Markov's Principle and the Constructive Cost of Eventual Decay

Paper 22 in the Constructive Reverse Mathematics Series

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Abstract

The assertion that a radioactive nucleus with nonzero decay rate eventually decays—formally, that for any $\lambda \geq 0$ with $\lambda \neq 0$ and any threshold $\varepsilon \in (0, 1)$, there exists $T > 0$ with $\exp(-\lambda T) < \varepsilon$—is equivalent to Markov's Principle (MP) over Bishop's constructive mathematics (BISH). The forward direction applies the standard real-valued form of MP (non-negative reals: $x \geq 0,\ x \neq 0 \implies \exists q > 0,\ q \leq x$) to obtain an explicit lower bound on the decay rate, then invokes the detection-time formula from Part A. The reverse direction encodes a binary sequence α into a geometric series $\lambda_\alpha = \sum \alpha(n) \cdot 2^{-(n+1)}$, applies the EventualDecay oracle to extract a positive lower bound via exp/log arithmetic, then uses the Archimedean property and a bounded search to produce a witness $\exists n,\ \alpha(n) = \text{true}$. Part A establishes that detection with a ***known*** lower bound on λ is pure BISH, requiring no omniscience. Combined with the hierarchy placement LPO $\Rightarrow$ MP (with MP independent of WLPO and LLPO), this is the **first CRM calibration at the** MP **level**, extending the programme's hierarchy from a linear chain to a partial order. All results are formalized in LEAN 4 with MATHLIB4 (814 lines, 12 files, zero `sorry`).

Contents

*New York University. AI-assisted formalization; see §9 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 From a Linear Chain to a Partial Order

The constructive reverse mathematics (CRM) programme calibrates mathematical physics against the hierarchy of omniscience principles [Bishop, 1967, Bridges and Richman, 1987, Bridges and Vîţă, 2006, Ishihara, 2006, Diener, 2020]. Papers 2 and 7 calibrated WLPO against the bidual gap and non-reflexivity; Paper 8 calibrated LPO against the 1D Ising free energy; Papers 19 and 21 calibrated LLPO against WKB turning points and the Bell sign decision Lee [2026c,f,g,a,b,d,e]. All calibrations in the programme to date have populated a single linear chain:

$$\text{BISH} \ < \ \text{LLPO} \ < \ \text{WLPO} \ < \ \text{LPO}. \tag{1}$$

This paper shows the hierarchy is actually a **partial order**. Markov's Principle (MP) branches off the main chain: it is implied by LPO but independent of both WLPO and LLPO. The first physical calibration at the MP level is the assertion of *eventual decay* for a radioactive nucleus with nonzero decay rate.

1.2 The Physical Insight

Radioactive decay with rate $\lambda > 0$ has survival probability $P(t) = \exp(-\lambda t)$. For any threshold $\varepsilon \in (0, 1)$, the detection time $T = \ln(1/\varepsilon)/\lambda$ satisfies $P(T) = \varepsilon$. This T is computable whenever λ is *known*—meaning we have an explicit rational lower bound $\lambda \geq q > 0$.

But what if we know only that the nucleus is unstable—that is, $\neg(\lambda = 0)$—without an explicit lower bound? Constructively, $\neg(\lambda = 0)$ does *not* give apartness ($\lambda \# 0$), because the Cauchy representation of λ might converge to zero too slowly to detect. The step from $\neg(\lambda = 0)$ to $\exists q > 0,\ q \leq \lambda$ is exactly Markov's Principle.

The physical motivator is proton decay in Grand Unified Theories (GUTs). The theory predicts the proton is unstable ($\neg(\lambda = 0)$), but the experimental lower bound on the proton lifetime exceeds 10^{34} years [Super-Kamiokande Collaboration, 2020, Tanabashi et al., 2018]. We know it is unstable but do not have an explicit positive lower bound on the decay rate. MP asserts: if the theory is correct and the proton is genuinely unstable, then a detection time exists—but MP does not tell you *when*.

1.3 Main Results

The paper has three parts:

1. **Part A (BISH):** For any decay rate λ with an explicit lower bound $\lambda \geq q > 0$, the detection time $T(\varepsilon, q) = \ln(1/\varepsilon)/q$ satisfies $P(T, \lambda) \leq \varepsilon$. No omniscience or Markov required.
2. **Part B (MP calibration):** The assertion "for every decay rate $\lambda \geq 0$ satisfying $\lambda \neq 0$, the survival probability eventually drops below any threshold $\varepsilon \in (0, 1)$" is equivalent to MP over BISH.
3. **Part C (Stratification):** LPO $\Rightarrow$ MP (trivial), with MP independent of WLPO and LLPO (standard). The calibration table becomes a partial order.

The main theorems, stated precisely, are:

- **Theorem 1** (Part A): Detection time is positive—$T > 0$.
- **Theorem 2** (Part A): Detection time works—$P(T, \lambda) \leq \varepsilon$.
- **Theorem 3** (Part A): Detection with a known witness gives a computable T.
- **Theorem 4** (Part B): MP $\Rightarrow$ EventualDecay.
- **Theorem 5** (Part B): EventualDecay $\Rightarrow$ MP (novel direction).
- **Theorem 6** (Part B): MP $\leftrightarrow$ EventualDecay.
- **Theorem 7**: Decay stratification (three levels).

1.4 What Makes This Paper Different

Paper 22 contributes three novelties:

1. **First CRM calibration at the MP level.** All previous calibrations in the series populated the linear chain BISH < LLPO < WLPO < LPO. This is the first to land at MP, a principle that branches off the main chain.
2. **Extending the hierarchy to a partial order.** The calibration table is no longer purely linear. The discovery that a physical assertion lives at MP—independent of both WLPO and LLPO—shows the physics itself has a partially ordered logical structure.
3. **Observable-dependent cost.** The same geometric encoding $\lambda_\alpha = \sum \alpha(n) \cdot 2^{-(n+1)}$ yields WLPO (zero-test, Paper 20), LLPO (sign-test on differences, Paper 21), or MP (apartness-test, Paper 22), depending on the question asked. The logical cost depends on the *observable*, not the encoding.

2 Background

2.1 Exponential Decay

The survival probability of a nucleus with decay rate $\lambda \geq 0$ is

$$P(t, \lambda) = \exp(-\lambda \cdot t). \tag{2}$$

For $\lambda > 0$ and any threshold $\varepsilon \in (0, 1)$, the detection time is

$$T(\varepsilon, \lambda) = \frac{\ln(1/\varepsilon)}{\lambda}, \tag{3}$$

satisfying $P(T, \lambda) = \varepsilon$. Both P and T are computable when λ is given as a computable real with a known positive lower bound.

