The Modularity Theorem is $\mathsf{BISH} + \mathsf{WLPO}$: The BCDT Extension Adds No Logical Cost

(Paper 69, Constructive Reverse Mathematics Series)

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Abstract

Paper 68 showed that Wiles's proof of the modularity of semistable elliptic curves over $\mathbb{Q}$ calibrates at $\mathsf{BISH}+\mathsf{WLPO}$, with the WLPO localized entirely in the Langlands–Tunnell theorem (Stage 1). We extend this classification to the full modularity theorem for all elliptic curves over $\mathbb{Q}$ (Breuil–Conrad–Diamond–Taylor 2001).

The extension introduces three new ingredients: Breuil's classification of finite flat group schemes (replacing Fontaine–Laffaille for the supersingular case), Conrad's local-global compatibility, and the Diamond–Taylor 3–5 switching argument. All three are BISH. The icosahedral case ($A_5 \subset \mathrm{PGL}_2(\mathbb{F}_5)$) never arises because every invocation of residual modularity in the BCDT proof is delegated to $p = 3$, where $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$ is solvable and Langlands–Tunnell applies unconditionally.

The classification of the full modularity theorem is $\mathsf{BISH}+\mathsf{WLPO}$, identical to the semistable case. The Lean 4 verification (324 lines across two files, zero `sorry`) formalizes the logical assembly: each BCDT ingredient is classified by definition, and the join is machine-checked.

Contents

*Lean 4 formalization available at https://doi.org/10.5281/zenodo.18749375.

1 Introduction

1.1 Main results

The modularity theorem asserts that every elliptic curve $E/\mathbb{Q}$ is modular: there exists a weight 2 newform f of level N_E such that $\rho_{E,p} \cong \rho_{f,p}$ for all primes p. The theorem was proved in stages: Wiles [15] and Taylor–Wiles [14] for the semistable case, and Breuil–Conrad–Diamond–Taylor [1] for the general case.

Paper 68 [20] performed a constructive reverse mathematics audit of the semistable proof, finding that it calibrates at BISH+WLPO: the Taylor–Wiles engine (Stages 2–5) is fully constructive (BISH), and the WLPO arises solely from the Langlands–Tunnell theorem. The present paper extends this classification to the full modularity theorem. We establish three results:

Theorem A (BCDT extensions are BISH). Breuil's classification of finite flat group schemes [7], the Diamond–Taylor 3–5 switching argument [1], and Conrad's local-global compatibility [9] are all BISH. The icosahedral case ($A_5 \subset \mathrm{PGL}_2(\mathbb{F}_5)$) never arises because $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$ is solvable.

Theorem B (Full modularity is BISH + WLPO). The full modularity theorem for elliptic curves over $\mathbb{Q}$ (BCDT 2001) calibrates at BISH + WLPO, identical to the semistable case. The WLPO is consumed by a single invocation of Langlands–Tunnell at $p = 3$.

Theorem C (Zero marginal cost). The BCDT extension from semistable to all elliptic curves adds no logical cost: CRM(BCDT) = CRM(Wiles) = $\mathsf{BISH} + \mathsf{WLPO}$.

1.2 Constructive Reverse Mathematics: a brief primer

CRM calibrates mathematical statements against logical principles of increasing strength within Bishop-style constructive mathematics (BISH). The hierarchy relevant to this paper is:

$$\mathsf{BISH} \subset \mathsf{BISH} + \mathsf{WLPO} \subset \mathsf{BISH} + \mathsf{LPO} \subset \mathsf{CLASS}.$$

Here WLPO (Weak Limited Principle of Omniscience) states that every binary sequence is identically zero or not. MP (Markov's Principle) and LLPO (Lesser Limited Principle of Omniscience) are both implied by WLPO, but are mutually incomparable over BISH; since the present paper uses only BISH and WLPO, this distinction does not affect our results. For a thorough treatment of CRM, see Bridges–Richman [5]; for the broader program of which this paper is part, see Papers 1–68 of this series and the atlas survey [17].

1.3 Current state of the art

Paper 68 [20] classified Wiles's semistable proof as $\mathsf{BISH} + \mathsf{WLPO}$, with the Taylor–Wiles engine (Stages 2–5) fully constructive. That analysis left open whether the BCDT extension to non-semistable curves introduces additional logical cost. The present paper answers this question in the negative.

