7.1 Discrete-Time Control Systems

Discrete-systems: There is one or more impulse series or digital signals in the system.

Types:
- Sampling systems: Discrete Time, Continuous Value
- Digital systems: Discrete Time, Quantized Value

7.2 Signal Sampling and Shannon Theorem

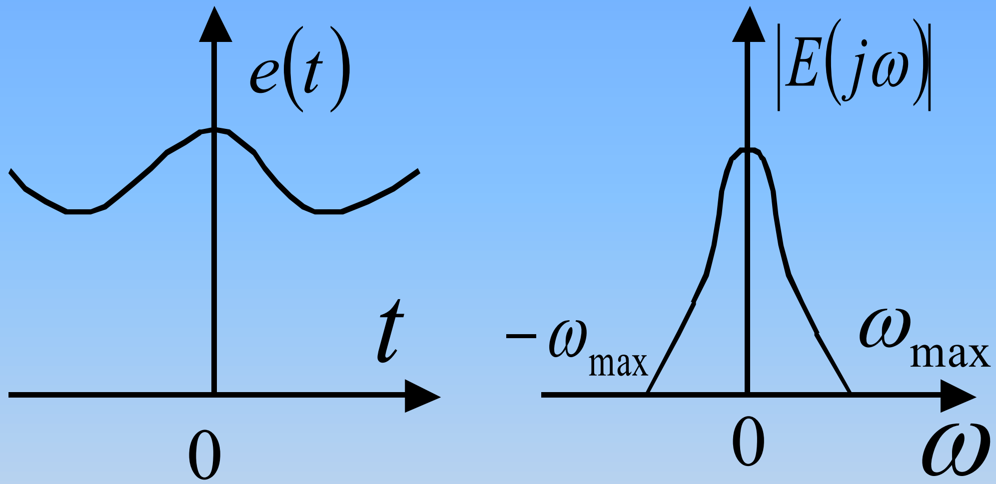
A/D:
- $\tau \ll T$
- Enough word length

- It is equivalent to an ideal sampling switch
- $e^*(t) = e(t) \cdot \delta_T(t)$

D/A: It can be realized by ZOH

Shannon theorem $\omega_s = \dfrac{2\pi}{T} > 2\omega_h$ or $T < \dfrac{\pi}{\omega_h}$

The continuous signal and its amplitude spectrum are



The Fourier-series expansion of $\delta_T(t)$:

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$

$\omega_s = \dfrac{2\pi}{T}$ is the sampling freq.

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta_T(t) e^{-jkw_s t} dt \overset{令 t=0}{=} \frac{1}{T}\int_{0^-}^{0^+} \delta(t) dt = \frac{1}{T}$$

$$\Downarrow$$

$$\delta_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

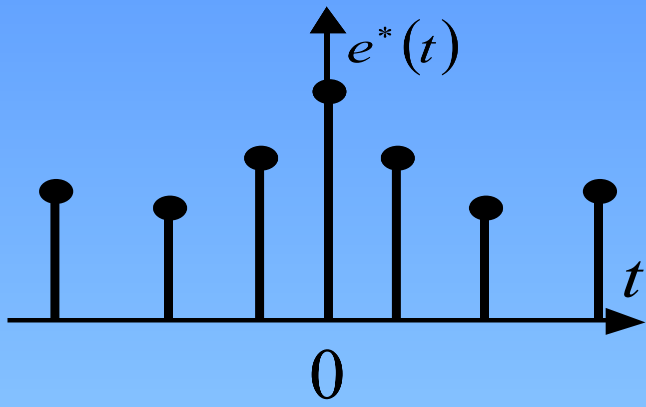
So the sampled signal is

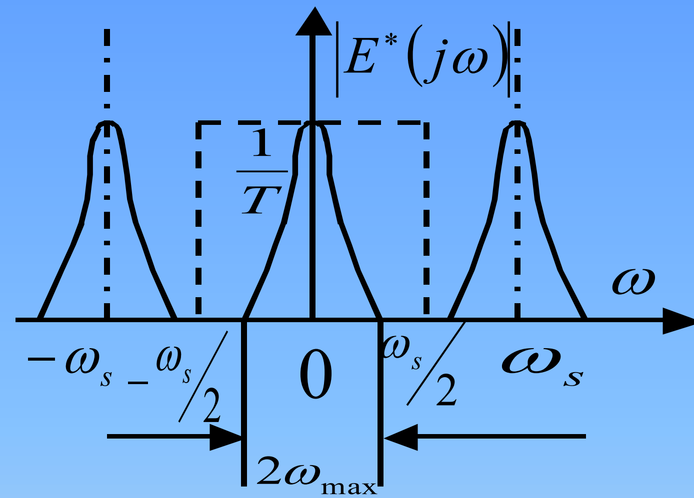
$$e^*(t) = e(t)\cdot\delta_T(t) = \frac{1}{T}\cdot\sum_{k=-\infty}^{\infty} e(t)\cdot e^{jk\omega_s t}$$