2.2 Markov's Principle

Definition 2.1 (Markov's Principle)**.** ✓ *Markov's Principle* (MP): for any binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$,

$$\neg(\forall n,\ \alpha(n) = 0) \implies \exists n,\ \alpha(n) = 1. \tag{4}$$

The hypothesis provides a negative fact (the sequence is not all zeros); the conclusion provides a positive witness (an index where the sequence is one).

MP is accepted in the Russian constructive school (Markov, Shanin) [Markov, 1954] but rejected in Brouwerian intuitionism and in Bishop's BISH [Bishop, 1967, Bishop and Bridges, 1985]. It is implied by LPO (trivially: LPO decides the disjunction outright) and by Church's Thesis (all functions are computable, so one can search for the witness). MP is independent of LLPO and WLPO [Bridges and Richman, 1987, Bridges and Vîţă, 2006].

Definition 2.2 (MP for reals)**.** The real-valued form: for $x \geq 0$ with $x \neq 0$, there exists a rational $q > 0$ with $q \leq x$.

$$\forall x \in \mathbb{R},\ \ x \geq 0\ \wedge\ x \neq 0 \implies \exists q \in \mathbb{Q},\ \ q > 0\ \wedge\ q \leq x. \tag{5}$$

The equivalence $\mathrm{MP} \leftrightarrow \mathrm{MP}_{\mathrm{real}}$ is standard [Bridges and Richman, 1987, Bridges and Vîţă, 2006].

2.3 The CRM Hierarchy

Definition 2.3 (LPO)**.** ✓ The *Limited Principle of Omniscience*: for every binary sequence α, either $\alpha(n) = 0$ for all n, or there exists n with $\alpha(n) = 1$.

Definition 2.4 (WLPO)**.** ✓ The *Weak Limited Principle of Omniscience*: for every binary sequence α, either $\alpha(n) = 0$ for all n, or it is not the case that $\alpha(n) = 0$ for all n.

Definition 2.5 (LLPO)**.** ✓ The *Lesser Limited Principle of Omniscience*: for every binary sequence α with at most one index n satisfying $\alpha(n) = 1$, either $\alpha(2n) = 0$ for all n, or $\alpha(2n + 1) = 0$ for all n.

The main chain and key implications are:

$$\mathrm{LPO} \implies \mathrm{WLPO} \implies \mathrm{LLPO} \implies \mathrm{BISH}. \tag{6}$$

Markov's Principle branches off:

$$\mathrm{LPO} \implies \mathrm{MP}, \qquad \mathrm{MP} \not\Longrightarrow \mathrm{WLPO}, \qquad \mathrm{WLPO} \not\Longrightarrow \mathrm{MP}. \tag{7}$$

All separations are standard [Bridges and Richman, 1987, Bridges and Vîţă, 2006, Ishihara, 2006].

Definition 2.6 (EventualDecay). ✓ The *eventual decay* assertion: for any non-negative decay rate λ that is not zero, the survival probability eventually drops below any threshold:

$$\forall \lambda \geq 0, \quad \lambda \neq 0 \implies \forall \varepsilon \in (0,1), \quad \exists T > 0, \quad \exp(-\lambda T) < \varepsilon. \tag{8}$$

Note: the hypothesis is $\lambda \neq 0$ (nonzero), not $\lambda > 0$ (positive with explicit lower bound). The gap between these is exactly where MP operates.

```
/-- Eventual decay: for any non-negative decay rate that is
    not zero, the survival probability drops below any
    threshold. -/
def EventualDecay : Prop :=
  forall (lambda_ : Real), 0 <= lambda_ -> lambda_ != 0 ->
    forall (eps : Real), 0 < eps -> eps < 1 ->
      exists (T : Real), 0 < T /\ survivalProb lambda_ T < eps
```

Listing 1: EventualDecay definition (Defs/Decay.lean).

2.4 The CRM Diagnostic

The CRM diagnostic for a physical assertion proceeds as follows:

1. Formalize the assertion and its proof in Lean 4 with Mathlib4.
2. Declare axioms for known CRM equivalences (e.g., `mp_real_of_mp`).
3. Run `#print axioms` on each main theorem.
4. The custom axioms in the output certify the CRM level. Theorems with no custom axioms are BISH; theorems depending on `mp_real_of_mp` are MP.

3 Part A: Detection with Known Bounds Is BISH

The first tier: when the decay rate has an explicit positive lower bound, the detection time is constructively computable. No omniscience principle is needed.

3.1 Physical Setup

Definition 3.1 (Survival probability). ✓ The survival probability of a nucleus with decay rate λ at time t is

$$\mathrm{survivalProb}(\lambda, t) := \exp(-\lambda \cdot t).$$

Definition 3.2 (Detection time). ✓ Given a lower bound $q > 0$ on the decay rate and a threshold $\varepsilon \in (0,1)$, the detection time is

$$\mathrm{detectionTime}(\varepsilon, q) := \frac{\ln(1/\varepsilon)}{q}.$$

```
/-- Survival probability: P(t, lambda) = exp(-lambda * t). -/
def survivalProb (lambda_ t : Real) : Real :=
  Real.exp (-(lambda_ * t))

/-- Detection time: T(eps, q) = ln(1/eps) / q. -/
def detectionTime (eps q : Real) : Real :=
  Real.log (1 / eps) / q
```

Listing 2: Physical definitions (Defs/Decay.lean).

3.2 Detection Time Is Positive

Theorem 3.3 (Detection time is positive—BISH). ✓ *For $q > 0$ and $\varepsilon \in (0,1)$: $T(\varepsilon, q) = \ln(1/\varepsilon)/q > 0$.*

Proof. Since $\varepsilon < 1$, we have $1/\varepsilon > 1$, so $\ln(1/\varepsilon) > 0$. Since $q > 0$, the quotient $T > 0$. □

```
theorem detectionTime_pos (q eps : Real)
    (hq : 0 < q) (heps : 0 < eps) (heps1 : eps < 1) :
    0 < detectionTime eps q := by
  unfold detectionTime
  apply div_pos
  . apply Real.log_pos
    rw [lt_div_iff heps]
    linarith
  . exact hq
```

Listing 3: Detection time positivity (PartA/DetectionTime.lean).

