An Arithmetic Proof That Bidual Gap Detection Is WLPO-Complete

Gödel Sequences, Explicit Reductions, and a Lean 4 Formalization

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February 2026

Abstract

We give a new proof that deciding membership in the bidual gap ℓ^∞/c_0 is WLPO-complete over Bishop-style constructive mathematics, independent of the functional-analytic proof in Paper 2 [Lee, 2026a]. The proof proceeds via arithmetic: we construct an injective, order-preserving map Φ from the Lindenbaum algebra of Π^0_1 sentences of Peano Arithmetic into ℓ^∞/c_0, using bounded proof-search sequences (*Gödel sequences*). We prove that $\Phi([\varphi]) = [0]$ if and only if φ is PA-refutable. Since Π^0_1 consistency decidability is known to be WLPO-equivalent, this yields that gap detection is WLPO-complete by a route entirely independent of Paper 2. That two unrelated methods—functional-analytic (Paper 2) and arithmetic (this paper)—both establish WLPO-completeness of gap detection is evidence that this classification is robust.

The construction is formalized in Lean 4 (1,213 lines, 30 standard axioms for PA metamathematics, Mathlib4-compatible).

Contents

*New York University. AI-assisted formalization; see §10 for methodology.

1 Introduction

Paper 2 of this series [Lee, 2026a] proved that detecting the bidual gap—deciding whether the canonical embedding $J_X : X \to X^{**}$ is surjective—is WLPO-complete over BISH. The proof used functional analysis: an Ishihara kernel extracted from a gap witness, reducing gap detection to tail-behavior decidability. This established gap detection as a constructive complexity benchmark.

Is Paper 2's result an artifact of its proof method, or is the WLPO-completeness of gap detection a robust classification? An independent proof via different methods would provide evidence for robustness.

We prove gap detection is WLPO-complete by an entirely different route—through arithmetic rather than functional analysis. The proof constructs an explicit reduction from Π^0_1 consistency (a known WLPO-complete problem) to gap membership. Each Π^0_1 sentence φ maps to a *Gödel sequence* v^φ in ℓ^∞, constructed from bounded proof search. The sequence lies in c_0 if and only if φ is refutable. The map is injective on PA-equivalence classes. This reduces Π^0_1 consistency to gap membership, and since Π^0_1 consistency is WLPO-complete (Ishihara [Ishihara, 2006]; Bridges–Richman), gap detection is WLPO-complete.

This proof shares no machinery with Paper 2. Paper 2 works with dual spaces, functionals, and thresholds. This paper works with Gödel numbering, proof predicates, and sentence equivalence. The two proofs arrive at the same classification via independent constructions, confirming that WLPO-completeness of gap detection is not an artifact of either approach.

The reduction maps into ℓ^∞/c_0, which is a mathematical abstraction. The Gödel sequences are engineered from the proof predicate. The paper establishes a complexity-class result, not a physical correspondence. Physical interpretations of WLPO are treated in other papers of the series.

1.1 Contributions

1. **Gödel sequence reduction.** A well-defined injection Φ from the Lindenbaum algebra $\Pi^0_1/{\sim_{\mathrm{PA}}}$ into ℓ^∞/c_0 (Theorem 5.4).
2. **Gap detection theorem.** $\Phi([\varphi]) = [0]$ iff φ is refutable (Theorem 5.5).
3. **Calibration chain.** WLPO $\iff$ Π^0_1 consistency decidable $\iff$ gap detection decidable (Theorem 6.4).
4. **Machine-checked formalization.** 1,213 lines of Lean 4, 7 modules, built on Mathlib4.

2 Arithmetic Side: Axiomatized PA

We axiomatize the properties of Peano Arithmetic needed for the construction. These are standard metamathematical facts that would be theorems given a formalization of Gödel numbering; axiomatizing them allows us to focus on the reduction itself.

