Fekete's Subadditive Lemma is Equivalent to LPO

A Lean 4 Formalization

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Abstract

We prove that Fekete's Subadditive Lemma—the assertion that every subadditive sequence (u_n) with u_n/n bounded below converges—is equivalent over Bishop-style constructive mathematics (BISH) to the Limited Principle of Omniscience (LPO). The backward direction encodes a binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$ into a mock free energy $F_n = -n \cdot x_n$, where $x_n = 1$ if $\exists k < n,\ \alpha(k) = 1$ and $x_n = 0$ otherwise. The sequence F is subadditive and $F_n/n \geq -1$; applying Fekete's lemma yields a limit whose value (0 or -1) decides α. The forward direction composes LPO $\to$ BMC (Bridges–Vîţă) with the classical Fekete proof via bounded monotone convergence. The entire equivalence is formalized in Lean 4 with Mathlib4 dependencies: 549 lines across 6 modules, zero `sorry`s, with the backward direction free of custom axioms. This resolves Problem 1 of the constructive calibration program (Paper 10) and establishes a three-tier hierarchy for thermodynamic-limit convergence: exact solvability (BISH), cluster expansions (BISH), and generic subadditivity (LPO).

Contents

*New York University. AI-assisted formalization; see §10 for methodology.

1 Introduction

1.1 The Thermodynamic Limit and Subadditivity

The thermodynamic limit is the foundational idealization of equilibrium statistical mechanics. For a lattice system $\Lambda \subset \mathbb{Z}^d$ with Hamiltonian H_Λ, the free energy density

$$f_\Lambda(\beta) = -\frac{1}{|\Lambda|} \log Z_\Lambda(\beta)$$

is well-defined for each finite volume. The thermodynamic limit asserts that $f_\infty(\beta) = \lim_{|\Lambda|\to\infty} f_\Lambda(\beta)$ exists. The standard proof, due to Ruelle [13] and developed in modern textbooks [5], proceeds via subadditivity: the partition function satisfies $\log Z_{\Lambda_1 \cup \Lambda_2} \geq \log Z_{\Lambda_1} + \log Z_{\Lambda_2}$ (for suitable boundary conditions), so $-\log Z_\Lambda$ is subadditive in the volume $|\Lambda|$. Fekete's classical lemma [4] then guarantees convergence of f_Λ to its infimum.

1.2 Fekete's Lemma

Fekete's Subadditive Lemma (1923) states: if $(u_n)_{n\geq 1}$ satisfies $u_{m+n} \leq u_m + u_n$ for all $m, n \geq 0$, then $\lim_{n\to\infty} u_n/n = \inf_{n\geq 1} u_n/n$ (where the limit may be $-\infty$). When u_n/n is additionally bounded below, the limit is a finite real number. This result is central to statistical mechanics, ergodic theory, combinatorics, and information theory [1].

The classical proof uses the monotone convergence theorem: the running minimum $v_n = \inf_{1\leq k\leq n} u_k/k$ is non-increasing and bounded below, hence convergent; a Euclidean division argument then shows $u_n/n \to \lim v_n$.

1.3 Constructive Status

From a constructive standpoint, the monotone convergence theorem is not available in Bishop-style constructive mathematics (BISH). It is equivalent over BISH to the Limited Principle of Omniscience (LPO), which asserts that for any binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, either $\alpha(n) = 0$ for all n or there exists n_0 with $\alpha(n_0) = 1$. The equivalence BMC $\leftrightarrow$ LPO was established by Bridges and Vîţă [3] as part of the systematic classification program of constructive reverse mathematics initiated by Ishihara [6].

Since the classical proof of Fekete's lemma uses BMC in a single step (extracting the limit of the running minimum), a natural question arises: *is* LPO *the exact logical cost of Fekete's lemma, or could a more clever proof avoid it?*

Recent work by Boche, Bock, and Deppe [1] established that the Fekete limit is not Banach–Mazur computable as a function of the sequence—it lies strictly above the arithmetical hierarchy. While non-computability and non-constructivity are distinct concepts (the former concerns Turing machines, the latter concerns proof principles), both point to a fundamental limitation in extracting the limit value.

1.4 Context: Papers 8, 9, and 10

This paper is part of a constructive calibration program for mathematical physics. Papers 8 and 9 [8, 9] established that the 1D Ising model provides a concrete arena for studying the logical cost of the thermodynamic limit.

