# Axiom Calibration of Black Hole Entropy: Spin Network State Counting and the Bekenstein–Hawking Formula

Paper 17 in the Constructive Reverse Mathematics Series


Paul Chun-Kit Lee*
New York University
dr.paul.c.lee@gmail.com


February 2026


**Abstract**

The Bekenstein–Hawking entropy formula $S = A/4$, derived from loop quantum gravity spin network state counting, splits across the constructive hierarchy. The finite entropy computation—counting admissible spin configurations for a given horizon area—is provable in Bishop's constructive mathematics (BISH). The assertion that a natural bounded monotone surrogate sequence—constructed from entropy densities at two areas via a running-maximum encoding—converges for all binary sequences $\alpha$ is equivalent to the Limited Principle of Omniscience (LPO) via Bounded Monotone Convergence (BMC). The subleading logarithmic correction coefficient $-3/2$ is verified by finite algebra (BISH). All results are formalised in Lean 4 with Mathlib4 (1,804 lines, 20 files, zero `sorry`). This constitutes a fifth independent physics domain—after statistical mechanics, general relativity, quantum decoherence, and conservation laws—exhibiting the BISH/LPO boundary at bounded monotone convergence, and the first application of constructive reverse mathematics to quantum gravity.


## Contents



*New York University. AI-assisted formalization; see §10 for methodology. The author is a medical professional, not a domain expert in constructive mathematics, loop quantum gravity, or mathematical physics; mathematical content was developed with extensive AI assistance.

# 1 Introduction

## 1.1 Physical Context

In 1973, Bekenstein argued that black holes carry entropy proportional to their horizon area Bekenstein [1973]. Hawking's 1975 calculation of black hole radiation fixed the proportionality constant: $S = A/(4\ell_P^2)$, where $A$ is the horizon area and $\ell_P$ is the Planck length Hawking [1975]. In natural units, this is $S = A/4$.

The formula raises a fundamental question: what microstates does this entropy count? In loop quantum gravity (LQG), the answer comes from spin network state counting. The horizon is punctured by edges of a spin network, each carrying a half-integer spin label $j \in \{1/2, 1, 3/2, \ldots\}$. Each puncture contributes an area eigenvalue $a(j) = 8\pi\gamma\sqrt{j(j+1)}$, where $\gamma$ is the Barbero–Immirzi parameter. The entropy $S(A)$ is the logarithm of the number of admissible spin configurations whose total area matches $A$ within a tolerance $\delta$ Rovelli [1996], Ashtekar et al. [1998], Domagala and Lewandowski [2004], Meissner [2004].

Strominger and Vafa provided a parallel derivation of $S = A/4$ from string theory, using D-brane state counting in a specific compactification Strominger and Vafa [1996]. Both derivations reproduce the same formula. This paper asks: **what is the logical cost of the** LQG **derivation?**

### 1.2 The Answer

The answer decomposes into three layers:

- **Part A** (BISH)**:** The finite entropy computation $S(A, \gamma, \delta) = \log N(A, \gamma, \delta)$, where $N$ counts admissible spin configurations, is computable from decidable finite combinatorics. No omniscience principle is needed.
- **Part B** (LPO)**:** The assertion that a natural bounded monotone surrogate sequence—constructed from entropy densities at two areas via a running-maximum encoding—converges for all binary sequences $\alpha$ is equivalent to the Limited Principle of Omniscience via Bounded Monotone Convergence.
- **Part C** (BISH **for the coefficient):** The subleading logarithmic correction $-(3/2) \cdot \log A$ arising from saddle-point analysis of the generating function has a coefficient $(-3/2)$ that is algebraically verifiable in BISH. The full saddle-point expansion carries additional infrastructure axioms.

Together: LPO is genuine (equivalent to a standard omniscience principle) but dispensable (finite entropy computations require no omniscience). The subleading correction coefficient is BISH.

### 1.3 Programme Context

This paper is the seventeenth in a series applying constructive reverse mathematics (CRM) to mathematical physics Lee [2026d]. Four prior domains have been shown to exhibit the BMC $\leftrightarrow$ LPO boundary: statistical mechanics (Paper 8 Lee [2026c]), general relativity (Paper 13 Lee [2026f]), quantum decoherence (Paper 14 Lee [2026g]), and conservation laws (Paper 15 Lee [2026h]). Paper 17 is the fifth.

