The Motive Is a Decidability Certificate: Constructive Reverse Mathematics in Arithmetic Geometry

(Paper 67 of the Constructive Reverse Mathematics Series)

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For Mimi

Abstract

This monograph synthesizes the arithmetic geometry phase (Papers 45–66) of the Constructive Reverse Mathematics program. Working over BISH (Bishop's constructive mathematics), we calibrate the major conjectures and constructions of Grothendieck's theory of motives against the hierarchy $\mathsf{BISH} \subset \mathsf{BISH} + \mathsf{MP} \subset \mathsf{BISH} + \mathsf{WLPO} \subset \mathsf{BISH} + \mathsf{LPO} \subset \mathsf{CLASS}$.

The program produces two principal outputs. The first is a *decidability classification*: three invariants—the analytic rank r, the Hodge level ℓ, and the effective Lang constant c—determine the logical strength of the cycle-search problem for any pure or mixed motive (Papers 50–62). The second is an *arithmetic identity*: for CM abelian fourfolds arising from cyclic Galois cubics, the self-intersection degree of the exotic Weil class satisfies $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$, where f is the conductor, $\mathfrak{A}$ is the Steinitz ideal class, and h is the $\mathcal{O}_K$-Hermitian self-pairing (Papers 56–58, 65). For non-cyclic (S_3) cubics, the scalar h is replaced by the $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of the trace-zero lattice form (Paper 66).

No new theorems are proved. This paper organizes, corrects, and contextualizes the results of Papers 45–66 as a single coherent argument. For the physics calibrations of Papers 1–44, we refer to the earlier synthesis (Paper 40).

Fifty-three papers in the series carry Lean 4 formalizations; the total verified codebase exceeds 86,000 lines, with twenty-two papers achieving zero `sorry`.

Contents

Part I
The Map

1 Introduction

Over the course of sixty-six papers, the Constructive Reverse Mathematics program has applied a single methodology to the theorems of arithmetic geometry: determine the exact logical principle each theorem requires. The methodology is Bishop's [1, 2]: work over intuitionistic logic with countable choice, identify where non-constructive principles enter, and classify the result by the weakest such principle.

The output is a calibration atlas—a map of theorems plotted against logical coordinates. This monograph presents the atlas for the arithmetic geometry phase of the program. The physics phase (Papers 1–44) is synthesized in Paper 40 [11]; we do not revisit it here beyond noting two results that anchor the hierarchy:

- Fekete's Subadditive Lemma is equivalent to LPO (Paper 29 [8]).
- The Fan Theorem and Dependent Choice are dispensable for all physical calibrations (Papers 30–31 [9, 10]).

These establish that empirical physics lives in $\mathsf{BISH} + \mathsf{LPO}$ and that the hierarchy's independent axes (FT, DC) are irrelevant to the physical calibrations. Everything that follows concerns the arithmetic geometry that begins at Paper 45.

1.1 The hierarchy

The logical principles form a partial order:

Definition 1.1. The *CRM hierarchy* consists of the following principles over BISH:

(i) MP (Markov's Principle): if a binary sequence is not identically zero, it contains a 1 at a specific index. Equivalently, $\neg\neg\exists n\, \alpha(n) = 1 \implies \exists n\, \alpha(n) = 1$.

(ii) LLPO (Lesser Limited Principle of Omniscience): for every binary sequence with at most one 1, the 1 (if it exists) occurs at an even index or at an odd index.

(iii) WLPO (Weak Limited Principle of Omniscience): every binary sequence is either identically zero or not identically zero. Equivalently, $\forall\alpha\, (\alpha = 0 \vee \alpha \neq 0)$.

(iv) LPO (Limited Principle of Omniscience): every binary sequence is either identically zero or contains a 1 at a specific index. Equivalently, $\forall\alpha\, (\alpha = 0 \vee \exists n\, \alpha(n) = 1)$.

These satisfy $\mathsf{BISH} \subset \mathsf{BISH} + \mathsf{MP} \subset \mathsf{BISH} + \mathsf{LLPO} \subset \mathsf{BISH} + \mathsf{WLPO} \subset \mathsf{BISH} + \mathsf{LPO} \subset \mathsf{CLASS}$, where MP and WLPO are incomparable over BISH and their join is LPO.

Remark 1.2. The Fan Theorem (FT) and Dependent Choice (DC_ω) are independent of the omniscience spine. Papers 30–31 demonstrated that neither is required for any calibration in the program. We suppress them henceforth.

1.2 What this monograph contains

Part I presents the decidability classification: the DPT framework (§2), the de-omniscientizing descent (§3), the dimension-4 wall (§4), the p-adic resolution (§5), and the mixed motive frontier (§6).

Part II presents the arithmetic identity: the $h = f$ discovery (§7), its Steinitz generalization (§8), the computational verification at scale (§9), and the form-class extension to non-cyclic cubics (§10).

Part III reflects on the program's methodology and future directions (§11, §13).

2 The DPT Framework

Paper 50 [17] proposed three axioms characterizing when Grothendieck's conjectural category of numerical motives admits constructive decidability. The framework is called *Decidable Polarized Tannakian* (DPT): the underlying categorical structure is expected to be Tannakian (after Grothendieck), and the three axioms add decidability and polarization.

Axiom 2.1 (Decidable Morphisms — DPT A1)**.** For objects M, N in the category of numerical motives, the $\mathbb{Q}$-vector space $\mathrm{Hom}(M, N)$ has decidable equality: given $f, g \in \mathrm{Hom}(M, N)$, the proposition $f = g$ is decidable in BISH.

This is equivalent to Standard Conjecture D (Lieberman [5]): numerical and homological equivalence coincide.

Axiom 2.2 (Algebraic Spectrum — DPT A2)**.** The eigenvalues of Frobenius on ℓ-adic cohomology are algebraic numbers, and the characteristic polynomial is independent of ℓ.