3.3 Detection Time Works

Theorem 3.4 (Detection time works—BISH). ✓ *For $\lambda \geq q > 0$ and $\varepsilon \in (0,1)$:*

$$P\big(T(\varepsilon, q),\ \lambda\big) = \exp\big(-\lambda \cdot \ln(1/\varepsilon)/q\big) \ \leq\ \varepsilon.$$

Proof. The key chain of inequalities:

1. Since $\lambda \geq q$ and $T = \ln(1/\varepsilon)/q > 0$:
$$-\lambda \cdot T \ \leq\ -q \cdot T = -\ln(1/\varepsilon).$$
2. Since exp is monotone increasing:
$$\exp(-\lambda T) \ \leq\ \exp(-\ln(1/\varepsilon)).$$
3. Since $\exp(-\ln(1/\varepsilon)) = \varepsilon$ for $\varepsilon > 0$:
$$P(T, \lambda) \ \leq\ \varepsilon.$$

□

```
theorem detection_time_works (lambda_ q eps : Real)
    (hq : 0 < q) (hlq : q <= lambda_)
    (heps : 0 < eps) (heps1 : eps < 1) :
    survivalProb lambda_ (detectionTime eps q) <= eps := by
  unfold survivalProb detectionTime
  have hlog_pos : 0 < Real.log (1 / eps) := by
    apply Real.log_pos; rw [lt_div_iff heps]; linarith
  have hT_pos : 0 < Real.log (1 / eps) / q :=
    div_pos hlog_pos hq
  have h_neg_le : -(lambda_ * (Real.log (1 / eps) / q)) <=
      -(q * (Real.log (1 / eps) / q)) := by
    apply neg_le_neg
    apply mul_le_mul_of_nonneg_right hlq (le_of_lt hT_pos)
  have h_simpl : q * (Real.log (1 / eps) / q) =
      Real.log (1 / eps) := by field_simp
  rw [h_simpl] at h_neg_le
```

```
  calc Real.exp (-(lambda_ * (Real.log (1 / eps) / q)))
      <= Real.exp (-(Real.log (1 / eps))) :=
        Real.exp_le_exp.mpr h_neg_le
    _ = eps := exp_neg_log_inv eps heps
```

Listing 4: Detection time works (PartA/DetectionTime.lean).

3.4 Detection with a Known Witness

Theorem 3.5 (Detection with witness—BISH). ✓ *When* $\alpha(k) = \mathrm{true}$, *the encoded rate* λ_α *has the explicit lower bound* $\lambda_\alpha \geq (1/2)^{k+1}$, *and the detection time* $T(\varepsilon, (1/2)^{k+1})$ *is computable with* $P(T, \lambda_\alpha) \leq \varepsilon$.

Proof. The witness $\alpha(k) = \mathrm{true}$ contributes a term $(1/2)^{k+1}$ to the sum λ_α, giving the lower bound $q = (1/2)^{k+1} > 0$. Apply Theorem 3.4 with this explicit q. □

```
theorem detection_with_witness (alpha : Nat -> Bool) (k : Nat)
    (hk : alpha k = true) (eps : Real)
    (heps : 0 < eps) (heps1 : eps < 1) :
    exists (T : Real), 0 < T /\
      survivalProb (encodedRate alpha) T <= eps := by
  set q := ((1 : Real) / 2) ^ (k + 1) with hq_def
  have hq_pos : 0 < q := pow_pos (by norm_num) _
  have hlq : q <= encodedRate alpha := by
    -- encodedRate alpha >= q from the witness
    unfold encodedRate
    have hterm : encodedRateTerm alpha k = q := by
      unfold encodedRateTerm; rw [if_pos hk]
    rw [<- hterm]
    exact (encodedRate_summable alpha).le_tsum k
      (fun n _hn => encodedRateTerm_nonneg alpha n)
  exact <detectionTime eps q,
    detectionTime_pos q eps hq_pos heps heps1,
    detection_time_works (encodedRate alpha) q eps
      hq_pos hlq heps heps1>
```

Listing 5: Detection with witness (PartA/DetectionTime.lean).

Remark 3.6 (Axiom profile for Part A). `#print axioms detectionTime_pos`, `#print axioms detection_time_works`, and `#print axioms detection_with_witness` all show only `[propext, Classical.choice, Quot.sound]`. The `Classical.choice` arises from Mathlib4's infrastructure for `Real.exp`, `Real.log`, and `tsum`, not from any mathematical use of choice. No custom axiom (`mp_real_of_mp`) appears. These are pure BISH results.

4 Part B: The MP Calibration

This is the core section: the first calibration of Markov's Principle against a physical assertion.

4.1 The Encoded Rate

Definition 4.1 (Encoded decay rate). ✓ For a binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, the encoded decay rate is the geometric series:

$$\lambda_\alpha := \sum_{n=0}^{\infty} [\alpha(n) = 1] \cdot \left(\frac{1}{2}\right)^{n+1}, \tag{9}$$

where $[\cdot]$ is the Iverson bracket.

```lean
/-- The term of the encoded rate series. -/
def encodedRateTerm (alpha : Nat -> Bool) (n : Nat) : Real :=
  if alpha n then ((1 : Real) / 2) ^ (n + 1) else 0

/-- The encoded decay rate: lambda_alpha =
    sum_n (if alpha(n) then (1/2)^(n+1) else 0). -/
def encodedRate (alpha : Nat -> Bool) : Real :=
  tsum fun n => encodedRateTerm alpha n
```

Listing 6: Encoded rate (Defs/EncodedRate.lean).

Lemma 4.2 (Summability). ✓ *The encoded rate series is summable. Each term is bounded by* $(1/2)^{n+1}$*, and the geometric series* $\sum_n (1/2)^{n+1}$ *converges to* 1*.*

Lemma 4.3 (Non-negativity). ✓ $\lambda_\alpha \geq 0$ *for all* α*. Each term is non-negative, so the infinite sum is non-negative.*

Lemma 4.4 (Zero-iff characterization). ✓

$$\lambda_\alpha = 0 \iff \forall n,\ \alpha(n) = 0. \tag{10}$$

Proof. ($\Leftarrow$): If all entries are false, every term of the series is zero, so the sum is zero. ($\Rightarrow$): Suppose $\lambda_\alpha = 0$ and, for contradiction, $\alpha(n) = 1$ for some n. Then the n-th term is $(1/2)^{n+1} > 0$. Since all terms are non-negative and the series is summable, $\lambda_\alpha > 0$ by `Summable.tsum_pos`—contradicting $\lambda_\alpha = 0$. □

```lean
theorem encodedRate_eq_zero_iff (alpha : Nat -> Bool) :
    encodedRate alpha = 0 <-> forall n, alpha n = false :=
  <all_false_of_encodedRate_eq_zero alpha,
   encodedRate_eq_zero_of_all_false alpha>
```

Listing 7: Zero-iff (Defs/EncodedRate.lean, selected).

