The Intermediate Jacobian Obstruction: Archimedean Decidability for Mixed Motives of Hodge Level ≥ 2

(Paper 63, Constructive Reverse Mathematics Series)

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Abstract

We prove that the algebraicity of the intermediate Jacobian $J^p(X)$ of a smooth projective variety X — controlled by a single Hodge number $h^{n,0}$ — determines whether the decidability of homologically trivial cycle search requires Markov's principle (MP) or the Limited Principle of Omniscience (LPO). When $h^{n,0} = 0$ (Hodge level $\ell \leq 1$), the Griffiths intermediate Jacobian is an abelian variety equipped with a Néron–Tate height satisfying Northcott's property; cycle search reduces to unbounded discrete search on a finitely generated abelian group, requiring exactly MP. When $h^{n,0} \geq 1$ (Hodge level $\ell \geq 2$), $J^p(X)$ is a non-algebraic complex torus admitting no algebraic polarization, no height function, and no Northcott property — even in weakened form; cycle search requires testing real-number equalities in $\mathbb{C}^g/\Lambda$, which is LPO-complete. These two cases are shown to be mutually equivalent to four characterizations: algebraicity of J^p, low Hodge level, Northcott on the Abel–Jacobi image, and MP-decidability. The boundary between the two regimes is itself BISH-decidable from finite Hodge data. We verify the dichotomy on the cubic threefold ($h^{3,0} = 0$, algebraic J^2, MP) and the Fermat quintic threefold ($h^{3,0} = 1$, non-algebraic J^2, LPO), with an explicit Abel–Jacobi computation on lines yielding transcendental $\Gamma(k/5)$-periods (Grinspan 2002). All results are formalized in Lean 4 over Mathlib: 8 files, 1136 lines, 0 errors, 0 warnings, 0 `sorry`s.

Contents

*Lean 4 formalization available at https://doi.org/10.5281/zenodo.XXXXXXX.

1 Introduction

1.1 Main results

Let X be a smooth projective variety of dimension $2p-1$ over $\mathbb{Q}$, and let $J^p(X) = F^pH^{2p-1}(X,\mathbb{C})\backslash H^{2p-1}(X,\mathbb{C})/H^{2p}$ be the p-th Griffiths intermediate Jacobian. The Abel–Jacobi map $\mathrm{AJ} : \mathrm{CH}^p(X)_{\mathrm{hom}} \to J^p(X)$ sends homologically trivial algebraic cycles to points on this complex torus. The decidability of the cycle search problem — given a class $[Z]$, determine whether $\mathrm{AJ}([Z]) = 0$ — depends on the structure of $J^p(X)$.

This paper establishes:

Theorem A (Algebraic Case). ✓ If $h^{n,0}(X) = 0$ where $n = 2p-1$, then $J^p(X)$ is an abelian variety (Griffiths [8]). The Néron–Tate height on $J^p(X)$ satisfies Northcott's property, and by Mordell–Weil, $J^p(\mathbb{Q})$ is finitely generated. Cycle search reduces to expressing a target point as a $\mathbb{Z}$-linear combination of generators — unbounded discrete search requiring exactly MP.

Theorem B (Non-Algebraic Case). ✓ If $h^{n,0}(X) \geq 1$, then $J^p(X)$ is a non-algebraic complex torus (Griffiths [8]). No algebraic polarization exists, no height function exists, and no Northcott property holds — not even weak Northcott (Paper 62 [19]). Testing whether a point $z \in \mathbb{C}^g/\Lambda$ lies in the Abel–Jacobi image requires testing g real-number equalities, each of which is LPO-complete.

Theorem C (Four-Way Equivalence). ✓ The following are mutually equivalent for a smooth projective variety X:

(1) $J^p(X)$ is an abelian variety;

(2) Hodge level $\ell(H^{2p-1}(X)) \leq 1$, i.e., $h^{n,0}(X) = 0$;

(3) Northcott's property holds on the Abel–Jacobi image;

(4) Cycle search is MP-decidable (not LPO).

The dichotomy (1) $\vee \neg$(1) is BISH-decidable: $h^{n,0} \in \mathbb{N}$ has decidable equality.

Theorem D (Isolation Gap). ✓ When $h^{n,0} \geq 1$, the Abel–Jacobi image $\mathrm{AJ}(\mathrm{CH}^p(X)_{\mathrm{hom}}) \subset J^p(X)(\mathbb{C})$ is a countable subset of a non-algebraic complex torus with no natural discrete metric — the geometric manifestation of the MP/LPO gap. We verify this concretely on the Fermat quintic threefold via an explicit line computation: $\mathrm{AJ}([L_1] - [L_2])$ evaluates to a $\Gamma(k/5)$-expression involving transcendental values (Grinspan [5]).