This is largely a theorem: algebraicity is Deligne's theorem (Weil II, [3]); ℓ-independence is known in many cases.

Axiom 2.3 (Archimedean Polarization — DPT A3)**.** The real fibre of the motive carries a positive-definite inner product (the Rosati involution composed with the polarization). This is the condition that makes the Hodge–Riemann bilinear relations constructive: positive-definiteness provides an *isolation gap* between zero and nonzero intersection numbers, converting LPO-level comparisons to BISH-level ones.

The key invariant is the u-invariant $u(k)$ of the base field: the maximal dimension of an anisotropic quadratic form over k. Since $u(\mathbb{R}) = 1$, every nonzero real quadratic form is either positive or negative—the Archimedean place provides the isolation gap.

Theorem 2.4 (DPT Decidability, Paper 50 Theorem C)**.** *Assume Axioms A1–A3. Then the equivalence relation defining* $\mathrm{Hom}(M, N)$ *in the category of numerical motives is decidable in* BISH*. The motive* $h(X)$ *is a* decidability certificate*: given a smooth projective variety* X*, the* BISH*-decidability of the intersection pairing on* $h(X)$ *follows from the three axioms.*

Proof sketch. Axiom A2 ensures the characteristic polynomial has algebraic coefficients, so comparison of eigenvalues reduces to algebraic number arithmetic (BISH). Axiom A1 ensures numerical equivalence is decidable (intersection numbers are integers). Axiom A3 provides the isolation gap: for nonzero classes α, the positive-definite pairing gives $\langle \alpha, \alpha \rangle > 0$ with a computable lower bound, so the zero/nonzero distinction is BISH-decidable. □

2.1 The two boundaries

Papers 51–54 tested the DPT axioms against specific conjectures and identified two boundaries where decidability breaks.

Definition 2.5 (Axiom 1 boundary)**.** The Axiom 1 boundary occurs at codimension ≥ 2: the Lefschetz ring generates all algebraic classes in codimension 1 (the Lefschetz $(1, 1)$-theorem), but exotic Hodge classes outside the Lefschetz ring appear at codimension 2 for abelian fourfolds (Schoen [7]). At this boundary, Axiom A1 provides no decision procedure for detecting exotic classes.

Definition 2.6 (Axiom 3 boundary)**.** The Axiom 3 boundary occurs at finite primes: since $u(\mathbb{Q}_p) = 4$, positive-definite quadratic forms over $\mathbb{Q}_p$ exist only in dimensions ≤ 3. For abelian fourfolds (cohomological dimension 4), the p-adic analogue of Axiom A3 fails. No isolation gap is available at finite primes.

Paper 54 [20] identified these boundaries by stress-testing the DPT framework against the Bloch–Kato conjecture. The fracture analysis posed two questions: *(i)* Does the framework require an additional "Axiom 5" for p-adic decidability? *(ii)* What computable structure exists at the Axiom 1 boundary? Papers 55–66 answer both.

3 The De-Omniscientizing Descent

Paper 50 [17] identified a structural pattern—the *de-omniscientizing descent*—recurring across the five major conjectures calibrated in the program. The pattern formalizes how geometric origin converts undecidable (LPO-level) data into decidable (BISH-level) data.

Definition 3.1 (De-omniscientizing descent)**.** A *de-omniscientizing descent* for a cohomological datum D is a four-stage process:

1. **Continuous Prison.** The datum D is initially computed in a complete topological field ($\mathbb{Q}_\ell$, $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{C}$) where equality testing requires LPO.
2. **Discrete Rescue.** A conjecture asserts that D descends to an algebraic field ($\mathbb{Q}$, $\overline{\mathbb{Q}}$, $\mathbb{Z}$) where equality is BISH-decidable.
3. **Geometric Mechanism.** The descent is mediated by *geometric origin*: algebraic cycles, Galois representations, or other algebro-geometric structures force the datum to be algebraic.
4. **Residual.** After LPO is eliminated, a Diophantine search cost (MP) may remain: finding explicit cycle representatives or generators requires an unbounded search that geometric origin does not eliminate.

Theorem 3.2 (Five Conjectures, Paper 50)**.** *The five conjectures exhibit the de-omniscientizing descent with the following residuals:*

Conjecture	***Paper***	***LPO source***	***Residual***
Weight-Monodromy	*45*	*Equality in* $\mathbb{Q}_\ell$	BISH *(none)*
Tate	*46*	*Galois-fixed subspace*	MP *(cycle search)*
Fontaine-Mazur	*47*	*Zero-testing in* $\mathbb{Q}_p$	MP *(variety search)*
Birch–Swinnerton-Dyer	*48, 51*	*L-function evaluation*	MP *(generator search)*
Hodge	*49*	*Hodge filtration*	MP *(cycle search)*

*The Weight-Monodromy Conjecture achieves full descent (*LPO $\to$ BISH*) with no* MP *residual. The remaining four retain* MP *from their respective search problems.*

Proof sketch. For each conjecture, identify the complete field where the datum initially lives, the algebraic field to which the conjecture asserts it descends, and the geometric mechanism mediating descent. The WMC descent is algebraicity of Frobenius eigenvalues—Deligne's theorem provides this unconditionally. For the remaining four, descent is conjectural, and the MP residual arises from the unbounded search for explicit algebraic representatives (cycles, varieties, or generators). See Paper 50, §§4–6, for the individual calibrations. □

Remark 3.3 (The motive as LPO-killer)*.* The uniform pattern leads to a precise slogan: the motive kills LPO but not MP. Grothendieck's universal cohomology—if it satisfies the DPT axioms—converts LPO-level questions (equality in complete fields) to BISH-level questions (algebraic number arithmetic), but cannot eliminate the Diophantine search for explicit witnesses.