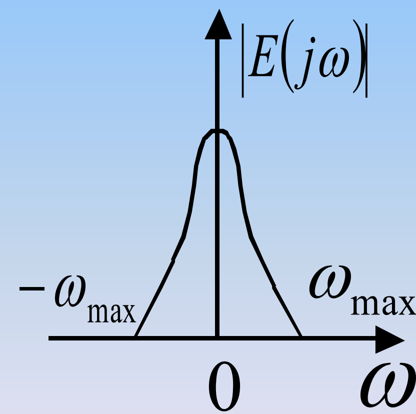
which Laplace transform is

$$E^*(\mathrm{j}\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} E\left[j\left(\omega + k\omega_s\right)\right]$$

where the operator s is replaced by $j\omega$







From the figure above, we can conclude that if $\omega_s > 2\omega_{\max}$

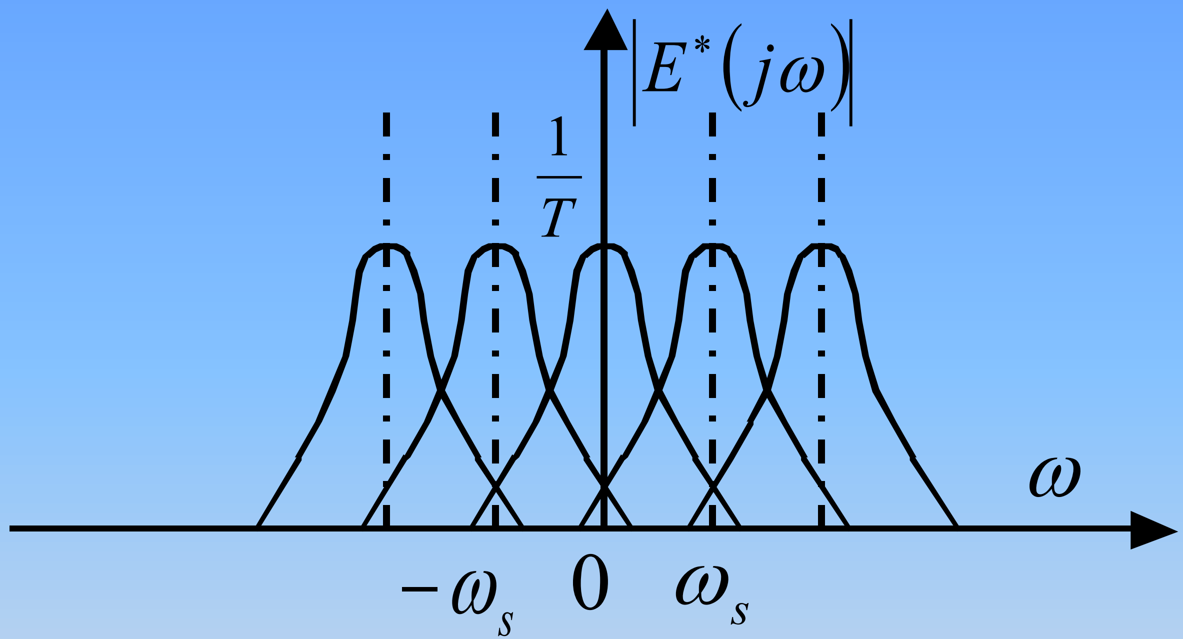
there are no overlap of each component, so the input signal can be recovered approximately. This is called sampling theorem or Shannon's Theorem

Shannon's Sampling Theorem:

Let $x(t)$ denote any continuous-time signal having a continuous Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Let $x^*(t)$ denote the samples of $x(t)$ at uniform intervals of T seconds. Then $x(t)$ can be exactly reconstructed from its samples $x^*(t)$ if and only if $X(j\omega) = 0$ **for all** $|\omega| \geq \pi / T$

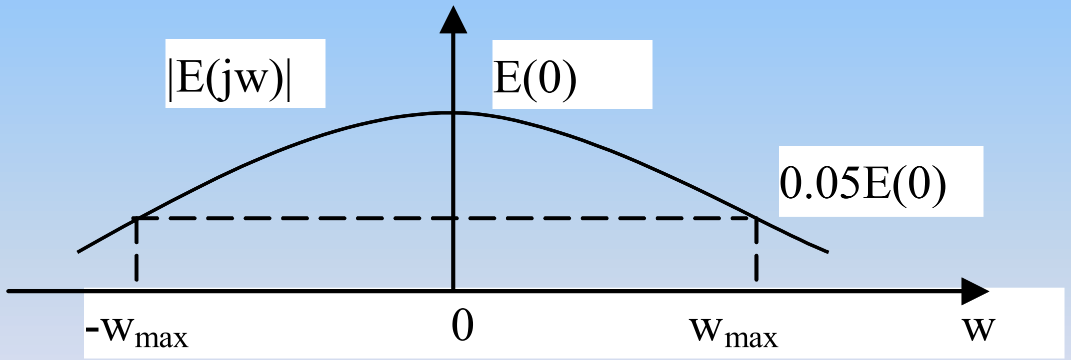


In the figure the input signal can't be recovered.

Problem: the maximum frequency ω_{max} is ***infinite*** for a non-periodic signal!

Then, how could we select the sampling frequency ω_s for it?

Solution:



Example 7-3 **Let $e(t)=e^{-t}$， select the sampling frequency according to Shannon's sampling therem.**

Solution.

