The Measurement Problem as a Logical Artefact:

Constructive Calibration of Quantum Decoherence

Paper 14 in the Constructive Reverse Mathematics Series

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February 2026

Abstract

We formalize in Lean 4 a decomposition of quantum decoherence into constructive content layers. A single qubit coupled to N environmental qubits via controlled-rotation unitaries $U(\theta)$ undergoes decoherence: the off-diagonal coherence $c(N) = \|\rho_0^{01}\| \cdot |\cos(\theta/2)|^N$ decays geometrically. For any $\varepsilon > 0$, there exists an explicitly computable N_0 such that $c(N) < \varepsilon$ for all $N \geq N_0$. All of this is BISH (Height 0). The abstract principle that *every* bounded antitone decoherence process converges to a definite real limit—the completed-limit formulation of exact decoherence—is equivalent to the Limited Principle of Omniscience (LPO), via the sign-flip equivalence ABC $\leftrightarrow$ BMC and the Bridges–Vîţă equivalence BMC $\leftrightarrow$ LPO. This is the third physical domain—after the 1D Ising thermodynamic limit (Paper 8) and Schwarzschild geodesic incompleteness (Paper 13)—producing the same BMC $\leftrightarrow$ LPO pattern. Three independent physical theories, one logical structure. The formalization comprises 805 lines across 9 modules with zero `sorry` statements. Two interface assumptions (`bmc_of_lpo` and `lpo_of_bmc`) are axiomatized with citation. The `Classical.choice` in the axiom profile arises from Mathlib4 infrastructure; the constructive calibration is established by proof-content analysis (see §8 and Paper 10).

Contents

*New York University. AI-assisted formalization; see §9 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or quantum foundations; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 Physical Context

The quantum measurement problem asks: what physical process causes a quantum superposition to become a definite classical outcome? The decoherence programme Zeh [1970], Zurek [1991], Joos and Zeh [1985], Schlosshauer [2007] provides a partial answer: interaction with the environment suppresses quantum coherence. A system initially in a superposition $\alpha|0\rangle + \beta|1\rangle$ evolves, through entanglement with environmental degrees of freedom, into a state whose reduced density matrix is approximately diagonal. The off-diagonal elements—the quantum coherence—decay exponentially with the number of environmental interactions.

The decoherence programme does not solve the measurement problem in full. It explains the loss of interference but does not explain why one outcome rather than another is observed. Different interpretations (Copenhagen, many-worlds, decoherent histories) agree on the decoherence dynamics but disagree on what happens "at the limit": does the wave function actually collapse to a definite state, or do all branches persist?

1.2 The CRM Question

From the standpoint of constructive reverse mathematics (CRM), the question becomes precise: what is the logical cost of asserting that decoherence reaches completion?

The answer decomposes into two layers:

- The **finite decoherence** content—the system is ε-close to classical after an explicitly computable number of interactions—is BISH. No omniscience principle is needed.
- The **exact decoherence** assertion—every bounded antitone decoherence process converges to a definite real limit—is equivalent to LPO.

This decomposition mirrors Paper 8's treatment of the 1D Ising model Lee [2026d], where finite-size bounds are BISH but the thermodynamic limit costs LPO, and Paper 13's treatment of Schwarzschild geodesic incompleteness Lee [2026h], where the specific cycloid reaches $r = 0$ constructively but the universal completeness principle costs LPO.

1.3 Contributions

1. Machine-verified proof that the assertion "every bounded antitone sequence converges" is equivalent to LPO (805 lines of Lean 4, zero `sorry`).
2. Explicit BISH content: geometric decay formula $c(N) = c_0 \cdot r^N$, constructive ε-bound, trace preservation, verification of the decoherence map against the physical definition—all Height 0.
3. The third domain-invariance result: quantum decoherence joins statistical mechanics (Paper 8) and general relativity (Paper 13) in exhibiting the same BMC $\leftrightarrow$ LPO pattern.
4. The ABC $\leftrightarrow$ BMC equivalence (antitone bounded convergence $\leftrightarrow$ monotone bounded convergence) fully proved in Lean 4 via sign-flip, with no custom axioms.

1.4 Related Work

Paper 8 Lee [2026d] proved that bounded monotone convergence, instantiated through the 1D Ising free energy, is equivalent to LPO. Paper 13 Lee [2026h] extended this to Schwarzschild interior geodesic incompleteness. The equivalence BMC $\leftrightarrow$ LPO is due to Bridges and Vîţă Bridges and Vîţă [2006] (Theorem 2.1.5). Paper 11 Lee [2026f] formalized partial trace and entanglement at BISH, providing infrastructure reused here.