Definition 2.1 (Axiomatized syntax of PA)**.** We postulate:

- A type `SentencePA` of sentences.
- An injective function `godelNum` : `SentencePA` $\to \mathbb{N}$.
- A decidable proof predicate `PrfPA` : $\mathbb{N} \to$ `SentencePA` $\to$ Prop.
- Syntactic operations: negation, conjunction, disjunction, implication.
- A bottom element $\bot$ (falsity).

Definition 2.2 (Provability and consistency)**.**

$$\texttt{ProvablePA}(\psi)\exists p,\ \texttt{PrfPA}(p,\psi)$$
$$\texttt{RefutablePA}(\psi)\texttt{ProvablePA}(\neg\psi)$$
$$\texttt{ConsistentPA}(\psi)\neg\,\texttt{RefutablePA}(\psi)$$

We axiomatize that PA is consistent ($\bot$ is not provable).

Definition 2.3 (PA-provable equivalence)**.** $\varphi \sim_{\mathrm{PA}} \psi$ iff PA $\vdash (\varphi \to \psi) \wedge (\psi \to \varphi)$. This is an equivalence relation that respects all syntactic operations and preserves consistency.

Definition 2.4 (The Lindenbaum algebra)**.** The type Π^0_1 consists of those sentences marked as arithmetically Π^0_1 (axiomatized as a predicate `isPi01` closed under $\wedge$ and $\vee$). The *Lindenbaum algebra* is `LindenbaumPi01`$\Pi^0_1/{\sim_{\mathrm{PA}}}$, with partial order $[\varphi] \le [\psi]$ iff PA $\vdash \varphi \to \psi$.

2.1 Canonical Gödel Numbers

Each equivalence class in `LindenbaumPi01` has a canonical Gödel number: the minimum Gödel number among all sentences in the class.

Definition 2.5 (Canonical Gödel number)**.** `canonGN`$(\varphi)\min\{n \mid \exists\psi \in \Pi^0_1,\ \texttt{godelNum}(\psi) = n \wedge \varphi \sim_{\mathrm{PA}} \psi\}$.

Lemma 2.6. *`canonGN` is well-defined on `LindenbaumPi01`: if $\varphi \sim_{\mathrm{PA}} \psi$, then `canonGN`$(\varphi) =$ `canonGN`(ψ).*

Proof. If $\varphi \sim_{\mathrm{PA}} \psi$, then for any witness χ with $\varphi \sim_{\mathrm{PA}} \chi$, we have $\psi \sim_{\mathrm{PA}} \chi$ by transitivity, and vice versa. So the sets of Gödel numbers are identical, and their minima agree. □

3 Analytic Side: The Bidual Gap

Definition 3.1 (Sequence spaces).
- ℓ^∞: bounded real sequences $\{f : \mathbb{N} \to \mathbb{R} \mid \exists M, \forall n, |f(n)| \le M\}$.
- c_0: sequences converging to 0.
- $\ell^\infty/c_0 = \ell^\infty/c_0$: the *bidual gap*.

Two bounded sequences f, g are c_0-*equivalent* if $f - g \to 0$.

3.1 Row-Based Structure

We use the Cantor pairing $\langle n, m \rangle : \mathbb{N} \to \mathbb{N}$ to decompose $\mathbb{N}$ into infinitely many infinite rows.

Definition 3.2 (Rows). $\mathrm{row}(n) = \{k \in \mathbb{N} \mid \pi_1(k) = n\}$.

Lemma 3.3. *Each row is infinite. Distinct rows are disjoint.*

Definition 3.4 (Row characteristic functions). $\texttt{rowChar}(n)(k) = \begin{cases} 1 & \text{if } \pi_1(k) = n \\ 0 & \text{otherwise} \end{cases}$.

Lemma 3.5. *$\texttt{rowChar}(n) \notin c_0$ for all n (infinitely many 1s).*

Lemma 3.6. *$[\texttt{rowChar}(n)] \neq [0]$ in ℓ^∞/c_0 for all n.*

The *logical gap sublattice* $L \subset \ell^\infty/c_0$ consists of $[0]$ together with $\{[\texttt{rowChar}(n)] \mid n \in \mathbb{N}\}$.

