9.4 the solution of linear time-invariant system state equation

Mathematical Representation

Quantitative(定量) Analysis

Qualitative(定性) Analysis

Solving the dynamic mathematical model equations;
The solutions analysis.

Transfer function

State Space – modern control theory

Matrices Operation (State transition Matrix)

Dynamic Response

State equation (Model) $\Longrightarrow$ Dynamic analysis (solve state equation)

Insuring the existence and uniqueness of the solution: the elements in A and B are bounded. 有界

9.4.1 Solution of Linear Time-invariant Continual System

1. The solution of homogeneous state equation(齐次状态方程):

$\dot{x} = Ax$ is homogeneous state equation, and there are general 2 solutions:

- Power Series Method (幂级数法)

Assume the solution of above equation is a vector power series (幂级数) of $\boldsymbol{t}$

$$\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{t} + \boldsymbol{b}_2\boldsymbol{t}^2 + \cdots + \boldsymbol{b}_k\boldsymbol{t}^k + \cdots$$

$\boldsymbol{x}, \boldsymbol{b}_0, \boldsymbol{b}_1, \cdots \boldsymbol{b}_k \cdots$ are n dimensional vectors.

Calculate the derivative of above equation:

$$\dot{x} = b_1 + 2b_2 t + \cdots + k b_k t^{k-1} + \cdots = A(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots)$$

Assume the coefficients with the same power are uniform.

$$b_1 = A b_0$$

$$b_2 = \frac{1}{2} A b_1 = \frac{1}{2} A^2 b_0$$

$$b_3 = \frac{1}{3} A b_2 = \frac{1}{3 \times 2} A^3 b_0$$

$$\vdots$$

$$b_k = \frac{1}{k} A b_{k-1} = \frac{1}{k!} A^k b_0$$

$$\vdots$$

$$\because \quad x(0) = b_0$$

$$\therefore \quad x(t) = (I + At + \frac{1}{2} A^2 t^2 + \cdots + \frac{1}{k!} A^k t^k + \cdots) x(0)$$

Define: $$e^{At} = I + At + \frac{1}{2} A^2 t^2 + \cdots + \frac{1}{k!} A^k t^k + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

$$x(t) = e^{At} x(0)$$

e^{At} —— Matrix exponential function, called state transition matrix: $\Phi(t)$.

状态转移矩阵.

- Laplace transformation for $\dot{x} = Ax$

$$sx(s) - x(0) = Ax(s)$$
$$(Is - A)x(s) = x(0)$$
$$x(s) = (Is - A)^{-1}x(0)$$

Inverse Laplace Transformation:

$$x(t) = L^{-1}[(sI - A)^{-1}]x(0)$$

Compare with the power series method:

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$$
$$\Uparrow$$

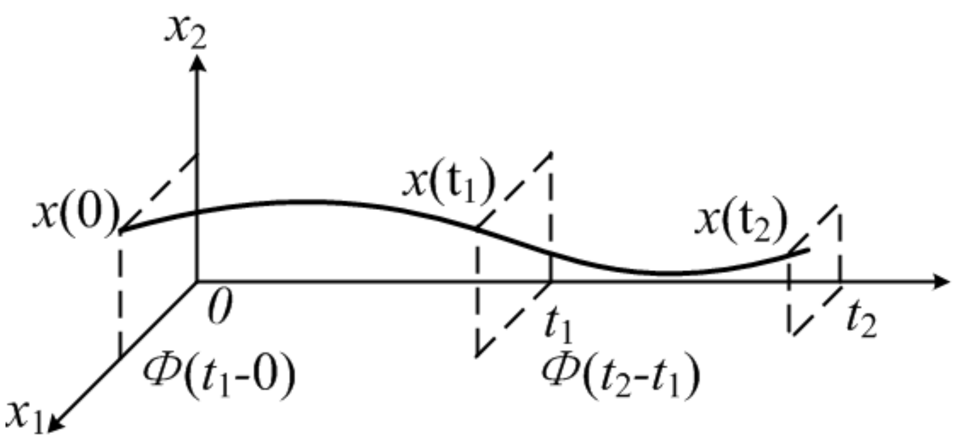
The closed form (闭合形式) (analytic form 解析形式) of the state transition matrix, which is convergent (收敛的).

