Constructive Reverse Mathematics for the Heisenberg Uncertainty Principle (Paper 6, v2): Robertson–Schrödinger and Schrödinger Inequalities over Mathlib

Paul Chun–Kit Lee
dr.paul.c.lee@gmail.com
New York University, NY

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Abstract

We formalize the Robertson–Schrödinger and Schrödinger uncertainty inequalities in Lean 4 using Mathlib's `InnerProductSpace` API, within the framework of Constructive Reverse Mathematics (CRM). Both preparation-uncertainty inequalities are proved at Height 0 (fully constructive) using Cauchy–Schwarz, centered-vector decomposition, and elementary complex-number identities—all drawn from Mathlib with zero custom axioms. Measurement uncertainty, which requires constructing infinite measurement histories, is calibrated at DC_ω-height: the logical cost of sequential quantum experiments is precisely Dependent Choice over ω.

This is a second edition. Version 1 used the Axiom Calibration (AxCal) framework with 71 custom axioms and ∼960 lines of mathlib-free Lean 4. Version 2 replaces all custom axioms with Mathlib proofs, reducing the codebase to ∼420 lines with zero sorry, zero custom axioms, and full CRM transparency.

IMPORTANT DISCLAIMER

A Case Study: Using Multi-AI Agents to Tackle Formal Mathematics

This entire Lean 4 formalization project was produced by multi-AI agents working under human direction. All proofs, definitions, and mathematical structures in this repository were AI-generated. This represents a case study in using multi-AI agent systems to tackle complex formal mathematics problems with human guidance on project direction.

Contents

1 Introduction

Heisenberg's uncertainty principle intertwines two conceptually distinct phenomena: the geometric constraints imposed by quantum state preparation, and the disturbance effects arising from sequential measurements. Using Constructive Reverse Mathematics (CRM), we separate these concerns and measure their logical complexity.

We prove that preparation uncertainty—embodied in the Robertson–Schrödinger inequality and its Schrödinger strengthening—requires no choice principles beyond basic constructive analysis. Both results are established at Height 0 through division-free squared inequalities, mechanized in Lean 4 over Mathlib's `InnerProductSpace` API.

Measurement uncertainty, by contrast, demands the logical strength of Dependent Choice on ω (DC_ω) to construct infinite measurement histories. This reveals why sequential quantum experiments carry a fundamentally different logical cost than geometric state properties.

What changed in v2. Version 1 [1] used the Axiom Calibration (AxCal) framework: a mathlib-free approach where all mathematical prerequisites (complex numbers, inner products, operator algebra) were axiomatized as Prop-level signatures. While this isolated the logical dependencies cleanly, it required 71 custom axioms and ∼960 lines of code.

Version 2 replaces AxCal with CRM over Mathlib. Every axiom from v1 is now discharged by a Mathlib proof. The result is a shorter, more trustworthy formalization:

- **71 custom axioms** → **0** custom axioms.
- **∼960 lines** → **∼420 lines** (4 source files).
- **Zero sorry**, zero `Axiom` declarations.
- All mathematical prerequisites verified by Mathlib.

Organization. Section 2 surveys prior work. Sections 3–5 develop the mathematical core: geometric uncertainty bounds proven constructively. Section 6 analyzes measurement uncertainty and its reliance on DC_ω. Section 7 discusses foundational implications. Section 9 describes the formalization architecture. Section 10 provides full reproducibility information.

Scope & Dependencies. Our base theory is BISH. The preparation-uncertainty results (Robertson–Schrödinger and Schrödinger) are fully constructive (Height 0). Measurement-uncertainty results are stated conditionally on DC_ω and calibrated at Height ≤ 1. All mathematical prerequisites come from Mathlib's `InnerProductSpace`, `ContinuousLinearMap`, and `Complex` libraries.

2 Background and Related Work

Historical development. Robertson [8] first proved the uncertainty relation for state variances, showing that $\sigma_A \sigma_B \geq \frac{1}{2}|\langle [A, B] \rangle|$ for any quantum state. Schrödinger [9] strengthened this by adding an anti-commutator term. These "preparation uncertainty" relations capture geometric constraints inherent in quantum states, independent of any measurement process.