4 The Gödel Sequence

The Gödel sequence is the bridge between the arithmetic and analytic sides.

Definition 4.1 (Gödel sequence). For a sentence φ with Gödel number $g = \texttt{godelNum}(\varphi)$:

$$v^\varphi(k) \begin{cases} 1 & \text{if } \pi_1(k) = g \text{ and } \forall p \le k,\ \neg\,\texttt{PrfPA}(p, \neg\varphi), \\ 0 & \text{otherwise.} \end{cases}$$

The Gödel sequence is $\{0, 1\}$-valued and bounded, hence in ℓ^∞.

Theorem 4.2 (Refutable $\Rightarrow$ null). *If φ is refutable, then v^φ is eventually zero, hence $v^\varphi \in c_0$ and $[v^\varphi] = [0]$ in ℓ^∞/c_0.* ✓

Proof. If p_0 is a proof of $\neg\varphi$, then for all $k \ge p_0$, the bounded proof search finds $p_0 \le k$, so $v^\varphi(k) = 0$. □

Theorem 4.3 (Consistent $\Rightarrow$ not null). *If φ is consistent, then $v^\varphi = \texttt{rowChar}(g)$ pointwise, hence $v^\varphi \notin c_0$ and $[v^\varphi] \neq [0]$.* ✓

Proof. If φ is consistent, no refutation proof exists, so the bounded search always fails, and $v^\varphi(k) = 1$ whenever $\pi_1(k) = g$. This gives $v^\varphi = \texttt{rowChar}(g)$ pointwise. Since $\texttt{rowChar}(g)$ has infinitely many 1s, it does not converge to 0. □

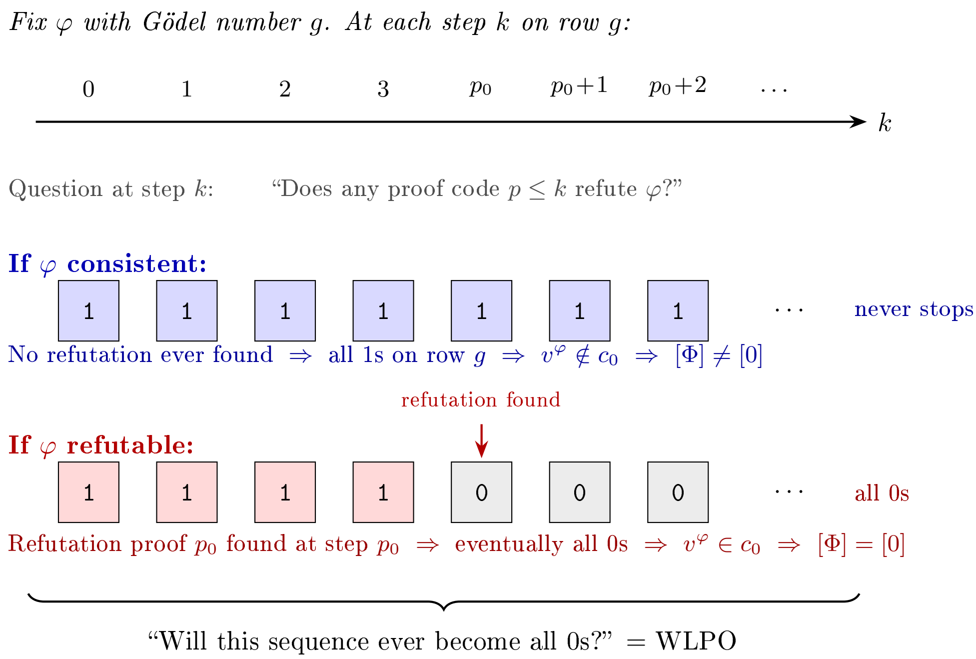

Figure 1: The Gödel sequence as a proof-search process. At each step k, bounded proof search checks whether any code $p \leq k$ refutes φ. If φ is consistent, no refutation is ever found and the sequence stays at 1 on row g indefinitely. If φ is refutable, the sequence drops to 0 once the refutation proof code p_0 is reached. The question “does this sequence eventually vanish?” is precisely WLPO.