To our knowledge, no prior work applies constructive reverse mathematics to quantum decoherence or the measurement problem.

2 Background

2.1 Constructive Reverse Mathematics

Constructive reverse mathematics (CRM) classifies mathematical theorems by the weakest omniscience principle needed to prove them Bishop [1967], Bridges and Vîţă [2006], Ishihara [2006]. Bishop's constructive mathematics (BISH) avoids all omniscience principles; every existential claim comes with a computable witness.

The key principles for this paper are:

- **LPO** (Limited Principle of Omniscience): For every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, either all terms are zero or some term is one. Equivalent to: every bounded monotone sequence of reals converges (BMC).
- **BMC** (Bounded Monotone Convergence): Every monotone sequence bounded above has a limit. Equivalent to LPO by Bridges and Vîţă [2006].
- **ABC** (Antitone Bounded Convergence): Every antitone (non-increasing) sequence bounded below has a limit. Equivalent to BMC by sign-flip (proved in this paper).

2.2 Quantum Decoherence

We model decoherence via the simplest non-trivial scenario: a single system qubit interacting with a single environment qubit. The system state is a 2×2 density matrix $\rho \in M_2(\mathbb{C})$. The environment starts in the ground state $|0\rangle$.

The interaction is modeled by a controlled-rotation unitary $U(\theta) \in M_4(\mathbb{C})$:

$$U(\theta) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes R_y(\theta), \tag{1}$$

where $R_y(\theta)$ is the rotation matrix $\left[\begin{smallmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{smallmatrix}\right]$. If the system qubit is $|0\rangle$, the environment is unchanged; if the system qubit is $|1\rangle$, the environment rotates by θ.

The decoherence map is the composite operation: embed $\rho \mapsto \rho \otimes |0\rangle\langle 0|$, conjugate by $U(\theta)$, and trace out the environment:

$$\Phi_\theta(\rho) = \mathrm{Tr}_E\big[U(\theta) \cdot (\rho \otimes |0\rangle\langle 0|) \cdot U(\theta)^\dagger\big]. \tag{2}$$

The result is:

$$\Phi_\theta(\rho)_{ij} = \begin{cases} \rho_{00} & (i,j) = (0,0), \\ \rho_{01} \cdot \cos(\theta/2) & (i,j) = (0,1), \\ \rho_{10} \cdot \cos(\theta/2) & (i,j) = (1,0), \\ \rho_{11} & (i,j) = (1,1). \end{cases} \tag{3}$$

Diagonal entries are preserved; off-diagonal entries are multiplied by $\cos(\theta/2) \in (-1, 1)$.

3 Finite Decoherence at BISH

3.1 The Geometric Decay Formula

Definition 3.1 (Coherence). ✓ The *coherence* after N decoherence steps is

$$c(N) = \|(\Phi_\theta^{[N]}(\rho_0))_{01}\|,$$

the norm of the $(0,1)$ entry of the N-fold iterated decoherence map applied to the initial state ρ_0.

Theorem 3.2 (Geometric decay). ✓ *For any initial state ρ_0, interaction angle θ, and step count N:*

$$c(N) = \|\rho_0^{01}\| \cdot |\cos(\theta/2)|^N.$$

Proof. By induction on N. The base case $N = 0$ is immediate: $c(0) = \|\rho_0^{01}\|$. For the inductive step, the `decoherenceMap_01` lemma gives $(\Phi_\theta(\sigma))_{01} = \sigma_{01} \cdot \cos(\theta/2)$ for any σ. Applying this at $\sigma = \Phi_\theta^{[n]}(\rho_0)$ and using the inductive hypothesis:

$$\begin{aligned} (\Phi_\theta^{[n+1]}(\rho_0))_{01} &= (\Phi_\theta^{[n]}(\rho_0))_{01} \cdot \cos(\theta/2) \\ &= \rho_0^{01} \cdot (\cos(\theta/2))^n \cdot \cos(\theta/2) \\ &= \rho_0^{01} \cdot (\cos(\theta/2))^{n+1}. \end{aligned}$$

Taking norms and using $\|a \cdot b\| = \|a\| \cdot |b|$ for $b \in \mathbb{R}$ gives the result. □

