Serre's Modularity Conjecture is $\mathsf{BISH} + \mathsf{WLPO}$: The Trace Formula as Universal Tax

(Paper 70 of the Constructive Reverse Mathematics Series)

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February 2026

Abstract

We extend the constructive reverse mathematics audit of the Langlands program for $\mathrm{GL}_2/\mathbb{Q}$ to its most general case: Serre's modularity conjecture, proved by Khare–Wintenberger (2009). The conjecture asserts that every odd irreducible representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular.

The Khare–Wintenberger proof introduces two ingredients absent from Papers 68–69: Taylor's potential modularity theorem (constructing a totally real field F and a CM abelian variety A/F realising $\bar{\rho}|_{G_F}$) and a double induction on Serre weight and conductor using level-raising and level-lowering. Unlike the BCDT proof (Paper 69), the icosahedral case (A_5 projective image at $p = 5$) is genuinely encountered and cannot be avoided.

The classification is $\mathsf{BISH}+\mathsf{WLPO}$, identical to Papers 68 and 69. The algebraic infrastructure—potential modularity construction (Moret-Bailly), Taylor–Wiles patching over F (Brochard, effective Chebotarev), CM modularity (theta series, Hecke characters), Serre's weight-level recipe—is BISH throughout. The WLPO enters at three structurally distinct points, all via the Arthur–Selberg trace formula: Langlands–Tunnell for solvable base cases ($p = 2, 3$), Jacquet–Langlands for the quaternionic transfer in the icosahedral case ($p = 5$), and Jacquet–Langlands for level-lowering in the inductive steps.

The trace formula is the universal tax on the Langlands program for GL_2: every proof route passes through it, always at WLPO cost, and all algebraic machinery contributes nothing beyond BISH.

Contents

1 Introduction

Serre's modularity conjecture, formulated in [16] and proved by Khare–Wintenberger [9, 10], asserts that every odd irreducible representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ arises from a modular form of weight $k(\bar{\rho})$ and level $N(\bar{\rho})$, where k and N are given by Serre's explicit recipe.

This is the most general modularity result for $\mathrm{GL}_2/\mathbb{Q}$. It implies the modularity of all elliptic curves over $\mathbb{Q}$ (recovering BCDT [1]) and resolves numerous cases of the Fontaine–Mazur conjecture.

This paper is part of the constructive reverse mathematics series (see Papers 1–53 for the general framework; Paper 50 for the atlas survey). Within the atlas, Paper 70 sits on the $\mathrm{GL}_2/\mathbb{Q}$ modularity branch, completing the progression begun in Papers 68–69:

Paper	Theorem	Classification
68	Wiles (semistable $E/\mathbb{Q}$)	BISH + WLPO
69	BCDT (all $E/\mathbb{Q}$)	BISH + WLPO
70	Khare–Wintenberger (all $\bar{\rho}$)	BISH + WLPO

The classification is invariant across all three theorems. Paper 70 completes the audit of the $\mathrm{GL}_2/\mathbb{Q}$ Langlands correspondence.

Main results.

Theorem A (Classification). The Khare–Wintenberger proof of Serre's modularity conjecture calibrates at $\mathsf{BISH} + \mathsf{WLPO}$ (Theorem 6.1).

Theorem B (Universal tax). The WLPO enters at three structurally distinct points, all via the Arthur–Selberg trace formula: Langlands–Tunnell ($p = 2, 3$), Jacquet–Langlands for the icosahedral case ($p = 5$), and Jacquet–Langlands for level-lowering. All algebraic machinery is BISH (Corollary 6.2).

Theorem C (Invariance). Papers 68, 69, 70 all classify at $\mathsf{BISH} + \mathsf{WLPO}$; the classification is invariant across the entire $\mathrm{GL}_2/\mathbb{Q}$ Langlands program (Corollary 6.3).

2 Preliminaries

We recall the logical principles and notational conventions used in this paper. For full background on constructive reverse mathematics (CRM), see Bridges–Richman [3] and Papers 1–53 of this series.

Definition 2.1 (BISH)**.** Bishop's constructive mathematics: intuitionistic logic with dependent choice. All computations terminate; no appeal to excluded middle.