4 The Dimension-4 Wall

Papers 52–53 [18, 19] established that the DPT framework has a sharp dimensional boundary.

Theorem 4.1 (Decidability below dimension 4, Paper 52)**.** *For abelian varieties of dimension $g \leq 3$, the Lefschetz ring generates all algebraic Hodge classes. Consequently, numerical equivalence is decidable in* BISH *(unconditionally, using Lieberman's theorem and the CM decidability oracle of Paper 53).*

Theorem 4.2 (Visibility failure at dimension 4, Paper 53)**.** *For CM abelian fourfolds ($g = 4$), exotic Weil classes exist outside the Lefschetz ring (Schoen [7]). These classes are:*

(i) algebraic (by Schoen's theorem),

(ii) computationally healthy: $\deg(w \cdot w) = 7 > 0$ *(verified in Lean 4),*

(iii) Hodge–Riemann compatible (positive self-intersection),

(iv) invisible *to the Lefschetz-based decidability oracle.*

The obstruction is geometric *(a visibility failure), not* logical *(an omniscience requirement).*

Remark 4.3*.* The distinction between logical and geometric obstructions is central to the program. A logical obstruction means the theorem genuinely requires an omniscience principle: no reformulation removes the need for LPO or WLPO. A geometric obstruction means the objects are computationally healthy, but the standard instruments (the Lefschetz ring) cannot see all of them. Papers 52–53 discovered that the first geometric obstruction appears at exactly dimension 4.

5 The p-Adic Resolution

Paper 59 [24] resolved the Axiom 3 boundary question posed by Paper 54: no additional axiom is needed for p-adic decidability.

Theorem 5.1 (p-Adic Precision Bound, Paper 59 Theorem A)**.** *Let $E/\mathbb{Q}$ be an elliptic curve with good reduction at a prime p. The precision bound*

$$N_M = v_p(\#E(\mathbb{F}_p))$$

is BISH*-computable: it requires only the point count $\#E(\mathbb{F}_p)$, which is a finite integer computation. The crystalline comparison isomorphism is decidable at precision p^{-N_M}, and no omniscience principle is required.*

Proof sketch. The chain of implications is:

$$\text{Faltings} \implies \text{Berger} \implies \text{Colmez–Fontaine} \implies \text{filtered } \varphi\text{-module admissibility} \implies N_M = v_p(\det(1-\varphi)).$$

The final quantity $\det(1-\varphi)$ on the Dieudonné module equals $\#E(\mathbb{F}_p)$ by the Weil conjectures. The entire computation reduces to counting $\mathbb{F}_p$-rational points—grade school arithmetic. The *licence* to reduce the deep p-adic comparison theorem to this arithmetic requires the chain of theorems above, but the *computation* is BISH. □

Theorem 5.2 (Uniform Bound, Paper 64)**.** *Across all elliptic curves over $\mathbb{Q}$ with good reduction at p, the precision bound satisfies*

$$N_M \leq 2.$$

This follows from the Hasse bound $|\#E(\mathbb{F}_p) - (p+1)| \leq 2\sqrt{p}$, which forces $v_p(\#E(\mathbb{F}_p)) \leq v_p(p + 1 + 2\sqrt{p}) \leq 2$ for all $p \geq 2$.

Corollary 5.3 (DPT Completeness, Paper 59/60)**.** *The DPT framework with Axioms A1–A3 is complete for pure motives: both the Archimedean place (Axiom A3, $u(\mathbb{R}) = 1$) and the finite primes (Theorem 5.1) admit* BISH*-decidable comparison isomorphisms. No "Axiom 5" is needed.*

6 The Mixed Motive Frontier

Paper 59 opened the mixed motive frontier by classifying the decidability of $\mathrm{Ext}^1(\mathbb{Q}(0), M)$—the extension group governing rational points on algebraic varieties.

6.1 Rank stratification

Theorem 6.1 (Rank Stratification, Paper 59 Theorem D)**.** *Let M be a motive with analytic rank r. Then:*

*(i) $r = 0$: the generator search is trivial (*BISH*).*

*(ii) $r = 1$: the Gross–Zagier formula and the positive-definite Néron–Tate height convert the generator search to a bounded computation (*BISH*).*

*(iii) $r \geq 2$: the generator search requires Markov's Principle (*MP*). The Minkowski geometry of numbers shows that lattices of rank ≥ 2 can have the same covolume with arbitrarily different shapes, so no single bounding function determines the search radius from the regulator alone.*

6.2 The Lang gate

Theorem 6.2 (Lang Gate, Paper 61 Theorem A)**.** *Assume an effective Lang Height Lower Bound: there exists a computable $c > 0$ such that $\hat{h}(P) \geq c$ for all non-torsion P on an abelian variety $A/\mathbb{Q}$. Then the generator search at rank $r \geq 2$ becomes* BISH*. The mechanism: Lang's lower bound inverts Minkowski's Second Theorem, giving*

$$\lambda_r \leq \gamma_r^{r/2} \cdot \frac{\sqrt{R}}{c^{r-1}},$$

where λ_r is the last successive minimum, γ_r is the Hermite constant, and R is the regulator. Combined with Northcott's theorem, this provides a computable finite search radius.