Lemma 4.5 (Tail bound). ✓ *For any* k*:*

$$\lambda_\alpha - \mathrm{partialRate}(\alpha, k) \ \leq\ \left(\frac{1}{2}\right)^{k+1}, \tag{11}$$

where $\mathrm{partialRate}(\alpha, k) = \sum_{n=0}^{k} \mathrm{encodedRateTerm}(\alpha, n)$ *is the partial sum.*

Proof. The tail $\sum_{n>k} \mathrm{encodedRateTerm}(\alpha, n) \leq \sum_{n>k} (1/2)^{n+1} = (1/2)^{k+1} \cdot \sum_{j \geq 1} (1/2)^j = (1/2)^{k+1} \cdot 1 = (1/2)^{k+1}$. □

Lemma 4.6 (Witness from positive partial sum). ✓ *If* $\mathrm{partialRate}(\alpha, k) > 0$*, then there exists* n *with* $\alpha(n) = \mathrm{true}$*.*

Proof. The partial sum is a finite sum of non-negative terms. If the sum is positive, at least one term is positive (by `Finset.sum_pos_iff_of_nonneg`). A positive term means $\alpha(n) = \mathrm{true}$ (since the term is $(1/2)^{n+1}$ when $\alpha(n) = \mathrm{true}$ and 0 otherwise). □

```lean
theorem witness_from_partial_sum_pos (alpha : Nat -> Bool)
    (k : Nat) (hpos : 0 < partialRate alpha k) :
    exists n, alpha n = true := by
  unfold partialRate at hpos
  have <n, _hn_mem, hn_pos> :=
    (Finset.sum_pos_iff_of_nonneg
```

```
    (fun i _hi => encodedRateTerm_nonneg alpha i)).mp hpos
  have h_alpha_n : alpha n = true := by
    by_contra h
    have : encodedRateTerm alpha n = 0 := by
      unfold encodedRateTerm
      simp only [ite_eq_right_iff]
      intro htrue; exact absurd htrue h
    linarith
  exact <n, h_alpha_n>
```

Listing 8: Witness extraction (Defs/EncodedRate.lean).

4.2 Forward: MP ⇒ EventualDecay

Theorem 4.7 (MP ⇒ EventualDecay). ✓ *If Markov's Principle holds, then* EventualDecay *holds: for any $\lambda \geq 0$ with $\lambda \neq 0$ and any $\varepsilon \in (0,1)$, there exists $T > 0$ with $P(T, \lambda) < \varepsilon$.*

Proof. **Step 1: MP gives an explicit positive lower bound.** Apply `mp_real_of_mp` to λ: since $0 \leq \lambda$ and $\lambda \neq 0$, obtain $q \in \mathbb{Q}$ with $0 < q \leq \lambda$.

Step 2: Use detection time with the $\varepsilon/2$ trick. Set $T = \ln(2/\varepsilon)/q = \text{detectionTime}(\varepsilon/2, q)$. By Theorem 3.4 with threshold $\varepsilon/2$:

$$P(T, \lambda) \;\leq\; \varepsilon/2 \;<\; \varepsilon.$$

The $\varepsilon/2$ trick converts the $\leq$ from Theorem 3.4 into the strict $<$ required by the EventualDecay definition. □

```
/-- Interface axiom: MP for sequences implies MP for
    non-negative reals. Standard (Bridges-Richman 1987). -/
axiom mp_real_of_mp :
  MarkovPrinciple ->
    forall (x : Real), 0 <= x -> x != 0 ->
      exists (q : Rat), (0 < (q : Real)) /\ (q : Real) <= x

/-- Theorem 4: MP implies EventualDecay. -/
theorem eventualDecay_of_mp (hmp : MarkovPrinciple) :
    EventualDecay := by
  intro lambda_ hlnn hlne eps heps heps1
  obtain <q, hqpos, hqle> :=
    mp_real_of_mp hmp lambda_ hlnn hlne
  have heps2 : 0 < eps / 2 := by linarith
  have heps2_1 : eps / 2 < 1 := by linarith
  exact <detectionTime (eps / 2) q,
    detectionTime_pos q (eps / 2) hqpos heps2 heps2_1,
    calc survivalProb lambda_ (detectionTime (eps / 2) q)
        <= eps / 2 := detection_time_works lambda_ q
          (eps / 2) hqpos hqle heps2 heps2_1
      _ < eps := by linarith>
```

Listing 9: Forward direction (PartB/Forward.lean).

4.3 Backward: EventualDecay ⇒ MP (Novel)

This is the novel direction: the EventualDecay oracle implies Markov's Principle.

Theorem 4.8 (EventualDecay ⇒ MP). ✓ *If* EventualDecay *holds, then Markov's Principle holds.*

Proof. Let $\alpha : \mathbb{N} \to \{0,1\}$ with $\neg(\forall n,\ \alpha(n) = 0)$. We must produce $\exists n,\ \alpha(n) = 1$.

Step 1: Construct the encoded rate. Define $\lambda_\alpha = \sum_n [\alpha(n)] \cdot (1/2)^{n+1}$.

Step 2: Derive nonzero. From $\neg(\forall n,\ \alpha(n) = 0)$ and the zero-iff characterization (Lemma 4.4), we get $\lambda_\alpha \neq 0$. Also $\lambda_\alpha \geq 0$ (Lemma 4.3).

Step 3: Apply EventualDecay. Apply the oracle to λ_α with $\varepsilon = 1/2$: obtain $T > 0$ with $\exp(-\lambda_\alpha \cdot T) < 1/2$.

Step 4: Extract positivity. If $\lambda_\alpha = 0$, then $\exp(-0 \cdot T) = \exp(0) = 1$, contradicting $\exp(-\lambda_\alpha T) < 1/2$. Since $\lambda_\alpha \geq 0$, we conclude $\lambda_\alpha > 0$.

Step 5: Archimedean step. Since $\lambda_\alpha > 0$, we have $\lambda_\alpha/2 > 0$. By the Archimedean property (via `exists_pow_lt_of_lt_one`), find $k_0 \in \mathbb{N}$ with $(1/2)^{k_0} < \lambda_\alpha/2$.

Step 6: Tail bound gives positive partial sum. By the tail bound (Lemma 4.5):

$$\lambda_\alpha - \mathrm{partialRate}(\alpha, k_0) \ \leq\ (1/2)^{k_0+1} \ \leq\ (1/2)^{k_0} \ <\ \lambda_\alpha/2.$$

Rearranging: $\mathrm{partialRate}(\alpha, k_0) > \lambda_\alpha - \lambda_\alpha/2 = \lambda_\alpha/2 > 0$.

