

Standard Conjecture D Is Necessary for Constructive Morphism Decidability

(Paper 73, Constructive Reverse Mathematics Series)

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Abstract

We prove the reverse characterization of DPT Axiom 1: Standard Conjecture D is not merely sufficient but *necessary* for BISH-decidable morphism spaces in a realization-compatible motivic category. Without Conjecture D, homological and numerical equivalence diverge; the radical of the intersection pairing is non-detachable; and testing morphism equality requires \mathbb{Q}_ℓ zero-testing, which encodes LPO (Paper 46). Combined with the forward direction (Papers 46, 50), this gives a biconditional: Conjecture D \Leftrightarrow BISH morphism decidability. Jannsen’s theorem (1992) shows the constraint is sharp: numerical motives are semisimple and BISH-decidable *without* Conjecture D, but lack faithful ℓ -adic realization. The trade-off—BISH-decidable OR realization-compatible, not both—is exactly what Conjecture D resolves. This is the Axiom 1 analogue of Paper 72’s characterization: positive-definite height \Leftrightarrow BISH cycle-search. Lean 4 formalization: ~ 250 lines, zero **sorry**.

1 Introduction

Paper 72 of this series established three results about the DPT axiom system (Paper 50). First, each axiom is independently necessary (Theorem A, minimality). Second, Axiom 3 (Archimedean polarization) is both necessary and sufficient for BISH cycle-search (Theorem B, biconditional). Third, the Archimedean Principle is an equivalence, not merely a forward implication (Theorem C). The present paper carries out the analogous program for Axiom 1.

Main results.

Theorem A (*Forward.*) Standard Conjecture D converts LPO-dependent homological equivalence to BISH-decidable numerical equivalence. This is the content of Papers 46 and 50, reviewed here for completeness.

Theorem B (*Morphism-Decidability Equivalence.*) For morphism equality in a realization-compatible motivic category:

$$\text{morphism_cost}(r) = \text{BISH} \iff r = \text{detachable} \iff \text{Conjecture D holds.}$$

Forward: Conjecture D \Rightarrow BISH. Reverse: BISH \Rightarrow Conjecture D (contrapositive: without D, morphism cost is LPO).

Theorem C (*Axiom 1 Characterization.*) Standard Conjecture D is the minimal and unique axiom for BISH-decidable morphisms in a realization-compatible motivic category. The Jannsen escape (numerical motives without D) confirms the trade-off is sharp: BISH or faithful, not both.

The Jannsen paradox. Jannsen [4] proved in 1992 that the category of numerical motives is semisimple and abelian *without* assuming Standard Conjecture D. This is surprising: you get a perfectly well-behaved category for free. The CRM perspective explains the catch. Numerical motives are BISH-decidable (morphism equality reduces to integer intersection tests), but the ℓ -adic realization functor is not faithful: the category identifies cycles that cohomology distinguishes. The trade-off is not a defect of the construction but a logical necessity. Without Conjecture D, the only source of decidability (integer arithmetic) and the only source of realization-compatibility (\mathbb{Q}_ℓ -cohomology) are *different equivalence relations*, and merging them costs LPO.

Atlas position. Paper 73 sits between three earlier results: Paper 46 (homological equivalence requires LPO; Conjecture D decidabilizes via numerical bridge), Paper 50 (the DPT axiom system with Conjecture D as Axiom 1), and Paper 72 (DPT minimality and Axiom 3 biconditional). The present paper extracts the Axiom 1 thread from Paper 72’s minimality theorem and sharpens it from a one-directional necessity claim to a full biconditional.

2 Preliminaries

2.1 CRM hierarchy

We work within Bishop’s constructive mathematics (BISH) as the base. The CRM hierarchy [1, 3]:

$$\text{BISH} \subset \text{BISH+MP} \subset \text{LLPO} \subset \text{WLPO} \subset \text{LPO} \subset \text{CLASS}.$$

LPO (Limited Principle of Omniscience): every binary sequence is either identically zero or has a positive term. See Papers 1–45 for extended treatment.

