



Chapter 8 Nonlinear Systems Theory

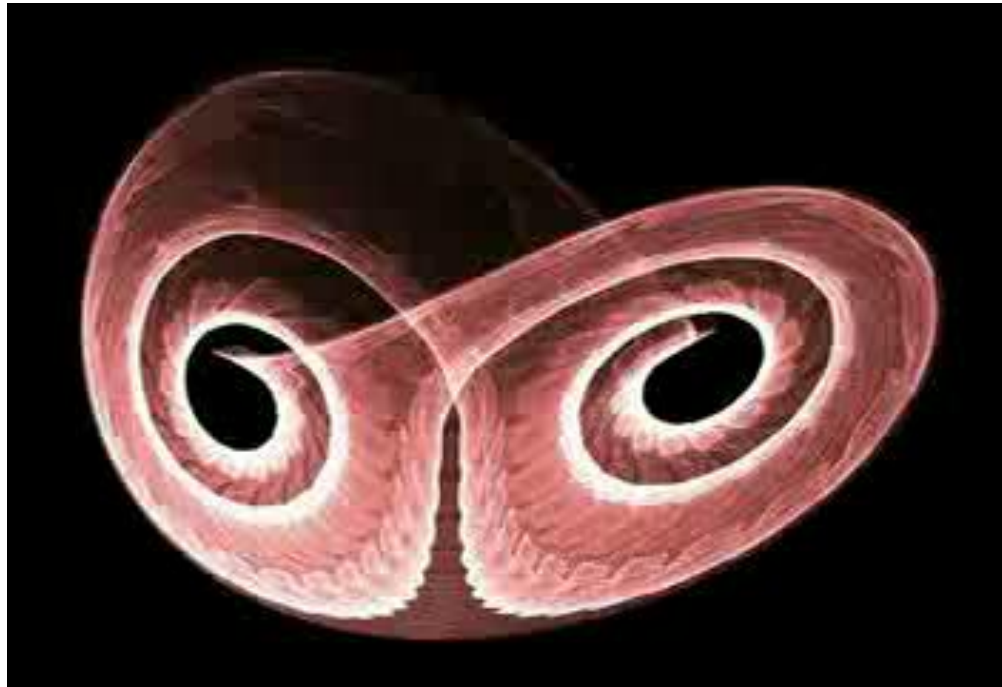
8.1 Overview of Nonlinear Systems

8.2 Typical Nonlinear Characteristics
and Mathematical Description

8.3 Describing Function Approach

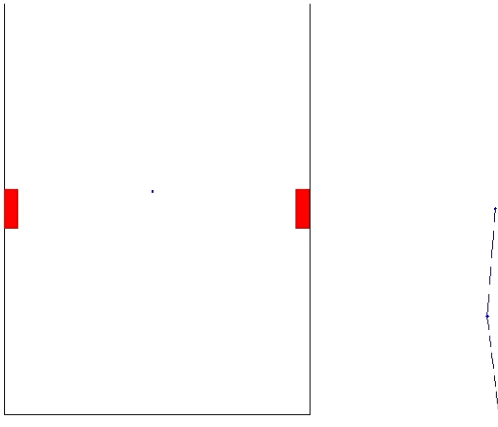
8.4 Phase Plane Analysis

Interesting Examples of Nonlinearity

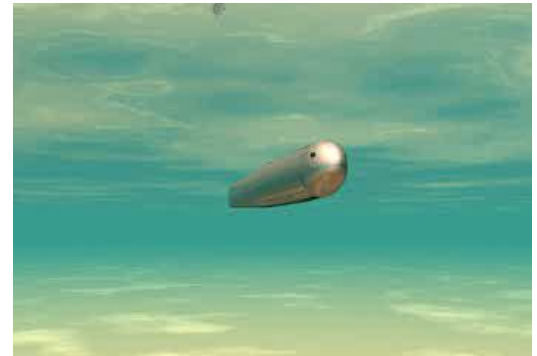
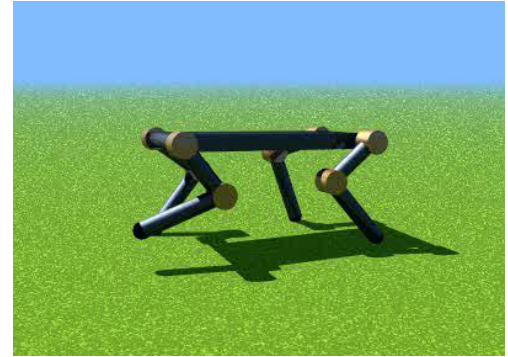