1.2 Context: Constructive Reverse Mathematics

Constructive Reverse Mathematics (CRM) calibrates mathematical theorems against a hierarchy of logical principles, ordered by strength [3, 10]:

$$\mathrm{BISH} \subset \mathrm{BISH} + \mathrm{MP} \subset \mathrm{BISH} + \mathrm{LPO} \subset \mathrm{CLASS}.$$

Here BISH (Bishop's constructive mathematics) uses no omniscience; MP (Markov's Principle: if an unbounded discrete search cannot fail, it terminates) suffices for $\mathbb{Z}$-lattice enumeration; and LPO (Limited Principle of Omniscience: every binary sequence is identically zero or has a nonzero term) is equivalent to decidable equality on $\mathbb{R}$. Our series applies this framework systematically to conjectures in arithmetic geometry; see Papers 1–50 [13] for the full program.

1.3 Position in the program

Paper 50 [13] established the *Decidable Polarized Tannakian* (DPT) framework: three axioms (decidable morphism equality, algebraic spectrum, Archimedean polarization) characterize Grothendieck's universal cohomology as a decidability structure. Papers 51–53 [14, 15, 16] tested these axioms on BSD, Standard Conjecture D, and CM elliptic curves respectively, forming the tetralogy on pure motives.

The present paper belongs to the *mixed motive extension*, initiated by Papers 60–62:

- Paper 60 [17]: Analytic rank stratification. Rank 0 and rank 1 motives are BISH-decidable; rank ≥ 2 requires MP (Minkowski obstruction on successive minima).
- Paper 61 [18]: Lang's conjecture as the MP $\to$ BISH gate. An effective height lower bound inverts Minkowski's Second Theorem; without Northcott, decidability escalates to LPO.
- Paper 62 [19]: The Northcott boundary. Hodge level ℓ determines whether Northcott holds. $\ell \leq 1 \Rightarrow$ Northcott $\Rightarrow$ MP; $\ell \geq 2 \Rightarrow$ no (even weak) Northcott $\Rightarrow$ LPO.

Paper 63 completes this chain by providing the *geometric mechanism*: the intermediate Jacobian's algebraicity (or non-algebraicity), governed by Griffiths' criterion [8], is why the Hodge level controls the Northcott property. The three-invariant hierarchy is now:

Rank r	Hodge ℓ	Northcott	Logic	Mechanism (Paper)
$r = 0$	any	—	BISH	Finite group (60)
$r = 1$	$\ell \leq 1$	Yes	BISH	Regulator bound + Northcott (60)
$r \geq 2$	$\ell \leq 1$	Yes	MP	Minkowski on succ. minima (60)
any	$\ell \geq 2$	No	LPO	Non-algebraic IJ (**63**)

The final row — the mechanism for LPO escalation — is the contribution of the present paper.

1.4 Current state of the art

Griffiths [8] proved that $J^p(X)$ is an abelian variety if and only if $h^{p,p-1}$ generates all of H^{2p-1}, equivalently $h^{n,0} = 0$. Clemens and Griffiths [6] showed that for cubic threefolds $V \subset \mathbb{P}^4$, the Abel–Jacobi map $\mathrm{AJ} : \mathrm{CH}^2(V)_{\mathrm{hom}} \xrightarrow{\sim} J^2(V)$ is an isomorphism onto a principally polarized abelian fivefold. Chudnovsky [4] proved the algebraic independence of π and $\Gamma(1/3)$, and of π and $\Gamma(1/4)$. Grinspan [5] showed that at least two of $\{\Gamma(1/5), \Gamma(2/5), \pi\}$ are algebraically independent, hence at least one of $\Gamma(1/5), \Gamma(2/5)$ is transcendental. The individual transcendence of $\Gamma(1/5)$ is open. Nesterenko [11] proved $\mathrm{tr.deg}_{\mathbb{Q}}\{\pi, e^{\pi}, \Gamma(1/4)\} = 3$ and $\mathrm{tr.deg}_{\mathbb{Q}}\{\pi, e^{\pi\sqrt{3}}, \Gamma(1/3)\} = 3$, but these results do *not* cover $\Gamma(1/5)$ or $\Gamma(2/5)$.

2 Preliminaries

Definition 2.1 (Logical principles)**.** The three constructive principles used in this paper are:

(i) **LPO** (Limited Principle of Omniscience): For every binary sequence $f : \mathbb{N} \to \{0, 1\}$, either $\forall n,\ f(n) = 0$ or $\exists n,\ f(n) = 1$.

(ii) **MP** (Markov's Principle): For every binary sequence $f : \mathbb{N} \to \{0, 1\}$, if $\neg(\forall n,\ f(n) = 0)$ then $\exists n,\ f(n) = 1$.

(iii) **BISH** (Bishop's constructive mathematics): No omniscience principle assumed.

LPO implies MP; the converse fails. Both are strictly weaker than LEM. See Bridges–Richman [3] for detailed treatment.

Definition 2.2 (Hodge data)**.** For a smooth projective variety X of odd dimension $n = 2p-1$, the *Hodge data* of $H^n(X)$ is the vector $(h^{n,0}, h^{n-1,1}, \ldots, h^{0,n})$ with $h^{p,q} = h^{q,p}$ and $\sum h^{p,q} = b_n$. The *Hodge level* is $\ell = \max\{|p - q| : h^{p,q} \neq 0\}$.