Paper 8 proved two complementary results: (A) the finite-size error bound $|f_N(\beta) - f_\infty(\beta)| \leq (1/N)\tanh(\beta)^N$ is BISH-valid with a constructively computed N_0, and (B) the existence of the thermodynamic limit as a completed real number is equivalent to LPO via BMC. Paper 9 refined the encoding by using a combinatorial coupling-sequence construction.

Paper 10 [10] synthesized these results into a "logical geography" of mathematical physics, listing 16 open problems. For the historical perspective on how constructive analysis recovers the empirical content of classical physics, see Paper 12 [11]. Problem 1 asked:

> *Is the* LPO *cost of the thermodynamic limit ineliminable for interacting (non-exactly-solvable) systems?*

The present paper resolves Problem 1 affirmatively at the level of Fekete's lemma itself: the generic subadditivity route to the thermodynamic limit costs *exactly* LPO, independent of any particular physical model.

1.5 Contribution

We prove:

Theorem 1.1 (Main, informal)**.** *Over* BISH*, Fekete's Subadditive Lemma is equivalent to* LPO*.*

The forward direction (LPO $\to$ FeketeConvergence) composes LPO $\to$ BMC [3] with the classical Fekete proof. The backward direction (FeketeConvergence $\to$ LPO) is the main new content: we encode binary sequences into mock free energies and extract decisions from the Fekete limit.

The entire equivalence is formalized in Lean 4 with Mathlib4 dependencies: 549 lines across 6 modules with zero `sorry`s. The backward direction depends only on Lean's foundational axioms (`propext`, `Classical.choice`, `Quot.sound`) with no custom axioms—the `Classical.choice` appearance is an infrastructure artifact of Mathlib's real number construction, not a logical dependency in the proof content.

2 Background

2.1 The Constructive Hierarchy

Bishop-style constructive mathematics (BISH) is mathematics with intuitionistic logic and dependent choice, but without the law of excluded middle (LEM). Within this framework, several "omniscience principles" have been identified that stratify the gap between BISH and classical mathematics:

Principle	Statement
LPO	$\forall \alpha : \mathbb{N} \to \{0,1\},\ (\forall n,\ \alpha(n) = 0) \vee (\exists n,\ \alpha(n) = 1)$
WLPO	$\forall \alpha : \mathbb{N} \to \{0,1\},\ (\forall n,\ \alpha(n) = 0) \vee \neg(\forall n,\ \alpha(n) = 0)$
LLPO	$\forall \alpha : \mathbb{N} \to \{0,1\},\ (\forall n,\ \alpha(2n) = 0) \vee (\forall n,\ \alpha(2n{+}1) = 0)$

The strict hierarchy BISH $\subsetneq$ LLPO $\subsetneq$ WLPO $\subsetneq$ LPO $\subsetneq$ LEM holds over BISH. Constructive reverse mathematics (CRM), as developed by Ishihara [6] and others, classifies mathematical theorems by which principle they require.

2.2 CRM for Physicists

For readers coming from physics rather than mathematical logic, the key idea of constructive reverse mathematics can be stated simply: *every mathematical theorem used in physics has a precise logical cost, and CRM measures that cost.*

In classical mathematics, all proofs are equally "allowed"—the law of excluded middle, the axiom of choice, and every omniscience principle are available for free. CRM strips away this blanket permission and asks: for each theorem T, what is the *weakest* logical principle P such that T is provable from BISH + P? The answer takes the form of an equivalence: $T \leftrightarrow P$ over BISH. This is analogous to how physicists classify phase transitions by universality class or symmetry-breaking pattern—different physical phenomena may share the same underlying logical structure.

For a working physicist, the hierarchy BISH $\subsetneq$ WLPO $\subsetneq$ LPO (with LLPO sitting between BISH and WLPO; see §2.1) can be read operationally:

- BISH theorems correspond to *explicitly computable* results—finite algorithms with known error bounds.
- WLPO theorems require *weak* infinite decisions—such as determining whether a sequence is eventually zero.
- LPO theorems require *strong* infinite decisions—such as deciding whether a sequence ever takes the value 1.

The CRM program for mathematical physics (this series of papers) asks: which physical idealizations live at which level? The answer reveals that the logical cost of a physical idealization is not arbitrary but is determined by the idealization's mathematical structure.

