$$z^{-1} = e^{-Ts}$$

$$E(z) = Z[e^*(t)] = E^*(s)\Big|_{z=e^{Ts}} = \sum_{n=0}^{\infty} e(kT)\cdot z^{-k}$$

7.4.2 Methods of z-Transform

- **By the definition.**
- **Partial fraction expansion.** Laplace $\xrightarrow{e}$ Z

7.4.3 Properties of z-Transform

1. linear property $$Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$$

2. Real shifting theorem 实位移定理

① Lag 延时定理 $$Z\left[e(t-nT)\right]=z^{-n}E(z)$$

Proof.
$$\begin{aligned}\text{LHS} &= \sum_{K=0}^{\infty} e(kT-nT)\cdot z^{-k} \\ &\quad j=k-n \\ &= \sum_{j=-n}^{\infty} e(jT)\cdot z^{-(j+n)} = z^{-n}\sum_{j=0}^{\infty} e(jT)\cdot z^{-j} \\ &= z^{-n}E(z)=\text{RHS}\end{aligned}$$

Left Hand Side (LHS); Right Hand Side (RHS)

2. Real shifting theorem 实位移定理

② Lead 超前定理

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

Proof.

$$\text{LHS}=\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-k}=z^{n}\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-(k+n)}$$

$$\Big\downarrow\; j=k+n$$

$$=z^{n}\sum_{j=n}^{\infty}e(jT)\cdot z^{-j}=z^{n}\left[\sum_{j=0}^{\infty}e(jT)\cdot z^{-j}-\sum_{j=0}^{n-1}e(jT)\cdot z^{-j}\right]$$

$$=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]=\text{RHS}$$

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

Example 5 $e(t)=t-T$

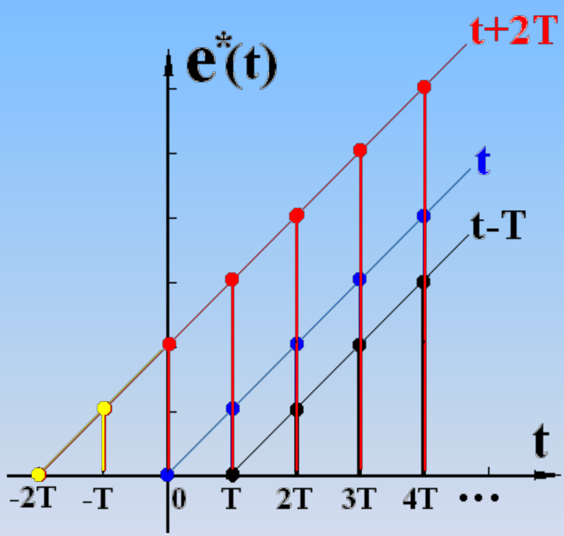
$$E(z)=Z\left[t-T\right]=z^{-1}Z[t]=z^{-1}\frac{Tz}{(z-1)^{2}}=\frac{T}{(z-1)^{2}}$$

Example 6 $e(t)=t+2T$

$$E(z)=Z\left[t+2T\right]$$

$$=z^{2}\left\{Z\left[t\right]-\sum_{k=0}^{1}kT\cdot z^{-k}\right\}$$

$$=z^{2}\left[\frac{Tz}{(z-1)^{2}}-0-Tz^{-1}\right]$$



3. Complex shifting theorem

复位移定理

$$Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)$$

Proof.

$$\mathrm{LHS}=\sum_{k=0}^{\infty}e(kT)\cdot e^{\mp akT}z^{-k}\quad=\sum_{k=0}^{\infty}e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k}$$

$$\Big\downarrow\ z_1=z\cdot e^{\pm aT}$$

$$=\sum_{k=0}^{\infty}e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k}=E(z_1)=E\left(z\cdot e^{\pm akT}\right)=\mathrm{RHS}$$

Example 7 $e(t)=t\cdot e^{-at}$

$$E(z_1)=Z\left[t\right]_{z_1=z\cdot e^{aT}}=\frac{Tz_1}{(z_1-1)^2}=\frac{T(z\cdot e^{aT})}{(z\cdot e^{aT}-1)^2}=\frac{Tz\cdot e^{-aT}}{(z-e^{-aT})^2}$$

4. Initial-value Theorem

$$\lim_{n\to 0} e(nT) = \lim_{z\to\infty} E(z)$$

Proof:

$$E(z) = \sum_{n=0}^{\infty} e(nT)\cdot z^{-n}$$

$$= \left[e(0) + e(1)\cdot z^{-1} + e(2)\cdot z^{-2} + e(3)\cdot z^{-3} + \cdots\right]$$

$$\lim_{z\to\infty} E(z) = e(0)$$

Example 8 $E(z) = \dfrac{0.792\cdot z^2}{(z-1)[z^2 - 0.416z + 0.208]}$

$$e(0) = \lim_{z\to\infty} E(z) = 0$$

Properties of z-Transform

1. linear property $\quad Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$

2. Real shifting theorem

$$\begin{cases} \text{Lag} \quad Z\left[e(t-nT)\right]=z^{-n}E(z) \\ \text{Lead} \quad Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right] \end{cases}$$

3. Complex shifting theorem $\quad Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)$

4. Initial-value theorem $\quad \lim_{n\to 0}e(nT)=\lim_{z\to\infty}E(z)$

5. Final-value theorem $\quad \lim_{n\to\infty}e(nT)=\lim_{z\to 1}(z-1)\cdot E(z)$

6. Convolution theorem $\quad c^*(t)=e^*(t)*g^*(t) \quad \Rightarrow \quad C(z)=E(z)\cdot G(z)$

7.4.4 Inverse z-Transform

$$Z^{-1}[X(z)] = x(nT)$$

Tips:

Inverse Z-transform can only provide discrete-time signal $x^*(t)$, instead of continuous signal x(t)。

- Long Division（长除法）
- Partial-Fraction expansion — Expansion of $\dfrac{E(z)}{z}$
- Residue（留数法） — $e(nT) = \sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right]$

1. Long Division（长除法）

$$E(z)=\frac{b_0 z^m+b_1 z^{m-1}+\cdots+b_m}{a_0 z^n+a_1 z^{n-1}+\cdots+a_n}$$

Numerator is divided by denominator ,we get

$$E(z)=c_0+c_1 z^{-1}+\cdots+c_k z^{-k}+\cdots=\sum_{k=0}^{\infty} c_k z^{-k}=\sum_{k=0}^{\infty} e(kT) z^{-k}$$

$$e^*(t)=c_0\delta(t)+c_1\delta(t-T)+\cdots+c_k\delta(t-kT)+\cdots$$

Example: $E(z)=\dfrac{10z}{(z-1)(z-2)}$, obtain e*(t)。

Solution: $E(z)=\dfrac{10z}{z^2-3z+2}=10z^{-1}+30z^{-2}+70z^{-3}+150z^{-4}+\cdots$

$$
\begin{array}{r|l}
 & 10z^{-1}+30z^{-2}+70z^{-3}+150z^{-4}\cdots \\
\hline
z^2-3z+2 & 10z \\
 & \underline{10z-30z^0+20z^{-1}} \\
 & \quad 30z^0-20z^{-1} \\
 & \quad \underline{30z^0-90z^{-1}+60z^{-2}} \\
 & \qquad 70z^{-1}-60z^{-2} \\
 & \qquad \underline{70z^{-1}-210z^{-2}+140z^{-3}} \\
 & \qquad\quad 150z^{-2}-140z^{-3} \\
 & \qquad\quad \cdots
\end{array}
$$

$$
\begin{aligned}
e^*(t) &= 10\delta(t-T) \\
&+30\delta(t-2T) \\
&+70\delta(t-3T) \\
&+150\delta(t-4T) \\
&+\cdots
\end{aligned}
$$

