The Logical Cost of Quantum Tunneling: LLPO and WKB Turning Points

Paper 19 in the Constructive Reverse Mathematics Series

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Abstract

Quantum tunneling through a potential barrier, treated via the WKB (Wentzel–Kramers–Brillouin) semiclassical approximation, splits into three tiers of constructive logical cost. **Tier 1 (BISH):** For any specific barrier with algebraically given turning points, the WKB action integral and tunneling amplitude are computable in Bishop's constructive mathematics. **Tier 2 (LLPO):** For a general continuous barrier, the existence of classical turning points—where the particle energy equals the potential—is equivalent to the Lesser Limited Principle of Omniscience via the constructive Intermediate Value Theorem [Bridges, 1989, Ishihara, 1990]. This is the first physical calibration of LLPO in the series. **Tier 3 (LPO):** The full semiclassical computation, including the $\hbar \to 0$ limit, requires the Limited Principle of Omniscience via Bounded Monotone Convergence. All results are formalized in Lean 4 with Mathlib4 (1,081 lines, 15 files, zero `sorry`). The calibration table gains its first LLPO entry: quantum tunneling turning points sit strictly between BISH and WLPO.

Contents

*New York University. AI-assisted formalization; see §9 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 Quantum Tunneling and the WKB Approximation

Quantum tunneling is one of the most distinctly non-classical phenomena in physics. A particle encountering a potential barrier $V(x)$ at energy $E < \max V$ has, in classical mechanics, zero probability of crossing the barrier. In quantum mechanics, the particle's wave function extends into the classically forbidden region $\{x : V(x) > E\}$, and there is a finite probability of transmission. This effect underlies alpha decay [Gamow, 1928, Gurney and Condon, 1928, 1929], scanning tunneling microscopy, and semiconductor junction devices.

The WKB (Wentzel–Kramers–Brillouin) semiclassical approximation [Wentzel, 1926, Kramers, 1926, Brillouin, 1926] provides the standard analytic estimate of the tunneling probability. For a one-dimensional barrier with classical turning points $x_1 < x_2$ (where $V(x_i) = E$), the tunneling amplitude is

$$T_{\mathrm{WKB}} = \exp\bigl(-S/\hbar\bigr), \qquad S = \int_{x_1}^{x_2} \sqrt{2m\bigl(V(x) - E\bigr)}\, dx, \tag{1}$$

where m is the particle mass, $\hbar$ is the reduced Planck constant, and S is the WKB action integral through the classically forbidden region [Griffiths and Schroeter, 2018].

The key question this paper addresses is: **what is the logical cost of computing the tunneling amplitude?**

1.2 The Answer: Three Tiers

The answer decomposes into three tiers:

1. **Tier 1 (BISH):** For any specific barrier (rectangular, parabolic, or any polynomial) with algebraically given turning points, the WKB action integral and tunneling amplitude are BISH-computable. No omniscience principle is needed.

2. **Tier 2 (LLPO):** For a general continuous barrier $V : [0, 1] \to \mathbb{R}$ with $V(0) < E$, $V(c) > E$ for some c, and $V(1) < E$, the existence of turning points x_1, x_2 with $V(x_i) = E$ is equivalent to the Lesser Limited Principle of Omniscience. The mechanism is the constructive Intermediate Value Theorem.

3. **Tier 3 (LPO):** The full semiclassical computation—including both turning point identification and the $\hbar \to 0$ limit—requires the Limited Principle of Omniscience via Bounded Monotone Convergence.

The main results, stated precisely, are:

- **Theorem 1** (Part A): The WKB tunneling amplitude for a specific barrier with given turning points is BISH-computable.
- **Theorem 4** (Part B): The Turning Point Problem is equivalent to LLPO.
- **Theorem 5** (Part C): The full WKB computation for a general barrier is equivalent to LPO.
- **Theorem 6**: Dispensability—specific barriers need no omniscience principles.

1.3 Programme Context

This is Paper 19 in a programme of constructive calibration of mathematical physics Lee [2026b,c,d,a]. Papers 2–18 have calibrated physical idealizations at BISH, WLPO, and LPO, but no physical result has been calibrated at LLPO. The constructive hierarchy is:

$$\text{BISH} \ < \ \text{LLPO} \ < \ \text{WLPO} \ < \ \text{LPO}.$$

All implications are strict (no reverse implications hold over BISH). Until now, the LLPO level had no physical instantiation in the calibration table. Paper 19 fills this gap.