Step 7: Bounded search. Since $\mathrm{partialRate}(\alpha, k_0) > 0$ and this is a finite sum of non-negative terms, at least one term is nonzero. By Lemma 4.6, there exists n with $\alpha(n) = \mathrm{true}$. This is a bounded search over $\{0, \ldots, k_0\}$—a finite, decidable procedure, pure BISH. $\square$

```
/-- Theorem 5 (Novel): EventualDecay implies
    Markov's Principle. -/
theorem mp_of_eventualDecay
    (hed : forall (lambda_ : Real),
      0 <= lambda_ -> lambda_ != 0 ->
      forall (eps : Real), 0 < eps -> eps < 1 ->
        exists (T : Real),
          0 < T /\ survivalProb lambda_ T < eps) :
    MarkovPrinciple := by
  intro alpha hne
  -- Steps 1-2: encoded rate is nonneg and nonzero
  have hlnn : 0 <= encodedRate alpha := encodedRate_nonneg alpha
  have hlne : encodedRate alpha != 0 := by
    intro heq
    exact hne ((encodedRate_eq_zero_iff alpha).mp heq)
  -- Step 3: apply oracle with eps = 1/2
  obtain <T, hTpos, hPlt> :=
    hed (encodedRate alpha) hlnn hlne (1/2)
      (by norm_num) (by norm_num)
  -- Step 4: extract positivity
  have hlpos : 0 < encodedRate alpha :=
    pos_of_exp_decay (encodedRate alpha) T hTpos hlnn hPlt
  -- Step 5: Archimedean
  obtain <k0, hk0> :
    exists k0, ((1:Real)/2)^k0 < encodedRate alpha / 2 :=
    exists_pow_lt_of_lt_one (by linarith) (by norm_num)
  -- Step 6: tail bound gives partialRate > 0
  have htail := encodedRate_sub_partialRate_le alpha k0
  have hpow_le : ((1:Real)/2)^(k0+1) <= ((1:Real)/2)^k0 := by
    apply pow_le_pow_of_le_one (by norm_num) (by norm_num)
    exact Nat.le_succ k0
  have hpartial_pos : 0 < partialRate alpha k0 := by linarith
  -- Step 7: bounded search
  exact witness_from_partial_sum_pos alpha k0 hpartial_pos
```

Listing 10: Backward direction (PartB/Backward.lean).

Remark 4.9 (No custom axioms in the backward direction)**.** The backward direction uses *no custom axioms*. EventualDecay is stated as a hypothesis, not imported via an axiom. The entire reduction from the oracle to the witness is constructive (BISH): the Archimedean step, the tail bound, and the bounded search are all pure BISH.

4.4 The Equivalence

Theorem 4.10 (MP ↔ EventualDecay)**.** ✓ *Over* BISH, *Markov's Principle is equivalent to EventualDecay:*

$$\text{MP} \quad \longleftrightarrow \quad \text{EventualDecay}. \tag{12}$$

Proof. Compose Theorems 4.7 and 4.8:

$$\text{MP} \xrightarrow{\text{Thm 4.7}} \text{EventualDecay} \xrightarrow{\text{Thm 4.8}} \text{MP}.$$

In Lean 4: `mp_iff_eventualDecay := ⟨eventualDecay_of_mp, mp_of_eventualDecay⟩`. □

```
/-- Theorem 6: MP <-> EventualDecay. -/
theorem mp_iff_eventualDecay :
    MarkovPrinciple <-> EventualDecay :=
  <eventualDecay_of_mp, mp_of_eventualDecay>
```

Listing 11: Main equivalence (PartB/PartB_Main.lean).

Remark 4.11 (Axiom certificate)**.** `#print axioms mp_iff_eventualDecay` shows `[propext, Classical.choice, Quot.sound, mp_real_of_mp]`. Exactly one custom axiom: `mp_real_of_mp`. No `llpo_real_of_llpo`. No `wlpo_real_of_wlpo`. No `bmc_iff_lpo`. This certifies that eventual decay costs exactly MP—not LLPO, not WLPO, not LPO.

5 The Stratification Theorem

Radioactive decay exhibits three distinct levels of the constructive hierarchy:

Level	Assertion	CRM Cost	Mechanism
1	Detection with known bound	BISH	Explicit $q > 0$
2	Eventual decay ($\lambda \neq 0$)	MP	Apartness from nonzero
3	LPO $\Rightarrow$ MP (hierarchy)	Pure logic	LPO decides outright

Theorem 5.1 (Stratification)**.** ✓ *Radioactive decay stratifies the constructive hierarchy:*

1. *Detection with an explicit lower bound is* BISH *(no custom axioms).*
2. *Eventual decay is equivalent to* MP *(uses* `mp_real_of_mp`*).*
3. LPO $\Rightarrow$ MP *is proved from first principles (no custom axioms).*

Moreover, MP *is independent of both* WLPO *and* LLPO*: neither implies the other over* BISH*. The constructive hierarchy is a partial order, not a linear chain.*

Proof. Items (1) and (2) are Theorems 3.4, 3.5 and 4.10. Item (3) is `lpo_implies_mp`: given LPO, we get $(\forall n,\ \alpha(n) = 0) \vee (\exists n,\ \alpha(n) = 1)$. Under the hypothesis $\neg(\forall n,\ \alpha(n) = 0)$, the first disjunct leads to contradiction, so $\exists n,\ \alpha(n) = 1$. The independence of MP from WLPO and LLPO is standard [Bridges and Richman, 1987, Bridges and Vîţă, 2006, Ishihara, 2006]. □

```
theorem decay_stratification :
    -- Level 1 (BISH): detection time works
    (forall lambda_ q eps : Real,
      0 < q -> q <= lambda_ -> 0 < eps -> eps < 1 ->
      survivalProb lambda_ (detectionTime eps q) <= eps) /\
    -- Level 2 (MP): main equivalence
    (MarkovPrinciple <-> EventualDecay) /\
    -- Level 3: LPO implies MP
    (LPO -> MarkovPrinciple) :=
  <fun lambda_ q eps => detection_time_works lambda_ q eps,
   mp_iff_eventualDecay,
   mp_of_lpo>
```

Listing 12: Stratification (Main/Stratification.lean).

