Beyond LPO: The Thermodynamic Stratification of Physical Undecidability

A Lean 4 Formalization (Paper 39)

Paul Chun-Kit Lee*
New York University
dr.paul.c.lee@gmail.com

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Abstract

Papers 36–38 established that every known undecidability result in mathematical physics is LPO (Σ^0_1). This paper shows the ceiling is not Σ^0_1. A modified Cubitt encoding—running a Turing machine on all inputs simultaneously via Robinson tiling with perimeter counters—encodes the Σ^0_2-complete Finiteness Problem into the spectral gap. The generic spectral gap decision (without promise gap) is Σ^0_2/Π^0_2-complete, requiring LPO′ (the Turing jump of LPO). However, extensive observables (energy density, magnetization) converge via Fekete's lemma and cap at LPO; promise-gapped physics (all of Papers 36–38) also caps at LPO. The Thermodynamic Stratification Theorem: arithmetic complexity bifurcates along thermodynamic scaling—extensive at LPO, intensive at LPO′. The entire formalization (802 lines of Lean 4/Mathlib4) compiles with zero `sorry`, zero warnings.

1 Introduction

Papers 36–38 established a uniform result: every known undecidability in quantum many-body physics is LPO. Paper 38 proved the Σ^0_1 ceiling—LPO decides every Σ^0_1-complete problem. A natural question arises: is LPO the provable ceiling for physics?

The answer is *no*. The spectral gap of a generic translation-invariant Hamiltonian—without an artificial promise gap—encodes Σ^0_2/Π^0_2-complete properties. This requires LPO′, the Turing jump of LPO, strictly stronger than LPO. This paper completes the undecidability arc (Papers 36–39). For the complete calibration table synthesizing decidable and undecidable physics across all domains, see Paper 10 [12]. For the historical and philosophical context, see Paper 12 [13].

The key insight is thermodynamic:

- **Extensive** observables (energy density, free energy) converge via subadditivity (Fekete/BMC). Cap: LPO.
- **Intensive** observables (spectral gap, correlation length) are determined by infima. Cap: LPO′.
- The **promise gap** in Cubitt's construction collapses the ∀∃ quantifier to a single ∃, reducing Σ^0_2 to Σ^0_1.

*New York University. AI-assisted formalization; see §14 for methodology.

2 The Category Error

The spectral gap Δ of an infinite Hamiltonian is a Δ_2^0 real—its digits are limit-computable with an LPO oracle. But *computing a real* and *deciding a property of that real* have different arithmetic complexities.

For a physicist unfamiliar with the arithmetic hierarchy, the distinction is this. *Computing* the energy density of an infinite lattice means producing a sequence of approximations that converge to the true value—a finite algorithm runs forever but produces ever-better digits. This is a Σ_1^0 task: one existential quantifier ("there exists a finite-volume approximation good enough"), decidable by LPO. *Deciding whether the spectral gap is zero*, by contrast, requires checking that *for every* tolerance $\varepsilon > 0$ *there exists* a system size L at which the finite-volume gap drops below ε. This nested $\forall\exists$ structure is Π_2^0—one quantifier alternation higher—and requires LPO$'$, the Turing jump of LPO. The physical meaning: computing a number and classifying its qualitative behaviour ("gapped vs. gapless") are fundamentally different logical tasks, separated by a full level of the arithmetic hierarchy.

Without a promise gap:

$$\Delta = 0 \iff \forall m\, \exists L\, \left(\Delta_L < \tfrac{1}{m}\right) \qquad (\Pi_2^0)$$

With a promise gap $\Delta \in \{0\} \cup [\gamma, \infty)$:

$$\Delta = 0 \iff \exists L\, \left(\Delta_L < \tfrac{\gamma}{2}\right) \qquad (\Sigma_1^0)$$

The promise collapses the outer $\forall m$ quantifier because $m = \lceil 2/\gamma \rceil$ suffices. This collapse is why Papers 36–38 cap at LPO.