Example

$$F(z)=\frac{z}{(z-2)(z-3)}$$ ，**obtain f*(t)**。

Solution：

Because $F(z)=\frac{z}{z^2-5z+6}=\frac{z^{-1}}{1-5z^{-1}+6z^{-2}}$

By long-division, we get that

$$F(z)=z^{-1}+5z^{-2}+19z^{-3}+65z^{-4}+\cdots$$

Thus

$$f(0)=0,\ f(T)=1,\ f(2T)=5,\ f(3T)=19,\ f(4T)=65,\cdots$$

Then

$$f^*(t)=\delta(t-T)+5\delta(t-2T)+19\delta(t-3T)+65\delta(t-4T)+\cdots$$

2. Partial fraction expansion

Note: here, we expand $\frac{X(z)}{z}$, instead of X(z)。

$$\frac{X(z)}{z} = \sum_{i=1}^{n} \frac{A_i}{z - z_i}$$

Consider

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}$$

Then

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{a_0\prod_{i=1}^{n}(z-z_i)}$$

If there is no repeated root for the denominator， it generates

$$X(z)=z\left(\frac{A_1}{z-z_1}+\frac{A_2}{z-z_2}+\cdots+\frac{A_n}{z-z_n}\right)$$

其中系数A_i，可由式决定：

$$A_i=\left[(z-z_i)\frac{X(z)}{z}\right]\Bigg|_{z=z_i}$$

Example Consider

$$F(z) = \frac{z}{(z-1)(z-e^{-T})}$$

Obtain f*(t)。

Solution:

$$\frac{F(z)}{z} = \frac{K_1}{z-1} + \frac{K_2}{z-e^{-T}}$$

$$K_1 = \lim_{z\to 1}\left(\frac{z-1}{z}\right)F(z) = \frac{1}{1-e^{-T}}$$

$$K_2 = \lim_{z\to e^{-T}}\left(\frac{z-e^{-T}}{z}\right)F(z) = -\frac{1}{1-e^{-T}}$$

$$F(z) = \frac{1}{1-e^{-T}}\left(\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right)$$

$$f(nT) = \frac{1}{1-e^{-T}}\left(1-e^{-nT}\right)$$

$$f^*(t) = \frac{1}{1-e^{-T}}\sum_{k=0}^{+\infty}(1-e^{-kT})\delta(t-kT)$$

Example 11 $E(z)=\dfrac{z^2}{(z-0.8)(z-0.1)}$ **Obtain e*(t). (PFE)**

PFE: $\dfrac{E(z)}{z}=\dfrac{z}{(z-0.8)(z-0.1)}=\dfrac{C_1}{(z-0.8)}+\dfrac{C_2}{(z-0.1)}$

$$C_1=\lim_{z\to 0.8}\frac{z}{(z-0.1)}=\frac{8}{7} \qquad C_2=\lim_{z\to 0.1}\frac{z}{(z-0.8)}=\frac{-1}{7}$$

$$=\frac{8/7}{(z-0.8)}-\frac{1/7}{(z-0.1)}$$

$$E(z)=\frac{8}{7}\cdot\frac{z}{(z-0.8)}-\frac{1}{7}\cdot\frac{z}{(z-0.1)}$$

$$e(t)=(8\times 0.8^{\frac{t}{T}}-0.1^{\frac{t}{T}})/7 \qquad e(nT)=(8\times 0.8^n-0.1^n)/7$$

$$e^*(t)=\sum_{n=0}^{\infty}\left[(8\times 0.8^n-0.1^n)/7\right]\cdot\delta(t-nT)$$

3、 Residue(留数法)

$$F(z)=\sum_{k=0}^{+\infty} f(kT)z^{-k}$$

$$F(z)z^{m-1}=\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}$$

Γ Encircle all the poles of $F(z)z^{k-1}$

$$\oint_{\Gamma} F(z)z^{m-1}dz=\oint_{\Gamma}\left[\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}\right]dz$$

$$\oint_{\Gamma} F(z)z^{m-1}dz=\sum_{k=0}^{+\infty} f(kT)\oint_{\Gamma} z^{m-k-1}dz$$

When m=k,

$$f(kT)=\sum_{i=1}^{n} res[F(z)z^{k-1},z_i]$$

$z_i, i=1,2,\cdots,n$ **are all the poles of** $F(z)z^{k-1}$

$$Res\left[z^{(k-1)}x(z)\right]=\lim_{z\to z_i}\frac{1}{(r-1)!}\frac{d^{r-1}}{dz^{r-1}}\left[(z-z_i)^r z^{k-1}x(z)\right]$$

其中***Res*[]**表示函数的留数, ***r***为极点的阶数。

<u>**Example**</u> For

$$F(z)=\frac{10z}{(z-1)(z-2)}$$

Obtain its inverse z-transform by residue Method.

Solution: $$F(z)z^{k-1} = \frac{10z^k}{(z-1)(z-2)}$$

Poles $z_1 = 1$ and $z_2 = 2$, and

$$res[F(z)z^{k-1},1] = \lim_{z\to 1}(z-1)F(z)z^{k-1} = -10$$

$$res[F(z)z^{k-1},2] = \lim_{z\to 2}(z-2)F(z)z^{k-1} = 10\cdot 2^k$$

Then $$f(kT) = 10(2^k - 1) \qquad (k = 0,1,2,\cdots)$$

Example 12 $E(z)=\dfrac{5}{(z-a)^2}$ **Obtain e*(t). (Residue)**

Solution.

$$e(nT)=\sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right]=\mathop{\mathrm{Res}}_{z=a}\left[\frac{5}{(z-a)^2}\cdot z^{n-1}\right]$$

$$e(nT)=\frac{1}{(2-1)!}\lim_{z\to a}\frac{d}{dz}\left[(z-a)^2\frac{5\cdot z^{n-1}}{(z-a)^2}\right]$$

$$=\lim_{z\to a}\frac{d}{dz}\left[5\cdot z^{n-1}\right]$$

$$=5\cdot\lim_{z\to a}\left[(n-1)\cdot z^{n-2}\right]$$

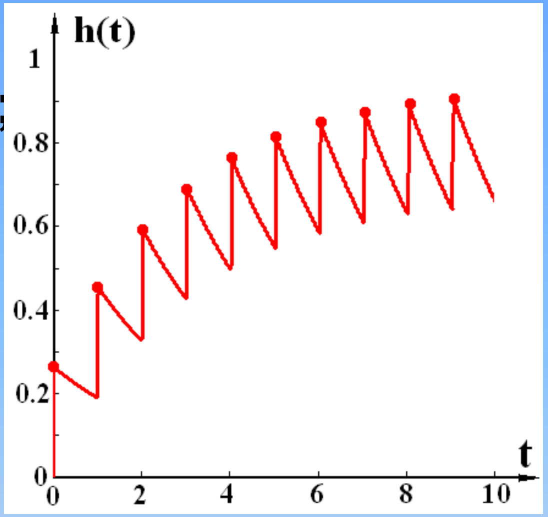
$$=5\cdot(n-1)\cdot a^{n-2}$$

$$e^*(t)=\sum_{n=0}^{\infty}\left(5(n-1)\cdot a^{n-2}\right)\cdot\delta(t-nT)$$

7.4.5 Limitations of z-Transform

(1) only shows the information of samples;

(2) In some cases, the continuous signal may jump on the sampling point.



Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

7.6 Performance Analysis of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

7.5 Mathematical Models of Discrete-Time Systems

- **Difference Equation** 差分方程.
- **Impulse Transfer function** 脉冲传递函数.

7.5.1 Linear Time-Invariant Difference Equations

线性时不变.

(1) Definition of difference e(kT) = e(k)

$$\lim_{T\to 0}\frac{\Delta e(k)}{T}=\frac{\mathrm{d}e(t)}{\mathrm{dt}}$$

差分替代微分.

Forward difference

First-order $\Delta e(k)=e(k+1)-e(k)$

Second-order $\Delta^2 e(k)=\Delta e(k+1)-\Delta e(k)$

$$=e(k+2)-2e(k+1)+e(k)$$

⋮

nth-order $\Delta^n e(k)=\Delta^{n-1}e(k+1)-\Delta^{n-1}e(k)$

Backward difference

First-order $\nabla e(k) = e(k) - e(k-1)$

$$\lim_{T \to 0} \frac{\nabla e(k)}{T} = \frac{\mathrm{d}e(t)}{\mathrm{dt}}$$

Second-order $\nabla^2 e(k) = \nabla e(k) - \nabla e(k-1)$

$= e(k) - 2e(k-1) + e(k-2)$

⋮

nth-order $\nabla^n e(k) = \nabla^{n-1} e(k) - \nabla^{n-1} e(k-1)$

(2) Difference equation

The equation of the input, output and their higher order differences.

The (forward) difference equation of nth-order linear time-invariant discrete system.

=n

$$c(k+n)+a_1c(k+n-1)+a_2c(k+n-2)+\cdots+a_{n-1}c(k+1)+a_nc(k)$$

输出

$$=b_0r(k+m)+b_1r(k+m-1)+\cdots+b_{m-1}r(k+1)+b_mr(k)$$

输入

The (backward) differential equation of n-order linear time-invariant discrete system.

$$c(k)+a_1c(k-1)+a_2c(k-2)+\cdots+a_{n-1}c(k-n+1)+a_nc(k-n)$$

$$=b_0r(k-n+m)+b_1r(k-n+m-1)+$$

$$\cdots+b_{m-1}r(k-n+1)+b_mr(k-n)$$

(3) To solve difference equations:

- Iteration method
- Z-transform method

Example 1 The differential equation of a continuous system is:

$$\begin{cases} \ddot{e}(t) - 4\dot{e}(t) + 3e(t) = r(t) = 1(t) \\ e(t) = 0 \qquad (t \le 0) \end{cases}$$

Obtain the corresponding forward difference equation and its solution.

Solution.

$$\dot{e}(t) \approx \frac{\Delta e(k)}{T} = \frac{e(k+1) - e(k)}{T} \overset{T=1}{=} e(k+1) - e(k)$$

$$\ddot{e}(t) \approx \frac{\Delta^2 e(k)}{T^2} = \frac{\Delta e(k+1)/T - \Delta e(k)/T}{T} \overset{T=1}{=} e(k+2) - 2e(k+1) + e(k)$$

$$\begin{array}{rr} & e(k+2) - 2e(k+1) + e(k) \\ -4\,[& e(k+1) - e(k)\,] \\ +3\,[& e(k)\,] \\ \hline & e(k+2) - 6e(k+1) + 8e(k) = 1(k) \end{array}$$

$$\begin{cases} e(k+2) - 6e(k+1) + 8e(k) = 1(k) \\ e(k) = 0 \qquad (k \le 0) \end{cases}$$

Solution I of the difference equation —— Iteration method

$$\begin{cases} e(k+2)-6e(k+1)+8e(k)=1(k) \\ e(k)=0 \qquad (k\le 0) \end{cases}$$

Solution $\quad e(k+2)=6e(k+1)-8e(k)+1(k)$

$$k=-1:\quad e(1)=6e(0)-8e(-1)+1(-1)=0$$

$$k=0:\quad e(2)=6e(1)-8e(0)+1(0)=0-0+1=1$$

$$k=1:\quad e(3)=6e(2)-8e(1)+1(1)=6-0+1=7$$

$$k=2:\quad e(4)=6e(3)-8e(2)+1(2)=6\times 7-8\times 1+1=35$$

$$\vdots \qquad \vdots$$

$$e^{*}(t)=\delta(t-2)+7\delta(t-3)+35\delta(t-4)+\cdots$$

$$\begin{cases} \text{Lag} \quad Z\left[e(t-nT)\right] = z^{-n}E(z) \\ \text{Lead} \quad Z\left[e(t+nT)\right] = z^{n}\left[E(z) - \sum_{k=0}^{n-1} e(kT)\cdot z^{-k}\right] \end{cases}$$

Solution II of difference equation — Z-transform method

$$e(k+2) - 6e(k+1) + 8e(k) = 1(k)$$

$$\begin{cases} e(k+2) - 6e(k+1) + 8e(k) = 1(k) \\ e(k) = 0 \qquad (k \le 0) \end{cases}$$

$$Z: \quad \begin{aligned} & z^2\left[E(z) - \underline{e(0)z^0 - e(1)z^{-1}}\right] \\ & -6\cdot z\left[E(z) - \underline{e(0)z^0}\right] \\ & +\ 8\left[E(z)\right] \end{aligned}$$

后项差分更好

$$(z^2 - 6z + 8)E(z) = Z[1(k)] = \frac{z}{z-1} \qquad E(z) = \frac{z}{(z-1)(z-2)(z-4)}$$

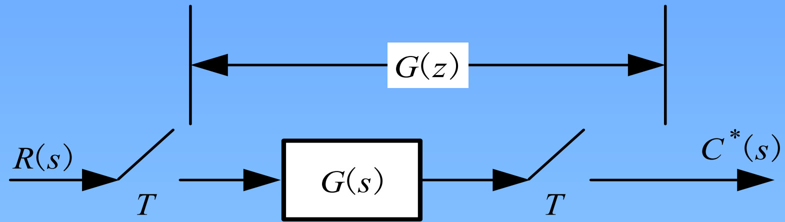
$$Z^{-1}: e(n) = \sum \text{Res}\left[\underline{E(z)\cdot z^{n-1}}\right]$$

$$= \lim_{z\to1}\frac{z\cdot z^{n-1}}{(z-2)(z-4)} + \lim_{z\to2}\frac{z\cdot z^{n-1}}{(z-1)(z-4)} + \lim_{z\to4}\frac{z\cdot z^{n-1}}{(z-1)(z-2)} = \frac{1}{3} - \frac{2^n}{2} + \frac{4^n}{6}$$

$$e^*(t) = \sum_{n=0}^{\infty} \underline{e(nT)\cdot\delta(t-nT)} = \sum_{n=0}^{\infty}\left(\frac{1}{3} - \frac{2^n}{2} + \frac{4^n}{6}\right)\cdot\delta(t-nT)$$