The Laplace transform of $e(t)$: $$E(s)=\frac{1}{s+1}$$

The Fourier transform of $e(t)$ is: $$E(j\omega)=\frac{1}{j\omega+1}$$

The amplitude frequency characteristics:

$$\left|E(j\omega)\right|=\frac{1}{\sqrt{\omega^2+1}}$$

Assume the maximum frequency of $e(t)$ satisfies: $\left|E(j\omega_{\max})\right|=0.05\left|E(0)\right|$

$$\frac{1}{\sqrt{\omega_{\max}^2+1}}=0.05,\quad \omega_{\max}=20rad/s$$

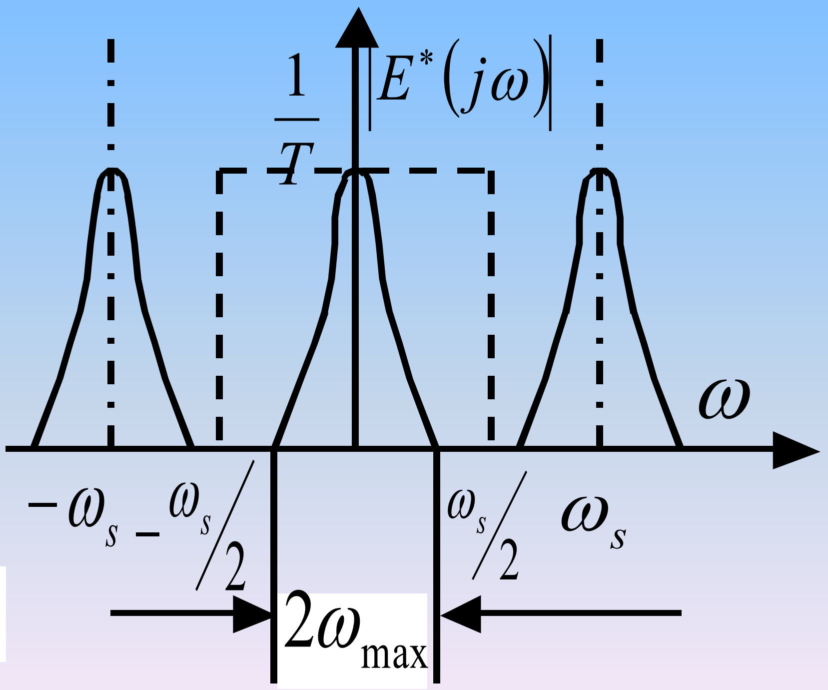
Then according to Shannon's theorem we have

$$\omega_s\geq 2\omega_{\max}=40rad/s$$

7.3 Signal Recovery and Zero-Order Hold

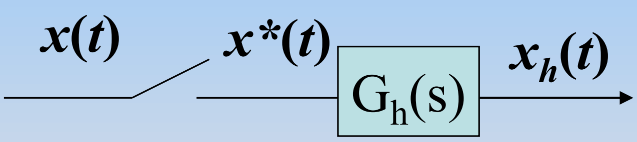
I.Signal recovery

The ideal filter is illustrated as the dotted line in the figure.

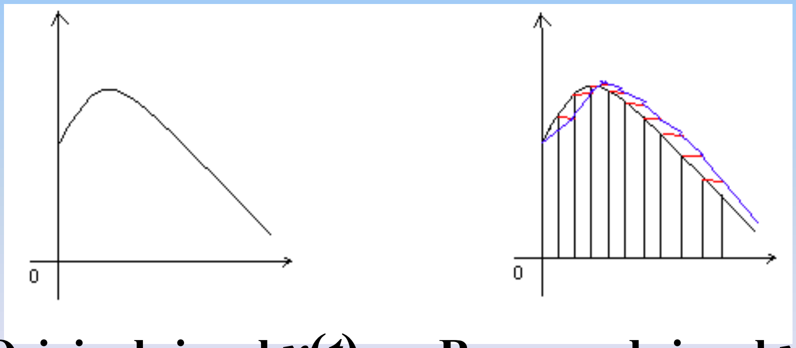


• The most common-used and simplest filter is zero-order hold filter.

• The zero-order hold (ZOH) is a mathematical model of the practical signal reconstruction done by a conventional digital-to-analog converter (DAC). That is, it describes the effect of converting a discrete-time signal to a continuous-time signal by ***holding each sample value for one sample interval***.



ZOH



Original signal $x(t)$ **Recovered signal $x_h(t)$**

The original signal $x(t)$ and recovered signal $x_h(t)$ satisfy :

$$x_h(t) = \sum_{k=0}^{\infty} x(kT)(1(t-kT)-1(t-kT-T))$$

Apply Laplace transform on both sides of the equation, we have

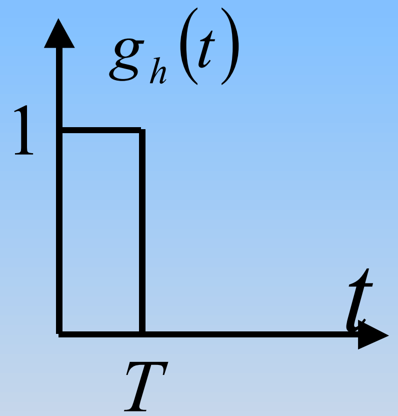
$$x_h(s) = \underbrace{\sum_{k=0}^{\infty} x(kT)e^{-kTs}}_{x^*(s)}\left[\frac{1}{s}-\frac{1}{s}e^{-Ts}\right]$$

