Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

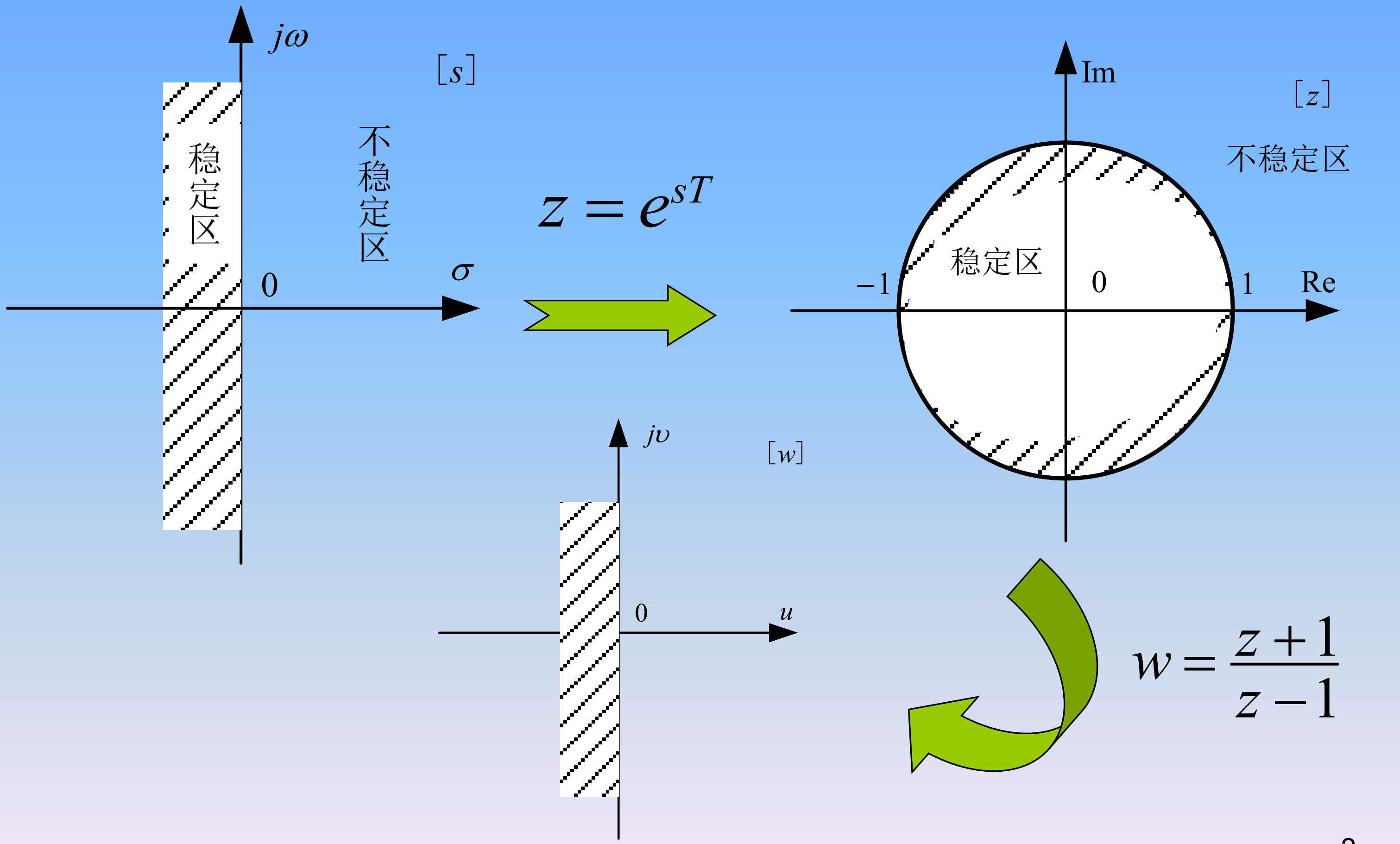
s-Domain to z-Domain Mapping

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

— All poles of $\Phi(z)$ lie in the unit circle of z plane

Routh criterion in w domain (Generalized Routh Criterion)

we’ve learned three methods to determine the stability of a discrete-time systems.



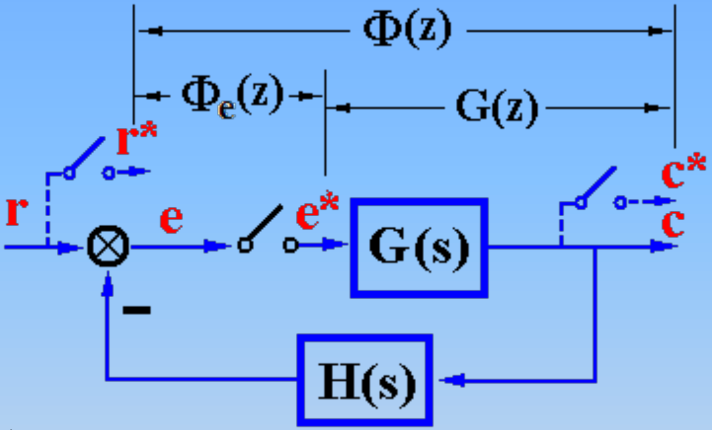
7.6 Performance Analysis of Discrete-Time Systems

- **Stability**
- **Dynamic Performance**
- **Steady-state Errors**

7.6.2 Dynamic Performance Analysis of Discrete-Time Systems

1. General algorithm to obtain the dynamic performance

(1) Obtain the impulse transfer function



Let
$$\begin{cases} GH(z) = Z[G(s)H(s)] \\ \Phi(z) = \dfrac{G(z)}{1+GH(z)} = \dfrac{M(z)}{D(z)} \end{cases}$$

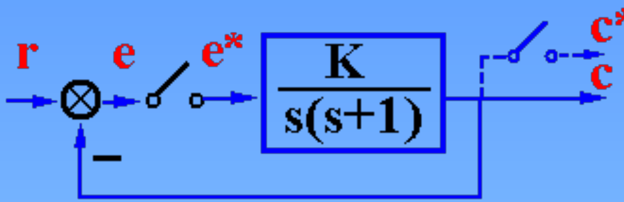
(2) Obtain $C(z) = \Phi(z)R(z) = \dfrac{M(z)}{D(z)} \cdot \dfrac{z}{z-1}$

$$= c(0) + c(T)z^{-1} + c(2T)z^{-2} + \cdots$$

(3) $c^*(t) = c(0)\delta(t) + c(T)\delta(t-T) + c(2T)\delta(t-2T) + \cdots$

(4) Determine the specifications $\sigma\%,\ t_s$.

Example 1 Consider the system shown in the figure, T=K=1. Obtain the dynamic specifications. (σ %, t_s).



