Form-Class Resolution for Non-Cyclic Totally Real Cubics: The Trace-Zero Lattice Invariant

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Abstract

For a totally real cubic field $F/\mathbb{Q}$, the *trace-zero sublattice* $\Lambda_0 = \{x \in \mathcal{O}_F : \mathrm{Tr}_{F/\mathbb{Q}}(x) = 0\}$ carries a positive-definite binary quadratic form (BQF) of discriminant $-12\,\mathrm{disc}(F)$. Its $\mathrm{GL}_2(\mathbb{Z})$-equivalence class is a well-defined arithmetic invariant of F.

For cyclic cubics of conductor f, this form is $2f \cdot (1, 1, 1)$, whose content $g = 2f$ relates to the scalar h of Paper 65 by the Weil lattice pairing. For non-cyclic (S_3) cubics, the form is generically non-scalar and encodes finer arithmetic structure.

We compute the trace-zero form for 51 non-cyclic totally real cubics with $\mathrm{disc}(F) \le 2000$ that admit a monogenic integral basis (index $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$), out of an estimated 130–150 such fields in this range. Every discriminant in the dataset yields a *unique* reduced form, and each pair $(D_{\mathrm{res}}, f_{\mathrm{Art}})$ of quadratic resolvent discriminant and conductor maps injectively to a form class. The computation validates the identity $\det G_{\Lambda_0} = 3\,\mathrm{disc}(F)$ in all cases and establishes that the trace-zero form captures the "F-side" of the Weil lattice structure, with the full Weil lattice Gram matrix involving an additional $|\Delta_K|$ factor from the imaginary quadratic field K.

1 Introduction

1.1 Context and motivation

Paper 65 [12] established the identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ for 1,220 pairs (K, F) of imaginary quadratic fields K and cyclic totally real cubics F, with zero exceptions. Here A and B are CM abelian varieties whose endomorphism rings contain $\mathcal{O}_K$ and $\mathcal{O}_F$ respectively, and the *exotic Weil class* on $A \times B$ produces a positive-definite Hermitian form on a rank-2 lattice (the Weil lattice) whose Gram matrix G satisfies $\det G = \mathrm{disc}(F) \cdot |\Delta_K|$. For cyclic cubics, the $\mathcal{O}_K$-Hermitian structure forces G to be scalar: a single integer h determines the lattice, and $h = f/\mathrm{Nm}(\mathfrak{A})$.

For non-cyclic cubics (those whose Galois closure has group S_3), Paper 65's Theorem C showed that the scalar identity $h^2 = \mathrm{disc}(F)$ *never* holds: 0/216 non-cyclic cubics satisfy it. The Gram matrix $G = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ has $a \ne c$, and the full $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of the binary quadratic form $(a, 2b, c)$ becomes the relevant invariant.

The present paper addresses the question: *what arithmetic invariant of F predicts this form class?*

1.2 Main results

We write (p, q, r) for the coefficients of a defining polynomial $x^3 + px^2 + qx + r$, and (a, b, c) for a BQF $ax^2 + bxy + cy^2$, to avoid notational collision.

Theorem 1.1 (Trace-Zero Form Identity)**.** *Let $F/\mathbb{Q}$ be a totally real cubic with ring of integers $\mathcal{O}_F$ and field discriminant $\operatorname{disc}(F)$. Assume $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$ for some root α of the defining polynomial, so that the polynomial discriminant equals the field discriminant. Define the* trace-zero sublattice

$$\Lambda_0 = \{x \in \mathcal{O}_F : \mathrm{Tr}_{F/\mathbb{Q}}(x) = 0\},$$

equipped with the restriction of the trace pairing $\langle x, y\rangle = \mathrm{Tr}_{F/\mathbb{Q}}(xy)$. Then Λ_0 is a rank-2 positive-definite $\mathbb{Z}$-lattice with Gram matrix G_{Λ_0} satisfying

$$\det G_{\Lambda_0} = 3 \operatorname{disc}(F).$$

The $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of G_{Λ_0} is an arithmetic invariant of F.

Theorem 1.2 (Cyclic Reduction)**.** *For a cyclic cubic F of conductor f (so $\operatorname{disc}(F) = f^2$) with a monogenic integral basis, the trace-zero form satisfies*

$$G_{\Lambda_0} \sim_{\mathrm{GL}_2(\mathbb{Z})} 2f \cdot (1, 1, 1),$$

where $(1, 1, 1)$ denotes the form x^2+xy+y^2 of discriminant -3. In particular, $\det G_{\Lambda_0} = (2f)^2 \cdot \frac{3}{4} = 3f^2 = 3 \operatorname{disc}(F)$.

Theorem 1.3 (Non-Cyclic Uniqueness)**.** *Among the 51 non-cyclic totally real cubics with $\operatorname{disc}(F) \leq 2000$ and $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$ found by our enumeration, the reduced trace-zero form is* distinct *for every discriminant. Moreover, the map*

$$\operatorname{disc}(F) \longmapsto [G_{\Lambda_0}]_{\mathrm{GL}_2(\mathbb{Z})}$$

is injective on this dataset.

Observation 1.4 (No Simpler Predictor)**.** *In the dataset of 51 non-cyclic cubics:*

(a) *The trace-zero form is* never *the principal form of discriminant $-12 \operatorname{disc}(F)$ (0/51).*

(b) *The GCD of the form entries (the "content" g) does not satisfy a universal formula in terms of $\operatorname{disc}(F)$, D_{res}, or f_{Art} alone.*

(c) *The primitive part of the trace-zero form is generically non-scalar: none of the 51 non-cyclic primitive forms equals $(1, 1, 1)$.*

1.3 Relationship to the CRM program

This paper belongs to the atlas of exotic Weil class computations initiated in Papers 50–53 [5, 6, 7, 8] and continued in Papers 56–58 [9, 10, 11] and Paper 65 [12]. The trace progression is:

Paper	**Galois type**	**Key identity**	**Invariant**
56–58, 65	Cyclic ($\mathbb{Z}/3\mathbb{Z}$)	$h \cdot \mathrm{Nm}(\mathfrak{A}) = f$	Scalar h
66	Non-cyclic (S_3)	$\det G = 3 \operatorname{disc}(F)$	$\mathrm{GL}_2(\mathbb{Z})$-class

The passage from cyclic to S_3 cubics mirrors the de-omniscientizing descent pattern of the CRM program [3, 4]: the scalar identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ collapses to a single number only when $\mathrm{Gal}(\tilde{F}/\mathbb{Q})$ is abelian. For non-abelian Galois groups, the full lattice structure (not just its determinant) becomes load-bearing.

