

## **7.5 Mathematical Models of Discrete-Time Systems**

### **7.5.1 Linear Time-Invariant Difference Equations**

- |   |                    |                           |
|---|--------------------|---------------------------|
| <b>(1) Definition of difference</b>                       | <b>① Forward</b>   | <b>② Backward</b>         |
| <b>(2) The difference equation and its solving method</b> | <b>① Iteration</b> | <b>② Z-transformation</b> |

### **7.5.2 Impulse-Transfer Function**

- (1) Definition (2) Properties (3) Limitation**

### **7.5.3 Impulse Transfer Function of Open-Loop Systems**

- { (1) Switch between factors  
(2) No switch between factors  
(3) With ZOH**

### **7.5.4 Impulse Transfer Function of Closed-Loop Systems**

- { (1) General Method  
(2) Mason's formula**

# **Chapter 7 Analysis and Design of Linear Discrete-Time System (Sampled-data System)**

## **7.1 Introduction**

## **7.2 The Sampling Process and Sampling Theorem**

## **7.3 Signal Recovery and Zero-Order Hold**

## **7.4 Z-Transform and Inverse Z Transform**

## **7.5 Mathematical Models of Discrete-Time Systems**

## **7.6 Performance Analysis of Discrete-Time Systems**

## **7.7 Digital Control Design for Discrete-Time Systems**

## 7.6 Performance Analysis of Discrete-Time Systems

- **Stability** 稳.
- **Dynamic Performance** 快
- **Steady-state Errors** 准.

## 7.6.1 Stability of Discrete systems

### 1. Preliminaries

**Stability is the most important performance of a system.**

**When we sampled a continuous systems, we still have a “continuous” system → the same properties hold as before:**

**A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.**

**Now, we introduced the variable  $z=e^{Ts}$ , how does stability look in the new variable?**

**First, we must understand the relationship between s-domain and z-domain.**

## **2. s-Domain to z-Domain Mapping**

**Because  $z = e^{sT}$ , let  $s = \sigma + j\omega$  then  $|z| = e^{\sigma T}$**

**$\angle z = \omega T$ , thus**

$$\begin{cases} \sigma > 0 & |z| > 1 \\ \sigma < 0 & |z| < 1 \\ \sigma = 0 & |z| = 1 \end{cases}$$

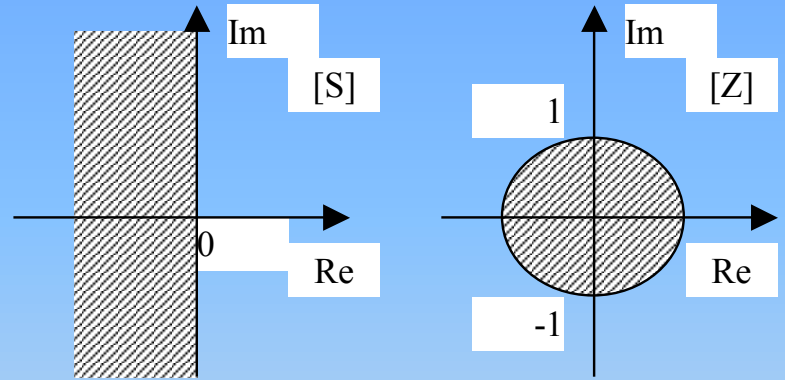
**The mapping relationship can be described as in the following figures.**

$$z = e^{Ts} = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{j\omega T}$$

$$\sigma < 0 \quad |z| = \frac{1}{e^{-T\sigma}} < 1$$

$$\sigma = 0 \quad |z| = 1$$

$$\sigma > 0 \quad |z| = e^{T\sigma} > 1$$



**For a C.L.discrete-time system with unit feedback, the impulse transfer function is:**

$$\Phi(z) = \frac{G(z)}{1 + G(z)}$$

**Its characteristic function is:**

$$1 + G(z) = 0$$


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## Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

— All poles of  $\Phi(z)$  lie in the unit circle of  $z$  plane.