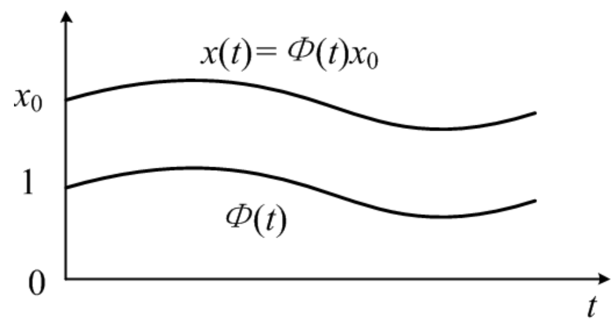
solve homogeneous state equation $\Longrightarrow$ calculate state transition matrix

Discussion:

$$\dot{x} = Ax \Rightarrow x(t) = e^{At}x(0) \quad \text{OR} \quad x(t) = e^{A(t-t_0)}x(t_0)$$

- The solution of homogeneous state equation describe a freedom motion (自由运动) of the system without the input $u(t)$, which is the transition of the initial state only based on the state transition matrix $\mathbf{e^{A(t-t0)}}$.





2. The solution of non-homogeneous state equation:

Give the non-homogeneous state equation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \qquad \boldsymbol{x}(t) \in R^n, \boldsymbol{u}(t) \in R^r, A \in R^{n\times n}, B \in R^{n\times r}$$

（1）Direct method（Integral method 积分法）

$$\dot{x}(t) - Ax(t) = Bu(t)$$

left multiply e^{-At} simultaneously: $\underbrace{e^{-At}[\dot{x}(t) - Ax(t)]}_{\boxed{\frac{d}{dt}[e^{-At}x(t)]}} = e^{-At}Bu(t)$

$$\frac{d}{dt}[e^{-At}x(t)] = e^{-At}Bu(t)$$

$$e^{-At}x(t) - x(0) = \int_o^t e^{-A\tau}Bu(\tau)d\tau$$

$$\boldsymbol{x}(t)\big|_{t=0} = \boldsymbol{x}(0)$$

$$x(t) = e^{At}x(0) + \int_o^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \Phi(t)x(0) + \int_o^t \Phi(t-\tau)Bu(\tau)d\tau$$

response of ~~zero~~ initial condition

零输入响应

Response of input u(t)

零状态响应

(2) Laplace transformation method

$$sX(s)-x(0)=AX(s)+Bu(s)$$

$$(sI-A)X(s)=x(0)+Bu(s)$$

$$X(s)=(sI-A)^{-1}x(0)+(sI-A)^{-1}Bu(s)$$

then $\quad x(t)=L^{-1}[(sI-A)^{-1}x(0)]+L^{-1}[(sI-A)^{-1}Bu(s)]$

(from $e^{At}=L^{-1}[(sI-A)^{-1}]$, we have)

$$x(t)=e^{At}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau=\Phi(t)x(0)+\int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

Discussion:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \Rightarrow x(t) = \Phi(t-t_0)x(t_0) + \int_{t_0}^{t} \Phi(t-\tau)Bu(\tau)d\tau$$

- The solution of non-homogeneous state equation is composed by two parts
  - the freedom motion of the initial state: $\boldsymbol{\Phi}(t\text{-}t_0)x(t_0)$, which is called zero-input response;
  - the controlled motion by the input: $\int_0^t \boldsymbol{\Phi}(t\text{-}\tau)\boldsymbol{Bu}(\tau)\mathrm{d}\tau$, which is called zero-state response.

9.4.2 Properties of state transition matrix

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

1. Initial state: $\Phi(0) = I$
2. $\dot{\Phi}(t) = A\Phi(t) = \Phi(t)A \qquad \dot{\Phi}(0) = A$
3. Linear relationship: $\Phi(t_1 \pm t_2) = \Phi(t_1)\Phi(\pm t_2) = \Phi(\pm t_2)\Phi(t_1)$
4. Reversibility: $\Phi^{-1}(t) = \Phi(-t), \ \Phi^{-1}(-t) = \Phi(t)$
5. $x(t_2) = \Phi(t_2 - t_1)x(t_1)$
6. Segmentation: $\Phi(t_2 - t_0) = \Phi(t_2 - t_1)\Phi(t_1 - t_0)$
7. $[\Phi(t)]^k = \Phi(kt)$
8. if $AB = BA$, $\quad e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}$;

   if $AB \neq BA$, $\quad e^{(A+B)t} \neq e^{At}e^{Bt} \neq e^{Bt}e^{At}$

9. if $\Phi(t)$ is state transition matrix of $\dot{x}(t) = Ax(t)$, the newly state tranition matrix after non-singular transform $x = P\overline{x}$ is:

$$\overline{\Phi}(t) = P^{-1}e^{At}P$$

10. Two common state transition matrices

If A is n-order Diagonal Matrix,

$$A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \qquad \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

If A is m-order Jordan Matrix,

$$A = \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}_{m\times m}, \quad \Phi(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & te^{\lambda t} \\ 0 & \cdots & 0 & e^{\lambda t} \end{bmatrix}$$

9.4.3 Calculation of matrix transition function e^{At}

- Method One: Direct method (matrix power function)

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

For any constant matrix A and limited t, the above infinite series is convergent.