5 The Gödel Sequence Reduction

5.1 Canonical Representatives

For each equivalence class in `LindenbaumPi01`, we select the *canonical representative*: the Π^0_1 sentence with the smallest Gödel number in the class. This is well-defined by the well-ordering of $\mathbb{N}$ and uses `Nat.find` with classical decidability.

Definition 5.1 (Gödel sequence reduction map)**.** Φ : `LindenbaumPi01` $\to \ell^\infty/c_0$ sends each class $[\varphi]$ to $[v^{\texttt{canonRep}(\varphi)}]$, the gap element of the canonical representative's Gödel sequence.

Lemma 5.2 (Well-definedness)**.** *Φ is well-defined: if $\varphi \sim_{\mathrm{PA}} \psi$, then `canonRep`$(\varphi)$ = `canonRep`(ψ) (they have the same canonical Gödel number, and `godelNum` is injective), so their Gödel sequences are identical.* ✓

Theorem 5.3 (Injectivity)**.** *Φ is injective.*

Proof. Suppose $\Phi([\varphi]) = \Phi([\psi])$. Four cases:

1. *Both consistent:* their Gödel sequences agree with `rowChar`(g_φ) and `rowChar`(g_ψ) respectively. If the gap elements are equal, the row characteristic functions are c_0-equivalent, forcing $g_\varphi = g_\psi$ (different rows give non-equivalent indicators). Since `godelNum` is injective, the canonical representatives are identical, so $[\varphi] = [\psi]$.
2. *φ consistent, ψ refutable:* $\Phi([\varphi]) \neq [0]$ but $\Phi([\psi]) = [0]$, contradiction. ✓
3. *φ refutable, ψ consistent:* symmetric. ✓
4. *Both refutable:* both are PA-equivalent to $\perp$, hence to each other, so $[\varphi] = [\psi]$. ✓

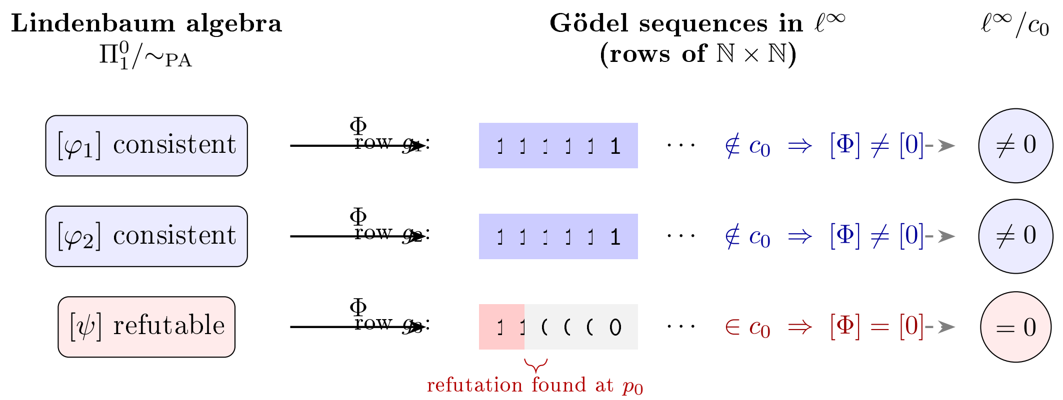

Figure 2: The Gödel sequence reduction map Φ. Each equivalence class $[\varphi]$ in the Lindenbaum algebra occupies a unique row of $\mathbb{N}\times\mathbb{N}$. Consistent sentences produce constant-1 rows (not in c_0, hence nonzero in ℓ^∞/c_0). Refutable sentences collapse to 0 once a refutation proof is found (eventually zero, hence in c_0, mapping to $[0]$ in ℓ^∞/c_0).

