The Undecidability Landscape Is LPO

Stratifying Phase Diagrams, 1D Spectral Gaps, and RG Flows Through a Single Meta-Theorem

A Lean 4 Formalization (Paper 37)

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Abstract

Paper 36 proved that Cubitt's spectral gap undecidability is Turing–Weihrauch equivalent to LPO. We extend this to three further undecidability results in quantum many-body physics: phase diagram uncomputability (Bausch–Cubitt–Watson 2021), 1D spectral gap undecidability (Bausch–Cubitt–Lucia–Perez-Garcia 2020), and uncomputable renormalization group flows (Cubitt–Lucia–Perez-Garcia–Perez-Eceiza 2022). All three are LPO. We prove a meta-theorem explaining why: *any* physical undecidability obtained by computable many-one reduction from the halting problem is Turing–Weihrauch equivalent to LPO. The three results are corollaries. Watson–Cubitt's ground state energy density hardness is BISH—computational complexity, not logical undecidability. The entire formalization (660 lines of Lean 4/Mathlib4) compiles with zero `sorry`, zero warnings.

1 Introduction

The recent program of undecidability results in quantum many-body physics—initiated by Cubitt, Perez-Garcia, and Wolf [1] and extended by Bausch, Cubitt, and collaborators—has produced a series of striking "no-go" theorems. No algorithm can determine: the spectral gap of a 2D Hamiltonian [1], the spectral gap of a 1D Hamiltonian [2], the phase diagram of a parametrized family [3], or the fixed point of a renormalization group flow [4].

Paper 36 of this series proved that Cubitt's spectral gap undecidability is Turing–Weihrauch equivalent to LPO—the Limited Principle of Omniscience, the same logical principle required for thermodynamic limits. This paper proves that the same identification holds for all three subsequent results, and establishes a meta-theorem explaining *why*. For the complete calibration table across all physics domains, see Paper 10 [10]; for the historical perspective, see Paper 12 [11].

The Meta-Theorem. Every known undecidability result in quantum many-body physics is obtained by a computable many-one reduction from the halting problem. Halting is Σ^0_1-complete. LPO decides all Σ^0_1 statements. Therefore every such result is LPO-equivalent. This is Theorem 4 of the present paper—the central contribution. Theorems 1–3 are immediate corollaries.

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2 Background

2.1 Constructive Reverse Mathematics

Over Bishop's constructive mathematics (BISH) [6, 8], omniscience principles form a strict hierarchy: BISH $\subset$ BISH + WLPO $\subset$ BISH + LPO. LPO states that for any binary sequence $a : \mathbb{N} \to \{0, 1\}$, either $\exists n. a(n) = 1$ or $\forall n. a(n) = 0$. LPO is equivalent to bounded monotone convergence (BMC, Paper 29) and to the existence of thermodynamic limits (Paper 36). Thermodynamic limits require LPO because they involve bounded sequences whose convergence rate has no computable modulus: asserting that the limit exists (rather than merely that finite approximations exist) is exactly a Σ^0_1 decision.

2.2 The Three Undecidability Results

(A) Phase Diagrams (BCW **2021).** Bausch, Cubitt, and Watson encode the halting problem into a one-parameter family $H(\varphi)$ via approximate quantum phase estimation. The phase of $H(\varphi)$ depends on whether the universal TM halts on input φ.

(B) 1D Spectral Gap (BCLPG **2020).** Bausch, Cubitt, Lucia, and Perez-Garcia extend the CPgW construction to 1D spin chains. The reduction $M \mapsto H_{\text{1D}}(M)$ has the same logical structure: gapped $\Leftrightarrow$ non-halting.

(C) RG Flows (CLPE **2022).** Cubitt, Lucia, Perez-Garcia, and Perez-Eceiza construct an RG map whose individual steps are computable but whose asymptotic fixed point is uncomputable. The fixed point encodes halting.