连续→离散

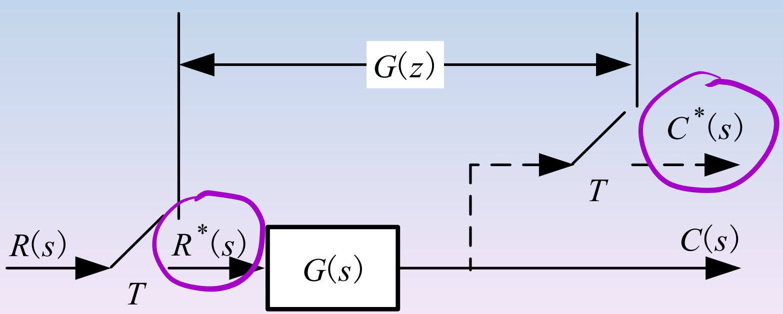
7.5.2 Mathematical Models in Complex Domain — Impulse Transfer Function（脉冲传递函数）

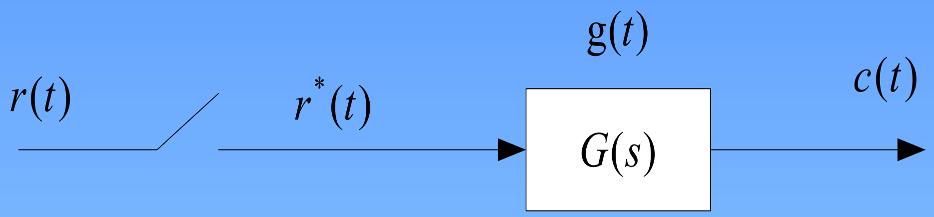


1. Definition

The ratio of the z-T. of the output to the z-T. of the input under zero initial condition.0初始条件.

$$G(z)=\frac{C(z)}{R(z)}$$





$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\delta(t-nT)$$

$$\because r^*(t) = r(0)\delta(t) + r(T)\delta(t-T) + \cdots r(nT)\delta(t-nT) + \cdots$$

$$\therefore c(t) = r(0)g(t) + r(T)g[t-T] + \cdots + r(nT)g[t-nT] + \cdots$$

$$c(kT) = r(0)g(kT) + r(T)g[(k-1)T] + \cdots + r(nT)g[(k-n)T] + \cdots$$

$$= \sum_{n=0}^{\infty} r(nT)g[(k-n)T]$$

$$c(kT)=\sum_{n=0}^{\infty} r(nT)g[(k-n)T]$$

$$C(z)=\sum_{k=0}^{\infty} c(kT)z^{-k} \quad =\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} r(nT)g[(k-n)T]z^{-k}$$

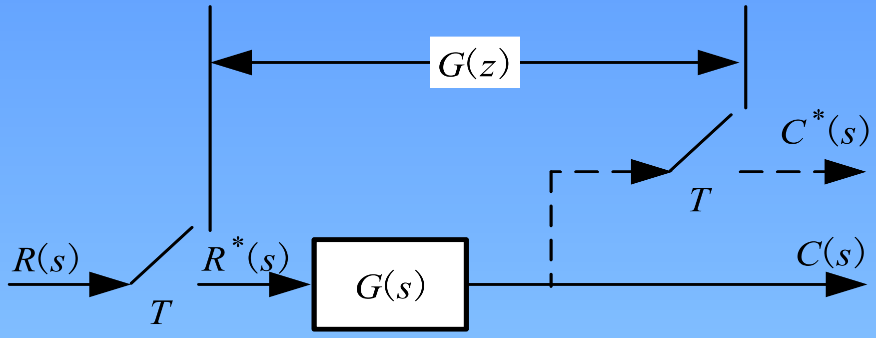
$$=\sum_{n=0}^{\infty} r(nT)z^{-n}\sum_{k=0}^{\infty} g[(k-n)T]z^{-(k-n)}$$

$$\therefore G(z)=\frac{C(z)}{R(z)}=\sum_{k=n}^{\infty} g[(k-n)T]z^{-(k-n)}=\boxed{\sum_{j=0}^{\infty} g(jT)z^{-j}}$$

The z-transform of unity impulse response sequence

G(Z)=Z[g(t)]=Z[G(s)]

冲激响应



Example 1 Consider the discrete system shown in the figure with

$$G(s) = \frac{1}{s(0.1s+1)}$$

Obtain the impulse-transfer function $G(z)$。

Solution:

Method I. **The impulse response is:**

$$g(t) = (1 - e^{-10t}) \qquad (t > 0)$$

$$g(kT) = 1 - e^{-10kT}$$

Then the impulse tranfer function is:

$$G(z) = \sum_{k=0}^{+\infty} g(kT) z^{-k} = \sum_{k=0}^{+\infty} \left(1 - e^{-10kT}\right) z^{-k}$$

$$= \frac{z}{z-1} - \frac{z}{z - e^{-10T}} = \frac{z(1 - e^{-10T})}{(z-1)(z - e^{-10T})}$$

Method II. **Because** $G(s) = \frac{1}{s} - \frac{1}{s+10}$

Then by G(Z)=Z[g(t)]=Z[G(s)], it derives

$$G(z) = \frac{z}{z-1} - \frac{z}{z - e^{-10T}} = \frac{z(1 - e^{-10T})}{(z-1)(z - e^{-10T})}$$

The properties of impulse transfer function:

(1) $G(z)$ is a complex function of complex variable z;

(2) $G(z)$ depends only on the structure and parameters of the system;

(3) $G(z)$ has a relation with the difference equation of the system;

(4) $G(z)$ is equal to $Z[\, g^*(t)\,]$;

(5) $G(z) \sim$ zero-pole location in z plane. 零极点分布

The limitation of impulse-transfer functions

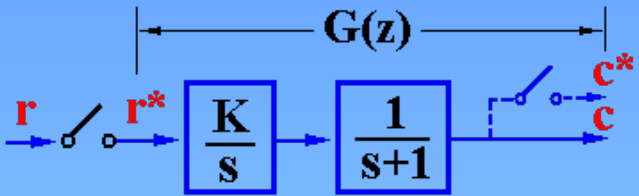
(1) It can not reflect the full information of the system response under non-zero initial conditions;

(2) It is only for SISO discrete systems; 单入单出

(3) It is only for linear time-invariant difference equations;

线性时不变.

Example 2 Consider the discrete system shown in the figure (T=1). Obtain



(1) Impulse-transfer function of the system

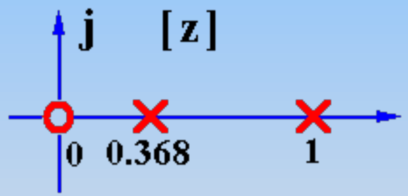
(2) Zero-poles location in z plane;

(3) Difference equation of the system.

