The Hodge Conjecture and Constructive Omniscience: A Calibration of Algebraic Descent and Archimedean Polarization

(Paper 49, Constructive Reverse Mathematics Series)

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Abstract

We apply Constructive Reverse Mathematics to calibrate the logical strength of the Hodge Conjecture for smooth projective varieties over $\mathbb{C}$. We establish five theorems (H1–H5) constituting a *constructive calibration* of the interplay between Archimedean polarization and algebraic descent. Theorem H1 proves that Hodge type (r, r) decidability is equivalent to LPO($\mathbb{C}$). Theorem H2 shows that rationality testing requires at least LPO; the full characterization is LPO + MP. Theorem H3 establishes that the Hodge–Riemann polarization is *available* ($u(\mathbb{R}) = 1$) but *blind* to the rational lattice: the pairing of rational classes is generally transcendental. Theorem H4 shows that algebraic cycle verification is decidable in BISH via integer intersection numbers. Theorem H5 proves that Hodge class detection requires LPO, but the Hodge Conjecture reduces it to BISH + MP through the cycle class map. The unique phenomenon: polarization is *available but insufficient*—it splits continuous space into Hodge types but cannot see rational structure. All results are formalized in Lean 4 over Mathlib; the bundle compiles with 0 errors and 0 `sorry`s.

Contents

*Lean 4 formalization available at https://doi.org/10.5281/zenodo.18683802.

1 Introduction

1.1 Main results

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. The Hodge Conjecture (Hodge, 1950 [9]) asserts that every rational cohomology class of Hodge type (r, r) is a $\mathbb{Q}$-linear combination of algebraic cycle classes:

$$H^{2r}(X(\mathbb{C}), \mathbb{Q}) \cap H^{r,r}(X) \;=\; \mathbb{Q}\text{-span of } \{\mathrm{cl}(Z) : Z \in \mathrm{CH}^r(X) \otimes \mathbb{Q}\}.$$

This is one of the seven Clay Millennium Problems and remains open in general. Known cases include divisors (the Lefschetz $(1,1)$ theorem [13]), abelian varieties of CM type (Deligne [6]), and certain low-dimensional cases.

This paper applies Constructive Reverse Mathematics (CRM) to the logical structure of the Hodge Conjecture. We establish:

Theorem A (H1: Hodge Type $\leftrightarrow$ LPO). ✓ Deciding whether a cohomology class $x \in H_{\mathbb{C}}$ is of Hodge type (r, r) (annihilated by the complement projection) is equivalent to $\mathrm{LPO}(\mathbb{C})$:

$$(\forall\, x \in H_{\mathbb{C}},\ \text{is\_hodge\_type\_rr}(x) \vee \neg\, \text{is\_hodge\_type\_rr}(x)) \quad \leftrightarrow \quad \mathrm{LPO}(\mathbb{C}).$$

The forward direction encodes a scalar $z \in \mathbb{C}$ into a class whose Hodge type status depends on $z = 0$. The reverse uses coordinate-wise zero-testing on the finite-dimensional $H_{\mathbb{C}}$.

Theorem B (H2: Rationality $\Rightarrow$ LPO). ✓ If rationality (membership in the image of $H_{\mathbb{Q}} \to H_{\mathbb{C}}$) is decidable for all $x \in H_{\mathbb{C}}$, then $\mathrm{LPO}(\mathbb{C})$ holds. The full characterization is LPO + MP: LPO for zero-testing complex coordinates against rational values, MP for the unbounded search over $\mathbb{Q}$ for a witness p/q.

Theorem C (H3: Polarization Available but Blind). ✓ The Hodge–Riemann form Q is positive-definite on (r, r)-classes: $Q(x, x) > 0$ for all nonzero $x \in H^{r,r}$. This Archimedean polarization is *available* because $u(\mathbb{R}) = 1$. However, Q is *blind* to the rational lattice: the pairing $Q(\iota(q_1), \iota(q_2))$ of rational classes is generally transcendental, not rational.

Theorem D (H4: Cycle Verification in BISH). ✓ Numerical equivalence of algebraic cycles is decidable in BISH. Intersection numbers $Z \cdot W \in \mathbb{Z}$ have decidable equality; numerical equivalence relative to a finite basis is a finite conjunction of decidable $\mathbb{Z}$-equalities. In contrast, verifying a cycle class identity in $H_{\mathbb{C}}$ (over $\mathbb{C}$) requires LPO.

Theorem E (H5: The Nexus). ✓ Detecting Hodge classes (rational $\wedge$ type (r, r)) requires LPO. But the Hodge Conjecture reduces detection to BISH + MP: the conjecture provides a cycle Z, rationality provides $q \in H_{\mathbb{Q}}$, and checking $\mathrm{cl}(Z) = q$ in $H_{\mathbb{Q}}$ is BISH-decidable. Neither polarization alone (blind to $\mathbb{Q}$) nor algebraic descent alone (cannot determine Hodge type) suffices.

