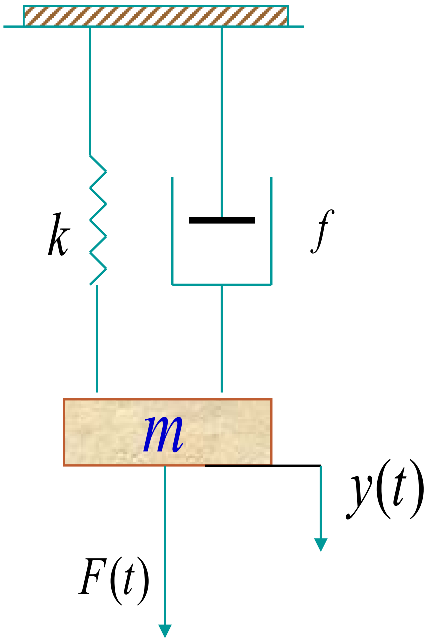
9.3 State-space Establishing of Linear System

General Methodology:

1. From Physics Mechanism of System
2. From Differential Equations of System
3. From Transfer Functions of System
4. From State-variable Diagram of System
5. Linear Transformation of State space

9.3.1 From Physics Mechanism of System

Ex.9-2 Mechanism system composed by force, spring(弹簧) and damper(阻尼器) without gravity(重力).



from Newton's law

$$m\frac{d^2y}{dt^2}+f\frac{dy}{dt}+ky=F(t)$$

in which, F(t) is Input, y(t) is Output.

Then, if the original displacement and velocity are available, the system's solution of the certain input is available as well.

Select the displacement and velocity as the state variables

$$x_1 = y, \quad x_2 = v = \dot{y}$$

Input is: $u(t) = F(t)$

State-equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{m}x_1 - \dfrac{f}{m}x_2 + \dfrac{1}{m}u \end{cases}$$

State space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{f}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u$$

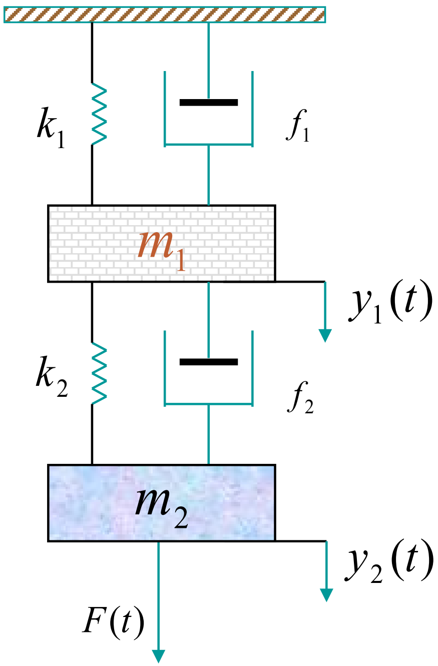
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ex.9-3 The State-space representation of Mechanism without gravity, the input is the pull F, the outputs are the mass: m_1 and m_2, and displacement y_1 and y_2.

From Newton's first Law, we have physics relationship of m_1 and m_2:

$$m_2\ddot{y}_2 = F(t) - k_2(y_2 - y_1) - f_2(\dot{y}_2 - \dot{y}_1)$$

$$m_1\ddot{y}_1 = k_2(y_2 - y_1) + f_2(\dot{y}_2 - \dot{y}_1) - k_1 y_1 - f_1\dot{y}_1$$



select 4 independent state variables:

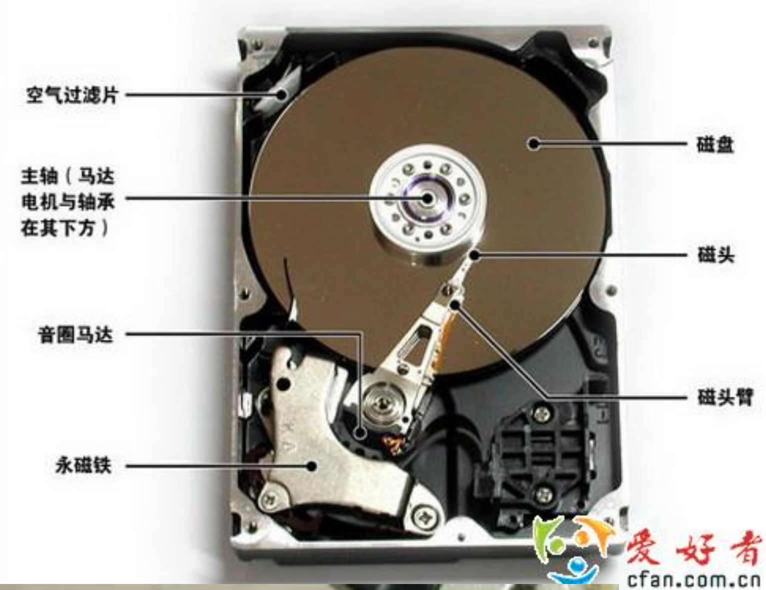
$$x_1 = y_1,\quad x_2 = y_2,\quad x_3 = \dot{y}_1,\quad x_4 = \dot{y}_2$$

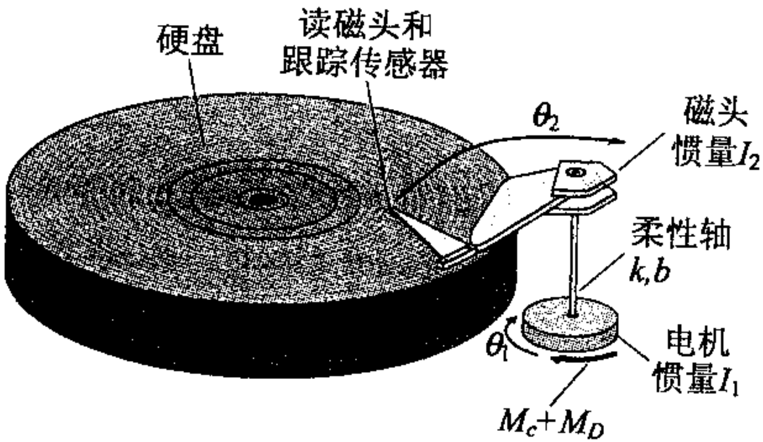
$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = \dfrac{k_2}{m_1}(x_2 - x_1) + \dfrac{f_2}{m_1}(x_4 - x_3) - \dfrac{k_1}{m_1}x_1 - \dfrac{f_1}{m_1}x_3 \\ \dot{x}_4 = \dfrac{1}{m_2}F(t) - \dfrac{k_2}{m_2}(x_2 - x_1) - \dfrac{f_2}{m_2}(x_4 - x_3) \end{cases}$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} & -\dfrac{f_1 + f_2}{m_1} & \dfrac{f_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & \dfrac{f_2}{m_2} & -\dfrac{f_2}{m_2} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix}}_{B} F
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

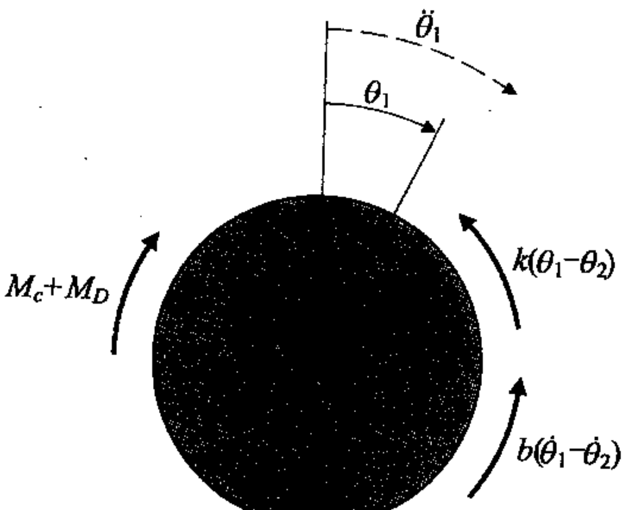

WD200
产品标签
安装螺丝
透气孔

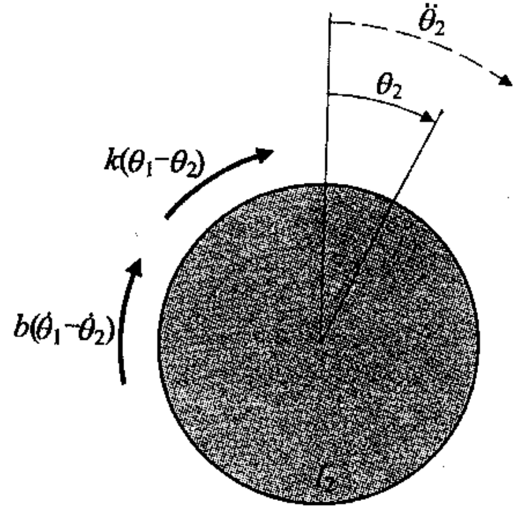

空气过滤片
主轴（马达
电机与轴承
在其下方）
音圈马达
永磁铁
磁盘
磁头
磁头臂
爱好者
cfan.com.cn



$$J_1\ddot{\theta}_1 = M_c + M_D - k(\theta_1 - \theta_2) - b(\dot{\theta}_1 - \dot{\theta}_2)$$

$$J_2\ddot{\theta}_2 = k(\theta_1 - \theta_2) + b(\dot{\theta}_1 - \dot{\theta}_2)$$





9.3 State-space Establishing of Linear System

General Methodology:

1. From Physics Mechanism of System
2. From Differential Equations of System
3. From Transfer Functions of System
4. From State-variable Diagram of System
5. Linear Transformation of State space

9.3.2 From Differential Equations of System

- **Methodology**:
  - Establish the differential/difference equations by the physics mechanism of system;
  - Establish the state equation focusing on the equations and a group of state-variables;
  - Establish the output function based on the relationship between system's outputs and states.