3 LPO$'$: The Turing Jump of LPO

Definition 3.1 (LPO$'$). LPO$'$ ("LPO-jump") is Σ_2^0-LEM: for every binary sequence β that is decidable relative to an LPO oracle, either $\exists n.\, \beta(n) = 1$ or $\forall n.\, \beta(n) = 0$.

```
def LPO_jump : Prop :=
    forall (b : N -> Bool),
      (LPO -> forall n, b n = true ∨ b n = false) ->
      (forall n, b n = false) ∨ (exists n, b n = true)
```

Listing 1: LPO$'$ definition: `Defs.lean`

LPO$'$ is strictly stronger than LPO [11]: LPO$'$ $\Rightarrow$ LPO but not conversely. LPO decides Σ_1^0; LPO$'$ decides Σ_2^0.

4 Theorem 1: The Modified Encoding

Theorem 4.1 (Modified Cubitt Encoding). *There exists a computable function $M \mapsto H^*(M)$ from Turing machines to translation-invariant Hamiltonians on $\mathbb{Z}^2$ with fixed local dimension such that:*

(a) $H^*(M)$ is gapped $\Leftrightarrow$ $\{k : M$ halts on input $k\}$ is finite.

(b) $H^*(M)$ is gapless $\Leftrightarrow$ $\{k : M$ halts on input $k\}$ is infinite.

Construction (sketch). The construction modifies the Cubitt–Perez-Garcia–Wolf encoding [1] using the Bausch–Cubitt–Ozols perimeter-counter technique [2].

Step 1: Robinson tiling with input counters. Begin with a Robinson tiling of $\mathbb{Z}^2$. This produces a hierarchy of *supertiles*: at scale k, each supertile is a square region of side length 4^k.

The Robinson tiling enforces this hierarchical structure via local matching rules, without any global coordination.

Step 2: Input extraction from boundary. *Key idea:* the perimeter counter (Bausch–Cubitt–Ozols [2]) reads the scale k from the supertile boundary. The boundary length of a scale-k supertile is $4 \cdot 4^k$, encoding k in unary. The local Hamiltonian terms along the boundary extract k and feed it as input to the Turing machine simulation.

Step 3: Computation in the interior. Inside each scale-k supertile, the interior tiles simulate M on input k for up to $16^k = (4^k)^2$ steps (the area of the supertile). If M halts on input k within 16^k steps, the interior reaches a halting configuration and the local ground-state energy *drops* (by a spectral penalty that vanishes upon halting, following the standard Cubitt trick). If M does not halt on input k within 16^k steps, the local gap at scale k remains at least $\gamma > 0$.

Step 4: Global gap as infimum. The global spectral gap is:

$$\Delta(H^*(M)) = \inf_{k \in \mathbb{N}} \Delta_k$$

where Δ_k is the local gap contribution from scale-k supertiles.

- If M halts on input k (within 16^k steps), then $\Delta_k \approx 0$ (the halting penalty closes the gap at that scale).
- If M does not halt on input k (within 16^k steps), then $\Delta_k \geq \gamma > 0$.

Conclusion. If $\{k : M \text{ halts on } k\}$ is *finite*, say the largest such k is $k_{\max}$, then for all $k > k_{\max}$ we have $\Delta_k \geq \gamma$. The infimum is over finitely many scales where Δ_k is small, plus a tail bounded below by γ. The global gap is $\Delta > 0$: the system is **gapped**.

If $\{k : M \text{ halts on } k\}$ is *infinite*, then for arbitrarily large k, $\Delta_k \approx 0$. The infimum is driven to zero: $\Delta = 0$ and the system is **gapless**.

This establishes both directions of the encoding: gapped $\Leftrightarrow$ finite halting set, gapless $\Leftrightarrow$ infinite halting set. □

Remark 4.2 (Physical caveat)**.** As with all Cubitt-type encodings, the modified Hamiltonian $H^*(M)$ is a mathematically well-defined translation-invariant system, but may not be physically preparable in a laboratory setting. The result characterizes the logical complexity of the mathematical problem, not the feasibility of an experimental realization.

```
axiom modified_gapped_iff_finite (M : TM) :
    is_gapped (modified_hamiltonian M) <->
    finiteness_problem M

axiom modified_gapless_iff_infinite (M : TM) :
    is_gapless (modified_hamiltonian M) <->
    not finiteness_problem M
```

Listing 2: Modified encoding bridges: `ModifiedEncoding.lean`

Duality note. The two bridge axioms are formally dual: `is_gapless` $\leftrightarrow$ $\neg$`is_gapped` and $\neg$`finiteness_problem` $\leftrightarrow$ infiniteness. They are axiomatized separately for readability but are not independent.