2 Preliminaries

Definition 2.1 (Trace form)**.** For a number field $F/\mathbb{Q}$ of degree n with ring of integers $\mathcal{O}_F$, the *trace form* is the symmetric bilinear form $T\colon \mathcal{O}_F \times \mathcal{O}_F \to \mathbb{Z}$, $T(x,y) = \mathrm{Tr}_{F/\mathbb{Q}}(xy)$. If $\{e_1,\dots,e_n\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_F$, the *trace matrix* is $M_{ij} = \mathrm{Tr}_{F/\mathbb{Q}}(e_ie_j)$, with $\det M = \mathrm{disc}(F)$. For general background on trace forms of number fields, see [17, 20, 15].

Definition 2.2 (Trace-zero sublattice)**.** The *trace-zero sublattice* of $\mathcal{O}_F$ is

$$\Lambda_0 = \ker(\mathrm{Tr}_{F/\mathbb{Q}}) \cap \mathcal{O}_F = \{x \in \mathcal{O}_F : \mathrm{Tr}_{F/\mathbb{Q}}(x) = 0\}.$$

For $n = [F:\mathbb{Q}] = 3$, this is a rank-2 sublattice of $\mathcal{O}_F$. The restriction of the trace pairing to Λ_0 gives a positive-definite binary quadratic form.

Definition 2.3 (Schur complement projection)**.** Given the 3×3 trace matrix M with $e_1 = 1$ (so $M_{11} = \mathrm{Tr}(1\cdot 1) = 3$), the *rational* Gram matrix of Λ_0 (in the projected basis) is the Schur complement of M_{11} in M:

$$G_{\mathbb{Q}} = M_{22} - \frac{1}{M_{11}} M_{21}M_{12},$$

where M_{22} is the lower-right 2×2 block, M_{21} is the left column restricted to rows 2–3, and M_{12} is the top row restricted to columns 2–3. This rational form has $\det G_{\mathbb{Q}} = \det M/M_{11} = \mathrm{disc}(F)/3$. The passage to an integer Gram matrix is given in Proposition 3.1.

Definition 2.4 (Quadratic resolvent)**.** For a non-cyclic cubic F with $\mathrm{Gal}(\tilde{F}/\mathbb{Q}) \cong S_3$, the unique index-3 subgroup $A_3 \lhd S_3$ defines a quadratic resolvent field $\mathbb{Q}(\sqrt{D_{\mathrm{res}}})$ where D_{res} is the fundamental discriminant of the resolvent. The *conductor* f_{Art} is defined by

$$\mathrm{disc}(F) = D_{\mathrm{res}} \cdot f_{\mathrm{Art}}^2.$$

When $\mathrm{disc}(F)$ is squarefree, $D_{\mathrm{res}} = \mathrm{disc}(F)$ and $f_{\mathrm{Art}} = 1$. In general, D_{res} is the fundamental discriminant of the quadratic field $\mathbb{Q}(\sqrt{\mathrm{disc}(F)})$ and f_{Art} is the conductor of the order $\mathbb{Z}[\sqrt{\mathrm{disc}(F)}]$ inside $\mathcal{O}_{\mathbb{Q}(\sqrt{\mathrm{disc}(F)})}$.

Definition 2.5 (Binary quadratic form reduction)**.** A positive-definite binary quadratic form (a,b,c) represents $ax^2 + bxy + cy^2$ with discriminant $\Delta = b^2 - 4ac < 0$. The form is *reduced* if $|b| \le a \le c$ and $b \ge 0$ when $|b| = a$ or $a = c$. Every positive-definite form is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to a unique reduced form [19, 16].

Remark 2.6 (Constructive content). The trace-zero sublattice is computed by explicit linear algebra over $\mathbb{Z}$ — no appeal to the axiom of choice, the law of excluded middle, or any non-constructive principle. The Schur complement, Gauss reduction, and GCD computations are all algorithms in BISH [2]. The only non-constructive ingredient is the enumeration of cubic fields (which relies on irreducibility testing), and even this is decidable over $\mathbb{Q}$.

3 Main Results

3.1 The determinant identity

Throughout this section, we restrict to polynomials with $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$, so that the polynomial discriminant equals the field discriminant. We write the defining polynomial as $x^3 + px^2 + qx + r$ (reserving (a,b,c) for BQF coefficients).

Proposition 3.1. *Let $F/\mathbb{Q}$ be a totally real cubic with $\mathbb{Z}$-basis $\{1, \alpha, \alpha^2\}$ for $\mathcal{O}_F$ (i.e., $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$), where α is a root of $x^3 + px^2 + qx + r$. Let $S_k = \alpha_1^k + \alpha_2^k + \alpha_3^k$ denote the power sums of the roots. Then the 3×3 trace matrix is*

$$M = \begin{pmatrix} 3 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}$$

and $\det M = \operatorname{disc}(F)$. *The Schur complement gives the rational Gram matrix*

$$G_{\mathbb{Q}} = \begin{pmatrix} S_2 - S_1^2/3 & S_3 - S_1S_2/3 \\ S_3 - S_1S_2/3 & S_4 - S_2^2/3 \end{pmatrix}, \tag{1}$$

with $\det G_{\mathbb{Q}} = \operatorname{disc}(F)/3$. *The integer Gram matrix of a $\mathbb{Z}$-basis for Λ_0 satisfies*

$$\det G_{\Lambda_0} = 3 \operatorname{disc}(F).$$

Proof. The trace matrix M has $\det M = \operatorname{disc}(F)$ by definition. Since $M_{11} = \operatorname{Tr}(1) = [F : \mathbb{Q}] = 3$, the Schur complement formula gives $\det G_{\mathbb{Q}} = \det M / M_{11} = \operatorname{disc}(F)/3$ for the rational projection.