Lorenz Chaotic Attractor

Interesting Examples of Nonlinearity



Acrobot Robots



Biomimetic robots



§ 8.1 Overview of Nonlinear Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y + \varepsilon \cdot f(y, y', \cdots y^{(n)}) = x$$

if $\varepsilon \rightarrow 0$, Linear Systems

if ε can't be ignored, Nonlinear systems

Such as: $\ddot{y} + \ddot{y}y + y = \sin \omega t$

$$(\ddot{y})^2 + 3\dot{y} + y = e^t$$

$$\ddot{y} + 3\dot{y} + y^2 = x$$



Thinking: How to distinguish nonlinearity?



1 Significance of Studying Non-linear Systems

1) There are no systems without nonlinearity.

(*Dead-zone* characteristics of measurement element、*saturation* characteristics of amplification element、Dead-zone and saturation characteristics of actuator、*Gap* Characteristics of actuating unit and so on) 。

Dead-zone 死区
saturation 饱和
Gap 间隙

2) The *intrinsic nonlinearities* make the linear system theory cannot be applied in analyzing the actual systems. The influences of nonlinear factors can not be explained by linear system theory.

3) The nonlinear characteristics do not always have negative impacts on systems. *Optimal control laws are often nonlinear laws*, such as the relay and waveform generator.

2. Features of Nonlinear Systems

Comparing with linear control systems, non-linear systems have many *new features*:

1. A linear systems satisfies the *principle of superposition*, while a non-linear system does not.

叠加原理

■ **(1) additivity:** $y = f(x)$

$$f(x_1 + x_2) = f(x_1) + f(x_2) \quad \text{additivity}$$

$f(x) = ax$ A linear function satisfies the principle of superposition obviously



Thinking: $f(x) = ax + b$

Superposition Theorem

■ (2) multiplicativity :

$$f(ax) = af(x)$$

Nonlinear systems may be additive, but it is not multiplicative.

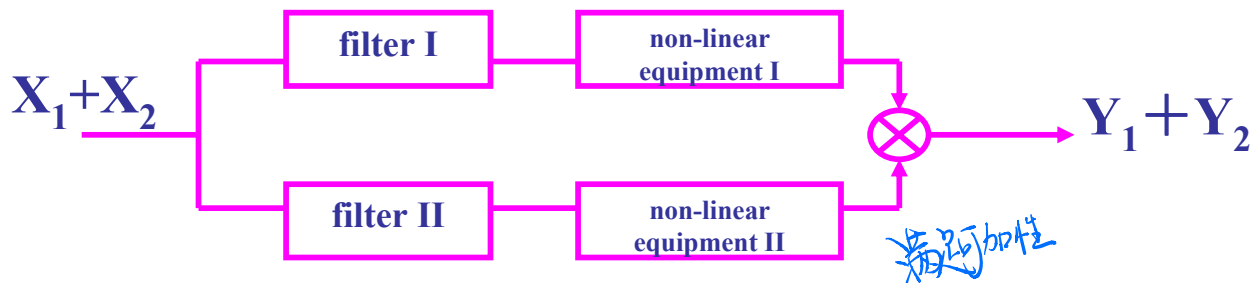



Fig. 8—1 Nonlinear systems with filters

$$X_1 \rightarrow Y_1, \quad X_2 \rightarrow Y_2$$

additivity: $X_1 + X_2 \rightarrow Y_1 + Y_2$ multiplicativity: $nX_1 \nrightarrow nY_1$



2. The stabilities of non-linear systems depend on not only the inherent structure and parameters of control systems, but also the initial conditions and the inputs.