2.3 Fekete's Subadditive Lemma

A sequence $(u_n)_{n\geq 0}$ of real numbers is *subadditive* if $u_{m+n} \leq u_m + u_n$ for all $m, n \geq 0$. Fekete's lemma asserts that if u_n/n is bounded below, then (u_n/n) converges. Classically, $\lim_{n\to\infty} u_n/n = \inf_{n\geq 1} u_n/n$.

In statistical mechanics, the log-partition function $-\log Z_\Lambda$ is typically subadditive in the volume (for translation-invariant interactions with suitable boundary conditions). Fekete's lemma is therefore the generic workhorse for establishing existence of the thermodynamic limit [5, 13].

2.4 Papers 8 and 9: The 1D Ising Test Case

Paper 8 [8] established that for the homogeneous 1D Ising model, the convergence of $f_N(\beta)$ to $f_\infty(\beta) = -\log(2\cosh\beta)$ is provable in BISH with an explicit rate: $|f_N - f_\infty| \leq (1/N)\tanh(\beta)^N$. The key insight is that the closed-form transfer-matrix solution provides an explicit Cauchy modulus—Fekete's lemma is not needed.

Paper 9 [9] proved the reverse direction by encoding binary sequences into disordered coupling sequences of the 1D Ising model, instantiating BMC $\to$ LPO through the specific physics.

These results show that the LPO cost is *dispensable* for the 1D Ising model because of exact solvability. But the generic route through Fekete's lemma remained unclassified—until now.

2.5 The Gap

The open question was: does Fekete's Subadditive Lemma, considered as a standalone logical principle, have a precise position in the constructive hierarchy? Since the classical proof uses BMC (which is equivalent to LPO), we know FeketeConvergence is provable from LPO. But is LPO *necessary*, or could a more subtle proof establish FeketeConvergence from a weaker principle?

3 Formal Definitions

We work in Lean 4 with Mathlib4 for the real number infrastructure. All definitions reside in `Defs.lean`.

Definition 3.1 (LPO)**.** LPO is the proposition that for every binary sequence $\alpha : \mathbb{N} \to \mathrm{Bool}$, either all terms are false or some term is true:

```
def LPO : Prop :=
  forall (a : Nat -> Bool),
    (forall n, a n = false) ∨ (exists n, a n = true)
```

Definition 3.2 (BMC)**.** Bounded Monotone Convergence: every bounded non-decreasing real sequence has a limit.

```
def BMC : Prop :=
  forall (a : Nat -> Real) (M : Real),
    Monotone a -> (forall n, a n <= M) ->
    exists L, forall eps, 0 < eps ->
      exists N0, forall N, N0 <= N -> |a N - L| < eps
```

Definition 3.3 (FeketeConvergence)**.** Fekete's Subadditive Lemma as a logical principle: every subadditive sequence with u_n/n bounded below converges.

```
def FeketeConvergence : Prop :=
  forall (u : Nat -> Real),
    (forall m n, u (m + n) <= u m + u n) ->
    (exists C, forall n, 0 < n -> C <= u n / n) ->
    exists L, forall eps, 0 < eps ->
      exists N0, forall N, N0 <= N -> 0 < N ->
        |u N / N - L| < eps
```

Definition 3.4 (The encoding)**.** Given $\alpha : \mathbb{N} \to \mathrm{Bool}$, define:

- $\mathrm{hasTrue}(\alpha, n) := \bigvee_{k<n} \alpha(k)$ (decidable bounded search, a Bool)
- $x_n := \mathrm{indicator}(\alpha, n) := \begin{cases} 1 & \text{if } \mathrm{hasTrue}(\alpha, n), \\ 0 & \text{otherwise} \end{cases}$

- $F_n :=$ mockFreeEnergy$(\alpha, n) := -n \cdot x_n$

```
def hasTrue (a : Nat -> Bool) (n : Nat) : Bool :=
  (List.range n).any (fun k => a k)

def indicator (a : Nat -> Bool) (n : Nat) : Real :=
  if hasTrue a n then 1 else 0

def mockFreeEnergy (a : Nat -> Bool) (n : Nat) : Real :=
  -(n : Real) * indicator a n
```

Remark 3.5 (Constructive design)**.** The indicator x_n is computed by decidable bounded search over $\{0, \dots, n-1\}$—it is a finite Boolean disjunction, not a real-number comparison. The encoding $F_n = -n \cdot x_n$ is pure arithmetic. No classical logic enters the construction.