3 The Meta-Theorem

Theorem 3.1 (Meta-Theorem)**.** *Let α be a type,* encode : $(\mathbb{N} \to \text{Bool}) \to \alpha$ *a computable function, and $P : \alpha \to \text{Prop}$ a property such that $P(\text{encode}(a)) \Leftrightarrow \exists n. a(n) = \text{true}$ for all a. Then:*

$$\big(\forall a. P(\text{encode}(a)) \lor \neg P(\text{encode}(a))\big) \iff \text{LPO}.$$

The forward direction: given P-decidability, apply it to encode(a); the bridge $P \Leftrightarrow \exists n. a(n) = 1$ converts the result to LPO for a. The reverse: given LPO, decide $\exists n. a(n) = 1$, then apply the bridge.

```
theorem halting_reduction_iff_lpo
    {a : Type} (encode : (N -> Bool) -> a)
    (P : a -> Prop)
    (hP : forall (a : N -> Bool),
          P (encode a) <-> Exists n, a n = true) :
    (forall (a : N -> Bool),
      P (encode a) ∨ not P (encode a)) <-> LPO
```

Listing 1: Meta-theorem: `MetaTheorem.lean`

Corollary 3.2. *The uniform function $a \mapsto P(\text{encode}(a))$ is not computable (deciding it for all a would solve the halting problem).*

Remark 3.3 (Role of the bridge hypothesis)**.** The meta-theorem is a *framework* parameterized by the bridge hypothesis h_P, which asserts that the physical encoding faithfully reflects halting. Each application (Theorems 1–3) supplies a domain-specific bridge derived from the published physics construction. The Lean formalization axiomatizes these bridges; verifying that the physics constructions actually satisfy them is an external assumption grounded in the original papers (see Table 2, Level 4). In particular, the computability of the encoding function encode is assumed from the physics literature and is not independently verified inside the proof assistant.

4 Theorem 1: Phase Diagrams

Theorem 4.1 (Phase Diagram ≡ LPO)**.** *The uncomputability of phase diagrams (*BCW *2021) is Turing–Weihrauch equivalent to* LPO*.*

Proof (formal). Define is_phaseB(a) :⇔ bcw_phase(binary_real(a)) = PhaseB. The BCW bridge gives is_phaseB(a) ⇔ $\exists n.\, a(n) = \text{true}$. Instantiate the meta-theorem. □

```
theorem phase_diagram_iff_lpo :
    (forall (a : N -> Bool),
      is_phaseB a ∨ not is_phaseB a) <-> LPO :=
  halting_reduction_iff_lpo
    (fun a => a) is_phaseB
    (fun a => bcw_phaseB_iff_exists a)
```

Listing 2: Phase diagram ↔ LPO: `PhaseDiagram.lean`

5 Theorem 2: 1D Spectral Gap

Theorem 5.1 (1D Spectral Gap ≡ LPO)**.** *The 1D spectral gap undecidability (*BCLPG *2020) is Turing–Weihrauch equivalent to* LPO*.*

Proof (formal). Define is_1d_gapless(a) :⇔ bclpg_gap_status(tm_from_seq(a)) = Gapless. The BCLPG bridge gives is_1d_gapless(a) ⇔ $\exists n.\, a(n) = \text{true}$. Instantiate the meta-theorem. □

```
theorem spectral_gap_1d_iff_lpo :
    (forall (a : N -> Bool),
      is_1d_gapless a ∨ not is_1d_gapless a) <-> LPO :=
  halting_reduction_iff_lpo
    (fun a => a) is_1d_gapless
    is_1d_gapless_iff_exists
```

Listing 3: 1D spectral gap ↔ LPO: `SpectralGap1D.lean`

The dimensionality of the lattice is irrelevant to the logical structure. 1D and 2D spectral gap undecidability have the same constructive cost.