Solution.

$$G(z)=Z\left[\frac{K}{s(s+1)}\right]=\frac{K(1-e^{-T})z}{(z-1)(z-e^{-T})}$$

$$\overset{K=T=1}{=}\frac{0.632z}{(z-1)(z-0.368)}$$

$$\Phi(z)=\frac{G(z)}{1+G(z)}=\frac{0.632z}{z^2-0.736z+0.368}$$

$$c(\infty T)=\lim_{z\to 1}(z-1)\cdot\Phi(z)\cdot\frac{z}{z-1}=1$$

终值定理

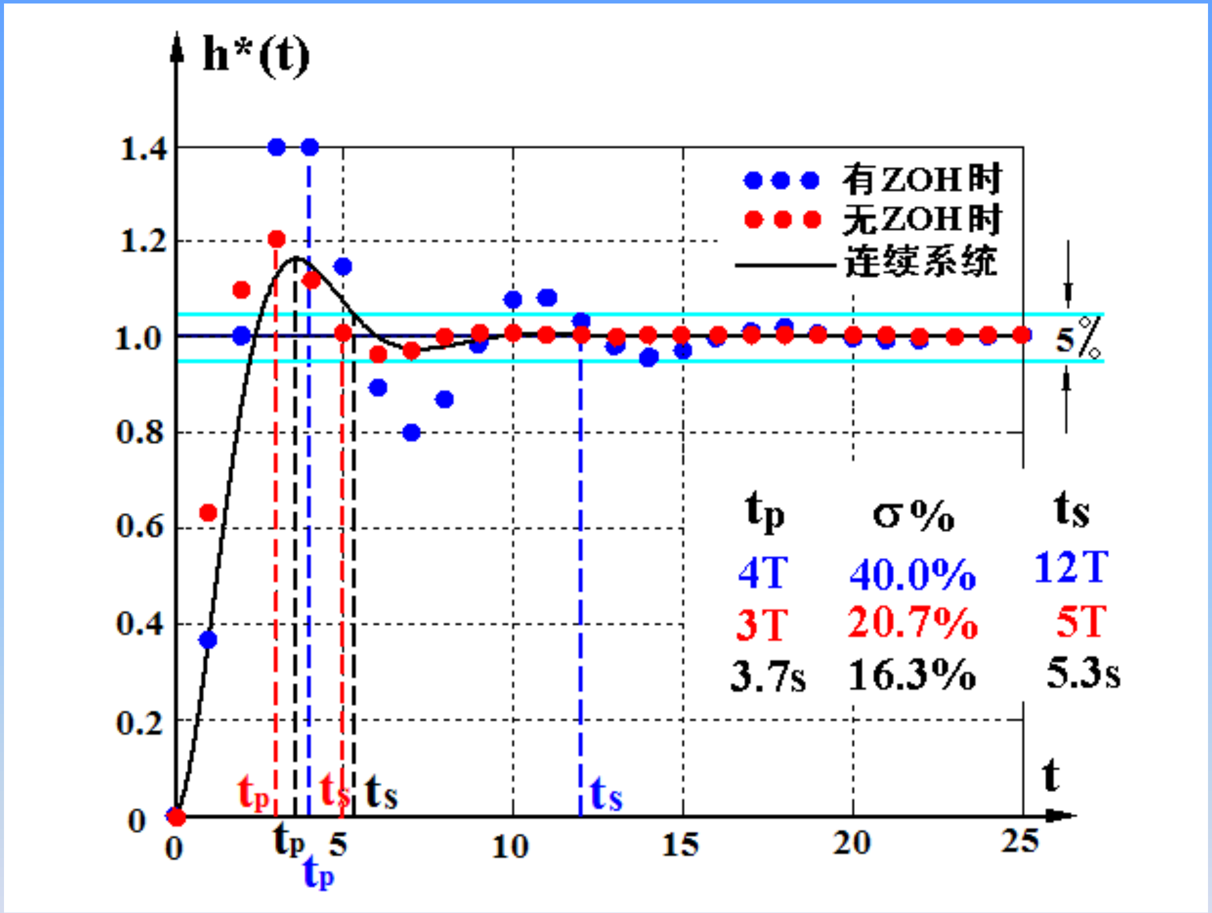
$$C(z)=\Phi(z)\cdot\frac{z}{z-1}=\frac{0.632z^2}{z^3-1.736z^2+1.104z-0.368}$$

h(0)= 0
h(1)= 0.632
h(2)= 1.097
h(3)= 1.207
h(4)= 1.117
h(5)= 1.014
h(6)= 0.964
h(7)= 0.970
h(8)= 0.991
h(9)= 1.004
h(10)=1.007
h(11)=1.003
h(12)=1.000
⋮

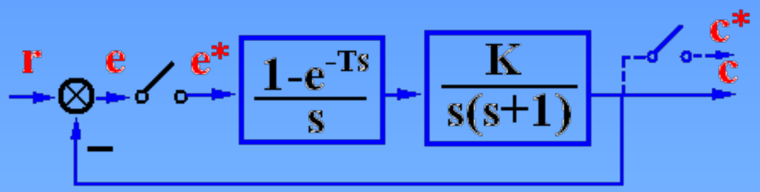
$t_P=3T$
$\sigma\%=20.7\%$

$t_s=5T$

Obtain the unit step response series h(k) by long division method.


h*(t)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
有ZOH时
无ZOH时
连续系统
5%
tp
σ%
ts
4T
40.0%
12T
3T
20.7%
5T
3.7s
16.3%
5.3s
t
tp
ts
ts
ts
tp
tp
0
5
10
15
20
25

Example 1 Consider the system shown in the figure, T=K=1. Obtain the dynamic specifications. (σ %, t_s).



Solution. $G(z)=K\dfrac{z-1}{z}Z\left[\dfrac{1}{s^2(s+1)}\right]$

$$=K\frac{(T-1+e^{-T})z+(1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}$$

$$\overset{K=T=1}{=}\frac{0.368z+0.264}{(z-1)(z-0.368)}$$

$$\Phi(z)=\frac{G(z)}{1+G(z)}=\frac{0.368z+0.264}{z^2-z+0.632}$$

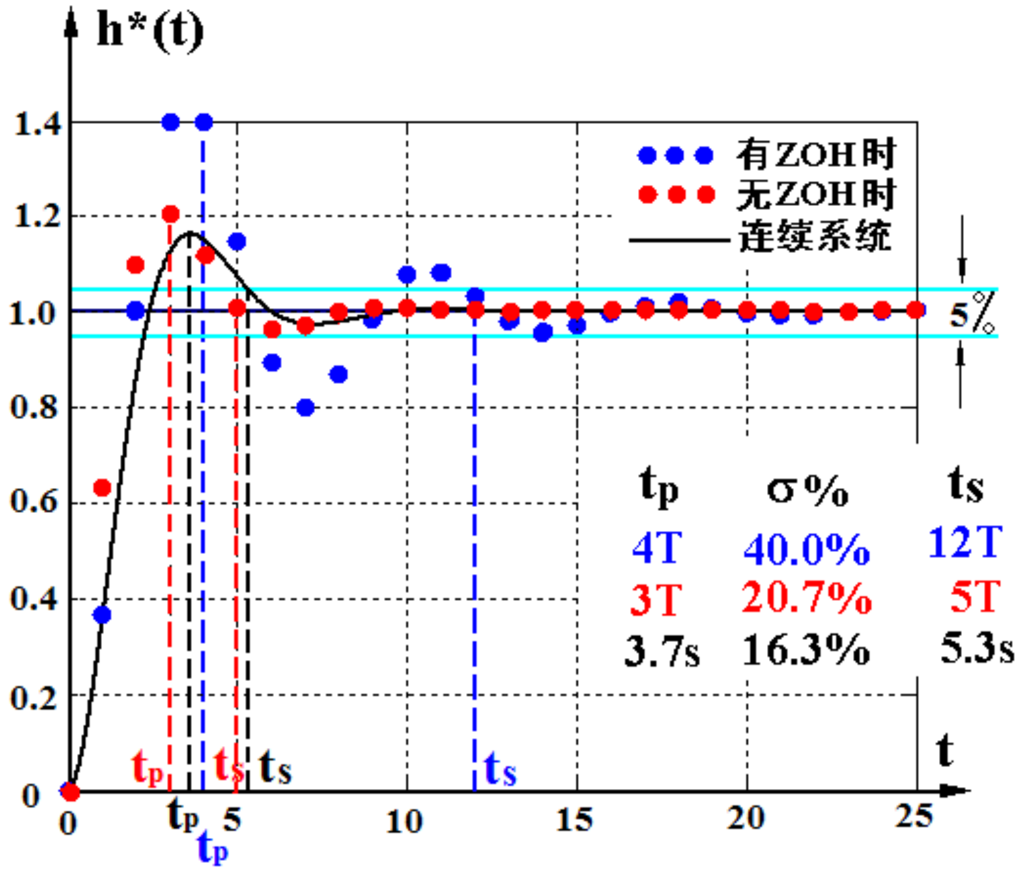
$$c(\infty T)=\lim_{z\to 1}(z-1)\cdot\Phi(z)\cdot\frac{z}{z-1}=1$$