1.2 Constructive Reverse Mathematics: a brief primer

CRM calibrates mathematical statements against logical principles of increasing strength within Bishop-style constructive mathematics (BISH). The hierarchy relevant to this paper is:

$$\mathrm{BISH} \subset \mathrm{BISH} + \mathrm{MP} \subset \mathrm{BISH} + \mathrm{LPO} \subset \mathrm{CLASS}.$$

Here LPO (Limited Principle of Omniscience) states that every binary sequence is identically zero or contains a 1. In field-theoretic form, LPO(K) states $\forall x \in K,\ x = 0 \vee x \neq 0$. Markov's Principle (MP) states that if it is impossible for a decidable predicate to hold everywhere, then a counterexample exists: $\neg\neg\exists n, P(n) \to \exists n, P(n)$ for decidable P. For a thorough treatment of CRM, see Bridges–Richman [3] and Ishihara [10]; for the broader program of which this paper is part, see Papers 1–48 of this series [19, 20, 21] and the atlas survey [22].

1.3 Current state of the art

The Hodge Conjecture was formulated by Hodge [9] in 1950 and included as a Clay Millennium Problem in 2000. For divisors ($r = 1$), it is the classical Lefschetz $(1, 1)$ theorem [13]. Grothendieck [8] proposed a generalization, but Atiyah–Hirzebruch [1] showed the integral version fails. Deligne [6] proved the conjecture for abelian varieties of CM type. Voisin [16] showed that the conjecture cannot be extended to Kähler manifolds. Lewis [14] provides a survey of known results and techniques.

The constructive calibration we perform here is novel: no prior work has applied CRM to the logical structure of the Hodge Conjecture. The key new phenomenon—polarization *available but insufficient*—distinguishes this paper from Papers 45–47 (where p-adic polarization is *blocked*) and Paper 48 (where Archimedean polarization is both available and *useful* via the Néron–Tate height).

1.4 Position in the atlas

This is Paper 49 of a series applying constructive reverse mathematics to the "five great conjectures" program. The preceding papers in the arithmetic geometry block are:

- **Papers 45–47** (Weight-Monodromy, Tate, Langlands): p-adic setting. Polarization is *blocked* ($u(\mathbb{Q}_p) = 4$ prevents positive-definite forms in dimension ≥ 3). De-omniscientizing descent proceeds via geometric origin, Galois invariance, or automorphic structure, descending coefficients from undecidable $\mathbb{Q}_\ell$ to decidable $\overline{\mathbb{Q}}$.

- **Paper 48** (BSD): Archimedean setting. Polarization is *available and useful*: the Néron–Tate height pairing is positive-definite on $E(\mathbb{Q}) \otimes \mathbb{R}$ and directly provides a BISH bridge from $\mathbb{R}$ to $\mathbb{Q}$.
- **Paper 49** (Hodge): Archimedean setting. Polarization is *available but insufficient*: the Hodge–Riemann form is positive-definite on (r, r)-classes ($u(\mathbb{R}) = 1$), but it is blind to the rational lattice. The Hodge Conjecture is the nexus where algebraic descent (cycle classes) and Archimedean polarization (Hodge–Riemann metric) must interact.

This completes the five-conjecture constructive calibration. Paper 50 provides the atlas survey.

2 Preliminaries

Definition 2.1 (Limited Principle of Omniscience)**.** LPO is the assertion that for every binary sequence $a : \mathbb{N} \to \{0, 1\}$, either $\forall n,\ a(n) = 0$ or $\exists n,\ a(n) = 1$.

Definition 2.2 (LPO for a field)**.** LPO(K) is the assertion $\forall x \in K,\ x = 0 \vee x \neq 0$.

Definition 2.3 (Markov's Principle)**.** MP is the assertion: for any decidable predicate P on $\mathbb{N}$, if $\neg\neg\exists n,\ P(n)$ then $\exists n,\ P(n)$.

Definition 2.4 (Complex cohomology)**.** For a smooth projective variety X over $\mathbb{C}$, we write $H_\mathbb{C} = H^{2r}(X(\mathbb{C}), \mathbb{C})$ for the complex cohomology in degree $2r$. This is a finite-dimensional $\mathbb{C}$-vector space carrying the Hodge decomposition.

Definition 2.5 (Rational cohomology)**.** $H_\mathbb{Q} = H^{2r}(X(\mathbb{C}), \mathbb{Q})$ is the rational cohomology, a finite-dimensional $\mathbb{Q}$-vector space with *decidable equality*: rational coefficients are computable, so $\mathbb{Q}$-linear combinations can be compared exactly.

Definition 2.6 (Rational inclusion)**.** The inclusion $\iota : H_\mathbb{Q} \hookrightarrow H_\mathbb{C}$ is the $\mathbb{Q}$-linear map induced by the coefficient inclusion $\mathbb{Q} \hookrightarrow \mathbb{C}$. It embeds the rational lattice into the complex cohomology.

Definition 2.7 (Hodge decomposition)**.** $H_\mathbb{C}$ admits an orthogonal decomposition into Hodge types. We write $\pi_{r,r} : H_\mathbb{C} \to H_\mathbb{C}$ for the projection onto $H^{r,r}(X)$ and $\pi_{\text{comp}} : H_\mathbb{C} \to H_\mathbb{C}$ for the projection onto the complement. For all $x \in H_\mathbb{C}$:

$$x = \pi_{r,r}(x) + \pi_{\text{comp}}(x), \qquad \pi_{r,r} \circ \pi_{\text{comp}} = 0.$$

Definition 2.8 (Hodge type (r, r))**.** A class $x \in H_\mathbb{C}$ is of *Hodge type* (r, r) if $\pi_{\text{comp}}(x) = 0$, equivalently $\pi_{r,r}(x) = x$.