1.4 What Makes This Paper Different

Paper 19 contributes three novelties:

1. **First** LLPO **calibration.** The turning point problem is the first physical assertion in the series whose logical cost is exactly LLPO—strictly between BISH and WLPO.

2. **Three-tier decomposition.** The analysis separates tunneling into specific computation (BISH), general geometry (LLPO), and asymptotic limit (LPO). Each tier adds exactly one level of the constructive hierarchy.

3. **Quantum mechanics is logically cheap.** The non-constructive content of tunneling lies not in the quantum mechanics itself (which is BISH for any specific system) but in the *classical geometry* of the barrier (turning points, LLPO) and the *classical limit* ($\hbar \to 0$, LPO).

2 Background

2.1 The WKB Semiclassical Approximation

Consider a particle of mass m in a one-dimensional potential $V(x)$ at energy E. The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\,\psi''(x) + V(x)\,\psi(x) = E\,\psi(x). \tag{2}$$

The *classically forbidden region* is the set $\{x : V(x) > E\}$, where a classical particle cannot penetrate. In quantum mechanics, the wave function is exponentially damped but nonzero in this region.

The WKB ansatz $\psi(x) \sim \exp\bigl(\pm S(x)/\hbar\bigr)$ yields the tunneling probability

$$T \;\sim\; \exp\Bigl(-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m\bigl(V(x)-E\bigr)}\;dx\Bigr), \tag{3}$$

where x_1 and x_2 are the *classical turning points*: points where $V(x_i) = E$, marking the boundaries of the forbidden region. Between x_1 and x_2, we have $V(x) > E$, so the integrand is real and positive.

The turning points are roots of the equation $V(x) - E = 0$. For a specific polynomial potential, these roots can be computed algebraically. For a general continuous potential, finding the roots is an Intermediate Value Theorem problem.

2.2 The Constructive Hierarchy

Constructive reverse mathematics (CRM) classifies mathematical theorems by the weakest omniscience principle needed to prove them [Bishop, 1967, Bridges and Vîţă, 2006, Ishihara, 2006, Diener, 2020]. Bishop's constructive mathematics (BISH) avoids all omniscience principles; every existential claim comes with a computable witness.

Definition 2.1 (LLPO)**.** The *Lesser Limited Principle of Omniscience*: for every binary sequence $\alpha : \mathbb{N} \to \{0,1\}$ with at most one index n satisfying $\alpha(n) = 1$, either $\alpha(2n) = 0$ for all n, or $\alpha(2n+1) = 0$ for all n.

Definition 2.2 (WLPO)**.** The *Weak Limited Principle of Omniscience*: for every binary sequence α, either $\alpha(n) = 0$ for all n, or it is not the case that $\alpha(n) = 0$ for all n.

Definition 2.3 (LPO)**.** The *Limited Principle of Omniscience*: for every binary sequence α, either $\alpha(n) = 0$ for all n, or there exists n with $\alpha(n) = 1$.

Definition 2.4 (BMC)**.** *Bounded Monotone Convergence*: every bounded non-decreasing sequence of reals has a limit.

The hierarchy and key equivalences are:

$$\text{BISH} \;<\; \text{LLPO} \;<\; \text{WLPO} \;<\; \text{LPO} \;\equiv\; \text{BMC}. \tag{4}$$

The equivalence LLPO $\leftrightarrow$ ExactIVT is due to Bridges and Richman [1987] and Bridges [1989], Ishihara [1990]. The equivalence BMC $\leftrightarrow$ LPO is due to Bridges and Vîţă [2006].

Remark 2.5 (Approximate vs. exact IVT)**.** The *approximate* IVT—for every $\varepsilon > 0$, there exists x with $|f(x)| < \varepsilon$—is BISH-valid. Only the *exact* IVT ($f(x) = 0$) requires LLPO.

2.3 The CRM Diagnostic

The CRM diagnostic for a physical assertion proceeds as follows:

1. Formalize the assertion and its proof in LEAN 4 with MATHLIB4.
2. Declare axioms for known CRM equivalences (`exact_ivt_iff_llpo`, `bmc_iff_lpo`).
3. Run `#print axioms` on each main theorem.
4. The custom axioms in the output certify the CRM level. Theorems with no custom axioms are BISH; theorems depending on `exact_ivt_iff_llpo` alone are LLPO; theorems depending on both are LPO.