➢ State-variable Selection

- ✓ Section of state variable is not unique.
- ✓ Methodology:
  - ☞1. Select variable in the initial conditions or related.
  - ☞2. Select characteristic variable of independent storage components (energy or information) with certain physical meaning, such as the electric current i of inductance, the voltage u_c of capacitor, mass m and velocity v, etc.

Scenario (1): No derivatives(微分) of input u contained in n-order linear differential equations

Assume the dynamic process of the SISO control system is described as follow:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

$y^{(n)}, y^{(n-1)}, \cdots, \dot{y}, y$ —— Derivatives of output

$\boldsymbol{u}$ —— Input

If initial conditions $y(0), y'(0), \ldots, y^{(n-1)}(0)$ of output and the input u(t) of $t \geq 0$, the behavior of system at any time can be confirmed.

Select state variables: $\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ \vdots \\ x_{n-1} = y^{(n-2)} \\ x_n = y^{(n-1)} \end{cases}$

then: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots a_1 x_n + bu \end{cases}$

therefore: $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$

State space:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

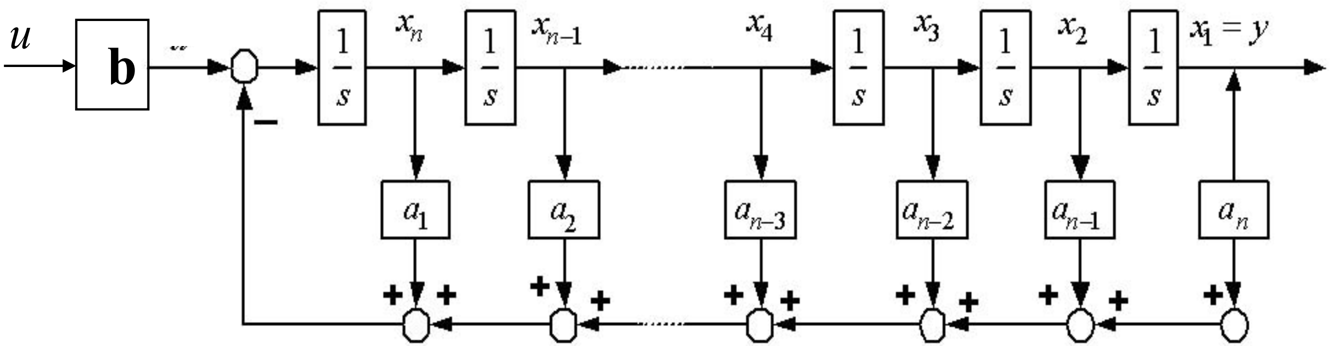
$$y = C\boldsymbol{x}$$

in which, the matrixes:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ b \end{bmatrix}$$

$$C = [1 \ \ 0 \ \ 0 \ \ \ldots \ \ 0]$$

- Then, draw the following block diagram (state variable diagram) among the state variables. 状态变量图
  - ✓ The output of each integrator corresponds to each state variable.
  - ✓ The state equations are decided by the relationship of I/O.
  - ✓ The output equation is on output part.



$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

Ex.9-4 assume the differential equation of the system's dynamic process is $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

in which, u and y are input and output.

Try to find the state space representation of the system.

Select state-variables: $x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y},$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + 6u \end{cases}$$

Standard Form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$

$$y = \boldsymbol{C}\boldsymbol{x}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}}_{\mathbf{B}} u$$

$$y = \underbrace{[1 \quad 0 \quad 0]}_{\mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Scenario(2): Derivatives of input u contained in n-order linear differential equation system

n-order linear differential equation representation:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

Reference scenario(1):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n y - a_{n-1}\dot{y} - \cdots - a_1 y^{(n-1)} + b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u \\ \quad = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u \end{cases} \Longleftarrow \begin{cases} x_1 = y \\ x_2 = \dot{y} \\ \vdots \\ x_{n-1} = y^{(n-2)} \\ x_n = y^{(n-1)} \end{cases}$$

However, the derivative of input u is still contained in state equation, which is INCONSEQUENCE(不合理).

Analysis: if input u is a limitary Step signal: Step Function, u' will be the Impulsive Function δ, $u^{(i)}(i=2,3,\ldots)$ will be the higher-order impulse function, and state trajectory will have infinite jump at t_0.

Hence, we cannot choose the output y and its derivatives to be state variables of the system. Such group of state variables cannot decide the future state of the system based on the known system input and original state conditions.

The principle of state variable selection:

状态方程不包含输入导数项

No derivative of the input/operation function could be included in any differential function in the system state equations represented by one-order differential equation sets.

Select the state variables:

The input is contained in the state variables

n-1 state equations

$$\begin{cases} x_1 = y - \beta_0 u \\ x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\ x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \\ \vdots \\ x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2} \dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u \end{cases}$$

then

$$\begin{cases} \dot{x}_1 = x_2 + \beta_1 u \\ \dot{x}_2 = x_3 + \beta_2 u \\ \vdots \\ \dot{x}_{n-1} = x_n + \beta_{n-1} u \\ \\ \dot{x}_n = ? \quad ? \quad ? \end{cases}$$

How to find the relationship between x'$_n$ and other states: x_1,x_2, …, x_{n-1}

? ?

Solution: $x'_n = f(x_1, x_2, \ldots, x_{n-1}, u)$

state equation of x_n: $x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u$

derivative of x_n: $\dot{x}_n = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \cdots - \beta_{n-2}\ddot{u} - \beta_{n-1}\dot{u}$

differential equation: $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$

and then: $y^{(n)} = -a_1 y^{(n-1)} - \cdots - a_{n-1}\dot{y} - a_n y + b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$

bring $y^{(n)}$ in x'_n:

$$\begin{aligned}
\dot{x}_n &= y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \cdots - \beta_{n-1}\dot{u} \\
&= (-a_1 y^{(n-1)} - a_2 y^{(n-2)} - \cdots - a_{n-1}\dot{y} - a_n y + b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u) \\
&\quad - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \cdots - \beta_{n-1}\dot{u} \\
&= -a_1 y^{(n-1)} - a_2 y^{(n-2)} - \cdots - a_{n-1}\dot{y} - a_n y \\
&\quad + (b_0 - \beta_0)u^{(n)} + (b_1 - \beta_1)u^{(n-1)} + \cdots + (b_{n-1} - \beta_{n-1})\dot{u} + b_n u
\end{aligned}$$

state variables:
$$\begin{cases} x_1 = y - \beta_0 u \\ x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u \\ x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u \\ \vdots \\ x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1} u \end{cases}$$

derivatives of y:
$$\begin{cases} y = x_1 + \beta_0 u \\ \dot{y} = x_2 + \beta_0 \dot{u} + \beta_1 u \\ \ddot{y} = x_3 + \beta_0 \ddot{u} + \beta_1 \dot{u} + \beta_2 u \\ \vdots \\ y^{(n-1)} = x_n + \beta_0 u^{(n-1)} + \beta_1 u^{(n-2)} + \cdots + \beta_{n-2}\dot{u} + \beta_{n-1} u \end{cases}$$

bring output y and it derivatives: y',...$y^{(n-1)}$ in x'_n:

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_2 x_{n-1} - a_1 x_n$$
$$+(b_0 - \beta_0) u^{(n)} + (b_1 - \beta_1 - a_1 \beta_0) u^{(n-1)} + (b_2 - \beta_2 - a_1 \beta_1 - a_2 \beta_0) u^{(n-2)} + \cdots$$
$$+(b_n - a_1 \beta_{n-1} - a_2 \beta_{n-2} - \cdots - a_{n-1} \beta_1 - a_n \beta_0) u$$

Principle:

No derivative of input $u(t)$ contained in state equations.

thus:
$$\begin{cases} \beta_0 = b_0 \\ \beta_1 = b_1 - a_1 \beta_0 \\ \beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0 \\ \vdots \\ \beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0 \end{cases}$$

State-space of the system is:

$$\begin{cases} \dot{x}_1 = x_2 + \beta_1 u \\ \dot{x}_2 = x_3 + \beta_2 u \\ \vdots \\ \dot{x}_{n-1} = x_n + \beta_{n-1} u \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots a_1 x_n + \beta_n u \end{cases}$$

Rewrite the system to the matrix representation:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

$$y = C\boldsymbol{x} + Du$$

In which:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_n \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \mathbf{D} = \beta_0 = b_0$$

Ex.9-5 Assume the dynamic equation of a control system can be written as the differential equation:

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 2y = 11\dot{u} + 6u$$

try to give its state space description.

