Physics Reaches Σ_2^0

The Thermodynamic Stratification of Physical Undecidability

A Lean 4 Formalization (Paper 39)

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Abstract

Papers 36–38 established that every known undecidability result in mathematical physics is LPO (Σ_1^0). This paper shows the ceiling is not Σ_1^0. A modified Cubitt encoding—running a Turing machine on all inputs simultaneously via Robinson tiling with perimeter counters—encodes the Σ_2^0-complete Finiteness Problem into the spectral gap. The generic spectral gap decision (without promise gap) is Σ_2^0/Π_2^0-complete, requiring LPO′ (the Turing jump of LPO). However, extensive observables (energy density, magnetization) converge via Fekete's lemma and cap at LPO; promise-gapped physics (all of Papers 36–38) also caps at LPO. The Thermodynamic Stratification Theorem: arithmetic complexity bifurcates along thermodynamic scaling—extensive at LPO, intensive at LPO′. The entire formalization (802 lines of Lean 4/Mathlib4) compiles with zero `sorry`, zero warnings.

1 Introduction

Papers 36–38 established a uniform result: every known undecidability in quantum many-body physics is LPO. Paper 38 proved the Σ_1^0 ceiling—LPO decides every Σ_1^0-complete problem. A natural question arises: is LPO the provable ceiling for physics?

The answer is *no*. The spectral gap of a generic translation-invariant Hamiltonian—without an artificial promise gap—encodes Σ_2^0/Π_2^0-complete properties. This requires LPO′, the Turing jump of LPO, strictly stronger than LPO.

The key insight is thermodynamic:

- **Extensive** observables (energy density, free energy) converge via subadditivity (Fekete/BMC). Cap: LPO.
- **Intensive** observables (spectral gap, correlation length) are determined by infima. Cap: LPO′.
- The **promise gap** in Cubitt's construction collapses the $\forall\exists$ quantifier to a single $\exists$, reducing Σ_2^0 to Σ_1^0.

2 The Category Error

The spectral gap Δ of an infinite Hamiltonian is a Δ_2^0 real—its digits are limit-computable with an LPO oracle. But *computing a real* and *deciding a property of that real* have different arithmetic complexities.

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Without a promise gap:

$$\Delta = 0 \iff \forall m\, \exists L\, \left(\Delta_L < \tfrac{1}{m}\right) \qquad (\Pi^0_2)$$

With a promise gap $\Delta \in \{0\} \cup [\gamma, \infty)$:

$$\Delta = 0 \iff \exists L\, \left(\Delta_L < \tfrac{\gamma}{2}\right) \qquad (\Sigma^0_1)$$

The promise collapses the outer $\forall m$ quantifier because $m = \lceil 2/\gamma \rceil$ suffices. This collapse is why Papers 36–38 cap at LPO.

3 LPO′: The Turing Jump of LPO

Definition 3.1 (LPO′)**.** LPO′ ("LPO-jump") is Σ^0_2-LEM: for every binary sequence β that is decidable relative to an LPO oracle, either $\exists n.\, \beta(n) = 1$ or $\forall n.\, \beta(n) = 0$.

```
def LPO_jump : Prop :=
    forall (b : N -> Bool),
      (LPO -> forall n, b n = true ∨ b n = false) ->
      (forall n, b n = false) ∨ (exists n, b n = true)
```

Listing 1: LPO′ definition: `Defs.lean`

LPO′ is strictly stronger than LPO: LPO′ $\Rightarrow$ LPO but not conversely. LPO decides Σ^0_1; LPO′ decides Σ^0_2.

4 Theorem 1: The Modified Encoding

Theorem 4.1 (Modified Cubitt Encoding)**.** *There exists a computable function $M \mapsto H^*(M)$ from Turing machines to translation-invariant Hamiltonians on $\mathbb{Z}^2$ with fixed local dimension such that:*

(a) $H^*(M)$ is gapped $\Leftrightarrow$ $\{k : M$ halts on input $k\}$ is finite.

(b) $H^*(M)$ is gapless $\Leftrightarrow$ $\{k : M$ halts on input $k\}$ is infinite.

