高阶微分

一、一阶微分形式不变性

设 $y=f(u),u=u(x)$ 可微，则 $y=f(u(x))$ 也可微，且

$$\mathrm{d}y=\left\{f(u(x))\right\}'\mathrm{d}x=f'(u(x))u'(x)\mathrm{d}x .$$

注意 $\mathrm{d}u=\mathrm{d}u(x)=u'(x)\mathrm{d}x$，所以 $\mathrm{d}y=f'(u(x))\mathrm{d}u(x)$，即

$$\boxed{df(u)=f'(u)du}$$

上式在 u 为自变量时自然成立，这表明一阶微分具有形式不变性.

二、高阶微分

设 $y=f(u),u=u(x)$ 二次可微，则 $y=f(u(x))$ 也二次可微，且

$$\mathrm{d}y=f'(u(x))u'(x)\mathrm{d}x ,$$

而二次微分为

$$\begin{aligned}\mathrm{d}^2y&=\mathrm{d}\left\{f'(u(x))u'(x)\mathrm{d}x\right\}=\left\{f'(u(x))u'(x)\right\}'(\mathrm{d}x)^2\\&=\left\{f''(u(x))(u'(x))^2+f'(u(x))u''(x)\right\}(\mathrm{d}x)^2\\&=f''(u(x))(u'(x)\mathrm{d}x)^2+f'(u(x))u''(x)(\mathrm{d}x)^2\end{aligned}$$

即 $$\mathrm{d}^2f(u)=f''(u)(\mathrm{d}u)^2+f'(u)\mathrm{d}^2u , \qquad (*)$$

这与 u 为自变量时，

$$\mathrm{d}^2f(u)=f''(u)(\mathrm{d}u)^2 ,$$

差了一项 $f'(u)\mathrm{d}^2u$，也就表明高阶微分不再具有形式不变性.

三、参数方程高阶导数计算的一个典型错误

设 $x=\varphi(t),y=\psi(t)$ 二次可微，且 $\varphi'(t)\neq 0$，则

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}\psi(t)}{\mathrm{d}\varphi(t)}=\frac{\psi'(t)\mathrm{d}t}{\varphi'(t)\mathrm{d}t}=\frac{\psi'(t)}{\varphi'(t)}.$$

这里

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi'(t)}{\varphi'(t)}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi'(t)}{\varphi'(t)}\right)\cdot\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$=\frac{\psi''(t)\varphi'(t)-\psi'(t)\varphi''(t)}{(\varphi'(t))^3}.$$

以上计算已经不能再简化了！

下面计算二阶导数的过程是错误的：

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\frac{\mathrm{d}^2\psi(t)}{(\mathrm{d}\varphi(t))^2}=\frac{\psi''(t)(\mathrm{d}t)^2}{(\varphi'(t))^2(\mathrm{d}t)^2}=\frac{\psi''(t)}{(\varphi'(t))^2},$$

你知道错误的原因吗？

错误原因是：t 不是自变量（事实上 t 为自变量 x 的函数 $t=\varphi^{-1}(x)$），却错用了 $\mathrm{d}^2\psi(t)=\psi''(t)(\mathrm{d}t)^2$. 要订正上面的错误，则需花费精力正确计算二阶微分 $\mathrm{d}^2\psi(t)$，利用(*)式有

$$\mathrm{d}^2\psi(t)=\psi''(t)(\mathrm{d}t)^2+\psi'(t)\mathrm{d}^2 t,$$

注意 $\mathrm{d}t=\dfrac{1}{\varphi'(t)}\mathrm{d}x$，二阶微分 $\mathrm{d}^2 t$ 也需如下计算，

$$\mathrm{d}^2 t=\mathrm{d}\left(\frac{1}{\varphi'(t)}\mathrm{d}x\right)=\left(\mathrm{d}\frac{1}{\varphi'(t)}\right)\cdot\mathrm{d}x=\left(\frac{-\varphi''(t)}{(\varphi'(t))^2}\mathrm{d}t\right)\cdot\varphi'(t)\mathrm{d}t=\frac{-\varphi''(t)}{\varphi'(t)}(\mathrm{d}t)^2,$$

故 $\mathrm{d}^2\psi(t)=\psi''(t)(\mathrm{d}t)^2+\psi'(t)\cdot\dfrac{-\varphi''(t)}{\varphi'(t)}(\mathrm{d}t)^2$.

综上有

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\frac{\mathrm{d}^2\psi(t)}{(\mathrm{d}\varphi(t))^2}=\frac{\psi''(t)-\psi'(t)\dfrac{\varphi''(t)}{\varphi'(t)}}{(\varphi'(t))^2}=\frac{\psi''(t)\varphi'(t)-\psi'(t)\varphi''(t)}{(\varphi'(t))^3}.$$