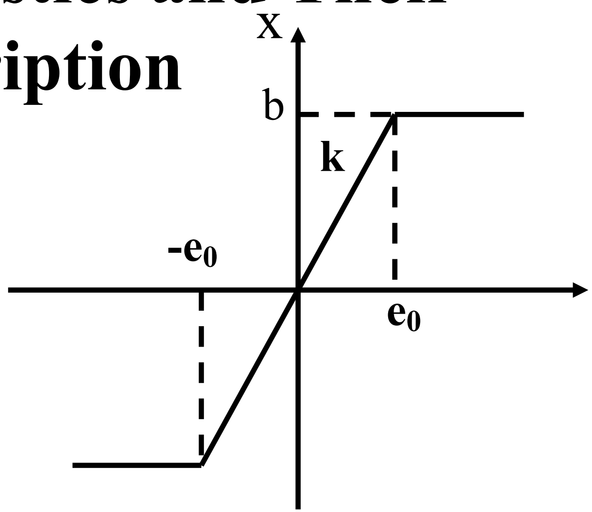
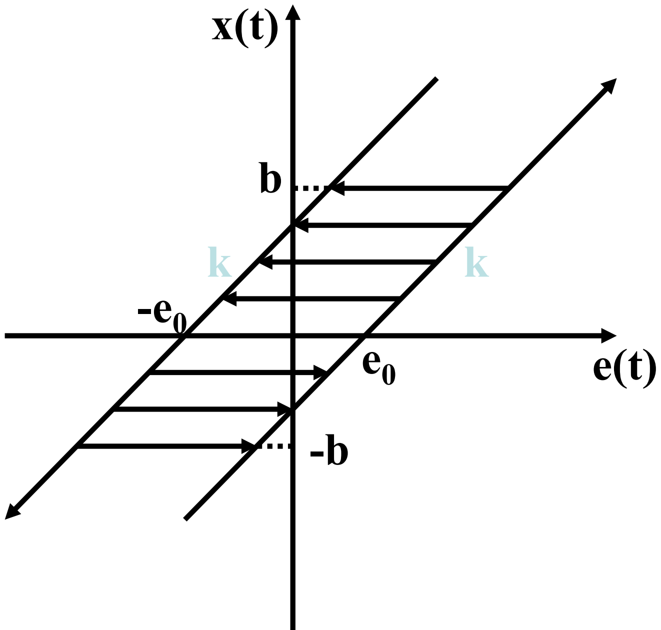
Review

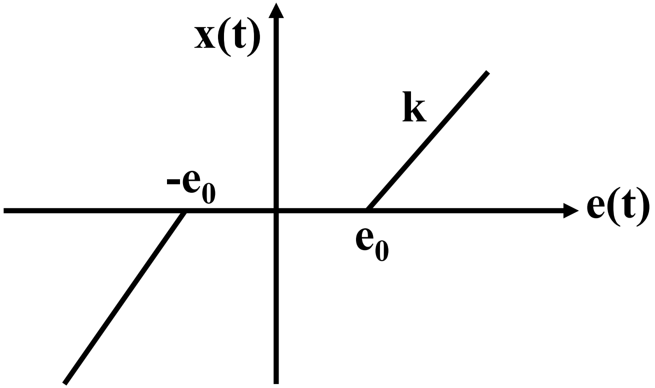
- Features of nonlinear systems
  - principle of superposition(叠加原理) is not available
  - The stability of a nonlinear system depends on not only the inherent structure and parameters of control systems, but also the initial conditions and the inputs.
  - Periodic oscillation
  - Jump resonance and Multi-valued response
  - harmonic oscillation
- Phase Plane Method
  - Singular Point and Limit Cycle
  - Time domain analysis of typical nonlinear systems

Typical Nonlinear characteristics and Their Mathematical Description

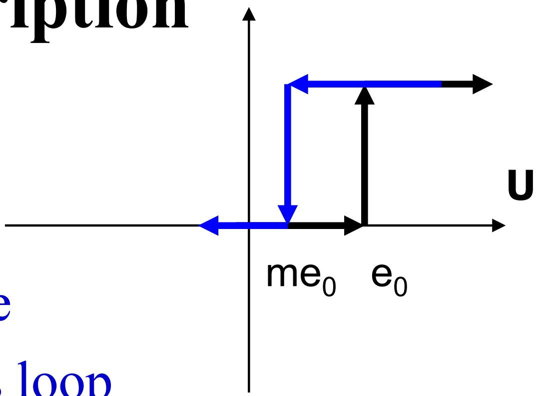
- Saturation characteristics
- Dead-zone characteristics
- Gap characteristics



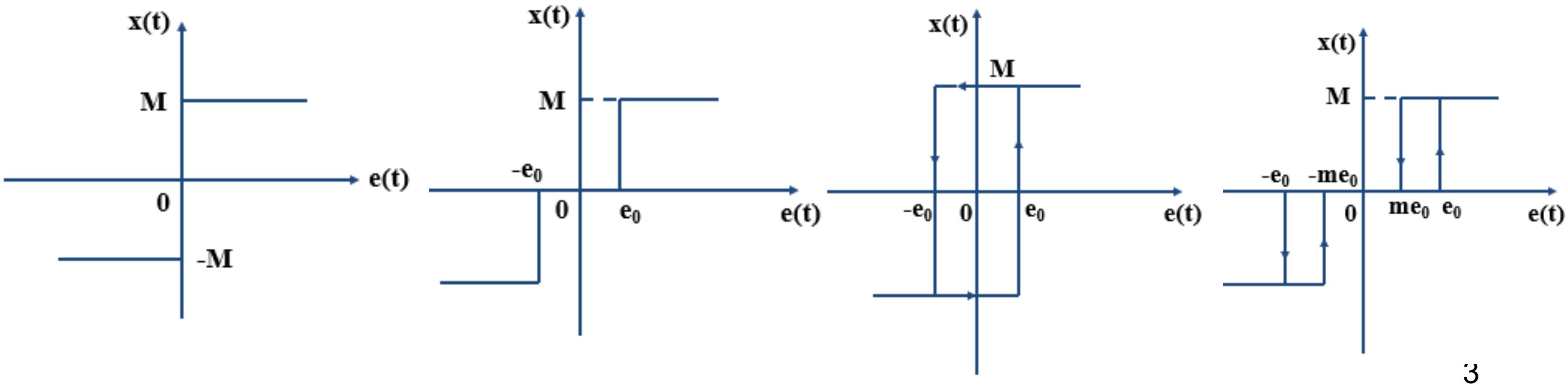




Typical Nonlinear characteristics and Their Mathematical Description



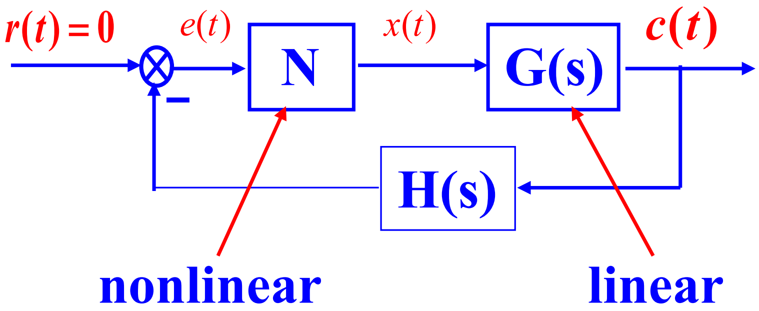
- Relay characteristics
  - Ideal relay characteristics
  - Relay characteristics with Dead-zone
  - Relay characteristics with Hysteresis loop
  - Relay characteristics with Dead-zone and Hysteresis loop



§ 8.3 Describing function method (Harmonic Linearizing method)

Basic idea

For the nonlinear system



Typical structure of the nonlinear systems

$If: e(t) = A\sin\omega t$ ⟹ a sinusoidal input,

$x(t)$, maybe it is not a sinusoidal but a periodic function, can be expressed as a Fourier series.

Assumption:

The harmonic of *x*(*t*) could be neglected, then:

$$x(t) \approx x_1 \sin(\omega t + \phi_1)$$

$\Longrightarrow$ **output frequency is equal to input frequency approximately.**

Similar to the *frequency analysis* of linear system, we can perform frequency analysis for the nonlinear system based on the assumption.

$\Longrightarrow$ **Describing function**

- The describing function method is mainly used to analyze the ***stability*** and ***self-oscillation*** of the nonlinear system without external excitation.

- Advantage:
  - It is not confined by ***the order of the system***. 无论阶次如何

- Disadvantages:
  - It is an ***approximate analysis method***. 近似方法
  - It can only be used to study the system ***frequency characteristics***. 频率特征

8.3.1 Concept of describing function

The describing function method can be applied to nonlinear systems with the following features:

1. The linear part and the nonlinear part can be separated.

Shown in Fig. 8-9，NL is a *nonlinear part*，G is the transfer function of the *linear part*.

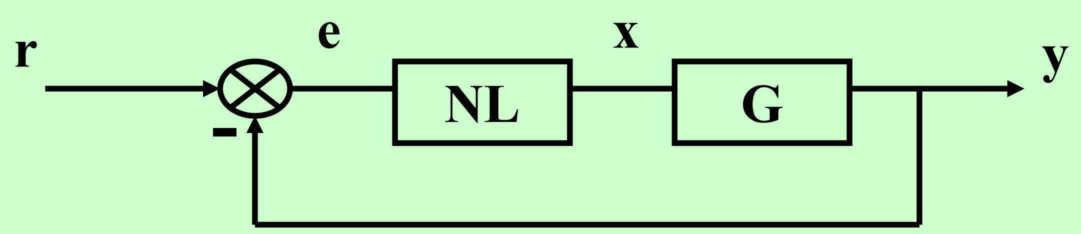


Fig. 8-9 Typical structure of nonlinear system

2. The system has an *odd-symmetric nonlinearity*, and the input-output relationship of the nonlinear part is static (without energy storage elements).