Solution. (1) $G(z)=\dfrac{C(z)}{R(z)}=Z\left[\dfrac{K}{s(s+1)}\right]=K\cdot Z\left[\dfrac{1}{s}-\dfrac{1}{s+1}\right]$

$$=K\left[\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right]=\frac{(1-e^{-T})Kz}{(z-1)(z-e^{-T})}=\frac{(1-e^{-T})Kz}{z^2-(1+e^{-T})z+e^{-T}}$$

$$=\frac{0.632Kz^{-1}}{1-1.368z^{-1}+0.368z^{-2}}$$

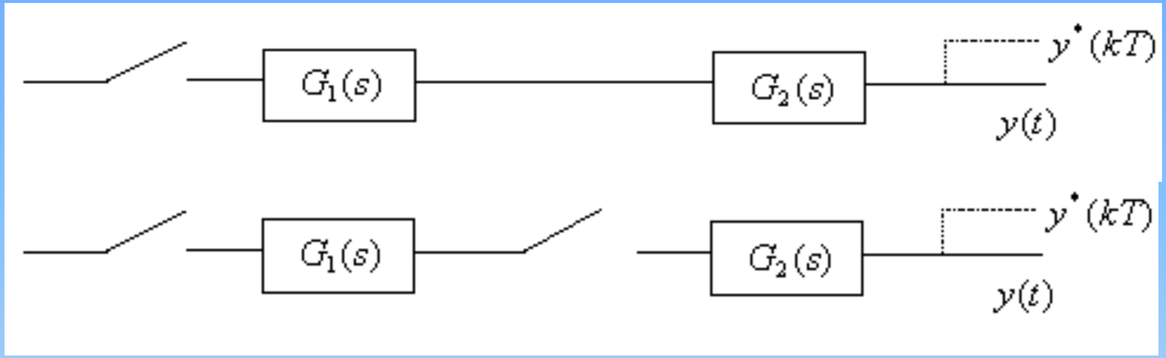


(2) Zero-poles location in z plane

(3) $$\left(1-1.368z^{-1}+0.368z^{-2}\right)C(z)=0.632Kz^{-1}R(z)$$

$$c(k)-1.368c(k-1)+0.368c(k-2)=0.632Kr(k-1)$$

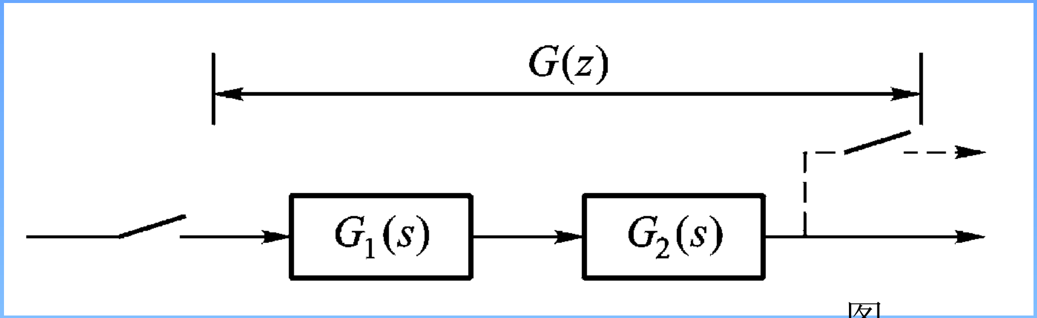
7.5.3 Impulse transfer function of Open-Loop Systems



(1) There is no sampler/switch between two components

$$G(s) = G_1(s)G_2(s)$$

$$G(z) = Z[G_1(s)G_2(s)] = G_1G_2(z)$$



图

Example 3 **Consider the discrete system shown in the above figure， where**

$$G_1(s)=\frac{1}{s+a} \qquad G_2(s)=\frac{1}{s+b}$$

Obtain the open-loop impulse transfer function。

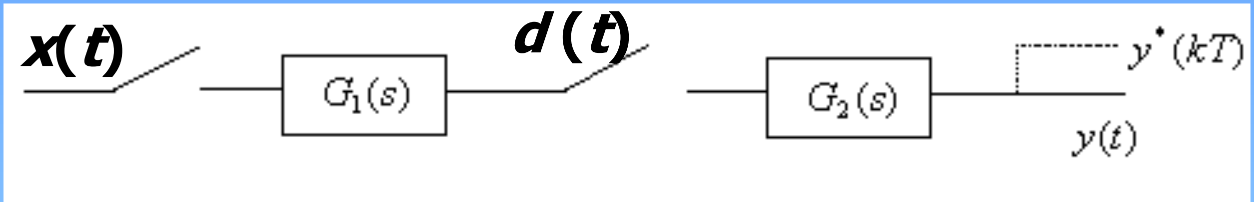
solution：

$$G_1(s)G_2(s)=\frac{1}{b-a}\left[\frac{1}{s+a}-\frac{1}{s+b}\right]$$

$$G(z)=G_1G_2(z)$$

$$=\frac{1}{b-a}\left[\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}\right]$$

（2）There is a sampler/switch between two components



$$D(z) = G_1(z)X(z)$$

$$Y(z) = G_2(z)D(z) = G_1(z)G_2(z)R(z)$$

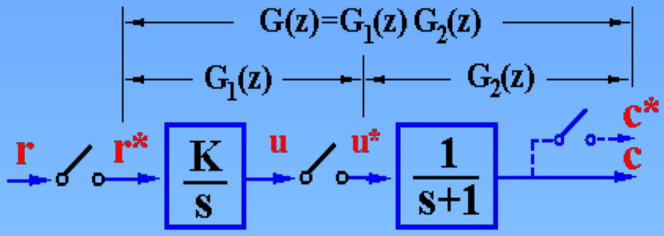
$$\therefore G(z) = G_1(z)G_2(z)$$

注 $G_1(z)G_2(z) \neq G_1G_2(z)$

(1) Switch between factors

$$G(z)=G_1(z)G_2(z)=Z\left[\frac{K}{s}\right]\cdot Z\left[\frac{1}{s+1}\right]$$

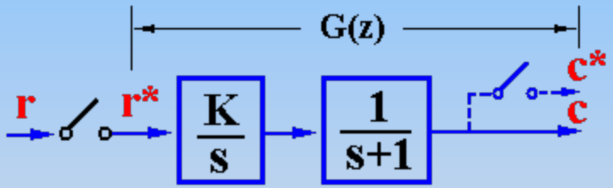
$$=\frac{Kz}{z-1}\cdot\frac{z}{z-e^{-T}}=\frac{Kz^2}{(z-1)(z-e^{-T})}$$



(2) No switch between factors

$$G(z)=Z[G_1(s)\cdot G_2(s)]=G_1G_2(z)$$

$$=K\left[\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right]=\frac{(1-e^{-T})Kz}{(z-1)(z-e^{-T})}$$



Note: the zeros of G(z), the poles of G(z).