Solution: compare with the standard differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

we have:

$$a_1 = 6,\quad a_2 = 11,\quad a_3 = 2,$$

$$b_0 = 0,\quad b_1 = 0,\quad b_2 = 11,\quad b_3 = 6$$

$$\boxed{\begin{aligned} x_1 &= y - \beta_0 u \\ x_2 &= \dot{x}_1 - \beta_1 u \\ x_3 &= \dot{x}_2 - \beta_2 u \end{aligned}}$$

and the coefficients:

$$\begin{aligned} \beta_0 &= b_0 = 0 \\ \beta_1 &= b_1 - a_1\beta_0 = 0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 = 11 \\ \beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = -60 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= y \\ x_2 &= \dot{x}_1 \\ x_3 &= \dot{x}_2 - 11u \end{aligned}$$

State equations are:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 + 11u$$

$$\dot{x}_3 = -a_3x_1 - a_2x_2 - a_1x_3 + \beta_3 u = -2x_1 - 11x_2 - 6x_3 - 60u$$

State space description of matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 11 \\ -60 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard form:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1}\dot{y}(t) + a_n y(t)$$
$$= b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}u' + b_n u(t)$$

while if $b_0=0$, we can select another group of state variable as follow:

$$x_n = y$$

$$x_i = \dot{x}_{i+1} + a_{n-i}y - b_{n-i}u \qquad i = 1, 2, 3, \cdots, n-1$$

$$x_{n-1} = \dot{x}_n + a_1 y - b_1 u \qquad \dot{x}_n = x_{n-1} - a_1 x_n + b_1 u$$

$$x_{n-2} = \dot{x}_{n-1} + a_2 y - b_2 u \qquad \dot{x}_{n-1} = x_{n-2} - a_2 x_n + b_2 u$$

$$\vdots \qquad \Rightarrow \qquad \vdots \qquad \dot{x}_1 ???$$

$$x_2 = \dot{x}_3 + a_{n-2}y - b_{n-2}u \qquad \dot{x}_3 = x_2 - a_{n-2}x_n + b_{n-2}u$$

$$x_1 = \dot{x}_2 + a_{n-1}y - b_{n-1}u \qquad \dot{x}_2 = x_1 - a_{n-1}x_n + b_{n-1}u$$

guess $\dot{x}_1 = -a_n x_n + b_n u$

Output equation: $y = x_n$

Furthermore:

$$x_{n-1} = \dot{x}_n + a_1 y - b_1 u$$
$$= \dot{y} + a_1 y - b_1 u$$
$$x_{n-2} = \dot{x}_{n-1} + a_2 y - b_2 u$$
$$= \ddot{y} + a_1 \dot{y} - b_1 \dot{u} + a_2 y - b_2 u$$

$\vdots$

$$x_2 = \dot{x}_3 + a_{n-2} y - b_{n-2} u$$
$$= y^{(n-2)} + a_1 y^{(n-3)} - b_1 u^{(n-3)} + a_2 y^{(n-4)} - b_2 u^{(n-4)} + \cdots + a_{n-2} y - b_{n-2} u$$
$$x_1 = \dot{x}_2 + a_{n-1} y - b_{n-1} u$$
$$= y^{(n-1)} + a_1 y^{(n-2)} - b_1 u^{(n-2)} + a_2 y^{(n-3)} - b_2 u^{(n-3)} + \cdots + a_{n-1} y - b_{n-1} u$$

$$
\begin{aligned}
x_1 &= \dot{x}_2 + a_{n-1}y - b_{n-1}u \\
&= y^{(n-1)} + a_1 y^{(n-2)} - b_1 u^{(n-2)} + a_2 y^{(n-3)} - b_2 u^{(n-3)} + \cdots + a_{n-1}y - b_{n-1}u
\end{aligned}
$$

calculate derivate of x_1

$$\dot{x}_1 = y^{(n)} + a_1 y^{(n-1)} - b_1 u^{(n-1)} + a_2 y^{(n-2)} - b_2 u^{(n-2)} + \cdots + a_{n-1}\dot{y} - b_{n-1}\dot{u}$$

bring $y^{(n)}$ into x'_1 according to:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1}\dot{y}(t) + a_n y(t) = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$\dot{x}_1 = -a_n x_n + b_n u$$

Matrix description:

$$\dot{x} = \mathbf{A}x + \mathbf{b}u$$
$$y = \mathbf{C}x + \mathbf{d}u$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{d} = \mathbf{0}$$

Ex.9-5(II) Differential equation of control system is

$$\dddot{y}+6\ddot{y}+11\dot{y}+2y=11\dot{u}+6u$$

try to give its state space description.

Solution: standard differential equation

$$y^{(n)}+a_1\,y^{(n-1)}+\cdots+a_{n-1}\dot{y}+a_n y=b_o\,u^{(n)}+b_1\,u^{(n-1)}+\cdots+b_{n-1}\dot{u}+b_n u$$

$$a_1=6,\quad a_2=11,\quad a_3=2,$$

$$b_0=0,\quad b_1=0,\quad b_2=11,\quad b_3=6$$

select state variable

$$\begin{aligned} x_3&=y\\ x_2&=\dot{x}_3+a_1y-b_1u\\ x_1&=\dot{x}_2+a_2y-b_2u \end{aligned} \quad\Rightarrow\quad \begin{aligned} x_3&=y\\ x_2&=\dot{x}_3+6y\\ x_1&=\dot{x}_2+11y-11u \end{aligned}$$

$$\dot{x}_1=-a_3x_3+b_3u=-2x_3+6u$$

State space description:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Conclusion: for a certain system, selection of state variables is not unique.

Ex.9-6 The equations of a 2input/2output 2nd-order system are:

$$\ddot{y}_1 + a_1\dot{y}_1 + a_2 y_2 = b_1\dot{u}_1 + b_2 u_1 + b_3 u_2$$

$$\dot{y}_2 + a_3 y_2 + a_4 y_1 = b_4 u_2$$

首先确定3个状态

try to give its state space description.

Solution: find derivative of y_1, y_2 with highest-order

$$\ddot{y}_1 = -a_1\dot{y}_1 + b_1\dot{u}_1 - a_2 y_2 + b_2 u_1 + b_3 u_2$$

$$\dot{y}_2 = -a_3 y_2 - a_4 y_1 + b_4 u_2$$

3-order

calculate their integration

$$y_1 = \iint\left[(-a_1\dot{y}_1 + b_1\dot{u}_1) + (-a_2 y_2 + b_2 u_1 + b_3 u_2)\right]dt^2$$

$$= \int(-a_1 y_1 + b_1 u_1)dt + \iint(-a_2 y_2 + b_2 u_1 + b_3 u_2)dt^2$$

$$\int[(-a_1 y_1 + b_1 u_1) + \int(-a_2 y_2 + b_2 u_1 + b_3 u_2)dt]dt$$

$$y_2 = \int[(-a_3 y_2 - a_4 y_1 + b_4 u_2)dt$$

Select state variables: $x_1 = y_1$

$$x_2 = y_2$$

From the equation of y_1:

$$\dot{x}_1 = -a_1 y_1 + b_1 u_1 + \int(-a_2 y_2 + b_2 u_1 + b_3 u_2)dt$$

$$= -a_1 x_1 + b_1 u_1 + \int(-a_2 x_2 + b_2 u_1 + b_3 u_2)dt$$

Select another state variable:

$$x_3 = \int(-a_2 x_2 + b_2 u_1 + b_3 u_2)dt$$

and $\dot{x}_3 = -a_2 x_2 + b_2 u_1 + b_3 u_2$

From the equation of y_2:

$$\dot{x}_2 = -a_3 x_2 - a_4 x_1 + b_4 u_2$$

The equation set: $\dot{x}_1 = -a x_1 + x_3 + b_1 u_1$

$$\dot{x}_2 = -a_4 x_1 - a_3 x_2 + b_4 u_2$$

$$\dot{x}_3 = -a_2 x_2 + b_2 u_1 + b_3 u_2$$

$$\dot{x}_1 = -ax_1 + x_3 + b_1 u_1$$
$$\dot{x}_2 = -a_4 x_1 - a_3 x_2 + b_4 u_2$$
$$\dot{x}_3 = -a_2 x_2 + b_2 u_1 + b_3 u_2$$

Rewrite the equations by the matrixes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 0 & 1 \\ -a_4 & -a_3 & 0 \\ 0 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \\ b_2 & b_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The output matrix equation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Review:

- **Differential Equations → State Space**
  - <u>No derivatives(微分) of input u</u> contained in n-order linear differential equations

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ \vdots \\ x_{n-1} = y^{(n-2)} \\ x_n = y^{(n-1)} \end{cases} \quad \mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ b \end{bmatrix}$$

$$C = [1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0]$$

– Derivatives of input u contained in n-order linear differential equation system

$$y^{(n)} + a_1\, y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_o\, u^{(n)} + b_1\, u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

– if $b_0 \neq 0$

$$\begin{cases} x_1 = y - \beta_0 u \\ x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\ x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \\ \vdots \\ x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u \end{cases}$$

$$\mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_n \end{bmatrix}$$

$$\begin{cases} \beta_0 = b_0 \\ \beta_1 = b_1 - a_1 \beta_0 \\ \beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0 \\ \vdots \\ \beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n \beta_0 \end{cases}$$

$$\mathbf{D} = \beta_0 = b_0$$

– Derivatives of input u contained in n-order linear differential equation system

$$y^{(n)}+a_1\, y^{(n-1)}+\cdots+a_{n-1}\dot{y}+a_n y=b_o\, u^{(n)}+b_1\, u^{(n-1)}+\cdots+b_{n-1}\dot{u}+b_n u$$

– if $b_0=0$

$$A=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_1 \end{bmatrix} \qquad \mathbf{b}=\begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{d}=\mathbf{0}$$

$$x_n=y$$

$$x_i=\dot{x}_{i+1}+a_{n-i}y-b_{n-i}u \qquad i=1,2,3,\cdots,n-1$$

9.3 State-space Establishing of Linear System

General Methodology:

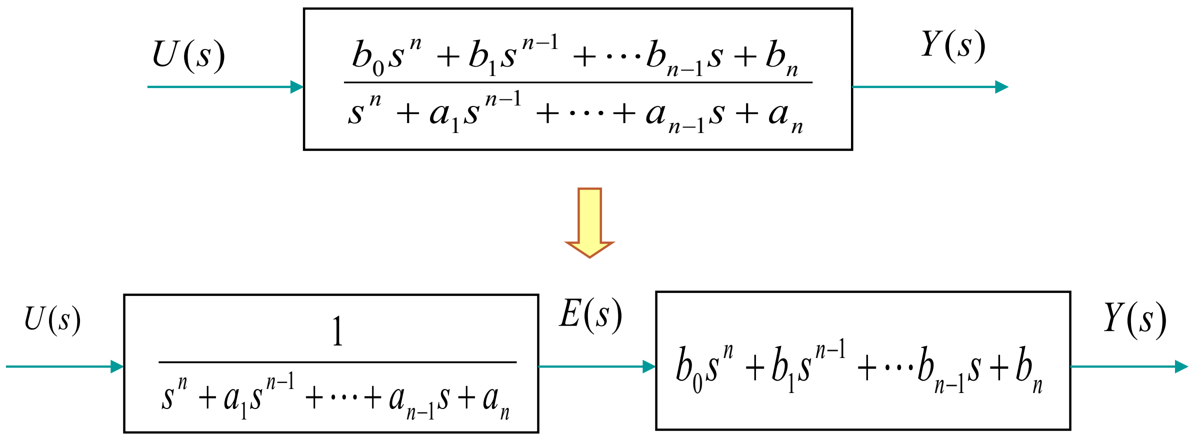
1. From Physics Mechanism of System
2. From Differential Equations of System
3. From Transfer Functions of System
4. From State-variable Diagram of System
5. Linear Transformation of State space