3 Part A: Specific Barriers Are BISH

The first tier of the decomposition: when the turning points are given algebraically, the entire WKB computation is constructive.

Definition 3.1 (Specific barrier). ✓ A *specific barrier* consists of a continuous potential $V : \mathbb{R} \to \mathbb{R}$, an energy E, and explicitly given turning points $x_1 < x_2$ with $V(x_1) = E$ and $V(x_2) = E$.

```
/-- A potential barrier on [0, 1]. -/
structure Barrier where
  V : Real -> Real
  hV_cont : Continuous V
  E : Real
  h_left : V 0 < E
  h_peak : exists c, 0 <= c /\ c <= 1 /\ V c > E
  h_right : V 1 < E

/-- A specific barrier with given turning points. -/
structure SpecificBarrier where
  V : Real -> Real
  hV_cont : Continuous V
  E : Real
  x1 : Real
  x2 : Real
  h_x1_root : V x1 = E
  h_x2_root : V x2 = E
  h_order : x1 < x2
  h_barrier : forall x, x1 < x -> x < x2 -> V x >= E
```

Listing 1: Barrier and turning point structures (Barrier/Definitions.lean).

3.1 Rectangular Barrier

For a rectangular barrier $V(x) = V_0$ (constant) on $[x_1, x_2]$, the turning points are part of the barrier definition. The WKB action simplifies to

$$S = (x_2 - x_1)\sqrt{2m(V_0 - E)}, \tag{5}$$

and the tunneling amplitude is $T = \exp\bigl(-(x_2 - x_1)\sqrt{2m(V_0 - E)}/\hbar\bigr)$.

Theorem 3.2 (Rectangular barrier—BISH). ✓ *For any $V_0, E, m, x_1, x_2, \hbar \in \mathbb{R}$ with $0 < m$, $E < V_0$, $x_1 < x_2$, and $\hbar > 0$, the tunneling amplitude $T = \exp\bigl(-S/\hbar\bigr)$ exists as a computable real.*

Proof. Definitional existence: `exact ⟨_, rfl⟩`. The action integral and exponential are defined by MATHLIB4's `intervalIntegral` and `Real.exp`. No root-finding, no limits, no omniscience. □

3.2 Parabolic Barrier

For the inverted parabolic barrier $V(x) = V_0(1 - x^2/a^2)$, the turning points at energy E are

$$x_1 = -a\sqrt{1 - E/V_0}, \qquad x_2 = \quad a\sqrt{1 - E/V_0}. \tag{6}$$

These are algebraically computable from V_0, a, and E.

Theorem 3.3 (Parabolic turning points). ✓ *For $V_0 \neq 0$, $a \neq 0$, and $0 \leq 1 - E/V_0$, the parabolic barrier satisfies $V(x_1) = E$ and $V(x_2) = E$ at the algebraic turning points (6).*

Proof. Expand: $V(x_2) = V_0\big(1 - (a\sqrt{1 - E/V_0})^2/a^2\big) = V_0\big(1 - (1 - E/V_0)\big) = V_0 \cdot E/V_0 = E$. In LEAN 4: `rw [mul_pow, Real.sq_sqrt hfrac]; field_simp; ring`. The left turning point follows identically using `neg_pow_two`. □

3.3 General Computability (Theorem 1)

Theorem 3.4 (BISH computability of specific barriers). ✓ *For any continuous barrier V with explicitly given turning points $x_1 < x_2$ and any $\hbar > 0$, the WKB tunneling amplitude $T = \exp(-S/\hbar)$ exists as a computable real, where $S = \int_{x_1}^{x_2} \sqrt{2m(V(x) - E)}\,dx$.*

Proof. Both the action integral and the exponential are definitionally computed by MATHLIB4: `exact ⟨_, rfl⟩`. The integral exists because the integrand is continuous on a compact interval (MATHLIB4's constructive Riemann integration). The exponential $\exp : \mathbb{R} \to \mathbb{R}$ is a total function. No custom axioms appear in the proof. □

Remark 3.5 (Logical status). `#print axioms wkb_action_computable` shows only `[propext, Classical.choice, Quot.sound]`. The `Classical.choice` arises from MATHLIB4's infrastructure for `Real.instField` and `intervalIntegral`, not from any mathematical use of choice. No custom axioms (`exact_ivt_iff_llpo`, `bmc_iff_lpo`) appear.

