Wang Tiling and the Origin of Physical Undecidability

Berger's Theorem Is LPO: Every Physical Undecidability Result Inherits Exactly One Omniscience Principle

A Lean 4 Formalization (Paper 38)

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Abstract

Every undecidability result in quantum many-body physics descends from a single ancestor: the undecidability of the Wang tiling problem (Berger 1966). We prove that this ancestor is Turing–Weihrauch equivalent to LPO. We establish: (1) the Wang tiling decision is LPO; (2) aperiodicity detection is LPO; (3) every known descendant—from Kanter (1990) through Cubitt (2015) to the present—inherits exactly LPO; (4) the Σ_1^0 ceiling: no Σ_1^0-complete reduction can exceed LPO. The entire formalization (573 lines of Lean 4/Mathlib4) compiles with zero `sorry`, zero warnings.

1 Introduction

Paper 36 proved Cubitt's spectral gap undecidability $\equiv$ LPO. Paper 37 proved a meta-theorem: any halting reduction $\equiv$ LPO. This paper goes to the source. For the complete calibration table positioning this result in the broader landscape of decidable and undecidable physics, see Paper 10 [12]; for the historical context, see Paper 12 [13].

The Wang tiling problem—given a finite set of colored unit squares, can they tile the infinite plane with matching edges?—was shown undecidable by Berger in 1966 [1]. (While Berger's original set required 20,426 tiles and Robinson [2] reduced this to 56, Jeandel and Rao [11] have shown that 11 tiles suffice—the smallest possible aperiodic set.) Every subsequent undecidability result in mathematical physics ultimately descends from this: Cubitt's construction encodes Turing machine computation into a Hamiltonian via aperiodic tilings; the Hamiltonian's ground state *is* a tiling.

We prove that the Wang tiling problem is Turing–Weihrauch equivalent to LPO. Since every descendant inherits its undecidability from Wang tiling, the entire genealogy of physical undecidability lives at LPO.

2 Wang Tiles

Definition 2.1. A *Wang tile* is a unit square with colored edges (top, bottom, left, right). A *Wang tileset* W is a finite set of such tiles. A *tiling* of $\mathbb{Z}^2$ by W assigns a tile from W to each cell such that adjacent tiles share colors on common edges.

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Definition 2.2. W *tiles the plane* if there exists a valid tiling of $\mathbb{Z}^2$. A tileset is *aperiodic* if it tiles the plane but every valid tiling is non-periodic.

The Berger–Robinson encoding maps a Turing machine M to a tileset $W(M)$ such that $W(M)$ tiles the plane if and only if M does not halt.

3 Theorem 1: Wang Tiling Decision

Theorem 3.1 (Wang Tiling $\equiv$ LPO)**.** *The Wang tiling problem is Turing–Weihrauch equivalent to* LPO*.*

Forward: tiling decidability $\Rightarrow$ LPO. Let $a : \mathbb{N} \to \{0, 1\}$ be an arbitrary binary sequence. Construct the Turing machine M_a that searches for the first n with $a(n) = 1$ and halts if found. Apply the Berger–Robinson encoding to obtain the tileset $W(M_a)$. By the encoding bridge: $W(M_a)$ tiles the plane if and only if M_a does not halt, which holds if and only if $\forall n.\ a(n) = 0$.

Now apply the tiling decision oracle to $W(M_a)$. *Case 1*: $W(M_a)$ tiles the plane. Then $\forall n.\ a(n) = 0$. *Case 2*: $W(M_a)$ does not tile. Then M_a halts, so $\exists n.\ a(n) = 1$. In either case, $(\forall n.\ a(n) = 0) \vee (\exists n.\ a(n) = 1)$, which is LPO for the sequence a. Since a was arbitrary, this yields LPO. □

Reverse: LPO $\Rightarrow$ *tiling decidability.* Given LPO and an arbitrary tileset W, define the *blocking sequence* $\beta : \mathbb{N} \to \{0, 1\}$ by $\beta(n) = 1$ if no valid $n \times n$ patch of W exists, and $\beta(n) = 0$ otherwise. Each value $\beta(n)$ is BISH-computable: enumerate all $|W|^{n^2}$ tile assignments for the $n \times n$ grid and check edge-matching.