4 Forward Direction: LPO $\to$ FeketeConvergence

The forward direction composes two known results:

1. LPO $\to$ BMC: Bridges and Vîţă [3], Theorem 2.1.5. Given a bounded non-decreasing sequence (a_n), LPO is used to decide, for each rational q, whether $a_n < q$ for all n or $a_n \geq q$ for some n. The supremum of the latter rationals gives the limit.
2. BMC $\to$ FeketeConvergence: The classical proof of Fekete's lemma, which is a BISH-valid implication (it uses BMC as a hypothesis, not LEM). Given subadditive (u_n) with u_n/n bounded below by C, define the running minimum $v_n = \min_{1\leq k\leq n} u_k/k$. Then (v_n) is non-increasing and bounded below by C, so $(-v_n)$ is non-decreasing and bounded above. Apply BMC to obtain a limit $L = \lim v_n$. The Euclidean division argument shows $u_n/n \to L$.

In the formalization, both steps are axiomatized:

```
axiom bmc_of_lpo : LPO -> BMC
axiom fekete_of_bmc : BMC -> FeketeConvergence

theorem fekete_of_lpo : LPO -> FeketeConvergence :=
  fun h => fekete_of_bmc (bmc_of_lpo h)
```

Remark 4.1 (Why axiomatize?)**.** The forward direction is standard and available in Mathlib as `Subadditive.tendsto_lim` [12]. Axiomatization follows the same pattern as Paper 8's treatment of LPO $\to$ BMC: the cited result is well-established, and the novel content of this paper lies entirely in the backward direction. Discharging the two axioms in Lean would require (i) formalizing the Bridges–Vîţă proof of LPO $\to$ BMC ([3], Theorem 2.1.5) and (ii) refactoring Mathlib's classical Fekete proof into a BMC-relative version—substantial but orthogonal work. The equivalence `fekete_iff_lpo` therefore depends on the correctness of these cited results; the machine-checked content is the novel backward implication.

5 Backward Direction: FeketeConvergence $\to$ LPO

This section contains the main new content of the paper.

Theorem 5.1 (FeketeConvergence $\to$ LPO)**.** ✓ *If every subadditive sequence with u_n/n bounded below converges, then* LPO *holds.*

The proof proceeds in four stages: encoding (§5.1), subadditivity (§5.2), lower bound (§5.3), and decision extraction (§5.4).

5.1 The Encoding

Given an arbitrary binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, we construct:

- The bounded-search indicator: $x_n = 1$ if $\exists k < n$, $\alpha(k) = 1$; otherwise $x_n = 0$.
- The mock free energy: $F_n = -n \cdot x_n$.

The sequence x_n is monotone non-decreasing (once a witness is found, it persists) and takes values in $\{0, 1\}$. There are exactly two possible asymptotic behaviors:

- If $\alpha \equiv 0$: $x_n = 0$ for all n, so $F_n = 0$ and $F_n/n = 0 \to 0$.
- If $\exists k$, $\alpha(k) = 1$: $x_n = 1$ for all $n > k$, so $F_n = -n$ and $F_n/n = -1 \to -1$.

The gap of 1 between the two possible limit values is the key feature that enables LPO extraction.

5.2 Subadditivity

Lemma 5.2. ✓ *F is subadditive: $F_{m+n} \le F_m + F_n$ for all $m, n \ge 0$.*

Proof. We have $F_{m+n} = -(m+n)\cdot x_{m+n}$, $F_m = -m\cdot x_m$, and $F_n = -n\cdot x_n$. Since x is monotone, $x_m \le x_{m+n}$ and $x_n \le x_{m+n}$. Since $m, n \ge 0$:

$$\begin{aligned} F_{m+n} &= -m \cdot x_{m+n} - n \cdot x_{m+n} \\ &\le -m \cdot x_m - n \cdot x_n = F_m + F_n. \end{aligned}$$

The last step uses $-m \cdot x_{m+n} \le -m \cdot x_m$ (from $x_m \le x_{m+n}$ and $m \ge 0$) and similarly for n. □

