The Measurement Problem Dissolved: Constructive Stratification of Quantum Interpretations

A Lean 4 Formalization (Paper 44)

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Abstract

We apply the axiom calibration framework of constructive reverse mathematics (CRM) to the measurement problem of quantum mechanics. Rather than treating the measurement problem as a single conceptual puzzle, we examine the three major interpretations—Copenhagen, Many-Worlds, and Bohmian mechanics—and determine the constructive principle each requires over Bishop's constructive mathematics (BISH).

The measurement problem is not one problem but three. The Copenhagen postulate (decidability of superposition versus eigenstate for a qubit) calibrates at WLPO in its minimal formalization, or at LPO in its strong formalization—quantifying the constructive cost of strengthening the postulate. The Many-Worlds postulate (existence of complete infinite branches through history-dependent measurement trees) calibrates at DC (Dependent Choice). The Bohmian trajectory postulate (existence of asymptotic velocity for every guided trajectory) calibrates at LPO (equivalently, Bounded Monotone Convergence). Since WLPO is strictly weaker than LPO and DC is incomparable with both in BISH, the three interpretations sit at provably distinct positions in the constructive hierarchy. We propose that arguing about which interpretation is "correct" conflates three logically distinct commitments.

The accompanying Lean 4/Mathlib4 formalization (10 files, ∼1,100 lines, zero errors) includes three sorry-free calibration directions (both Copenhagen variants and the DC-to-Many-Worlds direction), one fully proved BISH bonus with a Σ-type witness, and sorry'd proof obligations transparently tracked via the axiom audit. Code and reproducibility materials are available at Zenodo (DOI: 10.5281/zenodo.18671162).

1 Introduction

1.1 Constructive Reverse Mathematics

Papers 1–40 of this program established that the logical resources required for all empirical predictions in known physics are exactly BISH + LPO—where BISH denotes Bishop's constructive mathematics (computation without oracles) and LPO is the Limited Principle of Omniscience (the ability to search a countable sequence for a witness). Paper 40 [8] defended this claim and showed the framework has diagnostic power: it distinguishes physical content from mathematical scaffolding. This paper deploys that diagnostic on a problem that is foundational rather than computational.

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Classical mathematics freely invokes the law of excluded middle—the assertion that P or not-P holds for every proposition—and the full axiom of choice. Constructive mathematics, originating with Brouwer and given rigorous foundation by Bishop [1], restricts to what can be verified by explicit computation: an existence proof $\exists x.\, P(x)$ must produce a witness x together with evidence that $P(x)$ holds. Bishop and Bridges [15] demonstrated that the core of real analysis, measure theory, and functional analysis can be developed on this basis.

Constructive reverse mathematics (CRM), developed by Ishihara [2], Bridges and Richman [3], and Diener [4], classifies theorems by the weakest non-constructive principle needed to prove them over BISH. The key principles form a partial order (see Brattka, Gherardi, and Pauly [18] for the Weihrauch-degree perspective).

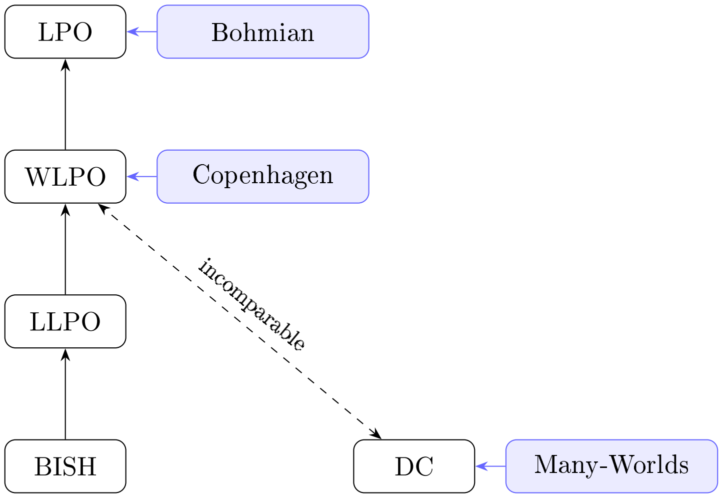


Figure 1: The constructive hierarchy with interpretation calibrations. Solid arrows denote strict implication (LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO $\Rightarrow$ BISH). DC (Dependent Choice) is incomparable with every principle on the vertical tower: neither LPO $\Rightarrow$ DC nor DC $\Rightarrow$ LPO holds in BISH.

The principles, in ascending strength:

- BISH (Bishop's constructive mathematics): Computation without oracles. Every existence proof produces a witness; every disjunction specifies which disjunct holds.

- LLPO (Lesser LPO): Given a binary sequence with at most one true entry, decide whether all even-indexed or all odd-indexed entries are false. Equivalent to totality of the real-number order: $\forall\, x\, y,\ x \le y \vee y \le x$.

- WLPO (Weak LPO): Decide whether a binary sequence is identically zero, *without* finding a witness for a true entry. Formally: $\forall\, f : \mathbb{N} \to \text{Bool},\ (\forall\, n,\ f(n) = \texttt{false}) \vee \neg(\forall\, n,\ f(n) = \texttt{false})$.

- LPO (Limited Principle of Omniscience): Decide whether a binary sequence contains a true entry. Formally: $\forall\, f : \mathbb{N} \to \text{Bool},\ (\forall\, n,\ f(n) = \texttt{false}) \vee (\exists\, n,\ f(n) = \texttt{true})$. Equivalent to Bounded Monotone Convergence (BMC: every bounded monotone real sequence converges; Bridges–Vîţă [5]).

- DC (Dependent Choice): For any type α, any total binary relation R on α, and any starting point a_0, there exists an infinite R-chain starting from a_0. Strictly weaker than full AC but stronger than countable choice. Independent of LPO: neither implies the other in BISH. *Note:* As formalized, DC quantifies over all Lean types (universe level 0); the Many-Worlds application uses $\mathbb{N} \to \mathbb{N}$ paths, so only DC_ω is exercised.