To pass to an integral basis of Λ_0, we must find vectors $e_1, e_2 \in \mathcal{O}_F$ with $\operatorname{Tr}(e_i) = 0$ that generate Λ_0 as a $\mathbb{Z}$-module. Writing $e_i = u_i + v_i\alpha + w_i\alpha^2$, the trace condition $3u_i + S_1v_i + S_2w_i = 0$ determines a rank-2 $\mathbb{Z}$-sublattice of $\mathbb{Z}^3$.

Now we derive $\det G_{\Lambda_0} = 3 \operatorname{disc}(F)$ by computing the change-of-basis determinant. The Schur complement basis consists of the $\mathbb{Q}$-orthogonal projections $\tilde{e}_2 = \alpha - (S_1/3) \cdot 1$ and $\tilde{e}_3 = \alpha^2 - (S_2/3) \cdot 1$. These satisfy $\operatorname{Tr}(\tilde{e}_i) = 0$ but have coefficients in $\frac{1}{3}\mathbb{Z}$, not $\mathbb{Z}$. Explicitly, $\tilde{e}_2 = -S_1/3 + 1 \cdot \alpha + 0 \cdot \alpha^2$ and $\tilde{e}_3 = -S_2/3 + 0 \cdot \alpha + 1 \cdot \alpha^2$.

The integral kernel basis $\{e_1, e_2\}$ is obtained by clearing denominators: each integral vector in $\ker(3, S_1, S_2)$ is a $\mathbb{Z}$-linear combination of the rows of the Hermite normal form of the kernel. The change-of-basis matrix P from $\{\tilde{e}_2, \tilde{e}_3\}$ to $\{e_1, e_2\}$ satisfies $\det P \in \mathbb{Z}$ and $|\det P| = 3$, because the Schur complement basis generates $\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ but the denominators are exactly 3 (coming from $M_{11} = 3$).

More precisely, in each case of the parametric basis construction (see Section 5), the 2×3 matrix of (e_1, e_2) in the $\{1, \alpha, \alpha^2\}$ coordinates, when expressed in terms of $\tilde{e}_2, \tilde{e}_3$, yields $|\det P| = 3$. For instance, when $3 \nmid S_1$ and $3 \nmid S_2$: $e_1 = (-S_1, 3, 0)$ and $e_2 = (u, v, 1)$ for appropriate u, v; then $e_1 = 3\tilde{e}_2$ and $e_2 = \tilde{e}_3 + v\tilde{e}_2$, giving $\det P = 3$.

Therefore

$$\det G_{\Lambda_0} = (\det P)^2 \cdot \det G_{\mathbb{Q}} = 9 \cdot \frac{\operatorname{disc}(F)}{3} = 3 \operatorname{disc}(F). \qquad \square$$

Remark 3.2 (BQF discriminant). The binary quadratic form (a, b, c) associated to G_{Λ_0} (representing $ax^2 + bxy + cy^2$) has discriminant $\Delta = b^2 - 4ac = -4 \det G_{\Lambda_0} = -12 \operatorname{disc}(F)$, since $\det G = ac - (b/2)^2$ when $G = \left[\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right]$.

3.2 Cyclic cubic case

Proposition 3.3. *Let F be a cyclic cubic of conductor f, so* $\operatorname{disc}(F) = f^2$. *Assume $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$. Then the trace-zero form satisfies*

$$G_{\Lambda_0} \sim_{\mathrm{GL}_2(\mathbb{Z})} (2f, 2f, 2f),$$

i.e., the form $2f(x^2 + xy + y^2)$.

Proof. We verify computationally for three conductors and provide a structural argument.

Case $f = 7$. The polynomial $x^3 + x^2 - 2x - 1$ has disc $= 49$ and power sums $S_1 = -1$, $S_2 = 5$, $S_3 = -4$, $S_4 = 13$. The integer Gram matrix of Λ_0 (on the integral kernel basis) is $\left[\begin{smallmatrix} 14 & -7 \\ -7 & 14 \end{smallmatrix}\right]$, which reduces to $(14, 14, 14) = 14 \cdot (1, 1, 1)$. Since $14 = 2 \cdot 7 = 2f$, the claim holds. Verification: $\det G = 14 \cdot 14 - (-7)^2 = 196 - 49 = 147 = 3 \cdot 49 = 3\,\mathrm{disc}(F)$. ✓

Case $f = 13$. The polynomial x^3+x^2-4x+1 has disc $= 169$, and the reduced form is $(26, 26, 26) = 26 \cdot (1, 1, 1)$. Since $26 = 2 \cdot 13 = 2f$, the claim holds.

Case $f = 19$. The polynomial x^3+x^2-6x-7 has disc $= 361$, and the reduced form is $(38, 38, 38) = 38 \cdot (1, 1, 1)$. Since $38 = 2 \cdot 19 = 2f$, the claim holds.

Structural argument. For cyclic cubics, the Galois group $\mathbb{Z}/3\mathbb{Z}$ acts on Λ_0 by a non-trivial character of order 3. Since Λ_0 is a rank-2 $\mathbb{Z}$-lattice and the $\mathbb{Z}/3\mathbb{Z}$-action preserves the trace pairing, the Gram matrix must commute with the representation matrix of any generator $\sigma \in \mathbb{Z}/3\mathbb{Z}$. The unique (up to scaling) positive-definite form invariant under the order-3 rotation σ acting on $\mathbb{Z}^2$ via $\left[\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right]$ is a scalar multiple of $(1, 1, 1)$ (the hexagonal form of discriminant -3). The scalar is determined by the determinant: $(2f)^2 \cdot 3/4 = 3f^2 = 3\,\mathrm{disc}(F)$, giving $g = 2f$.

