# Solving two-person zero-sum sequential games via efficient computation of Nash equilibria: proof-of-concept on Kuhn's 3-card Poker

DOHMATOB Elvis [1,2]

[1]Parietal Team, INRIA
[2]Université Paris-Sud

# Problem

Given a sequential two-person zero-sum game (e.g Texas Hold'em Poker), construct an optimal player.

# Two-person zero sum Sequential games

- There are 3 players: nature (player 0), Alice (player 1), and Bob (player 2).
- Players take turns in making moves
- Sets of moves of any two distinct players don't overlap (i.e $A_p \cap A_q = \emptyset$ if $p \neq q$)

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# Strategy profiles, simplexes and complexes

**Definition**

- The *n-simplex*, denoted $\Delta_n$, is the subset of $\mathbb{R}^{n+1}$ defined by

$$\Delta_n := \left\{ x \in \mathbb{R}^{n+1} \mid x_i \geq 0 \ \forall i, \text{ and } \sum_{i=1}^{n+1} x_i = 1 \right\} \quad (1)$$

- Each kink $\delta_i := (0, ..., 0, 1, 0, ..., 0)$ of $\Delta_n$ represents a *pure-strategy*. Any non-kink point $x$ of $\Delta_n$ represents a *mixed-strategy*. Thus $\Delta_n$ is simply the *convex hall* of $(n+1)$-dimensional pure strategies $\delta_i$.
- A *complex* is an abstract *polyhedron* whose kinks are themselves simplexes. Points on a complex correspond to *behavioural strategies*.

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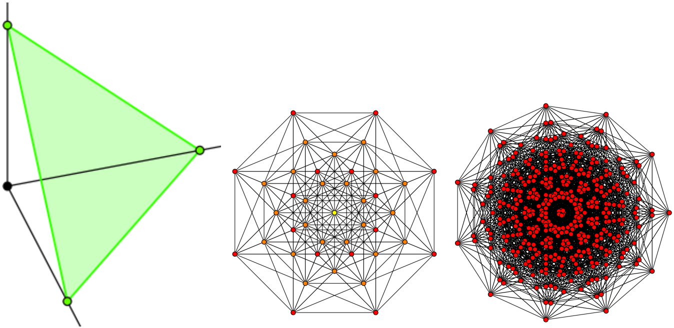
# Simplexes and complexes (examples of)



**Left**: 2-simplex **Middle**: 7-simplex **Right**: 10-simplex

# NE minimax and best-response in sequential-games

Thanks to the sequence-form representation, we have:

- NE problem for Alice and Bob is the the *saddle-point problem*

$$\underset{x \in Q_1}{\text{maximize}}\ \underset{y \in Q_2}{\text{minimize}}\ y^T A x, \tag{2}$$

where:

$$\left.\begin{aligned} Q_1 &:= \{x \in \mathbb{R}^n_+ \mid Ex = e\} \text{ is Alice's complex, and} \\ Q_2 &:= \{y \in \mathbb{R}^m_+ \mid Fy = f\} \text{ is Bob's complex.} \end{aligned}\right\} \tag{3}$$

A saddle-point $(x^*, y^*)$ corresponds to a NE for the game.

- Given a fixed behavioural strategy $y_0 \in Q_2$ for Bob, a *best-response* behavioural strategy $x_0$ for Alice [1] is a solution to the LCP

$$\underset{x \in Q_1}{\text{maximize}}\ y_0^T A x \tag{4}$$

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# Two-person zero sum Sequential games

The game tree T (from Alice's perspective) is defined as follows. T has a set V(T) of nodes (aka vertices) and a set E(T) of edges.

# Non-Linear Ledoit-Wolf: Why? How?

- ($p/n$ large) or ($\Sigma$ eigenvalues close to one another) $\Rightarrow$ linear shrinkage OK.
  ($p/n$ small) or ($\Sigma$ eigenvalues dispersed) $\Rightarrow$ non-linear shrinkage better.
- Marčenko-Pastur equation: relationship between $\boldsymbol{S}$ and $\boldsymbol{\Sigma}$ eigenvalues under large-dimensional asymptotics.