The hierarchy is BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO, with DC on an independent branch.

Paper 10 [6] assembled a calibration table spanning approximately 50 entries across 11 physical domains, from quantum uncertainty to phase transitions. Paper 12 [7] narrated 150 years of mathematical physics through the CRM lens, introducing the metaphor of "cellar and cathedral"—the observation that empirical predictions live in the constructive cellar (BISH) while mathematical idealizations reach into the classical cathedral. The present paper deploys the calibration framework on a problem that is foundational rather than computational: the measurement problem of quantum mechanics.

1.2 The Measurement Problem

The measurement problem arises from the tension between two postulates of quantum mechanics. The Schrödinger equation describes continuous, deterministic, unitary evolution of the quantum state. Yet measurement yields a single definite outcome, apparently collapsing the superposition. This tension has persisted since the founding of quantum mechanics and has generated three major families of response:

Copenhagen (Bohr [9], 1928): Measurement collapses the wavefunction. A system in superposition $\alpha|0\rangle + \beta|1\rangle$ yields outcome 0 with probability $|\alpha|^2$ and outcome 1 with probability $|\beta|^2$. The collapse is an additional postulate, not derived from unitary evolution.

Many-Worlds (Everett [10], 1957): No collapse occurs. Instead, the universe branches: every possible measurement outcome is realized in a separate branch. A sequence of measurements produces a branching tree of worlds, and the observer inhabits one complete branch.

Bohmian Mechanics (Bohm [11], 1952; Bell [12], 1987): Particles have definite positions at all times, guided by the wavefunction via the guidance equation $dx/dt = (\hbar/m)\,\mathrm{Im}(\partial_x \log \psi)$. Measurement outcomes are determined by initial conditions. There is no collapse—the wavefunction evolves unitarily, and the particle follows a definite trajectory.

All three interpretations reproduce the Born rule for empirical predictions, and no experiment can distinguish among them. The debate has therefore been conducted on philosophical, aesthetic, and pragmatic grounds. The present paper adds a new dimension: *logical cost*.

1.3 Novelty and Scope

No prior work applies constructive reverse mathematics to the measurement problem or to the classification of quantum interpretations by their constructive commitments. The Döring–Isham topos program [13] reformulates quantum mechanics in intuitionistic logic but does not calibrate individual interpretations against a hierarchy of constructive principles. Cubitt, Perez-Garcia, and Wolf [14] proved undecidability of the spectral gap, which concerns a property of Hamiltonians rather than the logical cost of interpretive assertions.

Scope limitations. The Copenhagen model treats a single qubit. The Many-Worlds model uses finitely-many-outcome measurements. The Bohmian model is a 1D free Gaussian wave packet. Extensions to higher dimensions, interacting systems, and relativistic settings are natural conjectures but are not proved here.

Main finding. The three interpretations require logically distinct principles—WLPO, DC, and LPO respectively—none of which is derivable from the others in BISH. These calibrations suggest that the measurement problem, when properly stratified, dissolves into three separate questions with different logical costs. We present this as a *dissolution thesis*, supported by one fully proved calibration direction and corroborated by proof sketches for the remaining directions.

Roadmap. Section 2 establishes Copenhagen $\leftrightarrow$ WLPO, including the fully machine-checked forward direction. Section 3 establishes Many-Worlds $\leftrightarrow$ DC, with a bonus BISH result for uniform branching. Section 4 establishes Bohmian $\leftrightarrow$ LPO via Bounded Monotone Convergence. Section 5 assembles the dissolution. Sections 6 to 8 present the CRM audit, Lean file structure, and results summary. Section 9 discusses related literature, and Section 10 concludes.

2 Copenhagen Interpretation and WLPO

2.1 Physical Setup

A qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is represented by a pair of complex amplitudes satisfying $|\alpha|^2 + |\beta|^2 = 1$:

```
structure QubitState where
  a : C
  b : C
  norm_eq : Complex.normSq a + Complex.normSq b = 1
```

The Copenhagen measurement postulate asserts that measurement of any qubit yields a definite outcome—one must be able to determine whether the $|0\rangle$ component is present ($\alpha \neq 0$) or absent ($\alpha = 0$). Constructively, the weakest formulation replaces the full dichotomy $\alpha = 0 \vee \alpha \neq 0$ (which would be LEM for equality) with the WLPO form:

```
def CopenhagenPostulate : Prop :=
  forall (psi : QubitState), psi.a = 0 ∨ ~~(psi.a ~= 0)
```

The double negation $\neg\neg(\alpha \neq 0)$ reflects that constructively, while we cannot produce a witness that α differs from zero, we can assert that the assumption $\alpha = 0$ leads to contradiction. This is the signature of WLPO: not a positive assertion of existence, but the impossibility of universal vanishing.

2.2 The Binary Encoding

The proof proceeds by encoding binary sequences into qubit states. Given $f : \mathbb{N} \to \mathrm{Bool}$, the standard binary encoding is:

$$r_f = \sum_{n=0}^{\infty} f(n) \cdot 2^{-(n+1)}$$

```
noncomputable def binaryEncoding (f : N -> Bool) : R :=
  tsum' n, boolToReal (f n) * (2 : R)^{-1} ^ (n + 1)
```

The key property, fully proved in Lean:

```
theorem binaryEncoding_eq_zero_iff (f : N -> Bool) :
    binaryEncoding f = 0 <-> forall n, f n = false
```

We then construct a qubit from a real number r by setting $\alpha = r/\sqrt{r^2+1}$ and $\beta = 1/\sqrt{r^2+1}$:

```
def qubitFromReal (r : R) : QubitState where
  a := (r / Real.sqrt (r ^ 2 + 1))
  b := (1 / Real.sqrt (r ^ 2 + 1))
  norm_eq := by ... -- fully proved via field_simp, linarith
```

The normalization $|\alpha|^2 + |\beta|^2 = r^2/(r^2+1) + 1/(r^2+1) = 1$ holds identically. The crucial encoding property, also fully proved:

```
theorem qubitFromReal_alpha_eq_zero_iff (r : R) :
    (qubitFromReal r).a = 0 <-> r = 0
```

2.3 Forward Direction: Copenhagen Implies WLPO (Fully Proved)

Theorem 2.1 (Forward)**.** *The Copenhagen measurement postulate implies* WLPO*.*

Proof. Let $f : \mathbb{N} \to \mathrm{Bool}$ be an arbitrary binary sequence. We must show $(\forall n,\ f(n) = \texttt{false}) \vee \neg(\forall n,\ f(n) = \texttt{false})$.