- Method Two: Linear transform method (diagonal form method and Jordan form method)

If the matrix A can be transited to the diagonal form, e^{At} can be given as:

$$e^{At} = Pe^{\Lambda t}P^{-1} = P \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

证明

In the above equation, P is the non-singular linear transform matrix for A.

P非奇异

Similarly, if matrix A can be transformed to Jordan form, e^{At} can be given as:

$$e^{At} = Se^{Jt}S^{-1}$$

➢ Method Three: Laplace transform method

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

For e^{At}, it is essential to calculate the inverse of (sI-A). Generally, the Recursive Algorithm (递推算法) can be used when the order of system matrix A is high.

<u>**Ex.9-13**</u> Consider following system matrix, try to find the proper eAt by linear transform method and Laplace transform method.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Solution:

<u>Linear transform method</u>: the eigenvalues of A is 0 and -2 (λ_1=0, λ_2=-2), thus, the transform matrix P is:

$$P = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

from $$e^{At} = Pe^{\Lambda t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$

we have, $$e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} e^{o} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

<u>Laplace transform method:</u>

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}$$

We have:

$$(sI-A)^{-1}=\begin{bmatrix}\dfrac{1}{s} & \dfrac{1}{s(s+2)}\\ 0 & \dfrac{1}{s+2}\end{bmatrix}$$

thus:

$$\mathrm{e}^{\mathrm{At}}=L^{-1}[(sI-A)^{-1}]=\begin{bmatrix}1 & \dfrac{1}{2}(1-e^{-2t})\\ 0 & e^{-2t}\end{bmatrix}$$

Ex.9-14 **find the state transition matrix Φ(t) and its inverse of following linear time-invariant system**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Then calculate the inverse of state transition matrix $\Phi^{-1}(t)$.

According to $\Phi^{-1}(t)=\Phi(-t)$, the inverse of state transition matrix is:

$$\Phi^{-1}(t) = e^{-At} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

Ex.9-15 try to find the time response relationship of following system, in which, the input $u(t)$=**1**(t), the unit step function at t=0.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$ (according to Ex.9-14)

$$x(t) = e^{At}x(0) + \int_o^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1(t) d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero: **x**(0)=**0**, **x**(t) can be simplified as:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Ex.9-16 Assume the dynamic equation is: $\ddot{y}+(a+b)\dot{y}+aby=\dot{u}+cu$

With a, b and c are real constants. Try to find:

(1) the state space equation of the system;

(2) the state transition matrix Φ(t).

Solution:

(1)
$$G(s)=\frac{Y(s)}{U(s)}=\frac{s+c}{s^2+(a+b)s+ab}$$
$$=\frac{s+c}{(s+a)(s+b)}$$
$$=\frac{c-a}{b-a}\cdot\frac{1}{s+a}+\frac{c-b}{a-b}\cdot\frac{1}{s+b}$$

并联形式

$$\begin{cases}\dot{x}=\begin{bmatrix}-a & 0\\ 0 & -b\end{bmatrix}x+\begin{bmatrix}1\\1\end{bmatrix}u\\ y=\begin{bmatrix}\dfrac{c-a}{b-a} & \dfrac{c-b}{a-b}\end{bmatrix}x\end{cases}$$

(2)
$$\Phi(t)=L^{-1}[(sI-A)^{-1}]$$
$$=L^{-1}\left[\begin{pmatrix}s+a & 0\\ 0 & s+b\end{pmatrix}^{-1}\right]=L^{-1}\begin{bmatrix}\dfrac{1}{s+a} & 0\\ 0 & \dfrac{1}{s+b}\end{bmatrix}=\begin{bmatrix}e^{-at} & 0\\ 0 & e^{-bt}\end{bmatrix}$$

9.4.4 Establishing and solution of linear discrete system state space representation

1. The state space description of discrete-time linear system:

The discrete-time linear time-variant system:

$$X(k+1) = A(k)x(k) + B(k)u(k)$$
$$Y(k) = C(k)x(k) + D(k)u(k)$$

The discrete-time linear time-invariant system:

$$X(k+1) = Ax(k) + Bu(k)$$
$$Y(k) = Cx(k) + Du(k)$$

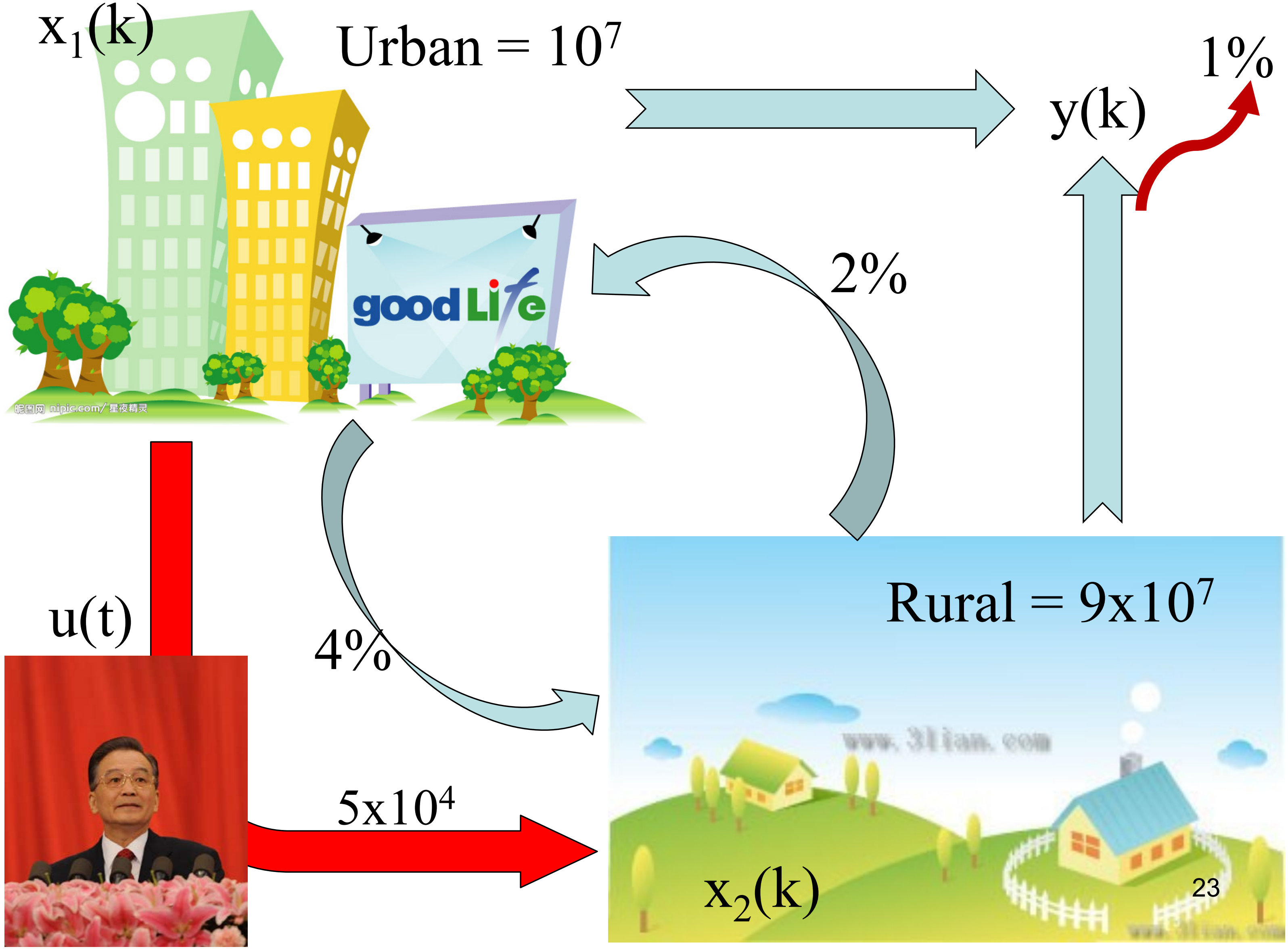
A_{nxn}: system matrix; B_{nxp}: input matrix
C_{qxn}: output matrix; D_{qxp}: transfer matrix

Population distribution Issue:

Assume the population condition of a country: Urban population is 10^7; Rural population is $9x10^7$; 4% of urban population transfer to countryside; 2% of rural population transfer to city; increasing rate of the whole country is 1%.

Assume: (1) k is a discrete-time variable; (2) $x_1(k)$ and $x_2(k)$ are urban and rural population of the k^{th} year; (3) u(k) is population control device of the government: a unit of positive control device can inspirit $5x10^4$ population move from city to the countryside, and v.v. ; (4) y(k) is the total population of the k^{th} year.

Try to describe such population distribution issue by discrete state space form.


