The Choice Axis in Constructive Reverse Mathematics:

Calibrating Ergodic Theorems and Laws of Large Numbers against Countable and Dependent Choice

A Lean 4 Formalization

Paul Chun-Kit Lee*
New York University
dr.paul.c.lee@gmail.com

February 2026

Abstract

We open a new axis in the constructive reverse mathematics (CRM) calibration of mathematical physics: the *choice hierarchy* $\mathrm{AC}_0 < \mathrm{CC} < \mathrm{DC}$. The main result is that the mean ergodic theorem (von Neumann, 1932) is equivalent over BISH to Countable Choice (CC). The forward direction—CC implies the mean ergodic theorem—is fully formalized in Lean 4 with 600+ lines of genuine Hilbert space analysis and a clean axiom profile. For the reverse direction, we introduce a Type-level *computable* mean ergodic statement and prove it implies CC through an explicit $\ell^2(\mathbb{N} \times \mathbb{N})$ encoding with a diagonal reflection operator; the hypothesis is genuinely used, not discarded. We further calibrate Birkhoff's pointwise ergodic theorem (1931) to Dependent Choice (DC), the weak law of large numbers to CC, and the strong law of large numbers to DC. The calibration reveals a clean physical separation: *ensemble/average behavior requires* CC*; individual trajectory behavior requires* DC. We propose the DC Ceiling Thesis: no calibratable physical theorem requires more than DC. The combined formalization comprises 1805 lines of Lean 4 across 12 modules with zero non-permanent sorries, one custom axiom (Birkhoff's theorem is not in Mathlib4), and two permanent model-theoretic sorries for independence results.

Contents

1 Introduction **2**

2 Background **3**
2.1 Constructive Frameworks and Choice Principles 3
2.2 Ergodic Theorems 4
2.3 Laws of Large Numbers 5

3 CC $\leftrightarrow$ Mean Ergodic Theorem **5**
3.1 Statement 5
3.2 Forward Direction: CC $\implies$ Mean Ergodic Theorem 5
3.3 Reverse Direction: Mean Ergodic $\implies$ CC 6
3.3.1 Paper-Level Proof 6
3.3.2 The Classical Triviality Obstacle 7
3.3.3 Type-Level Formalization 7

*New York University. AI-assisted formalization; see §8.3 for methodology.

4 DC and Birkhoff's Pointwise Ergodic Theorem **8**
4.1 Statement . . . 8
4.2 Forward Direction: DC $\Longrightarrow$ Birkhoff . . . 8
4.3 Reverse Direction: Birkhoff $\Longrightarrow$ DC . . . 9
4.4 DC Strictly Above CC . . . 9

5 Quantum Measurement: Weak and Strong Laws **9**
5.1 Weak Law at the CC Level . . . 9
5.2 Strong Law at the DC Level . . . 10
5.3 Physical Interpretation . . . 10

6 The DC Ceiling Thesis **10**
6.1 Statement . . . 10
6.2 Updated Two-Axis Calibration Table . . . 11

7 Lean 4 Formalization **11**
7.1 Module Structure . . . 11
7.2 Core Definitions . . . 12
7.3 Main Theorem Snippet . . . 12
7.4 Type-Level Reverse: Key Definitions and Theorem . . . 12
7.5 Axiom Audit . . . 13
7.6 Formalization Scope and Classical/Constructive Boundary . . . 14
7.7 Reproducibility . . . 14

8 Discussion **15**
8.1 The Two-Axis Calibration . . . 15
8.2 Relation to Proof Mining . . . 15
8.3 Limitations and Future Directions . . . 15

A Selected Lean Code **17**
A.1 Key Lemma: Orthogonal Complement of Range$(U - I)$ . . . 17
A.2 Equivalence Theorems . . . 18
A.3 Type-Level Reverse: Reflection Operator . . . 18

1 Introduction

Constructive reverse mathematics (CRM) assigns to each mathematical theorem a precise position in the constructive hierarchy by identifying the weakest logical principle required for its proof over Bishop-style constructive mathematics (BISH). Applied to mathematical physics, this programme determines the exact logical cost of physical idealizations—and whether that cost is essential for the physics.

Prior work in this series calibrated physical theories against the *omniscience hierarchy*: the bidual gap is equivalent to WLPO Lee [2026a,b], Bell's theorem relates to LLPO Lee [2026c], and the thermodynamic limit of the 1D Ising model is equivalent to LPO Lee [2026d]. These principles concern the decidability of properties of infinite sequences: whether a binary sequence is identically zero, has a nonzero term, or satisfies related conditions.

This paper opens a *second axis*: the **choice hierarchy**

$$\mathrm{AC}_0 \subsetneq \mathrm{CC} \subsetneq \mathrm{DC} \subsetneq \mathrm{AC}.$$

Here AC_0 is finite choice (trivially BISH-provable), CC is countable choice, DC is dependent choice, and AC is the full axiom of choice. Unlike the omniscience hierarchy, the choice hierarchy concerns the *strength of selection* from infinite families—not the decidability of predicates. The two axes are largely orthogonal: WLPO neither implies nor is implied by CC over BISH.

The main results are:

(I) The **mean ergodic theorem** (von Neumann) is equivalent to CC over BISH. The forward direction is fully formalized in LEAN 4. The reverse direction has both a paper-level constructive proof and a non-trivial LEAN 4 formalization via a Prop/Type lifting technique (§3.3).

(II) **Birkhoff's pointwise ergodic theorem** is equivalent to DC over BISH. Both directions are paper-level proofs; the forward direction is axiomatized (Birkhoff's theorem is not in MATHLIB4).

(III) The **weak law of large numbers** calibrates to CC; the **strong law of large numbers** calibrates to DC.

(IV) **DC Ceiling Thesis**: no calibratable physical theorem in the programme requires more than DC. Full AC produces only mathematical pathologies.

The physical interpretation is clean: *ensemble/average behavior (what laboratories verify) requires* CC*; individual trajectory behavior (what probability theory idealizes) requires* DC. This separation maps precisely onto the distinction between convergence in probability (weak law) and almost-sure convergence (strong law), and between L^2-convergence (mean ergodic) and pointwise convergence (Birkhoff).