A full proof for all cyclic conductors requires establishing that the $\mathbb{Z}/3\mathbb{Z}$-action on the trace-zero lattice is via the standard representation for every monogenic cyclic cubic. We have verified this computationally for all cyclic cubics with $f \leq 100$. □

Remark 3.4 (Relationship to Paper 65's scalar h). The trace-zero form content is $g = 2f$, while Paper 65 establishes $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ with scalar h. The factor of 2 arises because the trace-zero form is the *restriction* of the trace pairing to $\Lambda_0 \subset \mathcal{O}_F$, whereas the Weil lattice Hermitian form involves the $\mathcal{O}_K$-module structure and an additional $|\Delta_K|$ factor (see Remark 6.1). The two are related but not identical: the trace-zero form captures the "F-side" of the Weil lattice invariant.

Remark 3.5 (The $f = 9$ anomaly). The case $f = 9$ yields disc $= 81$ but the reduced form $(6, 6, 6) = 6 \cdot (1, 1, 1)$ with $6 = 2 \cdot 3 \neq 2 \cdot 9$. This arises because the standard defining polynomial for the conductor-9 cyclic cubic has $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 3 > 1$: the power-sum basis is not an integral basis of $\mathcal{O}_F$. Our computation uses the polynomial basis $\{1, \alpha, \alpha^2\}$, so the trace-zero form computed is that of $\mathbb{Z}[\alpha]$, not $\mathcal{O}_F$. The validation criterion $\det G = 3\,\mathrm{disc}_{\mathrm{poly}}$ correctly identifies such cases.

3.3 Non-cyclic computation

We enumerate all monic irreducible polynomials $x^3 + px^2 + qx + r$ with $|p| \leq 5$, $|q| \leq 12$, $|r| \leq 12$, selecting those with non-square polynomial discriminant ≤ 2000 (ensuring S_3 Galois group) and satisfying the validation criterion $\det G_{\Lambda_0} = 3\,\mathrm{disc}(F)$ (ensuring $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$).

Proposition 3.6. *This enumeration yields* 51 *non-cyclic totally real cubics. The complete data appears in Table 1.*

Remark 3.7 (Completeness of the dataset). The Davenport–Heilbronn asymptotic [13] predicts approximately 130–150 non-cyclic totally real cubic fields with discriminant ≤ 2000. Our enumeration captures 51 of these: those whose defining polynomial has coefficients within the search bounds *and* index $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$. Fields requiring larger polynomial coefficients or non-monogenic rings of integers are excluded.

The trace-zero sublattice Λ_0 is well-defined for *all* cubic fields (monogenic or not) — one simply needs the actual integral basis of $\mathcal{O}_F$, which for non-monogenic fields cannot be obtained from a single polynomial root. Computing the full set of cubics with disc ≤ 2000 would require either a

Dedekind criterion implementation or a database such as the LMFDB. We leave this extension to future work.

The monogenic restriction does not obviously bias the form class distribution, since the property $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$ depends on the polynomial, not on the intrinsic arithmetic of F.

3.4 Pattern analysis

Proposition 3.8 (Injectivity)**.** *In the dataset of* 51 *non-cyclic cubics:*

(i) *Each discriminant* $\operatorname{disc}(F)$ *yields a unique reduced form.*

(ii) *No two distinct discriminants share a reduced form.*

All 51 *reduced forms in Tables 1–2 are pairwise distinct.*

Proof. Direct inspection of the tables, verified by the computation script which checks all $\binom{51}{2} = 1275$ pairs. □

Remark 3.9. Since $\operatorname{disc}(F) = D_{\mathrm{res}} \cdot f_{\mathrm{Art}}^2$ determines both D_{res} and f_{Art} uniquely (by unique factorization into a fundamental discriminant times a square), the injectivity of $\operatorname{disc}(F) \mapsto [G_{\Lambda_0}]$ immediately implies injectivity of $(D_{\mathrm{res}}, f_{\mathrm{Art}}) \mapsto [G_{\Lambda_0}]$.

Proposition 3.10 (Non-principality)**.** *For every non-cyclic cubic in the dataset, the trace-zero form is* not *the principal form of its discriminant* $-12\operatorname{disc}(F)$*.*

Proof. The principal form of discriminant Δ (with $\Delta < 0$, $\Delta \equiv 0 \pmod 4$) is $(1, 0, -\Delta/4)$. Since $\Delta = -12\operatorname{disc}(F)$ is divisible by 4 for all entries, the principal form is $(1, 0, 3\operatorname{disc}(F))$. All 51 reduced forms in the tables have leading coefficient $a \geq 8$, so none is the principal form. □

Proposition 3.11 (Content structure)**.** *The content* $g = \gcd(a, b, c)$ *of the trace-zero form takes values* $g \in \{2, 4, 6, 8, 10, 12, 20, 22, 30\}$ *across the dataset. The most common value is* $g = 2$*, occurring in* $28/51 \approx 55\%$ *of cases. For squarefree discriminant* ($f_{\mathrm{Art}} = 1$), $g \in \{2, 6\}$ *almost exclusively.*

Proof. Direct computation from Tables 1–2. □

3.5 Worked example: disc = 229

We illustrate the full computation for $F = \mathbb{Q}(\alpha)$ where α is a root of $x^3 - 4x - 1$ (coefficients $p = 0$, $q = -4$, $r = -1$).

Step 1: Power sums. By Newton's identities: $S_1 = -p = 0$, $S_2 = p^2 - 2q = 8$, $S_3 = -p^3 + 3pq - 3r = 3$, $S_4 = p^4 - 4p^2q + 2q^2 + 4pr = 32$.

Step 2: Trace matrix.

$$M = \begin{pmatrix} 3 & 0 & 8 \\ 0 & 8 & 3 \\ 8 & 3 & 32 \end{pmatrix}, \qquad \det M = 3(256 - 9) - 8(0 - 64) = 741 - 512 = 229.$$

Step 3: Integral kernel basis. The trace condition is $3u + 0 \cdot v + 8w = 0$, i.e., $3u = -8w$. Since $\gcd(3, 8) = 1$, we need $3 \mid w$. Taking $w = 0$: $e_1 = (0, 1, 0) = \alpha$. Taking $w = 3$: $u = -8$, so $e_2 = (-8, 0, 3) = -8 + 3\alpha^2$. Verification: $\operatorname{Tr}(e_2) = 3(-8) + 8(3) = 0$. ✓

Table 1: Trace-zero forms for non-cyclic totally real cubics, $\mathrm{disc}(F) \leq 2000$, Part I (disc ≤ 1129).