3.2 Monotonicity and Boundedness

Theorem 3.3 (Coherence is antitone). ✓ *If $|\cos(\theta/2)| \le 1$, then c is antitone (non-increasing): $m \le n \implies c(n) \le c(m)$.*

Proof. From $c(N) = c_0 \cdot r^N$ where $r = |\cos(\theta/2)| \in [0, 1]$ and $c_0 = \|\rho_0^{01}\| \ge 0$. Since $r \le 1$, the power r^N is non-increasing: $r^{n+1} = r^n \cdot r \le r^n$. □

Theorem 3.4 (Coherence is non-negative). ✓ *$c(N) \ge 0$ for all N.*

Proof. $c(N) = \|\cdot\| \ge 0$ by the norm axiom. □

3.3 The BISH ε-Bound

Theorem 3.5 (Constructive ε-approximation). ✓ *For any $\varepsilon > 0$, if $0 < \theta < \pi$ (so each interaction produces genuine decoherence) and $\|\rho_0^{01}\| > 0$ (the initial state has non-trivial coherence), then there exists an explicit N_0 such that $c(N) < \varepsilon$ for all $N \geq N_0$.*

Proof. Set $r = |\cos(\theta/2)|$. Since $0 < \theta < \pi$, the half-angle $\theta/2 \in (0, \pi/2)$ satisfies $0 < \cos(\theta/2) < 1$, so $0 \leq r < 1$.

We need $c_0 \cdot r^N < \varepsilon$, i.e., $r^N < \varepsilon/c_0$ where $c_0 = \|\rho_0^{01}\| > 0$. By the MATHLIB4 lemma `exists_pow_lt_of_lt_one` (for $0 < \varepsilon/c_0$ and $r < 1$), there exists N_0 with $r^{N_0} < \varepsilon/c_0$. For any $N \geq N_0$:

$$c(N) = c_0 \cdot r^N \leq c_0 \cdot r^{N_0} < c_0 \cdot \frac{\varepsilon}{c_0} = \varepsilon.$$

The witness N_0 is constructive: it depends only on θ, ε, and c_0. □

Remark 3.6 (BISH status). All results in this section are BISH (Height 0). The geometric decay formula is explicit, the ε-bound provides a computable witness, and no omniscience principle is needed. In particular, *finite decoherence is constructive*: a quantum system interacting with a finite environment becomes indistinguishable from a classical system (to any finite-precision observation) after a computable number of interactions. No infinite environment, no completed limit, no LPO.

3.4 Verification of the Decoherence Map

Theorem 3.7 (Physical verification). ✓ *The explicit formula* (3) *equals the physical definition* (2)*:*

$$\Phi_\theta^{\text{explicit}}(\rho) = \mathrm{Tr}_E\big[U(\theta) \cdot (\rho \otimes |0\rangle\langle 0|) \cdot U(\theta)^\dagger\big].$$

Proof. Brute-force 4×4 matrix computation. The proof proceeds by `ext i j; fin_cases i; fin_cases j`, generating four goals (one per entry of the 2×2 output matrix). Each goal expands to a sum of products over Fin 2 × Fin 2 indices. Three of four cases close by `ring`; the $(1, 1)$ case requires the Pythagorean identity $\cos^2(\theta/2) + \sin^2(\theta/2) = 1$, which is BISH-valid. □

This verification bridges the gap between the algebraic formula and the physical model: the explicit decoherence map is *derived*, not assumed.

4 Exact Decoherence and LPO

4.1 The Sign-Flip Equivalence

Definition 4.1 (ABC). ✓ *Antitone Bounded Convergence* (ABC): every antitone sequence $f : \mathbb{N} \to \mathbb{R}$ bounded below converges to a definite limit.

Theorem 4.2 (ABC ↔ BMC). ✓ *Antitone bounded-below convergence is equivalent to monotone bounded-above convergence.*

Proof. **(ABC ⇒ BMC):** Given a monotone sequence a bounded above by M, define $g(n) = -a(n)$. Then g is antitone and bounded below by $-M$. By ABC, g converges to some L_{neg}. Then a converges to $-L_{\text{neg}}$.

(BMC ⇒ ABC): Given an antitone sequence f bounded below by B, define $g(n) = -f(n)$. Then g is monotone and bounded above by $-B$. By BMC, g converges to some L_{neg}. Then f converges to $-L_{\text{neg}}$.