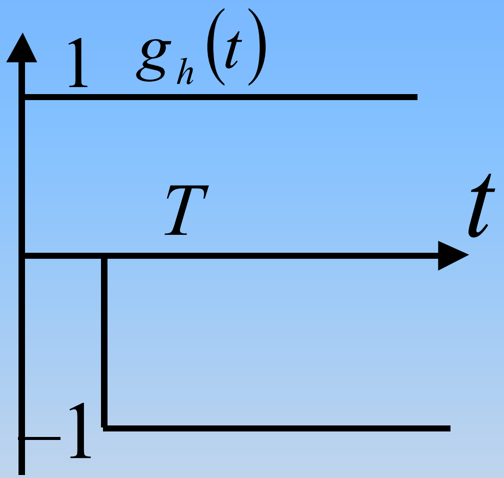
Then the ZOH equivalent transition function is

$$\frac{x_h(s)}{x^*(s)} = \frac{1-e^{-Ts}}{s} = G_h(s)$$

Corresponding time-domain function:

$$g_h(t) = 1(t) - 1(t - T)$$

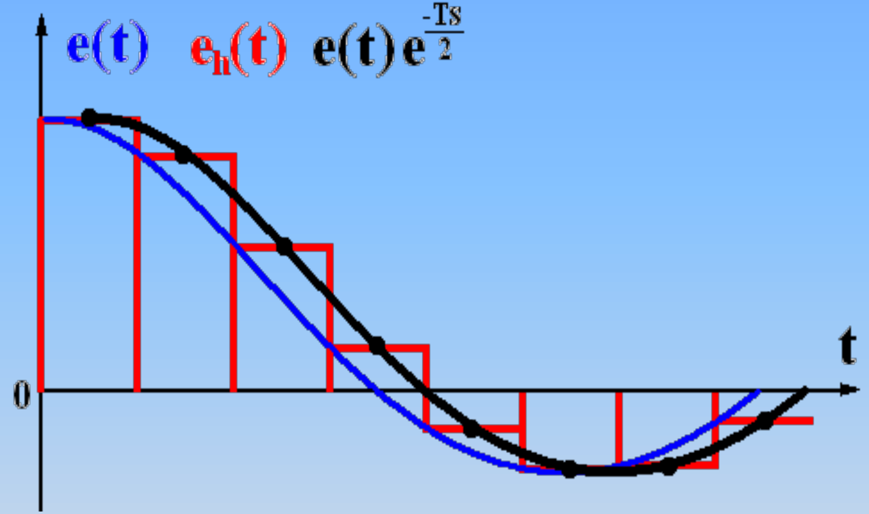




Effect of zero-order holder on the system

$$G_h(s) = \frac{1 - e^{-Ts}}{s}$$

$$\approx e^{-Ts/2}$$



Analysis of the frequency characteristics of ZOH filter :

$$G_h(j\omega) = \frac{1-e^{-j\omega T}}{j\omega} = \frac{e^{-\frac{1}{2}j\omega T}\left(e^{j\frac{1}{2}\omega T} - e^{-j\frac{1}{2}\omega T}\right)}{j\omega} = \frac{2e^{-\frac{1}{2}j\omega T}\left(\dfrac{e^{j\frac{1}{2}\omega T} - e^{-j\frac{1}{2}\omega T}}{2j}\right)}{\omega}$$

Considering we have, $\boxed{\sin x = \dfrac{e^{jx} - e^{-jx}}{2j}}$ **therefore**

$$G_h(j\omega) = \frac{2e^{-\frac{1}{2}j\omega T}\sin(\frac{1}{2}\omega T)}{\omega} \Rightarrow G_h(j\omega) = T\frac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}}e^{-\frac{1}{2}j\omega T}$$

$$G_h(j\omega)=\frac{1-e^{-j\omega T}}{j\omega}=T\cdot\frac{\sin\left(\omega T/2\right)}{\omega T/2}\cdot e^{-\frac{j\omega T}{2}}$$

$$\because T=\frac{2\pi}{\omega_s}$$

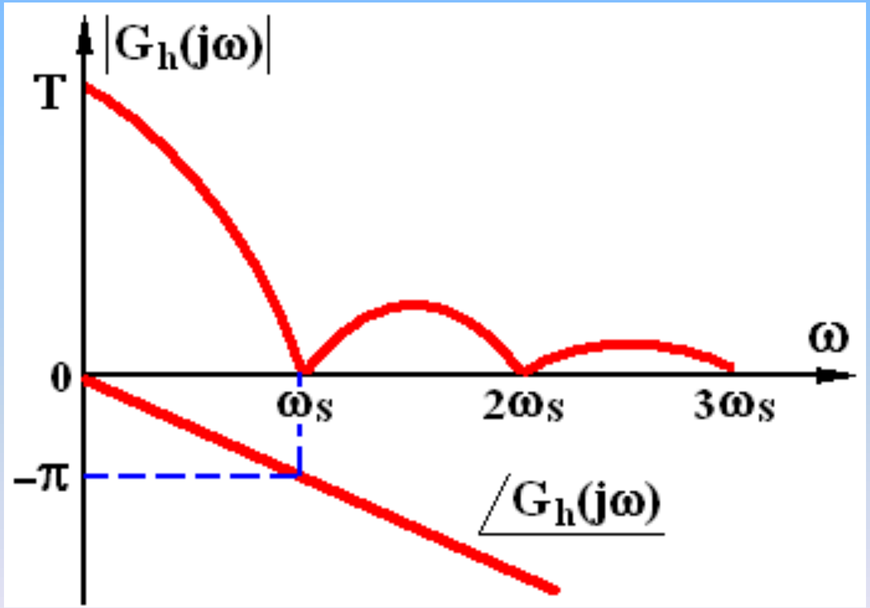
let $$S_a(x)=\frac{\sin x}{x}$$

then we have

$$G_h(j\omega)=\frac{2\pi}{\omega_s}\cdot S_a(\pi\omega/\omega_s)\cdot e^{-j\frac{\pi\omega}{\omega_s}}$$

Amplitude: $$|G_h(j\omega)| = \frac{2\pi}{\omega_s} \cdot |S_a(\pi\omega/\omega_s)|$$

Phase angle: $$\angle G_h(j\omega) = -\frac{\pi\omega}{\omega_s}$$



Note:

• The ZOH filter is not an ideal low-pass filter. ***Ripple error*** may occur after the filtering.