# Non-Linear Ledoit-Wolf: deeper into the "how?"

- $H_n$ empirical d.f. of sample ($\boldsymbol{S}_n$) eigenvalues $\boldsymbol{\tau}_n$.
  $H_n(\boldsymbol{\tau}) \equiv \frac{1}{p}\sum_{i=1}^{p} \mathbb{1}_{[\tau_i, +\infty[}(\tau)$
- $F_n$ e.d.f. of population ($\boldsymbol{\Sigma}_n$) eigenvalues $\boldsymbol{\lambda}_n$.
  $\boldsymbol{Y}_n = \boldsymbol{X}_n \Sigma_n^{1/2}$, $\boldsymbol{X}_n \sim \mathcal{N}(\mathbf{0}, \mathrm{Id})$
- $F_n$ (resp. $H_n$) converges almost surely to $F$ (resp. $H$).
- Stieltjes transform of a non-decreasing function $G$:
  $\forall z \in \mathbb{C}^+, m_G(z) \equiv \int_{-\infty}^{+\infty} \frac{1}{u-z} dG(u)$
- Marčenko-Pastur equation:
  $\forall z \in \mathbb{C}^+, m_F(z) = \int_{-\infty}^{+\infty} \frac{1}{\tau[1-\frac{p}{n}-\frac{p}{n} z m_F(z)]-z} dH(\tau)$

# Non-Linear Ledoit-Wolf: Implementation...

$$Q_{n,p} : [0, \infty[^p \to [0, \infty[^p$$
$$\boldsymbol{t} \equiv (t_1, \ldots, t_p)^\top \mapsto Q_{n,p}(\boldsymbol{t}) \equiv (q_{n,p}^1(t), \ldots, q_{n,p}^p(t))^\top$$

---

$$\hat{\boldsymbol{\tau}}_n = \underset{\boldsymbol{t} \in [0,\infty[^p}{\operatorname{argmin}} \frac{1}{p} \sum_{i=1}^p [q_{n,p}^i(\boldsymbol{t}) - \lambda_{n,i}]^2$$

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$$\breve{m}_{n,p}^{\hat{\tau}_n}(\lambda_i) \simeq \frac{n-p}{p\lambda_i} - \frac{n}{p}\frac{1}{\hat{\tau}_{n,i}\lambda_i}$$

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$$\hat{d}_i^* = \frac{\lambda_i}{|1 - \frac{p}{n} - \frac{p}{n}|^2 \breve{m}_{n,p}^{\hat{\tau}_n}(\lambda_i)}$$; Covariance: $S_n^* \boldsymbol{U}_n \hat{\boldsymbol{D}}_n^* \boldsymbol{U}_n^\top$, with $\boldsymbol{S}_n = \boldsymbol{U}_n \boldsymbol{D}_n \boldsymbol{U}_n^\top$

# About the $Q_{n,p}$ function

$$Q_{n,p} : [0,\infty[^p \to [0,\infty[^p$$
$$\boldsymbol{t} \equiv (t_1,\ldots,t_p)^\mathsf{T} \mapsto Q_{n,p}(\boldsymbol{t}) \equiv (q^1_{n,p}(t),\ldots,q^p_{n,p}(t))^\mathsf{T}$$

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$$\forall i = 1,\ldots,p \quad q^i_{n,p}(t) \equiv p\int_{(i-1)/p}^{i/p} (F^{\boldsymbol{t}}_{n,p})^{-1}(u)du$$

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$$(F^{\boldsymbol{t}}_{n,p})^{-1}(u) \equiv \sup\{x \in \mathbb{R} : F^{\boldsymbol{t}}_{n,p}(x) \leq u\}$$

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$$F^{\boldsymbol{t}}_{n,p}(x) \equiv \max\left\{1 - \frac{n}{p}, \frac{1}{p}\sum_{i=1}^p \mathbb{1}_{\{t_i=0\}}\right\} \text{ if } x = 0,$$
$$F^{\boldsymbol{t}}_{n,p}(x) \equiv \lim_{\eta\to 0^+} \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{Im}\left[m^{\boldsymbol{t}}_{n,p}(\zeta + i\,\eta)\right] d\zeta \text{ otherwise}$$

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$m \equiv m^{\boldsymbol{t}}_{n,p}(z)$ sol. in $\left\{m \in \mathbb{C} : -\frac{n-p}{nz} + \frac{p}{n}\right\}$ of

$$m = \frac{1}{p}\sum_{i=1}^p \frac{1}{t_i(1-\frac{p}{n}-\frac{p}{n}zm)-z}$$