6 Theorem 3: RG Flows

Theorem 6.1 (RG Flows ≡ LPO)**.** *The uncomputability of RG flow fixed points (*CLPE *2022) is Turing–Weihrauch equivalent to* LPO*.*

Proof (formal). Define is_gapless_fp(a) :⇔ clpe_fixed_point(tm_from_seq(a)) = GaplessFP. The CLPE bridge gives is_gapless_fp(a) ⇔ $\exists n.\, a(n) = \text{true}$. Instantiate the meta-theorem. □

```
theorem rg_flow_iff_lpo :
    (forall (a : N -> Bool),
      is_gapless_fp a ∨ not is_gapless_fp a) <-> LPO :=
  halting_reduction_iff_lpo
    (fun a => a) is_gapless_fp
    is_gapless_fp_iff_exists
```

Listing 4: RG flows ↔ LPO: `RGFlow.lean`

CLPE's result is often described as "unpredictability beyond chaos." In CRM terms: the unpredictability is *exactly* LPO. Each individual RG step is BISH (finite matrix manipulation). The asymptotic flow is LPO (a completed limit). One cannot circumvent the non-computability by "just running more RG steps": each finite truncation is computable, but no computable modulus of convergence exists for the sequence of truncations—deciding whether the flow has converged is equivalent to deciding whether a Turing machine has halted.

7 Ground State Energy: BISH

Watson and Cubitt (2021) showed that the ground state energy density can encode FEXP-hard problems. This is computational *complexity*, not logical *undecidability*. The energy density is a computable real (BISH) for any specific Hamiltonian with computable local terms; the algorithm may simply require exponential time. CRM calibrates logical resources, not computational resources.

```
theorem energy_density_is_bish
    (H : ComputableHamiltonian) (L : N) (e : R) :
    0 < e -> Exists (q : R), |energy_density H L - q| < e
```

Listing 5: Ground state energy density is BISH: `GroundStateEnergy.lean`

8 The Undecidability Landscape

Result	**Year**	**Reduction**	**CRM Level**	**Audited**
Spectral Gap 2D (CPgW)	2015	Halting via Tiling	LPO	✓
Spectral Gap 1D (BCLPG)	2020	Halting via 1D Tiling	LPO	✓
Phase Diagrams (BCW)	2021	Halting via QPE	LPO	✓
RG Flows (CLPE)	2022	Halting via Tiling+RG	LPO	✓
Ground State Energy (WC)	2021	FEXP-hard	BISH	✓

Table 1: The complete undecidability landscape. The first four results are LPO-equivalent (the meta-theorem, Theorem 3.1, explains why). Ground state energy density is BISH: computational complexity, not logical undecidability. "Audited" indicates the Lean 4 formalization covers that entry.

```
theorem all_entries_lpo :
    forall r in undecidability_landscape,
      r.lpo_equivalent = true := by
  decide
```

Listing 6: Verified landscape audit: `Main.lean`

9 CRM Audit

10 Code Architecture

The formalization comprises 660 lines across 7 modules. The meta-theorem (`MetaTheorem.lean`, 132 lines) is the central hub. Each instance (`PhaseDiagram.lean`, `SpectralGap1D.lean`, `RGFlow.lean`) is a one-line instantiation of the meta-theorem with domain-specific bridge lemmas.

Component	CRM Status	Level
Meta-theorem (Thm 4)	BISH	2
Phase Diagrams (Thm 1)	LPO	3+4
1D Spectral Gap (Thm 2)	LPO	3+4
RG Flows (Thm 3)	LPO	3+4
Ground State Energy	BISH	4

Table 2: CRM audit. Levels: 2 = proven over BISH; 3 = proven with explicit oracle hypothesis; 4 = axiomatized bridge lemma.