□

Theorem 5.4 (Gödel Sequence Reduction)**.** *The map Φ :* `LindenbaumPi01` $\to \ell^\infty/c_0$ *is injective, maps refutable classes to* $[0]$*, and maps consistent classes to nonzero elements of* ℓ^∞/c_0*.* ✓

Theorem 5.5 (Gap Detection)**.** $\Phi([\varphi]) = [0]$ *if and only if* φ *is refutable over* PA*.* ✓

6 Calibration Link

We connect the Gödel sequence reduction to the omniscience hierarchy.

Definition 6.1 (Π^0_1 consistency decidability)**.** Π^0_1 consistency decidable iff for every Π^0_1 sentence φ, either φ is consistent or φ is refutable.

Definition 6.2 (Gap detection decidability)**.** Gap detection is decidable iff for every Π^0_1 sentence φ, either $\Phi([\varphi]) = [0]$ or $\Phi([\varphi]) \neq [0]$.

Theorem 6.3 (WLPO $\Rightarrow$ Π^0_1 consistency decidable)**.** WLPO *implies* Π^0_1 *consistency is decidable.* ✓

Proof. Given WLPO and a Π^0_1 sentence φ, define the binary sequence $\alpha(n) = 1$ if $\exists p \leq n$, $\texttt{PrfPA}(p, \neg\varphi)$, else $\alpha(n) = 0$. Then $(\forall n.\ \alpha(n) = 0) \iff \texttt{ConsistentPA}(\varphi)$. WLPO decides whether α is identically zero. □

Theorem 6.4 (Calibration chain)**.**

$$\text{WLPO} \iff \Pi^0_1 \textit{ consistency decidable} \iff \textit{gap detection decidable.} \quad ✓$$

Proof. The first equivalence is WLPO $\Rightarrow$ (Theorem 6.3) and the reverse (axiomatized: given a binary sequence α, construct the Π^0_1 sentence "$\forall n.\ \alpha(n) = 0$" and decide its consistency). The second equivalence follows directly from the gap detection theorem (Theorem 5.5). □

Module	Content	Lines
`Basic.lean`	WLPO, ℓ^∞, c_0, ℓ^∞/c_0	110
`ArithmeticSide.lean`	Axiomatized PA, Lindenbaum algebra	291
`AnalyticSide.lean`	Rows, `rowChar`, logical gap sublattice	131
`GodelSequence.lean`	Gödel sequence v^φ, key properties	147
`Correspondence.lean`	Φ, injectivity, gap detection	220
`CalibrationLink.lean`	WLPO $\iff$ gap detection	137
`Main.lean`	Aggregator and axiom audit	177
Total		**1,213**

Table 1: Module structure of the Lean 4 formalization.

7 Formalization

7.1 Module Structure

7.2 Axiom Profile

The formalization uses 30 custom axioms, all encoding standard metamathematical facts about PA:

- **Type axioms (4):** `SentencePA`, `godelNum`, `PrfPA`, `botPA`.
- **Function axioms (4):** `negPA`, `andPA`, `orPA`, `implPA`.
- **Decidability (2):** `PrfPA_decidable`, `bounded_refutation_decidable`.
- **Structural (2):** `godelNum_injective`, `pa_consistent`.
- Π^0_1 **(4):** `isPi01`, `pi01_bot`, `pi01_and`, `pi01_or`.
- **PAEquiv (7):** reflexivity, symmetry, transitivity, consistency preservation, and congruences for $\neg$, $\wedge$, $\vee$.
- **PAImplies (4):** reflexivity, transitivity, equivalence-to-mutual-implication, congruence.
- **Refutability (1):** `refutable_equiv_bot`.
- **Order preservation (1):** `paImplies_preserves_consistency`.
- **Calibration (1):** `pi01_decidable_implies_wlpo`.