$$C(z)=\Phi(z)\cdot\frac{z}{z-1}=\frac{(0.368z+0.264)z}{z^3-2z^2+1.632z-0.632}$$

h(0)=0
h(1)=0.3679
h(2)=1.0000
h(3)=1.3996
h(4)=1.3996
h(5)=1.1470
h(6)=0.8944
h(7)=0.8015
h(8)=0.8682
h(9)=0.9937
h(10)=1.0770
h(11)=1.0810
h(12)=1.0323
h(13)=0.9811
h(14)=0.9607
⋮

$\begin{cases} t_P=4T \\ \sigma\%=40\% \end{cases}$

$t_s=12T$


h*(t)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
有ZOH时
无ZOH时
连续系统
5%
tp
σ%
ts
4T
40.0%
12T
3T
20.7%
5T
3.7s
16.3%
5.3s
tp
ts
ts
ts
t
0
tp
5
tp
10
15
20
25

2. Relationship between dynamic response and closed-loop poles

$$\Phi(z)=\frac{M(z)}{D(z)}=\frac{b_m}{a_n}\frac{\prod_{i=1}^{m}(z-z_i)}{\prod_{k=1}^{n}(z-p_k)} \qquad m\le n$$

$$C(z)=\Phi(z)R(z)=\frac{M(z)}{D(z)}\cdot\frac{z}{z-1}$$

$$=\underline{\frac{M(1)}{D(1)}\cdot\frac{z}{z-1}}+\underline{\sum_{k=1}^{n}\frac{c_k z}{z-p_k}}$$

C(z)的稳态值 动态

(1) Single closed-loop poles on the real axis

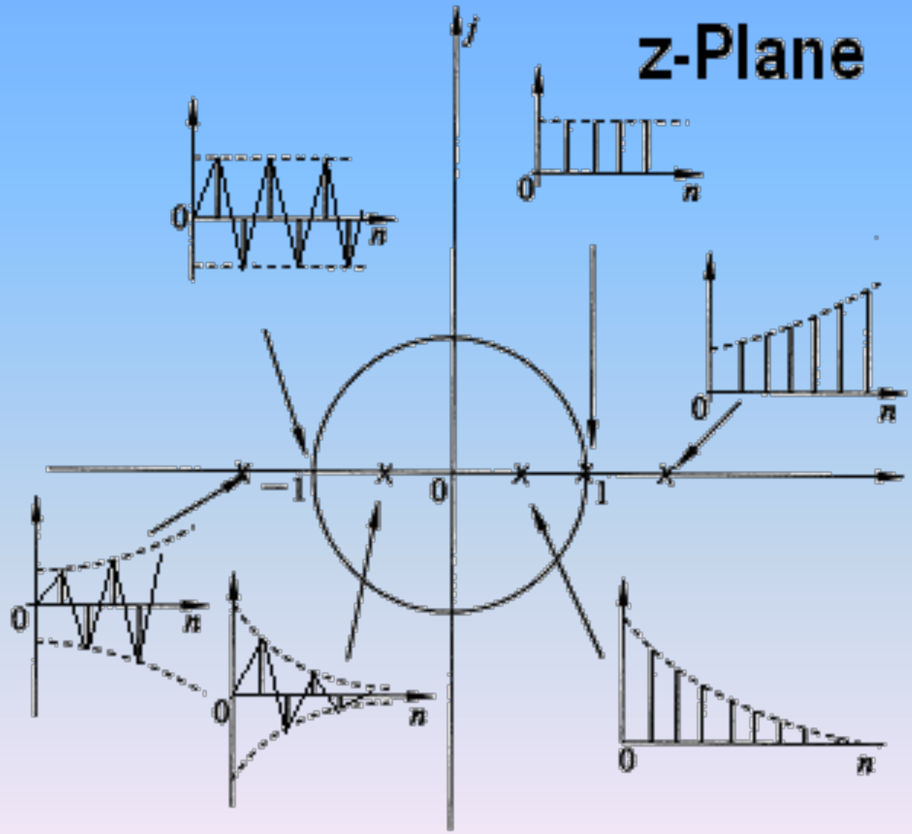
$$c_k^*(t)=Z^{-1}\left[\frac{c_k z}{z-p_k}\right] \quad k=1,2,\cdots n$$

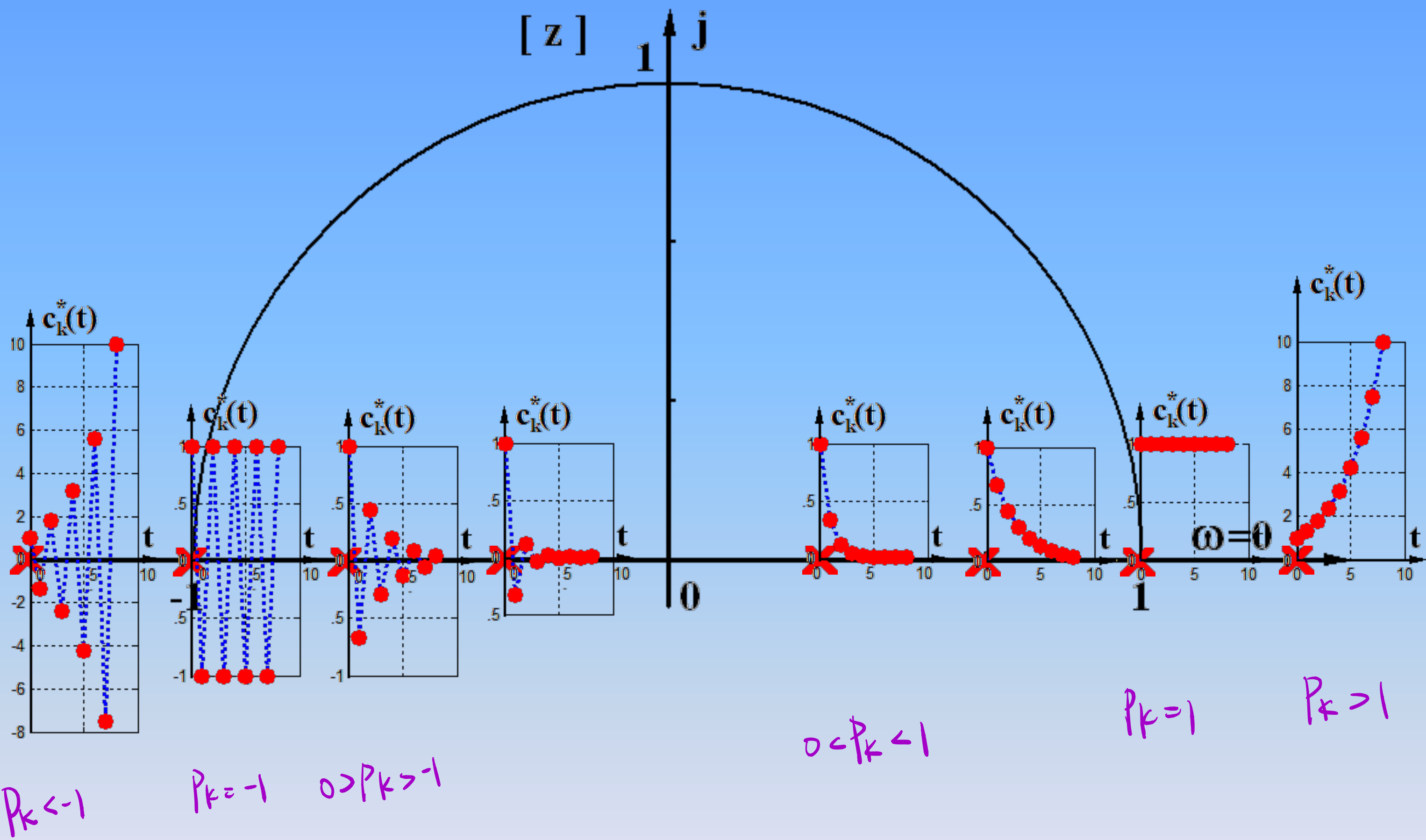
$$c_k(nT)=c_k p_k^n \qquad k=1,2,\cdots n$$