Exercise: **Consider** $G_1(s) = \frac{1}{s}$ **,** $G_2(s) = \frac{10}{s+10}$ **, obtain G(z).**

Solution:

If there is no switch between the components,

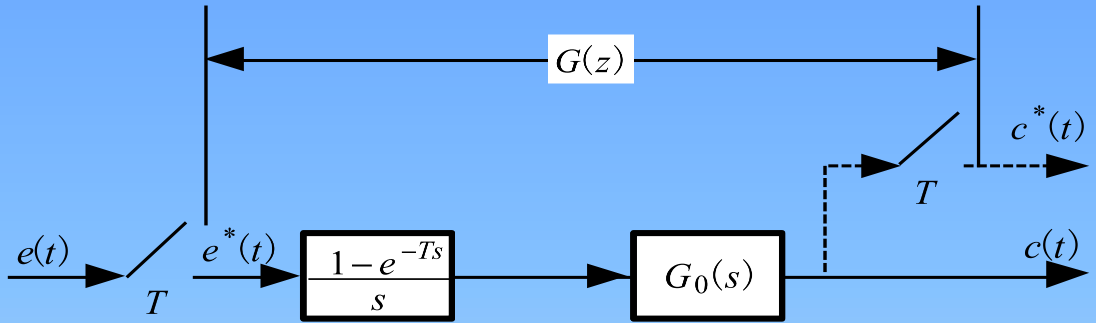
$$G(z) = G_1G_2(z) = Z[\frac{10}{s(s+10)}] = \frac{z(1-e^{-10T})}{(z-1)(z-e^{-10T})}$$

If there is a sampler between the components,,

$$G(z) = G_1(z)G_2(z) = Z[\frac{1}{s}]Z[\frac{10}{s+10}]$$

$$= \frac{z}{z-1}\frac{10z}{z-e^{10T}} = \frac{10z^2}{(z-1)(z-e^{-10T})}$$

（3）ZOH in the system



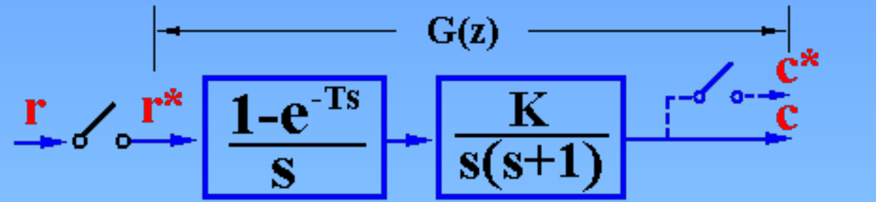
零阶保持 (ZOH)

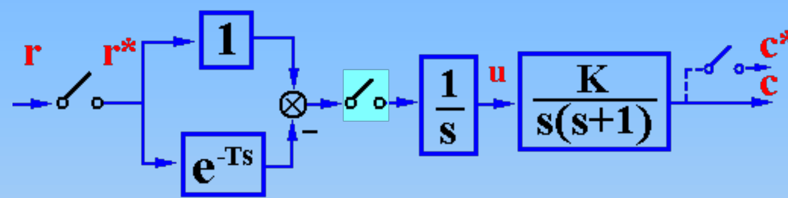
$$C(z) = Z[\frac{1-e^{-Ts}}{s} G_0(s)]R(z) = Z[\frac{1}{s} G_0(s) - \frac{e^{-Ts}}{s} G_0(s)]R(z)$$

$$Z\left[\frac{e^{-Ts}}{s} G_0(s)\right] = z^{-1} Z\left[\frac{G_0(s)}{s}\right] \qquad C(z) = (1-z^{-1}) Z[\frac{G_0(s)}{s}]R(z)$$

$$\underbrace{G(z) = \frac{C(z)}{R(z)} = (1-z^{-1}) Z[\frac{G_0(s)}{s}]}$$ 记住.

Example 4 Consider the discrete system shown in the following figure, obtain its impulse transfer function.





$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+1)}\right]$$

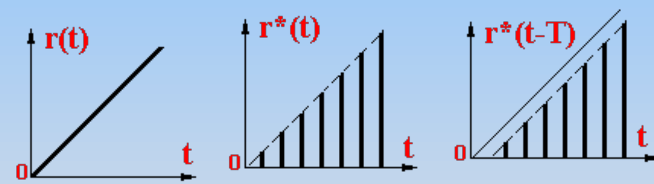
$$= K(1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right]$$

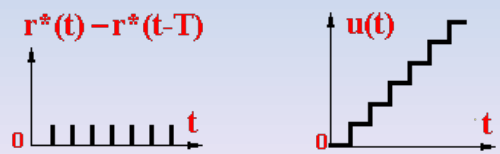
$$= K\frac{z-1}{z}Z\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right]$$

$$= K\frac{z-1}{z}\left[\frac{Tz}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-e^{-T}}\right]$$

$$= K\left[\frac{T}{z-1}-1+\frac{z-1}{z-e^{-T}}\right]$$

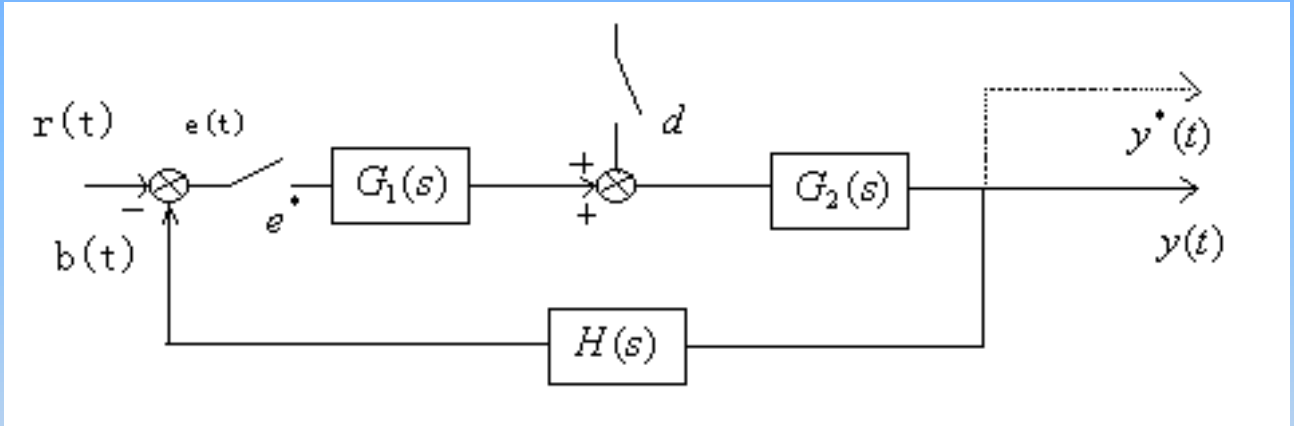
$$= K\frac{(T-1+e^{-T})z+(1-Te^{-T}-e^{-T})}{(z-1)(z-e^{-T})}$$





ZOH does not change the system order and O.-L. poles but changes the O.-L. zeros.