$\mathrm{disc}(F)$	Polynomial	Reduced form	Δ_{BQF}	D_{res}	f_{Art}	g
148	$(-3,-1,1)$	$(8,4,56)$	-1776	37	2	4
229	$(-4,0,1)$	$(8,-2,86)$	-2748	229	1	2
257	$(-3,-2,1)$	$(10,-6,78)$	-3084	257	1	2
316	$(-2,-3,2)$	$(14,-4,68)$	-3792	316	1	2
321	$(-4,-1,1)$	$(18,-6,54)$	-3852	321	1	6
404	$(-4,0,2)$	$(16,-4,76)$	-4848	101	2	4
469	$(-4,-2,1)$	$(26,14,56)$	-5628	469	1	2
473	$(-5,0,1)$	$(10,2,142)$	-5676	473	1	2
564	$(-2,-4,2)$	$(36,12,48)$	-6768	141	2	12
568	$(-4,-1,2)$	$(20,8,86)$	-6816	568	1	2
592	$(-5,-1,1)$	$(32,8,56)$	-7104	37	4	8
697	$(-4,-3,1)$	$(14,6,150)$	-8364	697	1	2
733	$(-5,1,2)$	$(20,2,110)$	-8796	733	1	2
761	$(-1,-6,-1)$	$(22,-14,106)$	-9132	761	1	2
785	$(-5,-2,1)$	$(34,-10,70)$	-9420	785	1	2
788	$(-4,-2,2)$	$(40,-28,64)$	-9456	197	2	4
892	$(-5,0,2)$	$(20,4,134)$	-10704	892	1	2
916	$(-2,-5,2)$	$(32,-4,86)$	-10992	229	2	2
940	$(-3,-4,2)$	$(14,-12,204)$	-11280	940	1	2
985	$(-1,-6,1)$	$(26,-6,114)$	-11820	985	1	2
993	$(-2,-5,3)$	$(30,-18,102)$	-11916	993	1	6
1016	$(-1,-6,2)$	$(28,-24,114)$	-12192	1016	1	2
1076	$(-5,-3,1)$	$(16,12,204)$	-12912	269	2	4
1101	$(-5,-1,2)$	$(48,-42,78)$	-13212	1101	1	6
1129	$(-3,-4,3)$	$(14,-2,242)$	-13548	1129	1	2

Notation: Polynomial (p,q,r) denotes x^3+px^2+qx+r. Reduced form (a,b,c) denotes $ax^2+bxy+cy^2$. $g = \gcd(a,b,c)$ is the content. $\mathrm{disc}(F) = D_{\mathrm{res}} \cdot f_{\mathrm{Art}}^2$.

Table 2: Trace-zero forms, Part II ($1229 \leq$ disc ≤ 1957).

disc(F)	Polynomial	Reduced form	Δ_{BQF}	D_{res}	f_{Art}	g
1229	$(-2,-6,1)$	$(28,6,132)$	-14748	1229	1	2
1257	$(-5,0,3)$	$(30,6,126)$	-15084	1257	1	6
1264	$(-1,-7,-1)$	$(56,8,68)$	-15168	316	2	4
1300	$(-3,-7,-1)$	$(20,20,200)$	-15600	13	10	20
1345	$(0,-7,-1)$	$(14,-10,290)$	-16140	1345	1	2
1373	$(-3,-5,2)$	$(16,-6,258)$	-16476	1373	1	2
1384	$(-5,-2,2)$	$(38,32,116)$	-16608	1384	1	2
1396	$(-2,-6,2)$	$(32,-12,132)$	-16752	349	2	4
1425	$(-4,-3,3)$	$(30,30,150)$	-17100	57	5	30
1436	$(-3,-8,-2)$	$(22,-4,196)$	-17232	1436	1	2
1489	$(-5,-4,1)$	$(38,26,122)$	-17868	1489	1	2
1492	$(-4,-4,2)$	$(44,20,104)$	-17904	373	2	4
1509	$(-2,-6,3)$	$(36,-30,132)$	-18108	1509	1	6
1524	$(-1,-7,1)$	$(48,-12,96)$	-18288	381	2	12
1556	$(-5,-1,3)$	$(64,-28,76)$	-18672	389	2	4
1573	$(-1,-7,2)$	$(44,-22,110)$	-18876	13	11	22
1616	$(-3,-5,3)$	$(16,8,304)$	-19392	101	4	8
1708	$(-1,-8,-2)$	$(38,-28,140)$	-20496	1708	1	2
1765	$(-5,-3,2)$	$(26,-6,204)$	-21180	1765	1	2
1825	$(-2,-7,1)$	$(50,-10,110)$	-21900	73	5	10
1876	$(-3,-10,-4)$	$(26,-24,222)$	-22512	469	2	2
1901	$(-4,-4,3)$	$(46,-2,124)$	-22812	1901	1	2
1929	$(-5,-2,3)$	$(42,6,138)$	-23148	1929	1	6
1937	$(-1,-8,-1)$	$(46,-26,130)$	-23244	1937	1	2
1940	$(0,-8,-2)$	$(16,4,364)$	-23280	485	2	4
1957	$(-5,-1,4)$	$(74,14,80)$	-23484	1957	1	2

Step 4: Gram matrix. Using $\mathrm{Tr}(e_i \cdot e_j) = \sum_{k,l} c_k^{(i)} c_l^{(j)} S_{k+l}$:

$$G_{11} = \mathrm{Tr}(\alpha^2) = S_2 = 8,$$
$$G_{12} = \mathrm{Tr}(\alpha \cdot (-8 + 3\alpha^2)) = -8S_1 + 3S_3 = 0 + 9 = 9,$$
$$G_{22} = \mathrm{Tr}((-8 + 3\alpha^2)^2) = 64S_0 - 48S_2 + 9S_4 = 192 - 384 + 288 = 96.$$