Example: A nonlinear system described by the non-linear differential equation:

$$\dot{x} = -x(1-x)$$

have two equilibrium points obviously, $x_1=0$ and $x_2=1$. The equation equals to $\dot{x}=0$

$$\frac{dx}{x(1-x)} = -dt$$

Integrating both sides: $\ln \frac{cx}{1-x} = -t \quad \therefore \frac{cx}{1-x} = e^{-t}$

Set the initial state of the system be x_0 ,
if $t = 0$, then:

$$c = \frac{1-x_0}{x_0}$$

$$\therefore x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$$

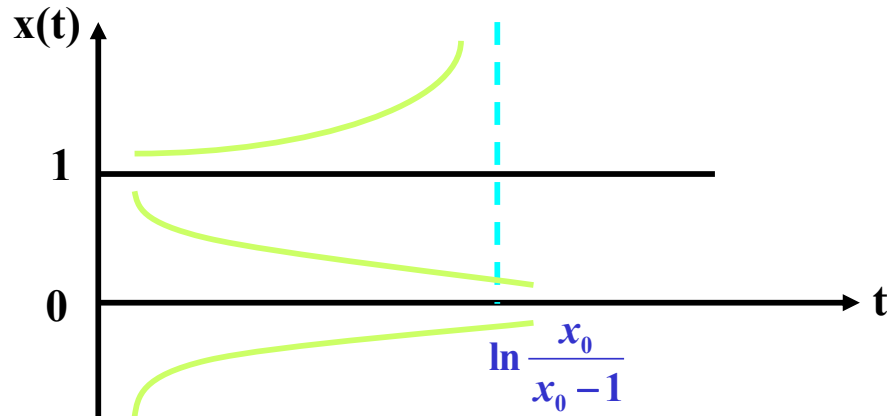
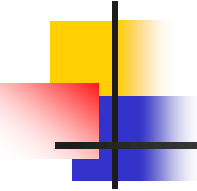


Fig. 8-2 First-order non-linear systems

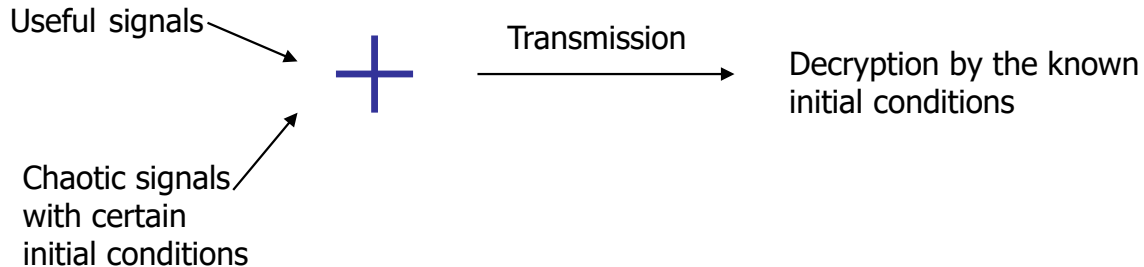
If $x_0 < 1$, $t \rightarrow \infty$, then $x \rightarrow 0$

**Initial conditions affect
stability of the system!**

If $x_0 > 1$, when $t = \ln \frac{x_0}{x_0 - 1}$, we have $x \rightarrow \infty$

- 
- The initial conditions can be even used as a key to the *encryption* of transmission signals in Chaotic systems

加密



3. *Periodic oscillation* does not exist in an actual physical linear systems, while it may occur in a nonlinear system.

4. A stable linear system under a *periodic input* → output with the same frequency; A nonlinear system under a periodic input → many complex cases of the outputs

(1) *Jump resonance* and *Multi-valued response*

Input signals with constant amplitude, then the *amplitude frequency characteristics* of the output is:

$\omega \uparrow: 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$

$\omega \downarrow: 5 \rightarrow 4 \rightarrow 4' \rightarrow 2' \rightarrow 1$

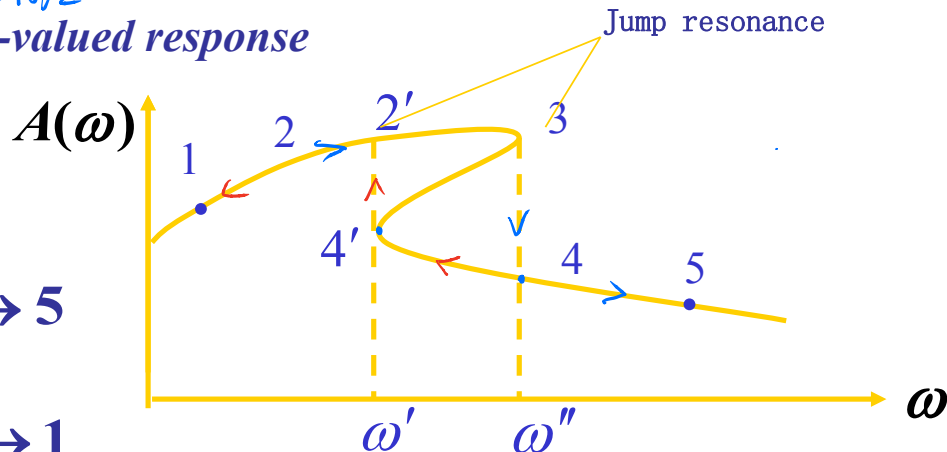


Fig. 8-3 Amplitude frequency characteristic of output of non-linear spring

(2) *subharmonic oscillation* and *harmonic oscillation*

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Steady-state outputs of non-linear systems can be divided into *subharmonic oscillation* and *harmonic oscillation*. When the input signal is sinusoidal, showed in Fig. 8-4

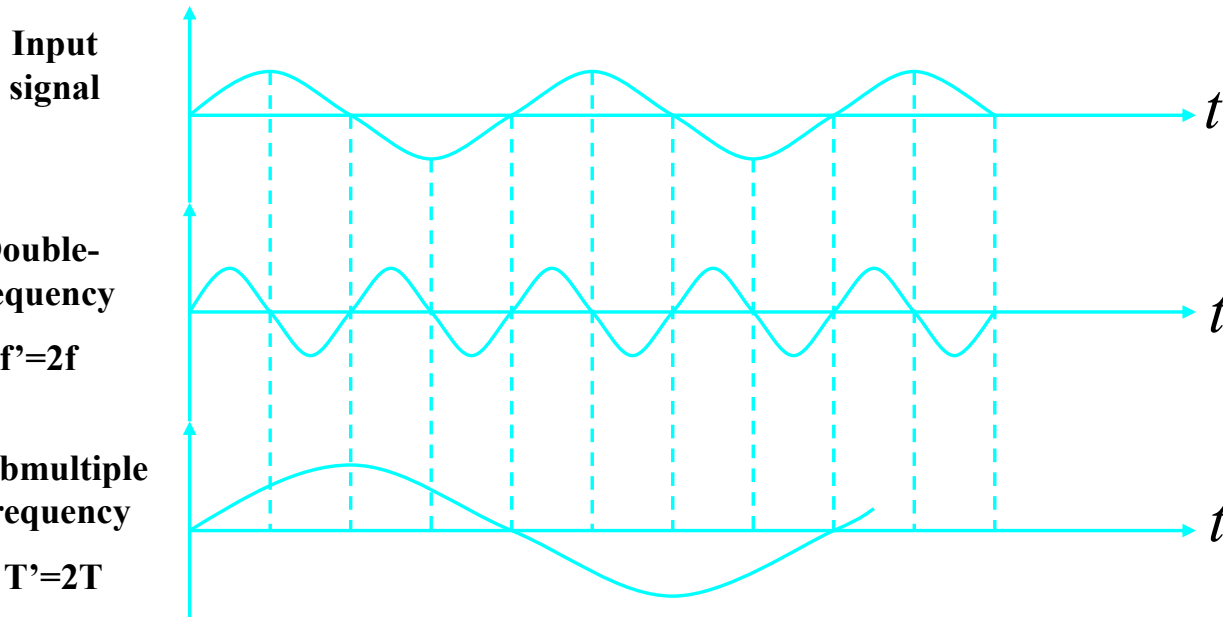


Fig. 8-4 *subharmonic oscillation* and *harmonic oscillation*



3. Methods of Studying Nonlinear Systems

1) ^{时域} Phase-Plane Analysis is the **graphical method** used to analyze **first-order** and **second-order** Nonlinear systems. It analyzes the features of Nonlinear systems through drawing *phase portrait* to find all the solutions of the differential equations in any initial condition. It is the generalization and application of **time-domain analyzing** method in non-linear systems. *It can only be used in the first- and second-order nonlinear systems.* ^{-一阶或二阶}