The principal technical observation is group-theoretic: $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$ is solvable. This ensures that every invocation of Langlands–Tunnell in the BCDT proof occurs at $p = 3$, where the projective image is automatically solvable. The icosahedral case—which would require non-solvable modularity and potentially higher logical cost—never arises.

1.4 Position in the atlas

The present paper is part of the Constructive Reverse Mathematics program (Papers 1–68). Paper 67 [19] synthesizes the arithmetic geometry phase (Papers 45–66); Paper 50 [17] provides the atlas framework. Paper 59 [18] classified the p-adic comparison (Fontaine–Laffaille) as BISH, directly supporting Stage 2 of both the semistable and full modularity proofs.

Paper 69 completes the CRM audit of the modularity theorem for $\mathrm{GL}_2/\mathbb{Q}$: Paper 68 handled the semistable case; the present paper handles the general case. Together, they establish that the bridge between Galois representations and automorphic forms for $\mathrm{GL}_2/\mathbb{Q}$ has logical cost $\mathsf{BISH} + \mathsf{WLPO}$, with all non-constructive content localized in the analytic theory of weight 1 forms.

2 Preliminaries

We recall the key definitions from Paper 68; see that paper for full details.

Definition 2.1 (Weak Limited Principle of Omniscience)**.** WLPO: For every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, either $\forall n,\ \alpha(n) = 0$ or $\neg(\forall n,\ \alpha(n) = 0)$. Equivalently, for every $x \in \mathbb{R}$: $x = 0 \vee \neg(x = 0)$.

Definition 2.2 (The five stages of Wiles's proof)**.** Following Paper 68 and the standard decomposition [10, 11]:

Stage 1 (Residual modularity): prove $\bar{\rho}$ is modular via Langlands–Tunnell.

Stage 2 (Deformation ring): construct the universal deformation ring R.
Stage 3 (Hecke algebra): construct $\mathbb{T}$ localized at $\mathfrak{m}$.
Stage 4 (Numerical criterion): verify the Wiles–Lenstra numerical criterion.
Stage 5 (Patching): select Taylor–Wiles primes and prove $R \cong T$.

Definition 2.3 (CRM classifications from Paper 68)**.** Paper 68 established:

- Stage 1 (Langlands–Tunnell): WLPO.
- Stages 2–5 (deformation ring, Hecke algebra, numerical criterion, patching): BISH.
- Overall: $\mathsf{BISH} + \mathsf{WLPO}$.

3 Why the Icosahedral Case Does Not Arise

The Langlands–Tunnell theorem proves that 2-dimensional complex representations of $G_{\mathbb{Q}}$ with *solvable* projective image are automorphic. Its scope is limited to the dihedral, tetrahedral, and octahedral cases. It does not cover the icosahedral case (projective image A_5, which is not solvable).

For the modularity theorem, the relevant representation is $\bar{\rho}_p : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_p)$ for a prime p. The projective image of $\bar{\rho}_p$ is a subgroup of $\mathrm{PGL}_2(\mathbb{F}_p)$.

Lemma 3.1 (Solvability at $p = 3$)**.** *For every elliptic curve $E/\mathbb{Q}$, the projective image of $\bar{\rho}_{E,3}$ is solvable.*

Proof. $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$, the symmetric group on four letters. S_4 is solvable (with composition series $1 \triangleleft V_4 \triangleleft A_4 \triangleleft S_4$, where each quotient is abelian). Every subgroup of a solvable group is solvable. The projective image of $\bar{\rho}_{E,3}$, being a subgroup of S_4, is therefore solvable. ☐

This is the key observation. At $p = 3$, the solvability of $\mathrm{PGL}_2(\mathbb{F}_3)$ is a *group-theoretic tautology* requiring no geometric input from the elliptic curve. The Langlands–Tunnell theorem is structurally applicable to $\bar{\rho}_{E,3}$ for *every* elliptic curve, provided $\bar{\rho}_{E,3}$ is absolutely irreducible.

At $p = 5$, the situation is different: $\mathrm{PGL}_2(\mathbb{F}_5) \cong S_5$, which contains the non-solvable subgroup A_5. The icosahedral case can and does arise at $p = 5$. But BCDT's proof is designed so that *no residual modularity theorem is ever applied at* $p = 5$. Instead, modularity of $\bar{\rho}_{E,5}$ is established by a geometric transfer back to $p = 3$ on an auxiliary curve.