- **If so, sinusoidal input → periodic output**
- **The output can be expanded into Fourier series with a *zero D. C. component*.**

0直流分量

$$f(x) = -f(-x)$$

3. The linear part is a good low-pass filter

We can suppose the *higher-oder harmonic* is filtered out.

⇓

There is only a *fundamental component* in the the output .

If all the conditions above are satisfied, we can describe the nonlinear components by the frequency response like as that we did in the linear systems.

So we have:

Definition of the describing function

The describing function $N(A)$ of the nonlinear element is: the *complex ratio* of the fundamental component of the output $x(t)$ and the sinusoidal input $e(t)$, that is:

$$For\ \ e(t) = A\sin\omega t,$$

$$\begin{aligned} x(t) &\approx A_1\cos\omega t + B_1\sin\omega t \\ &= x_1\sin(\omega t + \phi_1) \end{aligned} \Longrightarrow N(A) = \frac{x_1 e^{j\phi_1}}{A}$$

Assume the input of nonlinear is sinusoidal $e(t) = A\sin\omega t$

Normally, the output is periodic, which can be expressed as a ***Fourier series*****:**

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\omega t + B_n \sin n\omega t\right)$$

The nonlinearity is ***odd-symmetric*****(奇对称).**

$$\Rightarrow \quad A_0 = 0$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos n\omega t \; d(\omega t)$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin n\omega t \; d(\omega t)$$

For the fundamental component, we have

$$\boxed{A_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos\omega t \; d(\omega t)}$$

$$\boxed{B_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin\omega t \; d(\omega t)}$$

Thus, the fundamental component is

$$\boxed{x_1(t) = A_1\cos\omega t + B_1\sin\omega t = x_1\sin(\omega t + \varphi_1)}$$

where $\boxed{x_1 = \sqrt{A_1^2 + B_1^2}}$ $\boxed{\varphi_1 = arctg\frac{A_1}{B_1}}$

The describing function is then given by

$$N(A) = \frac{X_1}{A} e^{j\varphi_1}$$

Obviously, the describing function is a function of the input amplitude *A*. So we can regard it as a *variable gain amplifier*.

$$\boxed{N(A) = \frac{\sqrt{A_1^2 + B_1^2}}{A} e^{j\,arctg\frac{A_1}{B_1}} = \frac{B_1}{A} + j\frac{A_1}{A}}$$

Replacing the nonlinear part by N(A), we can extend the *frequency response method* of linear system to the nonlinear system so as to analyze the *frequency characteristics* of nonlinear system.

Remarks:

Normally, the describing function N is *a function of the amplitude and the frequency of input signal*, it should be expressed as $N(A,\omega)$.

In most of the nonlinear components, there are no energy storage elements. The frequencies of output and input are then independent. So the describing function N of common nonlinear components is *only a function of the amplitude* of input, which can be expressed as $N(A)$。

Remarks: (cont.)

If the nonlinearity is *single-valued odd-symmetric* ⇒

The output $x(t)$ is *an odd function*.

$$A_1 = 0 \qquad N(A) = B_1 / A$$

The describing function is a *real function* of input amplitude A.

If the nonlinearity is not *single-valued odd-symmetric* ⇒

The output $x(t)$ is *neither an odd nor even*.

$$A_1 \neq 0, B_1 \neq 0$$

The describing function is a *complex function* of input amplitude A.

8.3.2 Describing function of typical nonlinear characteristic

1. Saturation

Assume the input is $e(t) = A\sin\omega t$

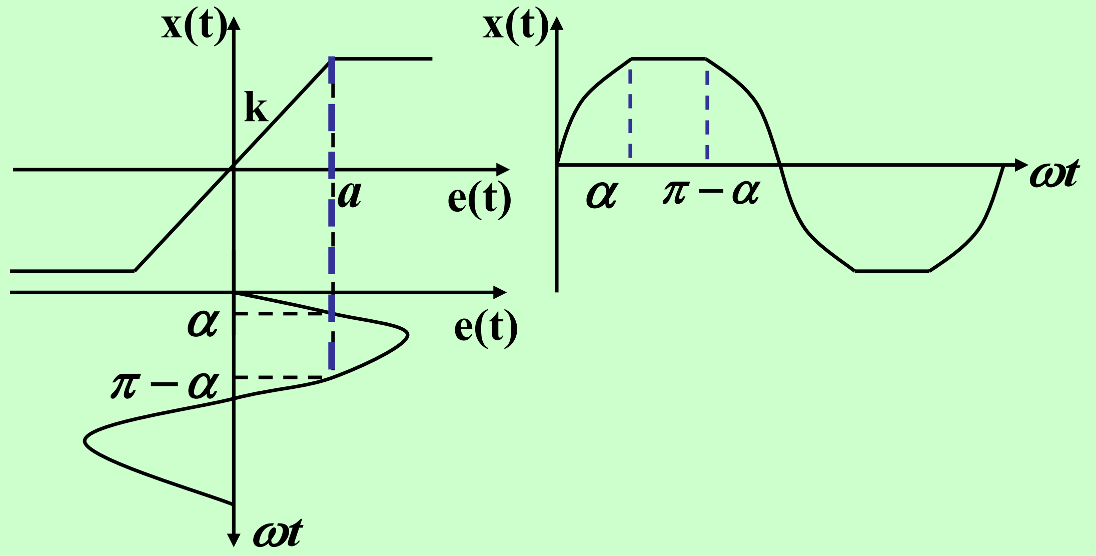


Fig 8-10 Input and output waveform of a saturation nonlinearity

when $A > a$, the output $x(t)$ is

$$x(t)=\begin{cases} KA\sin\omega t, & 0\le \omega t\le \alpha \\ Ka, & \alpha<\omega t\le \pi-\alpha \\ KA\sin\omega t, & \pi-\alpha<\omega t\le \pi \end{cases}$$

where,

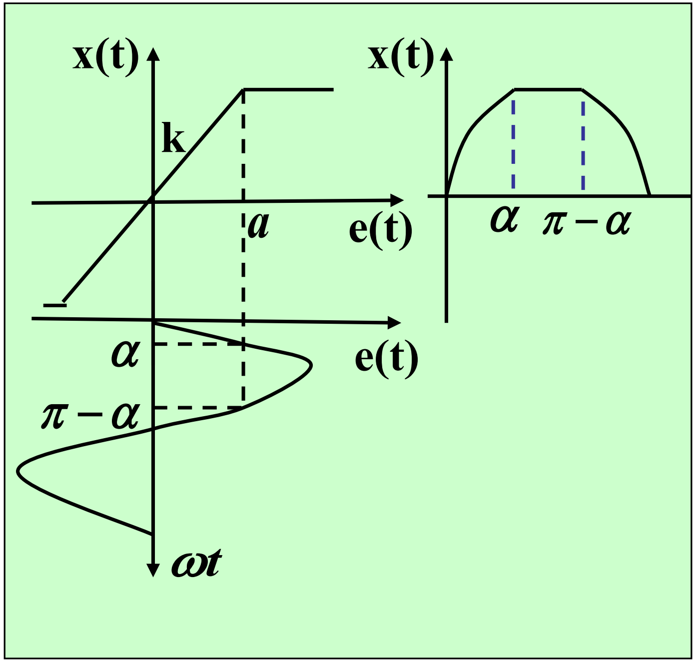
$$A\sin\alpha = a, \therefore \alpha=\sin^{-1}\frac{a}{A}$$

Because the output is odd,

$A_1 = 0$ (*single-valued odd-symmetric*)

$$\phi_1 = tg^{-1}\frac{A_1}{B_1}=0$$

$$\boxed{N(A)=\frac{B_1}{A}+j\frac{A_1}{A}=\frac{B_1}{A}}$$



$$B_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\sin\omega t \; d(\omega t)$$

$$= \frac{2}{\pi}\left[\int_0^{\alpha} KA\sin^2\omega t d(\omega t) + \int_{\alpha}^{\pi-\alpha} Ka\sin\omega t d(\omega t) + \int_{\pi-\alpha}^{\pi} KA\sin^2\omega t d(\omega t)\right]$$

$$= \frac{2}{\pi}KA\left[\sin^{-1}\frac{a}{A} + \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

The describing function of saturation nonlinearity is:

$$N(A) = \frac{B_1}{A} = \frac{2}{\pi}K\left[\sin^{-1}\frac{a}{A} + \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

$N(A)$ is a *nonlinear real function* of input amplitude A.
It can be regarded as a *variable gain amplifier*.