Apply LPO to β: *Case 1*: $\forall n.\ \beta(n) = 0$. Then every finite patch can be tiled. By König's lemma (the compactness bridge axiom `tiling_compactness`), W tiles the plane. *Case 2*: $\exists n.\ \beta(n) = 1$. Then there exists a patch size n with no valid tiling, so W does not tile the plane. □

Remark 3.2. The compactness step (valid patches at all scales $\Rightarrow$ global tiling) is *not* BISH-provable: it is an instance of König's lemma for finitely branching infinite trees, which requires a compactness principle (WKL or equivalent). The tree of partial tilings has branching factor $|W|^{n^2}$, and while each level is finite, extracting the infinite path is a *bridge axiom*—formally, `tiling_compactness` in the Lean formalization. The reverse direction therefore decomposes as: (i) a BISH-computable blocking sequence, (ii) an LPO application, and (iii) a compactness bridge axiom.

```
theorem wang_tiling_iff_lpo :
    (forall (W : WangTileset),
      tiles_plane W ∨ not tiles_plane W) <-> LPO :=
  ⟨tiling_to_lpo, lpo_to_tiling⟩
```

Listing 1: Wang tiling ↔ LPO: `TilingDecision.lean`

4 Theorem 2: Aperiodicity Detection

Theorem 4.1 (Aperiodicity $\equiv$ LPO)**.** *Deciding whether a tileset tiles the plane aperiodically is Turing–Weihrauch equivalent to* LPO*.*

Proof. Define the encoded aperiodicity predicate: for a binary sequence a, set $\mathrm{AperEnc}(a) :\Leftrightarrow \text{tiles\_aperiodically}(W(\mathrm{tm}(a)))$. By the Berger–Robinson bridge, $\mathrm{AperEnc}(a)$ holds if and only if $\mathrm{tm}(a)$ does not halt, which holds if and only if $\forall n.\ a(n) = 0$.

Forward (aperiodicity decidability $\Rightarrow$ LPO): Given a, apply the oracle to decide $\mathrm{AperEnc}(a)$. If $\mathrm{AperEnc}(a)$, then $\forall n.\ a(n) = 0$ by the bridge. If $\neg\mathrm{AperEnc}(a)$, suppose for contradiction that $\forall n.\ a(n) = 0$. Then by the bridge $\mathrm{AperEnc}(a)$, contradicting the hypothesis. So $\exists n.\ a(n) = 1$. Either case gives LPO.

Reverse (LPO $\Rightarrow$ aperiodicity decidability): Apply LPO to a. If $\forall n.\ a(n) = 0$, the bridge gives $\mathrm{AperEnc}(a)$. If $\exists n.\ a(n) = 1$, suppose $\mathrm{AperEnc}(a)$. Then the bridge gives $\forall n.\ a(n) = 0$, contradicting the witness. So $\neg\mathrm{AperEnc}(a)$. ☐

```
theorem aperiodicity_iff_lpo :
    (forall (a : N -> Bool),
      is_aperiodic_encoded a ∨
      not is_aperiodic_encoded a) <-> LPO :=
  ⟨aperiodicity_to_lpo, lpo_to_aperiodicity⟩
```

Listing 2: Aperiodicity $\leftrightarrow$ LPO: `Aperiodicity.lean`

5 Theorem 3: The Genealogy

Theorem 5.1 (The Genealogy Theorem)**.** *Every known undecidability result in mathematical physics factors through a computable many-one reduction from halting via Wang tiling. All are* LPO*-equivalent.*

Proof. Each result in the genealogy (Table 1) is established by a computable many-one reduction from the halting problem. Such a reduction has the form: given Turing machine M, compute (in BISH) a physical instance $I(M)$ such that the target property holds of $I(M)$ if and only if M halts (or does not halt). Since the halting problem is Σ^0_1-complete and LPO $\equiv$ Σ^0_1-LEM (Paper 37), every such reduction is LPO-equivalent.

Concretely: (1) Berger (1966) reduces halting to tiling; (2) Kanter (1990) encodes tile constraints as Potts ground states; (3) GWPN (2009) extends Potts to 2D Ising; (4) CPgW (2015) encodes tiling in Hamiltonian ground states; (5) BCLPG (2020) compresses to 1D via quantum tiling; (6) BCW (2021) reduces halting to phase diagram computation; (7) CLPE (2022) lifts CPgW to renormalization group flows; (8) Watson–Cubitt (2021) reduces halting to ground state energy density.