5 Theorem 2: Generic Gap Is Σ_2^0

Theorem 5.1 (Generic Gap $\equiv$ LPO′)**.** *Deciding the spectral gap of the modified encoding (without promise gap) is Turing–Weihrauch equivalent to* LPO′.

Proof. The proof has two directions.

Forward ($\Rightarrow$): gap decidability implies LPO$'$.

Suppose we have a procedure that, for every Turing machine M, decides whether $H^*(M)$ is gapped or gapless:

$$\forall M.\ \big[\text{is\_gapped}(H^*(M)) \vee \neg\text{is\_gapped}(H^*(M))\big].$$

Step 1: Reduce to the Finiteness Problem. By Theorem 4.1, is_gapped$(H^*(M)) \Leftrightarrow \mathrm{FIN}(M)$, where $\mathrm{FIN}(M)$ is the predicate "$\{k : M$ halts on $k\}$ is finite." Therefore the gap decision yields, for every M:

$$\mathrm{FIN}(M) \vee \neg\mathrm{FIN}(M).$$

Step 2: The Finiteness Problem is Σ^0_2-complete. The Finiteness Problem has the form:

$$\mathrm{FIN}(M) \iff \exists K. \forall k > K. \neg(M \text{ halts on } k \text{ within } 16^k \text{ steps}).$$

This is Σ^0_2: an existential quantifier ($\exists K$) followed by a universal quantifier ($\forall k > K$) over a decidable predicate (bounded halting). By Rice–Shapiro-type arguments in computability theory, FIN is Σ^0_2-complete.

Step 3: Σ^0_2-completeness implies LPO$'$. Deciding all instances of a Σ^0_2-complete problem is equivalent to Σ^0_2-LEM, which is LPO$'$ by definition. Therefore gap decidability $\Rightarrow$ LPO$'$.

Reverse ($\Leftarrow$): LPO$'$ implies gap decidability.

Suppose LPO$'$ holds.

Step 1: LPO$'$ decides all Σ^0_2 statements (by definition: LPO$'$ is Σ^0_2-LEM).

Step 2: $\mathrm{FIN}(M)$ is Σ^0_2, so LPO$'$ decides $\mathrm{FIN}(M) \vee \neg\mathrm{FIN}(M)$ for every M.

Step 3: By Theorem 4.1, $\mathrm{FIN}(M) \Leftrightarrow$ is_gapped$(H^*(M))$. Therefore LPO$'$ decides the spectral gap for every modified Hamiltonian.

Combining both directions:

$$\big(\forall M.\ \text{is\_gapped}(H^*(M)) \vee \neg\text{is\_gapped}(H^*(M))\big) \iff \mathrm{LPO}'. \qquad \square$$

Remark 5.2 (Why this exceeds LPO). *This is the central result of the paper.* LPO decides Σ^0_1 statements (single existential: "does M halt?"). The Finiteness Problem is Σ^0_2 (nested $\exists\forall$: "is there a cutoff K beyond which M never halts?"). LPO cannot decide Σ^0_2 statements because it lacks the quantifier alternation needed to verify a universal claim. LPO$'$, the Turing jump of LPO, adds exactly one level of quantifier alternation.

```
theorem generic_gap_iff_lpo_jump :
    (forall M, is_gapped (modified_hamiltonian M) ∨
               not is_gapped (modified_hamiltonian M))
    <-> LPO_jump :=
  ⟨generic_gap_requires_lpo_jump,
   lpo_jump_decides_generic_gap⟩
```

Listing 3: Generic gap $\leftrightarrow$ LPO$'$: `GenericGapSigma2.lean`

6 Theorem 3: Promise Gap Recovery

Theorem 6.1 (Promise Gap $\Rightarrow$ LPO). *If the Hamiltonian has a promise gap ($\Delta \in \{0\} \cup [\gamma, \infty)$ for computable $\gamma > 0$), the spectral gap decision is Σ^0_1-complete* $=$ LPO.