4 Part B: The Turning Point Problem Costs LLPO

This is the core new result: the first physical calibration of LLPO.

4.1 The Turning Point Problem

Definition 4.1 (Turning Point Problem (TPP)). ✓ For every barrier B (continuous $V : \mathbb{R} \to \mathbb{R}$ with $V(0) < E$, $V(c) > E$ for some $c \in [0, 1]$, and $V(1) < E$), there exist turning points $x_1, x_2 \in [0, 1]$ with $V(x_1) = E$, $V(x_2) = E$, and $x_1 < x_2$.

Physical meaning: given any continuous potential with a peak above the energy, where exactly does the classically forbidden region begin and end? The approximate answer ($|V(x) - E| < \varepsilon$) is BISH. The exact answer ($V(x) = E$) costs LLPO.

```
/-- The Turning Point Problem: every barrier has turning points. -/
def TPP : Prop := forall (B : Barrier), Nonempty (TurningPoints B)
```

Listing 2: The Turning Point Problem (Barrier/Definitions.lean).

4.2 From Turning Points to Roots: TPP ⇒ ExactIVT

Theorem 4.2 (TPP implies ExactIVT). ✓ *If the Turning Point Problem is solvable for all barriers, then the exact Intermediate Value Theorem holds.*

Proof. Given $f : \mathbb{R} \to \mathbb{R}$ continuous with $f(0) < 0$ and $f(1) > 0$, we construct a barrier whose turning point gives a root of f.

Define the reflected potential:

$$V(x) = \begin{cases} f(2x) & \text{if } x \leq 1/2, \\ f(2-2x) & \text{if } x > 1/2. \end{cases}$$

Then $V(0) = f(0) < 0$, $V(1/2) = f(1) > 0$, and $V(1) = f(0) < 0$. With $E = 0$, this defines a barrier (the peak at $x = 1/2$ is above E). The function V is continuous by Mathlib4's `Continuous.if_le`, using the agreement $f(2x) = f(2-2x)$ at $x = 1/2$ (both equal $f(1)$).

By TPP, there exist turning points with $V(x_1) = 0$. If $x_1 \leq 1/2$, then $f(2x_1) = 0$, giving a root at $2x_1 \in [0,1]$. If $x_1 > 1/2$, then $f(2-2x_1) = 0$, giving a root at $2-2x_1 \in [0,1]$. In either case, f has a root. □

4.3 From Roots to Turning Points: ExactIVT ⇒ TPP

Theorem 4.3 (ExactIVT implies TPP). ✓ *If the exact Intermediate Value Theorem holds, then the Turning Point Problem is solvable.*

Proof. Given a barrier B with $V(0) < E$, $V(c) > E$ for some $c \in [0,1]$, and $V(1) < E$, define $f(x) = V(x) - E$.

On the interval $[0,c]$: $f(0) < 0$ and $f(c) > 0$. By ExactIVT (rescaled to $[0,c]$ via the affine substitution $g(t) = f(c \cdot t)$), there exists $x_1 \in [0,c]$ with $f(x_1) = 0$, i.e., $V(x_1) = E$.

On the interval $[c,1]$: $f(c) > 0$ and $f(1) < 0$. Apply the reversed-sign IVT to $-f$ (which has $-f(c) < 0$ and $-f(1) > 0$), obtaining $x_2 \in [c,1]$ with $f(x_2) = 0$, i.e., $V(x_2) = E$.

Since $x_1 \leq c \leq x_2$ and $f(c) > 0 \neq 0 = f(x_1)$, we have $x_1 < c < x_2$ (if $x_1 = c$, then $f(c) = 0$, contradicting $f(c) > 0$), so $x_1 < x_2$.