Both directions use only the algebraic identity $|-x - L_{\text{neg}}| = |x - (-L_{\text{neg}})|$, which is BISH. □

Remark 4.3. The ABC $\leftrightarrow$ BMC equivalence is fully proved in Lean 4 with no custom axioms. The `#print axioms` output for `abc_iff_bmc` shows only `[propext, Classical.choice, Quot.sound]`—standard Mathlib4 infrastructure.

4.2 The Headline Theorem

Theorem 4.4 (Exact decoherence $\leftrightarrow$ LPO)**.** ✓

$$\big(\forall f : \mathbb{N} \to \mathbb{R},\ \mathit{Antitone}\ f \to (\exists B,\ \forall n,\ B \le f(n)) \to \exists L,\ \forall \varepsilon > 0,\ \exists N_0,\ \forall N \ge N_0,\ |f(N) - L| < \varepsilon\big) \quad \longleftrightarrow \quad \text{LPO}.$$

Proof. The left-hand side is definitionally equal to ABC. By Theorem 4.2, ABC $\leftrightarrow$ BMC. By the Bridges–Vîţă equivalence Bridges and Vîţă [2006], BMC $\leftrightarrow$ LPO. Composing: ABC $\leftrightarrow$ BMC $\leftrightarrow$ LPO.

In Lean 4: `abc_iff_bmc.trans lpo_iff_bmc.symm`. □

4.3 The Coherence Instance

Theorem 4.5 (Coherence is an instance of ABC)**.** ✓ *For any initial state ρ_0 and angle θ with $|\cos(\theta/2)| \le 1$, the coherence sequence c is antitone and bounded below by* 0.

Proof. Antitonicity is Theorem 3.3; the lower bound is Theorem 3.4. □

Therefore, asserting that the coherence converges *exactly* to a definite limit—for all initial states and all angle sequences, including those with no computable decay rate—is equivalent to LPO.

5 The Measurement Problem Dissolves

The measurement problem asks whether the decoherence process reaches completion: does the off-diagonal coherence actually reach zero, or does it merely approach zero?

Different interpretations answer differently:

- **Copenhagen:** The wave function collapses upon measurement. The limit is reached.
- **Many-worlds:** All branches persist. The limit is reached in each branch's reduced state.
- **Decoherent histories:** Decoherence is approximate. The limit is approached but never reached.

At BISH, these three interpretations agree: the system is ε-close to diagonal for any desired ε, after a computable number of interactions (Theorem 3.5). No experiment with finite precision can distinguish between "exactly diagonal" and "ε-close to diagonal."

The interpretations disagree only about the completed limit: does the coherence reach exactly zero? This is a question about bounded monotone convergence—specifically, about whether the assertion "every bounded antitone sequence converges" holds. That assertion is LPO.

The programme does not adjudicate between interpretations. It shows that the disagreement is about a mathematical assertion (ABC, equivalent to BMC and hence to LPO) that no finite experiment can distinguish from its constructive approximation. The "residual" measurement problem—why one outcome rather than another, whether collapse "really" happens—can only be formulated at the LPO level.

We do not claim that decoherence solves the measurement problem. We claim that the residual problem has a precise logical cost: LPO.

6 Domain Invariance

Paper 14 is the third physical domain producing the BMC $\leftrightarrow$ LPO pattern:

Domain	**Bounded Monotone Sequence**	**LPO Content**
Stat. Mech. (P8)	Free energy f_N	Thermodynamic limit exists
Gen. Rel. (P13)	Radial coordinate $r(\tau)$	Geodesic incompleteness
Quantum Meas. (P14)	Coherence $c(N)$	Decoherence completes (collapse)

In each case:

- The *finite* physics is BISH: finite-size partition functions, the cycloid geodesic, finite-step decoherence.
- The *infinite limit* assertion costs LPO: the thermodynamic limit, geodesic completeness, exact decoherence.
- The mechanism is identical: the physical quantity is a bounded monotone sequence, and asserting its completed limit is BMC, which is LPO.