$x_1(k)$
Urban = 10^7
$y(k)$
1%
2%
4%
$u(t)$
$5x10^4$
Rural = $9x10^7$
$x_2(k)$

Discrete state equations

$$x_1(k+1) = \left(1+0.01\right)\times\left[(1-0.04)x_1(k)+0.02x_2(k)-5\times10^4u(k)\right]$$

$$x_2(k+1) = \left(1+0.01\right)\times\left[(1-0.02)x_2(k)+0.04x_1(k)+5\times10^4u(k)\right]$$

Matrix representation:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9696 & 0.0202 \\ 0.0404 & 0.9898 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -5.05\times10^4 \\ 5.05\times10^4 \end{bmatrix}u(k)$$

Also standard state space description:

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

➢ 2. Discrete-time state space from differential equations

The difference equation and impulse transfer function are widely used to describe the discrete system in classic control theory. The general form of time-invariant differential equation of SISO system is:

$$
\begin{aligned}
&y(k+n)+a_1y(k+n-1)+\cdots+a_{n-1}y(k+1)+a_ny(k) \\
&=b_0u(k+n)+b_1u(k+n-1)+\cdots+b_{n-1}u(k+1)+b_nu(k)
\end{aligned}
$$

In which, k is time of kT; T is sampling period; u(k) and y(k) are input and output at time of kT; a_i and b_i are constant to describe system performance; consider the Z-transform with zero initial condition:

$$Z[y(k)]=y(z),\quad Z[(y(k+i)]=z^i y(z)$$

简化（前面的初始条件都为0）

$$G(z)=\frac{y(z)}{u(z)}=\frac{b_0 z^n+b_1 z^{n-1}+\cdots+b_{n-1}z+b_n}{z^n+a_1 z^{n-1}+\cdots+a_{n-1}z+a_n}$$

$$=b_0+\frac{\beta_1 z^{n-1}+\beta_2 z^{n-2}+\cdots+\beta_{n-1}z+\beta_n}{z^n+a_1 z^{n-1}+\cdots+a_{n-1}z+a_n}=b_0+\frac{N(z)}{D(z)}$$

Such G(z) is impulse transfer function, which is similar with the form of continual system:

$$Y(s)=(b_0 s^n+b_1 s^{n-1}+\cdots b_{n-1}s+b_n)E(s)$$

$$U(s)=(s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n)E(s)$$

The same way in continual system can be used in discrete situation: Such as intermediate variable method:

Import intermediate variable Q(z) in N(z)/D(z) serial decomposition:

$$D(z)=z^n Q(z)+a_1 z^{n-1}Q(z)+\cdots+a_{n-1}zQ(z)+a_n Q(z)$$

$$N(z)=\beta_1 z^{n-1}Q(z)+\cdots+\beta_{n-1}zQ(z)+\beta_n Q(z)$$

$$\begin{cases} u(z)=D(z) \\ y(z)=N(z)+b_0 u(z) \end{cases}$$

Define state variables:
$$\begin{cases} x_1(z) = Q(z) \\ x_2(z) = zQ(z) = zx_1(z) \\ \vdots \\ x_n(z) = z^{n-1}Q(z) = zx_{n-1}(z) \end{cases}$$

$$z^n Q(z) = -a_n x_1(z) - a_{n-1}x_2(z) \cdots - a_1 x_n(z) + u(z)$$
$$y(z) = \beta_n x_1(z) + \beta_{n-1}x_2(z) + \cdots + \beta_1 x_n(z) + b_0 u(z)$$

Z inverse transform $Z^{-1}[x_i(z)] = x_i(k), \qquad Z^{-1}[zx_i(z)] = x_i(k+1)$

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ \vdots \\ x_{n-1}(k+1) = x_n(k) \\ x_n(k+1) = -a_n x_1(k) - a_{n-1}x_2(k) \cdots - a_1 x_n(k) + u(k) \end{cases}$$
$$y(k) = \beta_n x_1(k) + \beta_{n-1}x_2(k) + \cdots + \beta_1 x_n(k) + b_0 u(z)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

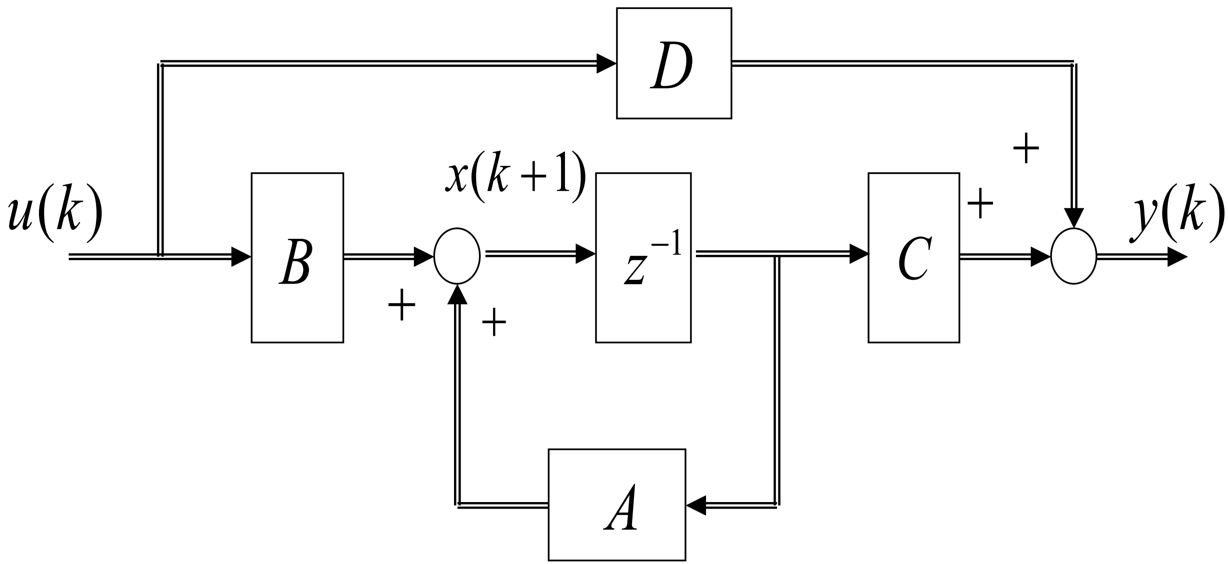
$$y(k) = [\beta_n \quad \beta_{n-1} \quad \cdots \quad \beta_1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + Du(k)$$

