Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

Design for discrete-time systems can be done in s-domain, z-domain and w-domain, respectively.

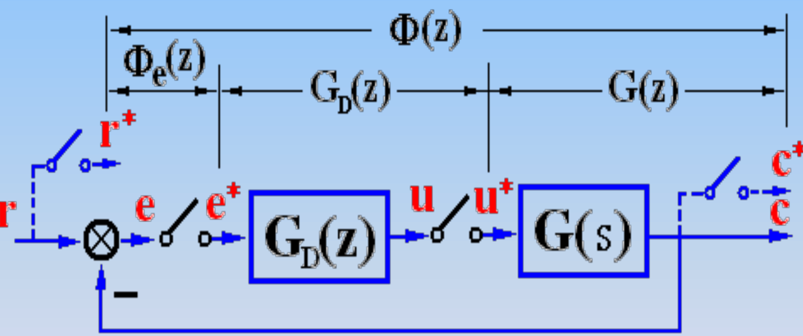
Design for discrete-time systems:

- **Design by discrete equivalent (Emulation)** $\Longrightarrow$ **Continuous compensation** Digitization
- **Discrete Design: Digital compensation**

7.7.1 The Impulse Transfer Function for the Digital Controller

$$\Phi(z)=\frac{G_D(z)\cdot G(z)}{1+G_D(z)\cdot G(z)}$$

$$\Phi_e(z)=\frac{1}{1+G_D(z)\cdot G(z)}=1-\Phi(z)$$



$$G_D(z)\cdot G(z)=\frac{\Phi(z)}{1-\Phi(z)}=\frac{\Phi(z)}{\Phi_e(z)}$$

$$\boxed{G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}}$$

$$\Phi_e(z)=1-\Phi(z),\ \mathrm{E(z)}=\Phi_e(z)R(z)$$

7.7.2 Deadbeat Control Design 最少拍控制

Deadbeat Control Systems: Matching a particular test input within a number of steps. —— No steady-state error on the sampling point.

(典型输入作用下，能在有限拍内结束响应过程且在采样点上无稳态误差的系统。)

1. A unified description of typical test inputs

$$r(t)=\begin{cases}1(t)\\ t\\ t^2/2\end{cases}\qquad R(z)=\left\{\begin{aligned}&\frac{z}{z-1}=\frac{1}{1-z^{-1}}\\ &\frac{Tz}{(z-1)^2}=\frac{Tz^{-1}}{(1-z^{-1})^2}\\ &\frac{T^2z(z+1)}{2(z-1)^3}=\frac{T^2z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}\end{aligned}\right\}\ \boxed{\frac{A(z)}{(1-z^{-1})^{\nu}}}$$

ν	$A(z)$
1	1
2	Tz^{-1}
3	$\frac{T^2z^{-1}(1+z^{-1})}{2}$

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z) \cdot G(z)}$$

Design Idea: **Obtain $G_D(z)$ by constructing $\Phi(z)$ so that the output can match the typical test signal within the minimum steps.**

No $\begin{cases} \text{Zeros} \\ \text{Poles} \end{cases}$ of G(z) on or beyond the unit circle, except for (1, j0)

零极点都在单位圆内

$$R(z) = \frac{A(z)}{(1 - z^{-1})^{\nu}}$$

$$E(z) = \Phi_e(z) R(z), \quad \Phi_e(z) = 1 - \Phi(z)$$

$$e(\infty) = \lim_{z \to 1} (z-1)\, \Phi_e(z)\, R(z) \quad \Rightarrow \Phi_e(z) = (1 - z^{-1})^{\nu} F(z^{-1})$$

To make the $G_D(z)$ simplest and of the lowest-order, we can choose $F(z^{-1})$ as 1.