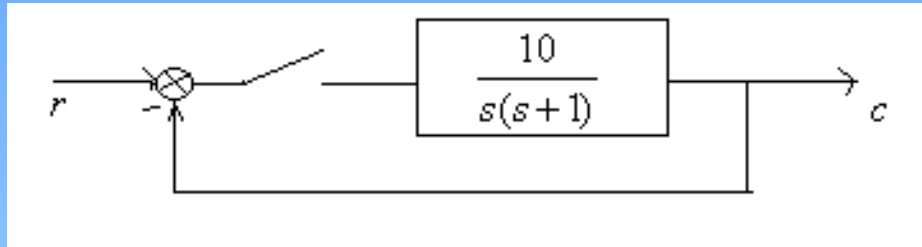
Prove:

$$\Phi(z) = \frac{M(z)}{D(z)} = \frac{\prod_{i=1}^m (z - \alpha_i)}{\prod_{j=1}^n (z - \beta_j)} = \sum_{j=1}^n \frac{C_j z}{z - \beta_j} = K(z)$$

$$c(k) = \sum_{j=1}^n C_j \beta_j^k \stackrel{k \rightarrow \infty}{=} 0 \quad \left| \beta_j \right| < 1 \quad \text{— Necessity}$$

$$c^*(t) = \sum_{k=0}^{\infty} \left( \sum_{j=1}^n C_j \beta_j^k \right) \cdot \delta(t - kT) \quad \text{— Sufficiency}$$

**Example** The discrete-time system is shown as the following figure, suppose  $T=1$ , is the system stable?



**Solution:** 
$$G(z) = Z\left[\frac{10}{s(s+1)}\right] = \frac{6.32z}{(z-1)(z-0.368)}$$

$$1 + G(z) = 0 \Rightarrow z^2 + 4.952z + 0.368 = 0$$

$$\Rightarrow z_1 = -0.076 \quad z_2 = -4.876$$

$\therefore |z_2| > 1$     So the system is unstable.



### **3. The Stability Criterion of Discrete-Time Systems**

- **For continuous-time systems, we can use Routh criterion to determine the stability of the system, where the stable area is on LHP (left-hand-plane) of  $[s]$ -domain.**
- **Unfortunately, for discrete-time systems, the stable area is unit circle, not LHP of  $[z]$ -domain, we cannot directly apply the Routh criterion as we have to test on something else than LHP.**

## w-transformation and Routh criterion in w-domain

We find a transformation that maps the unit circle back onto the LHP while maintaining the algebraic structure of rational functions.

A particular transformation that will accomplish this would be the bilinear transformation:

$$z = \frac{w+1}{w-1}, \quad w = \frac{z+1}{z-1}$$

Suppose  $z = x + jy \quad w = u + jv$

Then 
$$w = \frac{z+1}{z-1} = \frac{(x+1) + jy}{(x-1) + jy}$$

$$u + jv = \frac{(x^2 + y^2) - 1}{(x-1)^2 + y^2} - j \frac{2y}{(x-1)^2 + y^2}$$

$$\therefore u = \frac{x^2 + y^2 - 1}{(x-1)^2 + y^2}$$

$$w = \frac{z+1}{z-1} = \frac{x+1+jy}{x-1+jy} = \frac{x^2-1+y^2-j2y}{(x-1)^2+y^2} = u + jv$$