Each entry is Σ^0_1-complete and LPO-equivalent. The Lean formalization encodes all eight entries as a list of `GenealogyEntry` records and verifies the classification by `decide` (kernel computation). ☐

Result	**Year**	**Ancestor**	**LPO?**
Wang Tiling (Berger)	1966	Halting	✓
Potts Model (Kanter) [7]	1990	Tiling	✓
2D Ising (GWPN) [8]	2009	Potts	✓
Spectral Gap 2D (CPgW)	2015	Tiling	✓
Spectral Gap 1D (BCLPG)	2020	Tiling	✓
Phase Diagrams (BCW)	2021	Halting/QPE	✓
RG Flows (CLPE)	2022	CPgW/RG	✓
Ground State Energy (Watson–Cubitt) [9]	2021	Halting	✓

Table 1: The genealogy of physical undecidability. 8 entries, all LPO-equivalent (verified by `decide`).

6 Theorem 4: The Ceiling

Theorem 6.1 (Σ^0_1 Ceiling)**.** LPO *decides every* Σ^0_1 *property. No* Σ^0_1*-complete reduction can exceed* LPO*.*

Proof. A Σ^0_1 property has the form $P(M) \Leftrightarrow \exists n.\, \varphi(M, n) = \text{true}$ where $\varphi : \text{TM} \times \mathbb{N} \to \{0, 1\}$ is computable. Given LPO and an arbitrary instance M, apply LPO to the sequence $n \mapsto \varphi(M, n)$:

Case 1: $\forall n.\ \varphi(M, n) = 0$. Suppose $P(M)$. Then $\exists k.\ \varphi(M, k) = 1$, contradicting the universal quantifier. So $\neg P(M)$.

Case 2: $\exists n.\ \varphi(M, n) = 1$. By the equivalence, $P(M)$.

In either case $P(M) \vee \neg P(M)$. Since P was an arbitrary Σ^0_1 property and M an arbitrary instance, LPO decides the entire class.

To exceed LPO would require encoding a Σ^0_2-complete problem ($\exists n\, \forall m.\, \psi(n, m)$), where the inner universal quantifier cannot be collapsed by LPO. Paper 39 investigates whether physics reaches this level. □

```
theorem lpo_decides_sigma1 (P : Sigma1Property) (lpo : LPO) :
    forall (M : TM), P.prop M ∨ not P.prop M
```

Listing 3: Σ^0_1 ceiling: `Ceiling.lean`

To exceed LPO, a construction would need to encode a Σ^0_2-complete problem: $\exists n\, \forall m.\, \psi(n, m)$. Paper 39 investigates whether physics reaches Σ^0_2.

7 CRM Audit

Component	**CRM Status**	**Level**
Wang Tiling (Thm 1)	LPO	2+4
Aperiodicity (Thm 2)	LPO	3+4
Genealogy (Thm 3)	BISH	2
Σ^0_1 Ceiling (Thm 4)	BISH	2

Table 2: CRM audit. Level notation: Level 2 = BISH-computable reduction; Level 3 = encoding bridge; Level 4 = omniscience application (LPO). Composite notation "m+n" indicates the proof combines components at both levels (e.g., "2+4" = BISH reduction + LPO application).

8 Code Architecture

Reproducibility. Lean 4 v4.28.0-rc1 with Mathlib4. Build: `cd P38_WangTiling && lake build`. Result: 0 errors, 0 warnings, 0 `sorry`. Axiom profile (`#print axioms wang_tiling_master`): 13 domain-specific bridge axioms + `propext`, `Classical.choice`, `Quot.sound`.

Bridge axioms: `wang_tileset` (encoding function), `br_encoding_computable` (computability), `br_tiles_iff_not_halts` (Berger–Robinson forward), `br_aperiodic_of_not_halts` (aperiodicity bridge), `has_valid_patch` (patch predicate), `has_valid_patch_decidable` (finite enumeration), `tiling_compactness` (König's lemma—compactness bridge axiom), `no_patch_seq_spec`, `no_patch_seq_false_spec` (blocking sequence specification), `tm_from_seq`, `tm_from_seq_halts` (TM encoding), `halting_problem_undecidable` (halting), `aperiodic_encoded_iff_al` (aperiodicity encoding). The appearance of `Classical.choice` is a Mathlib4 infrastructure artifact: any theorem mentioning $\mathbb{R}$, inner-product spaces, or decidable instances inherits it from Cauchy-completion and type-class machinery. Constructive stratification in