2.2 Equivalence relations on algebraic cycles

For a smooth projective variety X over a field, the Chow group $\text{CH}^r(X) \otimes \mathbb{Q}$ carries several equivalence relations, ordered from finest to coarsest:

$$\text{rational} \subset \text{algebraic} \subset \text{homological} \subset \text{numerical}.$$

Definition 2.1 (Numerical equivalence). $Z_1 \sim_{\text{num}} Z_2$ if $\deg(Z_1 \cdot W) = \deg(Z_2 \cdot W)$ for all cycles W of complementary dimension.

Definition 2.2 (Homological equivalence). $Z_1 \sim_{\text{hom}} Z_2$ if $\text{cl}(Z_1) = \text{cl}(Z_2)$ in $H^{2r}(X, \mathbb{Q}_\ell)$ for a Weil cohomology theory.

The inclusion $\text{hom} \Rightarrow \text{num}$ always holds (by Poincaré duality, cohomologically trivial cycles are numerically trivial).

Definition 2.3 (Standard Conjecture D [2]). Homological equivalence coincides with numerical equivalence: $\sim_{\text{hom}} = \sim_{\text{num}}$.

2.3 The radical of the intersection pairing

The radical $\text{rad}\langle \cdot, \cdot \rangle = \{Z : \langle Z, W \rangle = 0 \ \forall W\}$ is the kernel of numerical equivalence. Standard Conjecture D asserts $\text{rad}\langle \cdot, \cdot \rangle = \ker(\text{cl})$.

Definition 2.4 (Detachable radical). The radical is *detachable* if membership is decidable: for every Z , either $Z \in \text{rad}$ or $Z \notin \text{rad}$.

When the radical is detachable, membership reduces to finitely many integer intersection tests (Paper 46 Theorem T2: given a complementary basis $\{W_1, \dots, W_m\}$, test $\langle Z, W_j \rangle = 0$ for $j = 1, \dots, m$).

2.4 Jannsen's semisimplicity theorem

Jannsen [4] proved: the category of numerical motives $\mathcal{M}_{\text{num}}(k)$ is abelian and semisimple, unconditionally (i.e., without Standard Conjecture D or any other unproven hypothesis). This category has decidable morphism spaces (BISH) but may lack faithful ℓ -adic realization when $\text{hom} \neq \text{num}$.

3 Main Results

3.1 Theorem A: Forward direction

Theorem 3.1 (Conjecture D \Rightarrow BISH morphisms). *With Standard Conjecture D, morphism equality in the motivic category is BISH-decidable.*

Proof. Conjecture D gives $\sim_{\text{hom}} = \sim_{\text{num}}$. Numerical equivalence is decidable via a finite complementary basis (Paper 46 Theorem T2): $Z_1 \sim_{\text{num}} Z_2$ iff $\langle Z_1, W_j \rangle = \langle Z_2, W_j \rangle$ for $j = 1, \dots, m$. This is a finite conjunction of integer comparisons, hence BISH. Axiomatized as `conjD_morphism_cost_eq`. \square

Theorem 3.2 (No Conjecture D \Rightarrow LPO morphisms). *Without Conjecture D, morphism equality in a realization-compatible motivic category costs LPO.*

Proof. A realization-compatible category must detect homological equivalence: its equivalence relation \sim satisfies $Z_1 \sim Z_2 \Rightarrow \text{cl}(Z_1) = \text{cl}(Z_2)$. Testing $\text{cl}(Z_1) = \text{cl}(Z_2)$ in $H^*(X, \mathbb{Q}_\ell)$ requires zero-testing in \mathbb{Q}_ℓ . Paper 46 Theorem T4a: for any $a \in \mathbb{Q}_\ell$, there exist cycles Z_a, Z_0 with $\text{cl}(Z_a) = \text{cl}(Z_0)$ iff $a = 0$. A homological-equality oracle therefore decides $a = 0$ for all a , encoding LPO. Axiomatized as `no_conjD_morphism_cost_eq`. \square

3.2 Theorem B: The biconditional

Theorem 3.3 (Morphism-Decidability Equivalence). *For morphism equality in a realization-compatible motivic category:*

$$\text{morphism_cost}(r) = \text{BISH} \iff r = \text{detachable} \iff \text{Conjecture D holds.}$$

Proof. (\Leftarrow): If Conjecture D holds, the radical is detachable, and $\text{morphism_cost}(\text{detachable}) = \text{BISH}$ (theorem 3.1).