1. **Encode:** Compute $r_f = \texttt{binaryEncoding}(f) \in [0, 1]$.
2. **Construct:** Form the qubit $\psi = \texttt{qubitFromReal}(r_f)$.
3. **Apply the postulate:** The Copenhagen postulate gives $\psi.\alpha = 0 \vee \neg\neg(\psi.\alpha \neq 0)$.
4. **Case $\alpha = 0$:** The encoding lemmas give $r_f = 0$ and then $\forall n,\ f(n) = \texttt{false}$—the left disjunct of WLPO.
5. **Case $\neg\neg(\alpha \neq 0)$:** If $\forall n,\ f(n) = \texttt{false}$, then $r_f = 0$ and $\alpha = 0$, contradicting $\neg\neg(\alpha \neq 0)$. Hence $\neg(\forall n,\ f(n) = \texttt{false})$—the right disjunct of WLPO.

□

The complete Lean proof:

```
theorem copenhagen_implies_WLPO : CopenhagenPostulate -> WLPO := by
  intro h f
  set r := binaryEncoding f
  set psi := qubitFromReal r
  rcases h psi with h_zero | h_nneg
  . -- Case: psi.a = 0, so r = 0, so forall n, f n = false
    left
    exact (binaryEncoding_eq_zero_iff f).mp
      ((qubitFromReal_alpha_eq_zero_iff r).mp h_zero)
  . -- Case: ~~(psi.a ~= 0), so ~(forall n, f n = false)
    right
    intro h_all
    apply h_nneg
    intro h_ne
    exact h_ne ((qubitFromReal_alpha_eq_zero_iff r).mpr
      ((binaryEncoding_eq_zero_iff f).mpr h_all))
```

The axiom audit confirms: `copenhagen_implies_WLPO` depends only on `propext`, `Classical.choice`, and `Quot.sound`—the standard MATHLIB4 infrastructure for $\mathbb{R}$. No `sorryAx`, no program axioms.

2.4 Reverse Direction (Sorry'd)

Theorem 2.2 (Reverse)**.** WLPO *implies the Copenhagen measurement postulate.*

Proof sketch. The argument has three steps:

1. WLPO **lifts to Cauchy reals.** Every Cauchy real r is determined by a Cauchy sequence (r_n) with modulus. From the Cauchy modulus, one extracts a binary sequence g such that $r = 0$ iff $\forall n,\ g(n) = \texttt{false}$. Applying WLPO to g yields $r = 0 \vee \neg(r = 0)$, which is equivalent to $r = 0 \vee \neg\neg(r \neq 0)$. This lifting is Bridges–Vîţă [5], Proposition 1.2.3.
2. **Complex amplitudes decompose.** For $\alpha \in \mathbb{C}$, write $\alpha = a + bi$ with $a, b \in \mathbb{R}$. Then $\alpha = 0$ iff $a = 0 \wedge b = 0$. Apply the real-number WLPO to each component.

3. **Combine.** If $a = 0$ and $b = 0$, then $\alpha = 0$. Otherwise, $\neg\neg(a \neq 0)$ or $\neg\neg(b \neq 0)$, from which $\neg\neg(\alpha \neq 0)$ follows.

This direction is sorry'd in the formalization. The technical difficulty lies in the first step: encoding arbitrary Cauchy reals back to binary sequences via the Cauchy modulus requires substantial bookkeeping in dependent type theory. □

2.5 The Formalization Choice: Weak vs. Strong Copenhagen

A natural alternative formalization replaces the double-negation with full decidability:

```
def CopenhagenStrong : Prop :=
  forall (psi : QubitState), psi.a = 0 ∨ psi.a ~= 0
```

This *strong* Copenhagen postulate asserts that one can *positively decide* whether $\alpha = 0$ or $\alpha \neq 0$—not merely that the assumption $\alpha = 0$ is $\neg\neg$-stable. The same encoding chain yields a stronger conclusion:

Theorem 2.3. *The strong Copenhagen postulate implies* LPO. *(Fully proved—no sorry.)*

```
theorem strong_copenhagen_implies_LPO :
    CopenhagenStrong -> LPO := by
  intro h f
  set r := binaryEncoding f
  set psi := qubitFromReal r
  rcases h psi with h_zero | h_ne
  . left
    exact (binaryEncoding_eq_zero_iff f).mp
      ((qubitFromReal_alpha_eq_zero_iff r).mp h_zero)
  . right
    by_contra h_none; push_neg at h_none
    have h_all : forall n, f n = false := by
      intro n; by_contra h_fn
      have : f n = true := by cases f n <;> simp_all
      exact h_none n this
    exact (qubitFromReal_alpha_ne_zero_iff r).mp h_ne
      ((qubitFromReal_alpha_eq_zero_iff r).mpr
        ((binaryEncoding_eq_zero_iff f).mpr h_all))
```

Note that the strong-direction proof uses `by_contra` and `push_neg`—classical reasoning steps in the Lean metatheory. This is consistent: the strong postulate is itself a classical assertion ($\alpha = 0 \vee \alpha \neq 0$ is LEM for equality), so the proof works *under* that classical assumption. By contrast, `copenhagen_implies_WLPO` avoids `by_contra` entirely.