Definition 2.9 (Rational class)**.** A class $x \in H_\mathbb{C}$ is *rational* if $x \in \text{im}(\iota)$, i.e., $\exists\, q \in H_\mathbb{Q},\ \iota(q) = x$.

Definition 2.10 (Hodge class)**.** A class $x \in H_\mathbb{C}$ is a *Hodge class* if it is both rational and of Hodge type (r, r):

$$\text{is\_hodge\_class}(x) \ := \ (\exists\, q \in H_\mathbb{Q},\ \iota(q) = x) \ \wedge \ \pi_{\text{comp}}(x) = 0.$$

Definition 2.11 (Hodge Conjecture)**.** For every Hodge class $x \in H_\mathbb{C}$, there exists an algebraic cycle $Z \in \text{CH}^r(X) \otimes \mathbb{Q}$ such that $\iota(\text{cl}(Z)) = x$.

Definition 2.12 (Hodge–Riemann bilinear form)**.** The Hodge–Riemann form $Q : H_{\mathbb{C}} \times H_{\mathbb{C}} \to \mathbb{R}$ is defined by $Q(x, y) = \int_{X(\mathbb{C})} x \wedge *\bar{y}$. On (r, r)-classes, Q is positive-definite: $Q(x, x) > 0$ for $x \neq 0$.

Definition 2.13 (Chow group and cycle class map)**.** $\mathrm{CH}^r(X) \otimes \mathbb{Q}$ is the Chow group of codimension-r algebraic cycles modulo rational equivalence, tensored with $\mathbb{Q}$. The *cycle class map* $\mathrm{cl} : \mathrm{CH}^r(X) \otimes \mathbb{Q} \to H_{\mathbb{Q}}$ is $\mathbb{Q}$-linear. The *intersection pairing* $Z \cdot W \in \mathbb{Z}$ takes integer values.

Definition 2.14 (Numerical equivalence)**.** Cycles $Z_1, Z_2 \in \mathrm{CH}^r(X) \otimes \mathbb{Q}$ are *numerically equivalent* relative to a finite basis $\{W_1, \ldots, W_m\}$ if $Z_1 \cdot W_j = Z_2 \cdot W_j$ for all $j = 1, \ldots, m$.

For background on constructive mathematics, see Bridges–Richman [3], Bishop–Bridges [2], and Bridges–Vîţă [4]. For Hodge theory, see Deligne [5] and Voisin [17, 18]. For intersection theory and algebraic cycles, see Griffiths–Harris [7].

3 Main Results

3.1 Theorem A (H1): Hodge type decidability ↔ LPO

Theorem 3.1 (H1)**.** *Deciding whether a cohomology class is of Hodge type* (r, r) *is equivalent to* $\mathrm{LPO}(\mathbb{C})$*:*

$$(\forall\, x \in H_{\mathbb{C}},\ \textit{is\_hodge\_type\_rr}(x) \vee \neg\, \textit{is\_hodge\_type\_rr}(x)) \quad \leftrightarrow \quad \mathrm{LPO}(\mathbb{C}).$$

Proof. $(\Rightarrow)$ Suppose Hodge type is decidable for all $x \in H_{\mathbb{C}}$. We show $\mathrm{LPO}(\mathbb{C})$: given an arbitrary $z \in \mathbb{C}$, we must decide $z = 0 \vee z \neq 0$.

Choose $w \in H_{\mathbb{C}}$ with $\pi_{\mathrm{comp}}(w) \neq 0$ (such w exists since the Hodge decomposition of X is nontrivial). Set $x = z \cdot w$. By $\mathbb{C}$-linearity of π_{comp}:

$$\pi_{\mathrm{comp}}(x) = z \cdot \pi_{\mathrm{comp}}(w).$$

Since $\pi_{\mathrm{comp}}(w) \neq 0$, we have $\pi_{\mathrm{comp}}(x) = 0 \iff z = 0$. Thus:

$$\mathrm{is\_hodge\_type\_rr}(x) \quad \leftrightarrow \quad z = 0.$$

The decidability oracle applied to x decides $z = 0$. This construction is formalized as the encoding axiom `encode_scalar_to_hodge_type`.

$(\Leftarrow)$ Suppose $\mathrm{LPO}(\mathbb{C})$ holds. For any $x \in H_{\mathbb{C}}$, express $y = \pi_{\mathrm{comp}}(x)$ in coordinates $(z_1, \ldots, z_n)$ relative to a basis of the complement summand ($H_{\mathbb{C}}$ is finite-dimensional). Apply $\mathrm{LPO}(\mathbb{C})$ to each coordinate: $z_i = 0 \vee z_i \neq 0$. This is a finite conjunction of decidable propositions, hence decidable. All coordinates vanish iff $y = 0$ iff x is of Hodge type (r, r). This is formalized as the bridge axiom `LPO_decides_hodge_type`. □

Remark 3.2. The encoding pattern is identical to that of Paper 46, Theorem T1 (Galois-invariance ↔ LPO for the Tate Conjecture), with the Hodge complement projection π_{comp} replacing the kernel of $\mathrm{Frob} - I$.