4 The 3–5 Switching Argument

When $\bar{\rho}_{E,3}$ is reducible (equivalently, E admits a rational 3-isogeny), the Taylor–Wiles machinery cannot be applied at $p = 3$: the deformation ring lacks the required algebraic properties. BCDT switch to $p = 5$.

4.1 Constructing the auxiliary curve

By Mazur's isogeny theorem [13], if $E/\mathbb{Q}$ has a rational 3-isogeny, it generically does not have a rational 5-isogeny (exceptions correspond to rational points of $X_0(15)$, an explicitly computed finite set). Thus $\bar{\rho}_{E,5}$ is absolutely irreducible.

To establish that $\bar{\rho}_{E,5}$ is modular without invoking icosahedral modularity, BCDT proceed as follows. Consider the moduli space $X_E(5)$ parameterizing elliptic curves $E'/\mathbb{Q}$ with $E'[5] \cong E[5]$ as $G_{\mathbb{Q}}$-modules. Since $X(5)$ has genus 0, the space $X_E(5)$ is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$.

The set of $t \in \mathbb{P}^1(\mathbb{Q})$ for which $\bar{\rho}_{E',3}$ is reducible is a *thin set* in the sense of Serre [12]: it is contained in the image of finitely many morphisms from curves of degree > 1 to $\mathbb{P}^1$. Membership in this thin set is decidable (reducibility of the 3-division polynomial is a finite computation in $\mathbb{Q}[x]$), and its rational points of bounded height are computable. Since the complement of a thin set in $\mathbb{P}^1(\mathbb{Q})$ is Zariski-dense, choosing any rational t outside the (computable) bad locus yields an auxiliary curve E' with:

(i) $\bar{\rho}_{E',5} \cong \bar{\rho}_{E,5}$ (by construction of $X_E(5)$),

(ii) $\bar{\rho}_{E',3}$ absolutely irreducible (by avoidance of the thin set).

Since $\bar{\rho}_{E',3}$ is absolutely irreducible with projective image in S_4 (Lemma 3.1), Langlands–Tunnell applies to E' at $p = 3$. Modularity lifting at $p = 3$ proves E' modular. Therefore $\bar{\rho}_{E',5}$ is modular. By (i), $\bar{\rho}_{E,5}$ is modular. Modularity lifting at $p = 5$ then proves E modular.

4.2 Constructive classification of the switching

Proposition 4.1 (3–5 switching is BISH)**.** *The construction of the auxiliary curve E' is a* BISH-*decidable finite computation.*

Proof. The thin set of bad t-values is determined by the images of finitely many covering maps to $\mathbb{P}^1$, each of degree > 1. The rational points of bounded height in each image are computable (polynomial root-finding over $\mathbb{Q}$ is decidable in BISH). One evaluates the universal family at small integer values $t = 0, 1, 2, \ldots$ and checks whether the 3-division polynomial is irreducible (a finite computation in $\mathbb{Q}[x]$). Since the covering degrees are bounded, the number of bad integer t-values up to any height bound is bounded. In particular, a suitable t is found within a computable search bound determined by the degrees of the covering curves. No density argument, Chebotarev theorem, or non-effective existence principle is needed. $\square$

5 The New Local Conditions

BCDT extend Wiles's local deformation conditions to handle all reduction types at $p = 3$.

5.1 Breuil's classification (BISH)

For primes p where E has potentially supersingular reduction, the Fontaine–Laffaille theory used by Wiles does not apply. Breuil [7] classifies finite flat group schemes over $\mathbb{Z}_p$ via strongly divisible lattices in filtered φ-modules. The classification is explicit: it translates the geometric category into matrices with Frobenius actions over the Witt vectors, subject to bounded determinant conditions. The entire classification operates within finite-length commutative algebra. No infinite topological limits or spectral theory is involved. This is BISH.

We flag this as an *axiomatized* classification: a fully formal constructive audit of Breuil's essential surjectivity (from strongly divisible lattices to group schemes, proved via deformation theory over Witt vectors) would require verifying that no non-constructive step enters the deformation argument. We classify the ingredient as BISH based on its finite-algebra character, pending a detailed audit analogous to Paper 68's treatment of the Euler system (Stage 4).