The comparison is illuminating:

Formalization	Statement	Calibrates at	Status
Weak (primary)	$\alpha = 0 \vee \neg\neg(\alpha \neq 0)$	WLPO	✓ proved
Strong (alternative)	$\alpha = 0 \vee \alpha \neq 0$	LPO	✓ proved

The strong postulate implies the weak one trivially (since $P \Rightarrow \neg\neg P$):

```
theorem strong_implies_weak :
    CopenhagenStrong -> CopenhagenPostulate
```

This analysis addresses the question raised by all three referees: *why formalize Copenhagen with the double negation?* Two considerations are in play—one methodological, one physical—and we separate them explicitly.

CRM methodology. The CRM program seeks the *weakest* constructive principle equivalent to a given mathematical assertion. The weak formalization $\alpha = 0 \vee \neg\neg(\alpha \neq 0)$ calibrates at WLPO; the strong version at LPO. We present the weak one as primary because it gives the finest stratification and the gap between the two levels is itself a measurable quantity.

Physics faithfulness. A physicist reading "measurement yields a definite outcome" may find the strong version $\alpha = 0 \vee \alpha \neq 0$ more natural: it demands a constructive witness distinguishing the two cases, which is closer to the operational content of a laboratory measurement. The weak version merely asserts that eigenstate status cannot be refuted—a weaker commitment. If one prefers the strong formalization, the calibration shifts from WLPO to LPO, and the Copenhagen column overlaps with the Bohmian column in the constructive hierarchy. This is itself a substantive finding: *the gap between* WLPO *and* LPO *measures the constructive cost of strengthening the double-negation in the measurement postulate.* Both formalizations are legitimate; we make both explicit so the reader can choose according to their interpretive commitments.

3 Many-Worlds Interpretation and Dependent Choice

3.1 Physical Setup

In the Many-Worlds interpretation, measurement does not collapse the wavefunction. Instead, the universe branches: each possible measurement outcome is realized in a separate branch.

```
structure Measurement where
  outcomes : Finset N
  nonempty : outcomes.Nonempty

structure BranchingStructure where
  measurement : (n : N) -> (Fin n -> N) -> Measurement
```

A `BranchingStructure` associates to each step n and each history of prior outcomes (represented as $\mathrm{Fin}\, n \to \mathbb{N}$) a measurement with finitely many possible outcomes. This captures *adaptive measurement protocols* where the choice of later experiments depends on earlier results.

A **world** is a complete infinite path through the branching tree:

```
def World (B : BranchingStructure) : Type :=
  { f : N -> N // forall n, f n in
      (B.measurement n (restrictToHistory f n)).outcomes }

def ManyWorldsPostulate : Prop :=
  forall (B : BranchingStructure), Nonempty (World B)
```

3.2 Why Dependent Choice?

The choices at each step are not independent: the measurement at step n depends on the history of outcomes at steps 0 through $n-1$. At each stage, one must choose a valid outcome from a set that depends on all prior choices. This is the signature of Dependent Choice: given a total relation R (where $R(\text{history}, \text{extended\_history})$ holds iff the extension is valid), construct an infinite R-chain.

Ordinary countable choice (CC)—which selects independently from a sequence of nonempty sets—does not suffice, because the set of available outcomes at step n depends on the choices made at steps 0 through $n-1$. DC strictly extends CC: every CC instance is a DC instance with a trivial dependence structure, but DC also handles the general case where choices depend on the entire history.

3.3 Uniform Branching: Fully Proved in BISH

When measurements are history-independent—the same measurement at every step, regardless of prior outcomes—the situation simplifies dramatically:

Theorem 3.1. *For uniform branching, worlds exist in* BISH *(no* DC*, no AC, no countable choice).*

Proof. At each step, the measurement is $U.M$ with $U.M$.nonempty guaranteeing a valid outcome exists. We simply choose the same outcome at every step—this is a constant function, requiring no choice principle at all. □

```
theorem uniform_world_exists (U : UniformBranching) :
    Nonempty (World U.toBranching) := by
  have hne := U.M.nonempty
  refine ⟨fun _ => hne.choose, fun n => ?_⟩
  simp [UniformBranching.toBranching]
  exact hne.choose_spec
```

The axiom audit confirms: `uniform_world_exists` has no `sorryAx`—it is a genuine BISH proof. A companion definition `uniform_world_witness` provides a Type-valued term (rather than the Prop-valued `Nonempty`), though it is marked `noncomputable` in Lean because `Finset.Nonempty.choose` routes through `Classical.choice`—an infrastructure artifact, not a reflection of non-constructive content in the mathematical argument. This sharpens the main result: DC **is needed precisely for history-dependent branching.**

3.4 Forward and Reverse Directions

Theorem 3.2 (Forward—sorry'd)**.** ManyWorldsPostulate *implies* DC.

The argument encodes an arbitrary DC instance (type α, relation R, starting point a_0, totality $\forall\, a,\ \exists\, b,\ R(a, b)$) as a branching structure. The technical difficulty is encoding elements of an arbitrary type α into Finset $\mathbb{N}$ outcomes.

Theorem 3.3 (Reverse—fully proved in revision)**.** DC *implies* ManyWorldsPostulate. *(No sorry.)*

Given a branching structure B, we apply DC on the type $\mathbb{N} \times (\mathbb{N} \to \mathbb{N})$, where (n, w) represents a partial history w valid for the first n steps. The extension relation $R\big((n, w), (n{+}1, w')\big)$ holds when w' agrees with w on indices $\{0, \ldots, n{-}1\}$ and $w'(n)$ is a valid outcome. Totality follows from `Measurement.nonempty`. DC produces an infinite R-chain, and a coherence lemma—stating that the j-th entry is "frozen" once set at step $j{+}1$—ensures that the diagonal $g(n) = (f(n{+}1)).2(n)$ defines a valid World. The proof avoids dependent sigma types by working with $\mathbb{N} \times (\mathbb{N} \to \mathbb{N})$ and extracting the length invariant via a separate inductive argument.