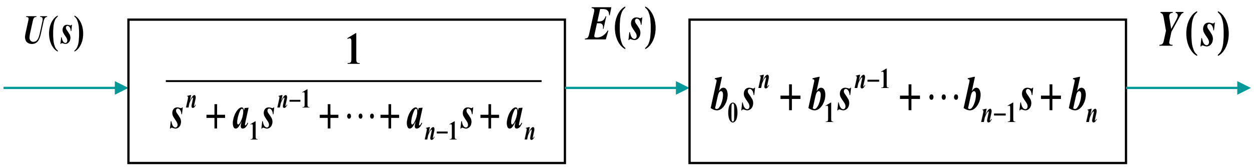
9.3.3 From Transfer Functions of System

- Transfer Functions ⟺ - State Space

1. Transfer Functions to State Space

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$





$$U(s) = (s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n) E(s)$$

$$Y(s) = (b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n) E(s)$$

Select state variables:

$$\begin{cases} x_1 = e(t) \\ x_2 = \dot{e}(t) \\ \vdots \\ x_n = e^{(n-1)}(t) \end{cases} \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \end{cases}$$

$$u = \dot{x}_n + a_1 x_n + a_2 x_{n-1} + \cdots + a_{n-1} x_2 + a_n x_1$$

$$y = b_0 \dot{x}_n + b_1 x_n + b_2 x_{n-1} + \cdots + b_{n-1} x_2 + b_n x_1$$

$$u = \dot{x}_n + a_1 x_n + a_2 x_{n-1} + \cdots + a_{n-1} x_2 + a_n x_1$$

$$y = b_0 \dot{x}_n + b_1 x_n + b_2 x_{n-1} + \cdots + b_{n-1} x_2 + b_n x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{B} u$$

$$\boldsymbol{y = b_0(-a_n \quad -a_{n-1} \quad \cdots \quad -a_1)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{b_0}_{D} u + \underbrace{(b_n \quad b_{n-1} \cdots b_1)}_{C} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

If b_0=0,the output equation will be simplified.

For the situation that the derivative of input u is included in the differential equations.

Homework: Comparing with the method before.

<u>Ex.9-7</u> Transfer function of a control system is:

$$\frac{Y(s)}{U(s)}=\frac{s^2+4s+1}{s^3+9s^2+8s}$$

transform it to the state space representation.

Solution:

$$a_1=9,\ \ a_2=8,\ \ a_3=0,\ \ b_0=0,\ \ b_1=1,\ \ b_2=4,\ \ b_3=1$$

The state equation is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -8 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

The output equations is:

$$y=\begin{bmatrix}1 & 4 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

➢ Serialization of the Transfer Function:

$$G(s)=\frac{Num(s)}{Den(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0} \qquad m\le n$$

The Numerator is: $Num=b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0$

The Denominator is: $Den=s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0$

If $z_1,z_2,\cdots,z_m$ are m zero-points of G(s),

and $p_1,\ p_2,\ \cdots,\ p_n$ are n pole-points of G(s).

then G(s) is：

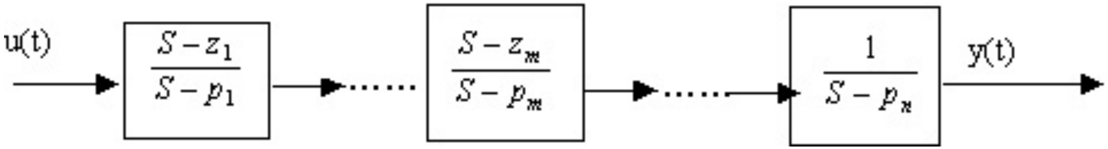
$$G(s)=\frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$=\frac{s-z_1}{s-p_1}\bullet\frac{s-z_2}{s-p_2}\bullet\cdots\bullet\frac{s-z_m}{s-p_m}\bullet\frac{b_m}{s-p_{m+1}}\bullet\cdots\bullet\frac{1}{s-p_n}$$

Therefore the system is composed serially by n items below:

$$\frac{s-z_1}{s-p_1},\ \frac{s-z_2}{s-p_2},\ \cdots,\ \frac{1}{s-p_n}$$

The structure of the system can be described by figure (a)。

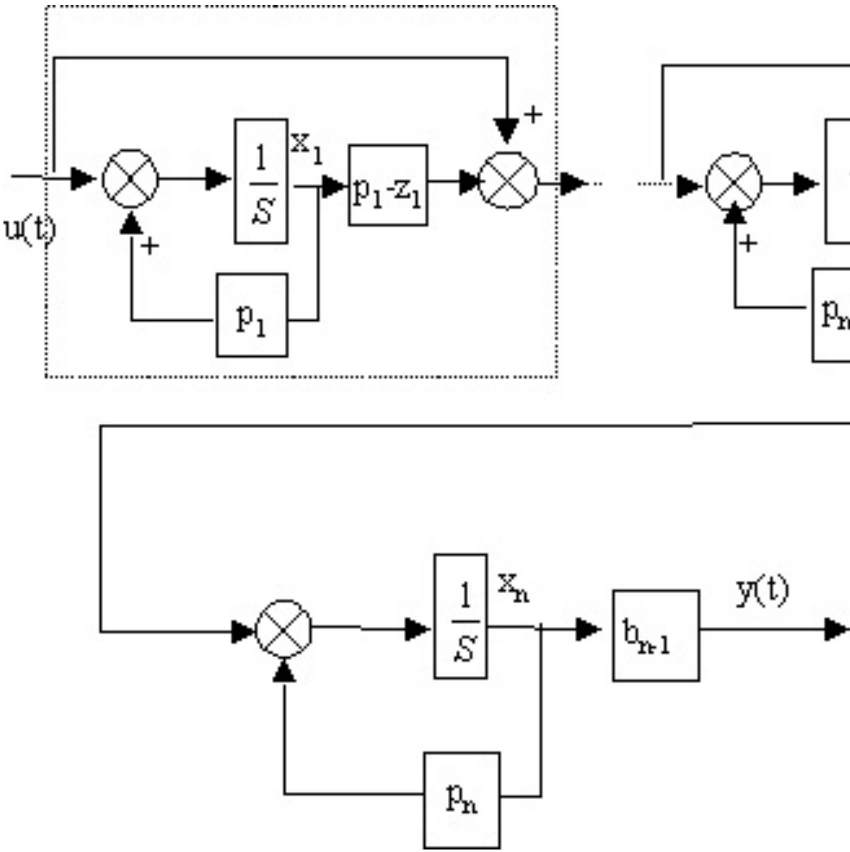


(a)

of which the first block is transformed as follow:

$$\frac{s-z_1}{s-p_1}=1+\frac{p_1-z_1}{s-p_1}=1+(p_1-z_1)\bullet\frac{\frac{1}{s}}{1-p_1\frac{1}{s}}$$

Thus its structure block will be recomposed as in figure (b).



(b) (m=n-1)

目的：选状态

Assume the outputs of integral items are required state variables

The state equations of the system are:

$$
\begin{cases}
\dot{x}_1 = p_1 x_1 + u \\
\dot{x}_2 = (p_1 - z_1)x_1 + u + p_2 x_2 = (p_1 - z_1)x_1 + p_2 x_2 + u \\
\vdots \\
\dot{x}_n = (p_1 - z_1)x_1 + (p_2 - z_2)x_2 + \cdots + (p_{n-1} - z_{n-1})x_{n-1} + p_n x_n + u
\end{cases}
$$

$$
y = b_m x_n = b_{n-1} x_n, (m = n-1)
$$

And the matrix representation:

$$
\begin{cases}
\dot{X} = \begin{bmatrix} p_1 & 0 & 0 & \cdots & 0 \\ p_1 - z_1 & p_2 & 0 & \cdots & 0 \\ p_1 - z_1 & p_2 - z_2 & p_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_1 - z_1 & p_2 - z_2 & p_3 - z_3 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 0 & 0 & \cdots & 0 & b_{n-1} \end{bmatrix} X
\end{cases}
$$

➢ Parallel of the Transfer Function:

$$G(s) = \frac{Num(s)}{Den(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \qquad m \le n$$

The Denominator:

$$Den(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

is the Characteristic Equations of the system.

Assume there are n characteristics roots: $p_i, i = 1,2,\cdots,n$

G(s) can be decomposed by the summation of n fractions:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n}$$

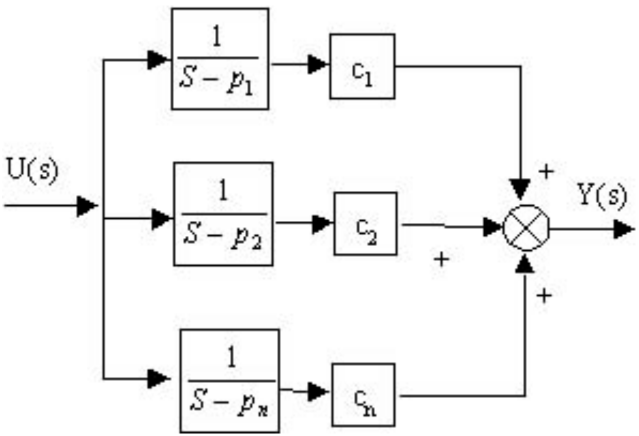
$$= \sum_{i=1}^{n} \frac{c_i}{s - p_i}$$

In which, $c_i = \lim_{s \to p_i} (s - p_i)G(s)$, is called Residue (留数) of pole p_i.