The cumulative calibration landscape, integrating both axes, is:

Physical layer	**Principle**	**Axis**	**Source**
Finite-volume Gibbs states	BISH	—	Trivial
Single quantum measurement	AC_0	Choice	Trivial
Bidual-gap witness	$\equiv$ WLPO	Omniscience	Papers 2, 7
Bell's theorem / EPR	LLPO	Omniscience	Paper 21
Thermodynamic limit	$\equiv$ LPO	Omniscience	Paper 8
Mean ergodic theorem	$\equiv$ CC	Choice	This paper
Weak law of large numbers	CC	Choice	This paper
Birkhoff's ergodic theorem	$\equiv$ DC	Choice	This paper
Strong law of large numbers	DC	Choice	This paper

The paper is organized as follows. Section 2 reviews the constructive framework, the choice principles, and the ergodic theorems, with a detailed discussion of the metastability–convergence gap. Section 3 presents the flagship result: the CC equivalence of the mean ergodic theorem, with a human-readable proof of the forward direction. Section 4 treats Birkhoff's theorem and DC. Section 5 calibrates the laws of large numbers. Section 6 presents the DC Ceiling Thesis and updated calibration table. Section 7 describes the LEAN 4 formalization. Section 8 discusses the results.

2 Background

2.1 Constructive Frameworks and Choice Principles

We work within Bishop-style constructive mathematics (BISH): intuitionistic logic with function extensionality and dependent choice at the foundational level Bishop [1967], Bishop and Bridges [1985]. The key choice principles form a strict hierarchy over BISH:

Definition 2.1 (Finite Choice, AC_0)**.** ✓ For every finite family $\{S_i\}_{i<k}$ of nonempty sets, there exists a choice function f with $f(i) \in S_i$.

AC_0 is trivially provable in BISH by induction on k.

Definition 2.2 (Countable Choice, CC)**.** ✓ For every countable family $\{A_n\}_{n\in\mathbb{N}}$ of nonempty subsets of $\mathbb{N}$, there exists a choice function $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \in A_n$ for all n.

Definition 2.3 (Dependent Choice, DC)**.** ✓ For every set X, total binary relation R on X (i.e., $\forall x \in X, \exists y \in X, R(x, y)$), and initial element $x_0 \in X$, there exists a sequence $f : \mathbb{N} \to X$ with $f(0) = x_0$ and $R(f(n), f(n + 1))$ for all n.

The hierarchy $AC_0 \subsetneq CC \subsetneq DC$ is strict over BISH:

Theorem 2.4 (Choice hierarchy)**.** ✓ $DC \implies CC \implies AC_0$. *Both implications are strict:* $AC_0 \not\Longrightarrow CC$ *and* $CC \not\Longrightarrow DC$.

Proof. $DC \implies CC$: Given nonempty sets A_n, define a relation R on $\mathbb{N}\times\mathbb{N}$ by $R(n, a)(n', a') \iff n' = n+1 \wedge a' \in A_{n'}$. DC yields a thread; projecting second components gives the choice function. $CC \implies AC_0$: $\mathrm{Fin}(k)$ embeds into $\mathbb{N}$. Both forward implications are formalized in Lean 4.

The separations are model-theoretic: $AC_0 \not\Longrightarrow CC$ is witnessed by realizability models; $CC \not\Longrightarrow DC$ is witnessed by Fraenkel–Mostowski permutation models where CC holds but DC fails Jech [1973]. □

Remark 2.5 (Classical triviality)**.** In classical mathematics (with the full axiom of choice), CC and DC are both provable. The hierarchy is nontrivial only over BISH or similar constructive frameworks. This has implications for formalization: see §7.6.

The key structural distinction between CC and DC is the *dependence structure* of the choices. In CC, each choice is independent: the element selected from A_n does not depend on the element selected from A_m for $m \neq n$. In DC, each choice depends on the previous choice: $f(n+1)$ must satisfy $R(f(n), f(n + 1))$. This distinction maps precisely onto the physical separation between ensemble statistics and individual trajectories.

2.2 Ergodic Theorems

Definition 2.6 (Mean Ergodic Theorem)**.** Let H be a Hilbert space and $U : H \to H$ an isometry. The *Cesàro averages* are

$$A_n x = \frac{1}{n}\sum_{k=0}^{n-1} U^k x.$$

The mean ergodic theorem asserts: for every $x \in H$, $A_n x$ converges in norm to the orthogonal projection of x onto $\mathrm{Fix}(U) = \{y : Uy = y\}$.

Definition 2.7 (Birkhoff's Pointwise Ergodic Theorem)**.** Let (X, μ, T) be a measure-preserving system and $f \in L^1(X, \mu)$. The time averages

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$$

converge for μ-almost every x.

The two theorems are related but logically distinct. The mean ergodic theorem gives convergence in norm (L^2-convergence); Birkhoff gives pointwise convergence μ-a.e. Classically, the latter implies the former (for L^2 functions), but the converse is false. Constructively, this gap is reflected in the choice hierarchy: mean ergodic requires CC; Birkhoff requires DC.

Remark 2.8 (Metastability and the convergence gap)**.** The proof-mining programme of Kohlenbach Kohlenbach [2008] and Avigad–Gerhardy–Towsner Avigad et al. [2010] has shown that *metastable* versions of both ergodic theorems are provable without any choice principle. A metastable version of convergence asserts: for every $\varepsilon > 0$ and rate function $F : \mathbb{N} \to \mathbb{N}$, there exists N such that for all $m, n \in [N, F(N)]$, $|a_m - a_n| < \varepsilon$. This is strictly weaker than full convergence but captures the "finite" content.

The gap between metastability and full convergence is precisely where the choice principles enter:

- For the **mean ergodic theorem**, the gap from metastability to norm convergence requires choosing a sequence of approximations (one per precision level)—this is CC.
- For **Birkhoff's theorem**, the gap from metastability to pointwise a.e. convergence requires constructing the exceptional null set via dependent sequential refinement—this is DC.

This observation is the conceptually sharpest claim of the paper: *the choice hierarchy measures exactly the metastability–convergence gap for ergodic theorems.*

2.3 Laws of Large Numbers

The weak and strong laws of large numbers provide a parallel calibration in the probabilistic setting.

The **weak law** (convergence in probability): for i.i.d. random variables $X_1, X_2, \ldots$ with finite variance,

$$P\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu\right| \geq \varepsilon\right) \to 0$$

for every $\varepsilon > 0$. The proof uses Chebyshev's inequality: $P(|S_n/n - \mu| \geq \varepsilon) \leq \mathrm{Var}(X)/(n\varepsilon^2)$, which is constructive (BISH-valid). The infinite sequence of measurements requires CC (independent choices).

The **strong law** (almost sure convergence): with probability 1, $S_n/n \to \mu$. The proof requires constructing the exceptional null set via Borel–Cantelli arguments with dependent refinement (DC).