Discrete system state equation describes the relationship between the state of the system at (k+1)T, and the state at time kT and input of the system.

Output equation describes the relationship between the output of the system at kT, and the state at kT and input of the system.

State space representation of linear time-invariant MIMO discrete system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k)$$



3. Discretization of continual system state space expression

The solution of the time-invariant continual state equation $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ under the input u(t)

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \Phi(t-t_0)x(t_0) + \int_{t_0}^t \Phi(t-\tau)Bu(\tau)d\tau$$

assume $t_0 = kT, \quad x(t_0) = x(kT) = x(k)$

$$t = (k+1)T, \quad x(t) = x[(k+1)T] = x(k+1)$$

at $t \in [k, k+1]$, u(k)=u(k+1) is constant

$$\boldsymbol{x}(k+1) = \Phi[(k+1)T - kT]\boldsymbol{x}(k) + \boxed{\left(\int_{kT}^{(k+1)T} \Phi[(k+1)T-\tau]Bd\tau\right)}\boldsymbol{u}(k)$$

$$G(T) = \int_{kT}^{(k+1)T} \Phi[(k+1)T-\tau]Bd\tau$$

Variable replacement $(k+1)T - \tau = \tau'$

then

$$G(T) = \int_0^T \Phi(\tau)Bd\tau$$

$$\boldsymbol{x}(k+1) = \varPhi(T)\boldsymbol{x}(k) + \left(\int_0^T \varPhi(\tau) B d\tau\right)\boldsymbol{u}(k)$$

State equation of discrete system is:

$$\boldsymbol{x}(k+1) = \varPhi(T)\boldsymbol{x}(k) + G(T)\boldsymbol{u}(k)$$

The relationship between $\Phi(T)$ and state transition matrix $\Phi(t)$ of continual system:

$$\Phi(T) = \Phi(t)\big|_{t=T}$$

The output equation of discrete system is:

$$y(k) = C\boldsymbol{x}(k) + D\mathrm{u}(k)$$

Ex.9-17 **find the discrete state equation with T=1s from following continual system**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution: from Ex.9-15, the state transition matrix Φ(t) of above continual system is:

$$\Phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\Phi(T) = \Phi(t)\big|_{t=T=1} = \begin{bmatrix} 0.6004 & 0.2325 \\ -0.4651 & -.0972 \end{bmatrix}$$

$$G(T) = \int_0^T \Phi(\tau) B d\tau = \int_0^T \begin{pmatrix} e^{-\tau} - e^{-2\tau} \\ -e^{-\tau} + 2e^{-2\tau} \end{pmatrix} d\tau = \begin{bmatrix} 1/2 - e^{-T} + 1/2e^{-2T} \\ e^{-T} - e^{-2T} \end{bmatrix}$$

$$G(T)\big|_{T=1} = \begin{bmatrix} 0.1998 \\ 0.2325 \end{bmatrix}$$

$$\boldsymbol{x}(k+1) = \boldsymbol{\Phi}(T)\boldsymbol{x}(k) + G(T)\boldsymbol{u}(k)$$

➢4. Solution of time-invariant discrete system dynamic equation

(1) Recurrence method (递推法)

(2) Z transform method

$$\boldsymbol{x}(k+1)=\Phi(T)\boldsymbol{x}(k)+G(T)\boldsymbol{u}(k) \quad \boldsymbol{k=0,1,\cdots,k-1,}$$