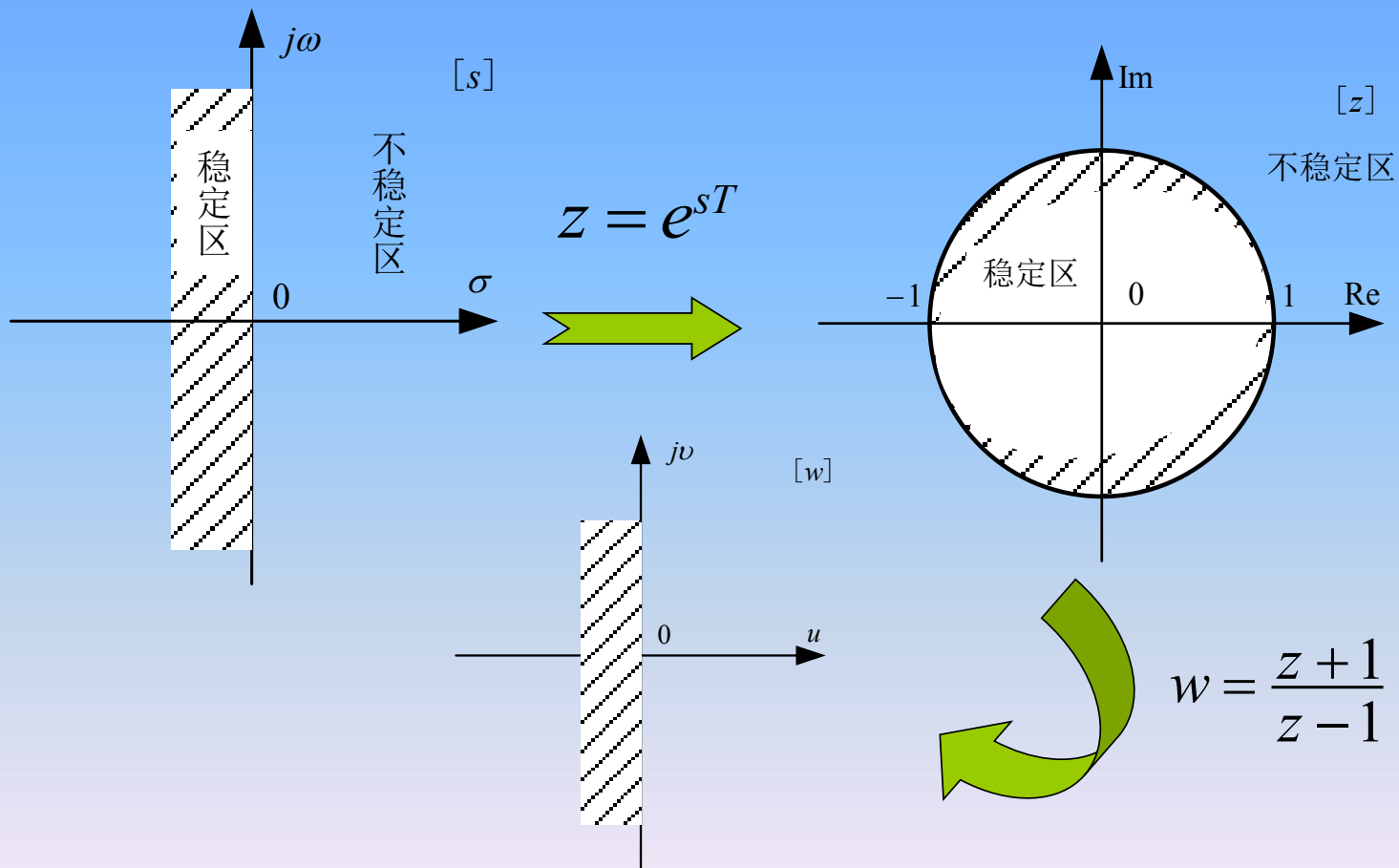
[w] imaginary axis  $u = 0 = \frac{x^2 + y^2 - 1}{(x-1)^2 + y^2} = 0$

$$\Rightarrow x^2 + y^2 = 1 \quad \text{[z] unit circle}$$

Points  $\begin{cases} \text{inside} \\ \text{outside} \end{cases}$  of the unit circle

$$\Rightarrow x^2 + y^2 \begin{cases} < 1 \\ > 1 \end{cases} \quad \text{in z-plane}$$

$$\Rightarrow \begin{cases} u < 0 \\ u > 0 \end{cases} \quad \text{in w-plane}$$



**Example 1** Determine the stability from the characteristic equation of a discrete system.

$$D(z) = 45z^3 - 117z^2 + 119z - 39 = 0$$

↓

$$z = (w + 1)/(w - 1)$$

$$= 45\left(\frac{w+1}{w-1}\right)^3 - 117\left(\frac{w+1}{w-1}\right)^2 + 119\left(\frac{w+1}{w-1}\right) - 39 = 0$$