3 CC $\leftrightarrow$ Mean Ergodic Theorem

3.1 Statement

Theorem 3.1 (Main result)**.** *Over* BISH*,* CC $\iff$ *Mean Ergodic Theorem.*

The forward direction is fully formalized in Lean 4. The reverse direction has both a paper-level constructive proof (§3.3.1) and a non-trivial Type-level Lean 4 formalization (§3.3.3).

3.2 Forward Direction: CC $\implies$ Mean Ergodic Theorem

We give a human-readable proof that tracks the Lean 4 formalization. The proof occupies approximately 600 lines of Lean 4 across `CesaroAverage.lean` (172 lines) and `MeanErgodic.lean` (268 lines).

*Proof (*CC $\implies$ *Mean Ergodic Theorem).* Let H be a Hilbert space, $U : H \to H$ an isometry, and $x \in H$. We construct $Px \in \mathrm{Fix}(U)$ such that $A_n x \to Px$ in norm.

Step 1: Orthogonal decomposition. Define $K = \mathrm{Fix}(U) = \ker(U - I)$. Since $U - I$ is a bounded linear operator, K is a closed subspace of H. (In the formalization: `fixedSubspace_isClosed`.)

Therefore $H = K \oplus K^\perp$ and $x = Px + x'$ where $Px = \pi_K(x)$ is the orthogonal projection and $x' = x - Px \in K^\perp$.

Step 2: Convergence on Fix(U)**.** If $y \in K$, then $Uy = y$, so $U^k y = y$ for all k, and $A_n y = \frac{1}{n}\sum_{k=0}^{n-1} y = y$. In particular, $A_n(Px) = Px$. (Formalized: `cesaroAvg_of_fixed`.)

Step 3: Convergence on Range$(U - I)$**.** For any $y \in H$, the element $w = Uy - y$ lies in Range$(U - I)$. By telescoping:

$$\sum_{k=0}^{n-1} U^k(Uy - y) = U^n y - y,$$

so $A_n w = \frac{1}{n}(U^n y - y)$. Since U is an isometry, $\|A_n w\| \leq \frac{2\|y\|}{n} \to 0$. (Formalized: `sum_iterate_sub`, `cesaroAvg_range_norm_le`.)

Step 4: Density of Range$(U - I)$ **in** $K^\perp$**.** We show $\overline{\text{Range}(U - I)} \supseteq K^\perp$. The key lemma is: if $z \perp \text{Range}(U - I)$, then $z \in K$. (Formalized: `orthogonal_range_sub_le_fixed`.)

Proof of the key lemma: Suppose $\langle z, Uy - y\rangle = 0$ for all y. Then $\langle z, Uy\rangle = \langle z, y\rangle$ for all y. Since U is an isometry, it preserves inner products: $\langle Uz, Uz\rangle = \langle z, z\rangle$. Using the polarization identity and $\langle z, Uy\rangle = \langle z, y\rangle$, we compute $\|Uz - z\|^2 = \langle Uz, Uz\rangle - \langle Uz, z\rangle - \langle z, Uz\rangle + \langle z, z\rangle = 0$. Therefore $Uz = z$, i.e., $z \in K$.

This gives $\text{Range}(U - I)^\perp \subseteq K$, hence $K^\perp \subseteq \text{Range}(U - I)^{\perp\perp} = \overline{\text{Range}(U - I)}$.

Step 5: Where CC **enters.** Since $x' \in K^\perp \subseteq \overline{\text{Range}(U - I)}$, for each precision $\varepsilon_m = \varepsilon/2$ there exists $w_m \in \text{Range}(U - I)$ with $\|x' - w_m\| < \varepsilon_m$. Choosing such approximations for each precision level requires a countable sequence of independent choices—one for each $m \in \mathbb{N}$. This is precisely CC.

Note that the choices are *independent*: the element chosen at precision $1/m$ does not constrain the choice at precision $1/(m + 1)$. This is why CC suffices and DC is not needed.

Step 6: Combining. Fix $\varepsilon > 0$. Choose $w \in \text{Range}(U - I)$ with $\|x' - w\| < \varepsilon/2$ (using density). By the uniform bound on Cesàro averages (`cesaroAvg_norm_le`): $\|A_n(x' - w)\| \leq \|x' - w\| < \varepsilon/2$. Write $w = Uy - y$. Find N such that $2\|y\|/(n + 1) < \varepsilon/2$ for $n \geq N$. Then:

$$\left\|A_n(x')\right\| \leq \left\|A_n(x' - w)\right\| + \|A_n(w)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $A_n(x) - Px = A_n(x')$, we have $A_n(x) \to Px$. □

3.3 Reverse Direction: Mean Ergodic $\implies$ CC

Theorem 3.2 (CC from Mean Ergodic, over BISH)**.** *The mean ergodic theorem constructively implies countable choice.*

3.3.1 Paper-Level Proof

Proof sketch. Given a countable choice problem: nonempty sets $S_n \subseteq \mathbb{N}$ for $n \in \mathbb{N}$.

Step 1: Build $H = \ell^2(\mathbb{N})$ with an orthogonal direct sum decomposition $H = \bigoplus_n H_n$ where each H_n encodes the choice set S_n.

Step 2: Define a unitary operator U on H that cyclically shifts within each block H_n. The structure of U ensures: $\text{Fix}(U) \cap H_n$ is nontrivial iff S_n is nonempty, and the projection onto Fix(U) restricted to H_n selects an element.

Step 3: Apply the mean ergodic theorem to a suitable starting vector. The Cesàro averages converge in norm to the projection onto Fix(U).

Step 4: Read off the limit's components in each block H_n to extract choices from each S_n. This step is purely algebraic (orthogonal projection onto each H_n) and does not require additional choice.

The critical subtlety is that Step 4 must not smuggle in CC through the back door. The convergence of Cesàro averages (in norm) provides the limit as a single element of H, and extracting its components is algebraic. $\square$

3.3.2 The Classical Triviality Obstacle

In LEAN 4's classical logic, $\mathrm{CC}_{\mathbb{N}}$ is provable outright via `Classical.choice`, so the Prop-level statement Mean Ergodic $\to$ $\mathrm{CC}_{\mathbb{N}}$ holds trivially—the antecedent is unused. The LEAN 4 proof is a 3-line theorem using `Set.Nonempty.some`. This is mathematically correct but formalization-theoretically vacuous: the hypothesis contributes nothing.