The calibration table for the full series, updated with Paper 14:

Physical layer	**Principle**	**Status**	**Source**
Finite-volume Gibbs states	BISH	Calibrated	Trivial
Finite-size approximations (Ising)	BISH	Calibrated	Paper 8
Schwarzschild exterior	BISH	Calibrated	Paper 1
Interior finite-time physics	BISH	Calibrated	Paper 13
Tsirelson bound (CHSH $\leq 2\sqrt{2}$)	BISH	Calibrated	Paper 11
Bell state entropy ($\log 2$)	BISH	Calibrated	Paper 11
Finite-step decoherence	BISH	Calibrated	Paper 14
Bidual-gap witness ($S_1(H)$)	$\equiv$ WLPO	Calibrated	Papers 2, 7
Thermodynamic limit (Ising)	$\equiv$ LPO	Calibrated	Paper 8
Geodesic incompleteness	$\equiv$ LPO	Calibrated	Paper 13
Exact decoherence (collapse)	$\equiv$ LPO	Calibrated	Paper 14

The pattern is consistent: all LPO costs arise from completed infinite limits; all finite-time and finite-size physics is BISH.

7 Lean Formalization

7.1 Architecture

The formalization is organized as a single Lean 4 project with 9 modules:

7.2 Key Design Decisions

Explicit formula first. The decoherence map is defined directly via the explicit formula (3) rather than through the Kronecker product–conjugation–partial trace chain. A separate verification theorem (`decoherenceMap_eq_physical`) connects it to the physical definition (2). This avoids the high risk of Kronecker product unfolding while preserving the physical connection.

Module	Lines	Content
Defs.lean	105	Core definitions: decoherence map, coherence, partial trace
PartialTrace.lean	37	Partial trace lemmas (re-stated from Paper 11)
DecoherenceMap.lean	105	Entry lemmas, trace preservation, physical verification
FiniteDecoherence.lean	85	N-step iteration, geometric decay formula
MonotoneDecay.lean	54	Antitonicity, non-negativity, initial bound
CauchyModulus.lean	101	BISH ε-bound (headline constructive result)
LPO_BMC.lean	62	LPO, BMC definitions + axiomatized equivalence
ExactDecoherence.lean	161	ABC $\leftrightarrow$ BMC (proved), headline LPO theorem
Main.lean	95	Assembly + `#print axioms` audit
Total	**805**	

Table 1: Module structure of Paper 14.

Standalone package. Paper 14 is a self-contained Lake package. It cannot import Paper 8 or Paper 11 as Lake dependencies, so LPO, BMC, and the partial trace are re-defined. The BMC $\leftrightarrow$ LPO equivalence is axiomatized with citation.

Abstract equivalence for the reverse direction. The LPO content arises from the abstract equivalence ABC $\leftrightarrow$ BMC, not from encoding binary sequences into physical quantities. The coherence sequence is an *instance* of the abstract framework; the logical structure provides the equivalence.

7.3 Core Definitions

```
/-- The single-step decoherence map on 2x2 density matrices. -/
def decoherenceMap (t : Real) (r : Matrix (Fin 2) (Fin 2) C) :
    Matrix (Fin 2) (Fin 2) C :=
  fun i j =>
    match i, j with
    | 0, 0 => r 0 0
    | 0, 1 => r 0 1 * (Real.cos (t / 2))
    | 1, 0 => r 1 0 * (Real.cos (t / 2))
    | 1, 1 => r 1 1

/-- Coherence after N decoherence steps. -/
def coherence (r0 : Matrix (Fin 2) (Fin 2) C) (t : Real)
    (N : Nat) : Real :=
  ||(decoherenceMap t ^[N] r0) 0 1||
```

Listing 1: Decoherence map (Defs.lean, excerpt).

```
/-- The controlled-rotation unitary on C2 x C2. -/
def controlledRotation (t : Real) :
    Matrix (Fin 2 * Fin 2) (Fin 2 * Fin 2) C :=
  fun i j =>
    if i = (0, 0) && j = (0, 0) then 1
    else if i = (0, 1) && j = (0, 1) then 1
    else if i = (1, 0) && j = (1, 0) then (Real.cos (t/2))
    else if i = (1, 0) && j = (1, 1) then (-Real.sin (t/2))
    else if i = (1, 1) && j = (1, 0) then (Real.sin (t/2))
    else if i = (1, 1) && j = (1, 1) then (Real.cos (t/2))
    else 0
```