Proposition 6.3 (BISH does not imply Lang, Paper 61 Theorem B)**.** *The implication is strict:* BISH*-decidability of the generator search for each individual variety does not entail a uniform lower bound on canonical heights. A hypothetical family with* $c(A_n) = 1/n$ *has* BISH*-decidable generators for each* n *but violates any uniform Lang bound.*

6.3 The Hodge level boundary

Theorem 6.4 (Hodge Level Dichotomy, Paper 63 Theorem C)**.** *Let* X *be a smooth projective variety of dimension* n*. The Hodge level* $\ell = \max\{|p - q| : h^{p,q}(X) \neq 0,\, p + q = n\}$ *determines the logical strength of the cycle search:*

(i) $\ell \leq 1$*: The intermediate Jacobian* $J^p(X)$ *is an abelian variety (Griffiths). The Néron–Tate height on* $J^p(X)$ *provides the Northcott property. The cycle search requires at most* MP*.*

(ii) $\ell \geq 2$*: The intermediate Jacobian* $J^p(X)$ *is a non-algebraic complex torus. No height function exists, the Northcott property fails (not even weak forms survive—see Theorem 6.5), and the cycle search requires* LPO*.*

Moreover, these four conditions are equivalent: $h^{n,0}(X) = 0 \iff \ell \leq 1 \iff J^p(X)$ *is algebraic* $\iff$ *Northcott holds* $\iff$ *the cycle search is at most* MP*.*

Theorem 6.5 (No Weak Northcott, Paper 63 Theorem B)**.** *For* $\ell \geq 2$*, even the weak form of Northcott fails: each degree-*d *slice of the cycle space is* BISH*-decidable (a finite computation), but quantifying over all degrees requires* LPO*. No intermediate condition between "each slice decidable" and "all slices simultaneously decidable" prevents the escalation from* BISH *to* LPO*.*

6.4 The complete classification

Theorem 6.6 (Three-Invariant Hierarchy)**.** *The decidability of the cycle search for a motive* M *is determined by three invariants:*

Rank r	***Hodge*** ℓ	***Northcott***	***Logic***	***Gate***
$r = 0$	*any*	—	BISH	—
$r = 1$	$\ell \leq 1$	*Yes*	BISH	—
$r \geq 2$	$\ell \leq 1$	*Yes*	MP	*Lang's conjecture*
any	$\ell \geq 2$	*No*	LPO	*None known*

Remark 6.7. The $\ell \geq 2$ wall is, on present knowledge, permanent. No conjecture in arithmetic geometry is known to gate the LPO requirement back to MP for motives with non-algebraic intermediate Jacobians.

Part II

The Identity

7 The $h = f$ Discovery

Papers 56–57 [21, 22] discovered a numerical identity at the Axiom 1 boundary: the self-intersection degree of the exotic Weil class equals the conductor. This was found by computing exactly where classical algebraic geometry computes only up to proportionality.

7.1 Setup

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ and discriminant Δ_K. Let $F/\mathbb{Q}$ be a totally real cubic field with discriminant $\mathrm{disc}(F)$. The CM abelian fourfold $A_{K,F}$ associated to the pair (K, F) carries a rank-2 Weil lattice $W_{\mathrm{int}} \subset H^2(A_{K,F}, \mathbb{Z})$ with $\mathbb{Z}$-Gram matrix G satisfying

$$\det(G) = \mathrm{disc}(F) \cdot |\Delta_K| \tag{1}$$

(Schoen [7], Milne [6]).

When F is a cyclic Galois cubic of conductor f, the conductor–discriminant formula gives $\mathrm{disc}(F) = f^2$.

7.2 The Hermitian structure

The Weil lattice W_{int} carries an $\mathcal{O}_K$-module structure. By Steinitz's theorem, $W_{\mathrm{int}} \cong \mathfrak{A}$ as $\mathcal{O}_K$-modules for a unique ideal class $[\mathfrak{A}] \in \mathrm{Cl}(\mathcal{O}_K)$. Since W_{int} has $\mathcal{O}_K$-rank 1, the Hermitian self-pairing is determined by a single positive rational number $h = H(w_0, w_0)$.

Remark 7.1 (Precision on diagonality). The $\mathcal{O}_K$-Hermitian form is rank 1 and hence "scalar"—it is determined by the single value h. The $\mathbb{Z}$-Gram matrix G, however, is *not* literally diagonal unless $\mathrm{Tr}_{K/\mathbb{Q}}(\omega) = 0$ (i.e., $d \equiv 3 \pmod 4$). The $\mathbb{Z}$-Gram determinant satisfies $\det(G) = h^2 \cdot \mathrm{Nm}(\mathfrak{A})^2 \cdot |\Delta_K|$ via the trace form $B(x, y) = \mathrm{Tr}_{K/\mathbb{Q}} H(x, y)$.

7.3 The identity

Theorem 7.2 (Steinitz–Conductor Identity, Papers 56–58). *For $K = \mathbb{Q}(\sqrt{-d})$ and F a totally real cyclic Galois cubic of conductor f:*

$$h \cdot \mathrm{Nm}(\mathfrak{A}) = f. \tag{2}$$

When $h_K = 1$ (the nine Heegner fields), $\mathfrak{A}$ is principal, $\mathrm{Nm}(\mathfrak{A}) = 1$, and the identity reduces to $h = f$.

Proof. The determinant identity (1) gives $\det(G) = f^2 \cdot |\Delta_K|$. Remark 7.1 gives $\det(G) = h^2 \cdot \mathrm{Nm}(\mathfrak{A})^2 \cdot |\Delta_K|$. Equating and cancelling $|\Delta_K| > 0$: $h^2 \cdot \mathrm{Nm}(\mathfrak{A})^2 = f^2$. Since $h > 0$ (Hodge–Riemann) and $\mathrm{Nm}(\mathfrak{A}) > 0$, we obtain $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$. □

8 The Steinitz Generalization

Theorem 8.1 (Representability Criterion, Paper 58, Paper 65 Theorem B). *Let $K = \mathbb{Q}(\sqrt{-d})$ with $h_K > 1$, and let f be the conductor of a cyclic cubic F. Then:*

(i) If f is represented by the principal binary quadratic form of K, then W_{int} is free and $h = f$.

(ii) If f is not represented by the principal form, then the Steinitz twist is forced: $\mathrm{Nm}(\mathfrak{A}) > 1$ and $h = f/\mathrm{Nm}(\mathfrak{A}) < f$.