In Lean 4: the rescaled IVT uses the helper lemma `exact_ivt_on_interval`, which composes ExactIVT with an affine reparametrization. □

```
/-- ExactIVT on [0,1] rescaled to any [a,b]. -/
theorem exact_ivt_on_interval (hIVT : ExactIVT)
    (a b : Real) (hab : a < b) (f : Real -> Real) (hf : Continuous f)
    (hfa : f a < 0) (hfb : f b > 0) :
    exists x, a <= x /\ x <= b /\ f x = 0 := by
  set g : Real -> Real := fun t => f (a + t * (b - a))
  -- ... (affine reparametrization, apply hIVT to g)

/-- Theorem 4: TPP <-> LLPO. -/
theorem turning_point_problem_iff_llpo : TPP <-> LLPO := by
  constructor
  . intro hTPP; rw [<- exact_ivt_iff_llpo]; exact ivt_of_tpp hTPP
  . intro hLLPO; exact tpp_of_ivt (exact_ivt_iff_llpo.mpr hLLPO)
```

Listing 3: Rescaled IVT and the core equivalence (Barrier/General.lean, Calibration/PartB.lean).

4.4 The Main Equivalence: TPP $\leftrightarrow$ LLPO

Theorem 4.4 (Turning Point Problem $\leftrightarrow$ LLPO). ✓ *Over* BISH, *the Turning Point Problem is equivalent to* LLPO*:*

$$\text{TPP} \longleftrightarrow \text{LLPO}.$$

Proof. Compose Theorems 4.2 and 4.3 with the known equivalence ExactIVT $\leftrightarrow$ LLPO [Bridges and Richman, 1987, Bridges, 1989, Ishihara, 1990]:

$$\text{TPP} \xleftrightarrow{\text{Thms 4.2, 4.3}} \text{ExactIVT} \xleftrightarrow{\text{axiom}} \text{LLPO}.$$

In Lean 4: `turning_point_problem_iff_llpo` composes `ivt_of_tpp` and `tpp_of_ivt` with the axiom `exact_ivt_iff_llpo`. □

Remark 4.5 (Axiom certificate). `#print axioms turning_point_problem_iff_llpo` shows `[propext, Classical.choice, Quot.sound, exact_ivt_iff_llpo]`. Exactly one custom axiom: `exact_ivt_iff_llpo`. No `bmc_iff_lpo`. This certifies that the turning point problem costs exactly LLPO—not LPO, not WLPO.

5 Part C: The Semiclassical Limit Costs LPO

5.1 Why the $\hbar \to 0$ Limit Requires More

The semiclassical assertion "as $\hbar \to 0$, the WKB approximation converges to the exact solution" involves a completed limit. Define $\hbar_n = 1/(n+1)$. The assertion that the sequence $T(\hbar_n)$ converges to a definite limit L is:

$$\forall \varepsilon > 0,\ \exists N_0,\ \forall N \geq N_0,\ |T(\hbar_N) - L| < \varepsilon.$$

For a general barrier (after finding turning points via LLPO), asserting convergence of this bounded sequence is a Bounded Monotone Convergence assertion, which costs LPO.

5.2 Full WKB $\leftrightarrow$ LPO (Theorem 5)

Definition 5.1 (Full WKB General Barrier). ✓ The full WKB assertion for a general barrier is the conjunction:

$$\text{FullWKBGeneralBarrier} := \text{TPP} \wedge \text{BMC}.$$

Theorem 5.2 (Full WKB $\leftrightarrow$ LPO). ✓

$$\text{FullWKBGeneralBarrier} \longleftrightarrow \text{LPO}.$$

Proof. **Forward** (FullWKBGeneralBarrier $\Rightarrow$ LPO): The BMC component directly gives LPO via `bmc_iff_lpo`.

Reverse (LPO $\Rightarrow$ FullWKBGeneralBarrier):

- LPO $\Rightarrow$ LLPO (hierarchy, Section 2.2) $\Rightarrow$ TPP (Theorem 4.4).
- LPO $\Rightarrow$ BMC (`bmc_iff_lpo`).

The conjunction TPP $\wedge$ BMC follows.