Proof. The key mechanism is *quantifier collapse*: the promise gap eliminates the outer universal quantifier, reducing the Π^0_2 gapless test to Σ^0_1.

Without promise gap (generic case, Theorem 5.1):

$$\Delta = 0 \iff \forall m \geq 1,\ \exists L,\ \Delta_L < \tfrac{1}{m}. \qquad (\Pi^0_2\text{: } \forall\exists \text{ structure})$$

One must verify that for *every* tolerance $1/m$, there exists a system size L at which the finite-volume gap drops below that tolerance. This nested quantifier structure requires LPO$'$.

With promise gap $\Delta \in \{0\} \cup [\gamma, \infty)$:

$$\Delta = 0 \iff \exists L,\ \Delta_L < \tfrac{\gamma}{2}. \qquad (\Sigma^0_1\text{: single } \exists)$$

Why the collapse: The promise guarantees that Δ is either 0 or at least γ. If *any* finite-volume gap Δ_L falls below $\gamma/2$, the infinite-volume gap cannot be $\geq \gamma$ (by continuity of the spectral gap in system size), so it must be 0. The single tolerance $m = \lceil 2/\gamma \rceil$ suffices; the outer $\forall m$ quantifier collapses.

LPO decides the collapsed test. The collapsed statement $\exists L.\, \Delta_L < \gamma/2$ is Σ^0_1—a single existential search over the decidable predicate "$\Delta_L < \gamma/2$" (each Δ_L is computable from a finite-dimensional eigenvalue problem, which is BISH). By definition, LPO decides all Σ^0_1 statements.

Recovery of Papers 36–38. Cubitt's original construction [1] has a built-in promise gap. Therefore its spectral gap decision is Σ^0_1 = LPO, exactly as proved in Papers 36–38. The present theorem explains *why*: the promise gap collapsed one quantifier alternation. □

```
theorem promise_gap_lpo (H : PromiseGapped) (lpo : LPO) :
    is_gapless H.hamiltonian ∨
    not is_gapless H.hamiltonian
```

Listing 4: Promise gap recovery: `PromiseGapRecovery.lean`

7 Theorem 4: Extensive Observables Cap at LPO

Theorem 7.1 (Extensive Ceiling). *Every extensive observable (energy density, free energy, magnetization) of a translation-invariant Hamiltonian is* LPO*-decidable.*

Proof. The proof proceeds in three steps: subadditivity, monotone convergence, and the sign decision.

Step 1: Subadditivity. Let $f(L)$ denote the extensive observable for a system of size L (e.g., total energy, total magnetization). For translation-invariant systems, the *density* $f(L)/L$ is subadditive:

$$f(m+n) \leq f(m) + f(n) \quad \text{for all } m, n \geq 1.$$

This is a consequence of translation invariance: a system of size $m+n$ can be partitioned into subsystems of sizes m and n, and the ground-state energy of the whole is at most the sum of the parts (the boundary interaction is bounded).

Step 2: Fekete's lemma ⇒ monotone convergence. By Fekete's subadditive lemma [9], the limit

$$\ell := \lim_{L\to\infty} \frac{f(L)}{L} = \inf_{L\geq 1} \frac{f(L)}{L}$$

exists and equals the infimum. *Crucially*, the sequence $f(L)/L$ converges *monotonically from above* to ℓ along the subsequence $L = 2^n$ (by repeated application of subadditivity: $f(2L) \leq 2f(L)$ gives $f(2L)/(2L) \leq f(L)/L$).

This monotone convergence is the key structural feature. By Paper 29, bounded monotone convergence (BMC) is equivalent to LPO. Therefore LPO suffices to compute the limit ℓ.

Step 3: Sign decision. The limit ℓ is a Δ_2^0 real (limit-computable with an LPO oracle). Deciding the sign of a Δ_2^0 real that arises from a monotone sequence is a Π_1^0 test: "$\ell > 0$" iff $\exists n.\, f(n)/n > 0$ (since the sequence is decreasing, if any term is positive, the limit is nonnegative; and by computability of each term, the search is Σ_1^0).