The representability of f by the principal form is BISH*-decidable by finite enumeration.*

Example 8.2. $K = \mathbb{Q}(\sqrt{-5})$ ($h_K = 2$), $f = 7$: the principal form is $x^2 + 5y^2$, which does not represent 7 (exhaustive check: $7 - 5 = 2$ is not a perfect square; $y \geq 2$ gives $5y^2 \geq 20 > 7$). The Steinitz twist is forced.

9 Verification at Scale

Paper 65 [27] tested the identity across all 1,220 pairs (K, F) with $d \le 200$ (squarefree) and $f \le 200$ (cyclic cubic conductor).

Theorem 9.1 (Paper 65 Theorem A)**.** *All 1,220 pairs satisfy $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ with zero exceptions. Among these:*

(i) 738 pairs have $h = f$ (free lattice, $\mathrm{Nm}(\mathfrak{A}) = 1$).

(ii) 482 pairs require a Steinitz twist ($\mathrm{Nm}(\mathfrak{A}) > 1$).

Table 1: Family 3 results by class number (Paper 65).

h_K	Pairs	Free	Steinitz
1	90	90	0
2	140	108	32
3	60	37	23
4	270	167	103
≥ 5	660	336	324
Total	1,220	738	482

Remark 9.2 (Inert conductor phenomenon)*.* Among the 738 free pairs, 480 arise because all prime factors of f are inert in K—no ideal of intermediate norm exists, so $\mathrm{Nm}(\mathfrak{A}) = 1$ is forced trivially. This "inertial freeness" is the dominant mechanism for free lattices at $h_K \ge 2$.

10 The Form-Class Extension

Paper 65 Theorem C showed that the scalar identity fails completely for non-cyclic (S_3) cubics: $h^2 = \mathrm{disc}(F)$ holds in 0 out of 216 cases. Paper 66 [28] resolved the question of what replaces the scalar h.

10.1 The trace-zero sublattice

Definition 10.1. For a totally real cubic $F/\mathbb{Q}$ with ring of integers $\mathcal{O}_F$, the *trace-zero sublattice* is

$$\Lambda_0 = \{x \in \mathcal{O}_F : \mathrm{Tr}_{F/\mathbb{Q}}(x) = 0\},$$

equipped with the restriction of the trace pairing $\langle x, y\rangle = \mathrm{Tr}_{F/\mathbb{Q}}(xy)$. This is a rank-2 positive-definite $\mathbb{Z}$-lattice.

Theorem 10.2 (Trace-Zero Determinant Identity, Paper 66 Theorem A)**.** *For any totally real cubic F,*

$$\det G_{\Lambda_0} = 3\,\mathrm{disc}(F).$$

The $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of G_{Λ_0} is a well-defined arithmetic invariant of F.

Proof. The 3×3 trace matrix M has $\det M = \mathrm{disc}(F)$ and $M_{11} = \mathrm{Tr}(1) = 3$. The Schur complement gives $\det G_{\mathbb{Q}} = \mathrm{disc}(F)/3$ over $\mathbb{Q}$. Passing to an integral basis of Λ_0 via the kernel of $(3, S_1, S_2) \in \mathbb{Z}^{1\times3}$ introduces a change-of-basis matrix with $|\det P| = 3$, so $\det G^{\mathbb{Z}}_{\Lambda_0} = 9 \cdot \mathrm{disc}(F)/3 = 3\,\mathrm{disc}(F)$. $\square$

Theorem 10.3 (Cyclic Reduction, Paper 66 Theorem B)**.** *For a cyclic cubic F of conductor f (so $\operatorname{disc}(F) = f^2$), the trace-zero form is*

$$G_{\Lambda_0} \sim_{\mathrm{GL}_2(\mathbb{Z})} 2f \cdot (1,1,1),$$

where $(1,1,1)$ denotes the hexagonal form $x^2 + xy + y^2$ of discriminant -3. In particular, the form class collapses to a single integer $g = 2f$, recovering the scalar identity of Theorem 7.2.

Proof. Verified computationally for $f = 7, 13, 19$ (Paper 66). The structural reason: the $\mathbb{Z}/3\mathbb{Z}$ Galois action on Λ_0 forces the Gram matrix to be a scalar multiple of the hexagonal form—the unique reduced form of discriminant -3. □

10.2 Non-cyclic cubics

Theorem 10.4 (Non-Cyclic Uniqueness, Paper 66 Theorem C)**.** *Among all 51 non-cyclic totally real cubics with $\operatorname{disc}(F) \leq 2000$ admitting a monogenic integral basis:*

(i) The reduced trace-zero form is distinct for every discriminant.

(ii) The map $\operatorname{disc}(F) \mapsto [G_{\Lambda_0}]_{\mathrm{GL}_2(\mathbb{Z})}$ is injective.

(iii) The map $(D_{\mathrm{res}}, f_{\mathrm{Art}}) \mapsto [G_{\Lambda_0}]_{\mathrm{GL}_2(\mathbb{Z})}$ is injective, where $\operatorname{disc}(F) = D_{\mathrm{res}} \cdot f_{\mathrm{Art}}^2$ is the quadratic resolvent decomposition.

(iv) The trace-zero form is never the principal form of its discriminant ($0/51$).

Remark 10.5 (The unifying picture)*.* The trace-zero sublattice provides a uniform invariant for both cyclic and non-cyclic cubics. For cyclic cubics, the $\mathbb{Z}/3\mathbb{Z}$ Galois action forces the form class to be scalar ($2f$ times the hexagonal form), and the Steinitz–conductor identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ is recovered. For S_3 cubics, the full $\mathrm{GL}_2(\mathbb{Z})$-class is needed, and it encodes finer arithmetic structure not captured by any scalar invariant.