2. Dead-zone

Assume the input is $e(t) = A\sin\omega t$,

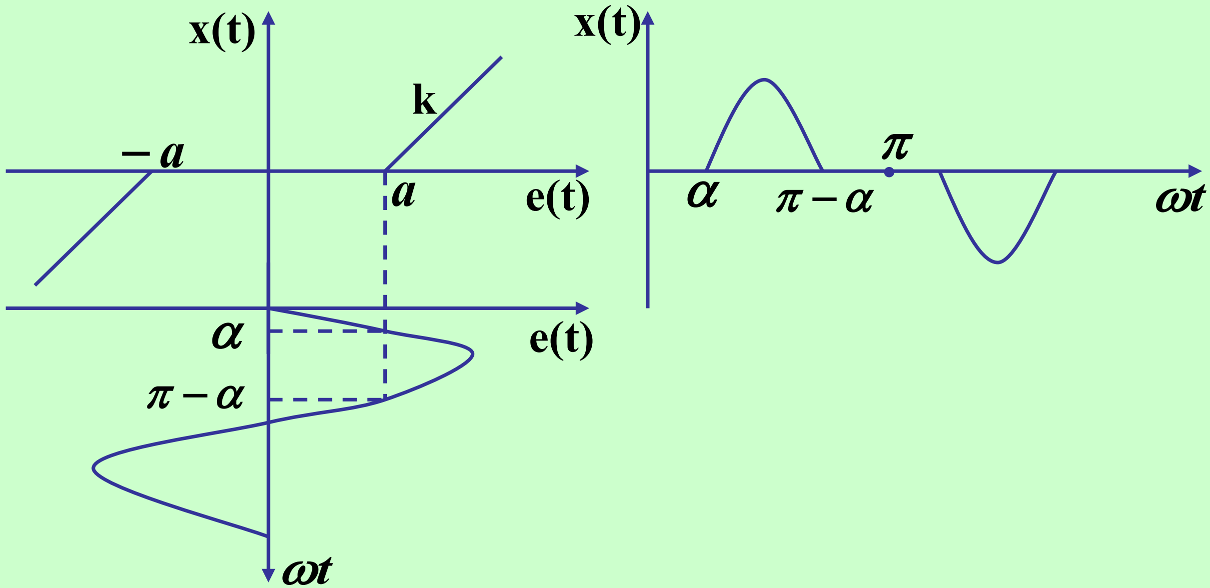


Fig8-11 input and output waveforms of dead zone nonlinearity

When $A > a$, the output of dead zone nonlinearity is

$$x(t)=\begin{cases}0, & 0\le \omega t\le \alpha\\ K(A\sin\omega t-a), & \alpha<\omega t\le \pi-\alpha\\ 0, & \pi-\alpha<\omega t\le \pi\end{cases}$$

where, $A\sin\alpha = a, \therefore \alpha=\sin^{-1}\dfrac{a}{A}$

The output is odd→$A_1 = 0$, $\phi_1 = 0$

$$B_1=\frac{2}{\pi}\int_0^{\pi}x(t)\sin\omega t\,d(\omega t)$$

$$=\frac{2}{\pi}\int_{\alpha}^{\pi-\alpha}K\left(A\sin\omega t-a\right)\sin\omega t\,d(\omega t)$$

$$B_1 = \frac{2}{\pi} KA\left[\frac{\pi}{2} - \sin^{-1}\frac{a}{A} - \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

The describing function of dead zone nonlinearity is

$$N(A) = \frac{B_1}{A} = \frac{2}{\pi} K\left[\frac{\pi}{2} - \sin^{-1}\frac{a}{A} - \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

Note:

(1) When a/A is very small, i.e., the non-sensible zone is small, $N(A)$ is approximately equal to K;

(2) $N(A)$ decreases as a/A becomes larger

(3) $a/A = 1 \qquad \Rightarrow \qquad N(A) = 0$

3. Gap

Assume the input is $e(t) = A\sin\omega t$,

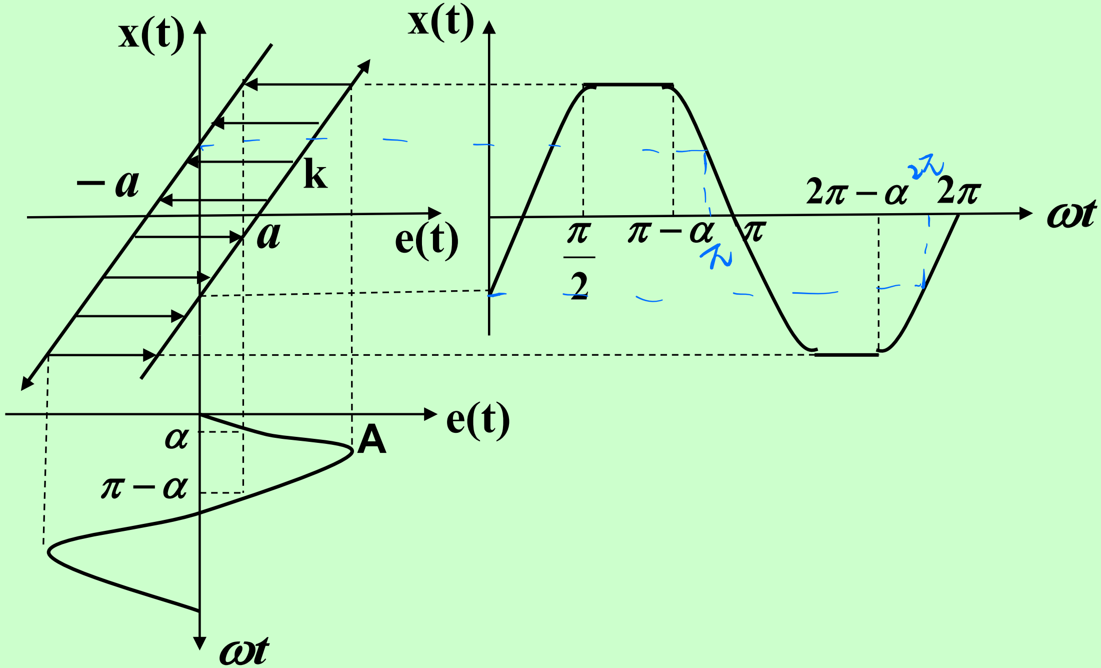


Fig 8-12 input and output waveforms of gap nonlinearity

From the mathematical description of gap nonlinearity, the $x(t)$ is given by

$$x(t)=\begin{cases} K(A\sin\omega t-a), & 0\le \omega t<\dfrac{\pi}{2} \\ K(A-a), & \dfrac{\pi}{2}\le \omega t<\pi-\alpha \\ K(A\sin\omega t+a), & \pi-\alpha\le \omega t\le \pi \end{cases}$$

where， $A\sin(\pi-\alpha)=A-2a,\ \therefore \alpha=\sin^{-1}\dfrac{A-2a}{A}$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\cos\omega t\ d(\omega t)$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} K(A\sin\omega t - a)\cos\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\alpha} K(A-a)\cos\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\pi-\alpha}^{\pi} K(A\sin\omega t + a)\cos\omega t d(\omega t) = \frac{4KA}{\pi}\left[\left(\frac{a}{A}\right)^2 - \frac{a}{A}\right]$$

$$B_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\sin\omega t\ d(\omega t) \quad = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} K(A\sin\omega t - a)\sin\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\alpha} K(A-a)\sin\omega t d(\omega t) + \frac{2}{\pi}\int_{\pi-\alpha}^{\pi} K(A\sin\omega t + a)\sin\omega t d(\omega t)$$

$$= \frac{KA}{\pi}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{A-2a}{A}\right) + \frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right]$$

So, we can obtain the describing function N(A) of gap nonlinearity as follows:

$$
\begin{aligned}
N(A) &= \frac{B_1}{A} + j\frac{A_1}{A} \\
&= \frac{K}{\pi}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{A-2a}{A}\right) + \frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right] + j\frac{4K}{\pi}\left[\frac{a(a-A)}{A^2}\right] \\
&= \left|N(A)\right| e^{j\varphi_1}
\end{aligned}
$$

$$
\left|N(A)\right| = \sqrt{\left[\frac{4K}{\pi}\left(\frac{a(a-A)}{A^2}\right)\right]^2 + \left[\frac{K}{\pi}\left(\frac{\pi}{2} + \sin^{-1}\frac{A-2a}{A} + \frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right)\right]^2}
$$

$$
\varphi_1 = tg^{-1}\frac{4\dfrac{a(a-A)}{A^2}}{\left[\dfrac{\pi}{2} + \sin^{-1}\left(\dfrac{A-2a}{A}\right) + \dfrac{A-2a}{A}\sqrt{1-\left(\dfrac{A-2a}{A}\right)^2}\right]}
$$

4. Relay

Assume the input is $e(t) = A\sin\omega t$,

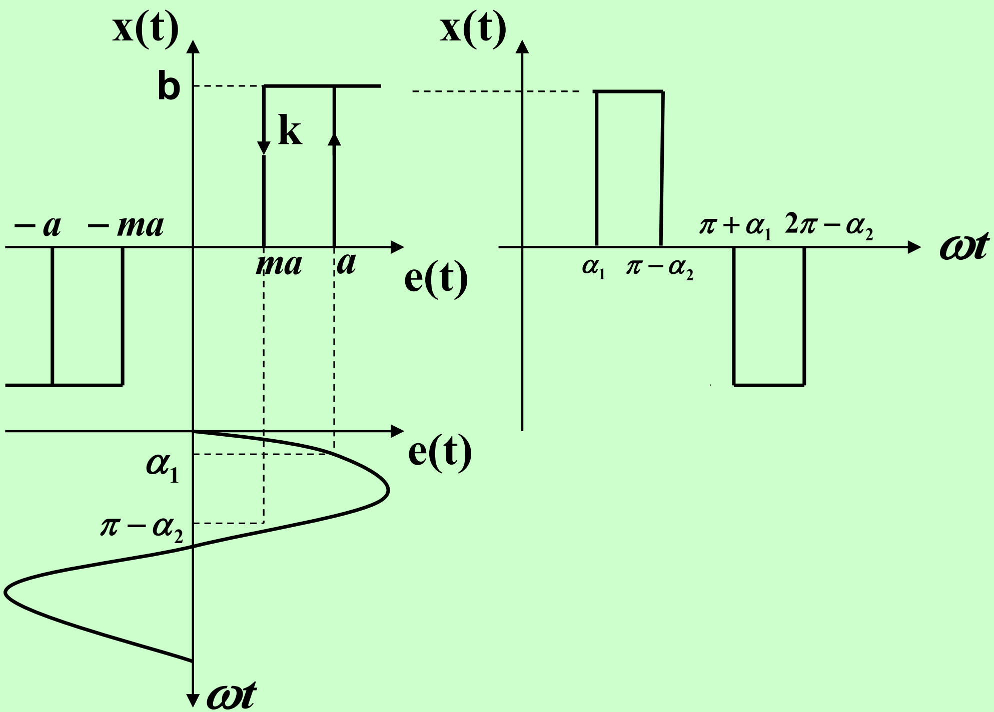


Fig 8-13 input and output waveforms of relay nonlinearity with dead zone and hysteresis ring