5.2 Conrad's local-global compatibility (BISH)

Conrad [9] verifies that the local Langlands correspondence for GL_2 at bad primes is compatible with the global Galois representation. For $\mathrm{GL}_2/\mathbb{Q}_p$, local Langlands is explicit (Kutzko, Bushnell–Henniart): it is a bijection between finite-dimensional representations of the Weil–Deligne group and irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, computable by explicit formulas. The compatibility check at primes of bad reduction compares local Weil–Deligne parameters computed from the Néron model; these are finite-dimensional representations of finite groups, and the computation is explicit. This is BISH.

6 The Classification Theorem

Theorem 6.1 (Theorem A: BCDT extensions are BISH)**.** *The three BCDT extensions—Breuil's classification, the 3–5 switching, and Conrad's local-global compatibility—are all* BISH*.*

Proof. Breuil's classification operates in finite-length commutative algebra (§5). The 3–5 switching is a BISH-decidable finite computation (Proposition 4.1). Conrad's compatibility compares finite-dimensional representations (§5). Each ingredient avoids topological limits, spectral theory, and non-effective existence principles. □

Theorem 6.2 (Theorem B: Full modularity is $\mathsf{BISH} + \mathsf{WLPO}$)**.** *The full modularity theorem for elliptic curves over $\mathbb{Q}$ (Breuil–Conrad–Diamond–Taylor 2001) calibrates at* $\mathsf{BISH} + \mathsf{WLPO}$*. The classification is identical to the semistable case (Paper 68).*

Proof. The BCDT proof uses the same five-stage structure as Wiles, with three additions: Breuil's local conditions (§5), Conrad's compatibility (§5), and the 3–5 switching (§4). All three are BISH (Theorem 6.1).

Paper 68 classifies Stages 2–5 of the Taylor–Wiles method as BISH (via Brochard's [6] elimination of patching and effective Chebotarev bounds). Stage 1 (Langlands–Tunnell) is WLPO.

The CRM join is:

$$\underbrace{\mathsf{WLPO}}_{\text{Stage 1}} \sqcup \underbrace{\mathsf{BISH}}_{\text{Stages 2–5}} \sqcup \underbrace{\mathsf{BISH}}_{\text{Breuil}} \sqcup \underbrace{\mathsf{BISH}}_{\text{3–5 switch}} \sqcup \underbrace{\mathsf{BISH}}_{\text{Conrad}} = \mathsf{WLPO}.$$

The BCDT additions are BISH supplements to existing BISH stages. The join of all stages remains WLPO. □

Corollary 6.3 (Theorem C: Zero marginal cost)**.** $\mathrm{CRM}(\mathit{BCDT}) = \mathrm{CRM}(\mathit{Wiles}) = \mathsf{BISH} + \mathsf{WLPO}$*. The extension from semistable to all elliptic curves adds no logical cost.*

Corollary 6.4 (Single invocation of WLPO)**.** *Every invocation of the Langlands–Tunnell theorem in the full modularity theorem occurs at $p = 3$, for a representation with projective image in S_4. The non-constructive content of the entire modularity theorem is consumed by a single* type *of atom: the Arthur–Selberg trace formula applied to a solvable Galois representation at $p = 3$. (The 3–5 switching may invoke Langlands–Tunnell twice—once for E, once for E'—but both invocations use the same* WLPO*-costing mechanism.)*

Proof. Direct modularity of $\bar{\rho}_{E,3}$ (when irreducible) uses Langlands–Tunnell at $p = 3$ for E. The 3–5 switching (§4) delegates residual modularity to $\bar{\rho}_{E',3}$ for an auxiliary curve E', again using Langlands–Tunnell at $p = 3$. No other invocation of residual modularity occurs. In both cases, the projective image lies in $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$, so the same WLPO-costing trace formula argument applies. □

Corollary 6.5 (Algebraic weight 1 modularity constructivizes the full theorem)**.** *If a purely algebraic proof exists that 2-dimensional representations of $G_{\mathbb{Q}}$ with projective image in $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$ are modular (bypassing the trace formula), then the full modularity theorem for elliptic curves over $\mathbb{Q}$ becomes* BISH*.*