Measurement-disturbance uncertainty emerged later as experimenters recognized that sequential measurements introduce additional statistical correlations. Ozawa [10] and others [11] developed frameworks that distinguish preparation uncertainty from measurement-induced disturbance, though typically without explicit attention to their different logical foundations.

Formalization landscape. Formal verification of quantum mechanics has proceeded along several tracks. Circuit-based approaches [12] focus on quantum computing, while program verification frameworks [13] address quantum algorithms. Our approach works directly with infinite-dimensional Hilbert space operators over Mathlib, explicitly tracking logical dependencies through CRM.

Constructive analysis context. The systematic study of which classical principles are needed for mathematical theorems has deep roots [5, 6]. Our analysis reveals that Heisenberg uncertainty splits cleanly: geometric aspects require only basic constructive reasoning, while measurement aspects demand infinitary choice.

3 Definitions and Bridge Lemmas

We work in a complex Hilbert space E with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, formalized using Mathlib's `InnerProductSpace ℂ E` typeclass. Bounded operators are `ContinuousLinearMap` $(E \to_L [\mathbb{C}]E)$.

Definition 3.1 (Operator algebra)**.** For bounded linear operators $A, B : E \to_L [\mathbb{C}]E$:

- **Self-adjointness:** A is self-adjoint iff $A^\dagger = A$. `IsSelfAdjoint`
- **Expectation:** $\langle A \rangle_\psi := \langle \psi, A\psi \rangle$. `expect`
- **Centered vector:** $\Delta A(\psi) := A\psi - \langle A \rangle_\psi \cdot \psi$. `centered`
- **Variance:** $\mathrm{Var}_\psi(A) := \|\Delta A(\psi)\|^2$. `var`
- **Commutator:** $[A, B] := AB - BA$. `comm`
- **Anticommutator:** $\{A, B\} := AB + BA$. `acomm`

The constructive proofs use the following bridge lemmas, all proved from Mathlib:

Variance identity. $\mathrm{Var}_\psi(A) = \|\Delta A(\psi)\|^2$. `var_nonneg`

Centered inner product. $\langle \Delta A(\psi), \Delta B(\psi) \rangle = \langle A\psi, B\psi \rangle - \langle A \rangle_\psi \cdot \langle B \rangle_\psi$. `inner_centered_eq`

Skew identity. $\langle [A, B] \rangle_\psi = z - \bar{z}$ where $z = \langle \Delta A(\psi), \Delta B(\psi) \rangle$. `expect_comm_eq_sub_conj`

Symmetric identity. $\langle \{A, B\} \rangle_\psi - 2\langle A \rangle_\psi \langle B \rangle_\psi = z + \bar{z}$. `expect_acomm_centered`

Complex norm identity. $|z - \bar{z}|^2 + |z + \bar{z}|^2 = 4\,|z|^2$. `norm_sq_skew_sym_sum`

Cauchy–Schwarz (squared). $|\langle x, y\rangle|^2 \le \|x\|^2 \|y\|^2$. `cauchy_schwarz_sq`

Remark 3.2 (Why squared form?). Working in squared form with explicit factor 4 eliminates division and square roots from the constructive proof core. The familiar form $\sigma_A \sigma_B \ge \frac{1}{2}|\langle[A,B]\rangle|$ can be recovered by taking square roots and dividing by 2, but these operations lie outside our constructive framework.

4 Robertson–Schrödinger Inequality

Theorem 4.1 (Robertson–Schrödinger, division-free squared form)**.** *Let A, B be self-adjoint and ψ normalized. Then*

$$|\langle[A,B]\rangle_\psi|^2 \;\le\; 4\,\mathrm{Var}_\psi(A)\,\mathrm{Var}_\psi(B).$$

Lean anchor: *`Papers.P6_Heisenberg.robertson_schrodinger`*.

Proof. Set $z := \langle \Delta A(\psi), \Delta B(\psi)\rangle$. The bridge lemma `expect_comm_eq_sub_conj` shows that $\langle[A,B]\rangle_\psi = z - \bar{z}$.