7.5.4、 Impulse transfer function of Closed-Loop Systems



（1）、Impulse Transfer Function for input to output.

$$d = 0$$

$$Y(z) = G_1G_2(z)E(z)$$

$$\left.\begin{aligned} & e(t) = r(t) - b(t) \\ & \Rightarrow E(z) = R(z) - B(z) \\ & B(z) = G_1G_2H(z)E(z) \end{aligned}\right\} \Rightarrow E(z) = \frac{R(z)}{1 + G_1G_2H(z)}$$

Error impulse transfer function (误差脉冲传递函数):

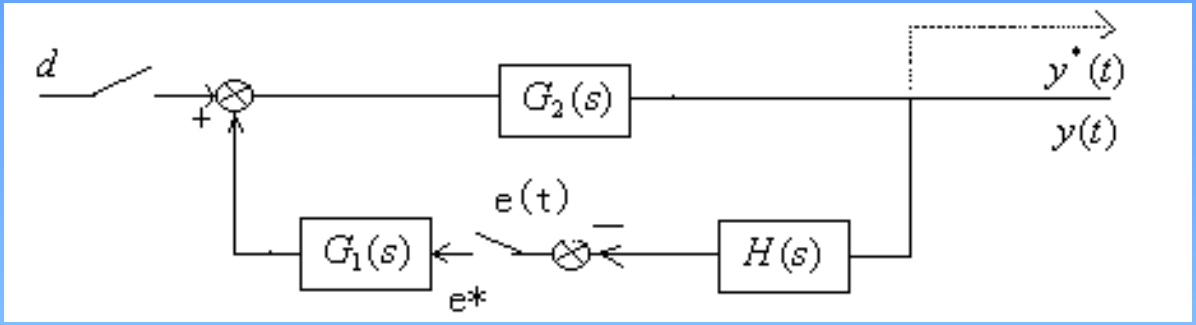
$$G_e(z) = \frac{E(z)}{R(z)} = \frac{1}{1 + G_1G_2H(z)}$$

$$\Rightarrow Y(z) = G_1G_2(z)\frac{R(z)}{1 + G_1G_2H(z)}$$

$$\therefore \Phi(z) = \frac{Y(z)}{R(z)} = \frac{G_1G_2(z)}{1 + G_1G_2H(z)}$$

（2） Impulse Transfer Function for disturbance to output

$$r(t)=0$$



$$Y(z)=G_2(z)D(z)+G_1G_2(z)E(z)$$

$$E(z)=-[G_2H(z)D(z)+G_1G_2H(z)E(z)]$$

$$\Rightarrow E(z)=-\frac{G_2H(z)}{1+G_1G_2H(z)}D(z)$$

$$\therefore Y(z)=G_2(z)D(z)-\frac{G_1G_2(z)G_2H(z)}{1+G_1G_2H(z)}D(z)$$

$$\Rightarrow \Phi_d(z)=\frac{Y(z)}{D(z)}=G_2(z)-\frac{G_1G_2(z)G_2H(z)}{1+G_1G_2H(z)}$$

E(z)：

D(z) passing through $G_2(z)$;

Loop of E(z)itself。

There is no switch/sampler for the error signal e(t)

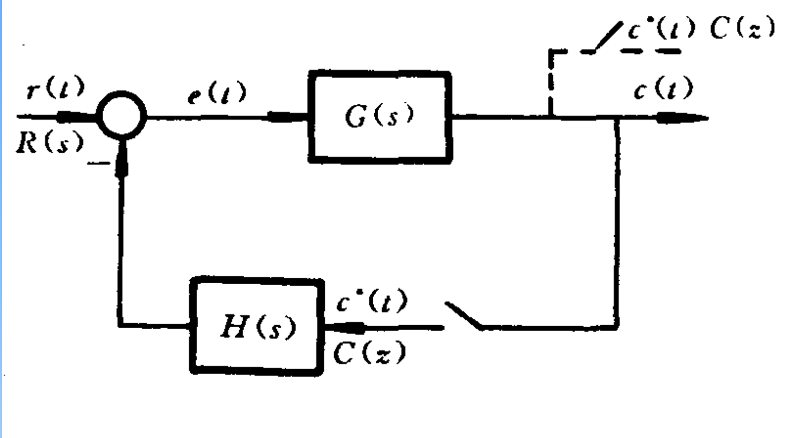


图 7-35 闭环离散系统

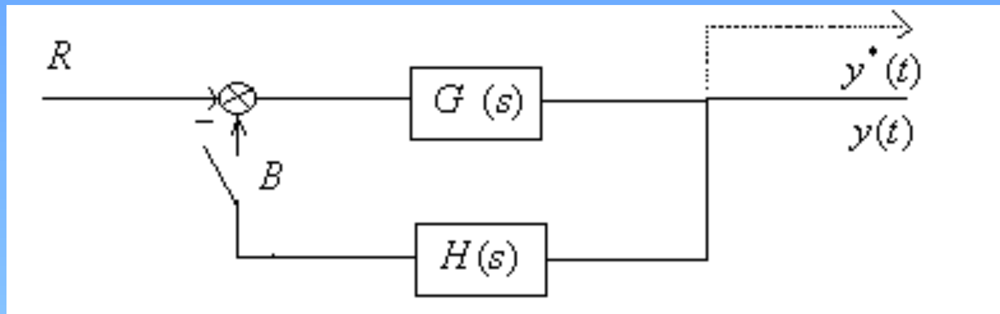
$$C(s) = G(s)R(s) - G(s)H(s)C^*(s)$$

$$C(z) = GR(z) - GH(z)C(z) \qquad \Rightarrow C(z) = \frac{GR(z)}{1+GH(z)}$$

不能除R(z)

Then, for this system, there exists no impulse transfer function.

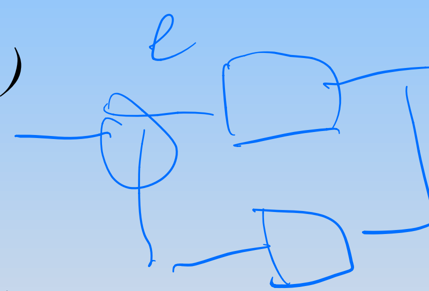
Example **Consider the discrete-time system as shown in the figure, find the z-transform of the output y(t).**



Solution:

$$Y(z) = GR(z) - G(z)B(z)$$

$$B(z) = GHR(z) - GH(z)B(z)$$



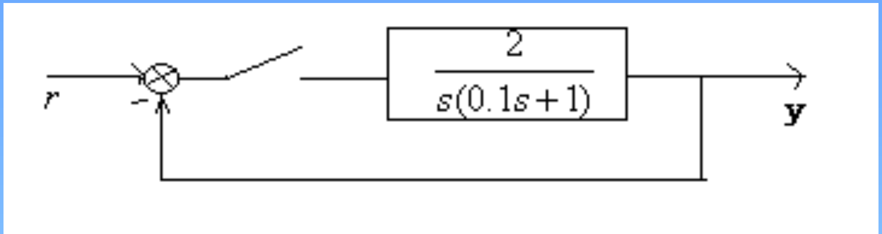
$$\therefore B(z) = \frac{GHR(z)}{1+GH(z)}$$

$$\therefore Y(z) = GR(z) - \frac{G(z)GHR(z)}{1+GH(z)}$$

$R - eGH = e$

There exists no impulse tranfer function.