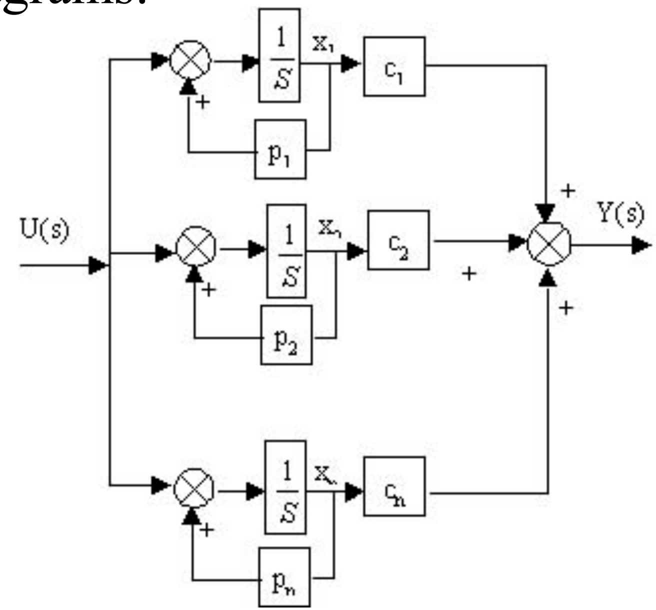
$$Y(s) = \frac{c_1}{s - p_1}U(s) + \frac{c_2}{s - p_2}U(s) + \cdots + \frac{c_n}{s - p_n}U(s)$$

$$= \sum_{i=1}^{n} \frac{c_i}{s - p_i}U(s)$$

The parallel connection diagrams:



(a)



(b) Parallel connection
(No repeated roots)

The state-equations of figure (b):

$$\begin{cases} \dot{x}_1 = p_1 x_1 + u \\ \dot{x}_2 = p_2 x_2 + u \\ \qquad \vdots \\ \dot{x}_n = p_n x_n + u \end{cases}$$

The output equation:

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Matrix representation:

$$\begin{cases} \dot{X} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ \\ Y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} X \end{cases}$$

PS: the system matrix A is a diagonal matrix.

2δ阳

If the denominator of G(s) Den(s)=0 has repeated roots:

$$Den(s) = (s-p_1)^q (s-p_{q+1}) \cdots (s-p_n)$$

$s = p_1$ is the only q times repeated root, we have G(s):

$$G(s) = \frac{Num(s)}{Den(s)}$$

$$= \frac{c_{11}}{s-p_1} + \frac{c_{12}}{(s-p_1)^2} + \cdots + \frac{c_{1q}}{(s-p_1)^q} + \frac{c_{q+1}}{s-p_{q+1}} + \cdots + \frac{c_n}{s-p_n}$$

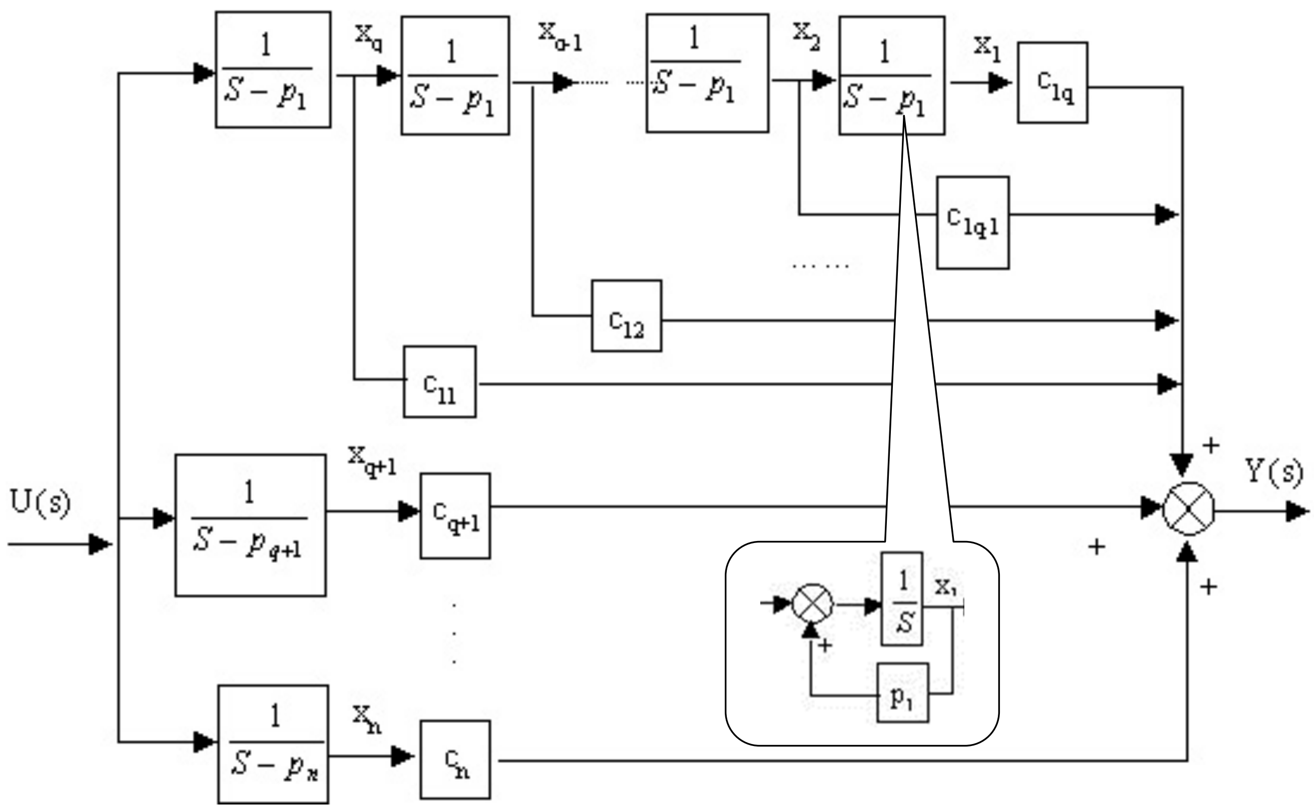
In which：

$$c_{1i} = \frac{1}{(q-i)!} \bullet \lim_{s \to p_1} \frac{d^{q-i}}{ds^{q-i}} [(s-p_1)^q G(s)] \qquad i = 1,2,\ldots,q$$

$$c_j = \lim_{s \to p_j} [(s-p_j) G(s)] \qquad j = q+1, q+2, \ldots, n$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{c_{11}}{s-p_1} + \frac{c_{12}}{(s-p_1)^2} + \cdots + \frac{c_{1q}}{(s-p_1)^q} + \frac{c_{q+1}}{s-p_{q+1}} + \cdots + \frac{c_n}{s-p_n}$$

$$Y(s)=\frac{c_{11}}{s-p_1}U(s)+\frac{c_{12}}{(s-p_1)^2}U(s)+\cdots+\frac{c_{1q}}{(s-p_1)^q}U(s)+\frac{c_{q+1}}{s-p_{q+1}}U(s)+\cdots+\frac{c_n}{s-p_n}U(s)$$



Parallel Connection (Repeated roots)

$$Y(s)=\frac{c_{11}}{s-p_1}U(s)+\frac{c_{12}}{(s-p_1)^2}U(s)+\cdots+\frac{c_{1q}}{(s-p_1)^q}U(s)+\frac{c_{q+1}}{s-p_{q+1}}U(s)+\cdots+\frac{c_n}{s-p_n}U(s)$$

Select the state-variables as the output of the integral items in the diagram:

$$\begin{cases}\dot{x}_1=p_1x_1+x_2\\ \dot{x}_2=p_1x_2+x_3\\ \vdots\\ \dot{x}_{q-1}=p_1x_{q-1}+x_q\\ \dot{x}_q=p_1x_q+u\\ \vdots\\ \dot{x}_n=p_nx_n+u\end{cases}$$

$$y=c_{1q}x_1+c_{1q-1}x_2+\cdots+c_{11}x_q+c_{q+1}x_{q+1}+\cdots+c_nx_n$$

$$
\begin{cases}
\dot{X} = \begin{bmatrix} p_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & p_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & p_{q+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} U \\
Y = \begin{bmatrix} c_{1q} & c_{1q-1} & \cdots & c_{11} & c_{q+1} & \cdots & c_n \end{bmatrix} X
\end{cases}
$$

System matrix A is Jordan Standard Form(约当标准型).

Ex.9-8 find the parallel connection of the follow system:

$$G(s)=\frac{4s^2+10s+5}{s^3+5s^2+8s+4}$$

Solution： the Denominator: $(s+2)^2(s+1)$

$$G(s)=\frac{4s^2+10s+5}{(s+2)^2(s+1)}=\frac{c_{11}}{s+2}+\frac{c_{12}}{(s+2)^2}+\frac{c_3}{s+1}$$

$$c_{12}=\lim_{s\to-2}(s+2)^2G(s)=-1$$

$$c_{11}=\frac{1}{(2-1)!}\lim_{s\to-2}\frac{d^{(2-1)}}{ds^{(2-1)}}[(s+2)^2G(s)]=\lim_{s\to-2}\frac{d}{dS}[\frac{4s^2+10s+5}{(s+1)}]=5$$

$$c_3=\lim_{s\to-1}(s+1)G(s)=\lim_{s\to-1}\frac{4s^2+10s+5}{(s+2)^2}=-1$$

The state equation of the system by parallel connection:

$$\begin{cases}\dot{X}=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & -1\end{bmatrix}X+\begin{bmatrix}0\\1\\1\end{bmatrix}u\\ y=\begin{bmatrix}-1 & 5 & -1\end{bmatrix}X\end{cases}$$

2. State Space to Transfer Functions

State Space (Dynamic Equations) VS Transfer Functions

- State Space denote both the input/output relationship and the internal state variables of the system;

  Transfer functions present the input/output relationship only.