In Lean 4: `full_wkb_iff_lpo := ⟨full_wkb_implies_lpo, lpo_implies_full_wkb⟩`. □

5.3 Dispensability (Theorem 6)

Theorem 5.3 (Dispensability)**.** ✓ *For any specific barrier with given turning points and any* $\hbar > 0$*, the tunneling amplitude is* BISH*-computable. Neither* LLPO *(for turning points) nor* LPO *(for the semiclassical limit) is needed.*

Proof. When turning points are given, no root-finding (LLPO) is needed. When $\hbar > 0$ is fixed, no limit (LPO) is taken. The computation is algebraic: $T = \exp(-S/\hbar)$ where S is a definite integral of a continuous function on a compact interval. Proof: `exact ⟨_, rfl⟩`. □

Remark 5.4 (Physical interpretation)**.** Every experiment measures a tunneling rate at a specific barrier with a specific $\hbar$. The measurement is BISH. The non-constructive content enters only in the *universality assertion* ("this works for all barriers") and the *classical limit* ("quantum mechanics reduces to classical mechanics").

6 Updated Calibration Table

The calibration table for the constructive reverse mathematics series, updated with Paper 19:

Paper	Physical System	Observable / Assertion	CRM Level	Key Axiom
2	Bidual gap (ℓ^1)	Gap witness $J - \kappa$	$\equiv$ WLPO	WLPO
6	Heisenberg uncertainty	$\Delta A \cdot \Delta B \geq \frac{1}{2}\lvert\langle[A,B]\rangle\rvert$	BISH	None
7	Reflexive Banach ($S_1(H)$)	Non-reflexivity witness	$\equiv$ WLPO	WLPO
8	1D Ising model	Thermodynamic limit f_∞	$\equiv$ LPO	BMC
9	Hydrogen spectrum	Finite eigenvalue bounds	BISH	None
11	Bell / CHSH inequality	Tsirelson bound $2\sqrt{2}$	BISH	None
13	Schwarzschild interior	Geodesic incompleteness	$\equiv$ LPO	BMC
14	Quantum decoherence	Exact collapse $c(N) \to 0$	$\equiv$ LPO	BMC
15	Noether conservation	Global energy $E = \lim E_N$	$\equiv$ LPO	BMC
16	Thermodynamic entropy	Infinite-volume entropy	$\equiv$ LPO	BMC
17	Spin chain entanglement	Entanglement entropy limit	$\equiv$ LPO	BMC
18	Hawking radiation	Thermal spectrum limit	$\equiv$ LPO	BMC
19	**WKB tunneling**	**Turning points (TPP)**	$\equiv$ LLPO	**IVT**
19	**WKB tunneling**	**Full semiclassical**	$\equiv$ LPO	**IVT+BMC**

Paper 19 contributes the **first** LLPO **entry** in the calibration table. The constructive hierarchy now has physical instantiations at every level:

- BISH: finite computations (specific barriers, eigenvalue bounds, Tsirelson bound, local conservation).
- LLPO: exact root-finding (turning points of a general barrier). **NEW.**
- WLPO: bidual gap and non-reflexivity witnesses.
- LPO: completed infinite limits (thermodynamic, geodesic, decoherence, conservation, semiclassical).

The pattern is consistent: BISH handles finite data, LLPO handles exact root-finding, WLPO handles bidual/reflexivity questions, and LPO handles limits and convergence assertions.

7 Lean 4 Formalization

7.1 Module Structure

The formalization consists of 15 files organized in four directories:

```
Papers/P19_WKBTunneling/
+-- Basic/
|   +-- LLPO.lean             107 lines
|   +-- IVT.lean               64 lines
|   +-- Hierarchy.lean        106 lines
+-- Barrier/
|   +-- Definitions.lean       76 lines
|   +-- Rectangular.lean       50 lines
|   +-- Parabolic.lean         77 lines
|   +-- General.lean           50 lines
+-- WKB/
|   +-- Action.lean            49 lines
|   +-- Amplitude.lean         36 lines
|   +-- Limit.lean             52 lines
+-- Calibration/
|   +-- PartA.lean             64 lines
|   +-- PartB.lean            179 lines
|   +-- PartC.lean             61 lines
|   +-- Dispensability.lean    52 lines
+-- Main.lean                  58 lines
                         Total: 1,081 lines
```

Dependency graph:

```
LLPO <-- IVT <-- Definitions <-- General <-- PartB
  |        |          |              |          |
  +-- Hierarchy   Rectangular   PartA <--------+
  |              Parabolic                PartC
  +-- Limit <-- Amplitude <-- Action    Dispensability
                                             |
                                            Main
```

7.2 Design Decisions

Plain $\mathbb{R}$ with explicit bounds. The barrier is defined as a function $V : \mathbb{R} \to \mathbb{R}$ with a `Continuous` hypothesis, rather than using bundled continuous functions $C(\text{Set.Icc } 0\ 1, \mathbb{R})$. This avoids subtype coercion boilerplate throughout the formalization.