All custom axioms would be theorems given a formalization of Gödel numbering. The construction works for any consistent, recursively axiomatized theory.

Remark 7.1. The axiom `pi01_decidable_implies_wlpo` encodes the reverse direction of the calibration: given a binary sequence α, construct the Π^0_1 sentence "$\forall n.\ \alpha(n) = 0$." This requires that primitive recursive predicates are representable in PA—a standard result (Gödel, 1931, §§5–6) but one that goes beyond the scope of this formalization.

7.3 Sorries

Three technical sorries remain:

1. `classGN_injective`: different equivalence classes have different canonical Gödel numbers. (Requires showing the `Nat.find` witnesses for the two classes agree.)

2. `rowChar_neq_mod_c0`: characteristic functions of different rows are not c_0-equivalent. (The difference is ± 1 on two disjoint infinite sets.)

3. `godelGapMap_injective` (both-consistent case): uses sorry 2 above.

These are all mathematically straightforward; the first two are essentially facts about well-orderings and indicator functions of disjoint infinite sets.

7.4 Standard Lean Axioms

The standard LEAN 4 axioms `propext`, `Classical.choice`, and `Quot.sound` appear throughout. `Classical.choice` enters via quotient operations, `Nat.find`, and MATHLIB4's filter/metric infrastructure. This is consistent with the project's use of classical metatheory.

8 CRM Calibration Summary

Result	Principle	Status	Paper
Bidual gap $\neq 0$	WLPO	Calibrated	2
Gap detection decidable	WLPO	Calibrated	26
Π^0_1 consistency decidable	WLPO	Calibrated	26
$\Pi^0_1/\sim_{\mathrm{PA}}$ embeds into ℓ^∞/c_0	(structural)	Verified	26

Table 2: CRM calibration results for the bidual gap and arithmetic consistency.

Gap detection is WLPO-complete, independently confirmed by two unrelated proofs (functional-analytic in Paper 2, arithmetic in this paper). The logical cost of detecting the bidual gap is exactly the cost of deciding Π^0_1 consistency.

9 Independence, Robustness, and the Constructive Complexity Classification

9.1 Two Proofs, One Result

Paper 2 and this paper independently prove that gap detection is WLPO-complete. Paper 2's proof is functional-analytic: it extracts an Ishihara kernel from a gap witness. This paper's proof is arithmetic: it reduces Π^0_1 consistency to gap membership via Gödel sequences. The methods share no common machinery. The shared conclusion is therefore robust—WLPO-completeness of gap detection is not an artifact of either proof technique.

9.2 Why Two Proofs Matter

In complexity theory, a single proof that a problem is NP-complete might depend on features of the specific reduction. Multiple independent reductions from different NP-complete problems provide stronger evidence that the classification is natural. Similarly, two independent proofs of WLPO-completeness from different mathematical domains (functional analysis and arithmetic) provide stronger evidence that gap detection is genuinely WLPO-complete, not just "provably equivalent to WLPO by a clever trick."