In the formalization, the proof is a single `nlinarith` call after establishing the monotonicity hypotheses:

```
theorem mockFreeEnergy_subadditive (a : Nat -> Bool)
    (m n : Nat) :
    mockFreeEnergy a (m + n)
      <= mockFreeEnergy a m + mockFreeEnergy a n := by
  unfold mockFreeEnergy
  have hm_nn : (0 : Real) <= m := Nat.cast_nonneg m
  have hn_nn : (0 : Real) <= n := Nat.cast_nonneg n
  have h_xm := indicator_le_add_left a m n
  have h_xn := indicator_le_add_right a m n
  push_cast
  nlinarith
```

5.3 Lower Bound

Lemma 5.3. ✓ *$F_n/n \ge -1$ for all $n \ge 1$.*

Proof. $F_n/n = -x_n$, and $x_n \le 1$ (since $x_n \in \{0, 1\}$), so $F_n/n = -x_n \ge -1$. □

5.4 Decision Extraction

We now prove the main theorem. Assume FeketeConvergence holds and let $\alpha : \mathbb{N} \to \{0, 1\}$ be arbitrary.

Step 1: Apply Fekete. Since F is subadditive (Lemma 5.2) and $F_n/n \ge -1$ (Lemma 5.3), FeketeConvergence yields a limit $L \in \mathbb{R}$ and a modulus: for every $\varepsilon > 0$ there exists N_0 such that $|F_N/N - L| < \varepsilon$ for all $N \ge N_0$ with $N > 0$.

Step 2: Choose ε and evaluate. Set $\varepsilon = 1/4$. Obtain N_0 from the modulus. Define $M = \max(N_0, 2)$, which ensures $M \ge N_0$ and $M \ge 2$.

Step 3: Case split on the Bool value. The key constructive step: $\mathrm{hasTrue}(\alpha, M)$ is a *Bool*—the case split is definitionally decidable. No real-number comparison is needed.

Case hasTrue(α, M) = `true`**:** By the witness extraction lemma, there exists $k < M$ with $\alpha(k) = 1$. We output $\exists n,\ \alpha(n) = 1$ with witness k.

Case hasTrue(α, M) = `false`**:** We prove $\forall n,\ \alpha(n) = 0$ by contradiction. Suppose there exists n_0 with $\alpha(n_0) = 1$. Define $K = \max(M, n_0 + 2)$, so $K \geq N_0$ and $n_0 < K$.

Since hasTrue(α, M) = `false`, we have $x_M = 0$, hence $F_M/M = 0$. Since $\alpha(n_0) = 1$ and $n_0 < K$, we have $x_K = 1$, hence $F_K/K = -1$.

From the Fekete modulus:

$$|F_M/M - L| = |0 - L| < 1/4, \tag{1}$$

$$|F_K/K - L| = |-1 - L| < 1/4. \tag{2}$$

From (1): $-1/4 < -L < 1/4$, i.e., $-1/4 < L < 1/4$.

From (2): $-1/4 < -1 - L < 1/4$, i.e., $-5/4 < L < -3/4$.

But the intervals $(-1/4, 1/4)$ and $(-5/4, -3/4)$ are disjoint (since $1/4 < 3/4$), so L cannot lie in both. This is a contradiction.

Since $\alpha(n) \in \{0, 1\}$ is decidable for each n, the conclusion $\neg(\exists n,\ \alpha(n) = 1)$ yields $\forall n,\ \alpha(n) = 0$.

Remark 5.4 (The gap)**.** The contradiction arises because the two possible limit values (0 and -1) are separated by a gap of 1, while $\varepsilon = 1/4$ forces the actual limit to lie within ε of both—an impossibility whenever $2\varepsilon <$ gap, i.e., $2 \times 1/4 = 1/2 < 1$. Any $\varepsilon < 1/2$ would work; we use 1/4 for clean arithmetic. The unit gap between the two limit values is a deliberate design feature of the encoding $F_n = -n \cdot x_n$: the choice of coefficient -1 (rather than, say, $-c$ for small c) ensures the gap is large enough for a simple ε-separation argument.

Remark 5.5 (Decidability of the case split)**.** As in Paper 8's BMC $\to$ LPO proof, the case split is on a *Bool*, not a real-number comparison. We compute hasTrue(α, M) from $\alpha(0), \ldots, \alpha(M-1)$ by finite recursion and branch on the result. This is the constructive core of the argument.