- From Transfer Functions to State Space: system realization process, which is complicated and non-unique.

  From State Space to Transfer Functions: simple and unique process.

状态变量图

❖ For SISO system: State-space to TF

The State-space representation of a SISO system:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \\ \boldsymbol{y} = C\boldsymbol{x} + Du \end{cases}$$

$x, \dot{x} \in R^{n\times 1}; A \in R^{n\times n}; B \in R^{n\times 1}; C \in R^{1\times n};$ D is a scalar quantity.

Assume the initial condition zero-input, and use Laplace Transform:

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1} BU(s)$$

The Transfer Function is:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

Ex.9-8 the state-space of the system is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Find its transfer function:

Solution: write the related matrices [A ,B ,C] at first:

$$A=\begin{bmatrix}0 & 1\\ -1 & -3\end{bmatrix},\qquad B=\begin{bmatrix}0\\ 1\end{bmatrix}\qquad C=\begin{bmatrix}1 & 0\end{bmatrix}$$

$$sI-A=\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1\\ -1 & -3\end{bmatrix}=\begin{bmatrix}s & -1\\ 1 & s+3\end{bmatrix}$$

$$(sI-A)^{-1}=\frac{1}{s(s+3)}\begin{bmatrix}s+3 & 1\\ -1 & s\end{bmatrix}$$

+1

$$\frac{\text{伴随矩阵}}{\text{行列式}}$$

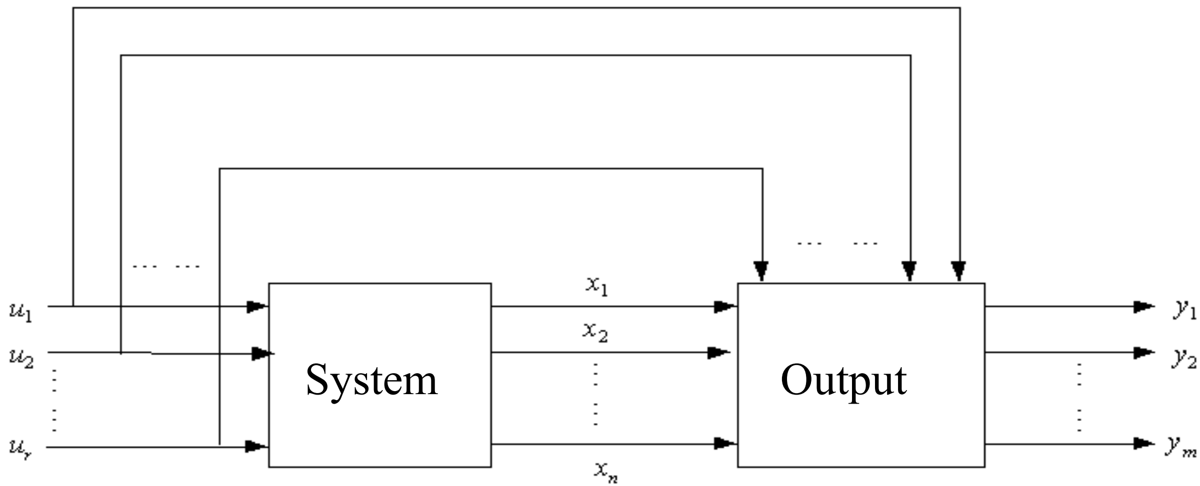
Transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{s+3}{s(s+3)+1} & \dfrac{1}{s(s+3)+1} \\ \dfrac{-1}{s(s+3)+1} & \dfrac{s}{s(s+3)+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s(s+3)+1} = \frac{1}{s^2 + 3s + 1}$$

❖ For MIMO system: State-space to TF



$u_1, u_2, \cdots, u_r$ —— input signals of the system

$y_1, y_2, \cdots, y_m$ —— output signals of the system

$x_1, x_2, \cdots, x_n$ —— state variables of the system

The dynamic equations:
$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u} \end{cases}$$

Equations have the same formation with SISO system, however, the matrices *B, C ,D* have different dimension.

$$\boldsymbol{x}, \dot{\boldsymbol{x}} \in \boldsymbol{R}^{n\times 1}, \quad \boldsymbol{U} \in \boldsymbol{R}^{r\times 1}, \quad \boldsymbol{Y} \in \boldsymbol{R}^{m\times 1},$$

$$\boldsymbol{A} \in \boldsymbol{R}^{n\times n}, \ \boldsymbol{B} \in \boldsymbol{R}^{n\times r}, \quad \boldsymbol{C} \in \boldsymbol{R}^{m\times n}, \quad \boldsymbol{D} \in \boldsymbol{R}^{m\times r}$$

Laplace transformation:

$$\boldsymbol{G}(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$G \in R^{m\times r}$ ———Transfer Function Matrix

Ex.9-9 The dynamic equations are:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the Transfer function of the system.

Solution: $A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad D = 0$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

- the Eigen-equation and Eigenvalue of the system matrix A

特征方程 特征值

The Eigen-equation of the system is:

$$|\lambda I - A| = 0$$

The Eigenvalue is one of its solutions.

Expand the equation: $|\lambda I - A| = 0$

we have the polynomial: $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots a_n = 0$

Then solve the equation we have its n eigenvalues.

e.g. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ then $|\lambda \ I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix}$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda+1)(\lambda+2)(\lambda+3) = 0$$

3 eigenvalues are －1、 －2 and －3.

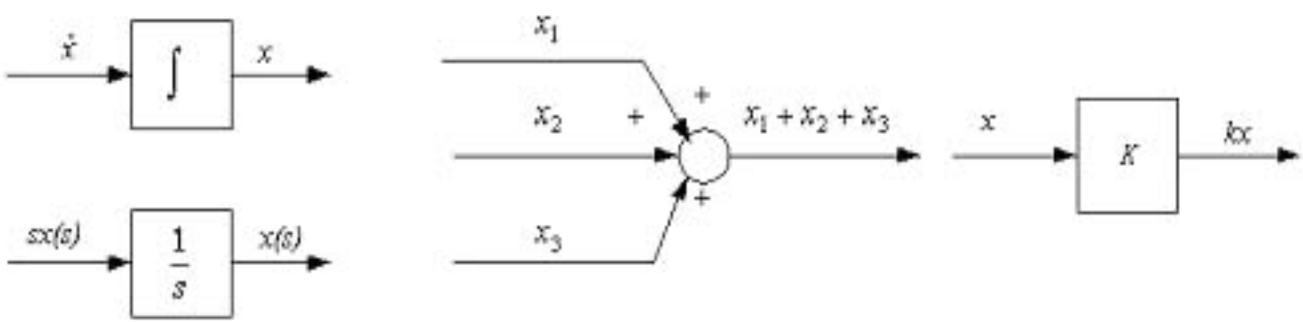
9.3.4 From State variable diagram of the System

- **State variable diagram:** the diagram description of the relationship of state variables, which is composed by the integral items, proportion items and sum symbols.

积分 比例 求和.

◆ The output of each integral item is one of the state variables of the system.

减的话 $K=-1$



(a) (b) (c)

Ex.9-10 **The Close-loop TF of the system is:**

$$\frac{Y(s)}{U(s)}=\frac{s^2+3s+2}{s(s^2+7s+12)}$$

Draw the State variable diagram and find the state space representation of the system.

Solution: rewrite the close-loop TF of the system:

$$\frac{Y(s)}{U(s)}=\frac{s^2+3s+2}{s^3+7s^2+12s}$$

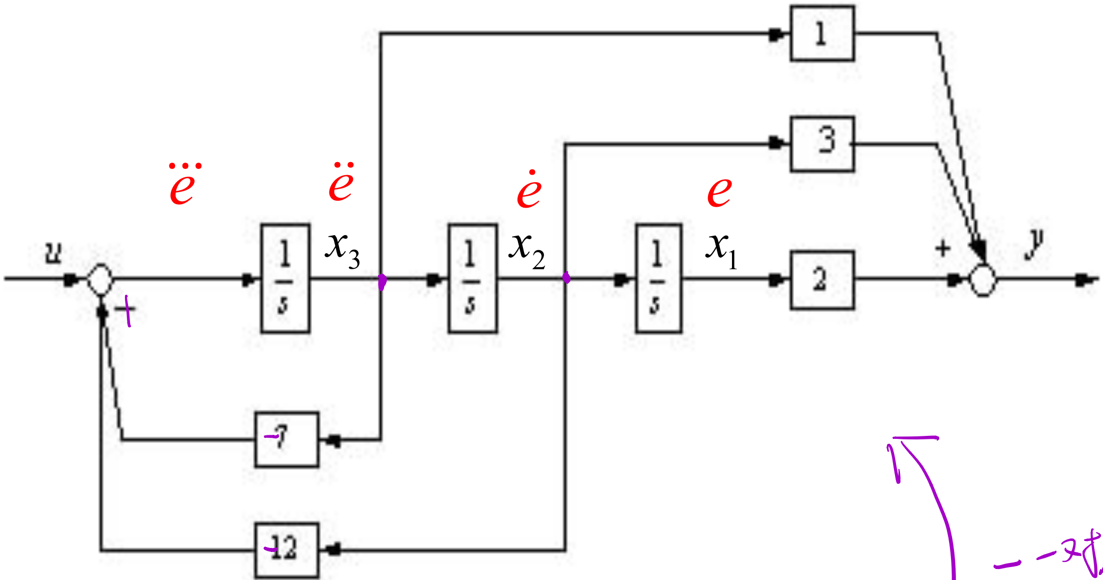
Assume:

$$Y(s)=(s^2+3s+2)E(s) \Rightarrow Y(s)=s^2E(s)+3sE(s)+2E(s)$$

$$\boxed{y(t)=\ddot{e}(t)+3\dot{e}(t)+2e(t)}$$

$$U(s)=(S^3+7s^2+12s)E(s) \Rightarrow S^3E(s)=U(s)-7s^2E(s)-12sE(s)$$

$$\boxed{\dddot{e}(t)=u(t)-7\ddot{e}(t)-12\dot{e}(t)}$$



一一对应

$$\begin{cases} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -12 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u \\ y = [2\ \ 3\ \ 1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \end{cases}$$