• There is a ***delayed phase angle*** when a signal is filtered by a ZOH.

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

7.6 Dynamic Performance of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

7.4 Z-Transform and Inverse Z Transform

7.4.1. Z-transform

Definition:

$$\because E^*(s) = \sum_{k=0}^{+\infty} e(kT) \cdot e^{-kTs}$$

Set

$$z = e^{Ts} \qquad s = \frac{1}{T} \ln z$$

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

$$E(z) = E^*(s) \big|_{z=e^{Ts}}$$

$$E^*(s) = \sum_{k=0}^{\infty} e(kT) \cdot e^{-kTs}$$

$$z^{-1} = e^{-Ts}$$

$$E(z) = Z[e^*(t)] = E^*(s)\Big|_{z=e^{Ts}} = \sum_{n=0}^{\infty} e(kT) \cdot z^{-k}$$

Rmk:

$$E(z) = Z[e^*(t)] = Z[E(s)] = Z[E^*(s)] = Z[e(t)]$$

The z-transform is only for discrete signal.

E(z) is only mapping to a unique e*(t), but not a unique e(t).

7.4.2 Methods of z-Transform

By the definition.

Partial fraction expansion.

1、By the Definition

只和 $e^*(t)$ 一一对应.

Example 1 $\boldsymbol{x_1(t) = 1(t)}$ **and** $x_2(t) = \sum_{k=0}^{\infty} \delta(t - kT)$, **obtain** $X_1(z)$ and $X_2(z)$。

Solution:

$$X_1(z) = \sum_{k=0}^{\infty} x_1(kT) z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$X_2(s) = \sum_{k=0}^{\infty} e^{-kTs} \quad \xrightarrow{\quad z = e^{Ts} \quad}$$

$$X_2(z) = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Tips: Though $x_1(t)$ and $x_2(t)$ are not same, they may have the same Z-transform.

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

Example 2 $e(t)=\sin\omega t=\frac{1}{2j}\left[e^{j\omega t}-e^{-j\omega t}\right]$

$$E(z)=\sum_{k=0}^{\infty}\frac{1}{2j}\left[e^{j\omega kT}-e^{-j\omega kT}\right]\cdot z^{-k} \quad =\frac{1}{2j}\sum_{k=0}^{\infty}\left[(e^{j\omega T}z^{-1})^{k}-(e^{-j\omega T}z^{-1})^{k}\right]$$

$$=\frac{1}{2j}\left[\frac{1}{1-e^{j\omega T}z^{-1}}-\frac{1}{1-e^{-j\omega T}z^{-1}}\right]=\frac{1}{2j}\left[\frac{z}{z-e^{j\omega T}}-\frac{z}{z-e^{-j\omega T}}\right]$$

$$=\frac{1}{2j}\cdot\frac{z(e^{j\omega T}-e^{-j\omega T})}{z^{2}-(e^{j\omega T}+e^{-j\omega T})z+1}=\frac{z\sin\omega T}{z^{2}-2\cos\omega T\cdot z+1}$$

Tip: $e^{j\omega t}=\cos\omega t+j\sin\omega t$

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

Example 3 $e(t) = t$

Solution. $E(z) = \sum_{k=0}^{\infty} kT \cdot z^{-k} = T\left[z^{-1} + 2z^{-2} + 3z^{-3} + \cdots\right]$

$$= Tz\left[z^{-2} + 2z^{-3} + 3z^{-4} + \cdots\right]$$

$$= -Tz\left[\frac{d}{dz}z^{-1} + \frac{d}{dz}z^{-2} + \frac{d}{dz}z^{-3} + \cdots\right]$$

$$= -Tz\frac{d}{dz}\left[z^{-1} + z^{-2} + z^{-3} + \cdots\right]$$

$$= -Tz\frac{d}{dz}z^{-1}\left[1 + z^{-1} + z^{-2} + \cdots\right]$$

$$= -Tz\frac{d}{dz}\left[\frac{1}{z} \cdot \frac{1}{1 - z^{-1}}\right] = -Tz\frac{d}{dz}\left[\frac{1}{z-1}\right] = \frac{Tz}{(z-1)^2}$$