2) ^{频域} Describing Function Approach is a kind of method for analyzing nonlinear systems inspired by frequency method of linear systems. It is the generalization of **frequency method** in nonlinear systems, and *is not restricted by the system order.* ^{只能处理较弱的非线性}

3) Numerical Solution is a kind of **numerical methods** to solve the nonlinear differential equation using high-speed computers. It is almost the only effective method for analyzing and designing *complex nonlinear systems.*



Note:

- It should be pointed out that, the above methods aim at solving the “*analysis*” problems of nonlinear systems based on analyzing **the system stability**.
- The achievement of “*synthesis*” methods in nonlinear systems is much less than stability problem. There are **NO** general approaches can be used to design arbitrary nonlinear systems so far.



§ 8.2 Typical Nonlinear characteristics and Their Mathematical Description

8.2.1 Saturation characteristics

8.2.2 Dead-zone characteristics

8.2.3 Gap characteristics

8.2.4 Relay characteristics

1. Saturation

A common nonlinearity in electronic amplifiers

Mathematical description of saturation features :

$$x(t) = \begin{cases} ke(t), & |e(t)| < e_0 \\ ke_0 \text{sign}[e(t)], & |e(t)| \geq e_0 \end{cases}$$

$\text{sign}[e(t)]$ is the *sign* function

$$\text{sign}[e(t)] = \begin{cases} 1, & e(t) \geq 0 \\ -1, & e(t) < 0 \end{cases}$$

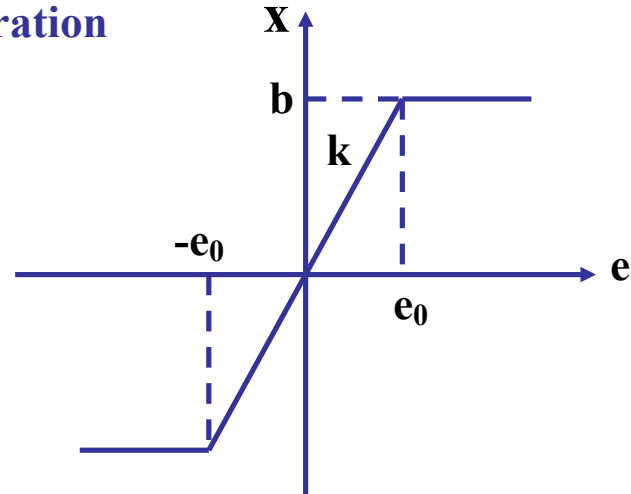


Fig. 8-5 Saturation characteristics

2. Dead-zone

Dead-zone can be also called neutral zone , its **mathematical description** is:

$$x(t) = \begin{cases} 0, & |e(t)| \leq e_0 \\ k[e(t) - e_0 \text{sign}[e(t)]], & |e(t)| > e_0 \end{cases}$$

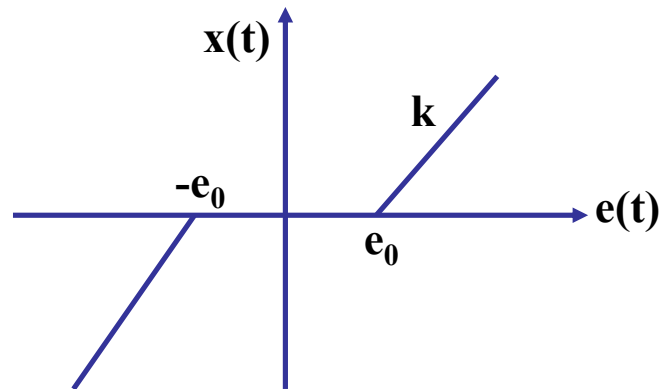


Fig. 8-6 Dead-zone characteristics

3. Gap

Mechanical transmission devices are based on gears, there must exist some gaps for sliding and reversing transmission, that means the gears have to pass a few distances when reversing transmission is needed.