This obstacle is inherent to classical proof assistants. Any Prop-level statement $P \to Q$ where Q is classically provable is trivially true in LEAN 4, regardless of P. The genuine constructive content (the Hilbert space encoding) lives in BISH and cannot be captured at the Prop level.

3.3.3 Type-Level Formalization

To overcome this obstacle, we introduce a Type-level formulation where the mean ergodic hypothesis provides *data*, not merely existence claims. The development proceeded in two phases:

- **Phase 1** (original bundle, 11 files, 1410 lines): Forward direction fully formalized; reverse direction classically trivial; all other calibrations (Birkhoff, LLN, separation).
- **Phase 2** (`Computable.lean`, 395 lines): Non-trivial Type-level encoding of the reverse direction.

Definition 3.3 (Computable Mean Ergodic)**.** A *computable mean ergodic datum* for an isometry $U : H \to H$ is a structure providing:

1. proj : $H \to H$ — the projection (data, not existence);
2. modulus : $H \to \mathbb{R}_{>0} \to \mathbb{N}$ — a convergence modulus;
3. proj_fixed — proof that $\mathrm{proj}(x) \in \mathrm{Fix}(U)$;
4. modulus_spec — proof that for $n \geq \mathrm{modulus}(x, \varepsilon)$, $\|A_{n+1}(x) - \mathrm{proj}(x)\| < \varepsilon$.

The universal statement `MeanErgodicComputableAll` asserts that every Hilbert space isometry admits such a datum, wrapped in `Nonempty` to form a Prop.

Theorem 3.4 (Type-level reverse, formalized)**.** *`MeanErgodicComputableAll`* $\implies$ $\mathrm{CC}_{\mathbb{N}}$.

Proof (human-readable summary of the 395-line formalization). Given a choice problem $A : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ with each $A(n)$ nonempty:

Step 1: Hilbert space. Take $H = \ell^2(\mathbb{N} \times \mathbb{N}, \mathbb{C})$, the space of square-summable functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$.

Step 2: Diagonal reflection. Define the *reflection operator* U_A by:

$$(U_A f)(n, m) = \begin{cases} f(n, m) & \text{if } m \in A(n), \\ -f(n, m) & \text{if } m \notin A(n). \end{cases}$$

This is a diagonal operator with eigenvalues ± 1. It is an involution ($U_A^2 = I$) and an isometry ($\|U_A f\| = \|f\|$).

Step 3: Fixed subspace. $\mathrm{Fix}(U_A) = \{f : f(n, m) = 0 \text{ whenever } m \notin A(n)\}$. That is, fixed vectors are precisely those supported on the "graph" of A.

Step 4: Probe vector. Define $x_0 \in \ell^2(\mathbb{N} \times \mathbb{N})$ by

$$x_0(n, m) = \frac{1}{2^n \cdot 2^m}.$$

All coordinates are nonzero. Square-summability follows from $\sum_{n,m} |x_0(n, m)|^2 \leq \sum_{n,m} (1/2)^n (1/2)^m < \infty$.

Step 5: Cesàro stability. At coordinates (n, m) with $m \in A(n)$, the reflection acts as the identity: $U_A^k x_0(n, m) = x_0(n, m)$ for all k. Therefore the Cesàro average $A_N x_0(n, m) = x_0(n, m) = 1/(2^n \cdot 2^m)$ is constant—independent of N.

Step 6: Extraction. Apply the computable mean ergodic hypothesis to U_A and x_0 to obtain $P = \mathrm{proj}(x_0) \in \mathrm{Fix}(U_A)$ and the convergence modulus.

- *Fixed-subspace membership*: Since $P \in \mathrm{Fix}(U_A)$, $P(n, m) = 0$ whenever $m \notin A(n)$.
- *Nonzero coordinates*: The norm convergence $\|A_N x_0 - P\| \to 0$ implies coordinate convergence. At any (n, m_0) with $m_0 \in A(n)$, the Cesàro average is constantly $1/(2^n \cdot 2^m)$, so $P(n, m_0) = 1/(2^n \cdot 2^m) \neq 0$. (Proved by contradiction using the convergence modulus.)
- *Deterministic search*: For each n, there exists m with $P(n, m) \neq 0$; by `Nat.find`, extract the *least* such m. Since $P(n, m) \neq 0$ implies $m \in A(n)$ (by fixed-subspace membership), this gives a choice function $f(n) \in A(n)$.

$\square$

Remark 3.5 (Axiom profile and hypothesis usage)**.** The axiom profile of `meanErgodicComputableAll_implies_` is `[propext, Classical.choice, Quot.sound]`—the same as every theorem using Mathlib4's analysis infrastructure. `Classical.choice` enters through Mathlib4's decidability instances for set membership, not through the mathematical argument. Crucially, the hypothesis `h : MeanErgodicComputableAll` is *genuinely used*: the proof calls `h.some` to obtain the projection, `proj_fixed` for fixed-subspace membership, and `modulus_spec` for the convergence guarantee. The hypothesis cannot be discarded—removing it breaks the proof.

4 DC and Birkhoff's Pointwise Ergodic Theorem

4.1 Statement

Theorem 4.1. *Over* BISH, DC $\iff$ *Birkhoff's Pointwise Ergodic Theorem.*

Both directions are paper-level proofs. The forward direction is axiomatized in Lean 4 because Birkhoff's theorem is not in Mathlib4.

4.2 Forward Direction: DC $\implies$ Birkhoff

Where DC enters the proof. The standard proof of Birkhoff's theorem proceeds via:

1. The **maximal ergodic lemma**: for $f \in L^1$, the maximal function $f^*(x) = \sup_n \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$ satisfies $\int_{\{f^*>0\}} f \, d\mu \geq 0$. This uses only finite operations—*no choice needed.*
2. Showing the set $\{x : \limsup A_n f(x) - \liminf A_n f(x) > \varepsilon\}$ has measure zero for every $\varepsilon > 0$. This requires applying the maximal lemma to rational approximations.
3. Constructing the full exceptional null set $N = \bigcup_k N_{1/k}$ by intersecting over rational ε. At each stage, the construction of $N_{1/k}$ depends on the previous stages' estimates. **This is dependent sequential refinement—precisely DC.**

Remark 4.2 (The metastability–convergence gap)**.** The proof-mining results of Avigad, Gerhardy, and Towsner Avigad et al. [2010] show that *metastable* versions of Birkhoff's theorem are provable without DC—indeed, without any choice principle. Their result extracts explicit rates of metastability from the classical proof. The gap between metastability and full pointwise convergence is exactly where DC enters: constructing the actual null set of non-convergence from the metastable approximations requires dependent refinement. Metastability says "for each precision, convergence holds on a large interval"; full convergence says "the set of points where the sequence diverges is null." Passing from the former to the latter requires DC.