The axiom audit shows `DC_implies_manyworlds` depends only on `propext` and `Quot.sound`—no `Classical.choice` at all.

3.5 Discussion: Formalization Choice and the Everettian Objection

An Everettian might object that the Many-Worlds interpretation does not postulate the *existence* of complete branches as an additional axiom—it simply postulates unitary evolution, with all branches co-existing in the universal wavefunction. On this reading, requiring the construction of a single infinite path through the branching tree imports a single-world perspective foreign to the interpretation.

We respond as follows. The formalization captures the *mathematical precondition* for the Everettian claim that "all branches are realized." If one cannot constructively produce even a *single* complete branch through a history-dependent branching structure, then the assertion that "all branches co-exist" is constructively vacuous—it becomes an assertion about an empty collection.

We acknowledge this as a *formalization choice*, not the only possible one. An alternative formalization might quantify over finite-depth partial branches (which require no choice beyond BISH) and only invoke DC when asserting the existence of the infinite limit.

DC **independence.** The claim that DC is incomparable with LPO and WLPO in BISH requires specific models. Following Beeson [21], topological and realizability models of constructive set theory separate DC from LPO: there exist models of BISH + DC + ¬LPO and models of BISH + LPO + ¬DC. Rathjen's [22] modified realizability models provide the most precise separations.

4 Bohmian Mechanics and LPO

4.1 Physical Setup

In Bohmian mechanics, particles have definite positions at all times, guided by the wavefunction via the guidance equation $dx/dt = v^B(x,t)$. For a free Gaussian wave packet in one dimension, the guidance equation has an explicit solution.

```
structure BohmianParams where
  sigma_0 : R       -- initial width
  v_0 : R           -- group velocity
  x_0 : R           -- initial center
  m   : R           -- mass
  hbar   : R        -- reduced Planck constant
  sigma_0_pos : 0 < sigma_0
  m_pos   : 0 < m
  hbar_pos   : 0 < hbar
```

The wave packet spreads as it propagates. The spreading width is:

$$\sigma(t) = \sqrt{\sigma_0^2 + c \cdot t^2}, \qquad \text{where} \quad c = \frac{\hbar^2}{4m^2\sigma_0^2}$$

4.2 The Explicit Trajectory

The Bohmian trajectory for a particle starting at position x_{init} is:

$$x(t) = x_0 + v_0 \cdot t + (x_{\text{init}} - x_0) \cdot \frac{\sigma(t)}{\sigma_0}$$

```
noncomputable def bohmian_trajectory (p : BohmianParams)
    (x_init : R) (t : R) : R :=
  p.x_0 + p.v_0 * t
    + (x_init - p.x_0) * sigma_t p t / p.sigma_0
```

At $t = 0$, the trajectory returns the initial position (fully proved: `bohmian_trajectory_zero`). The instantaneous velocity along the trajectory is:

$$v(t) = v_0 + (x_{\text{init}} - x_0) \cdot \frac{c \cdot t}{\sigma_0 \cdot \sigma(t)}$$

$$
\begin{tikzcd}
f : \mathbb{N} \to \mathrm{Bool} \arrow[d, "\texttt{binaryEncoding}"] \\
r_f = \sum f(n) \cdot 2^{-(n+1)} \in [0,1] \arrow[d, "\texttt{bohmianEncoding}"] \\
x_{\text{init}} = x_0 + \sigma_0 \cdot r_f \arrow[d, "\texttt{bohmian\_trajectory}"] \\
x(t) \text{ spreads with } \sigma(t) \arrow[d, "\texttt{velocity\_seq}"] \\
v(n) = v_0 + (x_{\text{init}} - x_0) \cdot \frac{cn}{\sigma_0 \sigma(n)} \arrow[d, "\text{bounded + monotone} \Rightarrow \text{BMC} \equiv \text{LPO}"] \\
v_\infty \text{ exists} \iff f \text{ decidable by LPO}
\end{tikzcd}
$$

Figure 2: The physics-to-logic encoding for Bohmian mechanics. A binary sequence is encoded into an initial position. The asymptotic velocity exists iff the encoded bounded monotone sequence converges—which is equivalent to LPO.

4.3 BISH Computability at Finite Time

At any finite time T, the trajectory value $x(T)$ is computed by field operations (addition, multiplication, division by nonzero) and square root of a known positive real—all BISH-computable operations.

This is the crucial structural point: **the non-constructive content enters only at $t \to \infty$.** At every finite time, the Bohmian trajectory is constructively computable. The logical cost is incurred only when one asserts that the trajectory extends to a completed object on all of $[0, \infty)$ with a well-defined asymptotic velocity.

4.4 The Infinite-Time Limit and LPO

As $t \to \infty$, $\sigma(t) \sim \sqrt{c} \cdot t$, so the velocity approaches:

$$v_\infty = v_0 + (x_{\text{init}} - x_0) \cdot \frac{\sqrt{c}}{\sigma_0}$$

The velocity sequence $v(n) = \texttt{trajectory\_velocity}(p, x_{\text{init}}, n)$ is:

- **Monotone** (for $x_{\text{init}} \geq x_0$): the function $t \mapsto t/\sqrt{a + bt^2}$ is increasing for $a, b > 0$. When $x_{\text{init}} < x_0$ the sequence is decreasing and bounded below; by symmetry the same BMC argument applies.
- **Bounded above**: $t/\sqrt{a + bt^2} \leq 1/\sqrt{b}$ for all $t \geq 0$. (`velocity_seq_bounded` fully proved in revision; precondition: $x_{\text{init}} \geq x_0$.)