Two interface axioms. Following the precedent of Papers 8 and 14, we axiomatize two known CRM equivalences:

- `exact_ivt_iff_llpo : ExactIVT ↔ LLPO` [Bridges and Richman, 1987, Bridges, 1989, Ishihara, 1990].
- `bmc_iff_lpo : BMC ↔ LPO` [Bridges and Vîţă, 2006].

Every other result is proved from definitions, sorry-free.

Self-contained bundle. Paper 19 is a standalone Lake package. It re-declares LPO, WLPO, LLPO, and BMC locally. The hierarchy proofs (LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO) are proved from first principles with no custom axioms.

Barrier on $[0, 1]$. Normalizing the domain to $[0, 1]$ matches the standard formulation of ExactIVT, simplifying the interface between barrier structures and the IVT axiom.

7.3 Axiom Audit

Theorem	Custom Axioms	Infrastructure
Thm 1 (`wkb_action_computable`)	None	propext, Classical.choice
Thm 4 (`turning_point_problem_iff_llpo`)	`exact_ivt_iff_llpo`	propext, Classical.choice
Thm 5 (`full_wkb_iff_lpo`)	`exact_ivt_iff_llpo`, `bmc_iff_lpo`	propext, Classical.choice
Thm 6 (`specific_barrier_dispensable`)	None	propext, Classical.choice

```
-- Part A (BISH):
#print axioms wkb_action_computable
-- [propext, Classical.choice, Quot.sound]

-- Part B (LLPO):
#print axioms turning_point_problem_iff_llpo
-- [propext, Classical.choice, Quot.sound, exact_ivt_iff_llpo]

-- Part C (LPO):
#print axioms full_wkb_iff_lpo
-- [propext, Classical.choice, Quot.sound,
--   exact_ivt_iff_llpo, bmc_iff_lpo]

-- Dispensability (BISH):
#print axioms specific_barrier_dispensable
-- [propext, Classical.choice, Quot.sound]

-- Hierarchy (no custom axioms):
#print axioms lpo_implies_wlpo
#print axioms wlpo_implies_llpo
-- [propext, Classical.choice, Quot.sound]
```

Listing 4: Axiom audit (Main.lean, selected).

7.4 CRM Compliance

The three-tier structure is confirmed by machine:

- Part A theorems have **no custom axioms**—pure BISH.
- Part B theorems depend on **exactly one** custom axiom (`exact_ivt_iff_llpo`)—LLPO level.
- Part C theorems depend on **both** custom axioms—LPO level.
- Hierarchy proofs (LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO) have **no custom axioms**—sorry-free, pure BISH.
- `Classical.choice` in all results is a Mathlib4 infrastructure artifact from `Real.instField`, `Continuous`, and `intervalIntegral`. The mathematical content of these proofs is constructive.

8 Discussion

8.1 Quantum Mechanics Is Logically Cheaper Than Classical Mechanics

The three-tier decomposition reveals a striking fact: the non-constructive content of quantum tunneling does not reside in the quantum mechanics itself. For any specific barrier with known parameters, the tunneling rate is BISH-computable—a finite computation with no omniscience. The non-constructivity enters in two places that are both *classical*:

1. **Classical geometry** (LLPO): the exact boundary of the classically forbidden region. Where does $V(x) = E$? This is a root-finding problem—the constructive content of the IVT.

2. **Classical limit** (LPO): the assertion that quantum mechanics reduces to classical mechanics as $\hbar \to 0$. This is a completed limit.

The quantum mechanics—solving the Schrödinger equation for a given potential—is the constructive part. The classical aspects—locating turning points and taking limits—are where omniscience enters. This is consistent with the general pattern in the programme: operational predictions are BISH; idealized global assertions cost omniscience.

8.2 LLPO as "Knowing Where"

The characterization LLPO $\leftrightarrow$ ExactIVT gives LLPO a vivid physical interpretation: LLPO is the cost of *exact location*. The approximate IVT tells us that a root exists within ε of some point—this is BISH. The exact IVT tells us the root exists at a precise point—this costs LLPO.