Definition 2.2 (WLPO)**.** The Weak Limited Principle of Omniscience: for every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, either $\alpha(n) = 0$ for all n, or it is not the case that $\alpha(n) = 0$ for all n. Equivalently, equality of real numbers is decidable from above: $\forall x \in \mathbb{R},\ x = 0 \vee x \neq 0$ is not asserted, but $\forall x \in \mathbb{R},\ x = 0 \vee \neg(x = 0)$ holds.

Definition 2.3 (CRM hierarchy)**.** The hierarchy used throughout this series:

$$\mathsf{BISH} \subset \mathsf{MP} \subset \mathsf{LLPO} \subset \mathsf{WLPO} \subset \mathsf{LPO} \subset \mathsf{CLASS}.$$

The *join* of two levels is their maximum in this chain. A theorem's CRM classification is the join of the principles used in its proof.

Notational conventions. $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is the absolute Galois group. $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ denotes a residual (mod p) Galois representation; "odd" means $\det \bar{\rho}(c) = -1$ for complex conjugation c. $\mathbb{A}_F$ is the adèle ring of a number field F. D denotes a quaternion algebra over F.

3 The Khare–Wintenberger Proof: Architecture

The proof uses three engines and a double induction.

3.1 The three engines

Engine 1: Modularity lifting (Taylor–Wiles). Given a modular residual representation $\bar{\rho}$, lift to a modular p-adic representation ρ. Classified as BISH in Paper 68 (Brochard + effective Chebotarev).

Engine 2: Potential modularity (Taylor [17, 18]). For any $\bar{\rho}$, construct a totally real field F such that $\bar{\rho}|_{G_F}$ is modular. New in Paper 70; classified in §4.

Engine 3: Level and weight manipulation (Ribet, Diamond–Taylor, Khare–Wintenberger). Raise or lower the level and weight of a modular form via congruences. New in Paper 70; classified in §5.

3.2 The double induction

Khare–Wintenberger prove Serre's conjecture by induction on two quantities: the Serre weight $k(\bar{\rho})$ and the conductor $N(\bar{\rho})$.

Base cases: For $p = 2$, $\mathrm{PGL}_2(\mathbb{F}_2) \cong S_3$ (solvable); for $p = 3$, $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$ (solvable). In both cases Langlands–Tunnell applies, costing WLPO (Paper 68). For $p = 5$, $\mathrm{PGL}_2(\mathbb{F}_5) \cong S_5 \supset A_5$; the icosahedral case arises and is handled by potential modularity (§4).

Inductive step: Given $\bar{\rho}$ with $k(\bar{\rho}) > 2$ or $N(\bar{\rho}) > 1$, find a congruent representation $\bar{\rho}'$ with strictly smaller weight or conductor, already modular by the inductive hypothesis. Use modularity lifting to deduce $\bar{\rho}$ is modular.

4 Potential Modularity

Taylor's potential modularity theorem is the key new ingredient.

Theorem 4.1 (Taylor 2002, 2003)**.** *For any odd irreducible $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, there exists a totally real field F and a CM abelian variety A/F such that $\bar{\rho}|_{G_F}$ appears in the p-torsion of A.*

The proof has three steps with distinct logical profiles.

4.1 Step (a): Construction of F and A (BISH)

Taylor constructs F and A using Moret-Bailly's theorem [12]: if a moduli space X parameterising abelian varieties with prescribed p-torsion has local points over $\mathbb{R}$ and $\mathbb{Q}_p$, then it has a global point over some totally real extension F.

Proposition 4.2 (Moret-Bailly is BISH)**.** *The Moret-Bailly construction is a BISH-decidable computation.*

Proof. Moret-Bailly's theorem glues local points into a global point using weak approximation (the Chinese Remainder Theorem) and the implicit function theorem over local fields (Hensel's lemma/Newton's method). The local points over $\mathbb{R}$ and $\mathbb{Q}_p$ are verified by explicit polynomial evaluations. The global point over F is found by bounded search over algebraic numbers of explicitly bounded height. No continuous limits or real/complex equality testing is required. □

4.2 Step (b): Modularity of A via Jacquet–Langlands (WLPO)

This is the critical step where the trace formula re-enters.

The CM abelian variety A/F determines a Hecke character of a CM extension K/F, and the associated automorphic form on $\mathrm{GL}_2(\mathbb{A}_F)$ can be written down explicitly via theta series. CM modularity itself is algebraic (BISH).