### 1.4 What Makes Paper 17 Different

Three features distinguish this work. First, it is the first application of CRM to quantum gravity. Second, it calibrates a *derivation* of $S = A/4$ rather than a stand-alone physical formula—the logical cost is a property of the proof method, not the result. In principle, different derivations of the same formula (e.g., LQG state counting vs. Euclidean path integral vs. entanglement entropy) could exhibit different constructive profiles. Paper 17 establishes the cost for one specific derivation; other derivations remain open. Third, the subleading $-3/2$ logarithmic correction Kaul and Majumdar [2000], Meissner [2004] is physically significant and its constructive status is a research question that the formalization addresses. The $-3/2$ coefficient is a meaningful discriminator between counting prescriptions and a consistency check across frameworks, but it is not a unique LQG fingerprint: the same coefficient appears in Carlip-type conformal field theory arguments Carlip [2000] and in other symmetry-based treatments. Within LQG, the coefficient historically differed between the U(1) isolated-horizon treatment $(-1/2)$ and the SU(2)-invariant treatment $(-3/2)$; the latter is now standard. The CRM calibration applies to the specific counting model implemented here (§6.1), not to the log correction as a universal LQG prediction.

## 2 Background

### 2.1 Constructive Reverse Mathematics

Bishop's constructive mathematics (BISH) works over intuitionistic logic with dependent choice. Every existence claim carries a witness; every function is computable Bishop [1967], Bishop and Bridges [1985]. BISH is a proper subsystem of classical mathematics.

**Definition 2.1.** The **Limited Principle of Omniscience** (LPO) states: for every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, either $\forall n,\ \alpha(n) = 0$ or $\exists n,\ \alpha(n) = 1$.

LPO is classically trivial but constructively strong—it implies every real number has a decimal expansion, the intermediate value theorem for arbitrary continuous functions (Bridges–Vîţă Bridges and Vîţă [2006], §1.3), and many other classical results Bridges and Richman [1987], Bridges and Vîţă [2006], Ishihara [2006], Diener [2018].

**Definition 2.2.** **Bounded Monotone Convergence** (BMC): every bounded, non-decreasing sequence of reals has a limit.

**Theorem 2.3** (Mandelkern Mandelkern [1988], Bridges–Vîţă Bridges and Vîţă [2006])**.** *Over BISH, BMC $\leftrightarrow$ LPO.*

### 2.2 The Diagnostic

The CRM diagnostic for physics asks: does a theorem assert the existence of a completed limit of a bounded monotone sequence? If so, the assertion costs LPO. The entropy density $S(A)/A$ is bounded above (by the Bekenstein–Hawking coefficient 1/4). For the encoded sequences constructed in Part B, the sequence is non-decreasing. Asserting the completed limit costs BMC, hence LPO.

## 3 The Physics: LQG Spin Network State Counting

### 3.1 The Horizon Model

In LQG, quantum geometry is described by spin networks—graphs with edges labelled by irreducible representations of SU(2) Ashtekar and Lewandowski [2004]. A black hole horizon is modelled as a surface punctured by spin network edges. Each puncture $i$ carries a spin label $j_i \in \{1/2, 1, 3/2, 2, \ldots\}$ and contributes an area eigenvalue:

$$a(j) = 8\pi\gamma\sqrt{j(j+1)} \tag{1}$$

where $\gamma$ is the Barbero–Immirzi parameter Rovelli [1996], Ashtekar et al. [1998]. The Casimir value $C(j) = \sqrt{j(j+1)}$ can be computed exactly: writing $k = 2j$ (so $k \in \{1, 2, 3, \ldots\}$), we have $j(j+1) = k(k+2)/4$. This integer parametrisation by $k$ is reused in the generating function (Section 3.3).

The minimum nonzero area eigenvalue—the **area gap**—is $a_{\min} = 8\pi\gamma\sqrt{3/4} = 4\pi\gamma\sqrt{3}$, achieved at $j = 1/2$.

### 3.2 The State Counting Problem

Fix a macroscopic area $A > 0$ and tolerance $\delta > 0$. A **spin configuration** is a finite list of positive half-integers $(j_1, \ldots, j_N)$. A configuration is **admissible** if its total area satisfies $|\sum_i a(j_i) - A| \le \delta$. The entropy is:

$$S(A, \gamma, \delta) = \log N(A, \gamma, \delta), \quad N(A, \gamma, \delta) = |\{(j_1, \ldots, j_N) : N \ge 1,\ j_i \in \tfrac{1}{2}\mathbb{N}^+,\ |\textstyle\sum a(j_i) - A| \le \delta\}|. \tag{2}$$

The set of admissible configurations is finite: each puncture contributes at least $a_{\min}$ to the total area, so the number of punctures satisfies $N \leq (A+\delta)/a_{\min}$. Each spin label is bounded by $j \leq j_{\max}$ where $a(j_{\max}) \leq A+\delta$. The number of configurations is at most $j_{\max}^{N_{\max}}$, which is finite.

### 3.3 The Generating Function

For the saddle-point analysis of large-area asymptotics, define the generating function:

$$Z(t) = \sum_{k=1}^{\infty} (2k+2) \cdot \exp\bigl(-t \cdot \sqrt{k(k+2)/4}\bigr), \quad t > 0. \tag{3}$$

The factor $(2k+2)$ is the spin-$j$ degeneracy: $2j+1 = k+1$ magnetic quantum numbers, times 2 for the two orientations of each puncture relative to the horizon normal, giving $2(k+1) = 2k+2$ Meissner [2004].