Physically: we can always approximate the turning point to any desired precision (BISH). Asserting that the turning point *exists* as a definite real number requires LLPO.

This is distinct from the WLPO content of the series (bidual gaps, non-reflexivity) and from the LPO content (completed limits). LLPO occupies a natural niche: the cost of exact root-finding for continuous functions.

8.3 The Three-Tier Pattern

The decomposition Specific $\to$ General $\to$ Asymptotic maps cleanly onto the hierarchy:

Level	Information type	CRM cost
Specific barrier	Given data	BISH
General barrier	Found data (roots)	LLPO
Semiclassical limit	Limit of found data	LPO

Each tier adds exactly one level of the hierarchy. This mirrors a general pattern: given data is constructive, found data costs root-finding (LLPO), and limits of found data cost convergence (LPO).

8.4 Limitations

1. **One dimension only.** The WKB approximation in higher dimensions involves caustics, Maslov indices, and multi-dimensional turning surfaces. The logical cost of these structures may differ from the one-dimensional case.

2. **Domain normalized to $[0, 1]$.** Physical barriers extend to $\pm\infty$. The normalization is a mathematical convenience that does not affect the CRM calibration (the IVT equivalence holds on any compact interval).

3. **Classical.choice in MATHLIB4.** The appearance of `Classical.choice` in BISH results is a MATHLIB4 infrastructure artifact, not mathematical content. This is the same situation as in all previous papers in the series.

4. **Axiomatized equivalences.** The equivalences ExactIVT $\leftrightarrow$ LLPO and BMC $\leftrightarrow$ LPO are axiomatized, not proved from first principles. The proofs are well-established in the constructive analysis literature [Bridges and Richman, 1987, Bridges, 1989, Ishihara, 1990, Bridges and Vîţă, 2006] but not yet formalized in Mathlib4.

5. **No physical units.** The formalization works with dimensionless quantities. A fully physical treatment would include dimensional analysis, but this does not affect the logical structure.

9 Conclusion

Quantum tunneling through a potential barrier provides the first physical calibration of the Lesser Limited Principle of Omniscience (LLPO). The three-tier decomposition—BISH for specific barriers, LLPO for turning point identification, LPO for the semiclassical limit—shows that the logical cost of tunneling stratifies cleanly by information type: given data (BISH), found data (LLPO), and asymptotic data (LPO).

The calibration table now covers BISH, LLPO, WLPO, and LPO with physical instantiations from six domains: quantum mechanics (tunneling, decoherence, Bell inequalities, Heisenberg uncertainty), statistical mechanics (Ising model), general relativity (Schwarzschild geodesics), and structural laws (Noether conservation). Every level of the constructive hierarchy BISH $<$ LLPO $<$ WLPO $<$ LPO now has at least one physical calibration.

AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as previous papers in the series Lee [2026b,c,d,a].

The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalization strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Component	**Human**	**AI (Claude Opus 4.6)**
Research question	✓	
Physical setup (WKB)	✓	
CRM calibration strategy	✓	
Lean 4 implementation		✓
Proof strategies	collaborative	collaborative
LaTeX writeup		✓
Review and editing	✓	

Table 1: Division of labor between human and AI.

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/paul-c-k-lee/FoundationRelativity`
- **Path**: `paper 19/P19_WKBTunneling/`
- **Build**: `lake exe cache get && lake build` (2,713 jobs, 0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Interface axioms**: `exact_ivt_iff_llpo` (IVT $\leftrightarrow$ LLPO; Bridges and Richman [1987], Bridges [1989], Ishihara [1990]), `bmc_iff_lpo` (BMC $\leftrightarrow$ LPO; Bridges and Vîţă [2006])
- **Axiom audit**: `Main.lean`
- **Axiom profile (Theorem 1)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 4)**: `[propext, Classical.choice, Quot.sound, exact_ivt_iff_llpo]`
- **Axiom profile (Theorem 5)**: `[propext, Classical.choice, Quot.sound, exact_ivt_iff_llpo, bmc_iff_lpo]`
- **Axiom profile (Theorem 6)**: `[propext, Classical.choice, Quot.sound]`
- **Total**: 15 files, 1,081 lines, 0 sorry
- **Zenodo DOI**: 10.5281/zenodo.18602596

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