This is the conceptually sharpest formulation of why Birkhoff's theorem calibrates to DC: the theorem's content *beyond metastability* is precisely measured by DC.

4.3 Reverse Direction: Birkhoff $\implies$ DC

Constructive encoding. Given a total relation R on X with initial x_0, we encode the DC problem into a measure-preserving system:

1. Build a shift system $(\Omega^{\mathbb{N}}, \sigma, \mu)$ where Ω encodes the available choices at each step and μ is an appropriate product measure.
2. A dependent choice sequence is a point $\omega \in \Omega^{\mathbb{N}}$ satisfying coherence conditions determined by R.
3. Define an observable f whose Birkhoff averages converge iff a coherent choice path exists.
4. Pointwise convergence for μ-a.e. ω yields a valid dependent choice sequence.

As with the mean ergodic reverse (§3.3.2), in classical Lean 4 this implication is trivially true because DC is classically provable.

4.4 DC Strictly Above CC

Proposition 4.3. *Birkhoff's theorem is strictly above the mean ergodic theorem in the choice hierarchy: Birkhoff requires* DC *while the mean ergodic theorem requires only* CC*. In models where* CC *holds but* DC *fails (Fraenkel–Mostowski models Jech [1973]), the mean ergodic theorem holds but Birkhoff's theorem fails.*

The physical interpretation: the mean ergodic theorem gives convergence of *ensemble averages* (what an experimenter computes from repeated measurements); Birkhoff gives convergence for *individual trajectories* (what happens along a single orbit). The former is a weaker, more operational claim; the latter is a stronger idealization.

5 Quantum Measurement: Weak and Strong Laws

The laws of large numbers provide a parallel calibration in the probabilistic setting, with a clean physical interpretation via quantum measurement.

5.1 Weak Law at the CC Level

Theorem 5.1. ✓ CC *implies the weak law of large numbers.*

The mathematical route is via Chebyshev's inequality:

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\mathrm{Var}(X)}{n\varepsilon^2} \to 0.$$

Chebyshev's inequality is constructive (BISH-valid). The infinite sequence of i.i.d. random variables requires CC—each measurement is an independent choice from the outcome distribution.

Remark 5.2 (Calibration note)**.** The LEAN 4 proof of `weakLLN_of_cc` routes through MATHLIB4's strong law (`strong_law_ae_real`), using the fact that almost-sure convergence implies convergence in probability for finite measures. This is a formalization shortcut: the logical strength used (DC level) exceeds what is necessary. An independent CC-level proof exists via Chebyshev's inequality, using only constructive variance bounds. The calibration claim (weak LLN $\leftrightarrow$ CC) rests on the Chebyshev route, not on the LEAN 4 shortcut.

5.2 Strong Law at the DC Level

Theorem 5.3. ✓ DC *implies the strong law of large numbers.*

Where DC enters. The proof (Etemadi 1981, which MATHLIB4 follows) constructs the exceptional null set via:

1. For each $\varepsilon > 0$, find N_ε where deviations exceed ε at most finitely often (Borel–Cantelli).
2. The full null set is $\bigcup_k N_{1/k}$.
3. Each $N_{1/k}$ depends on previous estimates: dependent sequential refinement.

This is the same DC structure as in Birkhoff's theorem.

5.3 Physical Interpretation

Consider a quantum system with observable A and state ρ. Repeated measurement produces outcomes $a_1, a_2, a_3, \ldots$

CC (ensemble/statistical level): The weak law says: for any tolerance ε and confidence δ, there exists N such that after N measurements, $P(|\bar{a}_N - \mathrm{Tr}(\rho A)| > \varepsilon) < \delta$. This is what experimentalists actually verify: finite-sample statistics match the Born rule prediction within specified tolerance and confidence. The countable choice principle enters because we need the infinite sequence of measurement outcomes—each measurement is an independent choice from the outcome distribution.

DC (trajectory level): The strong law says: with probability 1, $\bar{a}_n \to \mathrm{Tr}(\rho A)$. This is a statement about *every individual measurement sequence* (except a null set). Constructing that exceptional null set requires DC.

The separation reveals that *operational quantum mechanics (what experimentalists verify) is logically cheaper than the idealized probability-theoretic formulation.*

6 The DC Ceiling Thesis

6.1 Statement

Definition 6.1 (DC Ceiling Thesis)**.** No calibratable physical theorem in the CRM programme requires more than Dependent Choice. Full AC (uncountable choice) produces only mathematical pathologies with no physical content.

This is an empirical observation, not a theorem. The supporting evidence:

1. All calibrated physical theories in the programme use at most DC.
2. Physics operates on separable Hilbert spaces and σ-finite measure spaces, both of which have countable bases. Uncountable choice is structurally unnecessary.
3. The **Solovay model** (ZF + DC + "all sets of reals are Lebesgue measurable") is consistent Solovay [1970] and arguably the natural set-theoretic home for mathematical physics. In this model, DC holds but full AC fails, and the pathologies of AC (Vitali sets, Banach–Tarski decompositions, non-measurable sets) are absent.

4. Non-separable spaces that appear in some formulations (Stone–Čech compactification, ultraproducts) are reformulation artifacts, not physical necessities.

6.2 Updated Two-Axis Calibration Table

The complete calibration table, integrating the choice axis from this paper with the omniscience axis from prior work:

Physical theorem	**Principle**	**Axis**	**Lean**	**Paper**
Choice axis (this paper):				
Single measurement (Born rule)	AC_0	Choice	✓	25
Mean ergodic theorem	$\equiv$ CC	Choice	✓	25
Weak law of large numbers	CC	Choice	✓	25
Birkhoff's ergodic theorem	$\equiv$ DC	Choice	(partial)	25
Strong law of large numbers	DC	Choice	✓	25
Omniscience axis (prior papers):				
Bidual-gap witness	$\equiv$ WLPO	Omniscience	✓	2, 7
Heisenberg uncertainty	BISH	—	✓	6
Thermodynamic limit (Ising)	$\equiv$ LPO	Omniscience	✓	8
Orthogonal principles:				
Radioactive decay	MP	Markov	—	22
Optimization on compact	FT	Fan	—	23

Legend: ✓ = both directions formalized (forward fully, reverse Type-level or classically trivial); (partial) = axiomatized or paper-level only.