Its mathematical description is :

$$x(t) = \begin{cases} k[e(t) - e_0], & \dot{x}(t) > 0 \\ k[e(t) + e_0], & \dot{x}(t) < 0 \\ b \operatorname{sign}[e(t)], & \dot{x}(t) = 0 \end{cases}$$

↑ 主动轮
 ↓ 被动轮
 ↓ 速度

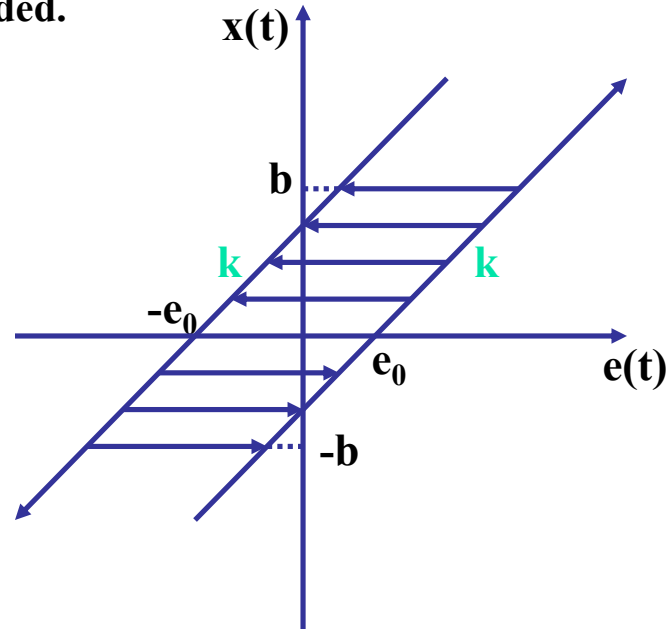
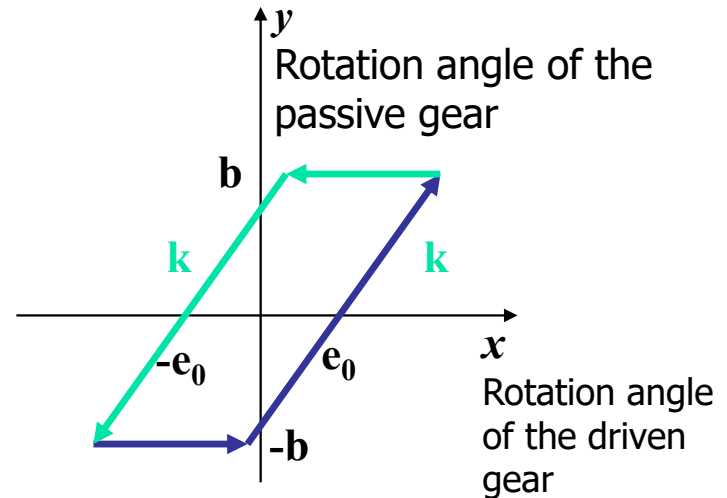
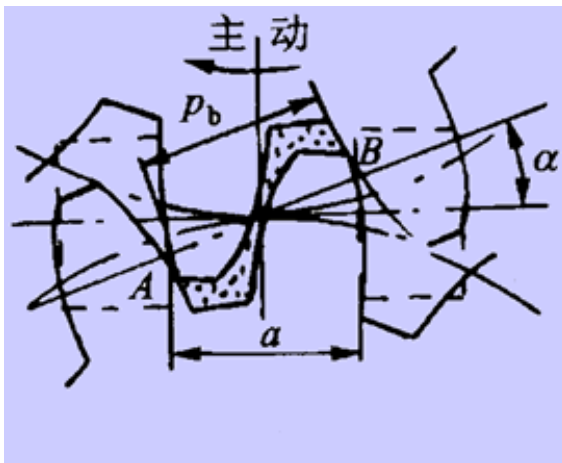


Fig. 8-7 Gap characteristics

3. Gap



Input x is the rotation angle of the driven gear
Output y is the rotation angle of the passive gear

4. Relay

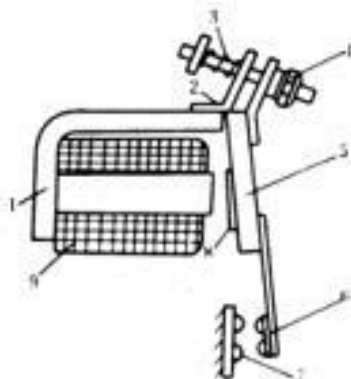


图1 电磁式继电器原理图

- 1- 铁心 2-旋转棱角 3-释放弹簧 4-调节螺母 5--衔铁
6-动触点 7-静触点 8-非磁性垫片 9-线圈

Principle of relay:

Input voltage \rightarrow Current in coil \rightarrow
generates the electromagnetic force
 \rightarrow Close the relay contact

If the input voltage is e_0 , the
electromagnetic force generated by
current in coil is enough to make
the switch to be closed, then e_0 is
called *Operation Voltage*.