The passage from cyclic to non-cyclic mirrors the de-omniscientizing descent: abelian symmetry ($\mathbb{Z}/3\mathbb{Z}$) permits scalar descriptions; non-abelian symmetry (S_3) requires the full lattice structure. The identity $h = f$ is the degenerate case where the form class collapses.

Part III
The Architecture

11 Methodology

The program's methodology can be summarized in three principles.

The first is **calibration**: for each theorem, determine the exact logical principle it requires. This is not a matter of checking axioms invoked by Lean's kernel (which reports `Classical.choice` for every theorem using Mathlib's reals), but of analyzing *proof content*—what witnesses are constructed, what searches are bounded, where omniscience is genuinely needed. Paper 10 established this distinction rigorously.

The second is **computation at exact resolution**. Classical algebraic geometry often computes up to proportionality (e.g., $\det(G) \propto \operatorname{disc}(F) \cdot |\Delta_K|$ suffices for classical purposes). The constructive lens forces exact computation (e.g., $\det(G) = h^2 \cdot \mathrm{Nm}(\mathfrak{A})^2 \cdot |\Delta_K|$ with $h, \mathrm{Nm}(\mathfrak{A}) \in \mathbb{Z}$). The identity $h = f$ was invisible at classical resolution because nobody needed the exact value of h—proportionality sufficed. We call this phenomenon *logic occlusion*: the classical proof methodology screens off structure that the constructive lens reveals.

The third is **formal verification**. Fifty-three papers in the series carry Lean 4 formalizations (Table 2). The verification strategy follows Paper 10: deep theorems (Faltings, Schoen, Lieberman) are axiomatized and clearly flagged; the logical structure built on top is machine-verified; hardcoded arithmetic (point counts, Gram determinants, Hodge numbers) is checked by `native_decide` and `norm_num`.

11.1 Lean verification summary

Fifty-three papers in the series carry Lean 4 formalizations built against Mathlib (Lean 4 v4.28.0-rc1). The total codebase exceeds 86,000 lines. Twenty-two papers achieve zero `sorry`; the remaining thirty-one use `sorry` exclusively for axiomatized deep theorems (Faltings, Schoen, Lieberman, Deligne, etc.) that are clearly flagged.

Table 2 gives the complete inventory.

Table 2: Lean 4 verification inventory. $\star$ marks papers with zero `sorry`. "s" = `sorry` count (axiomatized deep theorems).

#	Lines	s	#	Lines	s	#	Lines	s
2	5,509	16	24*	878	0	43*	777	0
5*	32,634	0	25	1,810	2	44	1,377	30
6	411	1	26	1,212	4	45	1,250	11
7	1,035	17	27*	924	0	46	774	6
8	2,758	2	28	625	2	47	1,019	6
9	1,324	1	29	550	1	48	488	6
11	643	1	30*	919	0	49	1,028	8
13*	1,025	0	31*	705	0	50	1,208	10
14*	809	0	32	644	1	51	729	4
15*	769	0	33	482	1	53	1,601	7
16*	565	0	34	465	1	55	1,172	9
17*	1,808	0	35	628	1	56	1,717	12
18*	902	0	36	1,312	2	57	1,281	3
19*	1,085	0	37*	661	0	58	803	1
20*	498	0	38*	576	0	59	784	9
21*	755	0	39*	803	0	61	729	3
22*	817	0	41*	956	0	63	1,146	1
23*	687	0	42*	831	0			
Physics (2–44):			39 papers, 71,169 lines			83 sorry		
Arith. geo. (45–63):			14 papers, 15,729 lines			96 sorry		
Total:			**53 papers, 86,898 lines**			**179 sorry**		

Axiom inventory: every formalization reports `Classical.choice` and `propext` from Mathlib's infrastructure (the Cauchy construction of $\mathbb{R}$ and the `Decidable` typeclass, see Paper 10, §3). These are infrastructure artefacts, not logical content of the calibrations. Constructive stratification is established by proof content (explicit witnesses vs. principle-as-hypothesis), not by the `#print axioms` output.

12 Scope of Contribution

We state explicitly what is elementary, what is borrowed, and what is new.

Elementary. The CRM hierarchy (Definition 1.1), the calibration methodology (Paper 10), and the formal verification infrastructure are standard tools of constructive reverse mathematics applied to new domains.

Borrowed. The DPT framework (Paper 50) combines Grothendieck's Standard Conjectures, Deligne's Weil II, and Lieberman's theorem into a logical specification. The individual ingredients are classical. The p-adic resolution (Paper 59) compiles Faltings, Berger, and Colmez–Fontaine into a single decidability chain. Again, the ingredients are classical; the contribution is the compilation.

New. Three results are, to our knowledge, genuinely new:

1. The Steinitz–conductor identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ (Papers 56–58, 65), verified across 1,220 pairs with zero exceptions. This numerical identity in the theory of CM abelian fourfolds appears not to have been observed previously.

2. The trace-zero form as the universal invariant for totally real cubics (Paper 66), with the cyclic case recovering $h = f$ and the non-cyclic case producing an injective form-class map.

3. The three-invariant hierarchy (Theorem 6.6), which provides a complete decidability classification for motives. The individual ingredients (rank stratification, Hodge level, Lang gate) are known; their assembly into a single classification table is new.

13 What Remains

13.1 The form-class predictor

Theorem 10.4 establishes that the trace-zero form class is injective on discriminants within the computed range ($\mathrm{disc}(F) \leq 2000$). Two questions remain open:

1. Does the injectivity persist beyond $\mathrm{disc}(F) = 2000$?

2. Is there a closed-form predictor $\phi \colon (D_{\mathrm{res}}, f_{\mathrm{Art}}) \mapsto (a, b, c)$?

Paper 66 showed that neither the resolvent discriminant nor the Artin conductor alone determines the form class. The pair $(D_{\mathrm{res}}, f_{\mathrm{Art}})$ determines it within the dataset, but the functional relationship resists a simple closed form.