The construction uses Robinson tiling with hierarchical regions: supertiles of scale k (side 4^k) run M on input k. The perimeter counter (Bausch–Cubitt–Ozols 2018) extracts the input from the boundary. The spectral gap—an intensive property—is the infimum of local gaps across all scales.

```
axiom modified_gapped_iff_finite (M : TM) :
    is_gapped (modified_hamiltonian M) <->
    finiteness_problem M

axiom modified_gapless_iff_infinite (M : TM) :
    is_gapless (modified_hamiltonian M) <->
    not finiteness_problem M
```

Listing 2: Modified encoding bridges: `ModifiedEncoding.lean`

5 Theorem 2: Generic Gap Is Σ^0_2

Theorem 5.1 (Generic Gap $\equiv$ LPO′)**.** *Deciding the spectral gap of the modified encoding (without promise gap) is Turing–Weihrauch equivalent to* LPO′*.*

Proof. The gap decision reduces to the Finiteness Problem (is $\{k : M(k) \text{ halts}\}$ finite?), which is Σ^0_2-complete. Deciding all Σ^0_2 statements requires LPO$'$; conversely, LPO$'$ decides all Σ^0_2 statements. ☐

```
theorem generic_gap_iff_lpo_jump :
    (forall M, is_gapped (modified_hamiltonian M) ∨
                 not is_gapped (modified_hamiltonian M))
    <-> LPO_jump :=
  ⟨generic_gap_requires_lpo_jump,
   lpo_jump_decides_generic_gap⟩
```

Listing 3: Generic gap $\leftrightarrow$ LPO$'$: `GenericGapSigma2.lean`

6 Theorem 3: Promise Gap Recovery

Theorem 6.1 (Promise Gap $\Rightarrow$ LPO)**.** *If the Hamiltonian has a promise gap ($\Delta \in \{0\} \cup [\gamma, \infty)$ for computable $\gamma > 0$), the spectral gap decision is Σ^0_1-complete $=$* LPO*.*

The promise collapses the $\forall\exists$ to a single $\exists$. This recovers Papers 36–38 as special cases.

```
theorem promise_gap_lpo (H : PromiseGapped) (lpo : LPO) :
    is_gapless H.hamiltonian ∨
    not is_gapless H.hamiltonian
```

Listing 4: Promise gap recovery: `PromiseGapRecovery.lean`

7 Theorem 5: Extensive Observables Cap at LPO

Theorem 7.1 (Extensive Ceiling)**.** *Every extensive observable (energy density, free energy, magnetization) of a translation-invariant Hamiltonian is* LPO*-decidable.*

Extensive observables are subadditive. By Fekete's lemma ($=$ BMC, Paper 29), the limit exists and converges monotonically from above. Monotone convergence with computable terms yields a Δ^0_2 real whose zero-test is Π^0_1—one level below the generic Π^0_2 of intensive properties.

```
theorem extensive_cap_lpo (O : ExtensiveObservable) :
    LPO -> (extensive_sign_positive O ∨
              not extensive_sign_positive O)
```

Listing 5: Extensive ceiling: `ExtensiveCeiling.lean`

8 Theorem 4: The Thermodynamic Stratification

Theorem 8.1 (Thermodynamic Stratification)**.** *The arithmetic complexity of physical observables bifurcates:*

(i) ***Extensive:*** LPO *(Σ^0_1).*
(ii) ***Intensive (generic):*** LPO$'$ *(Σ^0_2).*
(iii) ***Promise-gapped:*** LPO *(Σ^0_1).*
(iv) ***Empirical (finite precision):*** LPO *(Σ^0_1).*

```
theorem thermodynamic_stratification :
    -- (i) Extensive cap at LPO
    (forall (O : ExtensiveObservable), LPO ->
      (extensive_sign_positive O ∨
       not extensive_sign_positive O)) /\
    -- (ii) Intensive reach LPO_jump
    ((forall M, is_gapped (modified_hamiltonian M) ∨
                not is_gapped (modified_hamiltonian M))
      <-> LPO_jump) /\
    -- (iii) Promise-gapped cap at LPO
    (forall (H : PromiseGapped), LPO ->
      (is_gapless H.hamiltonian ∨
       not is_gapless H.hamiltonian)) /\
    -- (iv) Empirical cap at LPO
    (forall (H : ModifiedHamiltonian) (e : R),
      e > 0 -> LPO ->
      (gap_less_than H e ∨ not gap_less_than H e))
```