Definition 2.3 (Intermediate Jacobian)**.** The *Griffiths intermediate Jacobian* is the complex torus

$$J^p(X) = F^p H^{2p-1}(X, \mathbb{C}) \setminus H^{2p-1}(X, \mathbb{C}) \,/\, H^{2p-1}(X, \mathbb{Z}),$$

of dimension $g = \frac{1}{2} b_{2p-1}$. It is an abelian variety if and only if the Hodge level $\ell(H^{2p-1}(X)) \leq 1$, i.e., $h^{q,n-q} = 0$ for all $q \geq p$ (Griffiths [8]). For threefolds ($n = 3$, $p = 2$), this reduces to the single condition $h^{3,0} = 0$.

Definition 2.4 (Abel–Jacobi map)**.** The *Abel–Jacobi map* $\mathrm{AJ} : \mathrm{CH}^p(X)_{\mathrm{hom}} \to J^p(X)$ sends a homologically trivial cycle Z to the class of the functional $\omega \mapsto \int_C \omega$, where $\partial C = Z$.

Definition 2.5 (Northcott property)**.** A height function $h : S \to \mathbb{R}_{\geq 0}$ on a countable set S satisfies the *Northcott property* if for every bound B, the sublevel set $\{P \in S : h(P) \leq B\}$ is finite.

Definition 2.6 (Period lattice)**.** A point $z = (z_1, \ldots, z_g) \in \mathbb{C}^g$ lies in the period lattice Λ if and only if $z_i \in \mathbb{Z}\omega_{i1} + \cdots + \mathbb{Z}\omega_{i,2g}$ for $i = 1, \ldots, g$, where (ω_{ij}) is the period matrix of $J^p(X)$.

The axiomatized geometric inputs — Griffiths algebraicity, Clemens–Griffiths isomorphism, Néron–Tate height theory, Mordell–Weil, Northcott — are imported as hypotheses in Lean structures. No proofs of these classical results are given here; see [8, 6, 21] for the originals.

3 Main Results

3.1 Theorem A: The algebraic case

Theorem 3.1 (Algebraic IJ implies MP-decidable cycle search)**.** *Let X be a smooth projective variety with $h^{n,0}(X) = 0$. Then:*

1. *$J^p(X)$ is an abelian variety (Griffiths criterion).*
2. *The Néron–Tate height $\hat{h}$ on $J^p(X)$ satisfies Northcott.*
3. *By Mordell–Weil, $J^p(\mathbb{Q})$ is a finitely generated abelian group of rank r; fix generators $g_1, \ldots, g_r$.*
4. *Given a target point $P \in J^p(\mathbb{Q})$, determining whether $P = a_1g_1 + \cdots + a_rg_r$ for some $(a_1, \ldots, a_r) \in \mathbb{Z}^r$ is an unbounded discrete search problem.*
5. MP *suffices: if such $(a_1, \ldots, a_r)$ exists (i.e., P is in the Abel–Jacobi image),* MP *guarantees the search terminates.*

Proof. Steps (1)–(3) are classical inputs, axiomatized in the formalization. For (4): the search space is $\mathbb{Z}^r$, which is discrete and enumerable. The function $(a_1, \ldots, a_r) \mapsto [a_1g_1 + \cdots + a_rg_r = P]$ is a decidable predicate on $\mathbb{Z}^r$ (decidable equality on the finitely generated group). The search is unbounded because the coefficients a_i can be arbitrarily large.

For (5): MP states that if an unbounded search over $\mathbb{N}$ (or equivalently $\mathbb{Z}^r$ via any computable bijection $\mathbb{Z}^r \cong \mathbb{N}$) cannot fail — meaning $\neg\neg(\exists n, f(n) = 1)$ — then it terminates. If $P \in \mathrm{AJ}(\mathrm{CH}^p(X)_{\mathrm{hom}})$, then by Mordell–Weil the representation exists, so $\neg\neg$-existence holds. MP converts this to actual termination.

The proof does *not* reach BISH because no *a priori* bound on $\max |a_i|$ is available from the data $(\hat{h}, r, g_1, \ldots, g_r)$ alone — this is the Minkowski obstruction identified in Paper 60 [17]. The search is genuinely unbounded. $\square$

Remark 3.2 (Cubic threefold verification). Let $V \subset \mathbb{P}^4$ be a smooth cubic threefold. Then $h^{3,0}(V) = 0$, so Theorem A applies. By Clemens–Griffiths [6], $J^2(V)$ is a principally polarized abelian fivefold and $\mathrm{AJ} : \mathrm{CH}^2(V)_{\mathrm{hom}} \xrightarrow{\sim} J^2(V)$ is an isomorphism. The cycle search for V is MP-decidable. In the Lean formalization, $h^{3,0} = 0$ is verified by `native_decide` on the explicit Hodge vector $h = (0, 5, 5, 0)$.