The full formalization in `Decision.lean`:

```
theorem lpo_of_fekete (hFek : FeketeConvergence) :
    LPO := by
  intro a
  set F := mockFreeEnergy a with hF_def
  have hSub := mockFreeEnergy_subadditive a
  have hBdd : exists C : Real,
    forall n : Nat, 0 < n -> C <= F n / n :=
    ⟨-1, fun n hn => mockFreeEnergy_div_bdd_below a n hn⟩
  obtain ⟨L, hL⟩ := hFek F hSub hBdd
  obtain ⟨N0, hN0⟩ := hL (1 / 4) (by positivity)
  set M := max N0 2 with hM_def
  -- ... case split on hasTrue a M ...
  cases hx : hasTrue a M
  . -- false: contradiction via disjoint intervals
    left; apply bool_not_exists_implies_all_false
    intro ⟨n0, hn0⟩
    -- evaluate at K = max(M, n0 + 2)
    -- get |0 - L| < 1/4 and |-1 - L| < 1/4
    -- linarith closes the contradiction
    ...
  . -- true: extract witness
    right
    obtain ⟨k, _, hk⟩ :=
      hasTrue_witness (show hasTrue a M = true from hx)
    exact ⟨k, hk⟩
```

6 CRM Audit

6.1 Axiom Profile

The `#print axioms` command in LEAN 4 reports the logical dependencies of each theorem:

Theorem	Axioms	Status
`lpo_of_fekete`	`propext, Classical.choice, Quot.sound`	No custom axioms ✓
`fekete_of_lpo`	+ `bmc_of_lpo, fekete_of_bmc`	Two cited axioms (axiom)
`fekete_iff_lpo`	+ `bmc_of_lpo, fekete_of_bmc`	Two cited axioms (axiom)

6.2 Classical.choice in the Infrastructure

The appearance of `Classical.choice` in `lpo_of_fekete` is an infrastructure artifact: Mathlib's construction of $\mathbb{R}$ as the Cauchy completion of $\mathbb{Q}$ uses classical choice pervasively. Every theorem that mentions real numbers inherits this dependency. As discussed in Paper 10 [10], this does not reflect classical content in the *proof* but rather in the *ambient infrastructure*.

The constructive content of the backward direction is certified by the following properties:

1. The encoding (`hasTrue`, `indicator`, `mockFreeEnergy`) uses no classical logic—only decidable Boolean operations and arithmetic.
2. The decision extraction uses a Bool case split (decidable) and a contradiction argument via `linarith` (which is a verified decision procedure, not a classical axiom).
3. The implication $\neg\exists \to \forall$ false for Bool sequences is constructively valid (Remark 5.5).
4. No Markov's Principle, no `Classical.em`, and no `Classical.choice` appear in the proof *content*.

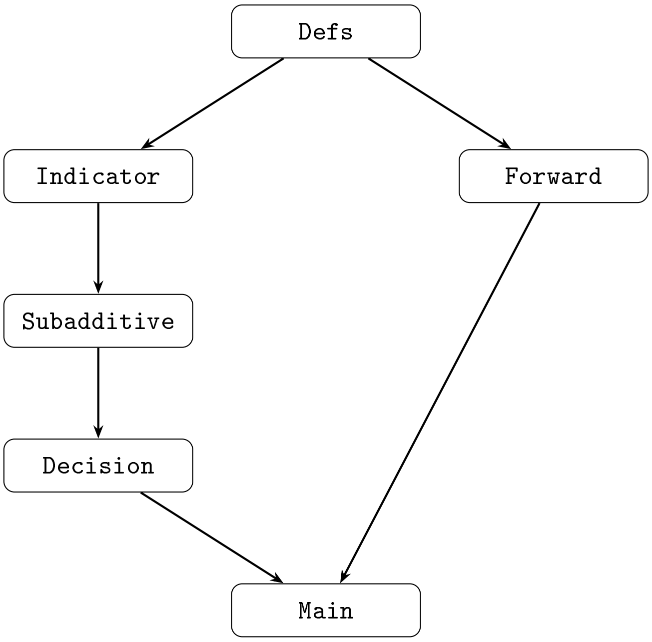
6.3 Constructive Certification Levels

Following Paper 10's methodology, the theorems fall into two certification levels:

Level	Description
Structurally verified	`lpo_of_fekete`: `Classical.choice` from $\mathbb{R}$ infrastructure only. No classical logic in proof content.
Intentionally classical	`fekete_of_lpo`: Uses BMC (which is LPO) by design—the whole point is to show that this use is necessary.