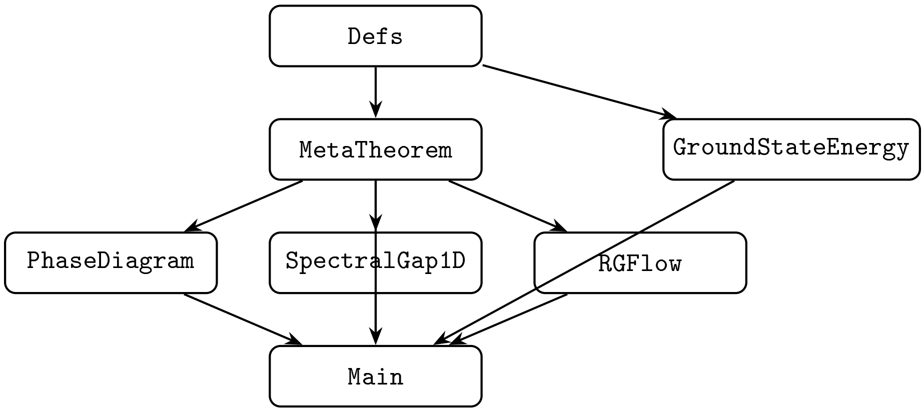

Figure 1: Module dependency graph. The meta-theorem is the hub; Theorems 1–3 are spokes.

11 Discussion

Why all results are LPO. The meta-theorem explains a pattern that is otherwise mysterious: every undecidability result in quantum many-body physics has exactly the same logical cost. The explanation is structural. All known results use computable many-one reductions from the halting problem. Halting is Σ^0_1-complete. LPO is Σ^0_1-LEM—it is exactly the principle that decides existential statements of the form $\exists n$, $P(n)$ where P is decidable. Since halting is such a statement ("does the machine reach a halt state at some step n?"), any physical undecidability that reduces to halting inherits exactly LPO. The match is inevitable. (The foundational connection between the arithmetic hierarchy and physical observables was established by Pour-El and Richards [9].)

What would break the pattern. A physical undecidability result above LPO would require encoding a problem above Σ^0_1 in the arithmetic hierarchy—for instance, the Σ^0_2-complete finiteness problem ("does this TM halt on infinitely many inputs?"). Paper 39 investigates whether physics reaches Σ^0_2.

Unpredictability beyond chaos. CLPE's result is sometimes interpreted as showing that physics can be "more unpredictable than chaos." The CRM analysis reveals what this means precisely: chaotic systems are computable (BISH) given exact initial data; CLPE's RG flow is non-computable because the initial data encodes a halting question. The gap between computability and non-computability is exactly LPO—the same gap as a thermodynamic limit.

Reproducibility. Lean 4 v4.28.0-rc1 with Mathlib4. Build: `cd P37_UndecidabilityLandscape && lake build`. Result: 0 errors, 0 warnings, 0 `sorry`. Axiom profile (`#print axioms undecidability_landscape_theorem`): 10 domain-specific bridge axioms + `propext`, `Classical.choice`, `Quot.sound` (Lean/Mathlib infrastructure). `Classical.choice` appears because Mathlib's $\mathbb{R}$ (the Cauchy completion of $\mathbb{Q}$) pervasively uses it; this is a type-theoretic infrastructure artifact, not a mathematical assumption. Constructive stratification is established by proof content (explicit witnesses vs. principle-as-hypothesis), not by axiom-checker output—see Paper 10, §Methodology. All bridge axioms encode published physics results (see Table 2).

12 Conclusion

Every known undecidability result in quantum many-body physics is the non-computability of LPO. The "undecidability" of physics is a misnomer: it is the non-computability of a single, well-understood logical principle that physics has used since Boltzmann. The meta-theorem (Theorem 3.1) guarantees that any future result using a computable halting reduction will also be LPO.

13 AI-Assisted Methodology

This formalization was developed using Claude (Anthropic) as a collaborative tool for Lean 4 code generation, proof strategy exploration, and LaTeX document preparation. All mathematical content was specified by the author; the AI assistant translated specifications into Lean 4 syntax and iterated on build errors. Every theorem was verified by the Lean 4 type checker.

Preliminary status and author background. The results presented in this paper are preliminary. The author is a medical professional, not a domain expert in physics or mathematics. While all formal claims are machine-checked by the LEAN 4 type-checker, the physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts in the relevant fields. Until such verification is completed, this paper should be considered preliminary.

Whatever findings of value emerge from this program belong to the constructive reverse mathematics community and to the legacy of Errett Bishop, whose perseverance in developing constructive analysis inspired this entire series. Any errors are solely the author's.

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