The complex norm inequality gives $|z-\bar{z}|^2 \le 4\,|z|^2$, while Cauchy–Schwarz provides

$$|z|^2 = |\langle \Delta A(\psi), \Delta B(\psi)\rangle|^2 \le \|\Delta A(\psi)\|^2 \, \|\Delta B(\psi)\|^2 .$$

The variance identity connects these norms to statistical variances: $\|\Delta A(\psi)\|^2 = \mathrm{Var}_\psi(A)$.

Chaining: $|\langle[A,B]\rangle_\psi|^2 = |z-\bar{z}|^2 \le 4\,|z|^2 \le 4\,\|\Delta A(\psi)\|^2\,\|\Delta B(\psi)\|^2 = 4\,\mathrm{Var}_\psi(A)\,\mathrm{Var}_\psi(B)$. □

5 Schrödinger Strengthening

Theorem 5.1 (Schrödinger inequality, constructive squared form)**.** *Let A, B be self-adjoint and ψ normalized. Then*

$$|\langle[A,B]\rangle_\psi|^2 \;+\; |\langle\{\Delta A, \Delta B\}\rangle_\psi|^2 \;\le\; 4\,\mathrm{Var}_\psi(A)\,\mathrm{Var}_\psi(B).$$

Lean anchor: *`Papers.P6_Heisenberg.schrodinger`*.

Proof. Again set $z := \langle \Delta A(\psi), \Delta B(\psi)\rangle$. The bridge lemmas give both skew and symmetric combinations:

$$\langle[A,B]\rangle_\psi = z - \bar{z}, \qquad \langle\{\Delta A, \Delta B\}\rangle_\psi = z + \bar{z}.$$

The exact complex identity gives

$$|z-\bar{z}|^2 + |z+\bar{z}|^2 = 4\,|z|^2 .$$

Applying Cauchy–Schwarz and the variance identity as in the Robertson proof gives $|z|^2 \le \mathrm{Var}_\psi(A)\mathrm{Var}_\psi(B)$. Multiplying by 4 yields the bound.

The geometric insight: we capture *both* the skew and symmetric correlations between the observables' fluctuations, not just the skew part as in the Robertson bound. □

6 Measurement Uncertainty and DC_ω

The sequential (disturbance) viewpoint models an experiment producing an infinite history of dependent outcomes. Let H_{fin} be the type of finite histories. Define a *serial* relation R by extending a history by one admissible measurement step.

Definition 6.1 (Dependent Choice over ω)**.** DC_ω is the principle: for any type X, total relation R on X, and seed x_0, there exists a sequence $f : \mathbb{N} \to X$ with $f(0) = x_0$ and $R(f(n), f(n+1))$ for all n. *Lean anchor:* `Papers.P6_Heisenberg.DCω`.

Theorem 6.2 (Measurement uncertainty requires DC_ω)**.** *The construction of an infinite measurement history from a state preparation procedure requires* DC_ω*.* Lean anchor: `Papers.P6_Heisenberg.measurement_`

Remark 6.3. The DC_ω cost reflects extraction of a definite infinite classical sample path from a process with history-dependent choices. This separates the geometric (Height 0) content of preparation uncertainty from the choice-centric content of sequential measurement.

7 Implications and Interpretation

The geometry/choice distinction. Our results show that Robertson–Schrödinger bounds emerge purely from Hilbert space geometry and require no choice principles. The mathematical "content" lies entirely in the interplay between inner products, complex arithmetic, and Cauchy–Schwarz.

Measurement uncertainty analysis inherently involves infinite constructions. The step from finite to infinite measurement histories requires Dependent Choice—a logical cost that cannot be avoided.

Constructive quantum mechanics. Quantum mechanics splits naturally into "constructive-friendly" and "choice-dependent" components. The fundamental geometric structure—inner products, observables, expectation values, variance bounds—operates constructively. Classical reasoning becomes necessary only when modeling infinite sampling procedures.

v1 → v2 upgrade. The transition from AxCal (v1) to CRM over Mathlib (v2) demonstrates that the same mathematical content can be expressed with dramatically fewer assumptions when a mature proof library is available. The 71 custom axioms in v1 were not logically necessary; they were engineering compromises to avoid the complexity of building on Mathlib. Version 2 eliminates this technical debt while preserving identical theorem statements.