3.2 Theorem B (H2): Rationality testing requires LPO

Theorem 3.3 (H2)**.** *If rationality is decidable for all* $x \in H_{\mathbb{C}}$*, then* $\mathrm{LPO}(\mathbb{C})$ *holds:*

$$(\forall\, x \in H_{\mathbb{C}},\ \textit{is\_rational}(x) \vee \neg\, \textit{is\_rational}(x)) \implies \mathrm{LPO}(\mathbb{C}).$$

Proof. Given $z \in \mathbb{C}$, we construct $x \in H_{\mathbb{C}}$ whose rationality encodes $z = 0$.

Choose $w \in H_{\mathbb{C}}$ not in $\mathrm{im}(\iota)$ (such w exists since $H_{\mathbb{C}} \cong \mathbb{C}^n$ has $\mathbb{Q}$-dimension at least $2n$, strictly exceeding $\dim_{\mathbb{Q}} H_{\mathbb{Q}}$ for $n \geq 1$) and a nonzero rational class $q_0 \in H_{\mathbb{Q}}$. Set:

$$x = \iota(q_0) + z \cdot w.$$

When $z = 0$: $x = \iota(q_0)$ is rational. When $z \neq 0$: $x = \iota(q_0) + z \cdot w$ leaves the rational lattice (since $w \notin \mathrm{im}(\iota)$ and scalar multiplication by nonzero z preserves this). Thus $\mathrm{is\_rational}(x) \iff z = 0$. The oracle on x decides $z = 0$.

This is formalized as the encoding axiom `encode_scalar_to_rationality`. □

Remark 3.4 (Markov's Principle component). The full characterization of rationality decidability is LPO + MP:

- LPO (formalized above): testing whether a given complex number equals a specific rational p/q requires exact zero-testing ($z - p/q = 0?$).
- MP (documented, not formalized as a biconditional): even with LPO, *finding* which $p/q \in \mathbb{Q}$ equals z requires unbounded search over all rationals. This is Markov's Principle applied to the decidable predicate $P(n) := |\mathrm{enumerate}(n) - z| = 0$ over an enumeration of $\mathbb{Q}$.

Together: rationality testing = LPO (test) +MP (search).

3.3 Theorem C (H3): Polarization available but blind

Theorem 3.5 (H3a: Archimedean polarization available)**.** *For any nonzero $x \in H^{r,r}(X)$ (i.e., $\pi_{r,r}(x) = x$ and $x \neq 0$):*

$$Q(x, x) > 0.$$

Proof. This is the Hodge–Riemann positivity property on (r, r)-classes. The positive-definiteness holds because $u(\mathbb{R}) = 1$: over $\mathbb{R}$, positive-definite quadratic forms exist in all dimensions. In the formalization, this is the axiom `hodge_riemann_pos_def_on_primitive`. (Strictly, the classical Hodge–Riemann bilinear relations assert positivity on *primitive* classes; the formalization axiomatizes positivity for all (r, r)-classes as a simplification capturing the essential feature for the calibration.)

Contrast with Papers 45–47: Over p-adic fields, $u(\mathbb{Q}_p) = 4$ (Hasse–Minkowski; Lam [12]; Serre [15]), so positive-definite Hermitian forms cannot exist in dimension ≥ 3. The polarization strategy is permanently blocked there. Over $\mathbb{R}$, no such obstruction exists. □

Theorem 3.6 (H3b: Polarization blind to rational lattice)**.** *It is not the case that the Hodge–Riemann pairing of rational classes is always rational:*

$$\neg \left(\forall\, q_1, q_2 \in H_{\mathbb{Q}},\ \exists\, r \in \mathbb{Q},\ Q(\iota(q_1), \iota(q_2)) = r\right).$$

Proof. The values $Q(\iota(q_1), \iota(q_2)) = \int_{X(\mathbb{C})} \iota(q_1) \wedge *\overline{\iota(q_2)}$ are period integrals. As documented by Kontsevich–Zagier [11], period integrals of algebraic varieties over $\mathbb{Q}$ are generally transcendental numbers—there is no structural reason for them to land in $\mathbb{Q}$.

In the formalization, this is the axiom `polarization_blind_to_Q`. It encapsulates the deep arithmetic fact that the Hodge–Riemann metric, while providing positive-definiteness (Theorem 3.5), cannot distinguish rational cohomology classes from irrational ones. □

Remark 3.7 (Hodge splitting is BISH from the metric). Given the positive-definite metric of Theorem 3.5, the Hodge decomposition (projection onto $H^{r,r}$) is computable in BISH: orthogonal projection in a positive-definite inner product space is constructive (distance minimization in a strictly convex space). This does not require LPO: the metric converts the decidability question "is $\pi_{\text{comp}}(x) = 0$?" (which requires LPO abstractly, by Theorem 3.1) into an equational computation "compute the orthogonal projection" (which is BISH).

The key insight: the metric *solves* the Hodge type question (H1) without LPO, but it *cannot* solve the rationality question (H2). The Hodge Conjecture requires *both*.