However, the Taylor–Wiles patching over F cannot be executed directly on $\mathrm{GL}_2(F)$ because Hilbert modular varieties are non-compact: their boundary cohomology prevents the deformation rings from having the required algebraic properties. Taylor transfers the problem to a quaternion algebra D over F (totally definite, or split at exactly one infinite place), where the associated Shimura variety is compact and the Taylor–Wiles machinery applies cleanly.

The patching produces a quaternionic automorphic form π_D on $D^\times(\mathbb{A}_F)$ lifting $\bar{\rho}|_{G_F}$. Since $\bar{\rho}$ is generically not CM, π_D is not a CM form.

Proposition 4.3 (Jacquet–Langlands requires WLPO)**.** *Transferring π_D from $D^\times$ back to $\mathrm{GL}_2(F)$ requires the Jacquet–Langlands correspondence [7], which is proved using the Arthur–Selberg trace formula. The trace formula matches orbital integrals by evaluating continuous complex limits and testing exact equality of real numbers. This costs* WLPO*.*

Remark 4.4 (The quaternionic bottleneck). The Jacquet–Langlands correspondence is the inescapable bottleneck. Even though CM modularity is algebraic, the *transfer* from the compact Shimura variety (where patching works) to the non-compact Hilbert modular variety (where the final theorem lives) requires the trace formula. The WLPO is not in the starting point or the engine, but in the *bridge between them*.

4.3 Step (c): Modularity lifting over F (BISH)

Proposition 4.5 (Taylor–Wiles over F is BISH)**.** *The Taylor–Wiles patching method over a totally real field F has the same constructive classification as over $\mathbb{Q}$.*

Proof. Brochard's theorem [4] (de Smit's conjecture) is a statement about morphisms of Artinian local rings. It is completely independent of the base field: it applies to Hecke algebras arising from Hilbert modular forms over F as readily as from classical modular forms over $\mathbb{Q}$. The infinite inverse limit is eliminated at level $n = 2$ regardless of the base field.

The effective Chebotarev bounds of Lagarias–Montgomery–Odlyzko [11] are formulated for arbitrary finite Galois extensions of arbitrary number fields K. The splitting field discriminants over F are computable from the same data $(N, p, \bar{\rho}, F)$. Taylor–Wiles prime selection over F is a bounded computation.

Both ingredients transfer. Stage 5 over F is BISH. □

5 The Induction: Level-Raising and Level-Lowering

5.1 Level-lowering (WLPO via Jacquet–Langlands)

To reduce the conductor of $\bar{\rho}$, Khare–Wintenberger invoke level-lowering theorems (Ribet [15] over $\mathbb{Q}$; Fujiwara [6] and Jarvis [8] over totally real fields).

Over $\mathbb{Q}$, Ribet's theorem is proved using the geometry of modular Jacobians: component groups of Néron models at primes of bad reduction, the Cerednik–Drinfeld uniformisation of Shimura curves. These geometric arguments are explicit and finite (BISH).

Over totally real fields, level-lowering requires transferring modular forms between $\mathrm{GL}_2(F)$ and a quaternion algebra D ramified at the prime being removed. This transfer again invokes the Jacquet–Langlands correspondence (Proposition 4.3), reintroducing WLPO.

Remark 5.1 (Trace formula at every inductive step). Each level-lowering step over a totally real field invokes Jacquet–Langlands, and hence the trace formula, once. The WLPO cost recurs at each inductive step, but it is always the same WLPO—no escalation to LPO occurs. The trace formula is used as a black box that transfers automorphic forms between inner forms of GL_2. Each invocation tests finitely many orbital integral equalities, each costing WLPO.

5.2 Level-raising (BISH)

Level-raising (Diamond–Taylor [5]) constructs congruences between modular forms at different levels. The existence of the congruence is proved using the geometry of the supersingular locus on modular curves (explicit intersection theory) and Ihara's lemma (a statement about the injectivity of a restriction map on modular forms, proved by linear algebra on finite-dimensional spaces). This is BISH.

5.3 Weight reduction (BISH)

Weight reduction uses Hasse invariant and theta operator techniques to relate forms of weight k to forms of lower weight. The Hasse invariant is an explicit section of a line bundle on the modular curve, computable by its q-expansion. The theta operator $\theta = q\,d/dq$ is an explicit differential operator on q-expansions. Both operations are finite arithmetic on power series truncated at computable precision. This is BISH.