The states at time of T, 2T, … ,kT time:

$$k=0 \qquad \boldsymbol{x}(1)=\Phi(T)\boldsymbol{x}(0)+G(T)\boldsymbol{u}(0)$$

$$k=1 \qquad \begin{aligned}\boldsymbol{x}(2)&=\Phi(T)\boldsymbol{x}(1)+G(T)\boldsymbol{u}(1)\\ &=\Phi^2(T)\boldsymbol{x}(0)+\Phi(T)G(T)u(0)+G(T)\boldsymbol{u}(1)\end{aligned}$$

$$k=2 \qquad \begin{aligned}\boldsymbol{x}(3)&=\Phi(T)\boldsymbol{x}(2)+G(T)\boldsymbol{u}(2)\\ &=\Phi^3(T)\boldsymbol{x}(0)+\Phi^2(T)G(T)u(0)+\Phi(T)G(T)u(1)+G(T)\boldsymbol{u}(2)\end{aligned}$$

$$\vdots \qquad \qquad \vdots$$

$$\boldsymbol{x}(k) = \Phi(T)\boldsymbol{x}(k-1) + G(T)\boldsymbol{u}(k-1)$$

$k = k-1$

$$= \Phi^k(T)\boldsymbol{x}(0) + \Phi^{k-1}(T)G(T)u(0) + \Phi^{k-2}(T)G(T)u(1) + \cdots$$

$$+ \Phi(T)G(T)\boldsymbol{u}(k-2) + G(T)\boldsymbol{u}(k-1)$$

$$= \Phi^k(T)\boldsymbol{x}(0) + \sum_{i=0}^{k-1} \Phi^{k-i-1}(T)G(T)\boldsymbol{u}(i)$$

It is the solution of discrete state equation, which is Discrete State Transition Equation.

when $u(i)=0, \quad (i=0,1,\cdots,k-1)$

$$\boldsymbol{x}(k)=\Phi^k\boldsymbol{x}(0)=\Phi(kT)\boldsymbol{x}(0)=\Phi(k)\boldsymbol{x}(0)$$

$\boldsymbol{\Phi}(k) \Longrightarrow$ State transition matrix of discrete system

The output equation:

$$\boldsymbol{y}(k)=C\boldsymbol{x}(k)+D\mathrm{u}(k)$$

$$=C\Phi^k(T)\boldsymbol{x}(0)+C\sum_{i=0}^{k-1}\Phi^{k-i-1}(T)G(T)\boldsymbol{u}(i)+D\boldsymbol{u}(k)$$

For the following discrete state equation:

$$\boldsymbol{x}(k+1)=A\boldsymbol{x}(k)+B\boldsymbol{u}(k)$$

$$\boldsymbol{y}(k)=C\boldsymbol{x}(k)+D\boldsymbol{u}(k)$$

Its solution is:

$$\boldsymbol{x}(k)=A^k\boldsymbol{x}(0)+\sum_{i=0}^{k-1}A^{k-i-1}B\boldsymbol{u}(i)$$

$$\boldsymbol{y}(k)=CA^k\boldsymbol{x}(0)+C\sum_{i=0}^{k-1}A^{k-i-1}B\boldsymbol{u}(i)+D\boldsymbol{u}(k)$$

Ex.9-18 **the state equation of the linear discrete system**

$$A=\begin{bmatrix}0 & 1\\ -0.16 & -1\end{bmatrix}\qquad B=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$x(k+1)=Ax(k)+Bu(k)$$

Find its solution by using the recurrence method with the initial state:

$$x(0)=\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}$$

and the input: $u(k)=1 \qquad (k\geq 0)$

Solution:

Recurrence method:

when $k=0$

$$x(1)=Ax(0)+Bu(0)$$

$$=\begin{bmatrix}0 & 1\\ -0.16 & -1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}\times 1=\begin{bmatrix}0\\ 1.84\end{bmatrix}$$

when $k=1$

$$x(2)=\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1.84 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}\times 1=\begin{bmatrix} 2.84 \\ -0.84 \end{bmatrix}$$

when $k=2$

$$x(3)=\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}\begin{bmatrix} 2.84 \\ -0.84 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}\times 1=\begin{bmatrix} 0.16 \\ 1.39 \end{bmatrix}$$

when $k=3$

$$x(4)=\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}\begin{bmatrix} 0.16 \\ 1.39 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}\times 1=\begin{bmatrix} 2.39 \\ -0.41 \end{bmatrix}$$

Iterate the operation, and we have $x(k)$ at any sampling time.

We cannot obtain the solutions of the linear discrete equation with closed-form by the recurrence method, rather than use state transition matrix.