操作电压

4. Relay

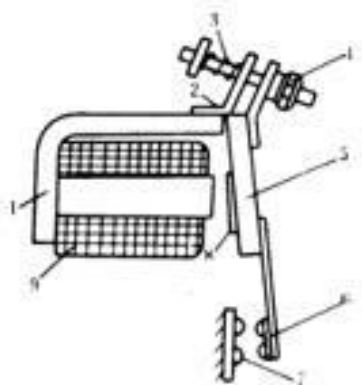


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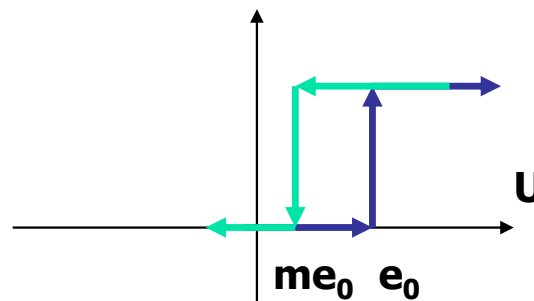
The relay contact will not release when the input voltage is reduced to e_0 because of the influence of Hysteresis.

When it is further reduced to me_0 ($m < 1$), the relay contact will be released.

Then me_0 is called Release Voltage.

磁吸

释放电压



There are four forms of relay characteristics:



1. Ideal relay characteristics

$$x(t) = \begin{cases} M, & e > 0 \\ -M, & e < 0 \end{cases}$$

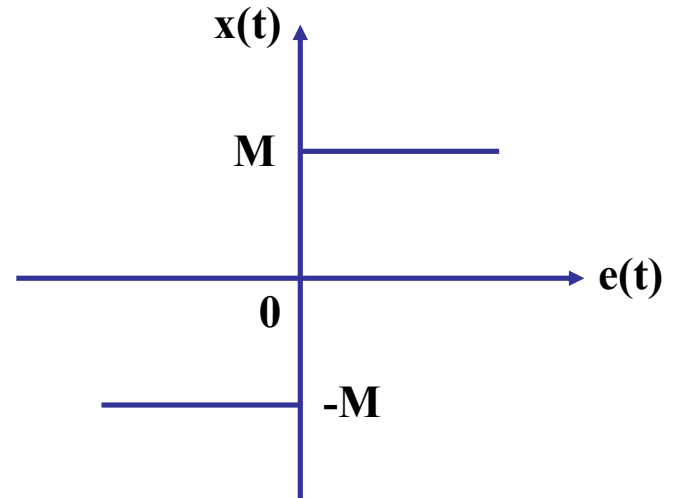


Fig. 8-8(a) Perfect relay characteristics

2. Relay characteristics with Dead-zone

$$x(t) = \begin{cases} M, & e(t) > e_0 \\ 0, & -e_0 \leq e(t) \leq e_0 \\ -M, & e(t) < -e_0 \end{cases}$$