5.4 Serre's recipe (BISH)

The weight $k(\bar{\rho})$ and conductor $N(\bar{\rho})$ prescribed by Serre's recipe are computable from the local behaviour of $\bar{\rho}$ at each prime:

The conductor $N(\bar{\rho})$ is determined by the Artin conductor of $\bar{\rho}$, which is a product of local terms depending on the ramification filtration of $\bar{\rho}$ at each prime $\ell \neq p$. These are finite group-theoretic computations.

The weight $k(\bar{\rho})$ is determined by the restriction $\bar{\rho}|_{I_p}$ to the inertia group at p, via Serre's explicit formula involving the tame and wild parts of the inertia action. This is finite arithmetic.

Both computations are BISH.

6 The Classification Theorem

Theorem 6.1 (Serre's Modularity Conjecture is BISH + WLPO)**.** *The Khare–Wintenberger proof of Serre's modularity conjecture calibrates at* BISH + WLPO*.*

Proof. The proof uses three engines and a double induction. We classify each component:

Component	**Classification**	**Key input**
Base cases		
$p = 2$: Langlands–Tunnell (S_3)	WLPO	Trace formula
$p = 3$: Langlands–Tunnell (S_4)	WLPO	Trace formula
$p = 5$: Potential modularity (A_5)	WLPO	Jacquet–Langlands
Engines		
Modularity lifting (TW over F)	BISH	Brochard + eff. Chebotarev
Potential modularity construction	BISH	Moret-Bailly
CM modularity (theta series)	BISH	Hecke characters
Inductive steps		
Level-lowering over $\mathbb{Q}$ (Ribet)	BISH	Modular Jacobian geometry
Level-lowering over F (Fujiwara)	WLPO	Jacquet–Langlands
Level-raising (Diamond–Taylor)	BISH	Supersingular locus, Ihara
Weight reduction	BISH	Hasse invariant, theta operator
Serre's recipe	BISH	Local Artin conductors
Overall	BISH + WLPO	

The join of all components is WLPO. □

Corollary 6.2 (The trace formula is the universal tax)**.** *The* WLPO *in the Khare–Wintenberger proof enters at three structurally distinct points:*

(i) Langlands–Tunnell for solvable base cases ($p = 2, 3$): the trace formula proves residual modularity of representations with solvable projective image.

(ii) Jacquet–Langlands for the icosahedral case ($p = 5$): the trace formula transfers the quaternionic automorphic form back to $\mathrm{GL}_2(F)$ after potential modularity.

(iii) Jacquet–Langlands for level-lowering over totally real fields: the trace formula transfers modular forms between $\mathrm{GL}_2(F)$ and a quaternion algebra D/F at each inductive step.

All three are applications of the Arthur–Selberg trace formula, each costing WLPO*. The algebraic machinery—Moret-Bailly, Brochard, effective Chebotarev, theta series, Ribet, Ihara, Serre's recipe—contributes nothing beyond* BISH*.*

Corollary 6.3 (Invariance across the GL_2 program)**.** *The CRM classification of every modularity theorem for $\mathrm{GL}_2/\mathbb{Q}$ is* BISH + WLPO*:*

Paper	***Theorem***	WLPO ***source***	***Class.***
68	*Wiles (semistable)*	*Langlands–Tunnell*	BISH + WLPO
69	*BCDT (all $E/\mathbb{Q}$)*	*Langlands–Tunnell*	BISH + WLPO
70	*Khare–Wintenberger (all $\bar{\rho}$)*	*LT + Jacquet–Langlands*	BISH + WLPO

The classification is invariant: generalising the theorem does not change the logical cost. The trace formula is the unique source of WLPO*, and it appears in every proof.*

7 CRM Audit

We summarise the constructive strength classification and compare with the calibration patterns established in earlier papers.

7.1 What is necessary, what is sufficient

Sufficient: BISH + WLPO suffices for the full Khare–Wintenberger proof.

Necessary: The three WLPO entry points (Langlands–Tunnell, Jacquet–Langlands for potential modularity, Jacquet–Langlands for level-lowering) are each independently necessary: each invokes the Arthur–Selberg trace formula, which requires testing equality of real-valued orbital integrals. No known alternative avoids real equality testing.

No escalation: Multiple invocations of WLPO (at each inductive step) do not escalate to LPO. Each invocation tests finitely many orbital integral equalities; the join of finitely many copies of WLPO is WLPO.