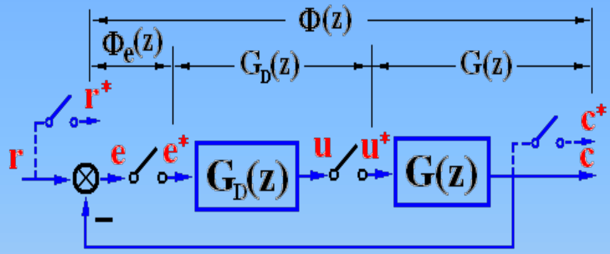
$$\Phi(z) = 1 - \Phi_e(z) = 1 - (1 - z^{-1})^{\nu}$$

From the design idea, we know that $e(\infty T) = 0$

$$E(z) = \Phi_e(z)\cdot R(z) = \frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)$$

$$e(\infty T) = \lim_{z\to 1}(1-z^{-1})\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z) = 0$$

$$\Phi_e(z) = (1-z^{-1})^{\nu}F(z) \overset{F(z)=1}{=} (1-z^{-1})^{\nu}$$



Hence:

$$\Phi(z) = 1-\Phi_e(z) = 1-(1-z^{-1})^{\nu} = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{\nu} z^{-\nu}$$

$$= \frac{b_1 z^{\nu-1} + b_2 z^{\nu-2} + \cdots + b_{\nu}}{z^{\nu}}$$

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

The rule to construct Φ(z): All poles of Φ(z) are located on the origin of z-plane.

2. Φ(z) for typical test inputs

(1) for $r(t) = 1(t)$

- **The C.L.impulse transfer function:**

$$\nu = 1 \qquad \boxed{\Phi(z) = z^{-1}}$$

$$\boxed{E(z) = 1} \xrightarrow{\text{时间域}} \delta(t).$$

The system can track the input by 1 step only。

一步之内

$v=1$

(2) for $r(t) = t \cdot 1(t)$

- **The C.L.impulse transfer function:**

$$\nu = 2 \quad \Phi(z) = 2z^{-1} - z^{-2}$$ $1-(1-z^{-1})^2$

$$\boxed{E(z) = Tz^{-1}}$$

The system can track the input by 2 step。

二步之内 $\nu=2$

(3) for $r(t) = \frac{1}{2} t^2 \cdot 1(t)$

- **The C.L.impulse transfer function:**

$$\nu = 3 \qquad \Phi(z) = 3z^{-1} - 3z^{-2} + z^{-3}$$

$$E(z) = \frac{1}{2} T^2 z^{-1} + \frac{1}{2} T^2 z^{-2}$$

The system can track the input by 3 step。

3步之内 $\nu=3$

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z) \cdot G(z)}$$

Deadbeat Control Design Table

$r(t)$	$R(z)$	$\Phi_e(z) = (1 - z^{-1})^v$	$\Phi(z) = 1 - \Phi_e(z)$	$G_D(z)$	t_s
$1(t)$	$\dfrac{1}{1 - z^{-1}}$	$1 - z^{-1}$	z^{-1}	$\dfrac{z^{-1}}{(1 - z^{-1}) \cdot G(z)}$	T
t	$\dfrac{Tz^{-1}}{(1 - z^{-1})^2}$	$(1 - z^{-1})^2$	$2z^{-1} - z^{-2}$	$\dfrac{z^{-1}(2 - z^{-1})}{(1 - z^{-1})^2 G(z)}$	$2T$
$\dfrac{t^2}{2}$	$\dfrac{T^2 z^{-1}(1 + z^{-1})}{2(1 - z^{-1})^3}$	$(1 - z^{-1})^3$	$3z^{-1} - 3z^{-2} + z^{-3}$	$\dfrac{z^{-1}(3 - 3z^{-1} + z^{-2})}{(1 - z^{-1})^3 G(z)}$	$3T$

3. Algorithm for Deadbeat Control Design

① **Obtain G(z) — Suppose there are no poles and zeros of G(z) on or beyond the unit circle.**

