots(n-k+1)x^{n-k}(\ln x + \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-k+1}).$$

特别地  $f^{(n)}(x) = n!(\ln x + \frac{1}{n} + \frac{1}{n-1} + \cdots + 1)$ . (记  $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ )

$$\frac{1}{n!} f^{(n)}\left(\frac{1}{n}\right) = H_n - \ln n$$

所以  $\lim_{n \rightarrow \infty} \frac{1}{n!} f^{(n)}\left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} (H_n - \ln n) = \gamma \approx 0.577$ . ( $\gamma$  为欧拉-马歇罗尼常数)

解法二 利用莱布尼兹法则有

$$\begin{aligned} f^{(n)}(x) &= \sum_{k=0}^n C_n^k (\ln x)^{(k)} (x^n)^{(n-k)} \\ &= n! \ln x + \sum_{k=1}^n C_n^k (-1)^{k-1} (k-1)! \frac{1}{x^k} \cdot n(n-1) \cdots (k+1) x^k \\ &= n! \ln x + n! \sum_{k=1}^n C_n^k (-1)^{k-1} \frac{1}{k}. \end{aligned}$$

下证等式  $\sum_{k=1}^n C_n^k (-1)^{k-1} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} (= H_n)$ . (用一点积分知识)

$$\begin{aligned} \sum_{k=1}^n C_n^k (-1)^{k-1} \frac{1}{k} &= \sum_{k=1}^n C_n^k (-1)^{k-1} \int_0^1 x^{k-1} dx = \int_0^1 \sum_{k=1}^n C_n^k (-x)^{k-1} dx \\ &= \int_0^1 \frac{1 - (1-x)^n}{x} dx = \int_0^1 (1+t+\cdots+t^{n-1}) dt = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \end{aligned}$$

所以  $f^{(n)}\left(\frac{1}{n}\right) = n!(H_n - \ln n)$ ,  $\lim_{n \rightarrow \infty} \frac{1}{n!} f^{(n)}\left(\frac{1}{n}\right) = \gamma$ .