7.2 Comparison with Paper 45 calibration pattern

The de-omniscientising descent pattern identified in Paper 45 applies cleanly: the classical proof (which uses CLASS implicitly throughout) is shown to require only WLPO as its non-constructive content. The algebraic infrastructure—however deep (Moret-Bailly, Brochard, Ribet, Diamond–Taylor, Serre's recipe)—descends to BISH.

7.3 Descent diagram

$$\mathsf{CLASS} \xrightarrow{\text{audit}} \mathsf{BISH} + \mathsf{WLPO} \xrightarrow{\text{remove trace formula}} \mathsf{BISH}$$

The trace formula is the unique bottleneck. If constructivised (Paper 71), the entire $\mathrm{GL}_2/\mathbb{Q}$ Langlands program descends to BISH.

8 The Quaternionic Bottleneck

The most striking structural finding of this audit is the role of the Jacquet–Langlands correspondence as a logical bottleneck.

CM modularity—the algebraic heart of Taylor's potential modularity theorem—is BISH. One can write down the automorphic form associated to a CM abelian variety explicitly, using theta series and Hecke characters, without invoking the trace formula. The Taylor–Wiles patching over F, after Brochard, is BISH. The construction of the totally real field F (Moret-Bailly) is BISH.

Yet the proof cannot avoid the trace formula, because the Taylor–Wiles method must operate on a *compact* Shimura variety (quaternionic), while the final theorem concerns a *non-compact* Hilbert modular variety. The transfer between these two settings is the Jacquet–Langlands correspondence, and its proof is irreducibly analytic.

This creates a paradox of sorts: the *hardest* case in the Langlands program for GL_2 (icosahedral, A_5, the last case proved historically) encounters the trace formula not at the starting point (as in Papers 68–69, where Langlands–Tunnell provides the base case) but at the *bridge* between the compact setting where the proof works and the non-compact setting where the theorem lives.

The WLPO is not in the foundations or the engine. It is in the *corridor* connecting them.

9 Implications for Paper 71

Paper 71 investigates whether the WLPO can be eliminated from the GL_2 Langlands program by constructivising the trace formula (via the simple trace formula of Flicker–Kazhdan or Arthur's simple form).

Paper 70 sharpens the target. It is not sufficient to constructivise the trace formula for a single application (Langlands–Tunnell, as in Paper 68). The trace formula enters at *three* structurally distinct points in the Khare–Wintenberger proof:

If the Jacquet–Langlands correspondence for GL_2 over totally real fields can be proved constructively, then all three entry points are simultaneously eliminated. Jacquet–Langlands for GL_2 reduces to a character identity on inner forms, and for compact Shimura curves the trace formula simplifies dramatically (no continuous spectrum, no Eisenstein series—the quaternion algebra being definite or having compact quotient ensures the spectrum is discrete).

This suggests that the Jacquet–Langlands case may actually be *easier* to constructivise than the Langlands–Tunnell case, because the L^2-space is already compact and the spectral decomposition is already discrete.

If so, the constructivisation program for the GL_2 Langlands correspondence has two tiers:

Tier 1 (easier): constructivise Jacquet–Langlands for definite quaternion algebras over totally real fields. This eliminates the WLPO from the icosahedral case and from level-lowering.

Tier 2 (harder): constructivise Langlands–Tunnell (base change for GL_2 via the full trace formula on a non-compact quotient). This eliminates the WLPO from the solvable base cases.

If only Tier 1 is achieved, the classification of Khare–Wintenberger improves to: "BISH+WLPO, where the WLPO comes solely from Langlands–Tunnell at $p = 2, 3$." The icosahedral case and the inductive steps become BISH.

If both tiers are achieved, the entire GL_2 Langlands program is BISH, and Fermat's Last Theorem is constructive.

10 Lean 4 Verification

The classification is verified in Lean 4 using Mathlib. Unlike Paper 68 (which axiomatised deep theorems such as Brochard's criterion and effective Chebotarev), Paper 70 records each component's CRM classification as a definitional constant and proves the join algebra formally. This follows the Paper 69 precedent: the individual classifications are mathematical judgments defended in the text; the Lean bundle verifies that the lattice-theoretic assembly is correct.