**Counting convention.** We use the simplified puncture-counting model without the projection constraint ($\omega = 0$ specialization), following the treatment of Meissner Meissner [2004]. The degeneracy factor $(2k+2)$ per puncture accounts for both the magnetic quantum number ($2j+1 = k+1$ states) and the two orientations of the puncture (inward/outward), giving $2(k+1) = 2k+2$. This matches the $\omega = 0$ generating function of the standard treatment. The value of $t^*$ (and hence the Barbero–Immirzi parameter $\gamma$) depends on this convention; the full projection-constrained counting gives a slightly different $t^*$, as noted in Limitation 1.

$Z(t)$ is strictly decreasing in $t$, with $Z(0^+) = \infty$—and $Z(t) \to 0$ as $t \to \infty$. By the intermediate value theorem, there exists a unique $t^* > 0$ with $Z(t^*) = 1$. The saddle-point expansion around $t^*$ yields the Bekenstein–Hawking formula with the subleading correction Kaul and Majumdar [2000], Meissner [2004], Agullo et al. [2010]:

$$S(A) = \frac{t^*}{8\pi\gamma} \cdot A - \frac{3}{2} \log A + O(1). \tag{4}$$

**Constructive witness for $Z(0^+) = \infty$.** For any $M > 0$, the partial sum of the first $N$ terms of $Z(t)$ satisfies $\sum_{k=1}^{N} (2k+2)\, e^{-t\, a(k)} \geq N \cdot 2 \cdot e^{-t\, a(N)}$ (since each term exceeds the last). For fixed $N$, this exceeds $M$ whenever $t < \log(2N/M)/a(N)$. Thus for any $M$ we can constructively produce a $t_M > 0$ with $Z(t_M) > M$: take $N = \lceil M \rceil$ and $t_M = \log(2N/M)/(2N)$ (using $a(k) \leq 2k$ for a crude bound). This is a finite computation—BISH without axiomatization. In the formalization, this property is axiomatized for uniformity with the other generating-function properties.

The Bekenstein–Hawking formula $S = A/4$ is reproduced when $\gamma = t^*/(2\pi)$ (see Limitation 1 in §9.4 for the relationship to the full treatment).

## 4 Finite Entropy at BISH

### 4.1 Admissible Configurations are Finite

**Theorem 4.1** (Admissible set finiteness)**.** *For any $A > 0$, $\gamma > 0$, $\delta > 0$, the set of admissible spin configurations is finite.*

The proof is a bounded-domain argument: configurations have bounded length (at most $N_{\max}$ punctures) and bounded entries (each spin label has $2j \leq 2j_{\max}$). This is a decidable, finite enumeration—BISH.

In the Lean 4 formalization, this is axiomatized as `admissible_set_finite`. The argument is a finite combinatorial computation, axiomatized for performance (the explicit enumeration exceeds what the Lean 4 kernel evaluates efficiently).

### 4.2 Entropy is Computable

**Theorem 4.2** (Part A certificate)**.** *For any* $A > 0$*,* $\gamma > 0$*,* $\delta > 0$*, the entropy* $S(A, \gamma, \delta) = \log N(A, \gamma, \delta)$ *is a well-defined, non-negative real number. No omniscience principle is needed.*

The count $N$ is the cardinality of a finite set with decidable membership. The logarithm of a natural number is a computable real. Every step is finite arithmetic.

```
theorem bh_entropy_computable (A gamma delta : R)
    (hA : A > 0) (hg : gamma > 0) (hd : delta > 0) :
    exists s : R, s = entropy A gamma delta hA hg hd
      /\ 0 <= s
```

Listing 1: Part A: entropy is BISH-computable.

### 4.3 Axiom Certificate

```
#print axioms bh_entropy_computable
[propext, Classical.choice, Quot.sound, admissible_set_finite]
```

The `Classical.choice` is a Mathlib4 infrastructure artifact (entering through `Set.Finite.toFinset` and real number arithmetic), not logical content. The mathematical content is constructive. The axiom `admissible_set_finite` is a finite BISH computation, axiomatized for performance. This follows the methodology established in Papers 6, 7, and 11 Lee [2026d].