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

2. Partial Fraction Expansion

Example 4 $E(s)=\dfrac{1}{(s+a)(s+b)}$ **Obtain** $E(z)$**=?**

Solution:

$$E(s)=\frac{1}{a-b}\cdot\frac{(s+a)-(s+b)}{(s+a)(s+b)}=\frac{1}{a-b}\left[\frac{1}{s+b}-\frac{1}{s+a}\right]$$

$$e(t)=\frac{1}{a-b}\left[e^{-bt}-e^{-at}\right]$$

$$E(z)=\frac{1}{a-b}\sum_{k=0}^{\infty}\left[e^{-bkT}-e^{-akT}\right]\cdot z^{-k}$$

$$=\frac{1}{a-b}\left[\sum_{k=0}^{\infty}(e^{-bT}\cdot z^{-1})^{k}-\sum_{k=0}^{\infty}(e^{-aT}\cdot z^{-1})^{k}\right]$$

$$=\frac{1}{a-b}\left[\frac{1}{1-e^{-bT}z^{-1}}-\frac{1}{1-e^{-aT}z^{-1}}\right]=\frac{1}{a-b}\left[\frac{z}{z-e^{-bT}}-\frac{z}{z-e^{-aT}}\right]$$

The z-transform of typical functions

$f(t)$	$F(s)$	$F(z)$
$\delta(t)$	1	1
$f(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{zT}{(z-1)^2}$
$t^2/2$	$\frac{1}{s^3}$	$\frac{z(z+1)T^2}{2(z-1)^3}$

e^{-at}	$\dfrac{1}{s+a}$	$\dfrac{z}{z-e^{-aT}}$
te^{-at}	$\dfrac{1}{(s+a)^2}$	$\dfrac{zTe^{-aT}}{(z-e^{-aT})^2}$
$a^{t/T}$	$\dfrac{1}{s-(1/T)\ln a}$	$\dfrac{z}{z-a}$ $(a>0)$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$	$\dfrac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$	$\dfrac{z^2-z\cos\omega T}{z^2-2z\cos\omega T+1}$
$1-e^{-at}$	$\dfrac{a}{s(s+a)}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$\dfrac{ze^{-at}\sin\omega T}{z^2-2ze^{-at}\cos\omega T+e^{-2aT}}$
$e^{-at}\cos\omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$\dfrac{z(z-e^{-aT}\cos\omega T)}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$

7.4.3 Properties of z-Transform

1. linear property $Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$

2. Real shifting theorem 实位移定理

① Lag 延时定理 $Z\left[e(t-nT)\right]=z^{-n}E(z)$

Proof. $$\mathrm{LHS}=\sum_{K=0}^{\infty}e(kT-nT)\cdot z^{-k}$$

$$\Big\downarrow\; j=k-n$$

$$=\sum_{j=-n}^{\infty}e(jT)\cdot z^{-(j+n)}=z^{-n}\sum_{j=0}^{\infty}e(jT)\cdot z^{-j}$$

$$=z^{-n}E(z)=\mathrm{RHS}$$

Left Hand Side (LHS); Right Hand Side (RHS)

2. Real shifting theorem 实位移定理

② Lead 超前定理

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

Proof.

$$\mathrm{LHS}=\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-k}=z^{n}\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-(k+n)}$$

$$\Big\downarrow\; j=k+n$$

$$=z^{n}\sum_{j=n}^{\infty}e(jT)\cdot z^{-j}=z^{n}\left[\sum_{j=0}^{\infty}e(jT)\cdot z^{-j}-\sum_{j=0}^{n-1}e(jT)\cdot z^{-j}\right]$$

$$=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]=\mathrm{RHS}$$

$$Z[e(t+nT)] = z^n\left[E(z) - \sum_{k=0}^{n-1} e(kT)\cdot z^{-k}\right]$$

Example 5 $e(t) = t - T$

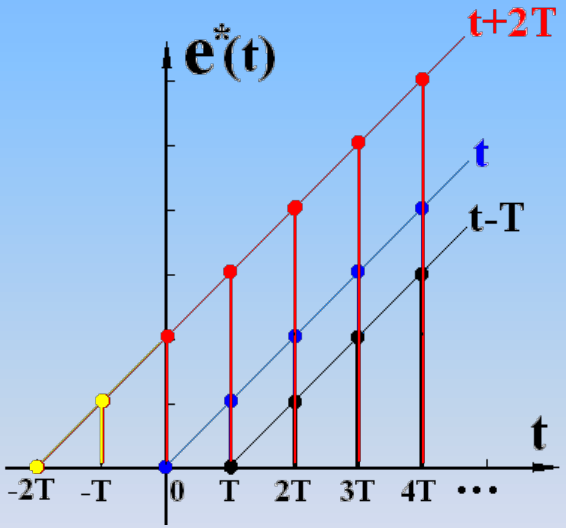
$$E(z) = Z[t-T] = z^{-1}Z[t] = z^{-1}\frac{Tz}{(z-1)^2} = \frac{T}{(z-1)^2}$$

Example 6 $e(t) = t + 2T$

$$E(z) = Z[t+2T]$$

$$= z^2\left\{Z[t] - \sum_{k=0}^{1} kT\cdot z^{-k}\right\}$$

$$= z^2\left[\frac{Tz}{(z-1)^2} - 0 - Tz^{-1}\right]$$



3. Complex shifting theorem
复位移定理

$$Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)$$

Proof.

$$\text{LHS}=\sum_{k=0}^{\infty}e(kT)\cdot e^{\mp akT}z^{-k} \quad =\sum_{k=0}^{\infty}e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k}$$

$$\Big\downarrow \quad z_1=z\cdot e^{\pm aT}$$

$$=\sum_{k=0}^{\infty}e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k} = E(z_1) = E\left(z\cdot e^{\pm aT}\right)=\text{RHS}$$