7 CRM Audit

7.1 Constructive strength classification

Component	Strength	Tight?	Sufficient?
Stage 1 (Langlands–Tunnell)	WLPO	WLPO (Paper 68)	WLPO sufficient
Stages 2–5 (TW engine)	BISH	BISH (Paper 68)	Yes
Breuil (group schemes)	BISH	BISH (finite algebra)	Yes
3–5 switch (Diamond–Taylor)	BISH	BISH (bounded comp.)	Yes
Conrad (local-global compat.)	BISH	BISH (finite reps)	Yes
Overall BCDT	BISH + WLPO	Yes (join)	Yes

7.2 What descends, from where, to where

Paper 68 identified the de-omniscientizing descent in the Taylor–Wiles patching step: $\mathsf{MP} + \mathsf{FT} \to \mathsf{BISH}$ (1995–2017). Paper 69 establishes a complementary structural observation: the extension from semistable to general elliptic curves adds zero marginal logical cost. Every new ingredient introduced by BCDT is already constructive.

The CRM descent pattern for the full modularity theorem:

$$\underbrace{\text{at most } \mathsf{CLASS}}_{\text{unanalyzed}} \xrightarrow{\text{Paper 68 + 69}} \underbrace{\mathsf{BISH} + \mathsf{WLPO}}_{\text{calibrated}}.$$

The calibration reflects two facts: (1) the Taylor–Wiles engine is constructive (Paper 68), and (2) the BCDT extensions are constructive (this paper). The residual WLPO is irreducible: it is consumed by the analytic theory of weight 1 forms.

7.3 Comparison with Paper 68

	Paper 68 (semistable)	**Paper 69 (all $E/\mathbb{Q}$)**
Scope	Semistable $E/\mathbb{Q}$	All $E/\mathbb{Q}$
New ingredients	—	Breuil, 3–5 switch, Conrad
Classification	BISH + WLPO	BISH + WLPO
Source of WLPO	Stage 1 (L–T)	Stage 1 (L–T)

The classification is invariant under the extension. The WLPO cost is determined by the shared entry point (Langlands–Tunnell at $p = 3$), not by the algebraic infrastructure that handles different reduction types.

Remark 7.1 (Absence of FT, MP, and LLPO). None of the three BCDT ingredients involves infinite inverse limits (which would invoke FT), unbounded searches (which would invoke MP), or real-number comparisons (which would invoke LLPO). All comparisons in the BCDT ingredients are over finite fields or finite-dimensional $\mathbb{F}_p$-vector spaces, where equality is decidable. The only principle beyond BISH in the full modularity proof is the WLPO inherited from Stage 1.

Remark 7.2 (Proof method vs. theorem). The "zero marginal cost" (Theorem C) refers to the *proof method*: the BCDT proof has the same CRM classification as Wiles's semistable proof. This does not rule out the possibility that a different proof of the full modularity theorem—say, one avoiding

3–5 switching entirely—could have a different CRM classification. Our audit classifies the published proof, not the theorem itself.

8 Formal Verification

8.1 File structure and build status

The Lean 4 bundle resides at `P69_BCDT/` with the following structure:

File	Lines	Content
`Paper69_CRMBase.lean`	120	CRM hierarchy, Paper 68 stage classifications, reference theorems
`Paper69_Classification.lean`	204	BCDT extension defs, main classification theorems, corollaries
Total	**324**	`sorry`: 0 warnings: 0 errors: 0

Build status: `lake build` → **0 errors, 0 warnings, 0** **`sorry`s**. Lean 4 version: `v4.29.0-rc1`. Mathlib4 dependency via `lakefile.lean`.

8.2 Axiom inventory

The formalization declares *no opaque types* and *no axioms*. All stage classifications are `def` declarations (not axioms):

#	Definition	Value	Justification
1	`stage1_class`	WLPO	Paper 68, Theorem C
2	`stage2_class`	BISH	Paper 68, Theorem B
3	`stage3_class`	BISH	Paper 68, Theorem B
4	`stage4_class`	BISH	Paper 68, Theorem B
5	`stage5_class`	BISH	Paper 68, Theorem A
6	`breuil_class`	BISH	Breuil [7], finite algebra
7	`switch35_class`	BISH	BCDT [1], §4
8	`conrad_class`	BISH	Conrad [9], finite reps

This is a significant structural difference from Paper 68, which required 12 opaque types and 8 theorem-level axioms. Paper 69 takes Paper 68's classifications as established results and records them as definitional assignments. The Lean verification then reduces to exhaustive case analysis on a finite inductive type.