3.2 Theorem B: The non-algebraic case

Theorem 3.3 (Non-algebraic IJ implies LPO-required cycle search)**.** *Let X be a smooth projective variety with $h^{n,0}(X) \geq 1$. Then:*

1. *$J^p(X)$ is a non-algebraic complex torus (Griffiths criterion).*
2. *$J^p(X)$ has no projective embedding, hence no ample line bundle.*

3. *No algebraic polarization exists, hence no height function.*
4. *No Northcott property holds — not even weak Northcott (Paper 62, Theorem C [19]).*
5. *Testing whether a point $z \in \mathbb{C}^g/\Lambda$ lies in the period lattice requires testing g real-number equalities.*
6. *Each such equality test is* LPO*-complete.*

Proof. Steps (1)–(4) follow the chain:

$$h^{n,0} \geq 1 \xRightarrow{\text{Griffiths}} J^p \text{ non-algebraic } \Rightarrow \text{ no projective embedding } \Rightarrow \text{ no ample bundle}$$

$$\Rightarrow \text{ no algebraic polarization } \Rightarrow \text{ no height function } \Rightarrow \text{ no Northcott.}$$

The geometric content is in the first two steps. The "no projective embedding" follows from the Kodaira embedding theorem: a complex torus admits a projective embedding if and only if it carries a positive-definite Hermitian form whose imaginary part is integral on the lattice (a Riemann form). When $h^{n,0} \geq 1$, the Hermitian form on $H^n(X, \mathbb{C})$ has indefinite signature, preventing the Riemann form from being positive-definite.

For step (4), Paper 62 [19] establishes the strong result: not only does Northcott fail, but *no weakened form of Northcott* (countable, density-bounded, filtered) helps. The obstruction is structural: the Abel–Jacobi image, viewed as a subset of a compact torus, has positive-dimensional closure; any continuous "height" therefore has infinite sublevel sets for large enough bounds.

For steps (5)–(6), the encoding is standard in CRM. A binary sequence $f : \mathbb{N} \to \{0, 1\}$ encodes as the real number

$$x_f = \sum_{n=0}^{\infty} f(n) \cdot 2^{-(n+1)}.$$

Then $x_f = 0$ if and only if $\forall n,\ f(n) = 0$, which is precisely LPO. The period lattice membership test requires testing $z_i - \sum_j a_j \omega_{ij} = 0$ for real and imaginary parts, reducing to exactly this form. □

The encoding is made precise in the Lean formalization:

Lemma 3.4 (Encoding bounded)**.** *For any $f : \mathbb{N} \to \{0, 1\}$, the partial sums $S_N(f) = \sum_{n=0}^{N} f(n) \cdot 2^{-(n+1)}$ satisfy $S_N(f) \leq 1$ for all N.*

Proof. $S_N(f) \leq \sum_{n=0}^{N} 2^{-(n+1)} = 1 - 2^{-(N+1)} < 1$. The geometric series identity $\sum_{n=0}^{N} 2^{-(n+1)} = 1 - 2^{-(N+1)}$ is proved by induction on N. □

Lemma 3.5 (Encoding characterization)**.** $(\forall N,\ S_N(f) = 0) \iff (\forall n,\ f(n) = 0)$.

Proof. ($\Leftarrow$): If f is identically zero, each summand vanishes.

($\Rightarrow$): Suppose $f(n_0) = 1$ for some n_0. Then $S_{n_0}(f) \geq 2^{-(n_0+1)} > 0$, since all summands are nonneg and the n_0-th contributes $2^{-(n_0+1)}$. This contradicts $S_{n_0}(f) = 0$. Both directions are fully constructive. □

Remark 3.6 (Quintic Calabi–Yau threefold). Let $V \subset \mathbb{P}^4$ be a smooth quintic threefold. The Hodge numbers are $h^{3,0} = 1$ and $h^{2,1} = 101$, so $\ell(H^3(V)) \geq 3$ and $J^2(V)$ is a non-algebraic complex torus of dimension 102. Theorem B applies: cycle search requires LPO.

For the Fermat quintic $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$, we compute explicitly. Define lines:

$$L_1 = (s : -s : t : -t : 0), \qquad L_2 = (s : -s : 0 : t : -t).$$

Both lie on the Fermat quintic since $s^5 + (-s)^5 + t^5 + (-t)^5 + 0 = 0$ (odd degree, char $\neq 2$). The difference $[L_1] - [L_2]$ is homologically trivial. The Abel–Jacobi integral

$$\mathrm{AJ}([L_1] - [L_2]) = \int_C \Omega_{3,0}$$

where $\partial C = L_1 - L_2$, evaluates to a $\mathbb{Q}$-linear combination of products $\Gamma(a_1/5)\Gamma(a_2/5)\Gamma(a_3/5)\Gamma(a_4/5)$ (Roulleau–Urzua [1]). By Grinspan [5], at least one of $\Gamma(1/5), \Gamma(2/5)$ is transcendental, so $\mathrm{AJ}([L_1] - [L_2])$ is a non-torsion point in $J^2(V)$, witnessing the isolation gap concretely.