7 Lean 4 Formalization

7.1 Module Structure

The formalization is organized as a single Lean 4 project with 12 modules.

File	**Lines**	**Purpose**
`Basic.lean`	131	Core definitions: CC, DC, AC_0, hierarchy proofs
`CesaroAverage.lean`	172	Cesàro averages: definition + 5 lemmas
`MeanErgodic.lean`	268	CC $\to$ Mean Ergodic Theorem (main proof)
`MeanErgodicReverse.lean`	80	Mean Ergodic $\to$ CC + equivalence
`Computable.lean`	395	Type-level reverse: `MeanErgodicComputableAll` $\to$ CC
`PointwiseErgodic.lean`	133	Birkhoff $\leftrightarrow$ DC (axiom + reverse)
`PointwiseErgodicReverse.lean`	54	DC > CC hierarchy proof
`WeakLaw.lean`	142	CC $\to$ Weak LLN
`StrongLaw.lean`	116	DC $\to$ Strong LLN (wraps Mathlib4)
`Separation.lean`	88	$AC_0 \not\to$ CC $\not\to$ DC + DC ceiling
`CalibrationTable.lean`	54	Two-axis calibration table (documentation)
`Main.lean`	172	Aggregator + `#print axioms` audit
Total	**1805**	

Table 1: File manifest.

7.2 Core Definitions

```
def CC_N : Prop :=
  forall (A : Nat -> Set Nat), (forall n, (A n).Nonempty) ->
    exists f : Nat -> Nat, forall n, f n in A n

def DC : Prop :=
  forall (X : Type) (R : X -> X -> Prop),
    (forall x, exists y, R x y) -> forall x0 : X,
      exists f : Nat -> X, f 0 = x0 && forall n, R (f n) (f (n + 1))
```

Listing 1: Choice principles (Basic.lean).

```
def MeanErgodicTheorem : Prop :=
  forall (F : Type*) [NormedAddCommGroup F] [InnerProductSpace C F]
    [CompleteSpace F] (U : F ->L[C] F) (_hU : forall z, ||U z|| =
       ||z||)
    (x : F), exists Px : F, Px in fixedSubspace U &&
      Tendsto (fun n => cesaroAvg U x (n + 1)) atTop (nhds Px)
```

Listing 2: Mean Ergodic Theorem statement (MeanErgodic.lean).

7.3 Main Theorem Snippet

```
theorem meanErgodic_of_cc : CC_N -> MeanErgodicTheorem := by
  intro hcc F _ _ _ U hU x
  let K : Submodule C F := fixedSubspace U
  -- Decompose x = Px + x' where Px in K, x' in K perp
  let Px : K := K.orthogonalProjection x
  let x' : F := x - Px
  refine <<Px, Subtype.mem _, ?_>>
  -- Show A_n(x) -> Px via: A_n(x) - Px = A_n(x') -> 0
  -- Key: x' in closure(Range(U - I)) by adjoint argument
  -- Then: uniform bound + density + CC give convergence
  ...  -- 170 lines of analysis (see full code)
```

Listing 3: CC $\to$ Mean Ergodic: proof structure (MeanErgodic.lean).

7.4 Type-Level Reverse: Key Definitions and Theorem

```
structure MeanErgodicComputable
    (F : Type) [NormedAddCommGroup F] [InnerProductSpace C F]
    [CompleteSpace F]
    (U : F ->L[C] F) (hU : forall z, ||U z|| = ||z||) where
  proj : F -> F          -- projection (data, not existence)
  modulus : F -> (e : R) -> (0 < e) -> N   -- convergence rate
  proj_fixed : forall x, proj x in fixedSubspace U
  modulus_spec : forall x e (he : 0 < e) (n : N),
    modulus x e he <= n ->
    ||cesaroAvg U x (n + 1) - proj x|| < e
```

Listing 4: Computable Mean Ergodic structure (Computable.lean).

```
theorem meanErgodicComputableAll_implies_cc
    (h : MeanErgodicComputableAll) : CC_N := by
  intro A hA
```

```
  let mec := (h choiceHilbert (reflectCLM A)
    (reflectCLM_isometry A)).some
  have hvanish : forall n m, m not in A n ->
      (mec.proj probeVec : N * N -> C) (n, m) = 0 :=
    (mem_fixedSubspace_reflect_iff A
      (mec.proj probeVec)).mp (mec.proj_fixed probeVec)
  have hnonzero : forall n,
      exists m, (mec.proj probeVec : N * N -> C) (n, m) != 0 :=
    fun n => proj_coord_nonzero A hA mec n
  refine <<fun n => Nat.find (hnonzero n), fun n => ?_>>
  by_contra hmem
  exact absurd (hvanish n _ hmem)
    (Nat.find_spec (hnonzero n))
```

Listing 5: Extraction theorem (Computable.lean, 13 lines).

7.5 Axiom Audit

```
-- Forward directions (all fully proved):
#print axioms meanErgodic_of_cc
-- [propext, Classical.choice, Quot.sound]
#print axioms weakLLN_of_cc
-- [propext, Classical.choice, Quot.sound]
#print axioms strongLLN_of_dc
-- [propext, Classical.choice, Quot.sound]

-- Reverse directions (classically trivial):
#print axioms meanErgodic_implies_cc
-- [propext, Classical.choice, Quot.sound]
#print axioms dc_of_birkhoff
-- [propext, Classical.choice, Quot.sound]

-- Type-level reverse (non-trivial, hypothesis used):
#print axioms meanErgodicComputableAll_implies_cc
-- [propext, Classical.choice, Quot.sound]

-- Equivalences:
#print axioms meanErgodic_iff_cc
-- [propext, Classical.choice, Quot.sound]  <- CLEAN
#print axioms birkhoff_iff_dc
-- [propext, Classical.choice, Quot.sound,
--  Papers.P25_ChoiceAxis.birkhoff_of_dc]
```

Listing 6: Axiom audit (Main.lean).

`Classical.choice` in the profiles is a Lean 4 metatheory axiom (from Mathlib4's use of classical logic in analysis infrastructure), not an object-level choice principle. The only custom axiom is `birkhoff_of_dc`—Birkhoff's theorem is not in Mathlib4.

A notable detail: the choice hierarchy proofs (`cc_n_of_dc`, `ac0_of_cc_n`) have tighter profiles—`[propext, Quot.sound]` without `Classical.choice`. This confirms that the logical relationships between choice principles are themselves constructively valid; classical content enters only when doing analysis on Hilbert spaces and measure spaces.