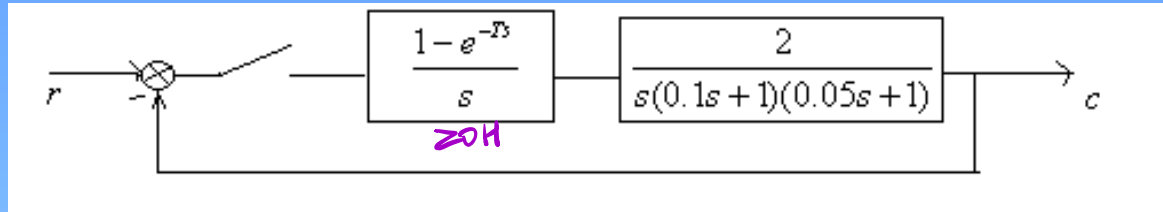
$$D(w) = 45(w+1)^3 - 117(w+1)^2(w-1) + 119(w+1)(w-1)^2 - 39(w-1)^3 = 0$$

$$D(w) = w^3 + 2w^2 + 2w + 40 = 0$$

<b>Routh</b>	$w^3$	1	2
	$w^2$	2	40
	$w^1$	-18	
	$w^0$	40	

**Unstable!**

**Example 2** Consider the discrete-time system as shown in the figure, if  $T=0.1$ , determine the stability of the system.



**Solution:**

$$G(z) = \frac{z-1}{z} Z \left[ \frac{2}{s^2 (0.1s+1)(0.05s+1)} \right] \quad \text{公式} \quad \text{有 } \checkmark - \text{ZOH} -$$

$$= \frac{z-1}{z} \left[ -\frac{0.3z}{z-1} + \frac{0.4z}{(z-1)^2} + \frac{0.4z}{z-e^{-10T}} - \frac{0.1z}{z-e^{-20T}} \right] \quad \text{则 } \frac{z-1}{z} \cdot Z \left[ \frac{1}{s^2 E(s)} \right]$$

$$1 + G(z) = 0$$

$$z^3 - 1.001z^2 + 0.3356z + 0.0535 = 0$$

$$z = \frac{w+1}{w-1}$$

$$2.33w^3 + 3.68w^2 + 1.65w + 0.34 = 0$$

$$w^3 \quad 2.33 \quad 1.65$$

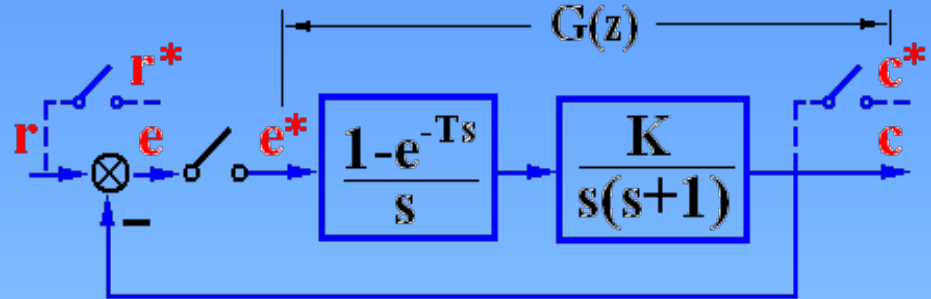
$$w^2 \quad 3.68 \quad 0.34$$

$$w \quad 1.43 \quad 0$$

$$w^0 \quad 0.34$$

**The elements in the first column are all positive, the system is stable.**

**Example 3** Consider the system shown in the figure ( $T=1$ ). Determine the stable range of  $K$ .



**Routh criterion in w domain**

$$\begin{aligned}
 G(z) &= Z\left[\frac{1-e^{-Ts}}{s} \cdot \frac{K}{s(s+1)}\right] = (1-z^{-1})K \cdot Z\left[\frac{1}{s^2(s+1)}\right] \\
 &= \frac{(z-1)K}{z} \cdot Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right] = \frac{(z-1)K}{z} \cdot \left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right] \\
 &= K \left[\frac{(T-1+e^{-T})z + (1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}\right] \stackrel{T=1}{=} \frac{0.368K(z+0.718)}{(z-1)(z-0.368)}
 \end{aligned}$$