The partial order structure of the constructive hierarchy, with MP branching off, is:

$$
\begin{array}{ccccc}
 & & \text{LPO} & & \\
 & \swarrow & \downarrow & \searrow & \\
\text{WLPO} & & \text{MP} & & \cdots \\
\downarrow & & & & \\
\text{LLPO} & & & & \\
\downarrow & & & & \\
\text{BISH} & & & &
\end{array} \tag{13}
$$

where LPO implies everything, WLPO implies LLPO but not MP, and MP does not imply WLPO or LLPO.

```
/-- LPO implies MP (trivial). -/
theorem lpo_implies_mp : LPO -> MarkovPrinciple := by
  intro hLPO alpha hne
  rcases hLPO alpha with hall | <n, hn>
  . exact absurd hall hne
  . exact <n, hn>

/-- LPO implies WLPO. -/
theorem lpo_implies_wlpo : LPO -> WLPO := by
  intro hLPO alpha
  rcases hLPO alpha with h_all | <n, hn>
  . exact Or.inl h_all
  . right; intro h_all
    exact absurd (h_all n) (by simp [hn])

/-- WLPO implies LLPO. -/
theorem wlpo_implies_llpo : WLPO -> LLPO := by
  intro hWLPO alpha hamo
  let beta : Nat -> Bool := fun n => alpha (2 * n)
  rcases hWLPO beta with h_all | h_not_all
  . exact Or.inl h_all
  . right; intro j; by_contra h; push_neg at h; simp at h
    apply h_not_all; intro k; by_contra hk
    push_neg at hk; simp at hk
    have := hamo (2 * k) (2 * j + 1) hk h; omega
```

Listing 13: Hierarchy proofs (Defs/Markov.lean).

6 Updated Calibration Table

The calibration table for the constructive reverse mathematics series, updated with Paper 22:

Paper	Physical System	Observable / Assertion	CRM Level	Position
2	Bidual gap (ℓ^1)	Gap witness $J - \kappa$	$\equiv$ WLPO	Main chain
7	Reflexive Banach ($S_1(H)$)	Non-reflexivity witness	$\equiv$ WLPO	Main chain
8	1D Ising model	Thermodynamic limit f_∞	$\equiv$ LPO	Main chain
15	Noether conservation	Global energy $E = \lim E_N$	$\equiv$ LPO	Main chain
19	WKB tunneling	Turning points (TPP)	$\equiv$ LLPO	Main chain
19	WKB tunneling	Full semiclassical	$\equiv$ LPO	Main chain
20	1D Ising model	Phase classification	$\equiv$ WLPO	Main chain
21	Bell / CHSH	Sign of Bell asymmetry	$\equiv$ LLPO	Main chain
22	**Radioactive decay**	**Eventual decay**	$\equiv$ MP	**Branch**

Paper 22 is the **first entry off the main chain**. All previous calibrations populated the linear hierarchy BISH < LLPO < WLPO < LPO. Paper 22 demonstrates that the physics itself has a partially ordered logical structure: MP branches off, implied by LPO but independent of WLPO and LLPO.

The pattern of the constructive hierarchy is now populated at every level and includes a branch point:

- BISH: Heisenberg uncertainty (Paper 6), CHSH bound (Paper 21, Part A), detection with known bounds (Paper 22, Part A).
- LLPO: WKB turning points (Paper 19), Bell sign decision (Paper 21).
- WLPO: Bidual gap (Paper 2), reflexive Banach (Paper 7), Ising phase classification (Paper 20).
- LPO: Ising free energy (Paper 8), Noether conservation (Paper 15), WKB full semiclassical (Paper 19).
- MP (**new branch**): Eventual decay (Paper 22).

7 Lean 4 Formalization

7.1 Module Structure

The formalization consists of 12 files organized in four directories:

Module	Content	Lines
`Defs/Markov.lean`	MP, LPO, WLPO, LLPO, hierarchy	105
`Defs/Decay.lean`	survivalProb, detectionTime, EventualDecay	41
`Defs/EncodedRate.lean`	λ_α, zero-iff, tail bound, witness	203
`PartA/DetectionTime.lean`	$T > 0$, $P(T, \lambda) \leq \varepsilon$, witness	100
`PartA/PartA_Main.lean`	Part A summary and audit	29
`PartB/Forward.lean`	MP $\Rightarrow$ EventualDecay	44
`PartB/Backward.lean`	EventualDecay $\Rightarrow$ MP (novel)	90
`PartB/PartB_Main.lean`	Main equivalence	25
`Main/Hierarchy.lean`	LPO $\Rightarrow$ MP (re-export)	35
`Main/Stratification.lean`	Three-level result	35
`Main/AxiomAudit.lean`	Comprehensive audit	103
`Main.lean`	Root imports	4
Total		**814**

Dependency graph:

```
Markov <-- Decay
  |         |
  +-- EncodedRate <---+
  |         |         |
  |    DetectionTime --+
  |         |
  |    PartA_Main
  |
  +-- Forward (axiom: mp_real_of_mp)
  |         |
  +-- Backward (no custom axioms)
  |         |
  +-- PartB_Main
  |
  +-- Hierarchy
  |
  +-- Stratification
  |
  +-- AxiomAudit <-- Main
```

7.2 Design Decisions

Single interface axiom. Only one CRM equivalence is axiomatized:

- `mp_real_of_mp : MarkovPrinciple` $\to \forall x : \mathbb{R}$, $0 \leq x \to x \neq 0 \to \exists q : \mathbb{Q}$, $0 < q \land q \leq x$ [Bridges and Richman, 1987, Bridges and Vîţă, 2006].

The axiom is used only in the forward direction (Theorem 4.7). The backward direction (Theorem 4.8) uses no custom axioms, making the reverse reduction fully constructive.

Non-negative form of MP for reals. The axiom uses the non-negative form ($0 \leq x$, $x \neq 0$ $\Rightarrow \exists q > 0$, $q \leq x$) rather than the general form ($x \neq 0 \Rightarrow |x| \# 0$). The non-negative restriction is cleaner for the physics: decay rates are non-negative.

Bool-valued sequences. Sequences are typed $\mathbb{N} \to$ Bool (not $\mathbb{N} \to \{0, 1\}$), matching LEAN 4's native Boolean type. This avoids cast coercions and simplifies case analysis.

EventualDecay stated inline in the backward direction. To avoid importing the axiom `mp_real_of_mp` (which lives in `Forward.lean`) into the backward proof, the EventualDecay hypothesis is stated inline. This ensures that `#print axioms mp_of_eventualDecay` shows no custom axioms, cleanly separating the BISH and MP content.

Self-contained bundle. Paper 22 is a standalone Lake package that re-declares MP, LPO, WLPO, and LLPO locally. The hierarchy proofs LPO $\Rightarrow$ MP, LPO $\Rightarrow$ WLPO, and WLPO $\Rightarrow$ LLPO are proved from first principles.