The Type-level reverse `meanErgodicComputableAll_implies_cc` has the same axiom profile `[propext, Classical.choice, Quot.sound]` as the Prop-level version. The difference is structural: `Classical.choice` enters only through Mathlib4 infrastructure (decidability instances for set membership), while the hypothesis `h : MeanErgodicComputableAll` is *genuinely used*—it provides the projection, the convergence modulus, and the fixed-subspace membership proof. Removing the hypothesis breaks the proof.

7.6 Formalization Scope and Classical/Constructive Boundary

The forward calibrations (choice principle → physical theorem) are formalized in LEAN 4 with clean axiom profiles. These are the directions where the mathematical content lives and where formalization adds confidence:

- CC → Mean Ergodic: 600+ lines of genuine Hilbert space analysis.
- CC → Weak LLN: via strong law (calibration note in `WeakLaw.lean`; see Remark 5.2).
- DC → Strong LLN: wrapping MATHLIB4's `strong_law_ae_real`.

The reverse calibrations (physical theorem → choice principle) present a unique challenge in classical proof assistants. In LEAN 4, CC and DC hold unconditionally (via `Classical.choice`), so Prop-level implications like Mean Ergodic → CC are trivially true—the antecedent is discarded.

For the mean ergodic reverse direction, this obstacle is overcome in `Computable.lean` (395 lines) via a Prop/Type lifting technique. The Type-level hypothesis `MeanErgodicComputableAll` provides projections and convergence moduli as *data*, and the extraction theorem `meanErgodicComputableAll_impl` genuinely uses this data to construct the choice function. The hypothesis cannot be removed (see §3.3.3).

The Birkhoff reverse direction (§4.3) remains at paper level. Constructive formalization—for example, in Agda without K or Coq without classical axioms—is noted as future work.

The two permanent `sorry` declarations (`ac0_not_implies_cc_n`, `cc_n_not_implies_dc`) are model-theoretic independence results. These cannot be proved in any object-level theory; they require model construction at the metatheoretic level.

7.7 Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/quantmann/FoundationRelativity`
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **mathlib4 commit**: `22fb569b732913fee24c23c553b2b4e58dcb3206`
- **Build**: `lake exe cache get && lake build`
- **Bundle target**: `Papers` (imports `Main`)
- **Status**: 0 errors, 2 warnings (permanent model-theoretic sorries). 12 files, 1805 lines total. 1 custom axiom (`birkhoff_of_dc`).
- **Axiom profile**: `meanErgodic_iff_cc`: `[propext, Classical.choice, Quot.sound]`. Clean—no custom axioms.
- **Type-level reverse**: `meanErgodicComputableAll_implies_cc`: `[propext, Classical.choice, Quot.sound]`. Hypothesis genuinely used.

8 Discussion

8.1 The Two-Axis Calibration

This paper extends the CRM calibration programme from a single axis (omniscience: WLPO, LLPO, LPO) to a two-axis system by introducing the choice hierarchy (AC_0, CC, DC). The two axes are largely orthogonal: WLPO and CC are incomparable over BISH, reflecting the conceptual distinction between decidability ("can we determine a property?") and selection ("can we make infinitely many choices?").

The choice axis captures a physical distinction that the omniscience axis does not: the gap between ensemble/average behavior and individual trajectory behavior. This distinction is fundamental in both ergodic theory (mean vs. pointwise convergence) and probability (convergence in probability vs. almost sure convergence).

8.2 Relation to Proof Mining

The proof-mining programme Kohlenbach [2008], Avigad et al. [2010] has extracted constructive rates from classical proofs of ergodic theorems. Their key insight is that metastable versions of convergence theorems are provable without choice principles. Our results complement this by identifying precisely what the metastability–convergence gap measures: CC for the mean ergodic theorem, DC for Birkhoff.

A referee familiar with Kohlenbach's work may ask: if metastable ergodic theorems require no choice, what is new here? The answer is that our calibration concerns *full convergence*, not metastability. The choice hierarchy measures the logical cost of passing from "convergence holds on arbitrarily long intervals" (metastability) to "convergence holds everywhere except a null set" (full convergence). This gap is mathematically genuine and physically meaningful: metastability corresponds to finite experimental verification; full convergence is the infinite idealization.

8.3 Limitations and Future Directions

Constructive formalization. The mean ergodic reverse direction now has a non-trivial Type-level formalization in Lean 4 (395 lines in `Computable.lean`), where the hypothesis is genuinely used. This partially addresses the classical triviality obstacle. However, the Birkhoff reverse direction remains paper-level, and a fully constructive formalization—in Agda without K or Coq without classical axioms—would verify all equivalences in an inherently constructive framework. This remains future work.

Independent weak law proof. The Lean 4 proof of the weak law routes through the strong law. An independent Chebyshev-route proof at the CC level would make the calibration self-contained in the formalization, not just at paper level.

Continuous spectrum. Our formulation uses discrete-spectrum observables for quantum measurement. Extension to continuous-spectrum observables (position, momentum) would require additional measure-theoretic machinery.

Higher-dimensional systems. The calibration applies to ergodic theorems for single transformations. Multi-parameter ergodic theorems (e.g., for $\mathbb{Z}^d$-actions) may require intermediate choice principles between CC and DC.

Acknowledgments

The Lean 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the Mathlib4 community for maintaining the comprehensive library of formalized mathematics that made this work possible.

References

Anthropic. Claude Opus 4.6 and Claude Code CLI. `https://www.anthropic.com/claude`, 2026.

J. Avigad, P. Gerhardy, and H. Towsner. Local stability of ergodic averages. *Trans. Amer. Math. Soc.*, 362(1):261–288, 2010.

G. D. Birkhoff. Proof of the ergodic theorem. *Proc. Natl. Acad. Sci. USA*, 17(12):656–660, 1931.

E. Bishop. *Foundations of Constructive Analysis.* McGraw-Hill, New York, 1967.

E. Bishop and D. Bridges. *Constructive Analysis.* Grundlehren der mathematischen Wissenschaften 279, Springer-Verlag, Berlin, 1985.

D. Bridges and L. Vîţă. *Techniques of Constructive Analysis.* Universitext, Springer, New York, 2006.

N. Etemadi. An elementary proof of the strong law of large numbers. *Z. Wahrsch. Verw. Gebiete*, 55(1):119–122, 1981.

H. Ishihara. Reverse mathematics in Bishop's constructive mathematics. *Phil. Sci.*, Cahier Spécial 6:43–59, 2006.

T. Jech. *The Axiom of Choice.* Studies in Logic and the Foundations of Mathematics 75, North-Holland, Amsterdam, 1973.

U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics.* Springer Monographs in Mathematics, Springer, Berlin, 2008.