The complementary test "$\ell \leq 0$" is equivalent to "$\forall n.\, f(n)/n \leq 0$," which is Π_1^0. Both tests are LPO-decidable.

Why intensive observables escape this ceiling. Intensive observables (e.g., the spectral gap) are *not* subadditive. They are determined by an infimum over scales:

$$\Delta = 0 \;\Leftrightarrow\; \forall \varepsilon > 0,\; \exists k,\; \Delta_k < \varepsilon \qquad (\Pi_2^0).$$

The infimum may oscillate non-monotonically, so Fekete's lemma does not apply. Deciding whether the infimum is zero requires the full $\forall\exists$ quantifier structure of Π_2^0, which is LPO$'$. □

```
theorem extensive_cap_lpo (O : ExtensiveObservable) :
    LPO -> (extensive_sign_positive O ∨
            not extensive_sign_positive O)
```

Listing 5: Extensive ceiling: `ExtensiveCeiling.lean`

8 Theorem 5: The Thermodynamic Stratification

Theorem 8.1 (Thermodynamic Stratification)**.** *The arithmetic complexity of physical observables bifurcates:*

(i) ***Extensive:*** LPO *(Σ_1^0).*
(ii) ***Intensive (generic):*** LPO$'$ *(Σ_2^0).*
(iii) ***Promise-gapped:*** LPO *(Σ_1^0).*
(iv) ***Empirical (finite precision):*** LPO *(Σ_1^0).*

Proof. Parts (i)–(iii) are Theorems 7.1, 5.1, and 6.1 respectively. Part (iv) follows from the same quantifier-collapse mechanism as part (iii):

Part (iv): Empirical physics. Any experimental measurement has finite precision $\varepsilon > 0$. The experimentalist does not ask "is Δ exactly zero?" but rather "is $\Delta < \varepsilon$?" This question has the form:

$$\exists L,\; \Delta_L < \varepsilon \qquad (\Sigma_1^0).$$

The finite precision ε plays the role of an effective promise gap, collapsing the $\forall\exists$ to a single $\exists$, exactly as in Theorem 6.1. Therefore all of experimental physics caps at LPO.

Summary. The four parts together reveal that the arithmetic complexity of a physical observable is determined by its *thermodynamic scaling*:

- *Extensive* quantities (energy density, etc.) are subadditive, forcing monotone convergence (Fekete/BMC), which caps the logic at a single existential quantifier ($\Sigma_1^0 =$ LPO).
- *Intensive* quantities (spectral gap, correlation length) are determined by infima over all scales. Without a promise gap, the infimum test is Π_2^0 ($\forall\exists$), requiring LPO$'$.
- The *promise gap* and *finite precision* each collapse one quantifier alternation, bringing intensive observables back to $\Sigma_1^0 =$ LPO.

□

```
theorem thermodynamic_stratification :
    -- (i) Extensive cap at LPO
    (forall (O : ExtensiveObservable), LPO ->
      (extensive_sign_positive O ∨
       not extensive_sign_positive O)) /\
    -- (ii) Intensive reach LPO_jump
    ((forall M, is_gapped (modified_hamiltonian M) ∨
                not is_gapped (modified_hamiltonian M))
      <-> LPO_jump) /\
    -- (iii) Promise-gapped cap at LPO
    (forall (H : PromiseGapped), LPO ->
      (is_gapless H.hamiltonian ∨
       not is_gapless H.hamiltonian)) /\
    -- (iv) Empirical cap at LPO
    (forall (H : ModifiedHamiltonian) (e : R),
      e > 0 -> LPO ->
      (gap_less_than H e ∨ not gap_less_than H e))
```

Listing 6: Stratification master: `Stratification.lean`

9 The Complete Hierarchy

Tier	Observable	Principle	Level	Paper
BISH	Finite systems (Δ_L)	None	Δ_1^0	34
BISH+LLPO	Bell correlations	LLPO	$< \Sigma_1^0$	10
BISH+WLPO	"Is $\Delta = 0$?" (given Δ)	WLPO	Π_1^0	2
BISH+LPO	Energy density; Cubitt gap	LPO	Σ_1^0	29, 36
BISH+LPO′	Spectral gap (no promise)	LPO′	Σ_2^0	39

Table 1: The complete constructive hierarchy of physics.