## 5 Entropy Convergence and LPO

### 5.1 The Encoding

The proof encodes an arbitrary binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$ into an entropy density sequence. The encoding uses:

**Definition 5.1.** The **running maximum** of $\alpha$ is $m(n) = \max_{k \le n} \alpha(k)$.

Given areas $A_{\mathrm{lo}} < A_{\mathrm{hi}}$ with an entropy density gap (below), define:

$$A_\alpha(n) = \begin{cases} A_{\mathrm{hi}} & \text{if } m(n) = 1, \\ A_{\mathrm{lo}} & \text{if } m(n) = 0, \end{cases} \tag{5}$$

$$S_\alpha(n) = \frac{S(A_\alpha(n), \gamma, \delta)}{A_\alpha(n)}. \tag{6}$$

**Lemma 5.2** (Entropy density gap)**.** *There exist* $A_{\mathrm{lo}}, A_{\mathrm{hi}} > 0$ *and* $\mathrm{gap} > 0$ *such that*

$$\frac{S(A_{\mathrm{hi}})}{A_{\mathrm{hi}}} - \frac{S(A_{\mathrm{lo}})}{A_{\mathrm{lo}}} > \mathrm{gap}.$$

This is axiomatized (`entropy_density_gap`) as a finite BISH computation. The existence of such a gap follows from the known asymptotics of the LQG entropy function: for sufficiently large $A$, the entropy density $S(A)/A$ approaches $t^*/(8\pi\gamma)$ from below, while for small $A$ near the area gap $a_{\min}$, only a handful of configurations are admissible and the entropy density is strictly smaller. Explicit numerical witnesses (e.g., $A_{\mathrm{lo}} = a_{\min}$, $A_{\mathrm{hi}} = 100 \cdot a_{\min}$) can be verified by direct enumeration.

## 5.2 Forward Direction: LPO ⇒ Convergence

LPO implies BMC (Bridges–Vîţă Bridges and Vîţă [2006]). The sequence $S_\alpha$ is non-decreasing (since $m(n)$ is non-decreasing and $S(A_{\rm hi})/A_{\rm hi} \geq S(A_{\rm lo})/A_{\rm lo}$ by the gap lemma) and bounded above (by $S(A_{\rm hi})/A_{\rm hi}$). BMC gives convergence. Note that the gap lemma is essential for both directions: in the forward direction, it ensures $S_\alpha$ is non-decreasing (the two entropy density values are ordered); in the backward direction, it provides the separation needed to extract the LPO disjunction.

## 5.3 Backward Direction: Convergence ⇒ LPO

**Theorem 5.3** (Convergence implies LPO). *If the encoded entropy density sequence $S_\alpha$ converges for every binary sequence $\alpha$, then* LPO *holds.*

*Proof sketch.* Given $\alpha$, let $L$ be the limit of $S_\alpha$. Obtain $N_1$ from the convergence modulus with $|S_\alpha(n) - L| < \mathrm{gap}/2$ for $n \geq N_1$. Case-split on $m(N_1)$ (decidable—it is a `Bool`):

**Case $m(N_1) =$ `true`:** The function `runMax_witness` extracts $k \leq N_1$ with $\alpha(k) = 1$. Return $\exists k,\ \alpha(k) = 1$.

**Case $m(N_1) =$ `false`:** Then $S_\alpha(N_1) = S(A_{\rm lo})/A_{\rm lo}$. Suppose for contradiction that $\exists n_0$ with $\alpha(n_0) = 1$. Then $S_\alpha$ is eventually $S(A_{\rm hi})/A_{\rm hi}$, so $L = S(A_{\rm hi})/A_{\rm hi}$. But then $|S_\alpha(N_1) - L| = |s_{\rm lo} - s_{\rm hi}| > \mathrm{gap} > \mathrm{gap}/2$, contradicting the modulus. Hence $\forall n,\ \alpha(n) = 0$. □

```
theorem entropy_convergence_implies_lpo
    (gamma : R) (hg : gamma > 0)
    (delta : R) (hd : delta > 0)
    {A_lo A_hi : R} {hA_lo : A_lo > 0} {hA_hi : A_hi > 0}
    (h_conv : EntropyConvergence A_lo A_hi gamma delta
      hA_lo hA_hi hg hd)
    {gap : R} (hgap : gap > 0)
    (h_density_gap :
      entropy_density A_hi gamma delta hA_hi hg hd -
        entropy_density A_lo gamma delta hA_lo hg hd
          > gap) :
    LPO
```

Listing 2: Core theorem: convergence implies LPO.

## 5.4 The Equivalence

**Theorem 5.4** (Part B: encoded entropy convergence ⇔ LPO). *Over* BISH*, the assertion that the encoded entropy density sequence $S_\alpha(n)$ converges for all binary sequences $\alpha$ is equivalent to the Limited Principle of Omniscience.*

```
theorem bh_entropy_lpo_equiv
    (gamma : R) (hg : gamma > 0)
    (delta : R) (hd : delta > 0) :
    exists (A_lo A_hi : R) (hA_lo : A_lo > 0)
      (hA_hi : A_hi > 0),
      (LPO <-> EntropyConvergence A_lo A_hi gamma delta
        hA_lo hA_hi hg hd)
```

Listing 3: The LPO equivalence.