9.3.5 Linear Transformation of State space

- State-space equation establishing method review:
  Physics Mechanism / Differential Equations/ Transfer
  物理机理 微分方程.
  Functions / State-variable Diagram
  传递函数. 状态空间图
- State-variables selection: Non-unique
- State-space equation: Non-unique
- The amount of the independent state variables in different state-space equation for a certain physics system: Uniform
- The connection of different state-space representations:
  Linear Transformation 线性变换

- Non-uniqueness of State-space Variables

Assume a State-space Equation:

$$\dot{x} = Ax + Bu$$

Linear Transform the state-variables: $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ into another group: $x_1, x_2, \ldots, x_n$

We have

$$\begin{cases} x_1 = p_{11}\bar{x}_1 + p_{12}\bar{x}_2 + \cdots + p_{1n}\bar{x}_n \\ x_2 = p_{21}\bar{x}_1 + p_{22}\bar{x}_2 + \cdots + p_{2n}\bar{x}_n \\ \cdots\cdots \\ x_n = p_{n1}\bar{x}_1 + p_{n2}\bar{x}_2 + \cdots + p_{nn}\bar{x}_n \end{cases}$$

and also

$$x = P\bar{x}$$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

If P is a non-singular(非奇异) constant matrix: $|P| \neq 0$

The vector $\overline{x}$ is the state-variable vector of the system: $\dot{x} = Ax + Bu$

Proof: $\dot{\overline{x}} = P^{-1}AP\overline{x} + P^{-1}Bu$

Symbolize: $A_1 = P^{-1}AP, \quad B_1 = P^{-1}B$

then $\dot{\overline{x}} = A_1\overline{x} + B_1u$

A_1 and A are similar matrices, the characteristic polynomial:

$$|sI - A_1| = |sI - A|$$

The state-equations from x and $\overline{x}$ have same eigenvalues

Result: for any control system, selection of state variables is non-unique. Any state variables from linear transformation, which satisfy the **non-singular condition** of the transform matrix P, are the appropriate state variables of the system.

Invariability(不变性) of linear transformation

System: $\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu \\ \mathbf{y} = C\mathbf{x} + D\mathbf{u} \end{cases}$ If $\mathbf{x} = P\overline{\mathbf{x}}$ $\begin{cases} \dot{\overline{\mathbf{x}}} = P^{-1}AP\overline{\mathbf{x}} + P^{-1}B\mathbf{u} \\ \mathbf{y} = CP\overline{\mathbf{x}} + D\mathbf{u} \end{cases}$

(1) Invariability of eigenequations and eigenvalues

Eigenvalues after transformation:

Analysis:

$$\left|\lambda I - P^{-1}AP\right| = \left|\lambda P^{-1}P - P^{-1}AP\right| = \left|P^{-1}\lambda P - P^{-1}AP\right|$$

$$= \left|P^{-1}(\lambda I - A)P\right| = \left|P^{-1}\right|\left|\lambda I - A\right|\left|P\right|$$

$$= \left|P^{-1}\right|\left|P\right|\left|\lambda I - A\right| = \left|P^{-1}P\right|\left|\lambda I - A\right|$$

$$= \left|I\right|\left|\lambda I - A\right| = \left|\lambda I - A\right|$$

Obviously, the eigenvalues of the system are same before and after the non-singular linear transformation.

(2) Invariant of system transfer function matrix

The transfer matrix after transformation:

$$
\begin{aligned}
G'(s) &= CP(sI - P^{-1}AP)^{-1}P^{-1}B + D \\
&= CP(P^{-1}sIP - P^{-1}AP)^{-1}P^{-1}B + D \\
&= CP\left[P^{-1}(sI - A)P\right]^{-1}P^{-1}B + D \\
&= CPP^{-1}(sI - A)^{-1}PP^{-1}B + D \\
&= C(sI - A)^{-1}B + D = G(s)
\end{aligned}
$$

The transfer matrix of the system is invariant before and after the non-singular linear transformation.

✓Why need Linear Transformation?

Although there are infinite forms of state-space equations according to a certain system, which satisfy the non-singular linear transformation, only several kind of canonical forms are benefit for us:

- ☐ **Controllability Canonical Form** 能控标准型
- ☐ **Observability Canonical Form** 能观标准型
- ☐ **Diagonal Canonical Form** 对角标准型
- ☐ **Jordan Canonical Form.** Jordan 标准型

□ Controllability Canonical Form(能控标准型)

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_n & c_{n-1} & \cdots & c_1 \end{bmatrix} x \end{cases}$$

Matrix A is also called companion matrix(友矩阵) .

□ Observability Canonical Form(能观标准型)

$$
\begin{cases}
\dot{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_1 \end{bmatrix} x + \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix} u \\
y = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} x
\end{cases}
$$

□ Diagonal Canonical Form (I)

$$\begin{cases} \dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} x \end{cases}$$

□ Diagonal Canonical Form (II)

Select state variables: $$X_i(s) = \frac{c_i}{s - \lambda_i} U(s)$$

System Output is: $$Y(s) = \sum_{i=1}^{n} X_i(s)$$

Inverse Laplace Transformation:

$$\begin{cases} \dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} x \end{cases}$$

Jordan Canonical Form

$$\begin{cases} \dot{x} = \left[\begin{array}{ccc|ccc} \lambda_1 & 1 & 0 & & & \\ & \lambda_1 & 1 & & 0 & \\ & & \lambda_1 & & & \\ \hline & & & \lambda_4 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{array}\right] x + \begin{bmatrix}0\\0\\1\\1\\\vdots\\1\end{bmatrix} u \\ y = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_4 & \cdots & c_n \end{bmatrix} x \end{cases}$$

约旦块

The Block of repeated poles $\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$ is called Jordan Block.

(1) Outlines:

$$\left.\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}\right\} \xrightarrow[\text{Equivalent transforming}]{x = P\bar{x}} \left.\begin{aligned}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u\\ \bar{y} &= \bar{C}\bar{x} + \bar{D}u = y\end{aligned}\right\}$$

Non-canonical form Canonical form

(2) Relationship between the coefficient matrices

$$\because \quad x = P\bar{x}, \quad \dot{x} = P\dot{\bar{x}},$$

P is NxN non-singular constant matrix

Substitute into the equations:

$$\left.\begin{aligned}P\dot{\bar{x}} &= AP\bar{x} + Bu\\ y &= CP\bar{x} + Du\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\dot{\bar{x}} &= P^{-1}AP\bar{x} + P^{-1}Bu\\ y &= CP\bar{x} + Du\end{aligned}\right\}$$

$$\Downarrow$$

$$\bar{A} = P^{-1}AP, \quad \bar{B} = P^{-1}B, \quad \bar{C} = CP, \quad \bar{D} = D$$

- The constraint satisfied systems $\{A,B,C,D\}$ and $\{\bar{A},\bar{B},\bar{C},\bar{D}\}$ are called Similar Systems;
- The related dynamic equations are called Equivalent Dynamic Equations;
- The linear transformation is called Equivalent Transformation.

The purpose of non-singular transformation is to transform the system matrix A into the canonical form $\bar{A}$

Some Common Linear transformation methods

(1) Transform *A* to Diagonal Form

(a) Assume a square matrix A with n different real eigenvalues:

$\lambda_1, \cdots, \lambda_n$ which satisfy the follow eigen-equation.

$$\det(\lambda I - A) = |\lambda I - A| = 0$$

$$\Rightarrow \quad \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

The non-singular transformation matrix **P** is composed by real eigenvectors: $\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n]$ 特征向量

And the eigenvectors satisfy the equations:

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i \;\text{ or }\; (\lambda_i I - A)\mathbf{p}_i = 0$$

(b) Assume a companion matrix(友矩阵) A with n different real eigenvalues: $\lambda_1, \cdots, \lambda_n$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

Then the Vander-mode matrix(范德蒙特矩阵) P

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \\ \lambda_1^{n-1} & & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

can transform A to the diagonal matrix:

$$\Rightarrow \quad \overline{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

(c) Assume matrix A has m repeated eigenvalues: $\lambda_1 = \cdots = \lambda_m$ and other (n-m) different eigenvalues. If A still has m independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ while solving the equation:

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i \qquad (i = 1 \sim m)$$

the matrix A can be transformed to the diagonal form:

$$\Longrightarrow \quad \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & & & \mathbf{O} \\ & \ddots & & & & \\ & & \lambda_m & & & \\ & & & \lambda_{m+1} & & \\ & & & & \ddots & \\ \mathbf{O} & & & & & \lambda_n \end{bmatrix}$$

$$P = [\mathbf{p}_1, \cdots, \mathbf{p}_m \,\vdots\, \mathbf{p}_{m+1}, \cdots, \mathbf{p}_n]$$

Ex.9-11: transform the follow state equation to the diagonal form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} u$$

Solution: characteristic equation

$$\det(\lambda I - A) = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -4 & -5 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 2)(\lambda - 1)^2 = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = 1$$

$$(\lambda_1 I - A)\mathbf{p_1} = 0 \quad \begin{bmatrix} 0 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} = 0 \qquad \mathbf{p_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(\lambda_2 I - A)\mathbf{p}_2 = 0 \quad \begin{bmatrix} -1 & -4 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} = 0 \qquad \mathbf{p}_2 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

$$(\lambda_2 I - A)\mathbf{p}_3 = 0 \quad \begin{bmatrix} -1 & -4 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} = 0 \qquad \mathbf{p}_3 = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \quad P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3] = \begin{bmatrix} 1 & 4 & 5 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\overline{A} = P^{-1} A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \overline{\mathbf{b}} = P^{-1}\mathbf{b} = \begin{bmatrix} 24 \\ -2 \\ -3 \end{bmatrix}$$

(2) transform A to Jordan form

(a) Assume matrix A has m repeated real eigenvalues:

$$\lambda_1 = \cdots = \lambda_m$$

others are (n-m) different real eigenvalues.