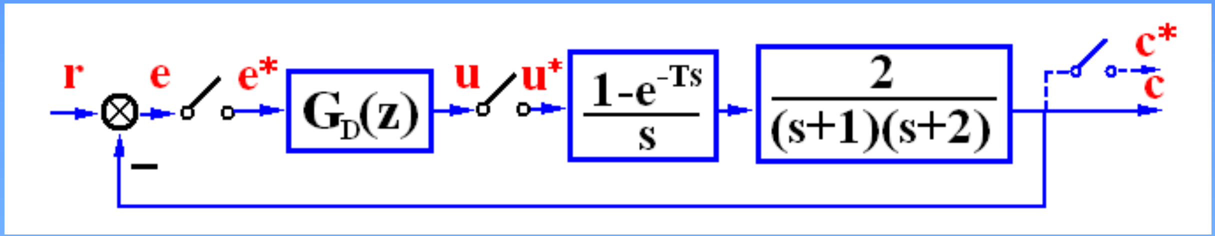
② **Determine $\Phi_e(z)$ for the particular test input**

$$r(t) \quad \Rightarrow \quad R(z) = \frac{A(z)}{(1-z^{-1})^{\nu}} \quad \Rightarrow \quad \Phi_e(z) = (1-z^{-1})^{\nu}$$

$$\Phi_e(z) = (1-z^{-1})^{\nu}\,\mathrm{F}(z)$$

③ **Obtain** $\Phi(z) = 1 - \Phi_e(z)$

④ **Achieve** $G_D(z) = \dfrac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$



Example 1. Consider the system shown in the above figure (T=1). Design deadbeat controllers $G_D(z)$ for r(t)=1(t)、t .

Solution.

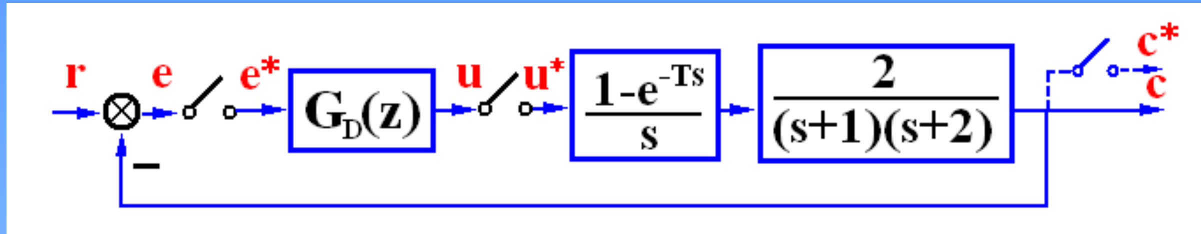
$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{2}{(s+1)(s+2)}\right] = 2(1-z^{-1})\cdot Z\left[\frac{C_0}{s}-\frac{C_1}{s+1}+\frac{C_2}{s+2}\right]$$

$$= 2\cdot\frac{z-1}{z}\cdot Z\left[\frac{1}{2}\cdot\frac{1}{s}-\frac{1}{s+1}+\frac{1}{2}\cdot\frac{1}{s+2}\right]$$

$$= \frac{z-1}{z}\left[\frac{z}{z-1}-\frac{2z}{z-e^{-T}}+\frac{z}{z-e^{-2T}}\right] = 1-\frac{2(z-1)}{z-e^{-T}}+\frac{z-1}{z-e^{-2T}}$$

$$= \frac{(1+e^{-2T}-2e^{-T})z+(e^{-3T}+e^{-T}-2e^{-2T})}{(z-e^{-T})(z-e^{-2T})}$$

$$\overset{T=1}{=} \frac{0.4(z+0.365)}{(z-0.368)(z-0.136)}$$



Referring to the result for $r(t)=1(t)$ in the Design Table

$$R(z)=\frac{z}{z-1} \qquad \text{Choose} \begin{cases} \Phi_e(z)=1-z^{-1} \\ \Phi(z)=1-\Phi_e(z)=z^{-1} \end{cases}$$