Listing 2: Controlled-rotation unitary (Defs.lean, excerpt).

```
def LPO : Prop :=
  forall (a : Nat -> Bool),
    (forall n, a n = false) ||| (exists n, a n = true)

def BMC : Prop :=
  forall (a : Nat -> Real) (M : Real),
    Monotone a -> (forall n, a n <= M) ->
    exists L : Real, forall e : Real, 0 < e ->
      exists N0 : Nat, forall N : Nat,
        N0 <= N -> |a N - L| < e

axiom bmc_of_lpo : LPO -> BMC
axiom lpo_of_bmc : BMC -> LPO

def ABC : Prop :=
  forall (f : Nat -> Real), Antitone f ->
    (exists B : Real, forall n, B <= f n) ->
    exists L : Real, forall e : Real, 0 < e ->
      exists N0 : Nat, forall N : Nat,
        N0 <= N -> |f N - L| < e
```

Listing 3: LPO, BMC, and ABC (LPO_BMC.lean and ExactDecoherence.lean).

7.4 Main Theorems

```
theorem coherence_eq_geometric (r0 : Matrix (Fin 2) (Fin 2) C)
    (t : Real) (N : Nat) :
    coherence r0 t N = ||r0 0 1|| * |Real.cos (t / 2)| ^ N := by
  simp only [coherence, decoherence_iterate_offdiag]
  rw [norm_mul, norm_pow, Complex.norm_real, Real.norm_eq_abs]
```

Listing 4: Geometric decay formula (FiniteDecoherence.lean).

```
theorem decoherence_epsilon_bound (r0 : Matrix (Fin 2) (Fin 2) C)
    (t : Real) (ht : 0 < t && t < pi) (hc : 0 < ||r0 0 1||)
    (e : Real) (he : e > 0) :
    exists N0 : Nat, forall N, N0 <= N ->
      coherence r0 t N < e := by
  set r := |Real.cos (t / 2)| with hr_def
  have hr_lt_one : r < 1 := abs_cos_half_lt_one ht
  set c0 := ||r0 0 1|| with hc0_def
  obtain <<N0, hN0>> :=
    exists_pow_lt_of_lt_one (div_pos he hc) hr_lt_one
  exact <<N0, fun N hN => by
    rw [coherence_eq_geometric]
    calc c0 * r ^ N
        <= c0 * r ^ N0 := by
          apply mul_le_mul_of_nonneg_left _ (le_of_lt hc)
          exact pow_le_pow_of_le_one (abs_nonneg _)
            (le_of_lt hr_lt_one) hN
      _ < c0 * (e / c0) := by
          apply mul_lt_mul_of_pos_left hN0 hc
      _ = e := by field_simp>>
```

Listing 5: BISH ε-bound (CauchyModulus.lean).

```
-- BMC -> ABC (the reverse direction)
intro hBMC f hf hB
obtain <<B, hB>> := hB
-- -f is monotone and bounded above by -B
have h_mono : Monotone (fun n => -f n) :=
  fun m n hmn => neg_le_neg (hf hmn)
have h_bdd : forall n, (fun n => -f n) n <= -B :=
  fun n => neg_le_neg (hB n)
obtain <<L_neg, hL>> :=
  hBMC (fun n => -f n) (-B) h_mono h_bdd
exact <<-L_neg, fun e he => by
  obtain <<N0, hN0>> := hL e he
  exact <<N0, fun N hN => by
    have := hN0 N hN
    rwa [show -f N - L_neg = -(f N - (-L_neg))
      from by ring, abs_neg] at this>>>>
```

Listing 6: ABC $\leftrightarrow$ BMC (ExactDecoherence.lean, BMC direction).

```
theorem exact_decoherence_iff_LPO :
    (forall (f : Nat -> Real), Antitone f ->
      (exists B : Real, forall n, B <= f n) ->
      exists L : Real, forall e : Real, 0 < e ->
        exists N0 : Nat, forall N : Nat,
          N0 <= N -> |f N - L| < e)
    <-> LPO := by
  rw [show (...) = ABC from rfl]
  exact abc_iff_bmc.trans lpo_iff_bmc.symm
```

Listing 7: Headline theorem (ExactDecoherence.lean).

7.5 Axiom Audit

The `Main.lean` module audits the axiom profile of each theorem:

```
#print axioms coherence_eq_geometric
-- [propext, Classical.choice, Quot.sound]

#print axioms decoherence_epsilon_bound
-- [propext, Classical.choice, Quot.sound]

#print axioms abc_iff_bmc
-- [propext, Classical.choice, Quot.sound]

#print axioms exact_decoherence_iff_LPO
-- [propext, Classical.choice, Quot.sound,
--   bmc_of_lpo, lpo_of_bmc]
```

Listing 8: Axiom audit (Main.lean, selected).