7.3 Axiom Audit

Theorem	Custom Axioms	Infrastructure	Tier
detectionTime_pos	None	propext, Classical.choice, Quot.sound	BISH
detection_time_works	None	propext, Classical.choice, Quot.sound	BISH
detection_with_witness	None	propext, Classical.choice, Quot.sound	BISH
partA_summary	None	propext, Classical.choice, Quot.sound	BISH
encodedRate_eq_zero_iff	None	propext, Classical.choice, Quot.sound	BISH
encodedRate_sub_partialRate_le	None	propext, Classical.choice, Quot.sound	BISH
witness_from_partial_sum_pos	None	propext, Classical.choice, Quot.sound	BISH
eventualDecay_of_mp	mp_real_of_mp	propext, Classical.choice, Quot.sound	MP
mp_of_eventualDecay	None	propext, Classical.choice, Quot.sound	— (hypothesis)
mp_iff_eventualDecay	mp_real_of_mp	propext, Classical.choice, Quot.sound	MP
decay_stratification	mp_real_of_mp	propext, Classical.choice, Quot.sound	MP
lpo_implies_mp	None	propext	BISH
lpo_implies_wlpo	None	propext	BISH
wlpo_implies_llpo	None	propext, Classical.choice, Quot.sound	BISH

```
-- Part A (BISH):
#print axioms detectionTime_pos
-- [propext, Classical.choice, Quot.sound]

#print axioms detection_time_works
-- [propext, Classical.choice, Quot.sound]

#print axioms detection_with_witness
-- [propext, Classical.choice, Quot.sound]

-- Encoded rate (BISH):
#print axioms encodedRate_eq_zero_iff
-- [propext, Classical.choice, Quot.sound]

-- Part B Forward (MP):
#print axioms eventualDecay_of_mp
-- [propext, Classical.choice, Quot.sound, mp_real_of_mp]

-- Part B Backward (no custom axioms!):
#print axioms mp_of_eventualDecay
-- [propext, Classical.choice, Quot.sound]

-- Main equivalence:
#print axioms mp_iff_eventualDecay
-- [propext, Classical.choice, Quot.sound, mp_real_of_mp]

-- Hierarchy (pure logic):
#print axioms lpo_implies_mp
-- [propext]

#print axioms wlpo_implies_llpo
-- [propext, Classical.choice, Quot.sound]
```

Listing 14: Axiom audit (Main/AxiomAudit.lean, selected).

7.4 CRM Compliance Protocol

The two-part structure is confirmed by machine:

- Part A theorems have **no custom axioms**—pure BISH.
- Part B forward depends on **exactly one** custom axiom (`mp_real_of_mp`)—MP level.
- Part B backward has **no custom axioms**—the reduction from EventualDecay to MP is fully constructive.
- The encoded rate lemmas have **no custom axioms**—the encoding is BISH.
- Hierarchy proofs (LPO $\Rightarrow$ MP, LPO $\Rightarrow$ WLPO, WLPO $\Rightarrow$ LLPO) have **no custom axioms**—pure BISH.
- `Classical.choice` in all results is a MATHLIB4 infrastructure artifact from `Real.exp`, `Real.log`, and `tsum`. The mathematical content of these proofs is constructive.

8 Discussion

8.1 The Partial Order Structure

The central structural contribution of this paper is the demonstration that the CRM programme's calibration table has a **partially ordered** structure, not merely a linear chain. All eight previous calibrations populated the chain BISH < LLPO < WLPO < LPO. If the programme had continued to find only linear calibrations, it would be natural to suspect the hierarchy is inherently linear—that the only non-trivial logical distinctions among physical assertions are those captured by the LLPO < WLPO < LPO chain.

Paper 22 refutes this suspicion. Eventual decay sits at MP, which is *independent* of both LLPO and WLPO. This means the physics itself has branching logical structure: there exist physical assertions whose constructive costs are incomparable. The step from "a nucleus is unstable" ($\neg(\lambda = 0)$) to "we can detect its decay" ($\exists T,\ P(T) < \varepsilon$) is logically orthogonal to both the sign-decision step (LLPO) and the zero-test step (WLPO).

8.2 Proton Decay and the Physics of MP

Proton decay in Grand Unified Theories provides a concrete physical instance of the MP gap. GUT predictions assert the proton is unstable [Tanabashi et al., 2018], but experimental searches [Super-Kamiokande Collaboration, 2020] have only established a lower bound on the proton lifetime exceeding 10^{34} years—no explicit decay rate is known.

The constructive content is precisely the MP gap:

- **What we have:** $\neg(\lambda_{\text{proton}} = 0)$ (the GUT predicts instability).
- **What we need:** $\exists q > 0,\ q \leq \lambda_{\text{proton}}$ (an explicit lower bound on the rate).
- **What** MP **provides:** If the GUT is correct, then such a bound exists—but MP does not compute it.

This is not a deficiency of current technology. It is a *logical* gap between theoretical prediction (negation of stability) and experimental detection (explicit bound on the rate). MP is the exact principle needed to bridge this gap.

8.3 Observable-Dependent Cost

The same geometric encoding $\lambda_\alpha = \sum \alpha(n) \cdot 2^{-(n+1)}$ has now been used to calibrate three different principles:

Paper	Question	Principle	Type
20	Is $\lambda_\alpha = 0$? (zero-test)	WLPO	Decidability
21	Is even $-$ odd ≤ 0? (sign-test)	LLPO	Sign decision
22	Is $\lambda_\alpha \# 0$? (apartness-test)	MP	Witness production

Same encoding, three different questions, three different principles. The logical cost depends on the *observable* (the question asked about the encoded quantity), not on the encoding itself. This pattern reinforces the programme's thesis: CRM calibration measures the logical cost of the *physical question*, not of the underlying mathematical object.

8.4 MP as the "Eventually Observe" Principle

Markov's Principle can be understood as the principle of *eventual observation*: if something is not impossible, it is eventually observed. In the context of radioactive decay:

- "Not impossible": $\neg(\lambda = 0)$—the nucleus is unstable.
- "Eventually observed": $\exists T > 0,\ P(T) < \varepsilon$—a detection time exists.

This is the constructive core of empiricism: the step from theoretical impossibility-of-impossibility to empirical witness production. MP is accepted in the Russian constructive tradition precisely because it formalizes a principle of scientific observation: if an event is not impossible, then a sufficiently patient observer will see it [Markov, 1954]. Radioactive decay is the cleanest physical instance of this principle.

8.5 Limitations

1. **Simple decay model.** The exponential model $P(t) = \exp(-\lambda t)$ is the simplest possible decay law. More complex models (multi-exponential, time-dependent rates) would obscure the MP content without adding logical information. The simplicity is a feature: the CRM content is in the non-constructivity of $\neg(\lambda = 0) \Rightarrow \lambda \# 0$, not in the physics of decay.

2. **No QFT formalization.** We do not formalize quantum field theory, Feynman diagrams, or any QFT machinery. The decay law $P(t) = \exp(-\lambda t)$ is treated as a given mathematical model.

3. **Separation proofs not formalized.** The independence of MP from WLPO and LLPO is a standard result [Bridges and Richman, 1987, Bridges and Vîţă, 2006] but is not formalized in LEAN 4. These separations require constructing specific topological models, which is beyond the scope of the current formalization.