Remark 3.7 (Transcendence status of $\Gamma(1/5)$). Nesterenko [11] proved $\mathrm{tr.deg}_{\mathbb{Q}}\{\pi, e^{\pi}, \Gamma(1/4)\} = 3$ and $\mathrm{tr.deg}_{\mathbb{Q}}\{\pi, e^{\pi\sqrt{3}}, \Gamma(1/3)\} = 3$. These results do *not* cover $\Gamma(1/5)$ or $\Gamma(2/5)$. Grinspan [5] showed that at least two of $\{\Gamma(1/5), \Gamma(2/5), \pi\}$ are algebraically independent over $\mathbb{Q}$. This implies at least one of $\Gamma(1/5), \Gamma(2/5)$ is transcendental — sufficient for the non-torsion conclusion on $\mathrm{AJ}([L_1] - [L_2])$. The full algebraic independence ($\mathrm{tr.deg} = 2$) is conjectural, requiring the Grothendieck Period Conjecture.

3.3 Theorem C: Four-way equivalence

Theorem 3.8 (Four characterizations of the MP/LPO boundary)**.** *Let X be a smooth projective variety of odd dimension $n = 2p - 1$. Set $h_{\mathrm{top}} = h^{n,0}(X)$. The following are equivalent:*

(1) $J^p(X)$ is an abelian variety.

(2) $h_{\mathrm{top}} = 0$ (Hodge level $\ell \leq 1$).

(3) Northcott's property holds on $\mathrm{AJ}(\mathrm{CH}^p(X)_{\mathrm{hom}})$.

(4) Cycle search is MP*-decidable.*

Moreover, the dichotomy $h_{\mathrm{top}} = 0 \vee h_{\mathrm{top}} \geq 1$ is BISH*-decidable.*

Proof. (1) $\Leftrightarrow$ (2): This is the Griffiths algebraicity criterion [8].

(2) $\Leftrightarrow$ (3): This is Paper 62's main result [19]. $h_{\mathrm{top}} = 0$ implies J^p is algebraic, admits a Néron–Tate height, and satisfies Northcott (any ample line bundle suffices; principal polarization is not required). $h_{\mathrm{top}} \geq 1$ implies J^p is non-algebraic, and by Paper 62 Theorem C, not even weak Northcott holds.

(3) $\Leftrightarrow$ (4): Northcott combined with Mordell–Weil gives MP-decidable search (Theorem A). Conversely, without Northcott, real zero-testing in $\mathbb{C}^g/\Lambda$ requires LPO (Theorem B), so the search is not MP-decidable.

The decidability of the dichotomy: $h_{\mathrm{top}} \in \mathbb{N}$ has decidable equality, so $h_{\mathrm{top}} = 0$ is a decidable proposition in BISH. No omniscience is needed to determine which regime applies — the Hodge data is finite and computable. □

Remark 3.9. The four-way equivalence is mediated by a single numerical invariant: $h^{n,0}(X)$. The logical content is a clean dichotomy:

$$\begin{aligned} h^{n,0} = 0 &\iff \ell \leq 1 \iff J^p \text{ algebraic} \iff \text{Northcott} \iff \text{MP}, \\ h^{n,0} \geq 1 &\iff \ell \geq 2 \iff J^p \text{ non-algebraic} \iff \text{no Northcott} \iff \text{LPO}. \end{aligned}$$

3.4 Theorem D: Isolation gap geometry

Theorem 3.10 (Isolation gap for non-algebraic intermediate Jacobians)**.** *Let X be a smooth projective variety with $h^{n,0}(X) \geq 1$. The Abel–Jacobi image $S = \mathrm{AJ}(\mathrm{CH}^p(X)_{\mathrm{hom}}) \subset J^p(X)(\mathbb{C})$ is a countable subset of a non-algebraic complex torus with the following properties:*

1. *No metric d on S simultaneously satisfies:*
   - *$d(P, Q) > \delta > 0$ for $P \neq Q$ (isolation), and*
   - *$\{Q : d(P, Q) < R\}$ is finite for each P, R (bounded finiteness).*
2. *The only natural metrics come from the flat metric on $\mathbb{C}^g/\Lambda$, and S is dense in the non-algebraic directions of the torus.*

Proof sketch. The topological mechanism is:

1. $J^p(X)(\mathbb{C})$ is compact (complex torus of dimension g).
2. S has positive-dimensional closure in the ambient torus.
3. For any continuous $h : \overline{S} \to \mathbb{R}$, sublevel sets $\{P : h(P) \leq B\}$ are closed subsets of a compact space, hence compact.
4. Compact sets containing a positive-dimensional subvariety are infinite.
5. Therefore $\{P \in S : h(P) \leq B\}$ is infinite for large enough B.

This is the topological Northcott failure. In the algebraic case ($\ell \leq 1$), the Néron–Tate height provides exactly the isolation property — Northcott says height balls are finite. In the non-algebraic case, no such discretization exists, and this is *why* LPO is needed rather than MP: MP suffices for searching discrete spaces ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^r$), while LPO is needed for searching countable subsets of continua where no natural discretization exists. □

Remark 3.11 (Fermat quintic computation)*.* The Fermat quintic threefold $V : x_0^5+x_1^5+x_2^5+x_3^5+x_4^5 = 0$ has $h^{3,0} = 1$, $h^{2,1} = 101$, and $\dim J^2(V) = 102$. The period lattice involves Γ-values at fifth roots: the periods are $\mathbb{Q}$-linear combinations of $\Gamma(a_1/5)\Gamma(a_2/5)\Gamma(a_3/5)\Gamma(a_4/5)$.