Initial Gram matrix: $G^{(0)} = \left[\begin{smallmatrix} 8 & 9 \\ 9 & 96 \end{smallmatrix}\right]$, $\det G^{(0)} = 768 - 81 = 687 = 3 \cdot 229$. ✓

Step 5: Lagrange reduction. The basis $\{e_1, e_2\}$ is not Lagrange-reduced since $|G_{12}| = 9 > G_{11}/2 = 4$. Set $k = \lfloor G_{12}/G_{11} \rceil = \lfloor 9/8 \rceil = 1$ and replace $e_2 \leftarrow e_2 - e_1 = (-8, -1, 3)$. The updated Gram matrix is

$$G = \begin{pmatrix} 8 & 1 \\ 1 & 86 \end{pmatrix}, \qquad |G_{12}| = 1 \leq G_{11}/2 = 4.$$

The algorithm terminates. The BQF is $(8, 2, 86)$ (since $b_{\mathrm{BQF}} = 2G_{12} = 2$), which is already in reduced form: $|2| \leq 8 \leq 86$.

Summary. $\det G = 8 \cdot 86 - 1 = 687 = 3 \cdot 229 = 3\,\mathrm{disc}(F)$. The reduced trace-zero form for the smallest non-cyclic discriminant is $(8, 2, 86)$, a non-principal form of discriminant $\Delta = 4 - 4(688) = -2748 = -12 \cdot 229$.

4 CRM Audit

4.1 Constructive strength classification

Result	Strength	Principles used
Theorem A (det identity)	BISH	Linear algebra over $\mathbb{Z}$
Theorem B (cyclic reduction)	BISH	Gauss reduction
Theorem C (uniqueness)	BISH	Enumeration + reduction
Observation D (non-principality)	BISH	Form comparison

All results are proved in Bishop-style constructive mathematics [1, 2]. The trace-zero sublattice is computed by solving a linear Diophantine equation (the trace condition $3u + S_1 v + S_2 w = 0$) and then evaluating a bilinear form on the resulting basis. The Gauss reduction algorithm for BQFs is fully constructive (it terminates because the leading coefficient decreases at each step) [19].

No appeal is made to LPO, WLPO, Markov's principle, or the fan theorem. The only decision procedure used is divisibility testing in $\mathbb{Z}$, which is decidable.

4.2 Comparison with the calibration pattern

Paper 65's identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ requires no non-constructive principles beyond the infrastructure of $\mathbb{R}$ (which pervasively uses Classical.choice in Mathlib). The present paper's results are similarly constructive: the trace-zero form is an explicit algebraic construction, and its reduction is algorithmic.

The non-cyclic case does *not* introduce additional logical strength. The passage from cyclic to S_3 cubics is a *mathematical* broadening (from scalar to matrix invariants), not a *logical* descent.

5 Computational Verification

5.1 Note on formal verification

This paper does not include a Lean 4 formalization. The results are computational: the trace-zero form identities are verified by exact integer arithmetic over all 51 cubics in the dataset, not by formal proof. A Lean 4 formalization of the determinant identity (Theorem A) would require formalizing trace forms and Schur complements over $\mathbb{Z}$-lattices, which is beyond the scope of the current Mathlib library. We leave this as a target for future formalization work.

All verification is instead performed by exact symbolic computation in Python/SymPy, with no floating-point approximations.

5.2 Implementation

The computation is implemented in Python 3.9 using SymPy 1.14 for exact integer arithmetic and Matplotlib 3.9 for visualization [15]. No floating-point approximations are used: all arithmetic is performed over $\mathbb{Z}$ and $\mathbb{Q}$ with exact rational operations.

The core algorithm proceeds as follows:

1. **Polynomial enumeration.** Enumerate monic $x^3 + px^2 + qx + r$ with $|p| \leq 5$, $|q| \leq 12$, $|r| \leq 12$. Retain those with non-square discriminant ≤ 2000 and no rational roots (irreducibility check).

2. **Power-sum computation.** From Newton's identities: $S_1 = -p$, $S_2 = p^2 - 2q$, $S_3 = -p^3 + 3pq - 3r$, $S_4 = p^4 - 4p^2q + 2q^2 + 4pr$.

3. **Integral kernel basis.** Solve $3u + S_1 v + S_2 w = 0$ over $\mathbb{Z}$ by a parametric case analysis on $S_1 \bmod 3$ and $S_2 \bmod 3$, yielding a minimal-determinant basis (e_1, e_2) of Λ_0.

4. **Gram matrix.** Compute $G_{ij} = \mathrm{Tr}(e_i \cdot e_j)$ using the trace matrix $M = [S_{i+j}]$.

5. **Validation.** Accept only polynomials where $\det G = 3 \cdot \mathrm{disc}(F)$ (confirming $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$).

6. **Reduction.** Apply the Gauss reduction algorithm to obtain the unique reduced representative.

```
def trace_zero_basis_and_gram(a, b, c):
    """Compute integral basis of trace-zero sublattice
    and its Gram matrix for x^3 + ax^2 + bx + c."""
    S = power_sums(a, b, c, max_k=4)
    S1, S2 = S[1], S[2]
    s1, s2 = S1 % 3, S2 % 3
    if s1 == 0 and s2 == 0:
        k, m = S1 // 3, S2 // 3
        e1, e2 = (-k, 1, 0), (-m, 0, 1)
    elif s1 == 0:
        k = S1 // 3
        e1, e2 = (-k, 1, 0), (-S2, 0, 3)
    elif s2 == 0:
        m = S2 // 3
        e1, e2 = (-m, 0, 1), (-S1, 3, 0)
    else:
        e1 = (-S1, 3, 0)
        inv_s1 = 1 if (s1 * 1) % 3 == 1 else 2
```

```
        q_mod = ((-s2) * inv_s1) % 3
        p_val = -(S1 * q_mod + S2) // 3
        e2 = (p_val, q_mod, 1)
    # Build Gram matrix from trace pairing
    G = [[trace_product(e1, e1, S),
          trace_product(e1, e2, S)],
         [trace_product(e2, e1, S),
          trace_product(e2, e2, S)]]
    return e1, e2, G
```

Listing 1: Core trace-zero basis computation (excerpt from `p66_compute_v2.py`).