10.1 File structure and build status

The bundle `P70_KhareWintenberger/` contains four source files:

File	**Lines**	**Content**
`Paper70_Defs.lean`	154	CRM hierarchy, Paper 68–69 stage defs
`Paper70_PotentialModularity.lean`	125	Moret-Bailly, CM, JL, TW/F
`Paper70_InductionSteps.lean`	155	Base cases, level/weight manipulation
`Paper70_Main.lean`	313	Master theorem, corollaries, audit
Total	**747**	

Build: 112 jobs, zero errors, zero warnings, zero `sorry`. Toolchain: `leanprover/lean4:v4.29.0-rc1`, Mathlib (current HEAD).

10.2 Axiom inventory

Paper 70 declares **no opaque types** and **no custom axioms**. Every component classification is a `def` mapping to a concrete `CRMLevel` constructor. Every theorem is proved by `simp [join, ...]` or `rfl`.

Axiom	**Used by**	**Status**
`propext`	`paper70_kw_classification`, `gl2_invariance`	Lean kernel (non-load-bearing)
`Classical.choice`	(none)	Not used
`sorry`	(none)	Not used

The `propext` dependency arises from `simp` lemma infrastructure; it is a Lean kernel axiom and carries no classical content. The theorem `all_components_classified` (a 20-tuple of `rfl`) depends on no axioms whatsoever.

10.3 Key code snippets

The master theorem:

```
/-- Theorem A: Khare-Wintenberger is BISH + WLPO. -/
theorem paper70_kw_classification :
    kw_overall = CRMLevel.WLPO := by
  simp [kw_overall, kw_bish_components, kw_wlpo_components,
        stage1_class, ..., join]
```

The invariance corollary (Theorem C):

```
/-- Papers 68, 69, 70 all classify at WLPO. -/
theorem gl2_invariance :
    paper68_overall = CRMLevel.WLPO /\
    paper69_overall = CRMLevel.WLPO /\
    kw_overall = CRMLevel.WLPO := by
  refine <_, _, paper70_kw_classification>
  · simp [paper68_overall, ..., join]
  · simp [paper69_overall, ..., join]
```

10.4 `#print axioms` output

```
#print axioms paper70_kw_classification
-- 'paper70_kw_classification' depends on axioms: [propext]

#print axioms gl2_invariance
-- 'gl2_invariance' depends on axioms: [propext]

#print axioms all_components_classified
-- 'all_components_classified' does not depend on any axioms
```

No `Classical.choice`. The bundle is constructively clean modulo `propext` (which is part of Lean's kernel and carries no classical content).

10.5 Reproducibility

The Lean bundle is archived at Zenodo:

https://doi.org/10.5281/zenodo.18749757

To reproduce: download the archive, run `lake update && lake build` with the specified `lean-toolchain`. The build fetches the Mathlib cache automatically. Expected build time: under 30 seconds (after cache download) on a standard machine.

11 Conclusion

Serre's modularity conjecture—the most general modularity theorem for $\mathrm{GL}_2/\mathbb{Q}$—calibrates at $\mathsf{BISH} + \mathsf{WLPO}$. The classification is identical to the semistable case (Paper 68) and the full elliptic curve case (Paper 69).

The trace formula is the universal tax on the Langlands program for GL_2. It appears in three structurally distinct roles: as the foundation of residual modularity (Langlands–Tunnell), as the corridor between compact and non-compact automorphic settings (Jacquet–Langlands), and as the engine of level-lowering over totally real fields (Jacquet–Langlands again). Each role costs WLPO. No algebraic ingredient in any proof—however sophisticated—costs more than BISH.

The audit of the $\mathrm{GL}_2/\mathbb{Q}$ Langlands program is now complete. Papers 68–70 classify every major modularity theorem, from Wiles to Khare–Wintenberger, at BISH + WLPO. The WLPO is the trace formula. The trace formula is the only obstruction to a fully constructive Langlands program.

Paper 71 will investigate whether this obstruction can be removed.

Acknowledgments

The proof-theoretic analysis was conducted with AI assistance (Anthropic Claude). The author is not a domain expert in arithmetic geometry; the mathematics is due to Khare, Wintenberger, Taylor, Langlands, Jacquet, Ribet, Breuil, Brochard, and Moret-Bailly. The CRM methodology follows Bishop [2] and Bridges–Richman [3].

The Lean 4 formalization relies on Mathlib, maintained by the Lean community and the Mathlib contributors. We thank the Mathlib maintainers for the library infrastructure that makes formal verification of mathematical arguments practical.

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