The output of the relay characteristic is :

$$x(t)=\begin{cases}0, & 0\le \omega t<\alpha_1\\ b, & \alpha_1\le \omega t<\pi-\alpha_2\\ 0, & \pi-\alpha_2\le \omega t\le \pi\end{cases}$$

where,

$$A\sin\alpha_1=a,\ \therefore \alpha_1=\sin^{-1}\frac{a}{A}$$

$$A\sin(\pi-\alpha_2)=ma,\ \therefore \alpha_2=\sin^{-1}\frac{ma}{A}$$

$$A_1 = \frac{2}{\pi}\int_{\alpha_1}^{\pi-\alpha_2} b\cos\omega t d(\omega t)$$

$$= \frac{2b}{\pi}(\sin\alpha_2 - \sin\alpha_1) = \frac{2ab(m-1)}{\pi A}$$

$$B_1 = \frac{2}{\pi}\int_{\alpha_1}^{\pi-\alpha_2} b\sin\omega t d(\omega t)$$

$$= \frac{2b}{\pi}(\cos\alpha_2 + \cos\alpha_1) = \frac{2b}{\pi}\left[\sqrt{1-\left(\frac{ma}{A}\right)^2} + \sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

The describing function N(A) of relay nonlinearity with dead zone and hysteresis ring is

$$N(A) = |N(A)|e^{j\phi_1} = \sqrt{\left(\frac{A_1}{A}\right)^2 + \left(\frac{B_1}{A}\right)^2}\, e^{jtg^{-1}\frac{A_1}{B_1}}$$

$$|N(A)| = \frac{2b}{\pi A}\sqrt{2\left[1 - m\left(\frac{a}{A}\right)^2 + \sqrt{1 + m^2\left(\frac{a}{A}\right)^4 - (m^2+1)\left(\frac{a}{A}\right)^2}\right]}$$

$$\phi_1 = tg^{-1}\frac{(m-1)\left(\dfrac{a}{A}\right)}{\sqrt{1 - m^2\left(\dfrac{a}{A}\right)^2} + \sqrt{1 - \left(\dfrac{a}{A}\right)^2}}$$

Corollary:

When a = 0, we can obtain the describing function of a *ideal relay nonlinearity*

$$N(A) = \frac{4b}{\pi A}$$

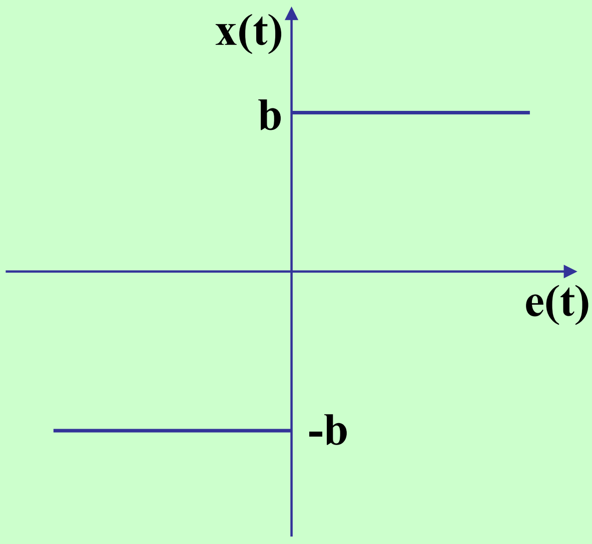


Fig 8-14 Ideal relay nonlinearity

When m = 1 and a≠0, we can obtain the describing function of a relay with dead zone

$$N(A)=\frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$

$4b\sqrt{1-\left(\frac{a}{x}\right)^2}$

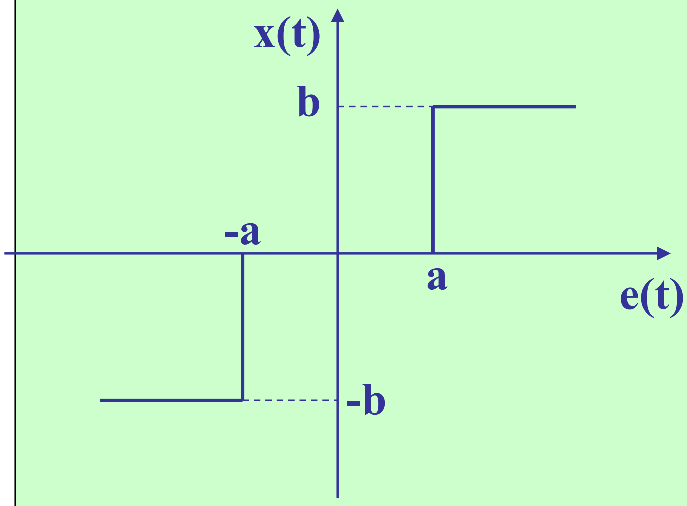


Fig 8-15 A relay with dead zone

When m = − 1, we can obtain the describing function of a relay with hysterics

$$N(A)=\frac{4b}{\pi A}e^{\,jtg^{-1}\frac{-\left(\frac{a}{A}\right)}{\sqrt{1-\left(\frac{a}{A}\right)^{2}}}}$$

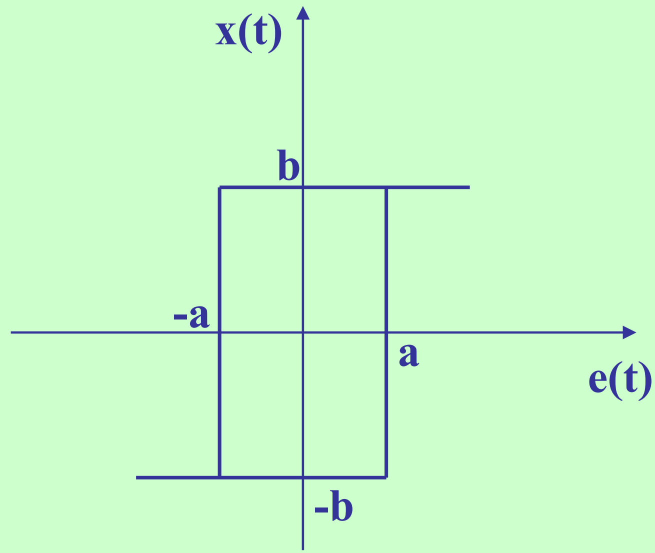


Fig 8-16 A relay with hysteresis

- Summary:
  - Nonlinear system analysis by the Describing Function Method:
    1. Draft of x-y, x-t, y-t;
    2. Decide the odd/even quality of y(t);
    3. Decide the symmetry property of y(t);
    4. Calculate A1, B1 - by integration;
    5. Calculate N(A).

8.3.3 Describing function of multiple nonlinearities

1. Cascaded nonlinear system

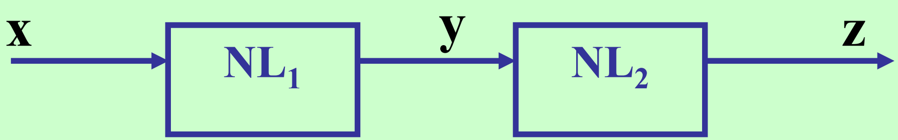
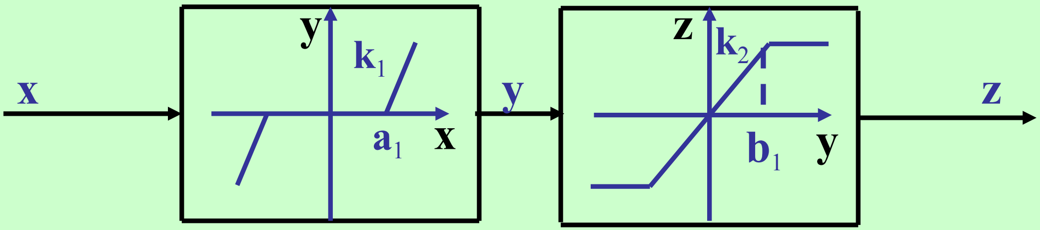


Fig 8-17 cascaded nonlinear system

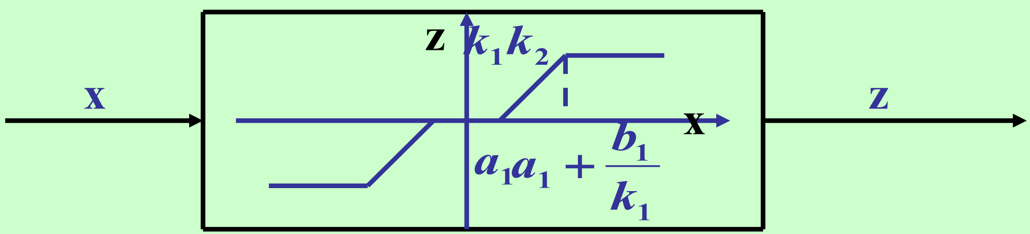
The describing function of a cascaded nonlinear system *is not equal to* the product of the two describing functions of each nonlinear elements.

$$N(A) \neq N_1(A) \cdot N_2(A)$$

assume NL_1 is a dead zone nonlinearity, NL_2 is a saturation nonlinearity, the composite nonlinearity of the cascaded nonlinear system is shown in Fig.8-18.