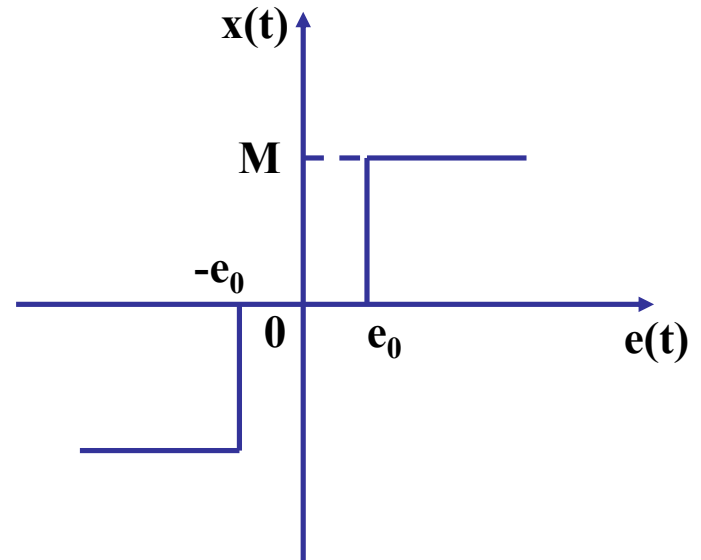


Fig. 8-8(b) Relay characteristics with Dead-zone

3. Relay characteristics with Hysteresis loop

$$x(t) = \begin{cases} M, & \dot{e} > 0, e > e_0 \\ & \dot{e} < 0, e > -e_0 \\ -M, & \dot{e} > 0, e < e_0 \\ & \dot{e} < 0, e < -e_0 \end{cases}$$

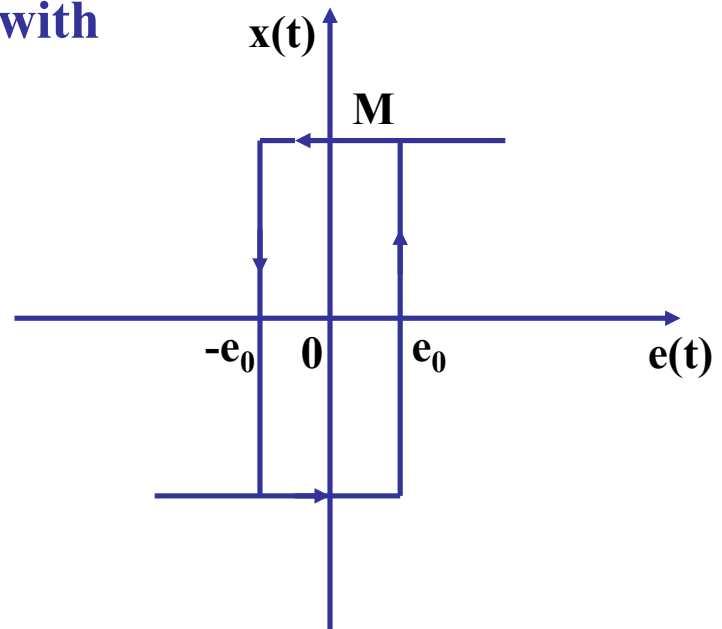


Fig. 8-8 (c) Relay characteristics with Hysteresis loop

4. Relay characteristics with Dead-zone and Hysteresis loop

$$x(t) = \begin{cases} M, & \dot{e} > 0, e \geq e_0 \\ & \dot{e} < 0, e > me_0 \\ 0, & \dot{e} > 0, -me_0 < e < e_0 \\ & \dot{e} < 0, -e_0 < e < me_0 \\ -M, & \dot{e} > 0, e < -me_0 \\ & \dot{e} < 0, e \leq -e_0 \end{cases}$$

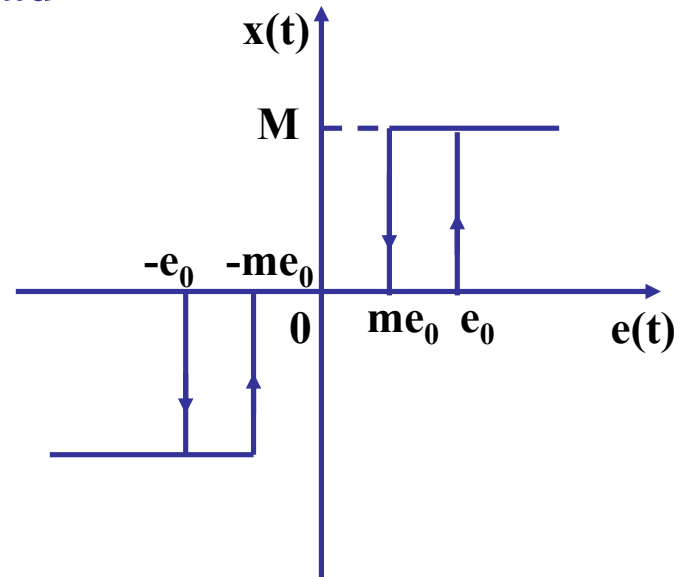


Fig. 8-8(d) Relay characteristics with Dead-zone and Hysteresis loop