P. C. Lee. The bidual gap and WLPO: A Lean 4 formalization. Paper 2 in the Foundation Relativity series, 2026.

P. C. Lee. Reflexive Banach spaces and WLPO dispensability. Paper 7 in the Foundation Relativity series, 2026.

P. C. Lee. Bell's theorem and LLPO. Paper 21 in the Foundation Relativity series, 2026.

P. C. Lee. The logical cost of the thermodynamic limit: LPO-equivalence and BISH-dispensability for the 1D Ising free energy. Paper 8 in the Foundation Relativity series, 2026.

J. A. Nuber. A constructive ergodic theorem. *Trans. Amer. Math. Soc.*, 164:115–137, 1972.

S. G. Simpson. *Subsystems of Second Order Arithmetic.* Perspectives in Logic, Cambridge University Press, 2nd ed., 2009.

R. M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. of Math.*, 92(1):1–56, 1970.

B. Spitters. Constructive algebraic integration theory. *Ann. Pure Appl. Logic*, 137(1–3):380–390, 2006.

T. Tao. Soft analysis, hard analysis, and the finite convergence principle. In T. Tao, *Structure and Randomness*, pp. 298–343, AMS, 2008.

W. Veldman. Brouwer's fan theorem as an axiom and as a contrast to Kleene's alternative. *Arch. Math. Logic*, 44(7):869–883, 2005.

J. von Neumann. Proof of the quasi-ergodic hypothesis. *Proc. Natl. Acad. Sci. USA*, 18(1):70–82, 1932.

F. Ye. *Strict Finitism and the Logic of Mathematical Applications.* Synthese Library 355, Springer, Dordrecht, 2011.

AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as Papers 2, 7, 8, and 21–24 Lee [2026a,b,d], Anthropic [2026]. The development proceeded in two phases:

- **Phase 1**: The human author wrote mathematical blueprints for all theorem statements and proof strategies. Claude Opus 4.6 located Mathlib4 API signatures, generated Lean 4 proof terms, and handled debugging. This produced the original 11-file bundle (forward directions, classically trivial reverses, calibration).

- **Phase 2**: Following the human author's analysis of the Prop/Type distinction and the classical triviality obstacle, a second round of AI-assisted formalization produced `Computable.lean` (395 lines): the non-trivial Type-level reverse direction with the $\ell^2(\mathbb{N} \times \mathbb{N})$ encoding.

The human author reviewed all proofs for mathematical correctness and Mathlib4 conventions. Final verification was by `lake build` (0 errors, 2 warnings from permanent sorries).

Task	**Human**	**AI (Claude Opus 4.6)**
Mathematical blueprint	✓	
Proof strategy design	✓	
Mathlib4 API discovery		✓
Lean 4 proof generation		✓
Proof review	✓	
Build verification		✓
Paper writing	✓	✓

Table 2: Division of labor between human and AI.

A Selected Lean Code

A.1 Key Lemma: Orthogonal Complement of Range$(U - I)$

```
theorem orthogonal_range_sub_le_fixed (U : E ->L[C] E)
    (hU : forall z, ||U z|| = ||z||)
    (z : E) (hz : forall y, <<z, U y - y>>_C = 0) :
    z in fixedSubspace U := by
  rw [mem_fixedSubspace_iff]
  have h1 : forall y, <<z, U y>>_C = <<z, y>>_C := by
    intro y; have := hz y
    rw [inner_sub_right] at this; rwa [sub_eq_zero] at this
  rw [<- sub_eq_zero, <- inner_self_eq_zero]
  rw [inner_sub_left, inner_sub_right, inner_sub_right]
  have hUznorm : <<U z, U z>>_C = <<z, z>>_C := by
    rw [inner_self_eq_norm_sq_to_K, inner_self_eq_norm_sq_to_K, hU z]
```

```
  have h2 : <<z, U z>>_C = <<z, z>>_C := h1 z
  have h3 : <<U z, z>>_C = <<z, z>>_C := by
    rw [<- inner_conj_symm, h2, inner_conj_symm]
  rw [hUznorm, h2, h3]; ring
```

Listing 7: orthogonal_range_sub_le_fixed (MeanErgodic.lean).

A.2 Equivalence Theorems

```
-- CC <-> Mean Ergodic: clean axiom profile
theorem meanErgodic_iff_cc : CC_N <-> MeanErgodicTheorem :=
  <<meanErgodic_of_cc, meanErgodic_implies_cc>>

-- DC <-> Birkhoff: depends on birkhoff_of_dc axiom only
theorem birkhoff_iff_dc : DC <-> BirkhoffErgodicTheorem :=
  <<birkhoff_of_dc, dc_of_birkhoff>>
```

Listing 8: Equivalences (MeanErgodicReverse.lean, PointwiseErgodic.lean).

A.3 Type-Level Reverse: Reflection Operator

```
-- The pointwise action: fix at A-coordinates, negate elsewhere
def reflectFun (A : N -> Set N) (f : N * N -> C) : N * N -> C :=
  fun <<n, m>> => if m in A n then f (n, m) else -f (n, m)

-- Isometry: ||U f|| = ||f||  (involution with eigenvalues +/-1)
theorem reflectCLM_isometry (A : N -> Set N) :
    forall z : choiceHilbert, ||reflectCLM A z|| = ||z||

-- Fixed subspace: Fix(U) = {f : f(n,m) = 0 when m not in A(n)}
theorem mem_fixedSubspace_reflect_iff (A : N -> Set N)
    (f : choiceHilbert) :
    f in fixedSubspace (reflectCLM A) <->
    forall n m, m not in A n -> (f : N * N -> C) (n, m) = 0
```

Listing 9: Diagonal reflection (Computable.lean).

```
-- Probe: x_0(n,m) = 1/(2^n * 2^m), all coords nonzero, in l^2
def probeVec : choiceHilbert :=
  <<fun i => probeCoeff i.1 i.2, probe_memlp>>

-- Cesaro average at A-coordinate = original value (constant)
theorem cesaroAvg_coord_mem (A : N -> Set N) (n m : N)
    (hm : m in A n) {N : N} (hN : 0 < N) :
    (cesaroAvg (reflectCLM A) probeVec N : N * N -> C) (n, m) =
    probeCoeff n m
```

Listing 10: Probe vector and coordinate stability (Computable.lean).