(a) cascaded nonlinear system



(b) composite nonlinearity

Fig 8-18 the cascaded nonlinear system and its composite nonlinearity

2. Parallel nonlinear system

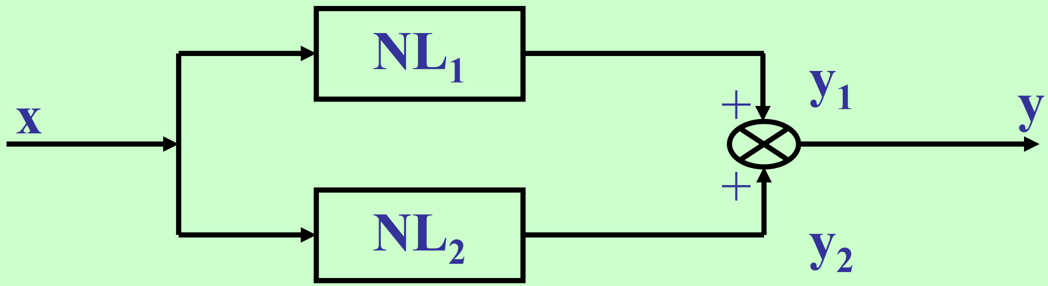


Fig 8-19 Parallel nonlinear system

According to the definition of describing function, the describing function N(A) of the output y and the input x is equal to the sum of two describing functions:

$$N(A) = N_1(A) + N_2(A)$$

[Example 1] **Obtain the describing function of the nonlinear system shown in the following figure.**

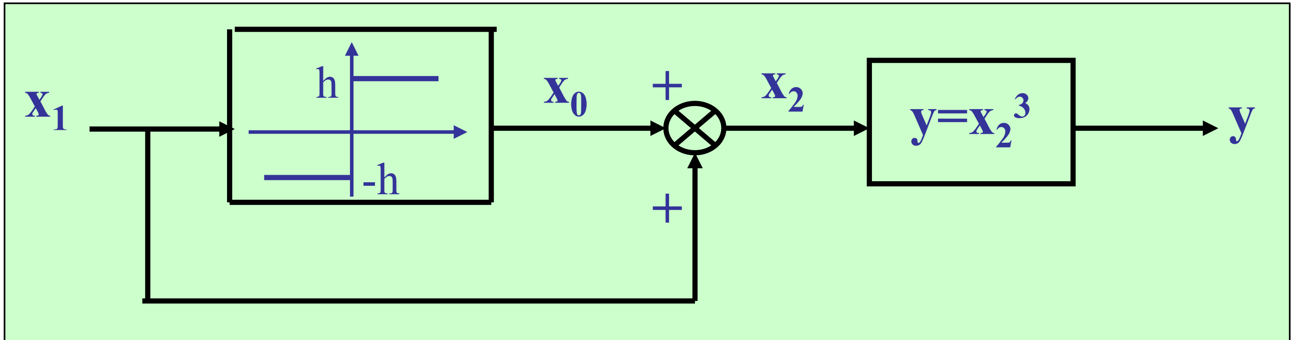


Fig 8-20 multiple nonlinearities

Solution: $y = x_2^3 = \left(x_0 + x_1\right)^3 = x_0^3 + 3x_0^2 x_1 + 3x_0 x_1^2 + x_1^3$

then $N(A) = N_1(A) + N_2(A) + N_3(A) + N_4(A)$

$assume\ x_1 = A\sin\omega t$

NL_1 is an ideal relay nonlinearity when a=0,

$\therefore\ A_1 = 0$

Obtain $N_1(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} h^3 \sin\omega t d(\omega t) = \frac{4h^3}{\pi}$$

$$\therefore \quad N_1(A) = \frac{B_1}{A} = \frac{4h^3}{\pi A}$$

Obtain $N_2(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} 3h^2 A \sin\omega t \cdot \sin\omega t d(\omega t) = 3h^2 A$$

$$\therefore \quad N_2(A) = 3h^2$$

Obtain $N_3(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} 3hA^2 \sin^2 \omega t \cdot \sin \omega t d(\omega t) \ = \frac{8hA^2}{\pi}$$

$$\therefore \quad N_3(A) = \frac{8hA}{\pi}$$

Obtain $N_4(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} A^3 \sin^3 \omega t \cdot \sin \omega t d(\omega t) \qquad \text{suppose } \theta = \omega t$$

$$= \frac{2}{\pi}\int_0^{\pi} - A^3 \sin^3 \theta \cdot d(\cos\theta)$$

$$= \frac{2A^3}{\pi}\left[\left(-\sin^3\theta\cos\theta\right)_0^{\pi} + \int_0^{\pi} 3\sin^2\theta\cos^2\theta d\theta\right] = \frac{3}{4}A^3$$

$$\therefore \quad N_4(A) = \frac{3}{4}A^2$$

Then, the describing function of the multiple nonlinearity is

$$N(A) = \frac{4h^3}{\pi A} + 3h^2 + \frac{8hA}{\pi} + \frac{3}{4}A^2$$

8.3.4 Analyze nonlinear system with describing function method

$$N(A) = \frac{\textit{Fundamental component} \text{ of the output } x(t)}{\text{sinusoidal input } e(t)}$$

It can only reflect ***partial dynamic characteristics*** of the system.

If a *self-excited oscillation* occurs in the system, any input $x(t)$ can be regarded as a *sinusoidal* signal because the linear part is a *low-pass filter*.

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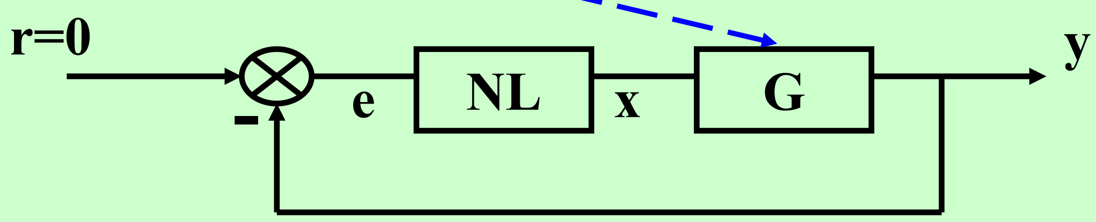
The describing function method is then *applicable*.



Typical structure of nonlinear system

Assume the input of a nonlinear system is zero, N(A) is the describing function of a nonlinear element, analyze the *condition for self-exited oscillation*.

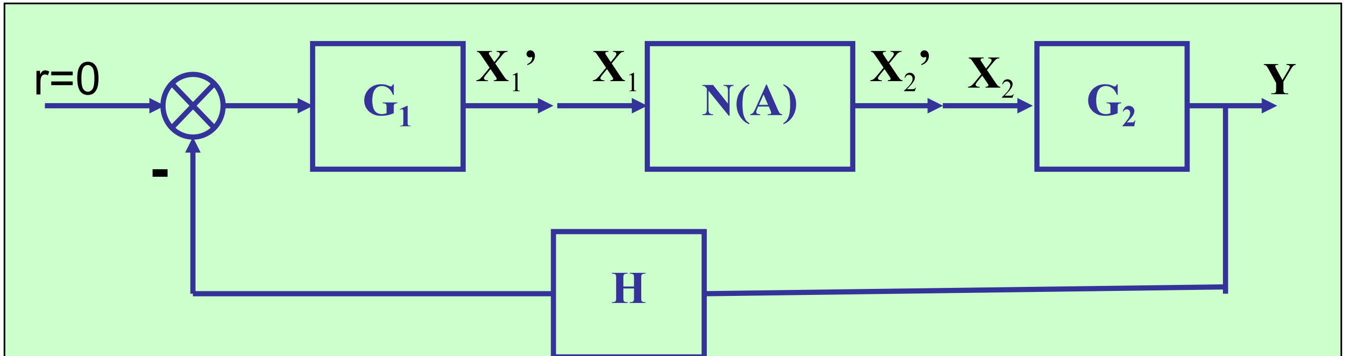


Fig. 8-21 nonlinear system

Assume $X_2 = A_2 \sin \omega t$

Then $X_1' = -\left|G_1(j\omega)G_2(j\omega)H(j\omega)\right| A_2 \sin(\omega t + \theta)$

where: $\boldsymbol{\theta = \angle G_1(j\omega) + \angle G_2(j\omega) + \angle H(j\omega)}$

Assume: $N(A) = |N(A)| e^{j\phi}$

then $x_2'(t) = -|N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)| A_2 \sin(\omega t + \theta + \phi)$

Note: $x_2'(t) = x_2(t)$ ⟺ **The self-oscillation occurs.**

Considering we have $x_2 = A_2 \sin \omega t$ **,**

the ***condition of self-oscillation*** **is**

$$\begin{cases} |N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)| = 1 \\ \theta + \phi = (2n+1)\pi \end{cases}$$

$$\begin{cases} |N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)| = 1 \\ \theta + \phi = (2n+1)\pi \end{cases}$$

Suppose the transfer function of linear part satisfies

$$G(s) = G_1(s)G_2(s)H(s)$$

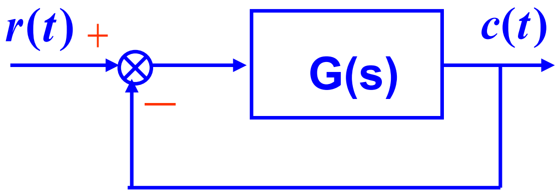
We can conclude that the *condition of self-excited oscillation* is

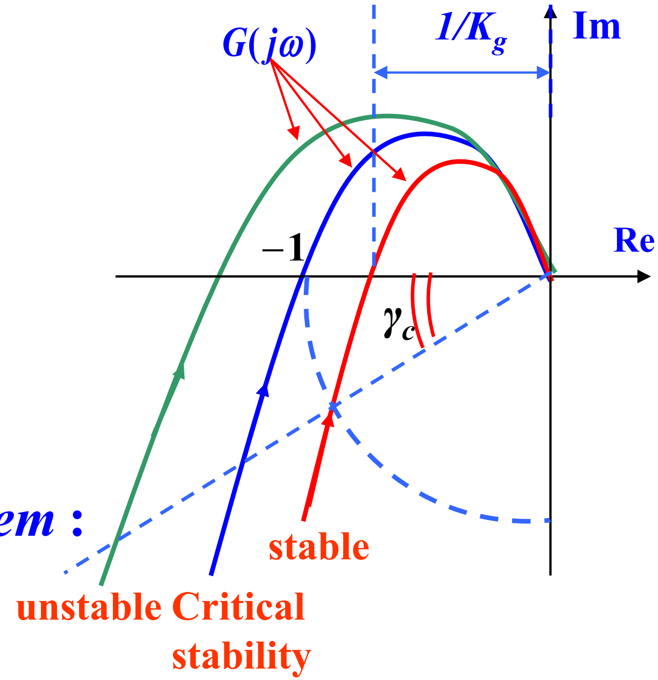
$$G(j\omega) = -\frac{1}{N(A)}$$

or $\quad 1 + N(A)G(j\omega) = 0$

Review of Nyquist criterion

For the linear system:





***The characteristic equation of the system* :**

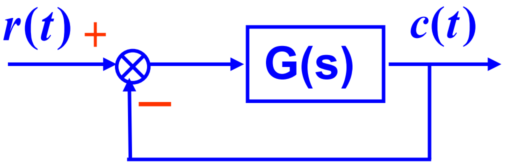
$$\mathbf{1} + \boldsymbol{G(j\omega)} = 0$$

$$\Rightarrow \boldsymbol{G(j\omega)} = -1 + j0$$

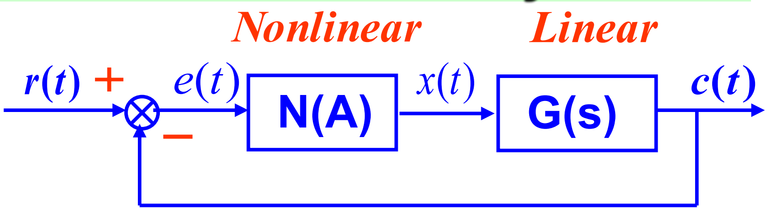
If $G(s)$ is a minimum phase transfer function, the necessary and sufficient condition of the stable system is:

$G(j\omega)$ *does not circle the point* $(-1, j0)$

Compare the nonlinear system with the linear system



Linear system



nonlinear system

Transfer function of the system:

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}$$

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{N(A)G(j\omega)}{1+N(A)G(j\omega)}$$

Characteristic equation:

$$1+G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-1$$

In the $G(j\omega)$ plane

A point

$$1+N(A)G(j\omega)=0$$

$$\Rightarrow \quad G(j\omega)=-\frac{1}{N(A)}$$

A curve

Because the describing function N(A) actually is a linearized frequency response, we can expand the Nyquist criterion to the nonliear system:

Stability analysis of the nonlinear system

(For example the minimum phase system)

(1) $G(j\omega)$ don't circle the $-\dfrac{1}{N(A)}$ curve, the nonlinear system is stable;

(2) $G(j\omega)$ circle the $-\dfrac{1}{N(A)}$ curve, the nonlinear system is unstable;

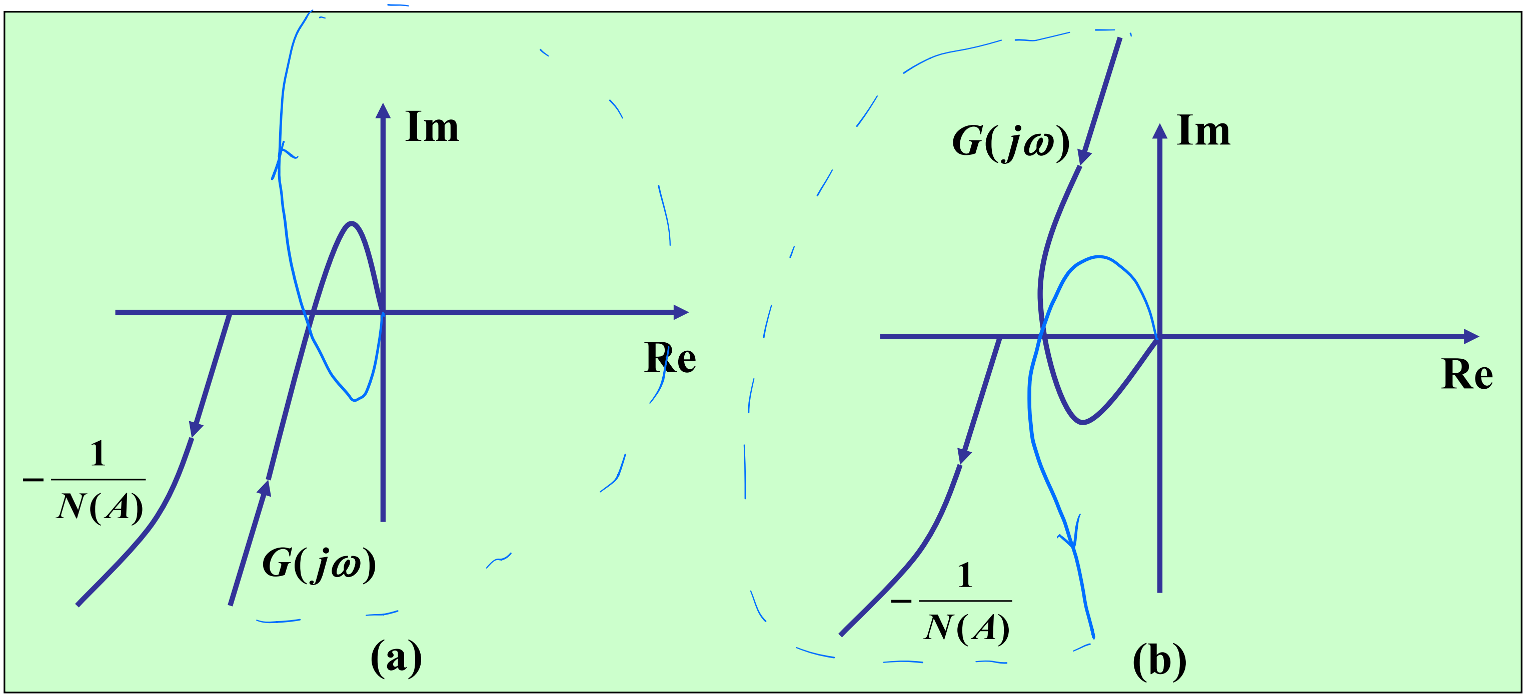
(3) $G(j\omega)$ intersect with the $-\dfrac{1}{N(A)}$ curve, there will be a self-oscillation in the nonlinear system .

compare with linear system

(1) $G(j\omega)$ don' t circle the point $(-1, j\omega)$, the system is stable;

(2) $G(j\omega)$ circle the point $(-1, j\omega)$, the system is unstable;

(3) $G(j\omega)$ intersect with the point $(-1, j\omega)$, the system is in the critical stability.

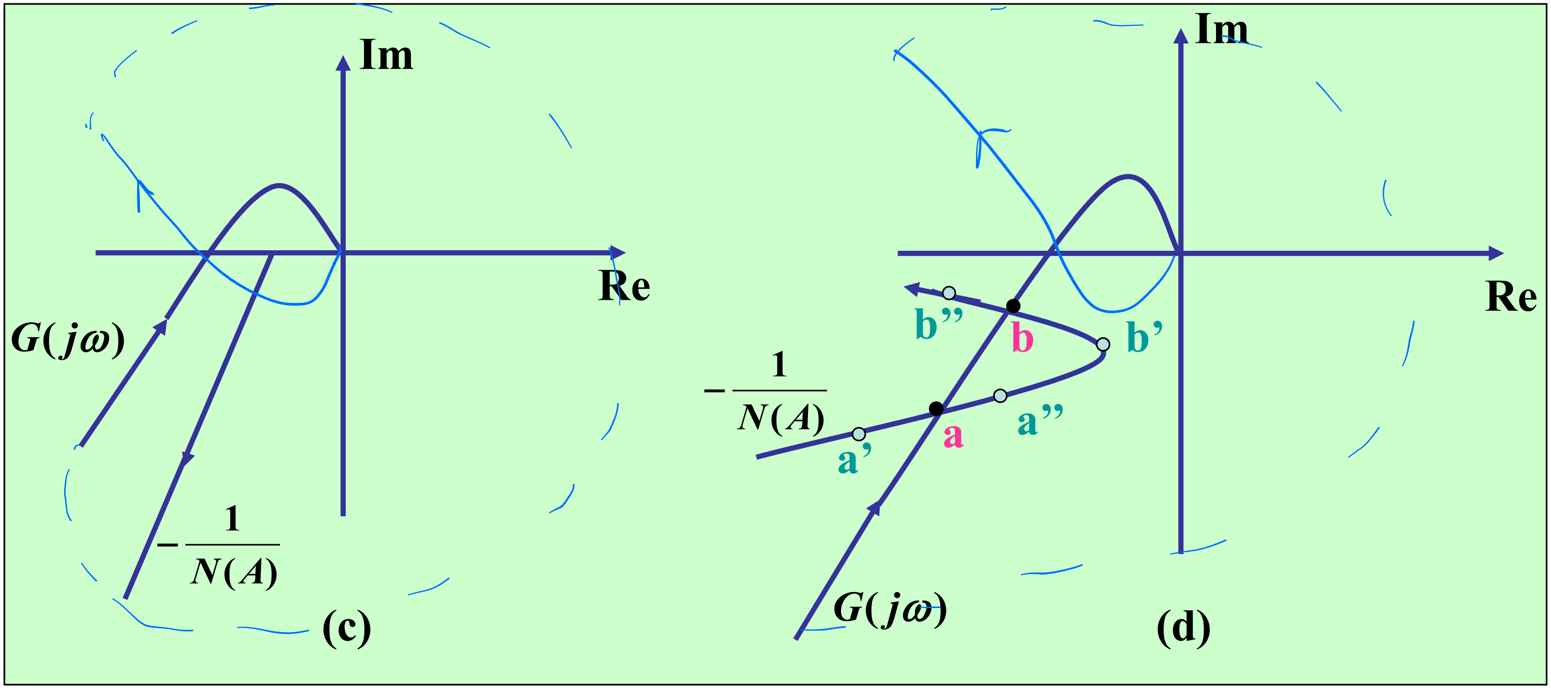


$G(j\omega)$ **do not circle —1/N(A)**

(Stable)