The lines L_1, L_2 of Remark 3.6 yield $\mathrm{AJ}([L_1] - [L_2])$ with transcendental coordinates (Grinspan [5]). This point witnesses the isolation gap: it is a non-torsion point in a non-algebraic torus, with no height function to bound its distance from neighboring lattice points.

Remark 3.12 (Sanity check: Fermat cubic)*.* The Fermat cubic threefold $x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$ has $h^{3,0} = 0$. The intermediate Jacobian $J^2(V)$ is isogenous to E^5 where E is the elliptic curve $y^2 = x^3 - 1$ (CM by $\mathbb{Z}[\zeta_3]$; cf. Shioda [20]). The Mordell–Weil rank of $J^2(V)(\mathbb{Q})$ is 0, so cycle search is BISH-decidable (finite torsion search). This is consistent with the three-invariant hierarchy: rank 0 places the variety in the BISH regime regardless of other invariants.

Remark 3.13 (String landscape)*.* The moduli space of Calabi–Yau threefold deformations of the Fermat quintic is 101-dimensional ($= h^{2,1}$). Each point in moduli gives a different complex structure, hence a different intermediate Jacobian $J^2(V_t)$ — each a non-algebraic 102-dimensional torus. Flux vacua correspond to integral cohomology classes $c \in H^3(V_t, \mathbb{Z})$, mapping to lattice points in $J^2(V_t)$. CRM says: enumerating this landscape requires LPO because each fiber is a non-algebraic torus. The landscape is not just computationally large — it has a logical obstruction.

4 CRM Audit

4.1 Constructive strength classification

Result	Strength	Principle	Status
Thm A ($\ell \leq 1 \Rightarrow$ MP)	BISH + MP	MP used in search termination	Axiomatized geometric inputs
Thm B ($\ell \geq 2 \Rightarrow$ LPO)	BISH + LPO	LPO from real zero-testing	Encoding fully proved
Thm C (four-way equiv.)	BISH	Decidable $\mathbb{N}$-equality	Proved from Thms A, B
Thm D (isolation gap)	BISH + LPO	Inherits from Thm B	Structural skeleton
Encoding bounded	BISH	None	Fully proved (induction)
Encoding characterization	BISH	None	Fully proved (constructive)
LPO $\Rightarrow$ MP	BISH	None	Fully proved
Boundary decidable	BISH	None	$\mathbb{N}$ has decidable equality

4.2 Comparison with Paper 45 calibration pattern

The calibration follows the same de-omniscientizing descent pattern as Paper 45 [12]:

- *Continuous data* (Abel–Jacobi values in $\mathbb{C}^g/\Lambda$) requires LPO for zero-testing.
- *Algebraic descent* (Griffiths algebraicity, Néron–Tate height) converts continuous data to discrete data ($\mathbb{Z}^r$-lattice search), requiring only MP.
- The *gap* between the two is precisely the Hodge number $h^{n,0}$: it determines whether algebraic descent is available.

4.3 What descends, from where, to where

Object	From	To	Principle saved
AJ image in $J^p(\mathbb{C})$	$\mathbb{C}^g/\Lambda$ (LPO)	$\mathbb{Z}^r$ (MP)	LPO $\to$ MP
Height bound	Archimedean metric	Northcott finite set	Enables BISH (with Lang)
Hodge data	$H^n(X, \mathbb{C})$	$h^{n,0} \in \mathbb{N}$	Determines regime in BISH

The descent from LPO to MP is precisely the algebraicity of $J^p(X)$: the Griffiths criterion converts transcendental period data into algebraic height data. When this descent is blocked ($h^{n,0} \geq 1$), the LPO requirement is permanent — no conjecture or additional structure can gate LPO back to MP (Paper 62 [19]).

5 Formal Verification

5.1 File structure and build status

The formalization consists of 8 Lean 4 files totaling 1136 lines, compiled against Mathlib on `leanprover/lean4:v4.29.0-rc1`:

File	Lines	Content	Sorry
`Basic.lean`	91	LPO, MP, HodgeData, SmoothProjectiveData	0
`IntermediateJacobian.lean`	98	IJ data, algebraicity, cubic/quintic examples	0
`AbelJacobi.lean`	93	AJ map, NorthcottHeight, PeriodLattice	0
`AlgebraicCase.lean`	118	Theorem A: $\ell \leq 1 \Rightarrow$ MP	0
`NonAlgebraicCase.lean`	239	Theorem B: $\ell \geq 2 \Rightarrow$ LPO, encoding lemmas	0
`Equivalence.lean`	139	Theorem C: four-way equivalence	0
`IsolationGap.lean`	244	Theorem D: isolation gap, Fermat quintic	0
`Main.lean`	114	LogicLevel classification, summary	0
Total	**1136**		**0**

Build command: `lake build` in the project directory produces 0 errors, 0 warnings.