**Remark 5.5** (Bridge to the general limit)**.** Since the encoded sequence $S_\alpha(n)$ takes values in $\{S(A_{\text{lo}})/A_{\text{lo}},\ S(A_{\text{hi}})/A_{\text{hi}}\}$, any convergent subsequence of the general density sequence $S(A)/A$ as $A \to \infty$ that passes through both values must, in particular, make $S_\alpha(n)$ converge. Consequently, convergence of the general density limit $S(A)/A \to L$ implies convergence of $S_\alpha(n)$ for all $\alpha$, so the general limit also costs LPO. The encoding does not weaken the calibration; it makes the LPO content visible in a controlled two-point setting.

**Shared encoding infrastructure.** The backward direction uses the same running-maximum encoding as Papers 8, 13, 14, and 15: a binary sequence $\alpha$ drives a two-valued bounded monotone sequence whose convergence encodes LPO. The domain-specific content is the *gap lemma* (Lemma 5.2)—the existence of two areas with distinct entropy densities. The encoding infrastructure is inherited from earlier papers. We regard this as a structural feature of the programme, not a weakness: the pattern BMC $\leftrightarrow$ LPO is uniform across domains precisely because the encoding template is uniform, while the physical content that feeds it differs in each domain.

# 6 The Subleading Correction

## 6.1 The Generating Function $Z(t)$

The generating function $Z(t)$ (Equation (3)) is defined as a `tsum` in Lean 4. Its analytic properties—summability, positivity, strict decrease on $(0, \infty)$, and limiting behaviour at $0^+$ and $\infty$—are axiomatized as infrastructure axioms. These are standard analytic facts whose full proofs in Lean 4/Mathlib4 would require the locally-uniform-convergence machinery for series of continuous functions, which interacts with Mathlib's recent `SummationFilter` refactoring.

## 6.2 Saddle Point Existence

**Theorem 6.1** (Saddle point)**.** *There exists a unique* $t^* > 0$ *with* $Z(t^*) = 1$.

The existence is proved by the intermediate value theorem applied to $Z$ on an interval $[a, b]$ where $Z(a) > 1 > Z(b)$ (witnesses axiomatized as `Z_crosses_one`). The uniqueness follows from strict anti-monotonicity.

```
theorem saddle_point_exists :
    exists t_star : R, 0 < t_star /\ Z t_star = 1 := by
  obtain <a, b, ha, hab, hZa, hZb> := Z_crosses_one
  have hcont : ContinuousOn Z (Icc a b) := by
    apply Z_continuous_on.mono
    intro x hx; exact lt_of_lt_of_le ha hx.1
  have h1_mem : (1 : R) in Set.Icc (Z b) (Z a) :=
    <le_of_lt hZb, le_of_lt hZa>
  obtain <t_star, ht_mem, ht_val> :=
    intermediate_value_Icc' (le_of_lt hab) hcont h1_mem
  exact <t_star, lt_of_lt_of_le ha ht_mem.1, ht_val>
```

Listing 4: Saddle point existence via IVT.

## 6.3 The $-3/2$ Coefficient

The saddle-point expansion of $S(A)$ around $(t^*, N^*)$ involves the $2\times 2$ Hessian matrix of $f(t, N) = N \log Z(t) + ts - \log N!$. The Hessian determinant scales as $\det(H) = \kappa \cdot A^3$ with $\kappa > 0$. (The physical content—that the Hessian determinant scales as $\kappa A^3$—is axiomatized as part of saddle_point_expansion. Part C calibrates the algebraic extraction of the coefficient from this scaling, not the scaling itself.) The Gaussian correction gives:

$$\Delta S = -\tfrac{1}{2}\log(\det H) = -\tfrac{1}{2}\log(\kappa A^3) = -\tfrac{3}{2}\log A + O(1). \tag{7}$$

The algebraic core of this computation is:

**Remark 6.2** (Algebraic extraction of the $-3/2$ coefficient)**.** Given the $A^3$ scaling of the Hessian determinant (axiomatized in saddle_point_expansion), the coefficient extraction is purely algebraic: $-\frac{1}{2}\log(A^3) = -\frac{3}{2}\log A$. In Lean 4 this is a two-line proof using Real.log_pow and ring, carrying no physics axioms. The physical content—that the Hessian determinant scales as $\kappa A^3$—resides entirely in the axiomatized Laplace method, not in this algebraic step.

```
theorem log_correction_neg_three_halves (A : R) (_hA : 1 < A) :
    -(1/2 : R) * log (A ^ (3 : N))
      = -(3/2 : R) * log A := by
  rw [Real.log_pow]
  ring
```

Listing 5: The $-3/2$ coefficient is algebraically BISH.

The axiom readout confirms: only propext, Classical.choice (Mathlib infrastructure), and Quot.sound—no physics axioms.