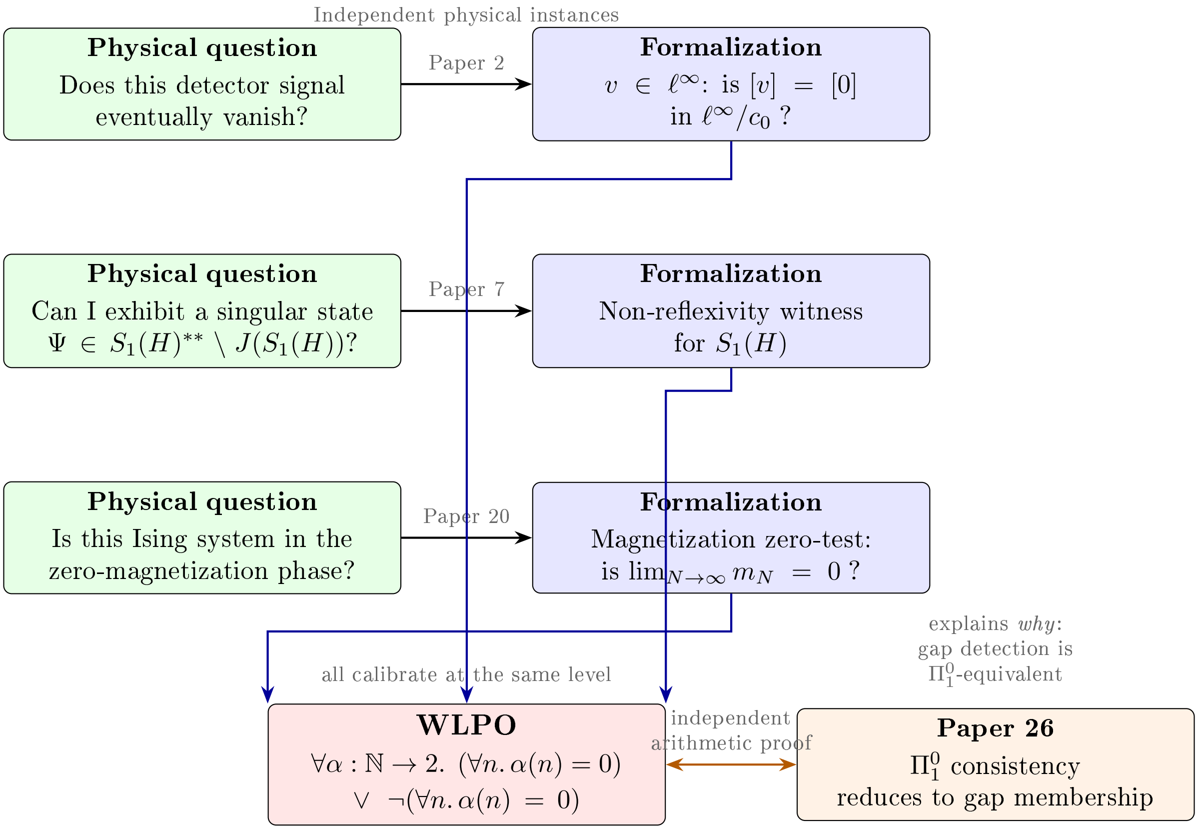

Figure 3: Multiple independent physical questions (left) formalize as mathematical problems (right) that all calibrate at WLPO. Paper 26 provides an arithmetic explanation: gap detection is WLPO-complete because ℓ^∞/c_0 admits a reduction from Π^0_1 consistency. The physical questions are real; the bidual gap is the mathematical obstruction that explains their shared logical cost; Paper 26's reduction explains why this obstruction sits at the Π^0_1 level of the arithmetic hierarchy.

9.3 The Reduction as Complexity-Class Identification

The Gödel sequence construction reduces Π^0_1 consistency to gap membership. Combined with the known WLPO $\iff$ Π^0_1 equivalence, this places gap detection in the Π^0_1 decidability class. Every WLPO-calibrated problem in the CRM programme (bidual gap, singular state witnessing, phase classification, etc.) is therefore Π^0_1-equivalent—it is exactly as hard as deciding consistency of arithmetic sentences. This gives the WLPO level of the calibration table an arithmetic characterization.

9.4 Relation to Cubitt et al.

Cubitt–Perez-Garcia–Wolf reduced the halting problem (Σ^0_1-complete) to spectral gap existence. That sits above LPO in the constructive hierarchy. This paper reduces Π^0_1 consistency (WLPO-complete) to gap membership in ℓ^∞/c_0. Both are reduction results placing analytic problems in arithmetic complexity classes, at different levels of the hierarchy.