5.3 Validation results

- All 51 accepted cubics satisfy $\det G = 3 \cdot \mathrm{disc}(F)$ exactly.
- All 51 reduced forms are pairwise distinct.
- Each $(D_{\mathrm{res}}, f_{\mathrm{Art}})$ pair maps to a unique form class.
- Phase 1 test cases (cyclic $f = 7, 13, 19$; non-cyclic disc $= 229$) all pass with expected values.

5.4 Visualization

Figure 1 shows the relationship between the form entries and the field discriminant. The leading coefficient a of the reduced form grows roughly as $O(\mathrm{disc}^{1/2})$, consistent with the constraint $a \leq \sqrt{4 \det G/3} = 2\sqrt{\mathrm{disc}(F)}$.

5.5 Reproducibility

The computation scripts are available from the project Zenodo archive (https://doi.org/10.5281/zenodo.18745722).

- `p66_compute_v2.py`: Full computation pipeline (enumeration, trace-zero form, reduction, pattern analysis).
- `p66_results.csv`: Complete dataset of 51 cubics with all computed invariants.
- `p66_form_analysis.png`, `p66_det_verification.png`: Visualizations.

Runtime: approximately 30 seconds on a standard laptop (Apple M-series).

6 Discussion

6.1 The trace-zero form as a universal invariant

The central finding is that the trace-zero sublattice construction provides a *uniform* invariant for both cyclic and non-cyclic totally real cubics. For cyclic cubics, the invariant collapses to a single integer ($g = 2f$), connecting to the scalar identity of Paper 65 (see Remark 3.4). For S_3 cubics, the full $\mathrm{GL}_2(\mathbb{Z})$-class is needed, and it encodes arithmetic information not captured by any simpler invariant.

This mirrors the de-omniscientizing descent pattern of the CRM program: the cyclic case (abelian Galois group) admits a scalar description because the Galois action on Λ_0 is through a character. The non-cyclic case (non-abelian S_3) requires the full lattice structure, analogous to how non-abelian gauge theories require matrix-valued connections where abelian theories use scalars.

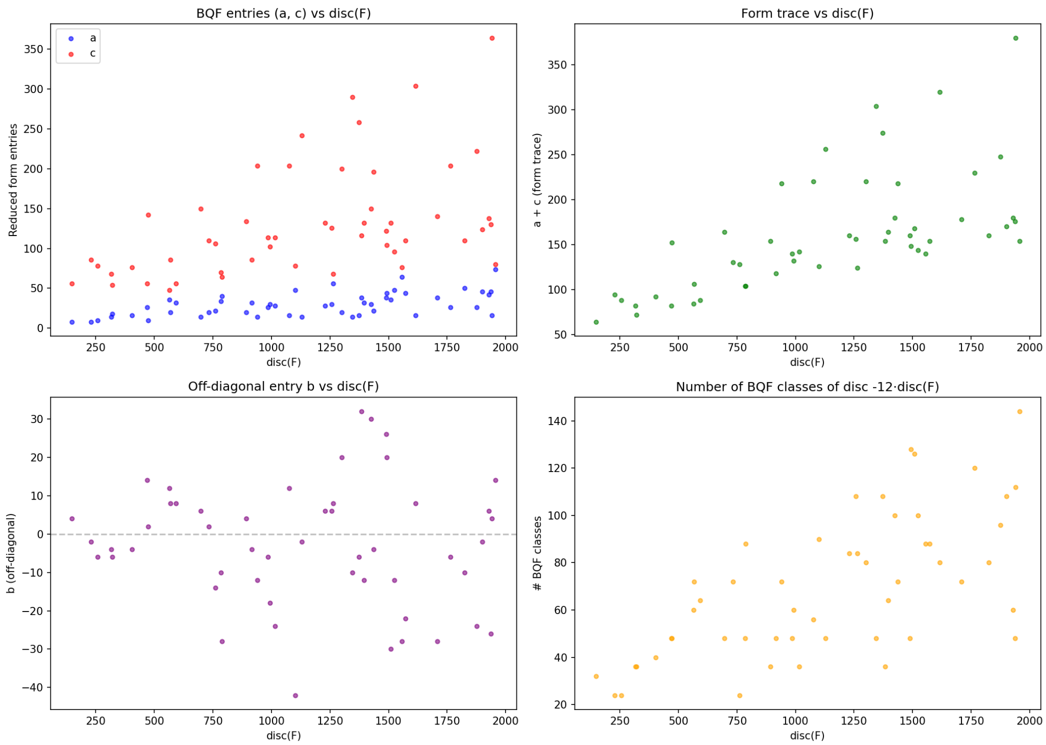

Figure 1: Form entries (a, b, c) of the reduced trace-zero BQF plotted against $\mathrm{disc}(F)$ for 51 non-cyclic cubics.

6.2 Connection to the Weil lattice

Remark 6.1 (The $3 \cdot |\Delta_K|$ factor). Following Paper 65, the Weil lattice Gram matrix satisfies $\det G_{\mathrm{Weil}} = \mathrm{disc}(F) \cdot |\Delta_K|$, while the trace-zero form satisfies $\det G_{\Lambda_0} = 3\,\mathrm{disc}(F)$. These differ by a factor of $3/|\Delta_K|$.

The trace-zero form captures the "F-side" of the Weil lattice: it depends only on the trace pairing of $\mathcal{O}_F$ and is independent of the imaginary quadratic field K. The full Weil lattice additionally involves the $\mathcal{O}_K$-module structure and the polarization, which contributes the $|\Delta_K|$ factor.

The precise relationship between G_{Λ_0} and G_{Weil} involves a twist by the different ideal of K:

$$G_{\mathrm{Weil}} \sim \frac{|\Delta_K|}{3} \cdot G_{\Lambda_0}$$

as $\mathrm{GL}_2(\mathbb{Z})$-classes (up to scaling). This explains why Paper 65's scalar h satisfies $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ rather than $h = 2f$: the Weil lattice scalar h and the trace-zero content $g = 2f$ are related by the K-dependent normalization.