$$c_k(nT) = c_k p_k^n \qquad k = 1, 2, \cdots\ n$$

$p_k > 0$: $\quad p_k > 1 \qquad p_k = 1 \qquad p_k < 1$

$p_k < 0$:




[z]
j
1
0
-1
1
$c_k^*(t)$
t
ω=0
$P_K < -1$
$P_K = -1$
$0 > P_K > -1$
$0 < P_K < 1$
$P_K = 1$
$P_K > 1$

(2) Closed-loop Complex conjugate poles

$$p_k = |p_k| e^{j\theta_k} \qquad \bar{p}_k = |p_k| e^{-j\theta_k}$$

$$c^*{}_{k,k}(k) = Z^{-1}\left[\frac{c_k z}{z - p_k} + \frac{\bar{c}_k z}{z - \bar{p}_k}\right]$$

$$\begin{cases} a = \dfrac{1}{T}\ln|p_k| \\ \omega = \dfrac{\theta_k}{T} \\ 0 < \theta_k < \pi \end{cases}$$

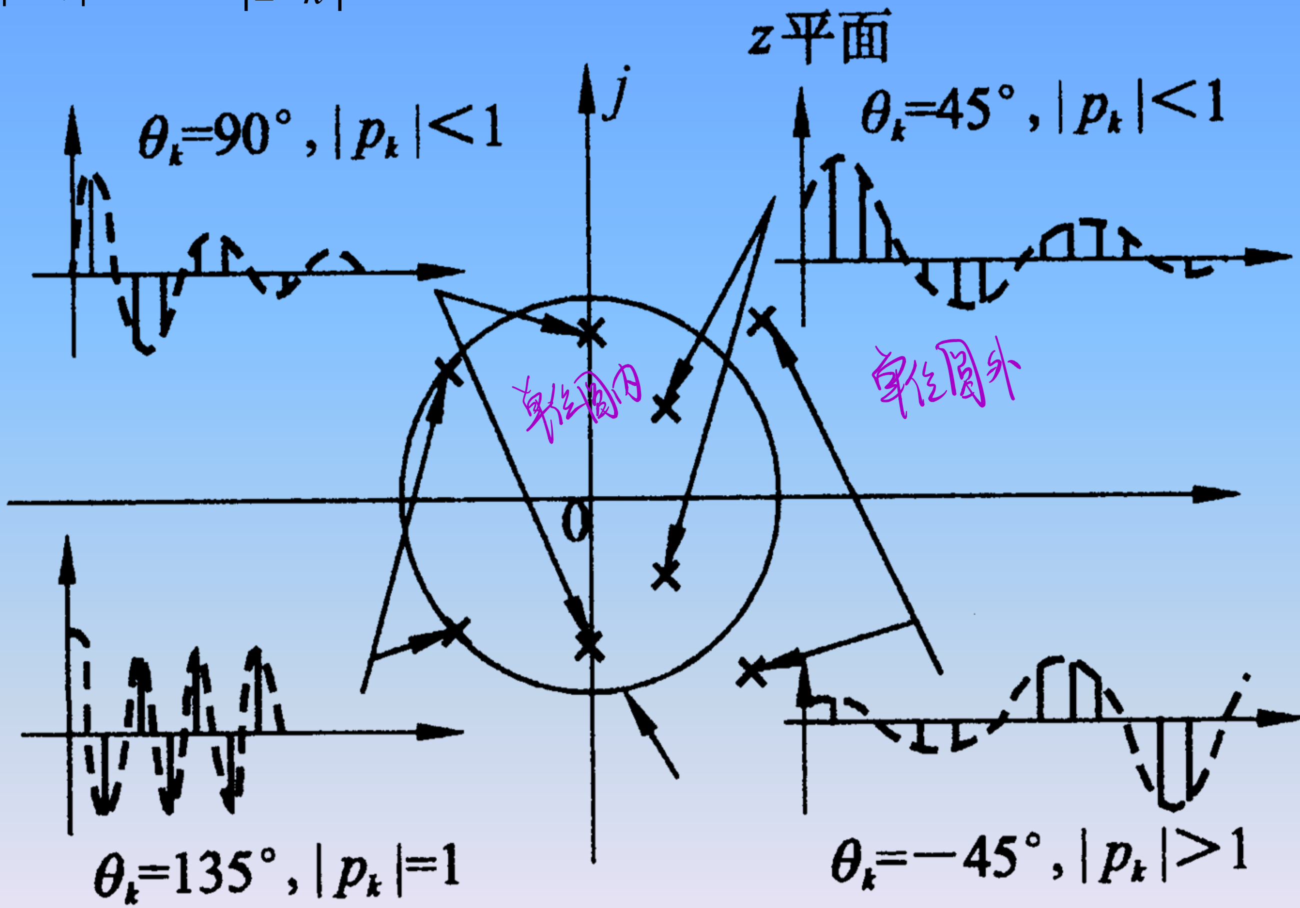
$$c_{k,k}(nT) = c_k p_k{}^n + \bar{c}_k \bar{p}_k{}^n$$

$$= c_k e^{a_k nT} + \bar{c}_k e^{\bar{a}_k nT}$$

$$= |c_k| e^{j\varphi_k} e^{(a+j\omega)nT} + |c_k| e^{-j\varphi_k} e^{(a-j\omega)nT}$$