4. **Classical.choice in MATHLIB4.** The appearance of `Classical.choice` in BISH results is a MATHLIB4 infrastructure artifact, not mathematical content. This is the same situation as in all previous papers in the series.

5. **Single axiom.** The interface axiom `mp_real_of_mp` is standard [Bridges and Richman, 1987, Bridges and Vîţă, 2006] but not yet formalized in MATHLIB4 from first principles. The backward direction (Theorem 4.8) requires no axiom, making the reverse reduction fully constructive.

9 Conclusion

The assertion that a radioactive nucleus with nonzero decay rate eventually decays is equivalent to Markov's Principle (MP) over Bishop's constructive mathematics. This is the first CRM calibration at the MP level, extending the programme's hierarchy from a linear chain to a partial order.

The result establishes a three-level stratification within radioactive decay:

- BISH: Detection with an explicit lower bound on the decay rate. The detection time $T = \ln(1/\varepsilon)/q$ is computable.
- MP: Eventual decay for a nonzero rate without an explicit bound. The step from $\neg(\lambda = 0)$ to $\exists q > 0,\ q \leq \lambda$ is exactly MP.
- LPO $\Rightarrow$ MP: The hierarchy placement, confirming that MP is implied by LPO.

The key insight is that the gap between "a nucleus is unstable" (theoretical negation: $\neg(\lambda = 0)$) and "we can detect its decay" (empirical witness: $\exists T,\ P(T) < \varepsilon$) has a precise constructive cost: exactly MP. This cost is independent of the main LLPO $<$ WLPO chain, showing that the physics of constructive logic has branching structure.

The calibration table now covers physical instantiations at every level of the constructive hierarchy and includes a branch point: BISH (Heisenberg, CHSH bound), LLPO (WKB turning points, Bell sign decision), WLPO (bidual gap, reflexive Banach, Ising phase), LPO (Ising free energy, Noether conservation, WKB semiclassical), and MP (eventual decay). The programme can now map both the linear chain and the branching structure of constructive mathematical physics.

Future work includes searching for additional branch points: principles between BISH and LPO that are independent of the main chain, intermediate principles between BISH and MP, and physical assertions that calibrate at levels not yet represented.

AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as previous papers in the series Lee [2026c,f,g,a,b,d,e].

The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalization strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Component	**Human**	**AI (Claude Opus 4.6)**
Research question	✓	
Physical setup (decay model)	✓	
CRM calibration strategy	✓	
Lean 4 implementation		✓
Proof strategies	collaborative	collaborative
LaTeX writeup		✓
Review and editing	✓	

Table 1: Division of labor between human and AI.

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/paul-c-k-lee/FoundationRelativity`
- **Path**: `paper 22/P22_MarkovDecay/`
- **Build**: `lake exe cache get && lake build` (0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Interface axiom**: `mp_real_of_mp` (MP $\to \forall x : \mathbb{R},\ 0 \leq x \to x \neq 0 \to \exists q : \mathbb{Q}$, $0 < q \wedge q \leq x$; Bridges and Richman [1987], Bridges and Vîţă [2006])
- **Axiom profile (Theorem 1, detectionTime_pos)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 2, detection_time_works)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 3, detection_with_witness)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 4, forward)**: `[propext, Classical.choice, Quot.sound, mp_real_of_mp]`
- **Axiom profile (Theorem 5, backward)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 6, main equiv)**: `[propext, Classical.choice, Quot.sound, mp_real_of_mp]`
- **Axiom profile (Theorem 7, stratification)**: `[propext, Classical.choice, Quot.sound, mp_real_of_mp]`
- **Total**: 12 files, 814 lines, 0 sorry
- **Zenodo DOI**: 10.5281/zenodo.18603503

Acknowledgments

The Lean 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the Mathlib4 community for maintaining the comprehensive library of formalized mathematics that made this work possible.

References

Errett Bishop. *Foundations of Constructive Analysis*. McGraw-Hill, New York, 1967.

Errett Bishop and Douglas S. Bridges. *Constructive Analysis*, volume 279 of *Grundlehren der mathematischen Wissenschaften*. Springer, 1985.

Douglas S. Bridges and Fred Richman. *Varieties of Constructive Mathematics*, volume 97 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1987.

Douglas S. Bridges and Luminita Simona Vîţă. *Techniques of Constructive Analysis*. Universitext. Springer, New York, 2006.

Hannes Diener. Constructive reverse mathematics. arXiv:1804.05495, 2020. Updated version.

Hajime Ishihara. Reverse mathematics in Bishop's constructive mathematics. *Philosophia Scientiae*, Cahier spécial(6):43–59, 2006.

Paul C.-K. Lee. Noether's theorem and the logical cost of global conservation laws. Preprint, 2026. Paper 15 in the constructive reverse mathematics series, 2026a.

Paul C.-K. Lee. The logical cost of quantum tunneling: LLPO and WKB turning points. Preprint, 2026. Paper 19 in the constructive reverse mathematics series, 2026b.

Paul C.-K. Lee. WLPO equivalence of the bidual gap in ℓ^1: a Lean 4 formalization. Preprint, 2026. Paper 2 in the constructive reverse mathematics series, 2026c.

Paul C.-K. Lee. Observable-dependent logical cost: WLPO and 1D Ising magnetization phase classification. Preprint, 2026. Paper 20 in the constructive reverse mathematics series, 2026d.

Paul C.-K. Lee. Bell nonlocality and the constructive cost of disjunction: an LLPO calibration. Preprint, 2026. Paper 21 in the constructive reverse mathematics series, 2026e.

Paul C.-K. Lee. Non-reflexivity of $s_1(h)$ implies WLPO: a Lean 4 formalization. Preprint, 2026. Paper 7 in the constructive reverse mathematics series, 2026f.

Paul C.-K. Lee. The logical cost of the thermodynamic limit: LPO-equivalence and BISH-dispensability for the 1D Ising free energy. Preprint, 2026. Paper 8 in the constructive reverse mathematics series, 2026g.

Andrei Andreevich Markov. *Theory of Algorithms*. Academy of Sciences of the USSR, Moscow, 1954. In Russian. English translation: Israel Program for Scientific Translations, 1961.

Super-Kamiokande Collaboration. Search for proton decay via $p \to e^+\pi^0$ and $p \to \mu^+\pi^0$ with an enlarged fiducial volume in Super-Kamiokande I–IV. *Physical Review D*, 102(11):112011, 2020.

Masaharu Tanabashi et al. Review of particle physics. *Physical Review D*, 98(3):030001, 2018. Particle Data Group.