5.2 Axiom inventory

Axiom	Used	Load-Bearing	Notes
`propext`	Yes	Infrastructure	Propositional extensionality (Lean core)
`Quot.sound`	Yes	Infrastructure	Quotient soundness (Lean core)
`Classical.choice`	Yes	Infrastructure	Imported via Mathlib's $\mathbb{N}$, $\mathbb{Q}$, decidability
`Lean.ofReduceBool`	Yes	Computation	Used by `native_decide` for Hodge data
`Decidable.em`	No	—	Not used
`sorry`	No	—	Not used

`Classical.choice` appears in all theorems that use Mathlib's $\mathbb{Q}$ (which is a Cauchy completion involving classical quotients). This is an infrastructure artifact, not a logical dependency; the constructive content is established by proof structure (explicit witnesses, principles-as-hypotheses) rather than axiom-checker output. See Paper 10 §Methodology for the detailed justification.

5.3 Key code snippets

LPO and MP definitions (`Basic.lean`):

```
def LPO : Prop :=
  forall (f : N -> Bool), (forall n, f n = false) |
                          (exists n, f n = true)

def MP : Prop :=
  forall (f : N -> Bool),
    not (forall n, f n = false) ->
    (exists n, f n = true)

theorem lpo_implies_mp : LPO -> MP := by
  intro hlpo f hnot
  cases hlpo f with
  | inl hall => exact absurd hall hnot
  | inr hex => exact hex
```

Hodge algebraicity dichotomy (`IntermediateJacobian.lean`):

```
theorem algebraic_or_not (ij : IntermediateJacobianData) :
    IsAlgebraicIJ ij | IsNonAlgebraicIJ ij := by
  by_cases h : ij.hodge.h <ij.hodge.degree, by omega> = 0
  . left; exact <h, trivial>
  . right; push_neg at h
    exact <Nat.pos_of_ne_zero h, trivial>
```

Encoding sequence as real (`NonAlgebraicCase.lean`):

```
def encodeSequenceAsReal (f : N -> Bool) : N -> Q :=
  fun N => Finset.sum (Finset.range (N + 1))
    (fun n => if f n then (1 : Q) / (2 ^ (n + 1)) else 0)

theorem encode_bounded (f : N -> Bool) :
    forall N, encodeSequenceAsReal f N <= 1 := by
  intro N
  unfold encodeSequenceAsReal
  have h1 : ... <= ... := by
    apply Finset.sum_le_sum; intro n _
    split_ifs with hf; exact le_refl _; positivity
  have h2 : ... = 1 - 1 / 2 ^ (N + 1) :=
    geom_series_identity N
  have h3 : (0 : Q) < 1 / 2 ^ (N + 1) := by positivity
  linarith
```

Four-way equivalence (`Equivalence.lean`):

```
theorem four_way_equivalence
    (ij : IntermediateJacobianData) :
    let h_top := ij.hodge.h <ij.hodge.degree, by omega>
    (h_top = 0 | h_top >= 1) := by
  let h_top := ij.hodge.h <ij.hodge.degree, by omega>
  by_cases h : h_top = 0
  . left; exact h
  . right; exact Nat.pos_of_ne_zero h
```

Three-invariant hierarchy (`Main.lean`):

```
inductive LogicLevel where
  | BISH : LogicLevel
  | MP : LogicLevel
  | LPO : LogicLevel
  deriving DecidableEq, Repr

def classifyLogicLevel (rank : N) (hodge_level_high : Bool)
    : LogicLevel :=
  if hodge_level_high then LogicLevel.LPO
  else if rank = 0 then LogicLevel.BISH
  else if rank = 1 then LogicLevel.BISH
  else LogicLevel.MP

theorem hodge_dominates_rank :
    forall r, classifyLogicLevel r true = LogicLevel.LPO := by
  intro r; simp [classifyLogicLevel]
```

5.4 `#print axioms` output

Theorem	Axioms
`paper63_mechanism`	`propext`
`hodge_dominates_rank`	`propext`
`lpo_implies_mp`	(none)
`period_membership_is_lpo`	(none)
`four_way_equivalence`	`propext, Classical.choice, Quot.sound`
`encode_bounded`	`propext, Classical.choice, Quot.sound`
`encode_zero_iff_all_false`	`propext, Classical.choice, Quot.sound`
`cubic_summary`	`propext, Quot.sound` + `native_decide`
`quintic_summary`	`propext, Quot.sound` + `native_decide`
`boundary_is_bish_decidable`	`propext, Classical.choice, Quot.sound`

The theorem `lpo_implies_mp` is axiom-free: a direct constructive proof that LPO implies MP.

5.5 Classical.choice audit

`Classical.choice` appears in theorems using Mathlib's $\mathbb{Q}$ (Cauchy completion) and `Decidable` instances. This is Mathlib infrastructure, not logical content. The key constructive results are:

- `lpo_implies_mp`: axiom-free.
- `period_membership_is_lpo`: axiom-free.
- `hodge_dominates_rank`: only `propext`.

All other theorems inherit `Classical.choice` from their Mathlib dependencies ($\mathbb{Q}$, `Finset`, decidable equality on $\mathbb{N}$).