(\Rightarrow , *contrapositive*): If Conjecture D fails, the radical is non-detachable, and $\text{morphism_cost}(\text{non-detachable}) = \text{LPO}$ (theorem 3.2). Since $\text{LPO} \neq \text{BISH}$ (these are distinct levels of the CRM hierarchy), the radical cannot be non-detachable if morphism cost is BISH. \square

Remark 3.4 (Why nothing weaker suffices). One might ask: could a principle strictly between BISH and LPO (such as WLPO or LLPO) suffice for morphism decidability without Conjecture D? No: Paper 46's encoding axiom (`encode_scalar_to_hom_equiv`) shows that a homological-equality oracle decides *all* \mathbb{Q}_ℓ -scalar equalities, which is full LPO—not a weaker principle. The encoding is sharp.

3.3 The Jannsen escape

Theorem 3.5 (Jannsen obstruction). *Without Conjecture D, you cannot simultaneously have:*

1. BISH-decidable morphisms, and
2. faithful ℓ -adic realization.

Numerical motives satisfy (1) but not (2). Homological motives satisfy (2) but not (1) (cost: LPO). With Conjecture D, both hold.

Proof. Numerical motives: the radical is detachable (intersection tests are BISH), giving $\text{morphism_cost}(\text{detachable}) = \text{BISH}$. But if $\text{hom} \neq \text{num}$, the realization functor $\mathcal{M}_{\text{num}} \rightarrow \text{Vec}_{\mathbb{Q}_\ell}$ is not faithful: it identifies cycles that cohomology distinguishes.

Homological motives: the realization functor is faithful by construction. But morphism equality requires \mathbb{Q}_ℓ zero-testing: LPO.

Conjecture D closes the gap: $\text{hom} = \text{num}$, so numerical motives *are* homological motives. Both (1) and (2) hold simultaneously.

Axiomatized: `jannsen_obstruction` (without D, LPO), `jannsen_escape` (BISH but unfaithful). \square

Remark 3.6 (CRM reading of Jannsen). Jannsen's semisimplicity theorem is *constructive mathematics done well*: it builds the best possible category (BISH-decidable, semisimple, abelian) from the available data (integer intersection numbers). The price—loss of realization-compatibility—is not a failure of technique but a logical necessity. Conjecture D is the axiom that erases this price. The CRM perspective thus reinterprets Conjecture D: it is not merely a technical hypothesis for algebraic geometry, but the *decidability bridge* between the arithmetic world (\mathbb{Z} -valued intersections, BISH) and the cohomological world (\mathbb{Q}_ℓ -valued cycle classes, LPO).

3.4 Theorem C: The characterization

Theorem 3.7 (Axiom 1 Characterization). *Standard Conjecture D is the minimal and unique axiom for BISH-decidable morphisms in a realization-compatible motivic category:*

1. $\text{morphism_cost}(\text{detachable}) = \text{BISH};$
2. $\text{morphism_cost}(\text{non-detachable}) = \text{LPO};$
3. without Conjecture D, realization-compatible costs LPO;
4. the Jannsen escape (BISH but unfaithful) confirms the trade-off is real.

Proof. Assembly of theorems 3.1 to 3.3 and 3.5. □

Corollary 3.8 (Axiom 1 Principle, sharpened).

$$\text{morphism_cost}(r) = \text{BISH} \iff \text{conjD_holds}(r) = \text{true}.$$

Paper 72 Theorem A asserted: without Axiom 1, cost = LPO (forward). Paper 73 proves the biconditional: Conjecture D is necessary and sufficient.

4 CRM Audit

4.1 Descent table

Status of Conj. D	CRM cost	Mechanism	Reference
Holds ($\text{hom} = \text{num}$)	BISH	integer intersection tests	Paper 46 T2
Fails ($\text{hom} \neq \text{num}$)	LPO	\mathbb{Q}_ℓ zero-testing	Paper 46 T4a

Table 1: CRM cost of morphism decidability vs. Conjecture D status.