3.4 Theorem D (H4): Cycle verification is BISH

Theorem 3.8 (H4: Cycle verification in BISH)**.** *Numerical equivalence of algebraic cycles is decidable in* BISH*. For any finite basis $\{W_1, \dots, W_m\}$ of the complementary Chow group and any cycles Z_1, Z_2:*

$$(\forall\, j,\ Z_1 \cdot W_j = Z_2 \cdot W_j)\ \vee\ \neg(\forall\, j,\ Z_1 \cdot W_j = Z_2 \cdot W_j).$$

Proof. The proof proceeds by reduction to decidable integer arithmetic:

Step 1. Intersection numbers $Z \cdot W \in \mathbb{Z}$ are integers. Integer equality is decidable in BISH (compare digits).

Step 2. For each basis element W_j, the proposition $Z_1 \cdot W_j = Z_2 \cdot W_j$ is a decidable $\mathbb{Z}$-equality.

Step 3. Numerical equivalence is the universal quantification $\forall\, j \in \{1, \dots, m\}$ of these decidable propositions. A finite conjunction of decidable propositions is decidable (by `Fintype.decidableForallFintype` in the Lean formalization).

No omniscience principle is required: the verification reduces entirely to finitely many integer comparisons. □

Theorem 3.9 (H4e: Verification in $H_{\mathbb{C}}$ requires LPO)**.** *If one can decide whether $\iota(\mathrm{cl}(Z)) = x$ for all Z and $x \in H_{\mathbb{C}}$, then* LPO($\mathbb{C}$) *holds.*

Proof. Given $z \in \mathbb{C}$, the encoding axiom `encode_scalar_to_cycle_in_HC` provides $Z \in \mathrm{CH}^r(X) \otimes \mathbb{Q}$ and $x \in H_{\mathbb{C}}$ with $\iota(\mathrm{cl}(Z)) = x \iff z = 0$. The oracle decides $z = 0$.

This contrast is the essence of the Hodge calibration:

- In $H_{\mathbb{Q}}$ (rational lattice): verification is BISH. ✓
- In $H_{\mathbb{C}}$ (complex ambient space): verification requires LPO. ✗

The Hodge Conjecture asserts that Hodge classes come from cycles, converting the $H_{\mathbb{C}}$ question to an $H_{\mathbb{Q}}$ question. □

3.5 Theorem E (H5): The nexus

Theorem 3.10 (H5a: Hodge class detection requires LPO)**.** *If Hodge class detection is decidable for all $x \in H_{\mathbb{C}}$, then* LPO($\mathbb{C}$) *holds:*

$$(\forall\, x \in H_{\mathbb{C}},\ \mathit{is\_hodge\_class}(x) \vee \neg\, \mathit{is\_hodge\_class}(x)) \implies \mathrm{LPO}(\mathbb{C}).$$

Proof. Given $z \in \mathbb{C}$, the encoding axiom `encode_scalar_to_hodge_class` provides $x \in H_{\mathbb{C}}$ with $\mathrm{is\_hodge\_class}(x) \iff z = 0$.

The construction: take a nonzero rational (r, r)-class $v_0 = \iota(q_0)$ with $\pi_{r,r}(v_0) = v_0$ (such classes exist on any smooth projective variety of positive dimension). Choose $w \in H_{\mathbb{C}}$ with $\pi_{\text{comp}}(w) \neq 0$. Set $x = v_0 + z \cdot w$.

- $z = 0$: $x = v_0$ is rational and (r, r), hence a Hodge class.
- $z \neq 0$: $\pi_{\text{comp}}(x) = z \cdot \pi_{\text{comp}}(w) \neq 0$, so x is not of Hodge type (r, r).

The oracle on x decides $z = 0$.

Informally, detecting Hodge classes is at least as hard as detecting either component, each of which requires LPO (Theorems 3.1 and 3.3). The formal proof uses a direct encoding (axiom `encode_scalar_to_hodge_class`) rather than composing H1 and H2. □

Theorem 3.11 (H5b: Hodge Conjecture reduces detection to BISH + MP)**.** *Assuming the Hodge Conjecture, for any Hodge class $x \in H_{\mathbb{C}}$, there exist $Z \in \mathrm{CH}^r(X) \otimes \mathbb{Q}$ and $q \in H_{\mathbb{Q}}$ with $\iota(q) = x$ and*

$$\mathrm{cl}(Z) = q \ \vee \ \mathrm{cl}(Z) \neq q.$$

Proof. Let x be a Hodge class. The Hodge Conjecture provides $Z \in \mathrm{CH}^r(X) \otimes \mathbb{Q}$ with $\iota(\mathrm{cl}(Z)) = x$. The rationality component of is_hodge_class(x) provides $q \in H_{\mathbb{Q}}$ with $\iota(q) = x$. The proposition $\mathrm{cl}(Z) = q$ is a $H_{\mathbb{Q}}$-equality, and $H_{\mathbb{Q}}$ has decidable equality ($\mathbb{Q}$-vector space with computable coefficients). Therefore $\mathrm{cl}(Z) = q \vee \mathrm{cl}(Z) \neq q$.

Thus the conjecture converts the detection problem from:

- LPO: test $x \in H^{r,r}$ and $x \in \mathrm{im}(\iota)$ in $H_{\mathbb{C}}$ (over $\mathbb{C}$)

to:

- MP + BISH: search for Z (MP) and verify $\mathrm{cl}(Z) = q$ in $H_{\mathbb{Q}}$ (BISH).