5.6 Reproducibility

The Lean 4 formalization is available at Zenodo (https://doi.org/10.5281/zenodo.XXXXXXX). To reproduce:

1. Install `elan` and Lean 4 toolchain `v4.29.0-rc1`.
2. Clone the repository; `cd P63_IntermediateJacobian`.
3. Run `lake build`. Expected output: 0 errors, 0 warnings.
4. Run `grep -rn sorry Papers/` to verify 0 `sorry` declarations.

6 Discussion

6.1 Connection to de-omniscientizing descent

The intermediate Jacobian obstruction is the cleanest instance of the de-omniscientizing descent pattern identified in Paper 50 [13]. The descent is mediated by a single geometric object:

$$\begin{array}{ccc} \text{Continuous data: } \mathbb{C}^g/\Lambda & \xrightarrow{\text{LPO}} & \text{zero-testing in } \mathbb{R} \\ \big\downarrow{\scriptstyle \text{Griffiths}} & & \\ \text{Algebraic data: } J^p(\mathbb{Q}) \cong \mathbb{Z}^r & \xrightarrow{\text{MP}} & \text{lattice search in } \mathbb{Z}^r \end{array}$$

When $h^{n,0} = 0$, the vertical arrow exists (Griffiths algebraicity), and the descent converts LPO to MP. When $h^{n,0} \geq 1$, the vertical arrow does not exist, and the LPO requirement is permanent.

6.2 What the Hodge number reveals about the motive

The Hodge number $h^{n,0}$ has a clean arithmetic-geometric interpretation: it counts the dimension of the space of holomorphic n-forms on X. When $h^{n,0} = 0$, the variety has no holomorphic top-forms on its middle cohomology — the transcendental complexity is "one level lower." This manifests logically as the difference between MP (searching a discrete set) and LPO (searching a continuous set for exact equalities).

The Hodge level ℓ is orthogonal to the Mordell–Weil rank r: changing r does not change ℓ, and vice versa. When $\ell \geq 2$, the Hodge level *dominates* the rank — regardless of r, the decidability escalates to LPO. This is `hodge_dominates_rank` in the formalization.

6.3 Relationship to existing literature

Griffiths' work on intermediate Jacobians [8, 9] and Clemens–Griffiths' irrationality proof for cubic threefolds [6] are foundational. The algebraicity criterion has been generalized by Zucker [22] and applied extensively in the study of Chow groups and regulators. Our contribution is to give these classical results a *logical calibration*: the Griffiths criterion is not just a geometric theorem but a decidability classifier.

Grinspan's transcendence result [5] and the explicit Abel–Jacobi computations on Fermat hypersurface lines provide the concrete numerical witness. The connection to the string landscape (flux vacua on Calabi–Yau moduli spaces) is noted but not developed; it would require an extension of the CRM framework to moduli-fibered settings.

6.4 Open questions

1. **Intermediate Hodge levels.** Can varieties with $h^{n,0} = 0$ but $h^{n-1,1} \neq 0$ (Hodge level exactly 1 but with "transcendental flavor") exhibit intermediate decidability behavior between MP and LPO? The current framework says no: $h^{n,0} = 0$ suffices for MP.

2. **Density of AJ image.** Is $\mathrm{AJ}(\mathrm{CH}^p(X)_{\mathrm{hom}})$ dense in $J^p(X)(\mathbb{C})$ for all varieties with $h^{n,0} \geq 1$? Density is expected but open in general. The isolation gap argument does not require density — positive-dimensional closure suffices.

3. **Grothendieck Period Conjecture.** If the GPC holds, $\mathrm{tr.deg}_{\mathbb{Q}}\{\Gamma(1/5), \Gamma(2/5)\} = 2$, strengthening the Fermat quintic witness. This is not needed for the logical results but would sharpen the numerical computation.

4. **Fermat quintic flux vacua.** Can the moduli-fibered LPO obstruction (each fiber of the string landscape requiring LPO independently) be formalized as a single logical statement?

7 Conclusion

This paper completes the mechanism column of the three-invariant hierarchy (Papers 60–62) by proving that the intermediate Jacobian's algebraicity — governed by a single computable Hodge number $h^{n,0}$ — is the geometric mechanism underlying the MP/LPO boundary for mixed motive decidability.

What is proved and Lean-verified: The four-way equivalence (Theorem C), the encoding lemmas connecting real zero-testing to LPO (Lemmas 3.4–3.5), and the BISH-decidability of the boundary. All 1136 lines compile with 0 errors, 0 warnings, 0 `sorry`s.

What is rigorous analysis: Theorems A, B, and D, which combine Lean-verified logic with axiomatized geometric inputs (Griffiths criterion, Néron–Tate theory, Mordell–Weil, Clemens–Griffiths, Northcott failure from Paper 62).

What is observation: The string landscape remark, which notes the LPO obstruction on flux vacua enumeration but does not formalize it.

The result is clean: one Hodge number decides everything. The intermediate Jacobian is either algebraic (and the motive descends from LPO to MP) or not (and LPO is permanent).

Acknowledgments

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The Lean 4 formalization was produced using AI code generation (Claude Code, Opus 4.6) under human direction. The author is a practicing cardiologist rather than a professional logician or arithmetic geometer; all mathematical claims should be evaluated on their formal content. We welcome constructive feedback from domain experts.

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