Example Consider the discrete-time system as shown in the figure, for T=0.1, find the unit step response of the system.



Solution: $G(z) = Z[\frac{2}{s(0.1s+1)}] = \frac{2z}{z-1} - \frac{2z}{z-e^{-10T}}$

$$= \frac{2z - 0.736z}{(z-1)(z-0.368)} = \frac{1.264z}{z^2 - 1.368z + 0.368}$$

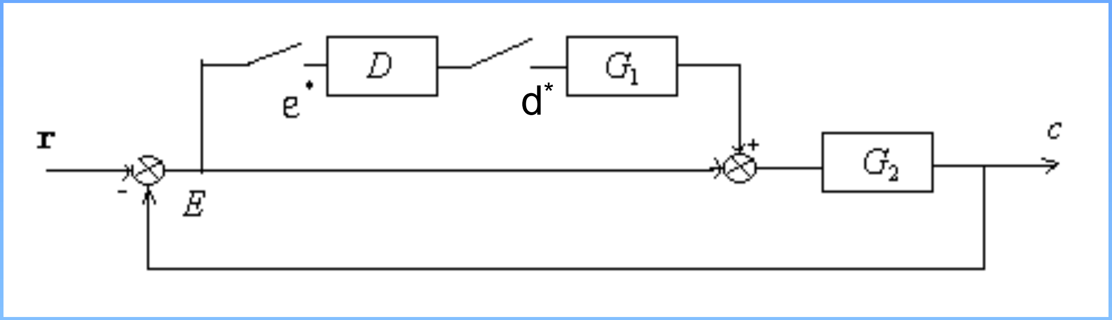
$$\therefore \Phi(z) = \frac{G(z)}{1+G(z)} = \frac{1.264z}{z^2 - 0.104z + 0.368}$$

$$\therefore Y(z) = \Phi(z)R(z) = \Phi(z)\frac{z}{z-1}$$

$$= 1.264z^{-1} + 1.396z^{-2} + 0.945z^{-3} + 0.849z^{-4} + \cdots$$

$$y^*(t) = 1.264\delta(t-0.1) + 1.396\delta(t-0.2) + \cdots$$

Example **Consider the discrete-time system as shown in the figure, find the z-transform of the output c(t).**



Solution: There exist both discrete and continuous signals, then employing L-Transform firstly,

$$C(s) = G_2(s)E(s) + G_1G_2D^*E^*$$

$$E(s) = R - C = R - G_2E - G_1G_2D^*E^* \qquad \therefore E = \frac{R}{1+G_2} - \frac{G_1G_2}{1+G_2}D^*E^*$$

Discretize e(t), then

$$E^* = [\frac{R}{1+G_2}]^* - [\frac{G_1G_2}{1+G_2}]^* D^*E^* \qquad \therefore E^* = \frac{[\frac{R}{1+G_2}]^*}{1+[\frac{G_1G_2}{1+G_2}]^* D^*}$$

Take E and E* into

$$C(s) = E(s)G_2(s) + G_1G_2D^*E^*$$

$$\boldsymbol{C = \frac{G_2R}{1+G_2} - \frac{G_1G_2^2}{1+G_2}D^*E^* + G_1G_2D^*E^*}$$

$$= \frac{G_2R}{1+G_2} + \left(-\frac{G_1G_2^2}{1+G_2} + G_1G_2\right)D^*E^*$$

$$= \frac{G_2R}{1+G_2} + \frac{G_1G_2}{1+G_2}D^*E^*$$

$X(z) = X^*(s)$

$E^* = E^*(s) = E(z) = Z(E(s))$

$(XZ^*)^* = X^*Z^*$

$(X^*Z^*)^* = X^*Z^*$

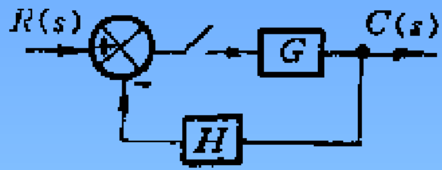
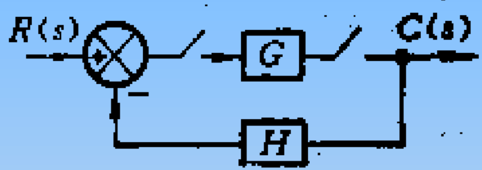
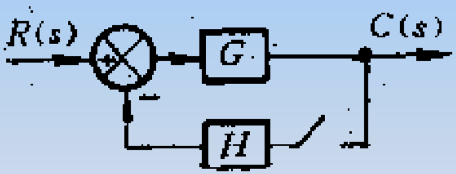
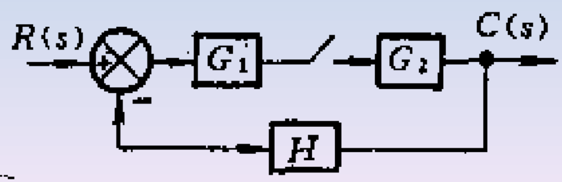
$$C^*(s)=[\frac{G_2R}{1+G_2}]^*+[\frac{G_1G_2}{1+G_2}]^*D^*E^*$$

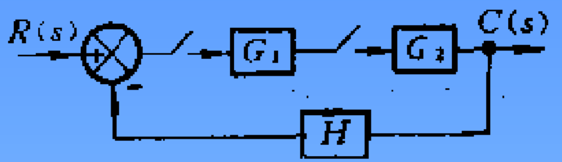
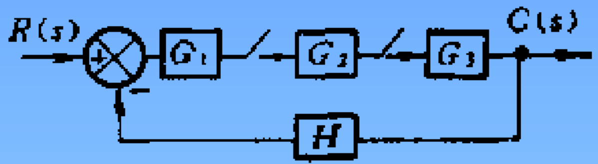
$$=\frac{[\frac{G_2R}{1+G_2}]^*+[\frac{G_1G_2}{1+G_2}]^*D^*[(\frac{G_2R}{1+G_2})^*+[\frac{R}{1+G_2})^*]}{1+[\frac{G_1G_2}{1+G_2}]^*D^*}$$

$$\because R=\frac{G_2R}{1+G_2}+\frac{R}{1+G_2} \qquad \therefore R^*=[\frac{G_2R}{1+G_2}]^*+[\frac{R}{1+G_2}]^*$$

$$\therefore C^*=\frac{[\frac{G_2R}{1+G_2}]^*+[\frac{G_1G_2}{1+G_2}]^*D^*R^*}{1+[\frac{G_1G_2}{1+G_2}]^*D^*}$$

Typical diagram of C.L.discrete-time systems

	系统方框图	$C(z)$
1		$C(z)=\dfrac{G(z)}{1+HG(z)}R(z)$
2		$C(z)=\dfrac{G(z)}{1+G(z)H(z)}R(z)$
3		$C(z)=\dfrac{RG(z)}{1+HG(z)}$
4		$C(z)=\dfrac{RG_1(z)G_2(s)}{1+G_1G_2H(z)}$

5		$C(z)=\dfrac{G_1(z)G_2(z)}{1+G_1(z)HG_2(z)}R(z)$
6		$C(z)=\dfrac{G_2(z)G_3(z)RG_1(z)}{1+G_2(z)G_1G_3H(z)}$