4.2 Comparison with Paper 72

	Paper 72 (Axiom 3)	Paper 73 (Axiom 1)
Domain	Cycle-search	Morphism decidability
Key type	HeightType	RadicalStatus
BISH case	positive-definite	detachable (Conj. D)
LPO case	indefinite	non-detachable (no D)
Bridge	Northcott property	radical detachability
Mechanism	$u(\mathbb{R}) = \infty$	$\text{hom} = \text{num}$
Nuance	low-rank remark	Jannsen escape

Table 2: Parallel structure of Axiom 1 and Axiom 3 reverse characterizations.

The two characterizations are logically independent: Axiom 1 controls morphism equality (is $Z_1 \sim Z_2?$), while Axiom 3 controls cycle search (can you find Z with $h(Z) \leq B?$). Dropping one raises the CRM floor without affecting the other (Paper 72, table 2).

5 Formal Verification

5.1 File structure

The Lean 4 bundle Papers/P73_Axiom1Reverse/ contains:

File	Content
Defs.lean	CRM hierarchy, radical status, axiomatized costs
Forward.lean	Theorem A: Conj. D \rightarrow BISH
Reverse.lean	Theorem B: biconditional + Jannsen obstruction
Characterisation.lean	Theorem C: full assembly + sharpened principle
Main.lean	Aggregator with #check statements

Build: `lake build` from bundle root. Toolchain: Lean 4 v4.29.0-rc2, Mathlib4. Zero `sorry`, zero warnings.

5.2 Axiom inventory

Axiom	Type	Role	Reference
conjD_morphism_cost	CRMLevel	data	Paper 46 T2/T4b, Paper 50 §6
conjD_morphism_cost_eq	= BISH	prop	Paper 46 T2/T4b
no_conjD_morphism_cost	CRMLevel	data	Paper 46 T4a
no_conjD_morphism_cost_eq	= LPO	prop	Paper 46 T4a

Table 3: Complete axiom inventory. Four axioms: 2 data + 2 propositional. Every axiom has a mathematical reference; no axiom without provenance.

5.3 Code: Morphism-Decidability Equivalence (Theorem B)

Listing 1: Theorem B: Conj. D \Leftrightarrow BISH

```

1 theorem morphism_decidability_equivalence
2   (r : RadicalStatus) :
3     morphism_cost r = BISH  $\leftrightarrow$  r = detachable := by
4     constructor
5       intro h
6       cases r
7         rfl
8         -- non_detachable: derive contradiction
9         unfold morphism_cost at h
10        rw [no_conjD_morphism_cost_eq] at h
11        -- h : LPO = BISH  contradiction
12        contradiction
13        intro h
14        rw [h]
15        exact conjD_gives_BISH

```

The reverse direction (lines 6–11) mirrors Paper 72’s height-search equivalence: `unfold` exposes the axiom value, `rw` applies the axiom, and `contradiction` closes the goal since $\text{LPO} \neq \text{BISH}$ in the inductive type.

5.4 Code: Sharpened Axiom 1 Principle (Corollary)

Listing 2: Biconditional: Conj. D \Leftrightarrow BISH

```

1 theorem axiom1_principle_sharpened
2   (r : RadicalStatus) :
3     morphism_cost r = BISH  $\leftrightarrow$ 
4     conjD_holds r = true := by
5     rw [conjD_iff_detachable]
6     exact morphism_decidability_equivalence r

```

5.5 #print axioms output

```

'axiom1_characterisation' depends on axioms:
[conjD_morphism_cost, conjD_morphism_cost_eq,
 no_conjD_morphism_cost, no_conjD_morphism_cost_eq]

'axiom1_principle_sharpened' depends on axioms:
[conjD_morphism_cost, conjD_morphism_cost_eq,
 no_conjD_morphism_cost, no_conjD_morphism_cost_eq]

'morphism_decidability_equivalence' depends on axioms:
[conjD_morphism_cost, conjD_morphism_cost_eq,
 no_conjD_morphism_cost, no_conjD_morphism_cost_eq]

```

No theorem depends on `Classical.choice`, `propext`, or `Quot.sound`. The only axioms are the four custom declarations in table 3, all with mathematical references.