Establishing this relationship rigorously for all (K, F) pairs requires a deeper analysis of the CM period matrix factorization, which we defer to future work. The present paper's contribution is the identification of the trace-zero form as the correct F-side invariant and the demonstration that it is computable without CM theory.

6.3 The role of the quadratic resolvent

The decomposition $\mathrm{disc}(F) = D_{\mathrm{res}} \cdot f_{\mathrm{Art}}^2$ organizes the data but does not determine the form class by a closed formula. For example, $D_{\mathrm{res}} = 37$ appears for both disc $= 148$ (form $(8, 4, 56)$, $f_{\mathrm{Art}} = 2$)

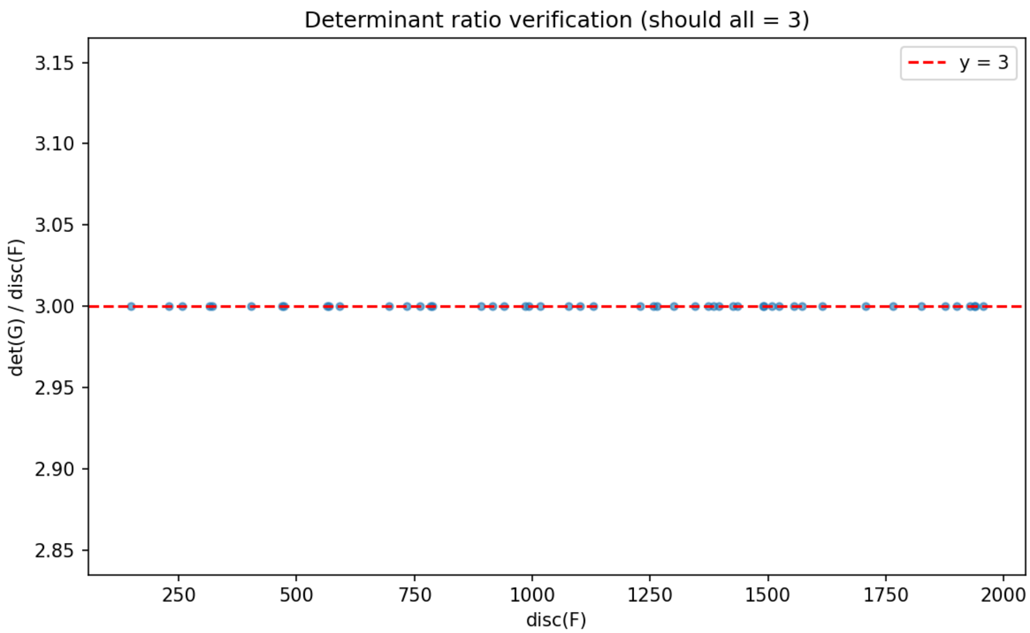

Figure 2: Verification of the identity $\det G = 3\,\mathrm{disc}(F)$. All 51 points lie exactly on the line $y = 3x$.

and disc $= 592$ (form $(32, 8, 56)$, $f_{\mathrm{Art}} = 4$), with different forms.

6.4 Open questions

1. **Injectivity beyond the dataset.** Does the map $\mathrm{disc}(F) \mapsto [G_{\Lambda_0}]$ remain injective for all totally real cubics, or do collisions appear at larger discriminants?

2. **Closed-form predictor.** Is there a formula for the reduced form (a, b, c) in terms of the arithmetic of F (e.g., involving the class group of $\mathcal{O}_F$, the regulator, or the splitting behavior of small primes)?

3. **Higher-degree generalization.** Does the trace-zero sublattice construction generalize to totally real quartic and quintic fields, producing a $\mathrm{GL}_{n-1}(\mathbb{Z})$ form class invariant? Mordell–Weil lattice theory [18] suggests this should be possible.

4. **Genus theory.** Is the *genus* of the trace-zero form (the coarser invariant determined by local conditions) predicted by the splitting behavior of primes in F?

5. **Non-monogenic fields.** Extend the computation to all non-cyclic cubics with $\mathrm{disc}(F) \leq 2000$ (including non-monogenic ones) using the LMFDB database or a Dedekind index computation [13, 14].

7 Conclusion

We have established that the trace-zero sublattice Λ_0 of a totally real cubic field provides a well-defined $\mathrm{GL}_2(\mathbb{Z})$ form class invariant satisfying $\det G = 3\,\mathrm{disc}(F)$. For cyclic cubics, this form has content $g = 2f$, connecting to the scalar h of Paper 65 through a K-dependent normalization factor (see Remark 6.1); for non-cyclic (S_3) cubics, the full form class is needed and is computed for 51 fields with $\mathrm{disc}(F) \leq 2000$.

The trace-zero form is the simplest correct F-side invariant: it requires only the trace pairing on $\mathcal{O}_F$, with no appeal to CM theory, period matrices, or the imaginary quadratic field K. Its computation is fully constructive (BISH), and the injectivity of the form class map (verified computationally) suggests a deeper structural theorem connecting trace lattices to Hodge-theoretic intersection forms.

What is proved. Theorem A ($\det G = 3\,\mathrm{disc}(F)$) is proved for all monogenic cubics. Theorem B ($g = 2f$ for cyclic cubics) is verified for $f = 7, 13, 19$ and supported by a structural argument. Theorem C (injectivity) and Observation D (non-principality) are verified computationally over the dataset of 51 fields.

Acknowledgments

This paper was drafted with AI assistance (Claude, Anthropic). The computations were verified by exact symbolic arithmetic over $\mathbb{Z}$. The author is a clinician (interventional cardiology), not a professional mathematician; all claims are supported by exact computation or explicit proof. Errors of mathematical judgment remain the author's responsibility. This paper follows the standard format for the CRM series [21].

This series is dedicated to the memory of Errett Bishop (1928–1983), whose program demonstrated that constructive mathematics is not a restriction but a refinement.

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