Example 7 $e(t)=t\cdot e^{-at}$

$$E(z_1)=Z[t]\big|_{z_1=z\cdot e^{aT}}=\frac{Tz_1}{(z_1-1)^2}=\frac{T(z\cdot e^{aT})}{(z\cdot e^{aT}-1)^2}=\frac{Tz\cdot e^{-aT}}{(z-e^{-aT})^2}$$

4. Initial-value Theorem

$$\lim_{n\to 0} e(nT) = \lim_{z\to\infty} E(z)$$

Proof:

$$E(z) = \sum_{n=0}^{\infty} e(nT)\cdot z^{-n}$$

$$= \left[e(0) + e(1)\cdot z^{-1} + e(2)\cdot z^{-2} + e(3)\cdot z^{-3} + \cdots\right]$$

$$\lim_{z\to\infty} E(z) = e(0)$$

Example 8 $E(z) = \dfrac{0.792\cdot z^2}{(z-1)[z^2 - 0.416z + 0.208]}$

$$e(0) = \lim_{z\to\infty} E(z) = 0$$

Properties of z-Transform

1. linear property $$Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$$

2. Real shifting theorem
$$\begin{cases}\text{Lag} \quad Z\left[e(t-nT)\right]=z^{-n}E(z)\\ \text{Lead} \quad Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]\end{cases}$$

3. Complex shifting theorem $$Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)$$

4. Initial-value theorem $$\lim_{n\to 0} e(nT)=\lim_{z\to\infty} E(z)$$

5. Final-value theorem $$\lim_{n\to\infty} e(nT)=\lim_{z\to 1}(z-1)\cdot E(z)$$

6. Convolution theorem $$c^*(t)=e^*(t)*g^*(t) \quad\Rightarrow\quad C(z)=E(z)\cdot G(z)$$

7.4.4 Inverse z-Transform

$$Z^{-1}[X(z)] = x(nT)$$

Tips:

Inverse Z-transform can only provide discrete-time signal $x^{*}(t)$, instead of continuous signal x(t)。

- **Long Division（长除法）**
- **Partial-Fraction expansion** — Expansion of $\dfrac{E(z)}{z}$
- **Residue（留数法）** — $e(nT) = \sum \mathrm{Res}\left[E(z) \cdot z^{n-1}\right]$

1. Long Division（长除法）

$$E(z)=\frac{b_0 z^m+b_1 z^{m-1}+\cdots+b_m}{a_0 z^n+a_1 z^{n-1}+\cdots+a_n}$$

Numerator is divided by denominator ,we get

分子 分母

$$E(z)=c_0+c_1 z^{-1}+\cdots+c_k z^{-k}+\cdots=\sum_{k=0}^{\infty} c_k z^{-k}=\sum_{k=0}^{\infty} e(kT) z^{-k}$$

$$e^*(t)=c_0\delta(t)+c_1\delta(t-T)+\cdots+c_k\delta(t-kT)+\cdots$$

Example: $E(z)=\dfrac{10z}{(z-1)(z-2)}$ ， obtain e*(t)。

Solution: $$E(z)=\frac{10z}{z^2-3z+2}=10z^{-1}+30z^{-2}+70z^{-3}+150z^{-4}+\cdots$$

$$\begin{array}{r|l}
 & 10z^{-1}+30z^{-2}+70z^{-3}+150z^{-4}\cdots \\
z^2-3z+2 & 10z \\
 & \underline{10z-30z^0+20z^{-1}} \\
 & \quad 30z^0-20z^{-1} \\
 & \quad \underline{30z^0-90z^{-1}+60z^{-2}} \\
 & \qquad 70z^{-1}-60z^{-2} \\
 & \qquad \underline{70z^{-1}-210z^{-2}+140z^{-3}} \\
 & \qquad\quad 150z^{-2}-140z^{-3} \\
 & \qquad\quad \cdots
\end{array}$$

$$\begin{aligned}
e^*(t) &= 10\delta(t-T) \\
&\quad +30\delta(t-2T) \\
&\quad +70\delta(t-3T) \\
&\quad +150\delta(t-4T) \\
&\quad +\cdots
\end{aligned}$$

Example

$$F(z)=\frac{z}{(z-2)(z-3)}$$ ， **obtain f*(t)。**

Solution：

Because $F(z)=\dfrac{z}{z^2-5z+6}=\dfrac{z^{-1}}{1-5z^{-1}+6z^{-2}}$

By long-division, we get that

$$F(z)=z^{-1}+5z^{-2}+19z^{-3}+65z^{-4}+\cdots$$

Thus

$$f(0)=0,\ \ f(T)=1,\ \ f(2T)=5,\ \ f(3T)=19,\ \ f(4T)=65,\cdots$$

Then

$$f^*(t)=\delta(t-T)+5\delta(t-2T)+$$
$$19\delta(t-3T)+65\delta(t-4T)+\cdots$$

2. Partial fraction expansion

Note: here, we expand $\frac{X(z)}{z}$, instead of X(z)。

$$\frac{X(z)}{z} = \sum_{i=1}^{n} \frac{A_i}{z - z_i}$$

Consider

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}$$

Then

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{a_0\prod_{i=1}^{n}(z-z_i)}$$

If there is no repeated root for the denominator，it generates

$$X(z)=z\left(\frac{A_1}{z-z_1}+\frac{A_2}{z-z_2}+\cdots+\frac{A_n}{z-z_n}\right)$$