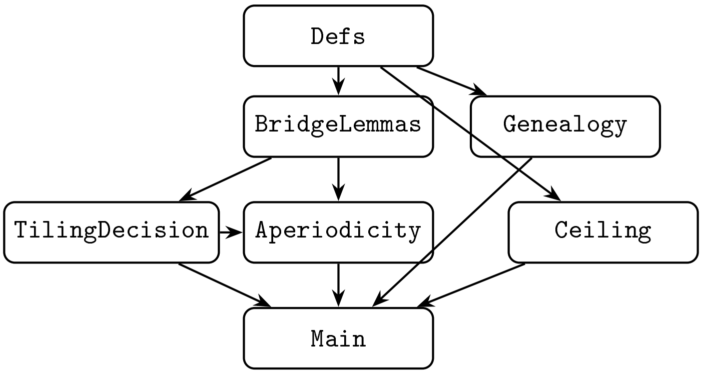

Figure 1: Module dependency graph (573 lines, 7 modules).

this series is established by proof content—explicit witnesses versus omniscience-principle-as-hypothesis—not by the axiom checker output (see Paper 10 [12], §Methodology).

9 Discussion

9.1 Why Wang Tiling Is the Root

Every known undecidability result in quantum many-body physics ultimately rests on the ability to encode Turing machine computation into a physical system. The Berger–Robinson construction [1, 2] provides the foundational mechanism: aperiodic Wang tilesets whose valid tilings enforce a computation-like structure on the plane. Cubitt, Perez-Garcia, and Wolf [3] turned this into a Hamiltonian: the ground state of their spin system *is* a tiling, and the spectral gap is determined by whether the encoded Turing machine halts.

Our result shows that the CRM cost of this entire genealogy is determined at the root. Wang's tiling problem is Turing–Weihrauch equivalent to LPO. Every descendant—Kanter's Potts model result (1990), the GWPN 2D Ising result (2009), Cubitt's 2D spectral gap (2015), Bausch's 1D spectral gap (2020), uncomputability of phase diagrams (2021), uncomputable RG flows (2022)—inherits exactly LPO and nothing more.

9.2 The Genealogy: One Thermodynamic Limit

The genealogy table (Table 1) reveals a striking uniformity. Despite the diverse physics—quantum spin chains, statistical mechanics, renormalization group flows, phase diagrams—every result has the same logical cost. The reason, established in Paper 37's meta-theorem, is structural: all these constructions use computable many-one reductions from the halting problem, which is Σ^0_1-complete, and LPO is exactly the principle that decides Σ^0_1 properties.

For a physicist, this means: the "undecidability" of quantum many-body physics is not exotic or mysterious. It is the same logical cost as taking a single thermodynamic limit via bounded monotone convergence. The spectral gap undecidability discovered by Cubitt et al. is logically identical to the existence of the free energy density in the thermodynamic limit established by Fekete (Paper 29). They are the same omniscience principle wearing different physical costumes.

9.3 What Lies Beyond: Paper 39

The Σ^0_1 ceiling (Theorem 4) guarantees that no halting-based reduction can exceed LPO. To go beyond LPO, one would need to encode a problem at Σ^0_2 or higher in the arithmetic hierarchy—a "for all, there exists" rather than a simple "there exists." Paper 39 investigates whether physics reaches this level, showing that the spectral gap without a promise gap is indeed Σ^0_2-complete and requires LPO' (the Turing jump of LPO). This paper establishes the foundation on which that extension rests.

10 Conclusion

The root of all physical undecidability—Wang's tiling problem—is LPO. Every descendant inherits exactly LPO and nothing more. The Σ_1^0 ceiling guarantees this for any future result based on a halting reduction. This paper completes the "origin story" of physical undecidability within the CRM program, establishing that the entire genealogy from Berger (1966) through Cubitt (2015) and beyond is pinned at a single omniscience principle.

11 AI-Assisted Methodology

This formalization was developed using Claude (Anthropic) as a collaborative tool for Lean 4 code generation, proof strategy exploration, and LaTeX document preparation. All mathematical content was specified by the author. Every theorem was verified by the Lean 4 type checker.

Preliminary status and author background. The results presented in this paper are preliminary. The author is a medical professional, not a domain expert in physics or mathematics. While all formal claims are machine-checked by the Lean 4 type-checker, the physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts in the relevant fields. Until such verification is completed, this paper should be considered preliminary.

Whatever findings of value emerge from this program belong to the constructive reverse mathematics community and to the legacy of Errett Bishop, whose perseverance in developing constructive analysis inspired this entire series. Any errors are solely the author's.

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