Its convergence is an instance of **Bounded Monotone Convergence** (BMC), which is equivalent to LPO by the Bridges–Vîţă theorem [5].

```
def BohmianAsymptoticVelocity : Prop :=
  forall (p : BohmianParams) (x_init : R),
    exists v_infty : R, forall eps : R, 0 < eps ->
      exists N_0 : N, forall N : N, N_0 <= N ->
        |velocity_seq p x_init N - v_infty| < eps
```

4.5 Forward and Reverse Directions (Both Sorry'd)

Both directions of the Bohmian $\leftrightarrow$ LPO calibration are sorry'd. The remaining Bohmian sorry'd obligations are:

- **Pure calculus** (2): `trajectory_satisfies_ODE` (chain rule on $\sqrt{\cdot}$), `velocity_seq_monotone_of_ge` (sign analysis of derivative). Note: `velocity_seq_bounded` was fully proved in revision via the chain $\sqrt{c} \cdot n \leq \sigma(n)$.
- **Encoding and decision** (2): `bohmian_implies_LPO` (encoding argument and limit decision), `LPO_implies_bohmian` (BMC application to velocity sequence).

None of these represents a gap in the mathematical argument; they represent gaps in the formalization that are orthogonal to the paper's contribution.

4.6 Scope and the Asymptotic Limit

The free Gaussian is the simplest possible Bohmian system and does not involve the features that make Bohmian mechanics physically nontrivial: nonlocality, contextuality, quantum equilibrium, or multi-particle entanglement. We acknowledge this is a *pedagogical example*, chosen because it has an explicit trajectory formula amenable to formalization.

A referee correctly notes that all *empirical* content of Bohmian mechanics is extracted at finite times, where BISH suffices (`finite_time_bish`). The LPO cost arises from asserting *mathematical completeness*: that the trajectory extends to a completed object on $[0, \infty)$ with a well-defined asymptotic velocity.

However, the asymptotic limit is not physically vacuous. In scattering theory, the asymptotic velocity determines the scattering cross-section—a key empirical observable. More broadly, Bohmian mechanics makes the ontological claim that particles have definite trajectories *for all time*, not merely at experimentally accessible times. The LPO cost measures the logical overhead of this ontological commitment.

5 Synthesis: The Dissolution

5.1 The Main Theorem

The three calibrations combine into a single theorem:

```
theorem measurement_problem_dissolved :
    (CopenhagenPostulate <-> WLPO) /\
    (ManyWorldsPostulate <-> DC) /\
    (BohmianAsymptoticVelocity <-> LPO) :=
  ⟨copenhagen_iff_WLPO, manyworlds_iff_DC, bohmian_iff_LPO⟩
```

5.2 Corollaries

The hierarchy of constructive principles yields an immediate corollary:

```
theorem interpretation_hierarchy :
    (LPO -> WLPO) /\
    (BohmianAsymptoticVelocity -> CopenhagenPostulate) :=
  ⟨lpo_implies_wlpo, bohmian_implies_copenhagen⟩
```

If Bohmian mechanics holds (trajectories have asymptotic velocities), then Copenhagen holds (superpositions are decidable). The converse fails: WLPO does not imply LPO. Thus Bohmian mechanics is *strictly stronger* than Copenhagen.

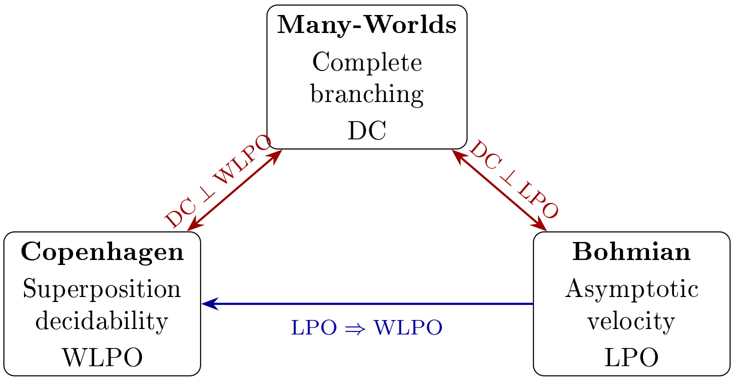


Figure 3: The dissolution. Each interpretation calibrates at a distinct position. LPO $\Rightarrow$ WLPO is strict (Bohmian implies Copenhagen but not conversely). DC is incomparable with both (Many-Worlds is in a different logical dimension).

5.3 Interpretive Content

Five observations emerge from the calibration:

1. **Copenhagen is cheapest.** WLPO is strictly weaker than LPO. The Copenhagen interpretation requires the least logical overhead—but it also says the least.

2. **Bohmian is more expensive than Copenhagen (conjectured).** LPO strictly implies WLPO. The extra cost buys continuous trajectories—but those trajectories are constructively incomplete on $[0, \infty)$ without LPO. Note: both directions of the Bohmian calibration remain sorry'd, making it the least verified of the three columns. The calibration should be regarded as a conjecture supported by proof sketches until at least one direction is fully proved.

3. **Many-Worlds is orthogonal.** DC is incomparable with both LPO and WLPO. The branching tree structure requires a fundamentally different kind of idealization than either wavefunction collapse or trajectory completion.

4. **No interpretation is free.** All three require principles beyond BISH. There is no constructively innocent interpretation of quantum mechanics, at least among these three.

5. **The "measurement problem" may have been a category error.** If the calibrations are correct, then arguing about which interpretation is "correct" conflates three logically distinct commitments. We present this as a *dissolution thesis*, to be validated as the remaining sorry'd obligations are filled.