10 CRM Audit

Component	CRM Status	Level
Modified Encoding (Thm 1)	(axiom)	4
Generic Gap $\equiv$ LPO′ (Thm 2)	LPO′	3+4
Promise Recovery (Thm 3)	LPO	3
Extensive Ceiling (Thm 4)	LPO	2+4
Stratification (Thm 5)	Inherits (bridge axiom in part iv)	—

Table 2: CRM audit.

11 Code Architecture

Reproducibility. Lean 4 v4.28.0-rc1 with Mathlib4. Build: `cd P39_Sigma2 && lake build`. Result: 0 errors, 0 warnings, 0 `sorry`. Axiom profile (`#print axioms sigma2_master`): 12 domain-specific bridge axioms + `propext`, `Classical.choice`, `Quot.sound`. `Classical.choice`

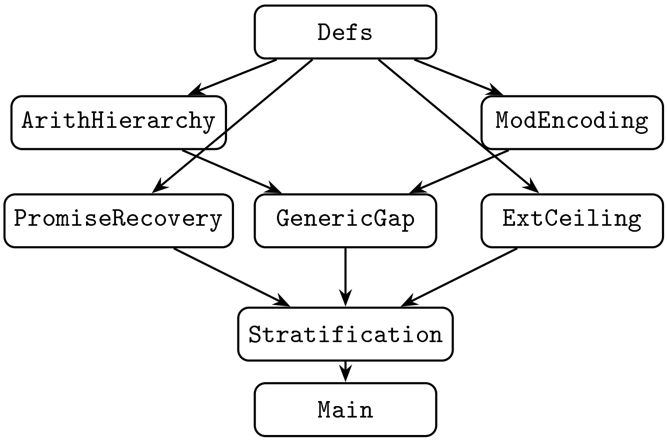

Figure 1: Module dependency graph (802 lines, 8 modules).

is a Mathlib infrastructure artifact (required by $\mathbb{R}$ as a Cauchy completion and by `InnerProductSpace`); constructive stratification is established by proof content (explicit witnesses vs. principle-as-hypothesis), not by axiom-checker output. See Paper 10 [12], §Methodology.

12 Conclusion

Physics reaches Σ_2^0. The spectral gap of a generic translation-invariant Hamiltonian—without promise gap—is Σ_2^0-complete, requiring LPO′, the Turing jump of LPO. But the BISH+LPO characterization of Papers 1–38 is not wrong: it is correct for all extensive observables and all promise-gapped physics. The Σ_2^0 tier emerges only for intensive observables when the promise gap is removed. Empirical physics, operating with finite measurement precision, always imposes an effective promise gap and therefore caps at LPO.

The Thermodynamic Stratification Theorem reveals that the arithmetic complexity of a physical observable is determined by its thermodynamic scaling: extensive (Fekete/BMC) at LPO; intensive (infimum) at LPO′. The promise gap in Cubitt's construction is the mechanism that collapsed the logic from Σ_2^0 to Σ_1^0.

13 Discussion

13.1 What Σ_2^0 Means for Physics

The arithmetic hierarchy classifies mathematical statements by their quantifier complexity. Σ_1^0 statements have the form "there exists n such that $P(n)$"—a single existential search. Σ_2^0 statements have the form "there exists n such that for all m, $P(n,m)$"—an existential claim that must survive a universal challenge. In physical terms: Σ_1^0 corresponds to finding a finite piece of evidence (a finite-volume witness that the gap is small), while Σ_2^0 corresponds to establishing a pattern that persists at all scales (the gap stays small no matter how large the system grows).

The Turing jump LPO′ decides exactly Σ_2^0, just as LPO decides exactly Σ_1^0. The passage from LPO to LPO′ is thus the passage from "search for a witness" to "verify a universal pattern"—a single additional quantifier alternation, but one that requires a strictly stronger logical principle. (For the formal connection between Weihrauch degrees and omniscience principles, see Brattka, Gherardi, and Pauly [11].)