$$= 2|c_k| e^{anT} \cos(n\omega T + \varphi_k)$$

$$\left|p_k\right|<1, \quad \left|p_k\right|>1$$



7.6 Performance Analysis of Discrete-Time Systems

- **Stability**
- **Dynamic Performance**
- **Steady-state Errors**

7.6.2 Analysis of discrete-time dynamic performance

(1)General method

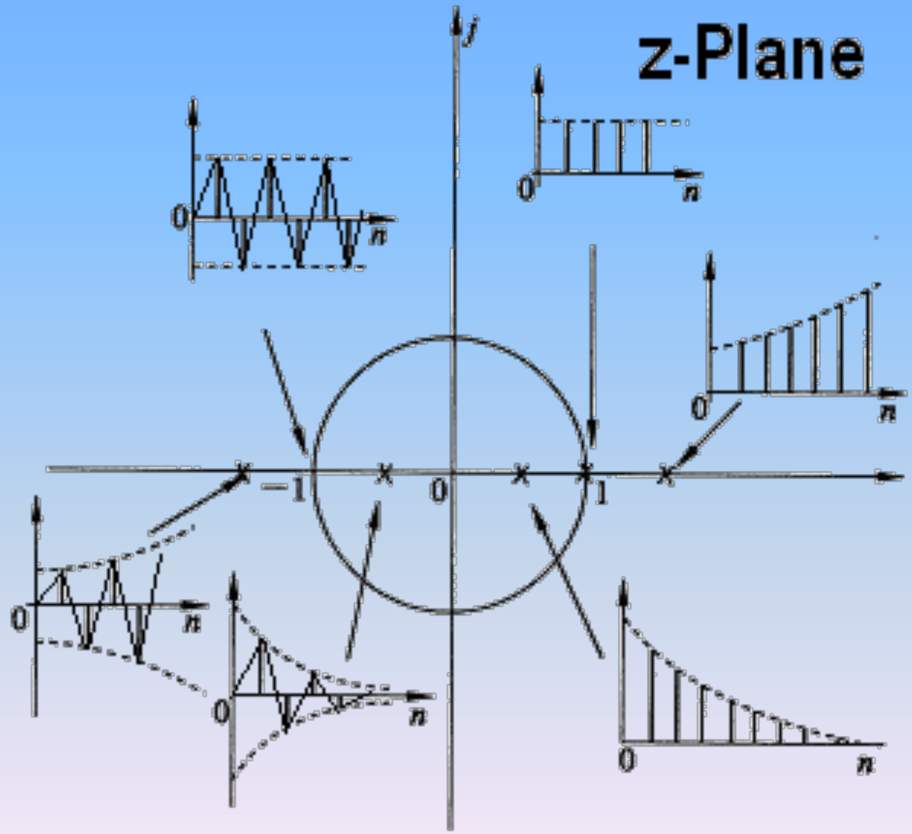
$$\begin{cases} G(z) \to \Phi(z) \longrightarrow C(z) = \sum_{n=0}^{\infty} c(nT) z^{-n} \\ c^*(t) = \sum_{n=0}^{\infty} c(nT)\delta(t - nT) \longrightarrow \text{Obtain s\%, } t_s \text{ by definition} \end{cases}$$

(2) Closed-loop poles $p_k \longrightarrow$ Response $c_k(nT) = C_k p_k^n$

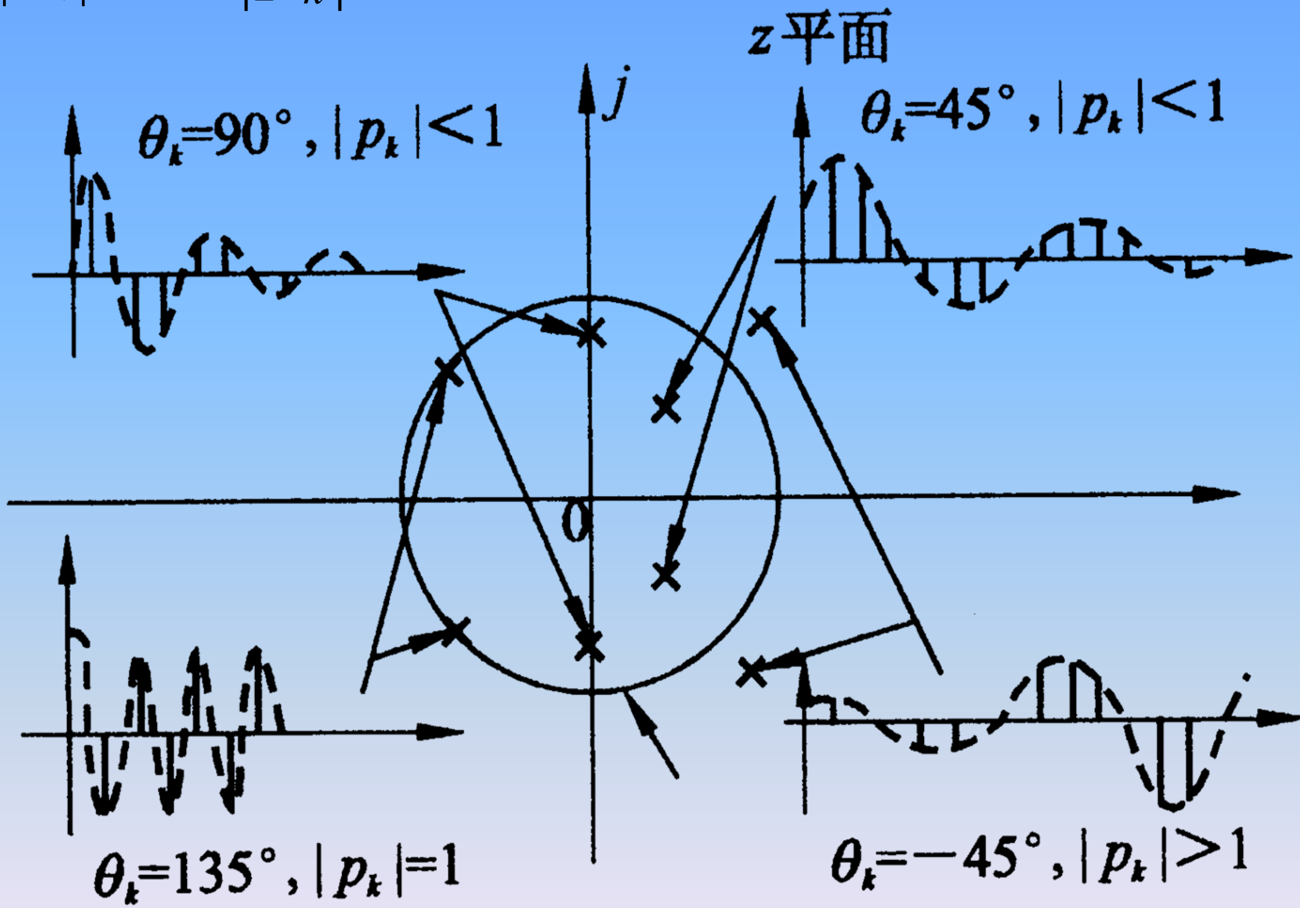
$$c_k(nT) = c_k p_k^n \qquad k = 1, 2, \cdots\ n$$

$p_k > 0$: $\quad p_k > 1 \qquad p_k = 1 \qquad p_k < 1$

$p_k < 0$:



$|p_k| < 1,\quad |p_k| > 1$

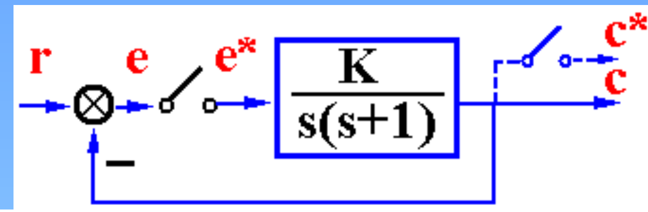


z平面
j
0
θ_k=90°, $|p_k|$<1
θ_k=45°, $|p_k|$<1
θ_k=135°, $|p_k|$=1
θ_k=−45°, $|p_k|$>1

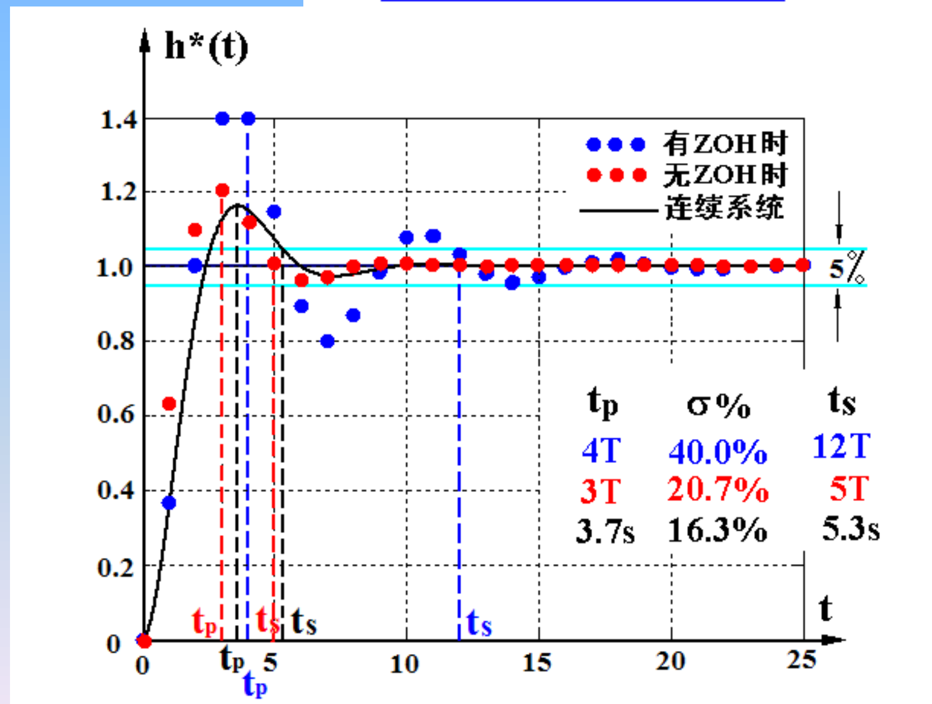
7.6.3 Steady-state error

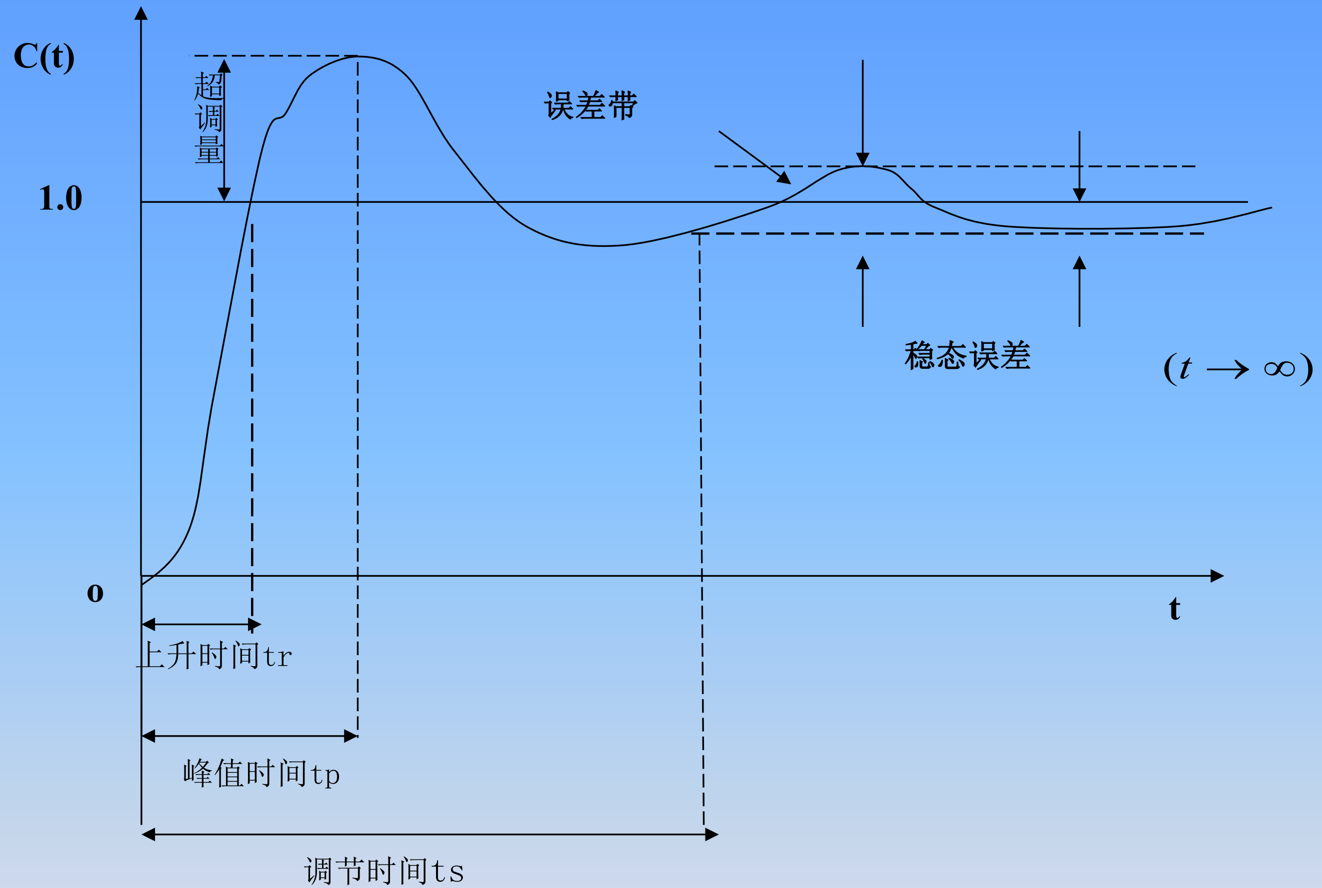
1. General method to obtain steady-state error

Example Consider the system shown in the figure, T=K=1. Obtain the dynamic specifications. (σ %, t_s).