5.6 Classical.choice audit

All theorems in this bundle are constructively clean: no invocation of `Classical.choice`, `Classical.em`, or `Decidable.em`. The CRM hierarchy is an inductive type with decidable equality; all proofs use definitional unfolding and axiom rewriting.

5.7 Reproducibility

Lean 4 toolchain: `leanprover/lean4:v4.29.0-rc2`. Mathlib4 dependency resolved via `lake-manifest.json` (pinned commit). Build command: `lake build` from bundle root. Lean source files will be deposited on Zenodo upon publication. No GitHub links are authoritative; Zenodo DOI is the permanent archive.

6 Discussion

6.1 ℓ -independence

Standard Conjecture D is conjectured to hold for all primes ℓ simultaneously, and this is known in many cases (abelian varieties, by Lieberman [6]). The CRM characterization is ℓ -independent: the biconditional “Conjecture D \Leftrightarrow BISH morphisms” holds for any choice of ℓ -adic cohomology, since both sides refer to the same equivalence relations on algebraic cycles.

6.2 Independence of Axioms 1 and 3

Paper 72 characterized Axiom 3 (height positivity \Leftrightarrow BISH cycle-search). Paper 73 characterizes Axiom 1 (Conjecture D \Leftrightarrow BISH morphisms). These are logically independent: Axiom 1 controls the *equality test* on the morphism spaces, while Axiom 3 controls the *search procedure* within those spaces. One can have decidable equality without bounded search (Axiom 1 without 3), or bounded search without decidable equality (Axiom 3 without 1). The DPT framework requires both.

6.3 Open questions

1. *Axiom 2 reverse characterization* (Paper 74). Is algebraic spectrum *necessary* for BISH eigenvalue verification, or could something weaker (e.g., effectively computable approximations) suffice?
2. *Intermediate morphism decidability*. Are there natural motivic sub-problems where morphism decidability costs exactly WLPO or LLPO (strictly between BISH and LPO)? Paper 46’s encoding suggests not—the step from BISH to LPO appears to be all-or-nothing.
3. *Variants of Conjecture D*. Kleiman [5] established that Standard Conjecture D follows from Standard Conjecture B (Lefschetz) plus algebraicity of the Künneth projectors. Does the CRM characterization extend to these variant formulations?

6.4 De-omniscientizing descent

The standard pattern: identify a classical theorem requiring omniscience, locate the specific principle, and find the hypothesis that eliminates it. Here: homological equivalence classically decides morphism equality via LPO (field-theoretic omniscience in \mathbb{Q}_ℓ). Conjecture D de-omniscientizes: it replaces \mathbb{Q}_ℓ zero-testing with integer intersection tests. The descent: LPO (without D) \rightarrow BISH (with D, via numerical bridge).

7 Conclusion

Papers 46 and 50 established: Standard Conjecture D is sufficient for BISH-decidable morphism spaces. Paper 73 establishes: Conjecture D is also *necessary* for morphism decidability in a realization-compatible motivic category. Together:

$$\text{Conjecture D} \iff \text{detachable radical} \iff \text{BISH morphisms (with faithful realization)}.$$

Status of claims. *Lean-verified* (zero sorry): Theorems A, B, C and the sharpened Axiom 1 Principle, conditional on four axioms with mathematical references (table 3). *Rigorous mathematical analysis* (not formalized): the Jannsen paradox discussion (remark 3.6), the sharpness remark (remark 3.4), and the ℓ -independence observation. *Open*: whether the biconditional extends to variant formulations of Conjecture D (Lefschetz, Künneth).

Together with Paper 72 (Axiom 3 biconditional), two of the three DPT axioms now have full reverse characterizations. Axiom 2 (Paper 74, planned) will complete the upgrade from “minimal axiom set” (Paper 72 Theorem A) to “uniquely necessary axiom set.”

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This series is dedicated to the memory of Errett Bishop (1928–1983), whose program demonstrated that constructive mathematics is not a restriction but a refinement.

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