- Z transform:

$$x(k+1) = \Phi x(k) + Gu(k)$$

$$\Downarrow$$

$$zX(z) - zx(0) = \Phi X(z) + GU(z)$$

$$\Downarrow$$

$$(zI - \Phi)X(z) = zx(0) + GU(z)$$

$$\Downarrow$$

$$X(z) = (zI - \Phi)^{-1}[zx(0) + GU(z)]$$

$$\Downarrow$$

$$x(k) = Z^{-1}[X(z)] = Z^{-1}\{(zI - \Phi)^{-1}[zx(0) + GU(z)]\}$$

$$= Z^{-1}[(zI - \Phi)^{-1} zx(0)] + Z^{-1}[(zI - \Phi)^{-1} GU(z)]$$

Analysis:

- $Z^{-1}[(zI-\Phi)^{-1}zx(0)]$

  For scale quantity a: $\quad Z^{-1}\left[\dfrac{1}{1-az^{-1}}\right]=a^k$

  Similarly for matrix Φ:

$$Z^{-1}\left[(zI-\Phi)^{-1}z\right]=Z^{-1}\left[(1-\Phi z^{-1})^{-1}\right]=\Phi^k$$

- $Z^{-1}[(zI-\Phi)^{-1}GU(z)]$

  For scale function w(k), and relative function W(z)

$$Z^{-1}\{W_1(z)W_2(z)\}=\sum_{i=0}^{k}w_1(k-i)w_2(i)$$

  Thus:

$$Z^{-1}[(zI-\Phi)^{-1}GU(z)]=Z^{-1}[(zI-\Phi)^{-1}z\bullet z^{-1}GU(z)]$$

$$=\cdots=\sum_{j=0}^{k-1}\Phi^{k-j-1}Gu(j)$$

- Result: $\quad x(t)=\Phi^k x(0)+\sum_{j=0}^{k-1}\Phi^{k-j-1}Gu(j)$

Ex.9-18 Solution:

Derive $(zI-A)^{-1}$:

$$|zI-A| = \begin{vmatrix} z & -1 \\ 0.16 & z+1 \end{vmatrix} = (z+0.2)(z+0.8)$$

$$(zI-A)^{-1} = \frac{adj(zI-A)}{|zI-A|} = \frac{\begin{bmatrix} z+1 & 1 \\ -0.16 & z \end{bmatrix}}{(z+0.2)(z+0.8)}$$

$$= \frac{1}{3}\begin{bmatrix} \frac{4}{z+0.2}-\frac{1}{z+0.8} & \frac{5}{z+0.2}-\frac{5}{z+0.8} \\ \frac{-0.8}{z+0.2}-\frac{0.8}{z+0.8} & \frac{-1}{z+0.2}+\frac{4}{z+0.8} \end{bmatrix}$$

$$\Phi(k) = A^k = Z^{-1}\left[(zI-A)^{-1}z\right]$$ discrete State transition matrix

$$= \frac{1}{3}\begin{bmatrix} 4(-0.2)^k-(-0.8)^k & 5(-0.2)^k-5(-0.8)^k \\ -0.8(-0.2)^k+0.8(-0.8)^k & -(-0.2)^k+4(-0.8)^k \end{bmatrix}$$

$u(k) = 1$ Z-transform $U(z) = \dfrac{z}{z-1}$

$$X(z) = (zI - A)^{-1}\left[zx(0) + BU(z)\right]$$

$$= \begin{bmatrix} \dfrac{(z^2+2)z}{(z+0.2)(z+0.8)(z-1)} \\ \dfrac{(-z^2+1.84z)z}{(z+0.2)(z+0.8)(z-1)} \end{bmatrix} = \frac{1}{18}\begin{bmatrix} \dfrac{-51z}{z+0.2} + \dfrac{44z}{z+0.8} + \dfrac{25z}{z-1} \\ \dfrac{10.2z}{z+0.2} + \dfrac{-35.2z}{z+0.8} + \dfrac{7z}{z-1} \end{bmatrix}$$

$$x(k) = Z^{-1}\{X(z)\} = \frac{1}{18}\begin{bmatrix} -51(-0.2)^k + 44(-0.8)^k + 25 \\ 10.2(-0.2)^k - 35.2(-0.8)^k + 7 \end{bmatrix}$$

Assume k=0,1,2,3, to double check the results before:

$$x(k) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1.84 \end{bmatrix}, \begin{bmatrix} 2.84 \\ -0.84 \end{bmatrix}, \begin{bmatrix} 0.16 \\ 1.384 \end{bmatrix}$$

- Two parallel sinks have the same cross section A in the bottom. Liquid from valve 0 flow into two sinks with the same Flow Q_0. The opening rate of valve 1 and 2 keep constant. Assume the resistance of the valve to the water R is constant near the balance point.
- Establish the state space equations of states: $x_1=\Delta h_1$, $x_2= \Delta h_2$, $u(\mathrm{t})= Q_0(\mathrm{t})$ and give its solution expression.