$G(j\omega)$ **circle —1/N(A)**

(Unstable)



Unstable

$G(j\omega)$ **intersect with —1/N(A)**

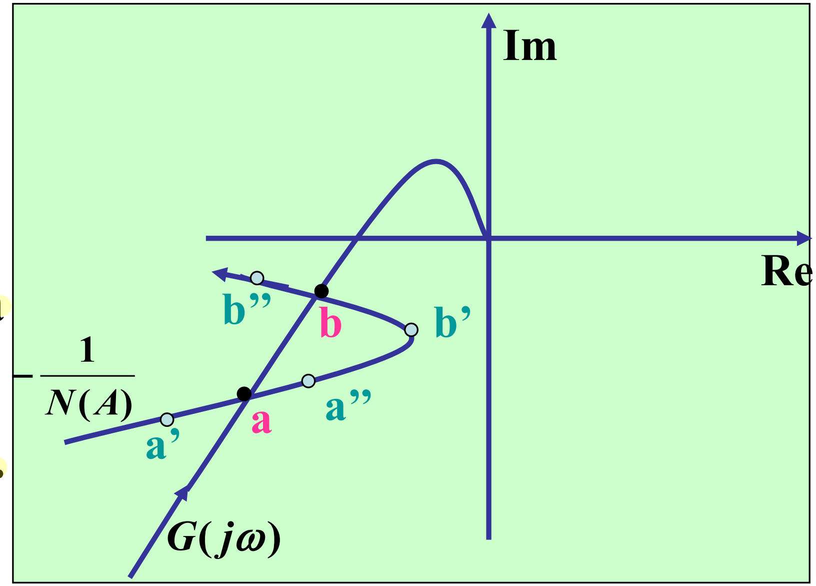
(Self-oscillation)

Analysis:

(1) Self-oscillation occurs at point a and point b.

(2) The self-oscillation at point a is unstable while the self-oscillation at point b is stable.

(3) There is only one stable self-oscillation in an actual physical system.



$a \to a'$ 收敛,反向→0. $A\downarrow$　　$b \to b'$ 发散,正向→b $A\uparrow$

$a \to a''$ 发散,正向→b $A\uparrow$　　$b \to b''$ 收敛,反向→b $A\downarrow$

The ***amplitude*** and ***frequency*** of the self-oscillation can be obtained by solving

$$|G(j\omega)N(A)| = 1$$

$$\theta + \phi = -\pi$$

[Example 2] A relay control system structure is shown in Fig. 8-23. Suppose a = 1, b = 3.

(1) Is there a self-oscillation in the system? If there is, obtain the amplitude and frequency of the oscillation.

(2) How to adjust the parameters if you want to eliminate the self-oscillation?

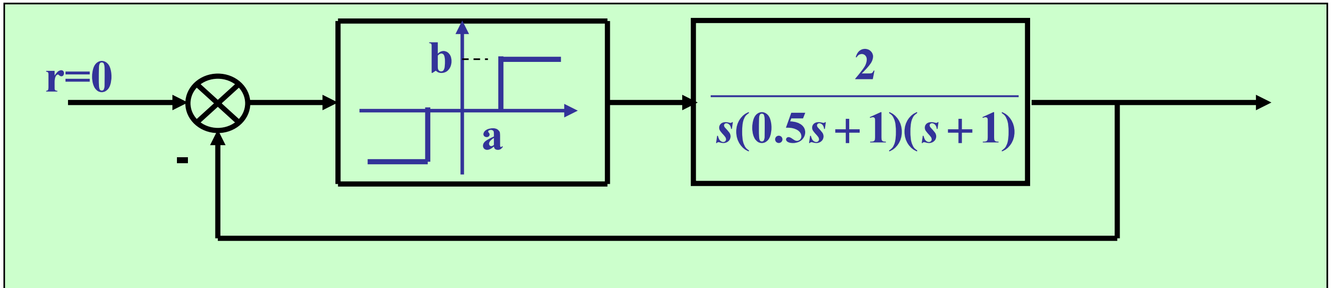


Fig.8-23 relay control sytem

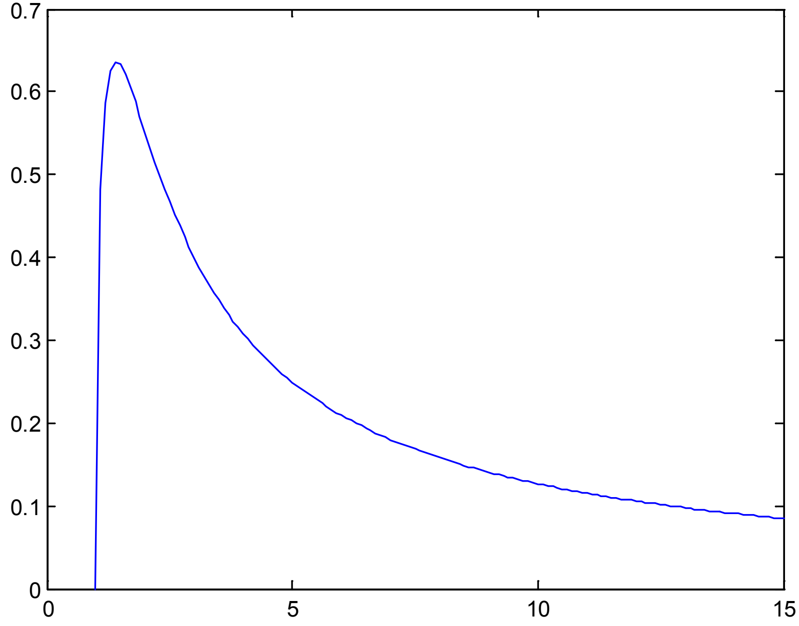
Solution: the describing function of a relay nonlinearity with dead zone is

$$N(A)=\frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$

Additional Summary of N(A)

- **Relay with dead zone**

$$N(A)=\frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$



$$\therefore\ -\frac{1}{N(A)} = -\frac{\pi A}{4b\sqrt{1-\left(\frac{a}{A}\right)^2}}$$

$$when \qquad A = a, -\frac{1}{N(A)} \to -\infty$$

$$when \qquad A \to \infty, -\frac{1}{N(A)} \to -\infty$$

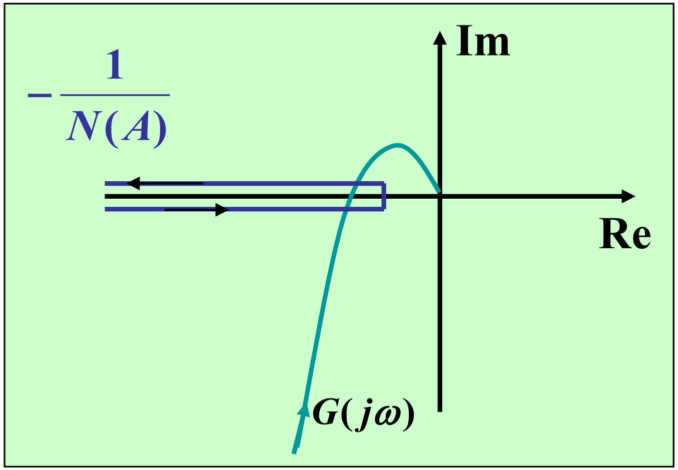
There is an *extreme value* of function $-\frac{1}{N(A)}$ on the real axis.

$$\frac{d}{dA}\left(-\frac{1}{N(A)}\right) = 0 \Rightarrow 1 - 2\left(\frac{a}{A}\right)^2 = 0 \qquad \therefore\ A = \sqrt{2}a$$

Substitute a = 1, b = 3 into above equations, we have

$$A = \sqrt{2}$$

$$-\frac{1}{N(A)}\bigg|_{A=\sqrt{2}} = -\frac{\pi}{6} \approx -0.52$$



$$G(s) = \frac{2}{s(0.5s+1)(s+1)}$$

$$G(j\omega) = -\frac{3\omega}{\omega(0.25\omega^4+1.25\omega^2+1)} - j\frac{2(1-0.5\omega^2)}{\omega(0.25\omega^4+1.25\omega^2+1)}$$

Set the imaginary part to zero, we have $\omega = \sqrt{2}$

Substituting $\omega = \sqrt{2}$ into the real part, we have

$$\mathrm{Re}\,G(j\omega)\Big|_{\omega=\sqrt{2}} = -\frac{1}{1.5} \approx -0.66$$

Let $$-\frac{1}{N(A)}=\frac{-\pi A}{12\sqrt{1-\left(\frac{1}{A}\right)^2}}=-\frac{1}{1.5}$$

We can obtain two amplitude: $A_1 = 1.11$, $A_2 = 2.3$

(not exist in the reality)

There is a self-oscillation in the system with the amplitude 2.3 and the frequency $\sqrt{2}$.

(2) To eliminate the self-oscillation, let

$$-\frac{1}{N(A)} = -\left.\frac{\pi A}{4b\sqrt{1-\left(\frac{a}{A}\right)^2}}\right|_{A=\sqrt{2}a} \le -\frac{1}{1.5}$$

We can get the ratio of relay parameters

$$\frac{b}{a} \le \frac{1.5\pi}{2} \approx 2.36$$

Adjust the ratio of a and b to b = 2a, we can eliminate the self-oscillation.

[Example 3] A multi-loop control system is shown in Fig. 8-25. If $G_1(s)=1$, the natural oscillation frequency and damping ratio of system is $\omega_n = 2$ and $\varsigma = 1$ when working on the linear area of the saturation. If we suppose $G_1(s) = 1 + \dfrac{1}{8s}$, try to find the minimal ratio T_1/T_2 when the system is stable.