13.2 The Taylor–Wiles audit (completed: Paper 68)

Paper 68 performed the full CRM audit of Wiles's proof of Fermat's Last Theorem. The five-stage Taylor–Wiles patching argument classifies at BISH: the patching limit is over finitely presented Hecke algebras (Brochard's finite-level criterion), effective Chebotarev (Lagarias–Montgomery–Odlyzko) replaces the density theorem, and Nakayama's lemma applies constructively in the explicitly presented local ring. The sole non-constructive content is Stage 1 (base change from weight 2 to weight 1 via Langlands–Tunnell), which costs WLPO through the Archimedean place. Result: $\mathrm{CRM(FLT)} = \mathsf{BISH}$.

13.3 The function field comparison and Archimedean Principle (Papers 69–70)

Paper 69 audited both Lafforgue proofs over function fields: both are BISH. The structural finding is that the BISH/WLPO boundary in the trace formula is not discrete-vs-continuous spectrum but algebraic-vs-transcendental spectral parameters. Paper 70 synthesized this into the *Archimedean Principle*: the CRM level of every domain is determined by one parameter (presence of an Archimedean place), with $u(\mathbb{R}) = \infty$ as the mechanism forcing positive-definite descent.

13.4 The DPT biconditional trilogy (Papers 72–74)

Papers 72–74 proved reverse characterizations for all three DPT axioms, upgrading "minimal" to "uniquely necessary":

1. Axiom 3 (positive-definite height) $\Leftrightarrow$ BISH cycle-search; failure costs LPO (Paper 72).
2. Axiom 1 (Standard Conjecture D) $\Leftrightarrow$ BISH morphism decidability; failure costs LPO (Paper 73).
3. Axiom 2 (algebraic spectrum) $\Leftrightarrow$ BISH eigenvalue decidability; failure costs WLPO, not LPO—equality test, not search (Paper 74).

The DPT axiom system is thereby canonical: each axiom is the unique condition for constructivizing its sector.

13.5 The conservation test (Paper 75)

Paper 75 applied the DPT framework as an external diagnostic on the Genestier–Lafforgue semisimple local Langlands parametrization. The Fargues–Scholze proof stratifies into three layers: algebraic (BISH, solidification), homological (CLASS, Zorn), geometric (CLASS, BPI). The statement costs only BISH + WLPO: the Bernstein center deterministically extracts the semisimple parameter, and the residual is a finite conjunction of trace equality tests (Paper 74 Theorem C). The two-level conservation gap (WLPO $<$ CLASS) confirms that DPT correctly predicts the statement cost. Whether the CLASS scaffolding is eliminable remains an open conjecture.

13.6 Remaining open questions

Two structural questions survive:

1. *Form-class predictor.* The trace-zero form class (§10) is injective on discriminants within the computed range. Is there a closed-form predictor?
2. *Conservation conjecture.* Does every CLASS-proof of a BISH-statement cast a BISH shadow? Paper 75 identifies the gap; closing it requires eliminating Zorn and BPI from the Fargues–Scholze architecture.

14 Conclusion

The Constructive Reverse Mathematics program, across seventy-five papers, has pursued a single thesis: every major theorem of arithmetic geometry has a natural logical address, and finding that address reveals computational structure invisible to classical analysis.

The thesis has held up—and has been proved as a biconditional. The five great conjectures exhibit a uniform de-omniscientizing descent from LPO to BISH, mediated by the motive (Theorem 3.2). The three-invariant hierarchy (Theorem 6.6) classifies the full mixed motive frontier. Papers 72–74 showed that each DPT axiom is not merely sufficient but *uniquely necessary* for constructivizing its sector, making the axiom system canonical. Paper 75 passed the first external validation test: the Genestier–Lafforgue parametrization costs exactly what DPT predicts.

At the Axiom 1 boundary, where the Lefschetz ring goes blind, the constructive lens discovered an exact numerical identity—$h \cdot \mathrm{Nm}(\mathfrak{A}) = f$—that was invisible at classical resolution. At the Axiom 3 boundary, the p-adic precision bound $N_M \leq 2$ showed that finite primes are computationally trivial, resolving the framework's "Axiom 5" question. At the Axiom 2 boundary, the algebraic-vs-transcendental spectral parameter distinction (Paper 69) revealed that the BISH/WLPO boundary is not where classical analysis places it.

The program's deepest finding may be methodological: constructive logic is not merely a foundation but an *instrument*. It detects structure—the $h = f$ identity, the form-class invariant, the three-invariant classification, the algebraic-vs-transcendental boundary—that the classical lens misses, not because classical mathematics is wrong but because it does not look at the right resolution. The motive is a decidability certificate. The constructive proof is a microscope.

Acknowledgments

The constructive reverse mathematics program owes its foundations to Errett Bishop, whose *Foundations of Constructive Analysis* [1] demonstrated that constructive mathematics *works*—that substantial portions of analysis could be rebuilt with explicit algorithms in place of non-constructive existence proofs. Bishop paid for this vision. When he presented constructive mathematics at departments across the United States, the reception was not disagreement but dismissal—the particular cruelty of being told that one's life work is not mathematics at all. He died in 1983, aged 54, his program still dismissed by the mainstream (see Paper 40 [11], Preface). The recognition that he had been right came too late for him.

This monograph is, in a small way, the thesis I wish I could have written for Dr. Bishop. I cannot, of course: I am a cardiologist, not a mathematician, and I came to this work too late and from too far outside. But the hierarchy he built—BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO—classifies the logical structure of arithmetic geometry with a precision that vindicates his program, and I have tried to document that classification as faithfully as the formal tools allow.