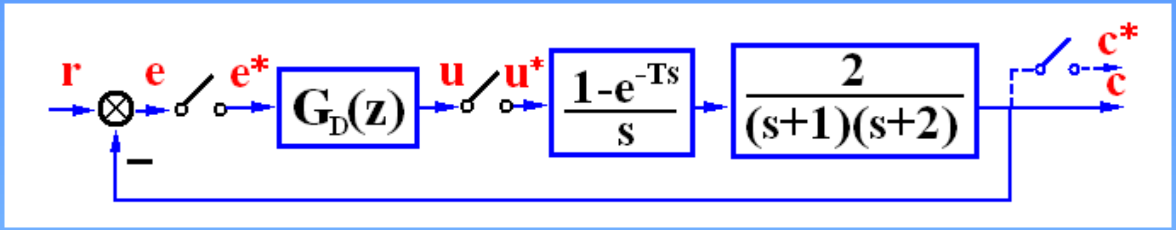
$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}=\frac{z^{-1}}{1-z^{-1}}\cdot\frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$=\frac{2.5(z-0.368)(z-0.136)}{(z-1)(z+0.365)}$$

$$C(z)=\Phi(z)R(z)=z^{-1}\cdot\frac{1}{1-z^{-1}}$$

$$=z^{-1}[1+z^{-1}+z^{-2}+\cdots]=z^{-1}+z^{-2}+z^{-3}+\cdots$$

$$E(z)=\Phi_e(z)R(z)=(1-z^{-1})\cdot\frac{1}{1-z^{-1}}=1$$ 在0时刻误差为1



For $r(t)=t$

$$R(z)=\frac{Tz^{-1}}{(1-z^{-1})^2}$$

Choose $\begin{cases}\Phi_e(z)=(1-z^{-1})^2\\ \Phi(z)=1-\Phi_e(z)=2z^{-1}-z^{-2}\end{cases}$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}=\frac{2z^{-1}-z^{-2}}{(1-z^{-1})^2}\cdot\frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$=\frac{5(z-0.5)(z-0.368)(z-0.136)}{(z-1)^2(z+0.365)}$$

$$E(z)=\Phi_e(z)\cdot R(z)=Tz^{-1}$$

在第1个时刻误差为T.

$$C(z)=\Phi(z)R(z)=(2z^{-1}-z^{-2})\cdot\frac{Tz^{-1}}{(1-z^{-1})^2}$$

$\frac{1}{(1-z^{-1})^2}=1+2z^{-1}+3z^{-2}+4z^{-3}$

$$=R(z)-E(z)=2Tz^{-2}+3Tz^{-3}+4Tz^{-4}+\cdots$$

Although the deadbeat control system tracks a particular test input accurately within a number of steps, it has the following disadvantages:

(1) It is designed only for a particular input.

(2) The output has ripples although there are no errors on the sampling points. 纹波

(3) The control input changes drastically.

4. G(z) has poles or zeros on or beyond the unit circle

suppose

$$G(z)=\frac{z^{-v}\prod_{i=1}^{L}(1-z_i z^{-1})}{\prod_{i=1}^{n}(1-p_i z^{-1})}$$

where z_i **is the zero of** ***G*****(z);**

p_i **is the pole of** ***G*** **(*z*) .**

Then

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{v}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi_e(z)}$$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{v}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi_e(z)}$$

①If there is z^{v} in $G_D(z)$， $G_D(z)$ is un-realizable.

Thus, we have to ensure that there exists z^{-v} in $\Phi(z)$,
which promises $G_D(z)$ is realizable.

② If there is z_i on or beyond the unit circle, $G_D(z)$ is unstable.

Then, those z_i will be designed as the zeros of $\Phi(z)$.