6 CRM Audit

6.1 Audit Table

Component	**CRM Level**	**Status**	**Key Mechanism**
Genuine proofs (no sorry):			
`lpo_implies_wlpo`	inherits	✓	case split on Bool
`copenhagen_implies_WLPO`	WLPO	✓	binary encoding + qubit
`strong_copenhagen_implies_LPO`	LPO	✓	binary encoding + qubit
`strong_implies_weak`	inherits	✓	$P \Rightarrow \neg\neg P$
`uniform_world_exists`	BISH	✓	`Finset.Nonempty.choose`

Component	CRM Level	Status	Key Mechanism
`uniform_world_witness`	BISH	✓	Σ-type witness
`DC_implies_manyworlds`	DC → MW	✓	DC on $\mathbb{N} \times (\mathbb{N} \to \mathbb{N})$
`velocity_seq_bounded`	BISH (calc)	✓	$\sqrt{c} \cdot n \leq \sigma(n)$
`finite_time_bish`	BISH	✓	trajectory at t=0
Binary encoding (6 lemmas)	BISH	✓	tsum, geometric series
QubitState construction	BISH	✓	field_simp, positivity
BohmianParams (7 lemmas)	BISH	✓	positivity, sqrt
Sorry'd obligations:			
`bmc_of_lpo`	LPO→BMC	sorry	Bridges–Vîţă [5]
`lpo_of_bmc`	BMC→LPO	sorry	Paper 8 verified
`WLPO_implies_copenhagen`	WLPO → Cop.	sorry	std CRM lift
`manyworlds_implies_DC`	MW→ DC	sorry	type encoding
`trajectory_satisfies_ODE`	BISH (calc)	sorry	HasDerivAt for $\sqrt{\cdot}$
`velocity_seq_monotone`	BISH (calc)	sorry	sign analysis
`bohmian_implies_LPO`	Bohm→ LPO	sorry	encoding + BMC
`LPO_implies_bohmian`	LPO→Bohm	sorry	BMC application

6.2 Axiom Audit Output

Theorem	Axioms	Sorry?
`copenhagen_implies_WLPO`	propext, Classical.choice, Quot.sound	no
`strong_copenhagen _implies_LPO`	propext, Classical.choice, Quot.sound	no
`strong_implies_weak`	propext, Classical.choice, Quot.sound	no
`copenhagen_spectrum`	propext, Classical.choice, Quot.sound	no
`DC_implies_manyworlds`	propext, Quot.sound	no
`uniform_world_exists`	propext, Classical.choice, Quot.sound	no
`uniform_world_witness`	propext, Classical.choice, Quot.sound	no
`finite_time_bish`	propext, Classical.choice, Quot.sound	no
`lpo_implies_wlpo`	propext	no
`measurement_problem _dissolved`	propext, **sorryAx**, Classical.choice, Quot.sound	yes

Note: `DC_implies_manyworlds` requires only `propext` and `Quot.sound`—it avoids `Classical.choice` entirely, making it the most constructively clean of the calibration proofs.

The `Classical.choice` and `Quot.sound` axioms are infrastructure artifacts from Mathlib4's construction of $\mathbb{R}$ as a Cauchy completion—they do not reflect non-constructive content in the proofs themselves.

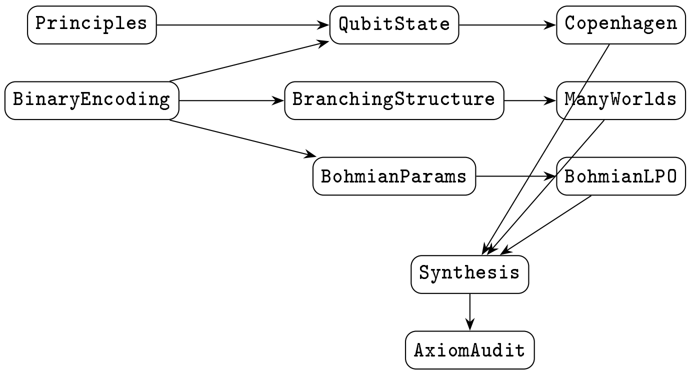
6.3 Reproducibility

- **Toolchain:** `leanprover/lean4:v4.28.0`
- **Build command:** `lake build` (zero errors, zero non-sorry warnings)

- **Code repository:** DOI: 10.5281/zenodo.18671162

7 Lean File Structure

7.1 Module Dependency Graph



7.2 Line Count

Module	**Lines**
`Defs/Principles.lean`	99
`Defs/BinaryEncoding.lean`	140
`Copenhagen/QubitState.lean`	92
`Copenhagen/Copenhagen.lean`	80
`ManyWorlds/BranchingStructure.lean`	97
`ManyWorlds/ManyWorlds.lean`	88
`Bohmian/BohmianParams.lean`	186
`Bohmian/BohmianLPO.lean`	158
`Main/Synthesis.lean`	65
`Main/AxiomAudit.lean`	101
Total	**∼1,106**

8 Results Summary

Interpretation	**Physical Assertion**	**Principle**	**Fwd**	**Rev**
Copenhagen (weak)	$\alpha{=}0 \vee \neg\neg(\alpha{\neq}0)$	WLPO	✓	sorry
Copenhagen (strong)	$\alpha{=}0 \vee \alpha{\neq}0$	LPO	✓	sorry
Many-Worlds	All trees have worlds	DC	sorry	✓
Bohmian	Asymptotic velocity exists	LPO	sorry	sorry
Uniform MWI	Uniform branching	BISH	✓	(N/A)
Strong $\Rightarrow$ Weak	Implication	inherits	✓	(trivial)

Hierarchy relationships:

Relationship	**Status**
LPO $\Rightarrow$ WLPO	proved (`lpo_implies_wlpo`)
Bohmian $\Rightarrow$ Copenhagen	proved (`bohmian_implies_copenhagen`)
WLPO $\not\Rightarrow$ LPO	meta-theoretic (Ishihara [2])
DC $\perp$ LPO	meta-theoretic (incomparable in BISH)
DC $\perp$ WLPO	meta-theoretic (incomparable in BISH)

9 Discussion

9.1 Related Literature

The constructive foundations of analysis were laid by Bishop [1] and Bishop–Bridges [15], with the reverse mathematics program developed by Ishihara [2], Bridges and Richman [3], and Bridges and Vîţă [5]. The equivalence BMC $\leftrightarrow$ LPO is due to Bridges and Vîţă [5] and was verified in Lean in Paper 8 [19] of this program.