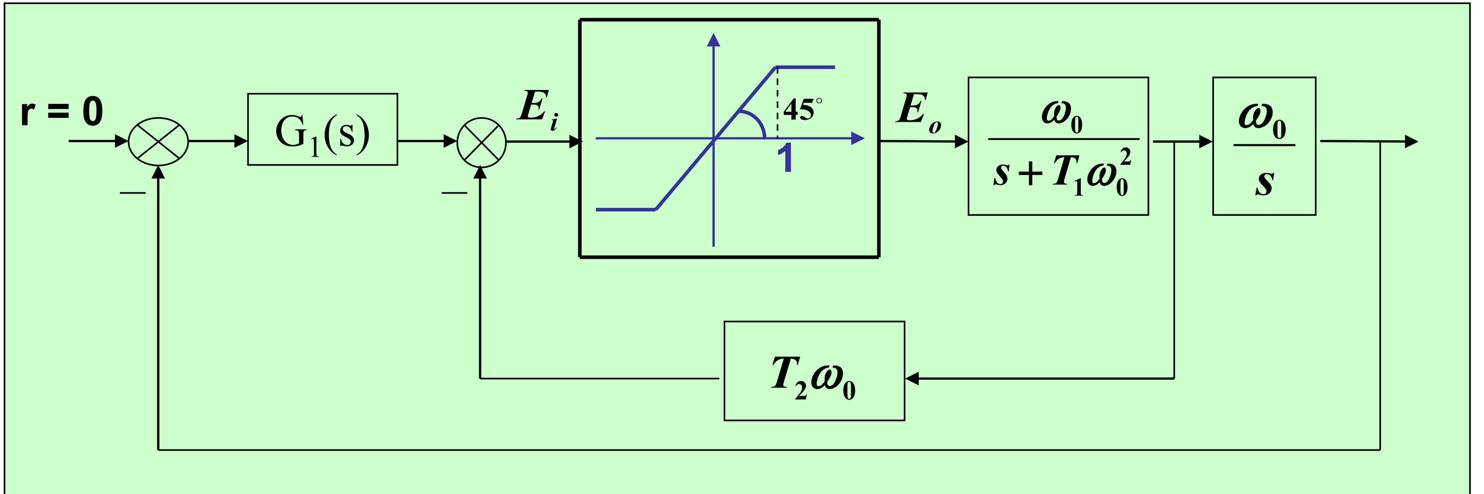


Fig.8-25 multi-loop control system

Solution: When $G_1(s) = 1$, the closed-loop transfer function of the inner loop is:

$$G_{内}(s) = \frac{\dfrac{\omega_0}{s+T_1\omega_0^2}}{1+\dfrac{T_2\omega_0^2}{s+T_1\omega_0^2}} = \frac{\omega_0}{s+(T_1+T_2)\omega_0^2}$$

The closed-loop transfer function of the whole system is

$$G_B(s) = \frac{\dfrac{\omega_0^2}{s[s+(T_1+T_2)\omega_0^2]}}{1+\dfrac{\omega_0^2}{s[s+(T_1+T_2)\omega_0^2]}} = \frac{\omega_0^2}{s^2+(T_1+T_2)\omega_0^2 s+\omega_0^2}$$

$$\therefore \begin{cases} \omega_0 = \omega_n = 2 \\ T_1 + T_2 = \varsigma = 1 \end{cases}$$

when $G_1(s) = 1 + \dfrac{1}{8s}$, the transfer function of inner loop is

$$G_{内}(s) = \frac{\dfrac{\omega_0}{s + T_1\omega_0^2} N(A)}{1 + \dfrac{T_2\omega_0^2}{s + T_1\omega_0^2} N(A)} = \frac{\omega_0 N(A)}{s + T_1\omega_0^2 + T_2\omega_0^2 N(A)}$$

The open-loop transfer function of the whole system

$$G(s) = G_1(s) G_{内}(s) \frac{\omega_0}{s} = \frac{\omega_0^2 (1 + \dfrac{1}{8s}) N(A)}{s[s + T_1\omega_0^2 + T_2\omega_0^2 N(A)]}$$

From the characteristic equation of closed-loop system $1 + G(s) = 0$, we have

$$s^2 + T_1\omega_0^2 s + T_2\omega_0^2 sN(A) + \omega_0^2\left(1 + \frac{1}{8s}\right)N(A) = 0$$

Substituting $\omega_0 = 2$ to the above equation, we can obtain

$$8s^3 + 32T_1 s^2 + (32T_2 s^2 + 32s + 4)N(A) = 0$$

特征方程

$$\therefore\ -\frac{1}{N(A)} = \frac{8T_2 s^2 + 8s + 1}{2s^3 + 8T_1 s^2} = \frac{8T_2 s^2 + 8s + 1}{s^2(2s + 8T_1)}$$

Note that the describing function of saturation nonlinearity is

$$N(A) = \frac{2}{\pi}\left[\sin^{-1}\frac{1}{A} + \frac{1}{A}\sqrt{1 - \left(\frac{1}{A}\right)^2}\right]$$

So the function of $-\dfrac{1}{N(A)}$ is given by

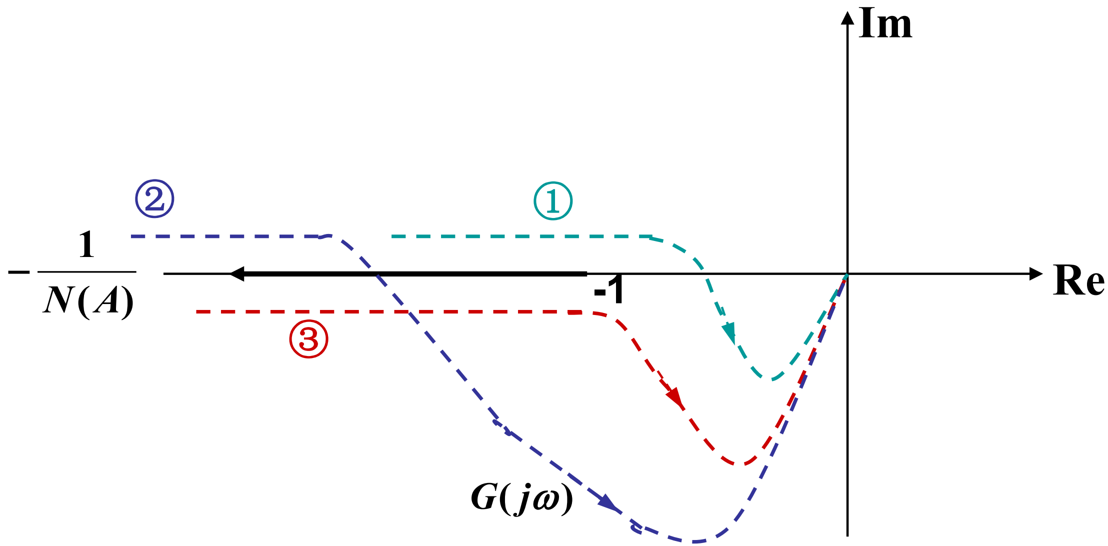
$$-\frac{1}{N(A)} = -\frac{\pi}{2}\frac{1}{\left[\sin^{-1}\frac{1}{A}+\frac{1}{A}\sqrt{1-\left(\frac{1}{A}\right)^2}\right]}$$

The open-loop transfer function of the whole system is

$$G(s) = -\frac{1}{N(A)} = \frac{8T_2s^2+8s+1}{s^2(2s+8T_1)}$$

$$\therefore\ G(j\omega) = \frac{-8T_2\omega^2+j8\omega+1}{-\omega^2(j2\omega+8T_1)} = u(\omega)+jv(\omega)$$

$$G(j\omega) = -\frac{4T_1 - 32\omega^2 T_1 T_2 + 8\omega^2}{2\omega^2(\omega^2 + 16T_1^2)} + j\frac{32T_1 - 1 + 8T_2\omega^2}{2\omega(\omega^2 + 16T_1^2)}$$



To guarantee the system to be stable, we should choose parameters T_1 and T_2 which make $G(j\omega)$ has no intersection with the negative real axis. So we choose curve ③

Let $$v(\omega)=\frac{32T_1-1+8T_2\omega^2}{2\omega(\omega^2+16T_1^2)}=0$$

Then we have $$\omega=\sqrt{\frac{1-32T_1}{8T_2}}$$

If $\boldsymbol{T_1>\frac{1}{32}}$ **holds,** $\boldsymbol{G(j\omega)}$ **has no intersection with the negative real axis.**

On the other hand, $T_1+T_2=1$

$$\therefore\ \boldsymbol{T_2<\frac{31}{32}} \Rightarrow \therefore\ \boldsymbol{\frac{T_1}{T_2}\geq\frac{1}{31}}$$

So, the minimal ratio of $\boldsymbol{\frac{T_1}{T_2}}$ **to guarantee the system to be stable is** $\boldsymbol{\frac{T_1}{T_2}=\frac{1}{31}}$

Additional Summary of N(A)

- **Relay with dead zone** $-\dfrac{1}{N(A)} = -\dfrac{\pi A}{4b\sqrt{1-\left(\dfrac{a}{A}\right)^2}}$

$$when \qquad A = a, -\frac{1}{N(A)} \to -\infty$$

$$when \qquad A \to \infty, -\frac{1}{N(A)} \to -\infty$$

There is an *extreme value* of function $-\dfrac{1}{N(A)}$ **on the real axis.**

$$\frac{d}{dA}\left(-\frac{1}{N(A)}\right) = 0 \;\Rightarrow\; 1-2\left(\frac{a}{A}\right)^2 = 0 \qquad \therefore\; \boldsymbol{A = \sqrt{2}a}$$