③Note that

$$\Phi(z) = G_D(z)G(z)\Phi_e(z)$$

If there are p_i on or beyond the unit circle,

$\Phi(z)$ will be unstable,

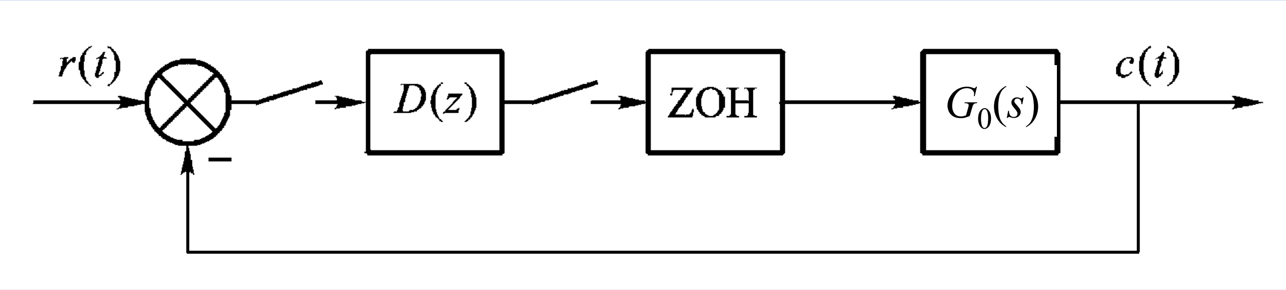
Then **those p_i will be designed as the zeros of $\Phi_e(z)$.**

Example **Given the discrete system described as in the Following figure, where**

$$G_0(s)=\frac{10}{s(0.1s+1)(0.05s+1)},\qquad G_h(s)=\frac{1-e^{-Ts}}{s}$$

with $$\mathrm{T}=0.2s$$

Design a deadbeat controller for $r(t)=1(t)$



Solution： the O. L. impulse transfer function is

$$G(z)=Z\left[G_h(z)G_0(z)\right]=\frac{0.76z^{-1}\left(1+0.05z^{-1}\right)\left(1+1.065z^{-1}\right)}{\underline{\left(1-z^{-1}\right)}\left(1-0.135z^{-1}\right)\left(1-0.0185z^{-1}\right)}$$

不算单位圆外or上

For $r(t)=1(t)$ **， we can design**

$$\Phi_e(z)=1-z^{-1} \qquad (1)$$

$$\Phi(z)=z^{-1} \qquad (2)$$

Because there exists z = - 1.065 (beyond the unit circle),

Thus, z should also be the zero of Φ(z)

There exist z^{-1} ***in G(z)*** **，** z^{-1} **should be in Φ(z), thus**

$$\Phi(z)=\ z^{-1}\left(1+1.065z^{-1}\right) \qquad (3)$$

Because that

$$\Phi(z) = 1 - \Phi_e(z) \qquad \textbf{(4)}$$

from (3), $\Phi(z)$ **is now a polynomial on** z^{-1} **of order 2,**
To satisfy （4）, $\Phi_e(z)$ **must be a polynomial on** z^{-1} **of order 2，thus based on (1), we redesign：**

$$\Phi_e(z) = \underline{(1 - z^{-1})}\ (1 + a_1 z^{-1}) \qquad \textbf{(5)}$$

必须包含

Where a_1 **is a constant to be chosen later.**

Thus multiplied by a constant b_1 to be designed later, we get

$$\Phi(z) = b_1 z^{-1}(1 + 1.065 z^{-1}) \qquad \textbf{(6)}$$

From (5) and (6), we get:

$$a_1 = 0.516 \qquad b_1 = 0.484$$

Thus,

$$\Phi_e(z) = (1 - z^{-1})\,(1 + 0.516z^{-1}) \qquad \textbf{(7)}$$

$$\Phi(z) = 0.484z^{-1}(1 + 1.065z^{-1}) \qquad \textbf{(8)}$$

Then the deadbeat controller is

$$D(z) = \frac{1 - \Phi_e(z)}{G(z)\Phi_e(z)}$$

$$= \frac{1 - (1 - z^{-1})\,(1 + 0.516z^{-1})}{\dfrac{0.76z^{-1}(1 + 0.05z^{-1})\,(1 + 0.065z^{-1})}{(1 - z^{-1})\,(1 - 0.135z^{-1})\,(1 - 0.0185z^{-1})}(1 - z^{-1})\,(1 + 0.516z^{-1})}$$

$$D(z) = \frac{0.637\left(1-0.0185z^{-1}\right)\left(1-0.135z^{-1}\right)}{\left(1+0.05z^{-1}\right)\left(1+0.516z^{-1}\right)}$$

Then the *Z*-transform is

$$C(z) = \Phi(z)R(z) = 0.484z^{-1}\left(1+1.085z^{-1}\right)\frac{1}{1-z^{-1}}$$
$$= 0.484z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-4} + \cdots$$

$(0.484z^{-1} + 0.516z^{-2})(1 + z^{-1} + z^{-2} + z^{-3} + \cdots)$

System can follow the input at the 2nd step, which is one step later.