$$\Phi(z) = \frac{G(z)}{1+G(z)} = \frac{0.368K(z+0.718)}{z^2 + (0.368K - 1.368)z + (0.264K + 0.368)}$$

$$\Phi(z) = \frac{G(z)}{1+G(z)} = \frac{0.368K(z+0.718)}{z^2 + (0.368K - 1.368)z + (0.264K + 0.368)}$$

$$D(z) = z^2 + (0.368K - 1.368)z + (0.264K + 0.368) = 0$$

$$\downarrow \quad z = \frac{w+1}{w-1}$$

$$= \left(\frac{w+1}{w-1}\right)^2 + (0.368K - 1.368)\left(\frac{w+1}{w-1}\right) + (0.264K + 0.368) = 0$$

$$(w+1)^2 + (0.368K - 1.368)(w+1)(w-1) + (0.264K + 0.368)(w-1)^2 = 0$$

$$D(w) = 0.632Kw^2 + (1.264 - 0.528K)w + (2.736 - 0.104K) = 0$$

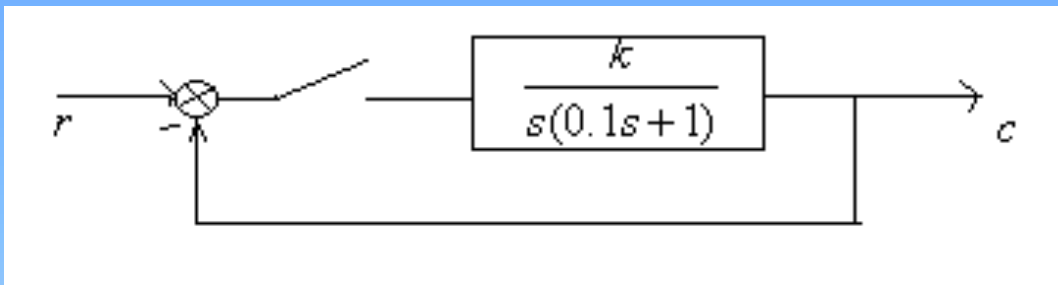
$$\begin{cases} K > 0 \\ 1.264 - 0.528K > 0 \\ 2.736 - 0.104K > 0 \end{cases} \quad \begin{cases} K > 0 \\ K < 2.394 \\ K < 26.3 \end{cases}$$

二次方程 简便 → 系数全同号则稳定

$$0 < K < 2.394$$



**Exercise** Consider the system shown in the figure ( $T=0.1$ ).  
 Determine the stable range of  $K$ . If it is a continuous-time system,  
 Determine the stable range of  $K$



**Solution:**

$$G(z) = Z \left[ \frac{k}{s(0.1s + 1)} \right] = \frac{0.632kz}{z^2 - 1.368z + 0.368}$$

$$1 + G(z) = 0 \quad z^2 + (0.632k - 1.368)z + 0.368 = 0$$

$$z = \frac{w + 1}{w - 1} \quad 0.632kw^2 + 1.264w + 2.736 - 0.632k = 0$$

For continuous-time system,  $0 < k$ , then system is stable;

For discrete-time system,  $0 < k < 4.33$ , stable.

稳定性边界

# Summary

**s-Domain to z-Domain Mapping**

**Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems**

— All poles of  $\Phi(z)$  lie in the unit circle of  $z$  plane

**Routh criterion in  $w$  domain (Generalized Routh Criterion)**