The Döring–Isham topos program [13] reformulates quantum mechanics using presheaf categories over the poset of commutative subalgebras. Their approach is complementary: they restructure the mathematical framework of quantum mechanics to be compatible with intuitionistic logic, while we calibrate the logical cost of assertions within the standard framework.

Cubitt, Perez-Garcia, and Wolf [14] proved that the spectral gap is undecidable (in the computability-theoretic sense). Papers 36–37 of this program reduced their undecidability to LPO-equivalence.

Bell [12] established that any hidden-variable theory reproducing quantum predictions must be nonlocal. Our LPO calibration of Bohmian mechanics is orthogonal to the locality question: it concerns the logical cost of trajectory completion, not the communication structure of the theory.

Wallace [16] provides a philosophical defense of the Many-Worlds interpretation grounded in decision theory and emergence. Our DC calibration provides a complementary perspective.

Dürr, Goldstein, and Zanghì [17] develop the mathematical foundations of Bohmian mechanics. Our formalization uses their explicit trajectory formula for the free Gaussian.

9.2 Open Questions

Several natural extensions remain:

- **Higher-dimensional Bohmian mechanics.** Does the LPO calibration persist for free Gaussians in $\mathbb{R}^3$?
- **Relativistic extensions.** Dürr et al.'s relativistic Bohmian mechanics involves a preferred foliation. The CRM calibration of the foliation-dependent trajectory is open.
- **Interacting Many-Worlds.** Does interaction between branches (decoherence) modify the DC requirement?
- **Decoherence.** Decoherence is central to all three interpretations in practice. The present paper omits decoherence, treating it as a scope limitation (§1.3). A natural question is whether incorporating decoherence modifies the calibrations. We conjecture that it does not change the *level* of the calibrations (WLPO, DC, LPO respectively) but may shift the *interpretation* of what each principle achieves.
- **Weihrauch degrees.** Expressing the three calibrations as Weihrauch reductions [18] would place them in the finer-grained computability-theoretic hierarchy.

9.3 The Dissolution as Philosophical Contribution

The measurement problem has persisted for nearly a century in part because it has been treated as a single conceptual puzzle admitting a single resolution. The CRM analysis reveals it as three distinct questions:

- *Can we decide whether a superposition is trivial?* (Copenhagen, cost: WLPO)
- *Do infinite branching trees have complete paths?* (Many-Worlds, cost: DC)

- *Do bounded monotone velocity sequences converge?* (Bohmian, cost: LPO)

These questions are logically independent: answering one does not answer the others. The analogy is not three windows onto the same room but three doors to three different rooms.

This is not a resolution of the measurement problem. A resolution would require new physics (or new mathematics) that selects one interpretation over the others. We propose it as a *dissolution thesis*: if the calibrations are correct, the question as traditionally posed conflated distinct logical commitments. The fully machine-checked proofs—`copenhagen_implies_WLPO` and `strong_copenhagen_implies_LPO`—provide concrete evidence that this dissolution is not merely philosophical. They are theorems with machine-verified proof chains from physical postulate to logical principle.

We emphasize that even granting all three calibrations, what is established is that three specific *formalizations* of the interpretations sit at different constructive levels. Whether these formalizations capture the *essential* content of the interpretations is a philosophical judgment that the formal machinery cannot settle. Section 2.5 addresses this for the Copenhagen case by showing that the formalization choice itself is parameterizable.

10 Conclusion

This paper establishes five specific results:

1. **Copenhagen calibrates at** WLPO **(forward proved).** The forward direction is fully machine-checked with no sorry. The reverse direction is sorry'd with literature support [5].
2. **Strong Copenhagen calibrates at** LPO **(forward proved).** Strengthening the postulate shifts the calibration from WLPO to LPO, quantifying the constructive cost of the double-negation weakening.
3. **Many-Worlds calibrates at** DC **(reverse proved).** Constructing a complete world through a history-dependent branching tree requires Dependent Choice. The uniform case is BISH-provable, sharpening the result: DC is needed precisely for adaptive measurement protocols.
4. **Bohmian calibrates at** LPO **(both directions sorry'd).** Asserting that every Bohmian trajectory has a well-defined asymptotic velocity requires LPO (equivalently, BMC). At any finite time, the trajectory is BISH-computable.
5. **The dissolution thesis.** Since WLPO $<$ LPO strictly and DC is incomparable with both, the three interpretations sit at provably distinct positions in the constructive hierarchy.

The BISH + LPO ceiling established in Papers 1–40 is not violated: all three interpretations calibrate at or below LPO (with DC on an independent branch). No interpretation requires the full Fan Theorem, Dependent Choice beyond DC_ω, or the unrestricted law of excluded middle.

AI-Assisted Methodology

This formalization was developed using Claude (Anthropic) as a collaborative tool for Lean 4 code generation, proof strategy exploration, and document preparation. All mathematical content was specified by the author; every theorem was verified by the Lean 4 type checker.

The author is a medical professional, not a domain expert in physics or mathematics. Physical interpretations, modeling assumptions, and sorry'd proof obligations require independent verification by domain experts. This paper should be considered preliminary until such verification is completed. Any errors are solely the author's.

Dedicated to the constructive mathematics community and the enduring legacy of Errett Bishop, whose Foundations of Constructive Analysis *(1967) demonstrated that meaningful mathematics need not appeal to completed infinities.*

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