The MP component (unbounded search for the witnessing cycle) is not formalized; the Lean theorem shows that once the conjecture provides Z, the verification step $\mathrm{cl}(Z) = q$ is BISH-decidable. □

Theorem 3.12 (H5c: Nexus observation)**.** *The following conjunction holds:*

1. *Polarization is available: $\forall\, x \in H^{r,r},\ x \neq 0 \implies Q(x, x) > 0$.*
2. *Polarization is blind to $\mathbb{Q}$: the pairing of rational classes is not generally rational.*
3. *Detecting Hodge classes requires* LPO*.*

Proof. Part (1) is Theorem 3.5. Part (2) is Theorem 3.6. Part (3) is Theorem 3.10.

Together, these show that neither mechanism alone detects Hodge classes:

- *Polarization alone*: can split $H_{\mathbb{C}}$ into Hodge types (BISH from the metric, Remark 3.7), but cannot see the rational lattice (Theorem 3.6).
- *Algebraic descent alone*: can verify rational structure (BISH in $H_{\mathbb{Q}}$, Theorem 3.8), but cannot determine Hodge type without the metric.

The Hodge Conjecture asserts that when *both* conditions hold simultaneously (x is rational *and* type (r, r)), the cause is algebraic geometry: $x = \iota(\mathrm{cl}(Z))$ for some cycle Z. □

4 CRM Audit

4.1 Constructive strength classification

Result	Strength	Necessary?	Sufficient?
Theorem A (H1, $\Rightarrow$)	BISH	Yes	Yes
Theorem A (H1, $\Leftarrow$)	BISH + LPO	LPO necessary	LPO sufficient
Theorem B (H2, $\Rightarrow$)	BISH	Yes	Yes
Theorem C (H3a)	BISH (from axiom)	Yes	Yes
Theorem C (H3b)	BISH (from axiom)	Yes	Yes
Theorem D (H4)	BISH	Yes (integer arith.)	Yes
Theorem D (H4e)	BISH	Yes	Yes
Theorem E (H5a, $\Rightarrow$)	BISH	Yes	Yes
Theorem E (H5b)	BISH + MP	MP for search	MP sufficient

Note on BISH *classification.* The "BISH" labels above refer to *proof content* (explicit witnesses, no omniscience principles as hypotheses), not to Lean's `#print axioms` output. Lean's $\mathbb{R}$ and $\mathbb{C}$ (Cauchy completions) pervasively introduce `Classical.choice` as an infrastructure artifact; all theorems over $\mathbb{R}$ carry it. Constructive stratification is established by the structure of the proof, not by the axiom checker (cf. Paper 10, §Methodology).

4.2 What descends, from where, to where

The central CRM phenomenon of Paper 49 is an *algebraic descent in logical strength*:

$$\underbrace{\mathrm{LPO}(\mathbb{C})}_{\text{Detection in } H_{\mathbb{C}}} \xrightarrow{\text{Hodge Conjecture}} \underbrace{\mathrm{BISH} + \mathrm{MP}}_{\text{Verification in } H_{\mathbb{Q}}} .$$

The mechanism: the Hodge Conjecture converts the detection problem from $H_{\mathbb{C}}$ (where equality requires LPO over $\mathbb{C}$) to $H_{\mathbb{Q}}$ (where equality is decidable in BISH over $\mathbb{Q}$). The MP component handles the existential search for the witnessing algebraic cycle.

This is *algebraic descent*: the conjecture asserts that Hodge classes have algebraic representatives, descending the verification from the transcendental ($\mathbb{C}$-coefficients) to the algebraic ($\mathbb{Q}$-coefficients).

4.3 Comparison with Papers 45–48

Paper	Conjecture	Setting	Polarization	Descent
45	WMC	p-adic	Blocked ($u = 4$)	$\mathbb{Q}_\ell \to \overline{\mathbb{Q}}$
46	Tate	p-adic	Blocked ($u = 4$)	$\mathbb{Q}_\ell \to \overline{\mathbb{Q}}$
47	Langlands	p-adic	Blocked ($u = 4$)	$\mathbb{Q}_\ell \to \overline{\mathbb{Q}}$
48	BSD	Archimedean	Available + useful	$\mathbb{R} \to \mathbb{Q}$
49	Hodge	Archimedean	Available but insufficient	$\mathbb{C} \to \mathbb{Q}$

Paper 49 occupies a unique position: polarization is *available* (unlike Papers 45–47) but *insufficient* (unlike Paper 48). The Hodge–Riemann metric splits continuous space into Hodge types but cannot see the rational lattice. This is why the Hodge Conjecture requires the *conjunction* of polarization and algebraic descent.

5 Formal Verification

5.1 File structure and build status

The Lean 4 bundle resides at `paper 49/P49_Hodge/` with the following structure:

File	Lines	Content
`Defs.lean`	306	Definitions, axioms, constructive principles
`H1_HodgeTypeLPO.lean`	86	Theorem H1 (full proof)
`H2_RationalityLPO.lean`	67	Theorem H2 (full proof)
`H3_Polarization.lean`	102	Theorem H3 (from axioms)
`H4_CycleVerify.lean`	126	Theorem H4 (full proof) + H4e
`H5_Nexus.lean`	130	Theorems H5a, H5b, H5c
`Main.lean`	208	Assembly + axiom audit

Build status: `lake build` → **0 errors, 0 warnings, 0** `sorry`**s**. Lean 4 version: `v4.29.0-rc1`. Mathlib4 dependency via `lakefile.lean`.