Listing 6: Stratification master: `Stratification.lean`

9 The Complete Hierarchy

Tier	Observable	Principle	Level	Paper
BISH	Finite systems (Δ_L)	None	Δ_1^0	34
BISH+LLPO	Bell correlations	LLPO	$< \Sigma_1^0$	10
BISH+WLPO	"Is $\Delta = 0$?" (given Δ)	WLPO	Π_1^0	2
BISH+LPO	Energy density; Cubitt gap	LPO	Σ_1^0	29, 36
BISH+LPO′	Spectral gap (no promise)	LPO′	Σ_2^0	39

Table 1: The complete constructive hierarchy of physics.

10 CRM Audit

Component	CRM Status	Level
Modified Encoding (Thm 1)	(axiom)	4
Generic Gap ≡ LPO′ (Thm 2)	LPO′	3+4
Promise Recovery (Thm 3)	LPO	3
Stratification (Thm 4)	Inherits	—
Extensive Ceiling (Thm 5)	LPO	2+4

Table 2: CRM audit.

11 Code Architecture

Reproducibility. Lean 4 v4.28.0-rc1 with Mathlib4. Build: `cd P39_Sigma2 && lake`

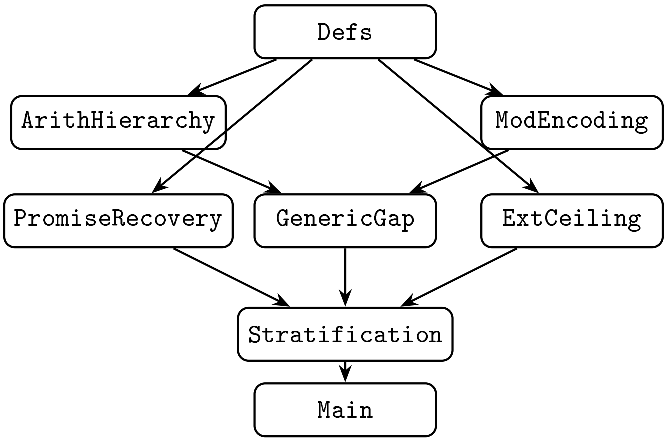

Figure 1: Module dependency graph (802 lines, 8 modules).

`build`. Result: 0 errors, 0 warnings, 0 `sorry`. Axiom profile (`#print axioms sigma2_master`): 12 domain-specific bridge axioms + `propext`, `Classical.choice`, `Quot.sound`.

12 Conclusion

Physics reaches Σ_2^0. The spectral gap of a generic translation-invariant Hamiltonian—without promise gap—is Σ_2^0-complete, requiring LPO′, the Turing jump of LPO. But the BISH+LPO characterization of Papers 1–38 is not wrong: it is correct for all extensive observables and all promise-gapped physics. The Σ_2^0 tier emerges only for intensive observables when the promise gap is removed. Empirical physics, operating with finite measurement precision, always imposes an effective promise gap and therefore caps at LPO.

The Thermodynamic Stratification Theorem reveals that the arithmetic complexity of a physical observable is determined by its thermodynamic scaling: extensive (Fekete/BMC) at LPO; intensive (infimum) at LPO′. The promise gap in Cubitt's construction is the mechanism that collapsed the logic from Σ_2^0 to Σ_1^0.

AI-Assisted Methodology

This formalization was developed using Claude (Anthropic) as a collaborative tool for Lean 4 code generation, proof strategy exploration, and LaTeX document preparation. All mathematical content was specified by the author. Every theorem was verified by the Lean 4 type checker.

Domain-Expert Disclaimer. Bridge axioms connecting Lean formalizations to published physics results must be validated by domain experts. In particular, the modified Cubitt encoding (Theorem 1) axiomatizes a construction that extends the Bausch–Cubitt–Ozols perimeter counter technique to input-dependent computation; its physical validity requires verification by specialists in Hamiltonian complexity.

References

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