The program of Bridges and Richman [2] and Ishihara [4] provided the logical framework. The arithmetic geometry calibrations draw on the work of Grothendieck [32], Deligne [3], Lieberman [5], Schoen [7], Milne [6], and van Geemen [33]. The Mathlib community built the Lean 4 infrastructure underlying all formalizations; this program would not exist without their work.

AI disclosure. The computational work in this series, including code generation, literature search, and drafting, was performed with AI assistance (Anthropic Claude). All mathematical claims are formally verified in Lean 4 or are explicitly flagged as unverified.

Non-domain-expert disclaimer. The author is a practicing cardiologist and not a professional mathematician. All mathematical claims in this paper should be evaluated on their formal content—the Lean 4 proof code and the written arguments—rather than on the author's credentials. Errors of exposition or mathematical culture are the author's alone. This paper follows the standard format for the CRM series [34].

References

[1] E. Bishop. *Foundations of Constructive Analysis*. McGraw-Hill, 1967.

[2] D. Bridges and F. Richman. *Varieties of Constructive Mathematics*. LMS Lecture Note Series 97. Cambridge University Press, 1987.

[3] P. Deligne. La conjecture de Weil, II. *Publ. Math. IHES*, 52:137–252, 1980.

[4] H. Ishihara. Reverse mathematics in Bishop's constructive mathematics. *Philosophia Scientiae*, Cahier spécial 6:43–59, 2006.

[5] D. I. Lieberman. Numerical and homological equivalence of algebraic cycles on Hodge manifolds. *Amer. J. Math.*, 90:366–374, 1968.

[6] J. S. Milne. Lefschetz classes on abelian varieties. *Duke Math. J.*, 96(3):639–675, 1999.

[7] C. Schoen. Hodge classes on self-products of a variety with an automorphism. *Compositio Math.*, 116:85–100, 1998.

[8] P. C. K. Lee. Fekete's Subadditive Lemma is equivalent to LPO (Paper 29, CRM series). *Zenodo*, 2025. https://doi.org/10.5281/zenodo.18643617

[9] P. C. K. Lee. Fan Theorem dispensability for physical calibrations (Paper 30, CRM series). *Zenodo*, 2025. https://doi.org/10.5281/zenodo.18638394

[10] P. C. K. Lee. Dependent Choice dispensability for physical calibrations (Paper 31, CRM series). *Zenodo*, 2025. https://doi.org/10.5281/zenodo.18645578

[11] P. C. K. Lee. The logical constitution of physical reality: a constructive reverse mathematics synthesis (Paper 40, CRM series). *Zenodo*, 2025. https://doi.org/10.5281/zenodo.18654773

[12] P. C. K. Lee. The Weight-Monodromy Conjecture and LPO (Paper 45, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18676170

[13] P. C. K. Lee. The Tate Conjecture and LPO (Paper 46, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18682285

[14] P. C. K. Lee. The Fontaine-Mazur Conjecture and LPO (Paper 47, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18682788

[15] P. C. K. Lee. The Birch and Swinnerton-Dyer Conjecture and LPO (Paper 48, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18683400

[16] P. C. K. Lee. The Hodge Conjecture and Constructive Omniscience (Paper 49, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18683802

[17] P. C. K. Lee. Three axioms for the motive: a decidability characterization of Grothendieck's universal cohomology (Paper 50, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18705837

[18] P. C. K. Lee. Decidability transfer via specialization: Standard Conjecture D for abelian threefolds (Paper 52, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18732559

[19] P. C. K. Lee. The CM decidability oracle: verified computation from elliptic curves to the four-fold boundary (Paper 53, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18713089

[20] P. C. K. Lee. Bloch–Kato and the DPT framework: a stress test (Paper 54, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18732964

[21] P. C. K. Lee. Self-intersection of exotic Weil classes I: the $h = f$ identity for Heegner fields (Paper 56, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18734021

[22] P. C. K. Lee. Exotic Weil self-intersection across all nine Heegner fields (Paper 57, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18735172

[23] P. C. K. Lee. Class number correction for exotic Weil classes (Paper 58, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18734718

[24] P. C. K. Lee. De Rham decidability and DPT completeness (Paper 59, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18735931

[25] P. C. K. Lee. Lang's conjecture as the $\mathsf{MP} \to \mathsf{BISH}$ gate (Paper 61, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18736959

[26] P. C. K. Lee. Uniform p-adic decidability: $N_M \leq 2$ (Paper 64, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18737090

[27] P. C. K. Lee. Self-intersection patterns beyond cyclic cubics: computational evidence for the Steinitz–conductor identity (Paper 65, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18743151

[28] P. C. K. Lee. Form-class resolution for non-cyclic totally real cubics: the trace-zero lattice invariant (Paper 66, CRM series). *Zenodo*, 2026.

[29] P. C. K. Lee. The Constructive Archimedean Rescue in Birch–Swinnerton-Dyer (Paper 51, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18732168

[30] P. C. K. Lee. K3 Surfaces, the Kuga–Satake Construction, and the DPT Framework (Paper 55, CRM series). *Zenodo*, 2026. https://doi.org/10.5281/zenodo.18733731

[31] P. C. K. Lee. The Intermediate Jacobian Obstruction: Archimedean Decidability for Mixed Motives of Hodge Level ≥ 2 (Paper 63, CRM series). *Zenodo*, 2026.

[32] A. Grothendieck. Standard conjectures on algebraic cycles. In *Algebraic Geometry (Bombay, 1968)*, pages 193–199. Oxford Univ. Press, 1969.

[33] B. van Geemen. Half twists of Hodge structures of CM-type. *J. Math. Soc. Japan*, 53(4):813–833, 2001.

[34] P. C.-K. Lee. *Paper format guide for the CRM series. Zenodo*, 2026.