Solving two-person zero-sum sequential games via efficient computaion of Nash equilibria: proof-of-concept on Kuhn's 3-card Poker

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Problem

Given a sequential two-person zero-sum game (e.g Texas Hold'em Poker), construct an optimal player.

- There are 3 players: nature (player 0), Alice (player 1), and Bob (player 2).
- Players take turns in making moves
- Set of moves of any two distinct players don't overlap (i.e $A_p \cap A_q = \emptyset$ if $p \neq q$)

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$$\underset{x \in Q_1}{maximize} \; \underset{y \in \mathbb{R}_+^m, Fy=f}{minimize} \; y^T Ax, \qquad (1)$$

where:

$$\left. \begin{array}{l} Q_1 := \{x \in \mathbb{R}_+^n \mid Ex = e\} \text{ is Alice's complex, and} \\ Q_2 := \{y \in \mathbb{R}_+^m \mid Fy = f\} \text{ is Bob's complex.} \end{array} \right\} \qquad (2)$$

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The game tree T (from Alice's perspective) is defined as follows. T has a set $V(T)$ of nodes (aka vertices) and a set $E(T)$ of edges.

Non-Linear Ledoit-Wolf: Why? How?

- (p/n large) or (Σ eigenvalues close to one another) $\Rightarrow$ linear shrinkage OK.
  (p/n small) or (Σ eigenvalues dispersed) $\Rightarrow$ non-linear shrinkage better.
- Marčenko-Pastur equation: relationship between $\boldsymbol{S}$ and Σ eigenvalues under large-dimensional asymptotics.

- H_n empirical d.f. of sample ($\boldsymbol{S}_n$) eigenvalues $\boldsymbol{\tau}_n$.
  $H_n(\boldsymbol{\tau}) \equiv \frac{1}{p} \sum_{i=1}^{p} \mathbb{1}_{[\tau_i, +\infty[}(\tau)$
- F_n e.d.f. of population ($\boldsymbol{\Sigma}_n$) eigenvalues $\boldsymbol{\lambda}_n$.
  $\boldsymbol{Y}_n = \boldsymbol{X}_n \Sigma_n^{1/2}$, $\boldsymbol{X}_n \sim \mathcal{N}(\boldsymbol{0}, \mathsf{Id})$
- F_n (resp. H_n) converges almost surely to F (resp. H).
- Stieltjes transform of a non-decreasing function G:
  $\forall z \in \mathbb{C}^+, m_G(z) \equiv \int_{-\infty}^{+\infty} \frac{1}{u-z} dG(u)$
- Marčenko-Pastur equation:
  $\forall z \in \mathbb{C}^+, m_F(z) = \int_{-\infty}^{+\infty} \frac{1}{\tau[1-\frac{p}{n}-\frac{p}{n}zm_F(z)]-z} dH(\tau)$

$$Q_{n,p} : [0, \infty[^p \to [0, \infty[^p$$
$$\boldsymbol{t} \equiv (t_1, \ldots, t_p)^\mathsf{T} \mapsto Q_{n,p}(\boldsymbol{t}) \equiv (q_{n,p}^1(t), \ldots, q_{n,p}^p(t))^\mathsf{T}$$

$$\hat{\boldsymbol{\tau}}_n = \underset{\boldsymbol{t} \in [0, \infty[^p}{\operatorname{argmin}} \frac{1}{p} \sum_{i=1}^p [q_{n,p}^i(\boldsymbol{t}) - \lambda_{n,i}]^2$$

$$\breve{m}_{n,p}^{\hat{\tau}_n}(\lambda_i) \simeq \frac{n-p}{p\lambda_i} - \frac{n}{p}\frac{1}{\hat{\tau}_{n,i}\lambda_i}$$

$$\hat{d}_i^* = \frac{\lambda_i}{|1 - \frac{p}{n} - \frac{p}{n}|^2 \breve{m}_{n,p}^{\hat{\tau}_n}(\lambda_i)} ; \text{ Covariance: } S_n^* \boldsymbol{U}_n \hat{\boldsymbol{D}}_n^* \boldsymbol{U}_n^\mathsf{T}, \text{ with } \boldsymbol{S}_n = \boldsymbol{U}_n \boldsymbol{D}_n \boldsymbol{U}_n^\mathsf{T}$$

About the $Q_{n,p}$ function

$$Q_{n,p} : [0, \infty[^p \to [0, \infty[^p$$
$$\boldsymbol{t} \equiv (t_1, \ldots, t_p)^\top \mapsto Q_{n,p}(\boldsymbol{t}) \equiv (q_{n,p}^1(t), \ldots, q_{n,p}^p(t))^\top$$

$$\forall i = 1, \ldots, p \quad q_{n,p}^i(t) \equiv p \int_{(i-1)/p}^{i/p} (F_{n,p}^{\boldsymbol{t}})^{-1}(u) du$$

$$(F_{n,p}^{\boldsymbol{t}})^{-1}(u) \equiv \sup\{x \in \mathbb{R} : F_{n,p}^{\boldsymbol{t}}(x) \leq u\}$$

$$F_{n,p}^{\boldsymbol{t}}(x) \equiv \max\left\{1 - \frac{n}{p}, \frac{1}{p} \sum_{i=1}^p \mathbb{1}_{\{t_i = 0\}}\right\} \text{ if } x = 0,$$
$$F_{n,p}^{\boldsymbol{t}}(x) \equiv \lim_{\eta \to 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} \text{Im}\left[m_{n,p}^{\boldsymbol{t}}(\zeta + i\eta)\right] d\zeta \text{ otherwise}$$

$$m \equiv m_{n,p}^{\boldsymbol{t}}(z) \text{ sol. in } \left\{m \in \mathbb{C} : -\frac{n-p}{nz} + \frac{p}{n}\right\} \text{ of}$$
$$m = \frac{1}{p} \sum_{i=1}^p \frac{1}{t_i(1 - \frac{p}{n} - \frac{p}{n}zm) - z}$$