$h(0)= 0$
$h(1)= 0.632$
$h(2)= 1.097$
$h(3)= 1.207$
$h(4)= 1.117$
$h(5)= 1.014$
$h(6)= 0.964$
$h(7)= 0.970$
$h(8)= 0.991$
$h(9)= 1.004$
$h(10)=1.007$
$h(11)=1.003$
$h(12)=1.000$
⋮



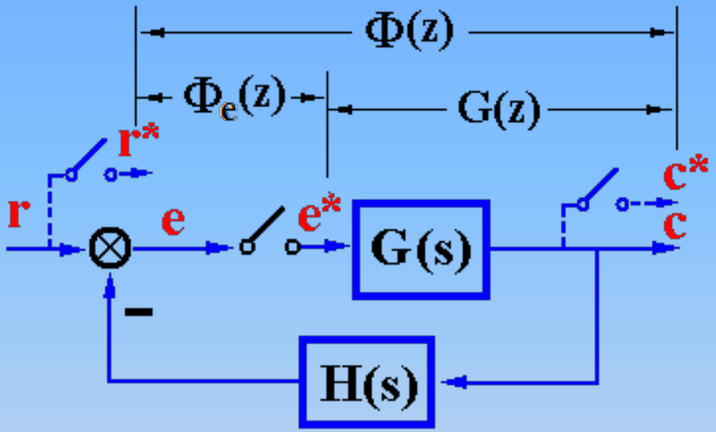

C(t)
超调量
1.0
误差带
稳态误差
$(t \rightarrow \infty)$
0
t
上升时间tr
峰值时间tp
调节时间ts

控制系统性能指标

2. Using final value theorem to obtain steady-state error

Let $\begin{cases} GH(z) = Z[G(s)H(s)] = \dfrac{1}{(z-1)^v} GH_0(z) \\ \lim\limits_{z\to 1} GH_0(z) = K \end{cases}$

v: System type



Algorithm:

(1) Determine the stability

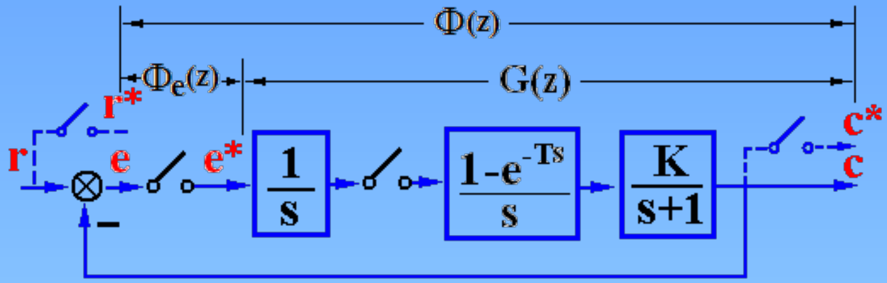
(2) Obtain the impulse transfer function from E(z) to R(z).

$$\Phi_e(z) = \frac{E(z)}{R(z)} = \frac{1}{1+GH(z)}$$

(3) Obtain $e(\infty)$ by the final value theorem

$$e(\infty) = \lim_{z\to 1}(z-1)\,\Phi_e(z)\,R(z) = \lim_{z\to 1}(z-1)\cdot R(z)\cdot\frac{1}{1+GH(z)}$$

Example 1 Consider the discrete system shown in the figure, K=2, T=1; Obtain e(∞) for r(t)=1(t), t, t²/2 .



$$G(z) = Z\left[\frac{1}{s}\right]\cdot Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s+1}\right]$$

$$= \frac{K(1-e^{-T})z}{(z-1)(z-e^{-T})} \qquad v = 1$$

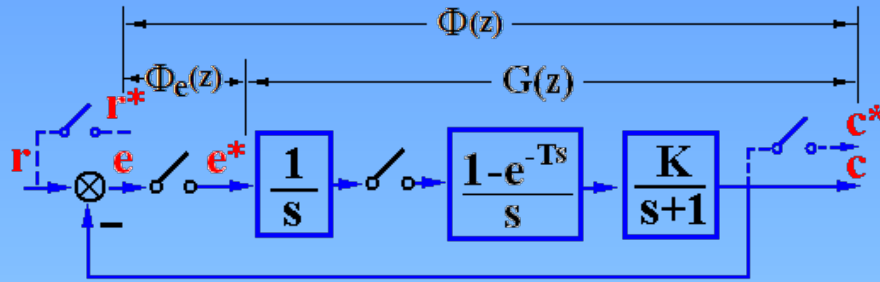
$$\Phi_e(z) = \frac{1}{1+\dfrac{K(1-e^{-T})z}{(z-1)(z-e^{-T})}} = \frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T})+K(1-e^{-T})z}$$

$$D(z) = z^2 + [K(1-e^{-T})-(1+e^{-T})]z + e^{-T} = 0$$

$$0 < K < \frac{2(1+e^{-T})}{(1-e^{-T})} \overset{T=1}{=} 4.33$$

$$w = \frac{z+1}{z-1}$$

Example 1 Consider the discrete system shown in the figure, K=2, T=1; Obtain e(∞) for r(t)=1(t), t, $t^2/2$.



$$0 < K < 4.33$$

$$e(\infty) = \lim_{z\to 1}(z-1)R(z)\Phi_e(z)$$

$$\Phi_e(z) = \frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z}$$

$r_1(t) = 1(t)$ $$e_1(\infty) = \lim_{z\to 1}\cancel{(z-1)}\frac{z}{\cancel{z-1}}\cdot\frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = 0$$

$v=1$

$r_2(t) = t$ $$e_2(\infty) = \lim_{z\to 1}\cancel{(z-1)}\frac{Tz}{\cancel{(z-1)^2}}\cdot\frac{\cancel{(z-1)}(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = \frac{T}{K}$$

$r_3(t) = \frac{t^2}{2}$ $$e_3(\infty) = \lim_{z\to 1}\cancel{(z-1)}\frac{Tz(z+1)}{2(z-1)^3}\cdot\frac{\cancel{(z-1)}(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = \infty$$

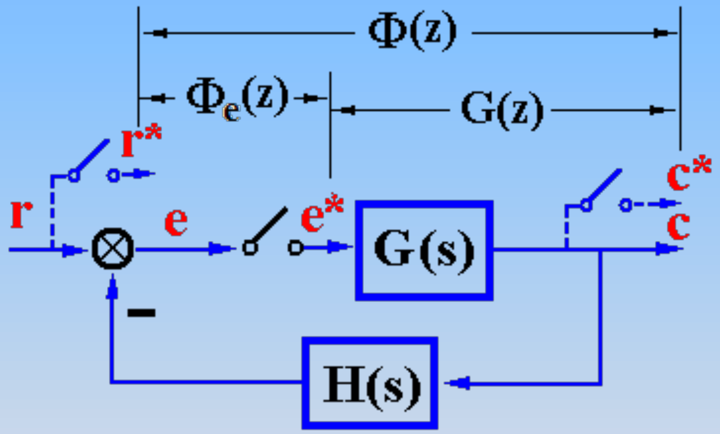
3. Static Error Constant Method

shows how $e(\infty)$ changes with r(t)

(For stable linear discrete systems subject to r(t) and sampled at the error signal)

Let $\begin{cases} GH(z) = Z\left[G(s)H(s)\right] = \dfrac{1}{(z-1)^{v}} GH_0(z) \\ \lim\limits_{z\to 1} GH_0(z) = K \end{cases}$ v: System type

$$\Phi_e(z) = \frac{E(z)}{R(z)} = \frac{1}{1+GH(z)}$$



$$e(\infty) = \lim_{z\to 1}(z-1)\,\Phi_e(z)\,R(z)$$

$$= \lim_{z\to 1}(z-1)\cdot R(z)\cdot \frac{1}{1+GH(z)}$$

$$e(\infty T) = \lim_{z\to 1}(z-1)\,\Phi_e(z)\,R(z) = \lim_{z\to 1}(z-1)\cdot R(z)\cdot\frac{1}{1+GH(z)}$$