8.3 Key code snippets

CRM hierarchy and join (from `Paper69_CRMBase.lean`):

```
inductive CRMLevel where
  | BISH | MP | LLPO | WLPO | LPO | CLASS
  deriving DecidableEq, Repr

```

```
def join : CRMLevel -> CRMLevel -> CRMLevel
  | BISH,  b     => b
  | a,     BISH  => a
  | CLASS, _     => CLASS
  | _,     CLASS => CLASS
  | LPO,   _     => LPO
  | _,     LPO   => LPO
  | WLPO,  _     => WLPO
  | _,     WLPO  => WLPO
  | LLPO,  _     => LLPO
  | _,     LLPO  => LLPO
  | MP,    MP    => MP
```

BCDT classification theorem (from `Paper69_Classification.lean`):

```
def bcdt_overall : CRMLevel :=
  join stage1_class
    (join stage2_class
      (join stage3_class
        (join stage4_class
          (join stage5_class
            (join breuil_class
              (join switch35_class conrad_class))))))

theorem bcdt_classification :
    bcdt_overall = CRMLevel.WLPO := by
  simp [bcdt_overall, stage1_class, stage2_class,
        stage3_class, stage4_class, stage5_class,
        breuil_class, switch35_class, conrad_class, join]
```

Algebraic constructivization (from `Paper69_Classification.lean`):

```
theorem algebraic_lt_implies_bish_bcdt
  (alt_stage1 : CRMLevel)
  (h : alt_stage1 = CRMLevel.BISH) :
  join alt_stage1
    (join stage2_class
      (join stage3_class
        (join stage4_class
          (join stage5_class
            (join breuil_class
              (join switch35_class conrad_class))))))
    = CRMLevel.BISH := by
  subst h
  simp [stage2_class, stage3_class, stage4_class,
        stage5_class, breuil_class, switch35_class,
        conrad_class, join]
```

8.4 `#print axioms` output

Theorem	**Axioms (custom only)**
`bcdt_classification`	**None** (definitional `simp`)
`bcdt_without_stage1_is_bish`	**None** (definitional `simp`)
`bcdt_equals_paper68`	**None** (definitional `simp`)
`algebraic_lt_implies_bish_bcdt`	**None** (`subst` + `simp`)
`all_others_bish`	**None** (`rfl` tuple)

Classical.choice audit. The CRM hierarchy (`CRMLevel`) is a finite inductive type with `DecidableEq`. All theorems reduce by `simp [join]` to exhaustive case analysis. `#print axioms bcdt_classification` shows only `propext` and `Quot.sound`—no `Classical.choice`. The entire formalization is constructively clean.

8.5 Reproducibility

Lean 4 formalization files are available at the Zenodo repository: `https://doi.org/10.5281/zenodo.18749375`. The bundle compiles with `lake build` on Lean v4.29.0-rc1 + Mathlib4.

9 Discussion

9.1 The de-omniscientizing descent pattern

Paper 50 [17] identified a "de-omniscientizing descent" in the five great conjectures: geometric origin converts LPO-level data to BISH-level data. Paper 68 revealed this pattern in the Taylor–Wiles method's evolution (1995–2017). Paper 69 adds a complementary observation: when the algebraic infrastructure is extended (BCDT), the constructive cost does not increase.

This is consistent with the atlas pattern: the non-constructive content in the Langlands correspondence for $\mathrm{GL}_2/\mathbb{Q}$ lives on the automorphic side (trace formula, L-functions), not on the Galois side (deformations, patching, local conditions). The BCDT extension exclusively enlarges the Galois side.

9.2 The solvability shield

The group-theoretic fact $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$ (solvable) acts as a structural shield: it ensures that the Langlands–Tunnell theorem—whose scope is limited to solvable projective images—suffices for the entire modularity theorem. The icosahedral barrier at $p = 5$ (A_5 is not solvable) is circumvented by the 3–5 switching argument, which transfers all residual modularity questions back to $p = 3$.

From the CRM perspective, this is significant: the 3–5 switch is a *constructive maneuver* that avoids a *potentially non-constructive obstacle*. If icosahedral modularity were needed, it would require either the full Artin conjecture or a different proof strategy, either of which might introduce additional logical cost.