5. Ripple-free deadbeat control design

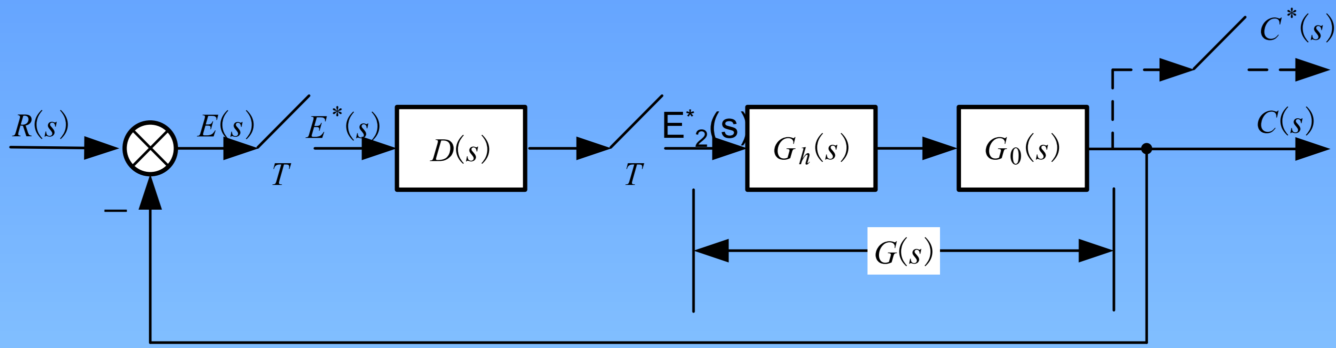
Ripple: though the system outputs are stable at the sampling time, they are varying between two sampling time, 见p251,图7-50。

Objective: Not only tracking the input at the sampling time, the outputs are ripple-free.

Necessary Condition: Given a input signal

$$r(t) = R_0 + R_1 t + \frac{1}{2} R_2 t^2 + \cdots + \frac{1}{(q-1)!} R_{q-1} t^{q-1}$$

Then $G_0(s)$ exists (q-1) z .



$$E_2(z) = D(z)E(z)$$

solution: ensure $E_2(z)$ being a polynomial on z^{-1} of a finite order。

Condition: $E_2(z)$ is a polynomial on $z^{-1}$of finite order。

$$E_2(z) = D(z)E(z) = D(z)\Phi_e(z)R(z), \quad D(z)\Phi_e(z) = \frac{\Phi(z)}{G(z)}$$

➔ **the zero of G(z) must be a zero of** $\Phi(z)$

最少拍设计中，$\Phi(z)$和$\Phi_e(z)$选取时应遵循的原则：

1。$G_D(z)$零点的数目不能大于极点的数目；

2。 $\Phi_e(z)$应把G(z)在单位圆上及单位圆外的极点作为自己的零点；

3。 $\Phi(z)$应把G(z)在单位圆上及单位圆外的零点作为自己的零点；

4。当G(z)含有z^{-1}因子时，要求$\Phi(z)$也含有z^{-1}的因子；

5。 因为$\Phi(z)=1-\Phi_e(z)$，他们应该是关于z^{-1}同样阶次的多项式，而且$\Phi_e(z)$还应包含常数项1。

6。当最小拍系统还有无纹波要求时，闭环脉冲传函$\Phi(z)$的零点应抵消G(z)的全部零点（因为最少拍系统设计中G（z）单位圆上及单位圆外的零极点已经被补偿，因此在无纹波的设计中只需抵消G(z)单位圆内的零点）。