7 Code Architecture

7.1 Module Dependency Graph



7.2 Line Counts

File	Content	Lines
`Defs.lean`	Core definitions (LPO, BMC, FeketeConvergence, encoding)	87
`Indicator.lean`	hasTrue/indicator properties (monotonicity, witnesses)	118
`Subadditive.lean`	Subadditivity and lower bound proofs	103
`Decision.lean`	Main theorem: FeketeConvergence $\to$ LPO	117
`Forward.lean`	Axiomatized forward direction	43
`Main.lean`	Assembly: `fekete_iff_lpo`, axiom audit	81
Total		**549**

7.3 Key Design Decisions

1. **Bool encoding.** The binary sequence α is typed as $\mathbb{N} \to$ Bool (not $\mathbb{N} \to \{0,1\} \subset \mathbb{N}$). This makes case splits definitionally decidable without any lemma.
2. **Decidable bounded search via `List.any`.** The indicator uses `(List.range n).any` to compute x_n. This is computable and avoids any classical content in the encoding.
3. **Noncomputable section.** The `indicator` and `mockFreeEnergy` are marked `noncomputable` because they produce `Real` values (Mathlib's $\mathbb{R}$ is noncomputable). The *decision* (left/right disjunct of LPO) is computed from Bool operations, not from the real-valued functions.
4. **Monotonicity does the work.** The subadditivity proof requires only that the indicator is monotone—no case split on Bool is needed. This yields a clean `nlinarith` proof.

8 Reproducibility

Reproducibility box.

Component	Version / Commit
Lean 4	`v4.28.0-rc1`
Mathlib4	`2598404fe9e0a5aee87ffca4ff83e2d01b23b024`

Build instructions:

```
lake exe cache get     # download prebuilt Mathlib (~5 min)
lake build             # compile Paper 29 (~2-5 min)
```

Verification: A successful build produces 0 errors, 0 warnings, 0 `sorry`s. The axiom audits in `Main.lean` confirm the axiom profiles reported in §6.

All dependency versions are pinned in `lake-manifest.json` for exact reproducibility.

9 Discussion

9.1 Why Should a Physicist Care?

Fekete's subadditive lemma is not an obscure technicality—it is the standard tool by which statistical mechanics establishes the existence of the thermodynamic limit for interacting systems where no exact solution is available. Our result, FeketeConvergence $\leftrightarrow$ LPO, assigns a precise logical cost to this foundational step: the generic route to the thermodynamic limit requires exactly one infinite decision (LPO), no more, no less.

For a physicist, this means the following. Whenever one invokes Fekete's lemma to argue that a free energy density "exists in the thermodynamic limit," one is implicitly invoking LPO—the ability to decide, for an arbitrary binary sequence, whether it is identically zero. This is unavoidable for the generic argument. Note that the mock free energy $F_n = -n \cdot x_n$ used in our backward proof is a logical encoding, not a physical model: the equivalence FeketeConvergence $\leftrightarrow$ LPO is a statement about the *proof tool* (Fekete's lemma), not about any particular thermodynamic system. However, for specific models with explicit solutions (transfer matrices, cluster expansions), the LPO cost is *bypassed*: these models achieve convergence in pure BISH without any omniscience principle. The take-away is that the logical cost of the thermodynamic limit is not a single number but a spectrum, determined by the mathematical structure of the model.

9.2 Resolution of Paper 10, Problem 1

The main consequence of FeketeConvergence $\leftrightarrow$ LPO is a resolution of Problem 1 from Paper 10 [10]: the LPO cost of the thermodynamic limit is ineliminable *at the level of Fekete's lemma*—the generic tool used for arbitrary interacting systems. This yields a three-tier hierarchy for thermodynamic-limit convergence:

Route	Logical cost	Example
Exact solution (closed-form modulus)	BISH	1D Ising (Papers 8, 9)
Cluster expansion (exponential decay)	BISH	High-T lattice gases
Generic subadditivity (Fekete)	LPO	This paper

The first two tiers provide constructive convergence by exhibiting explicit Cauchy moduli. The third tier—Fekete's lemma—extracts convergence from a non-constructive infimum, and this extraction is unavoidably LPO.

9.3 Physical Interpretation

The three-tier hierarchy has a natural physical interpretation.