9.5 Limitations

1. ℓ^∞/c_0 is not physical—it is a mathematical abstraction.
2. The Gödel sequences are engineered from the proof predicate.

3. The paper establishes a complexity-class result, not a physical result.
4. The order-embedding is not a lattice isomorphism—honestly downgraded from earlier claims.
5. The 30 PA metamathematics axioms are standard but would require significant formalization effort to eliminate.

9.6 What the Two Proofs Together Suggest

Paper 2 showed that gap detection is hard for functional-analytic reasons (dual space structure forces WLPO). This paper shows that gap detection is hard for arithmetic reasons (Π^0_1 sentences embed into the gap). That both routes yield WLPO suggests that WLPO-completeness captures something intrinsic about the gap, not something imposed by a particular proof strategy. Whether this intrinsic character can be made precise (e.g., via a completeness theorem for WLPO reductions) is an open question.

10 AI-Assisted Methodology

The formalization was developed with AI assistance (Claude, Anthropic). The division of labor:

- **Human:** mathematical conception, proof strategy, axiom design, verification of correctness.
- **AI:** Lean 4 code generation, debugging, iterative build-fix cycles.

All mathematical content was designed by the human author. The AI served as a programming assistant for the Lean 4 implementation.

11 Conclusion

We have given an arithmetic proof, independent of Paper 2 [Lee, 2026a], that bidual gap detection is WLPO-complete, via explicit reduction from Π^0_1 consistency through Gödel sequences, formalized in Lean 4.

The Gödel sequence construction embeds the Lindenbaum algebra of Π^0_1 sentences into ℓ^∞/c_0, with $\Phi([\varphi]) = [0]$ if and only if φ is refutable. Combined with the known WLPO $\iff$ Π^0_1 equivalence, this yields the calibration chain WLPO $\iff$ Π^0_1 consistency decidable $\iff$ gap detection decidable by a route that shares no machinery with Paper 2's functional-analytic proof.

That two independent methods—one from functional analysis, one from arithmetic—both establish WLPO-completeness of gap detection is evidence that this classification is robust: it captures something intrinsic about the gap, not something imposed by a particular proof strategy.

Data Availability

The Lean 4 source code is available at:

`https://doi.org/10.5281/zenodo.XXXXXXX`

Verified with `leanprover/lean4:v4.28.0-rc1` and current Mathlib4.

A Selected Lean Code

A.1 The Gödel Sequence

```
noncomputable def godelSeq (phi : SentencePA) : N -> R :=
  fun k =>
    if k.unpair.1 = godelNum phi &&
       forall p, p <= k -> not (PrfPA p (negPA phi))
    then 1
    else 0
```

Listing 1: The Gödel sequence (GodelSequence.lean).

A.2 Refutable Implies Null

```
theorem godelSeq_refutable_eventually_zero (phi : SentencePA)
    (href : RefutablePA phi) :
    exists N, forall k >= N, godelSeq phi k = 0 := by
  obtain <<p0, hp0>> := href
  exact <<p0, fun k hk => by
    simp only [godelSeq]
    rw [if_neg]
    push_neg
    intro _hrow
    exact <<p0, hk, hp0>>>>
```

Listing 2: Refutable sentences produce null sequences (GodelSequence.lean).

A.3 Gap Detection Theorem

```
theorem godelGap_detects_refutability (phi : Pi01) :
    godelGapMap (LindenbaumPi01.mk phi) = BidualGap.zero
    <-> RefutablePA phi.val := by
  constructor
  . intro h
    by_contra hcon
    exact godelGapMap_consistent_ne_zero phi hcon h
  . exact godelGapMap_refutable phi
```

Listing 3: Gap detection (Correspondence.lean).

A.4 Calibration Chain

```
theorem wlpo_iff_gap_detection :
    WLPO <-> GapDetectionDecidable :=
  wlpo_iff_pi01_decidable.trans pi01_decidable_iff_gap_detection
```

Listing 4: WLPO iff gap detection (CalibrationLink.lean).

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