其中系数A_i可由式决定：

$$A_i=\left[(z-z_i)\frac{X(z)}{z}\right]\Bigg|_{z=z_i}$$

Example **Consider**

$$F(z)=\frac{z}{(z-1)(z-e^{-T})}$$

Obtain $f^*(t)$。

Solution：

$$\frac{F(z)}{z}=\frac{K_1}{z-1}+\frac{K_2}{z-e^{-T}}$$

$$K_1=\lim_{z\to 1}\left(\frac{z-1}{z}\right)F(z)=\frac{1}{1-e^{-T}}$$

$$K_2=\lim_{z\to e^{-T}}\left(\frac{z-e^{-T}}{z}\right)F(z)=-\frac{1}{1-e^{-T}}$$

$$F(z)=\frac{1}{1-e^{-T}}\left(\overset{1}{\frac{z}{z-1}}-\overset{e^{-t}}{\frac{z}{z-e^{-T}}}\right)$$

$$\Rightarrow\quad f(nT)=\frac{1}{1-e^{-T}}\left(1-e^{-nT}\right)$$

$$f^*(t)=\frac{1}{1-e^{-T}}\sum_{k=0}^{+\infty}(1-e^{-kT})\delta(t-kT)$$

3、 Residue(留数法)

$$F(z)=\sum_{k=0}^{+\infty} f(kT)z^{-k}$$

$$F(z)z^{m-1}=\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}$$

Γ Encircle all the poles of $F(z)z^{k-1}$

$$\oint_{\Gamma} F(z)z^{m-1}dz=\oint_{\Gamma}\left[\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}\right]dz$$

$$\oint_{\Gamma} F(z)z^{m-1}dz=\sum_{k=0}^{+\infty} f(kT)\oint_{\Gamma} z^{m-k-1}dz$$

When m=k,

$$f(kT)=\sum_{i=1}^{n} res[F(z)z^{k-1},z_i]$$

$z_i, i=1,2,\cdots,n$ **are all the poles of** $F(z)z^{k-1}$

$$Res\left[z^{(k-1)}x(z)\right]=\lim_{z\to z_i}\frac{1}{(r-1)!}\frac{d^{r-1}}{dz^{r-1}}\left[(z-z_i)^r z^{k-1}x(z)\right]$$

其中***Res*[]**表示函数的留数, ***r***为极点的阶数。

Example For

$$F(z)=\frac{10z}{(z-1)(z-2)}$$

Obtain its inverse z-transform by residue Method.

Solution：

$$F(z)z^{k-1}=\frac{10z^{k}}{(z-1)(z-2)}$$

Poles $z_1=1$ and $z_2=2$ ， and

$$res[F(z)z^{k-1},1]=\lim_{z\to 1}(z-1)F(z)z^{k-1}=-10$$

$$res[F(z)z^{k-1},2]=\lim_{z\to 2}(z-2)F(z)z^{k-1}=10\cdot 2^{k}$$

Then

$$f(kT)=10(2^{k}-1) \qquad (k=0,1,2,\cdots)$$

Example 11 $E(z)=\dfrac{z^2}{(z-0.8)(z-0.1)}$ **Obtain e*(t). (PFE & Residue)**

PFE: $$\frac{E(z)}{z}=\frac{z}{(z-0.8)(z-0.1)}=\frac{C_1}{(z-0.8)}+\frac{C_2}{(z-0.1)}$$

$$C_1=\lim_{z\to 0.8}\frac{z}{(z-0.1)}=\frac{8}{7}\qquad C_2=\lim_{z\to 0.1}\frac{z}{(z-0.8)}=\frac{-1}{7}$$

$$=\frac{8/7}{(z-0.8)}-\frac{1/7}{(z-0.1)}$$

$$E(z)=\frac{8}{7}\cdot\frac{z}{(z-0.8)}-\frac{1}{7}\cdot\frac{z}{(z-0.1)}$$

$$e(t)=(8\times 0.8^{\frac{t}{T}}-0.1^{\frac{t}{T}})/7\qquad e(nT)=(8\times 0.8^{n}-0.1^{n})/7$$

$$e^*(t)=\sum_{n=0}^{\infty}\left[(8\times 0.8^{n}-0.1^{n})/7\right]\cdot\delta(t-nT)$$

算 $e^*(t)$.

Example 12 $E(z) = \dfrac{5}{(z-a)^2}$ **Obtain e*(t). (Residue)**

Solution.

$$e(nT) = \sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right] = \mathop{\mathrm{Res}}_{z=a}\left[\frac{5}{(z-a)^2}\cdot z^{n-1}\right]$$

$$e(nT) = \frac{1}{(2-1)!}\lim_{z\to a}\frac{d}{dz}\left[(z-a)^2\frac{5\cdot z^{n-1}}{(z-a)^2}\right]$$

$$= \lim_{z\to a}\frac{d}{dz}\left[5\cdot z^{n-1}\right]$$

$$= 5\cdot\lim_{z\to a}\left[(n-1)\cdot z^{n-2}\right]$$

$$= 5\cdot(n-1)\cdot a^{n-2}$$

$$e^*(t) = \sum_{n=0}^{\infty}\left(5(n-1)\cdot a^{n-2}\right)\cdot\delta(t-nT)$$

7.4.5 Limitations of z-Transform

(1) only shows the information of samples;

(2) In some cases, the continuous signal may jump on the sampling point.

跳变.