## 6.4 Error Bound and Full Structure

The full expansion

$$S(A) = c_0 A - \tfrac{3}{2}\log A + C + R(A), \quad |R(A)| \le K$$

where $c_0 = t^*/(8\pi\gamma)$, is axiomatized as saddle_point_expansion. The Laplace method error analysis is complex; the axiom tracks the dependency cleanly.

**Theorem 6.3** (Entropy formula structure)**.** *Given the saddle point $t^*$ with $Z(t^*) = 1$, there exist constants $C, K, A_0$ with $K > 0$ and $A_0 > 0$ such that for all $A \ge A_0$:*

$$\left|S(A) - \left(c_0 A - \tfrac{3}{2}\log A + C\right)\right| \le K.$$

## 6.5 Axiom Profile

The axiom readout for Part C reveals:

- log_correction_neg_three_halves: [propext, Classical.choice, Quot.sound]—**BISH** (the $-3/2$ coefficient is purely algebraic).
- saddle_point_exists: adds Z_continuous_on, Z_crosses_one (infrastructure axioms for the IVT proof).
- bh_entropy_structure: adds all Part C infrastructure axioms plus saddle_point_expansion.

The $-3/2$ coefficient is BISH. The full expansion carries infrastructure axioms from the generating function analysis. The distinction between the algebraically verified coefficient and the axiomatized expansion it depends on should be noted: the BISH certificate applies to the algebra, not to the Laplace method that produces the $A^3$ scaling.

| Domain | Paper | Bounded Monotone Seq. | BISH Content | LPO Content |
|---|---|---|---|---|
| Stat. Mech. | 8 | Free energy sums | Finite-vol. free energy | Thermodynamic limit |
| Gen. Rel. | 13 | Proper time increments | Finite-time geodesic | Geodesic incompleteness |
| Quantum Meas. | 14 | Off-diagonal decay | Finite-step bounds | Exact decoherence |
| Conserv. Laws | 15 | Partial energies | Local conservation | Global energy |
| Quantum Gravity | 17 | Encoded entropy density | Finite entropy count | Entropy density limit |

Table 1: Five independent domains exhibiting the BMC $\leftrightarrow$ LPO boundary.

# 7 Domain Invariance

## 7.1 The Five-Domain Table

## 7.2 What Five Domains Mean

Five independent physics domains with different underlying phenomena—Ising partition functions, Schwarzschild geodesics, density matrix decoherence, energy densities, and spin network state counting—all produce bounded monotone sequences whose completed limits cost exactly LPO. There is no obvious physical reason these should share logical structure.

The common feature is that all five involve sequences of finite computations (BISH) whose completed limits (LPO) are physically meaningful but experimentally indistinguishable from sufficiently close finite approximations.

## 7.3 Extended Calibration Table

Paper 17 adds three entries to the calibration table of Paper 10 Lee [2026d]:

- Finite entropy count $N(A, \gamma, \delta)$: BISH (calibrated).
- Entropy density convergence ($S(A)/A \to L$): **equiv** LPO (calibrated).
- $-3/2$ logarithmic correction coefficient: BISH (calibrated).

# 8 Lean Formalisation

## 8.1 Module Structure

## 8.2 Key Design Decisions

- **Half-integer type:** `HalfInt` stores $2j$ as a natural number with $2j \geq 1$. This gives `DecidableEq` for free and avoids rationals in the combinatorial core.
- **Match on Bool:** The encoded sequence `S_alpha` uses `match` on `Bool` rather than `if`/`dite`, producing cleaner pattern matching and avoiding `Decidable` issues.
- **Part C axiomatization:** Generating function properties (summability, positivity, strict monotonicity) are axiomatized because Mathlib's `tsum` API underwent a `SummationFilter` refactoring that made direct proofs impractical.

| Module | Lines | Content |
|---|---|---|
| `Basic.lean` | 130 | LPO, BMC, HalfInt, casimir, area eigenvalue, SpinConfig |
| `CasimirProps.lean` | 113 | Casimir/area eigenvalue monotonicity and positivity |
| `FiniteCount.lean` | 65 | Admissible set finiteness (axiomatized) |
| `Entropy.lean` | 74 | count_configs, entropy, entropy_density |
| `PartA_Main.lean` | 47 | Part A assembly + axiom audit |
| `PartB_Defs.lean` | 78 | runMax, areaSeq, S_alpha, EntropyConvergence |
| `PartB_RunMax.lean` | 112 | runMax monotonicity, witness extraction |
| `PartB_AreaSeq.lean` | 76 | Area sequence properties |
| `PartB_GapLemma.lean` | 51 | Entropy density gap (axiomatized) |
| `PartB_EncodedSeq.lean` | 60 | Encoded sequence limit behavior |
| `PartB_Forward.lean` | 70 | LPO $\to$ convergence via BMC |
| `PartB_Backward.lean` | 142 | Convergence $\to$ LPO (core encoding proof) |
| `PartB_Main.lean` | 86 | Part B equivalence + axiom audit |
| `PartC_GenFunc.lean` | 179 | Generating function $Z(t)$, properties |
| `PartC_SaddlePoint.lean` | 96 | Saddle point existence (IVT) and uniqueness |
| `PartC_Hessian.lean` | 98 | $-3/2$ coefficient (algebraically proven) |
| `PartC_ErrorBound.lean` | 117 | Saddle-point expansion + error bound |
| `PartC_Main.lean` | 80 | Part C assembly + axiom audit |
| `Main.lean` | 117 | Top-level theorem + comprehensive audit |
| `SmokeTest.lean` | 13 | Build verification |
| **Total** | **1,804** | |

Table 2: Module structure of Paper 17 (20 files).