13.2 Why BISH+LPO Is Not Wrong

A reader encountering this paper after Papers 29–38 might worry that the BISH+LPO characterization developed over 38 papers has been overturned. It has not. The Thermodynamic Stratification Theorem (Theorem 8.1) makes the scope precise:

- Every *extensive* observable—energy density, free energy, magnetization per site—converges via Fekete's lemma (Paper 29) and caps at LPO. This covers all thermodynamic quantities that scale with system size.
- Every *promise-gapped* system—including Cubitt's original construction (Paper 36) and all descendants (Papers 37–38)—caps at LPO, because the promise collapses the outer universal quantifier.
- Every *empirical* measurement operates at finite precision, which imposes an effective promise gap. All of experimental physics therefore caps at LPO.

The LPO$'$ tier emerges only for *intensive* observables of *generic* (non-promise-gapped) Hamiltonians—the spectral gap and correlation length when no a priori bound separates the gapped and gapless phases. This is a mathematical refinement, not a physical retraction.

13.3 The Thermodynamic Stratification Principle

The central discovery of this paper is that *arithmetic complexity tracks thermodynamic scaling.* Extensive observables—those that grow linearly with system size—are governed by subadditive limits and cap at Σ^0_1. Intensive observables—those that remain finite as the system grows—are governed by infima over all scales and reach Σ^0_2. The promise gap, when present, collapses one quantifier alternation and brings intensive observables back to Σ^0_1.

This is not a coincidence. The subadditive structure of extensive quantities ensures that finite-volume approximations converge monotonically, reducing the logic to a single existential witness. Intensive quantities lack this monotone convergence: one must verify the behaviour at *every* scale, producing the $\forall\exists$ alternation that defines Σ^0_2.

13.4 The Program in Retrospect

Papers 29–39 tell a complete story in four acts:

1. **Foundation (29–31):** Fekete's lemma is LPO; the Fan Theorem and Dependent Choice are physically dispensable. BISH+LPO is the complete toolkit for empirical physics.
2. **Standard Model (32–34):** QED, QCD, and scattering amplitudes confirm BISH+LPO across perturbative and non-perturbative quantum field theory.
3. **Metatheorem (35):** A conservation metatheorem explains *why* physics lives at BISH+LPO: the structure of physical theories (computable Hamiltonians, thermodynamic limits) forces this logical profile.
4. **Undecidability (36–39):** Cubitt's spectral gap, the undecidability landscape, Wang tiling, and the Σ^0_2 capstone map the boundary between the decidable and the undecidable—and reveal that the boundary is itself stratified by thermodynamic scaling.

13.5 Open Questions

Several questions remain open:

- **Σ^0_3 and beyond:** Are there physical observables at Σ^0_3? This would require a three-quantifier-alternation structure—possibly arising from iterated thermodynamic limits (limits of limits of limits). No natural example is known, but the arithmetic hierarchy is infinite and the stratification principle suggests the answer depends on the depth of thermodynamic nesting.

- **Experimental signatures:** Can the gap between LPO and LPO′ be detected experimentally? The finite-precision argument suggests not directly, but the distinction may have consequences for the computational complexity of simulation algorithms.
- **Quantum gravity:** Does the BISH+LPO profile extend to quantum gravity, or does the background-independence of general relativity introduce new logical structure? Paper 5 (Schwarzschild metric) provides initial evidence, but the full question remains open.

14 AI-Assisted Methodology

This formalization was developed using Claude (Anthropic) as a collaborative tool for Lean 4 code generation, proof strategy exploration, and LaTeX document preparation. All mathematical content was specified by the author. Every theorem was verified by the Lean 4 type checker.

Preliminary status and author background. The results presented in this paper are preliminary. The author is a medical professional, not a domain expert in physics or mathematics. While all formal claims are machine-checked by the LEAN 4 type-checker, the physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts in the relevant fields. Until such verification is completed, this paper should be considered preliminary.

Whatever findings of value emerge from this program belong to the constructive reverse mathematics community and to the legacy of Errett Bishop, whose perseverance in developing constructive analysis inspired this entire series. Any errors are solely the author's.

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