Then solve the equation: $A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad (i = 1 \sim m)$

and receive only one independent eigenvector: p_1

Matrix A can be transformed to Jordan Canonical Form only.

$$J = \overline{A} = P^{-1}AP = \left[\begin{array}{cccc|ccc} \lambda_1 & 1 & & & & & \\ & \lambda_1 & \ddots & & & 0 & \\ & & \ddots & 1 & & & \\ & & & \lambda_1 & & & \\ \hline & & & & \lambda_{m+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \lambda_n \end{array}\right]$$

Jordan Block

只能找一个p_1

$$P = [\mathbf{p}_1, \cdots, \mathbf{p}_m \,\vdots\, \mathbf{p}_{m+1}, \cdots, \mathbf{p}_n]$$

Here, $\mathbf{p}_1$, $\mathbf{p}_2$, …, $\mathbf{p}_m$ are generalized eigenvectors (广义特征向量) and satisfy:

增广特征向量.

$$[\mathbf{p}_1, \cdots, \mathbf{p}_m]\begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{bmatrix} = A[\mathbf{p}_1, \cdots, \mathbf{p}_m]$$

$\mathbf{p}_{m+1}$, …, $\mathbf{p}_n$ are the eigenvectors corresponding to the (n-m) different eigenvalues.

(b) Assume companion matrix A, which is controllability canonical form matrix, and its m repeated eigenvalues:

$$\lambda_1 = \cdots = \lambda_m$$

There is only one independent real eigenvector:

$$\mathbf{p}_1 = [1 \ \lambda_1 \ \lambda_1^2 \ \cdots \ \lambda_1^{n-1}]^T$$

satisfy:

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad (i = 1 \sim m)$$

which can transform A to Jordan form:

$$P = [\mathbf{p_1} \ \frac{\partial \mathbf{p_1}}{\partial \lambda_1} \ \frac{\partial^2 \mathbf{p_1}}{\partial \lambda_1^2} \cdots \frac{\partial^{m-1} \mathbf{p_1}}{\partial \lambda_1^{m-1}} \vdots \mathbf{p_{m+1}} \cdots \mathbf{p_n}]$$

(c) Assume a matrix A has 5 repeated eigenvalues:

$$\lambda_1 = \cdots = \lambda_5$$

Satisfy: $A\mathbf{p}_i = \lambda_i \mathbf{p}_i \quad (i = 1 \sim 5)$

With 2 independent real eigenvectors: p_1 and p_2.

Other (n-5) eigenvalues are different.

The matrix A can be transformed to the Jordan form:

$$J = \overline{A} = P^{-1}AP = \left[\begin{array}{ccc|cc|ccc} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & 1 & & & & & \\ & & \lambda_1 & & & & & \\ \hline & & & \lambda_1 & 1 & & & \\ & & & & \lambda_1 & & & \\ \hline & & & & & \lambda_{m+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \end{array}\right]$$

There are 2 upper Jordan blocks in J, in which:

$$P = \left[p_1 \quad \frac{\partial p_1}{\partial \lambda_1} \quad \frac{\partial^2 p_1}{\partial \lambda_1^2} \;\vdots\; p_2 \quad \frac{\partial p_2}{\partial \lambda_1} \;\vdots\; p_6 \quad \cdots \quad p_n \right]$$

(3) Transform the controllable system to Controllability Canonical Form

The controllability canonical form of a single input linear time-invariant system state equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

- The corresponding controllability matrix S is a Right Lower Triangular matrix with the main diagonal elements 1.
- Thus det S $\neq 0$, 满秩 the system is controllable, and A, b are called controllability canonical form.
- The controllability matrix S is:

$$S = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & \times & \times \\ 0 & 1 & -a_{n-1} & \cdots & \times & \times \\ 1 & -a_{n-1} & -a_{n-2} & \cdots & \times & \times \end{bmatrix}$$

- Any controllable system, if its A, b are not controllability canonical form, they can be transformed to the canonical form by appropriate transforming method.
- Assume a dynamic system: $\dot{x} = Ax + bu$
- Execute the P^{-1} transformation:

$$x = P^{-1}z$$

and we have

$$\dot{z} = PAP^{-1}z + Pbu$$

Satisfy:

$$PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad Pb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Analyze the transformation matrix: P

Assume P is

$$P = \begin{bmatrix} p_1^T & p_2^T & \cdots & p_n^T \end{bmatrix}^T$$

Based on the matrix A, P should satisfy:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}$$

$$\begin{cases} p_1 A = p_2 \\ p_2 A = p_3 \\ \vdots \\ p_{n-1} A = p_n \\ p_n A = -a_0 p_1 - a_1 p_2 - \cdots - a_{n-1} p_n \end{cases}$$

Then:

$$\begin{cases} p_1 A = p_2 \\ p_2 A = p_1 A^2 = p_3 \\ \vdots \\ p_{n-1} A = p_1 A^{n-1} = p_n \end{cases}$$

Transformation matrix:

$$P = \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix}$$

Then based on vector b, we have:

$$Pb = \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix} b = p_1 \begin{bmatrix} b \\ Ab \\ \vdots \\ A^{n-1} b \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$p_1 \begin{bmatrix} b & Ab & \cdots & A^{n-1} b \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

Thus $p_1 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}^{-1}$

It seems that p_1 is the last row of the inverse controllability matrix

Therefore, the solution of transformation matrix P^{-1}:

(i) Find the controllability matrix $S = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$

(ii) Find the inverse matrix S^{-1}, which is:

$$S^{-1} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

(iii) Take out the last row of S^{-1}, (the n^{th} row) and compose the vector:

$$p_1 = \begin{bmatrix} S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

(iv) Construct matrix P

$$P = \begin{bmatrix} p_1 \\ p_1 A \\ \vdots \\ p_1 A^{n-1} \end{bmatrix}$$

(v) Then, P^{-1} is required transforming matrix from non-canonical form to controllability canonical form.

(4) Canonical Form of SISO system– From TF

The state space description of the dynamic system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad x = P\overline{x}$$

- ✓ Consequently, according to transformation matrix P, we can transform the above system to the canonical forms we need.
- ✓ Here, we consider about the transformation using the transfer functions.

The required transfer function is:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

The canonical state space forms of **controllability form**, **observability form** and **diagonal form** (or **Jordan form**) are given as follow.

(1) Controllability Canonical Forms

$$
\begin{cases}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} b_n - a_n b_o & b_{n-1} - a_{n-1} b_o & \cdots & b_1 - a_1 b_o \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u
\end{cases}
$$

(2) Observability Canonical Forms

$$\begin{cases}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 0 & \cdots & 0 & -a_n\\ 1 & 0 & \cdots & 0 & -a_{n-1}\\ 0 & 1 & \cdots & 0 & -a_{n-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 1 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+\begin{bmatrix}b_n-a_nb_0\\ b_{n-1}-a_{n-1}b_0\\ b_{n-2}-a_{n-2}b_0\\ \vdots\\ b_1-a_1b_0\end{bmatrix}u\\ y=[0\ 0\ \cdots\ 0\ \ 1]\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+b_ou\end{cases}$$

(3) Diagonal Canonical Form

If the roots of denominate polynomial are different, transfer function can be written as follow:

$$\frac{Y(s)}{U(s)} = \frac{b_o s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)} = b_o + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$

The canonical form is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ 0 & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [c_1 \ \ c_2 \ \ \cdots \ \ c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_o u$$

(4) Jordan Canonical Form

If there are repeated eigenvalues in denominator, transform the diagonal form to Jordan form. Assume 3 repeated roots are $-p_1=-p_2=-p_3$ and others are different.

$$\frac{Y(s)}{U(s)}=\frac{b_0 s^n+b_1 s^{n-1}+\cdots+b_{n-1}s+b_n}{(s+p_1)^3(s+p_4)(s+p_5)\cdots(s+p_n)}$$

$$=b_0+\frac{c_1}{(s+p_1)^3}+\frac{c_2}{(s+p_1)^2}+\frac{c_3}{(s+p_1)}+\frac{c_4}{s+p_4}+\cdots+\frac{c_n}{s+p_n}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \vdots\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}-p_1 & 1 & 0 & 0 & \cdots & 0\\ 0 & -p_1 & 1 & \vdots & & \vdots\\ 0 & 0 & -p_1 & 0 & & 0\\ 0 & \cdots & 0 & -p_4 & & 0\\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & & -p_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\\ 1\\ \vdots\\ 1\end{bmatrix}u \qquad y=[c_1\ \ c_2\ \ \cdots\ \ c_n]\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+b_o u$$

Ex.9-12 consider the follow transfer function of a certain system:

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

try to find its controllability canonical form, observability canonical form, and diagonal form.

Solution:

Its controllability canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [3 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Observability canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonal canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [2 \quad -1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$