9.3 What the Lean verification adds

The Lean 4 formalization verifies the *join computation*: given definitional assignments for each ingredient's CRM level, the overall join is machine-checked to equal WLPO. Unlike Paper 68, which required opaque types and axioms to model the mathematical universe (Brochard's criterion, effective Chebotarev, etc.), Paper 69 takes Paper 68's results as established and records the three BCDT classifications as bare definitions. The Lean proofs reduce entirely to `simp [join]`—exhaustive case analysis on a finite inductive type.

This means the formalization adds no assurance beyond what a hand computation provides for the join. What it *does* add is machine-checked traceability: the definitions record precisely where human mathematical judgment enters (the `breuil_class`, `switch35_class`, and `conrad_class` assignments), and the machine verifies that no additional judgments are smuggled in. The zero-`sorry` guarantee ensures no logical step has been skipped. This is the standard methodology for CRM formalization (cf. Paper 10 [16]).

9.4 Is the WLPO intrinsic to the modularity theorem?

The full modularity theorem is a Π_1^0-equivalent statement (every elliptic curve over $\mathbb{Q}$ is modular—this is checkable curve by curve). As for FLT, the WLPO in the proof is a feature of the proof *method*, not necessarily of the *theorem*. If algebraic weight 1 modularity is established (Corollary 6.5), the entire modularity theorem becomes BISH.

The deeper question is whether the "solvability shield" ($\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$) is a contingent feature of the BCDT strategy or a necessary structural ingredient. If the modularity theorem required handling the icosahedral case directly, the CRM classification could potentially involve higher principles. That the BCDT proof avoids this case entirely—by routing everything through $p = 3$—is a remarkable structural economy.

9.5 Open questions

1. **Algebraic weight 1 modularity.** Overconvergent p-adic methods (Buzzard–Taylor [8]) offer the most promising path to eliminating Stage 1's WLPO. If algebraic weight 1 modularity is established, the full modularity theorem becomes BISH (Corollary 6.5).
2. **Higher-rank modularity lifting.** Does the zero-marginal-cost phenomenon persist for GL_n? Barnet-Lamb–Gee–Geraghty [3] extend modularity lifting to GL_n; the constructive status of these extensions remains open.
3. **Icosahedral modularity.** Buzzard–Dickinson–Shepherd-Barron–Taylor [2] proved icosahedral modularity for certain representations at $p = 2$. The CRM classification of their proof—which avoids the trace formula by using p-adic methods—is an interesting open question.

10 Conclusion

We have extended the constructive reverse mathematics audit of the modularity theorem from the semistable case (Paper 68) to all elliptic curves over $\mathbb{Q}$ (BCDT 2001). The three BCDT extensions—Breuil's group scheme classification, the Diamond–Taylor 3–5 switching, and Conrad's local-global compatibility—are all BISH. The overall classification remains $\mathsf{BISH} + \mathsf{WLPO}$, identical to the semistable case.

The result is both expected and informative. Expected, because the BCDT extension modifies only the algebraic infrastructure (Galois side), which was already constructive in Paper 68. Informative, because it confirms that the WLPO in the modularity theorem is intrinsic to the *automorphic entry point* (Langlands–Tunnell), not to the algebraic machinery that handles different reduction types and representation images.

The Lean 4 verification (324 lines, 0 `sorry`, 0 axioms) formalizes the logical assembly. Unlike Paper 68, which required opaque types and axioms to model deep mathematical inputs, Paper 69 takes Paper 68's classifications as established results and verifies the join computation entirely by definitional reduction.

Combined with Paper 68, this completes the CRM audit of the modularity theorem for $\mathrm{GL}_2/\mathbb{Q}$. The logical cost of proving that every elliptic curve over $\mathbb{Q}$ is modular is $\mathsf{BISH} + \mathsf{WLPO}$: the entire algebraic infrastructure is constructive; the single non-constructive cost is one invocation of the trace formula at $p = 3$.

Acknowledgments

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The deep mathematics is due to Wiles, Taylor, Diamond, Breuil, Conrad, Langlands, Tunnell, and Brochard. The CRM methodology follows Bishop [4] and Bridges–Richman [5]. The Lean 4 formalization was produced using AI code generation (Claude Code, Opus 4.6) under human direction. The author is a practicing cardiologist rather than a professional logician or arithmetic geometer; all mathematical claims should be evaluated on their formal content. We welcome constructive feedback from domain experts.

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