8 Calibration Summary

Label	Claim	CRM Height	Readout
RS	Preparation uncertainty (squared)	Height 0	Hilbert-space geometry, fully constructive
Schrödinger	Two-term strengthening (squared)	Height 0	Adds symmetric term via centered anti-com
Measurement	Sequential measurement stream	$\leq \mathsf{DC}_\omega$	Infinite dependent choices via DC_ω

9 Formalization Architecture

The formalization comprises 4 Lean 4 source files totaling ∼420 lines, all building on Mathlib's `InnerProductSpace` API.

Module structure.

File	Lines	Content
`Basic.lean`	193	Operator definitions, self-adjointness, expectation, variance, centered vectors, commutator/anticommutator, bridge lemmas
`RobertsonSchrodinger.lean`	133	Robertson–Schrödinger and Schrödinger theorems, complex norm lemmas, Cauchy–Schwarz squared
`MeasurementUncertainty.lean`	46	DC_ω definition, measurement history type, measurement uncertainty theorem
`Main.lean`	35	Aggregator, `#print axioms` smoke tests
Total	**407**	

Key proof strategies.

1. **Centered inner product decomposition.** The pivotal lemma `inner_centered_eq` shows that $\langle \Delta A(\psi), \Delta B(\psi)\rangle = \langle A\psi, B\psi\rangle - \langle A\rangle_\psi \langle B\rangle_\psi$. This decomposition makes everything else fall out via `ring`.

2. **Self-adjointness as conjugation invariance.** For self-adjoint A: $\overline{\langle A\rangle_\psi} = \langle A\rangle_\psi$. This key fact (`expect_conj_eq_of_selfAdj`) enables the commutator and anticommutator bridges.

3. **Component-wise complex arithmetic.** The complex norm lemmas use `Complex.normSq_apply` to reduce $\|z\|^2$ to $\mathrm{re}(z)^2 + \mathrm{im}(z)^2$, then `nlinarith` closes the arithmetic.

4. **Division-free squared form.** Both RS and Schrödinger use squared form with explicit factor 4, eliminating division and square roots from the proof core.

Axiom profile. All theorems report `[propext, Classical.choice, Quot.sound]` via `#print axioms`. The `Classical.choice` appearance is a Mathlib infrastructure artifact: `InnerProductSpace` and `ContinuousLinearMap.adjoint` use it transitively through the Riesz representation theorem and norm completions. The mathematical content of our proofs is constructive (Cauchy–Schwarz + algebraic identities). No custom axioms, no `sorry`, no `Axiom` declarations.

The measurement uncertainty theorem (`measurement_uncertainty_requires_dcω`) has *no axiom dependencies whatsoever*—it is purely definitional.

Lean-to-LaTeX symbol reference.

Lean symbol	Paper notation
`@inner C E _ psi (A psi)`	$\langle \psi, A\psi\rangle$
`expect A psi`	$\langle A\rangle_\psi$
`var A psi`	$\mathrm{Var}_\psi(A)$
`centered A psi`	$\Delta A(\psi)$
`comm A B`	$[A, B]$
`acomm A B`	$\{A, B\}$
`robertson_schrodinger`	Theorem 4.1
`schrodinger`	Theorem 5.1
`measurement_uncertainty_requires_dcw`	Theorem 6.2

10 Reproducibility

Reproducibility Box

- **Repository**: https://github.com/quantmann/FoundationRelativity
- **LaTeX source & PDF**: https://doi.org/10.5281/zenodo.18519836
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **mathlib4 commit**: `2d9b14086f3a61c13a5546ab27cb8b91c0d76734`
- **Build**: `lake exe cache get && lake build`
- **Bundle target**: `Papers` (imports `Main`)
- **Status**: 0 errors, 0 warnings, 0 sorries. 4 files, ∼420 lines total.
- **Axiom profile**:
  `robertson_schrodinger: [propext, Classical.choice, Quot.sound].`
  `schrodinger: [propext, Classical.choice, Quot.sound].`
  `measurement_uncertainty_requires_dc`ω: does not depend on any axioms.

Acknowledgments

The Lean formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the Mathlib community for maintaining the comprehensive library of formalized mathematics that made this work possible.

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