$r(t) = A\cdot 1(t)$ $$e(\infty T) = \lim_{z\to 1}(z-1)\cdot\frac{Az}{z-1}\cdot\frac{1}{1+GH(z)} = \frac{A}{1+\lim\limits_{z\to 1}GH(z)} \boxed{=\frac{A}{K_p}}$$

Static position error constant $$K_p = 1+\lim_{z\to 1}GH(z)$$

$r(t) = A\cdot t$ $$e(\infty T) = \lim_{z\to 1}(z-1)\cdot\frac{ATz}{(z-1)^2}\cdot\frac{1}{1+GH(z)} = \frac{AT}{\lim\limits_{z\to 1}(z-1)GH(z)} \boxed{=\frac{AT}{K_v}}$$

Static velocity error constant $$K_v = \lim_{z\to 1}(z-1)GH(z)$$

$r(t) = \frac{A}{2}t^2$ $$e(\infty T) = \lim_{z\to 1}(z-1)\cdot\frac{AT^2z(z+1)}{2(z-1)^3}\cdot\frac{1}{1+GH(z)} = \frac{AT^2}{\lim\limits_{z\to 1}(z-1)^2GH(z)} \boxed{=\frac{AT^2}{K_a}}$$

Static acceleration error constant $$K_a = \lim_{z\to 1}(z-1)^2GH(z)$$

Similar to the continuous system, we can divide the discrete-time system as type 0, type I, type II,… according to the numbers of the pole z=1 of the impulse transfer function.

$$r(t) = A \cdot 1(t)$$

Static position error constant $$K_p = 1 + \lim_{z \to 1} GH(z)$$

Type 0: K_p=constant

Type >=1: Kp= ∞, e(∞)=0

$$r(t) = A \cdot t$$

Static velocity error constant $$K_v = \lim_{z \to 1} (z-1) GH(z)$$

Type 0: K_v=0, e(∞)= ∞

Type =1: K_v= constant,

Type >=2: K_v= ∞, e(∞)=0

$r(t) = \frac{A}{2} t^2$ **Static acceleration error constant** $K_a = \lim_{z \to 1} (z-1)^2 GH(z)$

Type 0,1: K_a=0, e(∞)= ∞

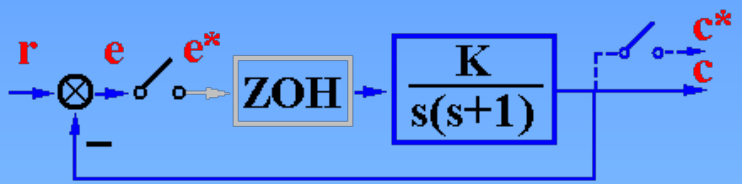
Type =2: K_a= constant,

Type >=3: K_a= ∞, e(∞)=0

$$\begin{cases} GH(z) = \dfrac{1}{(z-1)^v} GH_0(z) \\ \lim\limits_{z\to 1} GH_0(z) = K \end{cases}$$

型别	Static Error Constant			Steady-State Error		
V	$K_p = \lim GH(z)$	$K_v = \lim (z-1)GH(z)$	$K_a = \lim (z-1)^2 GH(z)$	$r = A\cdot 1(t)$ $e(\infty) = \dfrac{A}{K_p}$	$r = A\cdot t$ $e(\infty) = \dfrac{AT}{K_v}$	$r = A\cdot t^2/2$ $e(\infty) = \dfrac{AT^2}{K_a}$
0	K_p	0	0	$\dfrac{A}{K_p}$	∞	∞
I	∞	K_v	0	0	$\dfrac{AT}{K_v}$	∞
II	∞	∞	K_a	0	0	$\dfrac{AT^2}{K_a}$

Example 2 Consider the stable discrete system shown in the figure. When $r(t)=2t$, obtain $e(\infty)$ with/without ZOH.



Solution.

no ZOH

$$\begin{cases} G(z) = Z\left[\dfrac{K}{s(s+1)}\right] = \dfrac{K(1-e^{-T})z}{(z-1)(z-e^{-T})} \\ K_v = \lim\limits_{z\to1}(z-1)G(z) = \lim\limits_{z\to1}\dfrac{K(1-e^{-T})z}{(z-e^{-T})} = K \end{cases}$$

$$e(\infty) = \frac{AT}{K_v} = \frac{2T}{K}$$

— dependent of T

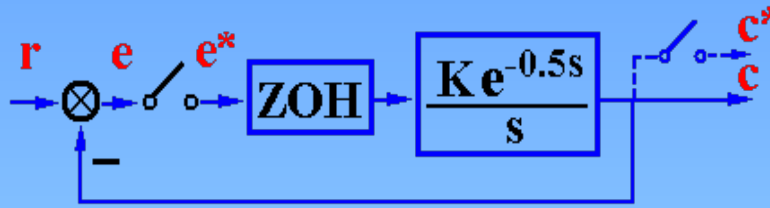
with ZOH

$$\begin{cases} G(z) = Z\left[\dfrac{1-e^{-Ts}}{s}\cdot\dfrac{K}{s(s+1)}\right] = K\,\dfrac{z-1}{z}\cdot Z\left[\dfrac{1}{s^2(s+1)}\right] \\ \quad = K\,\dfrac{(T-1+e^{-T})z+(1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})} \\ K_v = \lim\limits_{z\to1}(z-1)G(z) = \lim\limits_{z\to1}\dfrac{K(T-Te^{-T})}{z-e^{-T}} = KT \end{cases}$$

$$e(\infty) = \frac{AT}{K_v} = \frac{A}{K} = \frac{2}{K}$$

— independent of T

Example 3 Consider the system shown in the figure, T=0.25. When r(t)=2·1(t)+t, obtain the range of K for e(∞)<0.5.



Solution. The stable range of K is

$$0 < K < 2.472$$

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s} \cdot \frac{Ke^{-2Ts}}{s}\right]$$

$$= K(1-z^{-1})z^{-2}Z\left[\frac{1}{s^2}\right] = Kz^{-2}\frac{z-1}{z} \cdot \frac{Tz}{(z-1)^2} = \frac{KT}{z^2(z-1)} \qquad v = 1$$

$$K_v = \lim_{z\to 1}(z-1)G(z) = \lim_{z\to 1}(z-1)\frac{KT}{z^2(z-1)} = KT$$

$$r_1(t) = 2\cdot 1(t) \qquad e_1(\infty) = 0$$

$$r_2(t) = t \qquad e_2(\infty) = TA/K_v = 1/K$$

$$e(\infty) = e_1(\infty) + e_2(\infty) = 1/K < 0.5 \quad \Rightarrow \quad K > 2$$

$$\boxed{2 < K < 2.472}$$

7.6.3 Steady-state error of discrete systems

(1) General method: obtain system response

(2) Final value theorem
$$\begin{cases} G(z) \to \Phi_e(z) \\ D(z) \to \text{Stability} \\ e(\infty) = \lim\limits_{z\to 1}(z-1)R(z)\Phi_e(z) \end{cases}$$

(3) Static error constant
$$\begin{cases} G(z) \to v,\ K_p,\ K_v,\ K_a \\ \text{Obtain } e(\infty) \end{cases}$$