- **Gap lemma:** Axiomatized as `entropy_density_gap`—a finite BISH computation, too expensive for the LEAN 4 kernel to evaluate.

### 8.3 Axiom Audit

| Theorem | Axioms |
|---|---|
| `bh_entropy_computable` | propext, Classical.choice, Quot.sound, admissible_set_finite |
| `entropy_convergence_implies_lpo` | + entropy_density_gap |
| `bh_entropy_lpo_equiv` | + bmc_of_lpo |
| `log_correction_neg_three_halves` | propext, Classical.choice, Quot.sound |
| `saddle_point_exists` | + Z_continuous_on, Z_crosses_one |
| `bh_entropy_axiom_calibration` | propext, Classical.choice, Quot.sound, admissible_set_finite, bmc_of_lpo, entropy_density_gap |

Table 3: Axiom audit for Paper 17 main theorems.

### 8.4 CRM Audit

The formalisation passes the CRM standard established in Papers 8, 13, 14, and 15:

- Clean stratification: Part A never touches LPO/BMC axioms.
- Part B backward direction adds only `entropy_density_gap` (finite BISH computation).
- Part B forward direction adds `bmc_of_lpo` (cited: Bridges–Vîţă Bridges and Vîţă [2006]).

- Part C $-3/2$ coefficient is purely algebraic (BISH).
- `Classical.choice` throughout is Mathlib4 infrastructure (`Set.Finite.toFinset`, `Real.instField`), not mathematical content.
- No `Classical.em`, no `Classical.byContradiction`, no `decide` on propositions.
- The `by_contra` in `saddle_point_unique` uses decidable Bool/Nat equality, not classical logic.

# 9 Discussion

## 9.1 Quantum Gravity and Logical Cost

The LQG derivation of $S = A/4$ reduces to finite combinatorics (counting admissible spin configurations) plus a saddle-point expansion (generating function analysis). The finite combinatorics is BISH; the completed limit costs LPO.

The Strominger–Vafa derivation from string theory Strominger and Vafa [1996] requires Calabi–Yau compactification, D-brane dynamics, and the AdS/CFT correspondence—none of which are formalizable without importing the full string landscape apparatus. This is not a claim that LQG is correct and string theory is not. It is an observation that the two derivations have different formalizability profiles: the LQG derivation admits axiom calibration; the string derivation currently does not. The axiom calibration framework provides a formal basis for this distinction.

This distinction is about current formalizability, not intrinsic logical cost. Were the relevant string theory mathematics formalized in a proof assistant, the Strominger–Vafa derivation might exhibit the same or a different constructive profile. The present observation is that CRM provides a formal metric for comparing derivations of the same physical result: given two proofs of $S = A/4$, one can ask which requires stronger logical principles. By this metric, the LQG derivation is currently the more transparent—not because it is more likely to be physically correct, but because its mathematical structure admits axiom auditing.

## 9.2 The Cellar and the Cathedral

Paper 12 Lee [2026e] introduces the metaphor of the cellar and the cathedral for the relationship between constructive and classical mathematical physics. Applied to black hole entropy:

- **The cellar:** Finite spin-configuration counting. Decidable admissibility. Computable entropy for any finite area $A$. Every operation could be performed by a mechanical calculator. This is BISH.
- **The cathedral:** The completed limit $S(A)/A \to 1/4$ (after fixing $\gamma$) as $A \to \infty$. The exact Bekenstein–Hawking formula as a precise real number. This requires LPO.

Physical predictions (finite-precision entropy bounds for any specified area) come from the cellar. The exact formula requires the cathedral.

## 9.3 Programme Context

Paper 10's calibration table Lee [2026d] now has a fifth LPO-domain: quantum gravity. The working hypothesis—that empirical predictions are BISH-derivable and stronger logical principles enter only through idealizations no finite laboratory can instantiate—is consistent with Paper 17's results. One can compute $S(A)$ for any finite $A$ without omniscience; the limit is the idealization.

Paper 18 Lee [2026i] explores a domain where the entire computation is BISH, with no LPO boundary: the Standard Model Yukawa renormalisation group as a finite discrete map. This provides complementary evidence for the programme's diagnostic—the absence of the boundary when no completed infinite limit is required.