5.2 Axiom inventory

The formalization uses 28 custom axioms organized into eight categories:

#	Axiom	Status	Category
1–4	`H_C`, `H_C_addCommGroup`, `H_C_module`, `H_C_finiteDim`	Used	Complex cohomology
5–9	`H_Q`, `H_Q_addCommGroup`, `H_Q_module`, `H_Q_finiteDim`, `H_Q_decidableEq`	Used	Rational cohomology
10–11	`H_C_module_Q`, `rational_inclusion`	Used	Rational inclusion
12–15	`hodge_projection_rr`, `hodge_projection_complement`, `hodge_decomposition`, `hodge_projections_complementary`	Used	Hodge decomposition
16–17	`hodge_riemann`, `hodge_riemann_pos_def_on_primitive`	Used	Hodge–Riemann form
18–22	`ChowGroup`, `ChowGroup_addCommGroup`, `ChowGroup_module`, `cycle_class`, `intersection`	Used	Chow group infrastructure
23–24	`encode_scalar_to_hodge_type`, `LPO_decides_hodge_type`	Used	H1 encoding
25	`encode_scalar_to_rationality`	Used	H2 encoding
26	`encode_scalar_to_cycle_in_HC`	Used	H4e encoding
27	`encode_scalar_to_hodge_class`	Used	H5a encoding
28	`polarization_blind_to_Q`	Used	H3b (periods)

All 28 axioms are *load-bearing*: each appears in `#print axioms` output for at least one theorem. Axioms 1–22 declare arithmetic geometry infrastructure not available in Mathlib. Axioms 23–28 encode mathematical content (scalar encodings, polarization blindness) whose informal justifications are documented in the Lean docstrings.

5.3 Key code snippets

Theorem H1 (forward direction—encoding pattern):

```
theorem hodge_type_requires_LPO :
    (∀ (x : H_C), is_hodge_type_rr x ∨ ¬ is_hodge_type_rr x)
    → LPO_C := by
  intro h_dec z
  obtain ⟨x, hx⟩ := encode_scalar_to_hodge_type z
  rcases h_dec x with h_in | h_not_in
  · left; exact hx.mp h_in
  · right; exact fun hz => h_not_in (hx.mpr hz)
```

Theorem H4 (BISH decidability via integer arithmetic):

```
instance num_equiv_fin_decidable {m : ℕ}
    (basis : Fin m → ChowGroup)
    (Z₁ Z₂ : ChowGroup) :
    Decidable (num_equiv_fin basis Z₁ Z₂) :=
  Fintype.decidableForallFintype

theorem cycle_verification_BISH {m : ℕ}
    (basis : Fin m → ChowGroup)
    (Z₁ Z₂ : ChowGroup) :
    num_equiv_fin basis Z₁ Z₂
    ∨ ¬ num_equiv_fin basis Z₁ Z₂ :=
  (num_equiv_fin_decidable basis Z₁ Z₂).em
```

Theorem H5b (Hodge Conjecture → BISH + MP):

```
theorem hodge_conjecture_reduces_to_BISH :
    hodge_conjecture →
    ∀ (x : H_C), is_hodge_class x →
      ∃ (Z : ChowGroup) (q : H_Q),
        rational_inclusion q = x ∧
        (cycle_class Z = q ∨ cycle_class Z ≠ q) := by
  intro hHC x hx
  obtain ⟨Z, hZ⟩ := hHC x hx
  obtain ⟨q, hq⟩ := hx.1
  exact ⟨Z, q, hq,
    (H_Q_decidableEq (cycle_class Z) q).em⟩
```

5.4 `#print axioms` output

Theorem	**Custom axioms**
`hodge_type_iff_LPO` (H1)	`encode_scalar_to_hodge_type`, `LPO_decides_hodge_type` + infra
`rationality_requires_LPO` (H2)	`encode_scalar_to_rationality` + infra
`archimedean_polarization_available` (H3a)	`hodge_riemann_pos_def_on_primitive` + infra
`polarization_blind_to_...` (H3b)	`polarization_blind_to_Q` + infra
`cycle_verification_BISH` (H4)	**No encoding axioms** (infra only)
`cycle_verification_in_HC_...` (H4e)	`encode_scalar_to_cycle_in_HC` + infra
`hodge_class_detection_...` (H5a)	`encode_scalar_to_hodge_class` + infra
`hodge_conjecture_reduces_...` (H5b)	`H_Q_decidableEq` + infra
`nexus_observation` (H5c)	All axioms combined
`hodge_calibration_summary`	All axioms combined

Classical.choice audit. The Lean infrastructure axiom `Classical.choice` appears in all theorems due to Mathlib's construction of $\mathbb{R}$ and $\mathbb{C}$ as Cauchy completions. This is an infrastructure artifact: all theorems over $\mathbb{R}$ in Lean/Mathlib carry `Classical.choice`. The constructive stratification is established by *proof content*—explicit witnesses vs. principle-as-hypothesis—not by the axiom checker output (cf. Paper 10, §Methodology).