The `Classical.choice` appearing in the BISH results arises from MATHLIB4's real number and complex number infrastructure—specifically, the constructions of $\mathbb{R}$ and $\mathbb{C}$ via Cauchy completion, which pervasively use `Classical.choice` as a metatheoretic convenience. The mathematical content of these proofs is constructive: they involve only explicit trigonometric computation and induction. The constructive calibration is established by proof-content analysis, following the methodology described in Paper 10.

8 Certification Methodology

8.1 Axiom Profile

All theorems carry `[propext, Classical.choice, Quot.sound]` from MATHLIB4, plus `bmc_of_lpo` and `lpo_of_bmc` for the LPO equivalence. The BISH results have `Classical.choice` from MATHLIB4 infrastructure only.

Notably, `abc_iff_bmc` has *no custom axioms*—the sign-flip equivalence is fully proved.

8.2 Certification Level

Paper 14 contains two certification levels (Paper 10 terminology):

- **Level 0** (BISH)**:** Geometric decay formula, ε-bound, trace preservation, physical verification. Finite-dimensional explicit computation.
- **Level 1 (intentional classical):** The LPO equivalence via ABC $\leftrightarrow$ BMC $\leftrightarrow$ LPO. The two axiomatized lemmas are the sole classical content.

8.3 The Axiomatized Equivalences

Two axioms are used:

- `bmc_of_lpo`: LPO $\Rightarrow$ BMC. From Bridges–Vîţă Bridges and Vîţă [2006], Theorem 2.1.5.
- `lpo_of_bmc`: BMC $\Rightarrow$ LPO. Fully verified in Paper 8 Lee [2026d] via encoding into the 1D Ising free energy sequence.

These are the same interface assumptions used in Paper 13. They are conservative: they introduce no new logical content beyond what LPO already provides.

8.4 Methodological Limitations

1. **Classical.choice is a Mathlib infrastructure artifact.** The proof-content analysis methodology for handling this is described in Paper 10 Lee [2026e].
2. **Single-qubit model.** We formalize the simplest decoherence scenario: one system qubit interacting with single environmental qubits. Extension to multi-qubit systems or continuous environments would require additional infrastructure.
3. **The LPO cost is on the universal principle, not any specific process.** The uniform-angle decoherence $c(N) = c_0 \cdot r^N$ converges constructively (Theorem 3.5). The LPO cost attaches to the assertion that *every* bounded antitone sequence converges.
4. **The formalization abstracts from the full quantum formalism.** We do not formalize Hilbert spaces, operator algebras, or CPTP maps in full generality. The decoherence map is defined via explicit 2×2 matrix formulae, verified against the physical definition by brute force.

9 AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as Papers 2, 7, 8, 11, and 13 Lee [2026b,c,d,f,h], Anthropic [2026].

The author is a medical professional, not a domain expert in constructive mathematics or quantum foundations. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalization strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Task	**Human**	**AI (Claude Opus 4.6)**
Research direction	✓	
Mathematical blueprint	✓	✓
Proof strategy design	✓	✓
Mathlib4 API discovery		✓
Lean 4 proof generation		✓
Proof review	✓	
Build verification		✓
Paper writing	✓	✓

Table 2: Division of labor between human and AI.

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/AICardiologist/FoundationRelativity`
- **Path**: `Papers/P14_Decoherence/`
- **Build**: `lake exe cache get && lake build` (1,950 jobs, 0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Mathlib version**: commit `7091f0f6`
- **Interface axioms**: `bmc_of_lpo` (Bridges–Vîţă Bridges and Vîţă [2006]), `lpo_of_bmc` (Paper 8 Lee [2026d])
- **Axiom audit**: `Main.lean`
- **Axiom profile (main theorem)**: `[propext, Classical.choice, Quot.sound, bmc_of_lpo, lpo_of_bmc]`
- **Axiom profile (BISH content)**: `[propext, Classical.choice, Quot.sound]` (Mathlib infra only)
- **Axiom profile (abc_iff_bmc)**: `[propext, Classical.choice, Quot.sound]` (no custom axioms)
- **Total**: 9 files, 805 lines, 0 sorry
- **Zenodo DOI**: 10.5281/zenodo.18569068

Acknowledgments

The Lean 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the Mathlib4 community for maintaining the comprehensive library of formalized mathematics that made this work possible.

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