Tier 1 (exact solvability): Systems like the 1D Ising chain have closed-form partition functions (via transfer matrices, Bethe ansatz, etc.) that yield explicit error bounds. The convergence rate is typically exponential in the system size, with the rate depending on the spectral gap of the transfer matrix. No omniscience is needed.

Tier 2 (cluster expansions): For high-temperature or low-density systems, the cluster expansion [2, 7] provides an explicit convergent series for $\log Z_\Lambda$. The convergence is controlled by a fugacity parameter or inverse temperature and provides explicit moduli. The expansion converges when the activity is below a critical threshold—precisely the regime where correlations decay exponentially.

Tier 3 (generic Fekete): When neither exact solutions nor convergent expansions are available—typically near phase transitions where the correlation length diverges—the only route to the thermodynamic limit is the generic subadditivity argument. Our result shows this route costs exactly LPO.

The physical picture is suggestive: LPO becomes necessary precisely when the system loses its explicit finite-range structure. Phase transitions, where correlations become long-range and cluster expansions diverge, are the physical locus where the LPO cost becomes ineliminable.

9.4 Implications for Paper 10

Paper 10 [10] articulated the "logical geography hypothesis": each physical idealization occupies a specific position in the constructive hierarchy. The present result prompted a refinement of Paper 10's thesis (now incorporated in Paper 10 v5.0).

Paper 10's original formulation stated that the thermodynamic limit costs LPO. This is correct for the *generic* route through Fekete's lemma, but not for specific models with explicit solutions. The refined thesis is:

> *The thermodynamic limit costs* LPO *via the generic subadditivity route (Fekete's lemma). Exactly solvable models and models within the convergence radius of cluster expansions bypass Fekete and achieve convergence in* BISH*. The* LPO *cost becomes ineliminable at or near phase transitions.*

Paper 10 (v5.0) has been revised to incorporate this refinement, with its calibration table (Table 1) now distinguishing between the generic route ($\equiv$ LPO) and the model-specific routes (BISH).

9.5 The Boche–Bock–Deppe Connection

Boche, Bock, and Deppe [1] proved that the Fekete limit, as a functional from subadditive sequences to $\mathbb{R}$, is not Banach–Mazur computable—it lies above the arithmetical hierarchy. Our result shows a related but distinct fact: extracting the *existence* of the limit (not computing its value) already requires LPO.

Non-computability (a statement about Turing machines) and non-constructivity (a statement about proof principles) are different concepts, but they point in the same direction: the Fekete limit resists effective extraction at multiple levels of the mathematical hierarchy.

9.6 Open Questions

1. **Multi-dimensional subadditivity.** For $d > 1$, the subadditivity argument requires careful treatment of boundary effects (van Hove limits). Does the multi-dimensional version have the same LPO cost?

2. **Pressure functionals.** The thermodynamic formalism defines pressure as a supremum over measures. What is the constructive cost of this variational principle?

3. **Superadditivity.** Fekete's lemma applies equally to superadditive sequences ($u_{m+n} \geq u_m + u_n$). Is the LPO equivalence symmetric, or does one direction admit a weaker principle?

4. **Quantitative calibration.** Can the encoding be refined to give quantitative information about the relationship between the convergence rate of u_n/n and the decidability speed of the binary sequence?

10 AI-Assisted Methodology

The encoding $F_n = -n \cdot x_n$ and the proof strategy for FeketeConvergence $\to$ LPO were proposed by Gemini 3.0 DeepThink (Google DeepMind) in response to a detailed prompt about the constructive status of Fekete's lemma. The formalization in LEAN 4, including all tactic proofs, was carried out by Claude (Anthropic). The author supervised both stages, verified the mathematical content, and wrote the paper.

This workflow—mathematical insight from one AI system, formalization by another, supervision by a human—proved effective for bridging the gap between conceptual mathematics and machine-checked proof.

Preliminary status and author background. The results presented in this paper are preliminary. The author is a medical professional, not a domain expert in physics or mathematics. While all formal claims are machine-checked by the LEAN 4 type-checker, the physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts in the relevant fields. Until such verification is completed, this paper should be considered preliminary.

Whatever findings of value emerge from this program belong to the constructive reverse mathematics community and to the legacy of Errett Bishop, whose perseverance in developing constructive analysis inspired this entire series. Any errors are solely the author's.

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