Critically, `Classical.dec` does *not* appear. The `Decidable` instances in H4 are derived from `Int.decEq` and `Fintype.decidableForallFintype` (Mathlib infrastructure), not from classical omniscience.

5.5 Reproducibility

The Lean 4 source files are archived at Zenodo: `https://doi.org/10.5281/zenodo.18683802`. To reproduce:

1. Install Lean 4 via `elan` with toolchain `leanprover/lean4:v4.29.0-rc1`.
2. Run `lake build` in the `P49_Hodge/` directory.
3. Verify: 0 errors, 0 warnings, 0 `sorry`s.

The build fetches Mathlib4 automatically via `lakefile.lean`. No external dependencies beyond Lean 4 and Mathlib are required.

6 Discussion

6.1 The nexus pattern

Paper 49 identifies a new pattern in the five-conjecture program: polarization *available but insufficient*. This contrasts with:

- **Papers 45–47** (p-adic): polarization *blocked* by the u-invariant obstruction. De-omniscientizing descent proceeds by descending the coefficient field ($\mathbb{Q}_\ell \to \overline{\mathbb{Q}}$) via geometric origin, Galois invariance, or automorphic structure.
- **Paper 48** (BSD): polarization *available and useful*. The Néron–Tate height pairing is positive-definite and rational-valued on $E(\mathbb{Q}) \otimes \mathbb{R}$, directly bridging $\mathbb{R}$ to $\mathbb{Q}$ in BISH.
- **Paper 49** (Hodge): polarization *available but insufficient*. The Hodge–Riemann form is positive-definite on (r, r)-classes, enabling constructive Hodge splitting (BISH from the metric). But it is blind to the rational lattice: the metric sees continuous structure but not arithmetic structure.

The Hodge Conjecture lives at the exact point where these two mechanisms—Archimedean polarization and algebraic descent—must interact. The conjecture asserts that when both conditions coincide (rational *and* type (r, r)), the cause is algebraic geometry.

6.2 Connection to existing literature

The Hodge–Riemann bilinear relations are due to Hodge [9]; the modern treatment follows Griffiths [7] and Voisin [17]. The transcendence of periods connects to the Kontsevich–Zagier period conjecture [11]: period integrals $\int_{X(\mathbb{C})} \alpha \wedge *\bar{\beta}$ are generally transcendental numbers, which is precisely why the metric cannot see the rational lattice.

Grothendieck's standard conjectures [8] aim to provide a framework where algebraic cycles control cohomological data. The constructive calibration adds a new dimension: the standard conjectures would provide algebraic descent from transcendental ($\mathbb{C}$-valued) to algebraic ($\mathbb{Q}$-valued) computations, which is precisely the logical descent from LPO to BISH.

6.3 Open questions

1. Can the LPO calibration for H1 be sharpened to WLPO by considering approximate Hodge type ("$\|\pi_{\mathrm{comp}}(x)\| < \varepsilon$ for all $\varepsilon > 0$" instead of "$\pi_{\mathrm{comp}}(x) = 0$")?

2. Is there a constructive proof that the Hodge–Riemann form restricted to $\mathrm{im}(\iota)$ has special algebraic structure, i.e., that periods of rational classes satisfy additional constraints beyond transcendence?

3. Can the MP component of H2 be eliminated if one restricts to bounded-height rational classes (replacing unbounded search over $\mathbb{Q}$ with bounded search over $\{p/q : |p|, |q| \leq N\}$)?

4. What is the precise CRM strength of the Hodge Conjecture itself—as a proposition, does it require LPO, MP, or neither?

7 Conclusion

We have applied constructive reverse mathematics to the Hodge Conjecture and established that:

- Hodge type (r, r) decidability is *exactly* LPO($\mathbb{C}$) (Lean-verified, full proof from encoding axioms).
- Rationality testing requires at least LPO; the full characterization is LPO+MP (Lean-verified for the LPO component).
- The Hodge–Riemann polarization is available ($u(\mathbb{R}) = 1$) but blind to the rational lattice (Lean-verified from axioms).
- Algebraic cycle verification is decidable in BISH (Lean-verified, full proof via integer arithmetic).
- Hodge class detection requires LPO, but the Hodge Conjecture reduces it to BISH + MP (Lean-verified from axioms + conjecture hypothesis).

The constructive calibration does not resolve the Hodge Conjecture, but it reframes the problem: the conjecture provides *algebraic descent*, converting detection in $H_{\mathbb{C}}$ (requiring LPO over $\mathbb{C}$) to verification in $H_{\mathbb{Q}}$ (decidable in BISH over $\mathbb{Q}$). The unique position of Paper 49 in the five-conjecture atlas—polarization available but insufficient—shows that the Hodge Conjecture is fundamentally about the *nexus* of two mechanisms, neither of which suffices alone.

Acknowledgments

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The Lean 4 formalization was produced using AI code generation (Claude Code, Opus 4.6) under human direction. The author is a practicing cardiologist rather than a professional logician or algebraic geometer; all mathematical claims should be evaluated on their formal content. We welcome constructive feedback from domain experts.

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