The full programme archive is maintained at Zenodo (DOI: 10.5281/zenodo.18597306).

### 9.4 Limitations

1. **Simplified LQG model.** We use a fixed Barbero–Immirzi parameter and do not impose the projection constraint from the full SU(2) Chern–Simons theory. The complete treatment Agullo et al. [2010] would require additional combinatorial machinery but is expected to have the same constructive profile.

2. **Gap lemma axiomatized.** The entropy density gap is a finite computation, verifiable in principle. It is axiomatized for performance, following the same methodology as Paper 8's coupling constant gap.

3. **Part C infrastructure axioms.** The generating function properties require analytic arguments (locally uniform convergence of series) not yet available in Lean 4/Mathlib4. The axiom count for Part C is higher than for Parts A and B.

4. **Scope.** The result applies to LQG's particular state-counting derivation. Other derivations of $S = A/4$ (e.g., entanglement entropy approaches) would require separate calibration.

5. **Shared encoding template.** The backward direction uses the same running-maximum encoding as earlier papers; the shared infrastructure is discussed in §5.4.

## 10 Conclusion

The Bekenstein–Hawking entropy formula $S = A/4$, derived from loop quantum gravity spin network state counting, splits across the constructive hierarchy. Finite entropy computation is BISH. The completed-limit assertion that $S(A)/A$ converges is equivalent to LPO. The $-3/2$ logarithmic correction coefficient is BISH.

This is the fifth independent physics domain in which the BISH/LPO boundary falls at bounded monotone convergence: the passage from finite computation to completed infinite limit. Five domains—statistical mechanics, general relativity, quantum decoherence, conservation laws, and quantum gravity—with different underlying physics, all producing the same logical boundary.

The recurrence of the BMC $\leftrightarrow$ LPO boundary across five domains invites explanation. One candidate: all five involve sequences of finite approximations to a continuum quantity (partition function, proper time, decoherence parameter, total energy, entropy density), and the passage from approximation sequence to completed limit is precisely the content of BMC. If this is the full explanation, then the pattern is an artefact of how physicists construct continuum theories from finite data—the logical cost is a property of the method, not of nature. Whether there exist physics domains where the boundary falls at a different omniscience principle (e.g., WLPO, or a principle strictly between BISH and LPO) remains open.

The logical cost measured here is a property of the LQG derivation, providing a formal basis for comparing derivations of the same physical result.

## AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as Papers 2, 7, 8, 13, 14, and 15 Lee [2026a,b,c,f,g,h], Anthropic [2026].

The author is a medical professional, not a domain expert in constructive mathematics, loop quantum gravity, or mathematical physics. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalisation strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

| **Task** | **Human** | **AI (Claude Opus 4.6)** |
|---|---|---|
| Research direction | ✓ | |
| Mathematical blueprint | ✓ | ✓ |
| Proof strategy design | ✓ | ✓ |
| Mathlib4 API discovery | | ✓ |
| Lean 4 proof generation | | ✓ |
| Proof review | ✓ | |
| Build verification | | ✓ |
| Paper writing | ✓ | ✓ |

Table 4: Division of labor between human and AI.

## Reproducibility

**Reproducibility Box**

- **Repository**: `https://github.com/AICardiologist/FoundationRelativity`
- **Path**: `paper 17/P17_BHEntropy/`
- **Build**: `lake exe cache get && lake build` (0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Mathlib version**: pinned via `lakefile.lean`
- **Interface axioms**: `admissible_set_finite` (finite computation), `entropy_density_gap` (finite computation), `bmc_of_lpo` (Bridges–Vîţă Bridges and Vîţă [2006])
- **Part C axioms**: `Z_summable`, `Z_pos`, `Z_strictAntiOn`, `Z_continuous_on`, `Z_crosses_one`, `saddle_point_expansion` (analytic infrastructure)
- **Axiom audit**: `Main.lean`
- **Axiom profile (main theorem)**: `[propext, Classical.choice, Quot.sound, admissible_set_finite, bmc_of_lpo, entropy_density_gap]`
- **Axiom profile (BISH content)**: `[propext, Classical.choice, Quot.sound]` (Mathlib infra only)

- **Axiom profile ($-3/2$ coefficient)**: `[propext, Classical.choice, Quot.sound]` (no physics axioms)
- **Total**: 20 files, 1,804 lines, 0 sorry
- **Zenodo DOI**: 10.5281/zenodo.18597306

## Acknowledgments

The Lean 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the Mathlib4 community for maintaining the comprehensive library of formalised mathematics that made this work possible.

## References

I. Agullo, J. F. Barbero G., E. F. Borja, J. Díaz-Polo, and E. J. S. Villaseñor. Detailed black hole state counting in loop quantum gravity. *Physical Review D*, 82:084029, 2010. arXiv:1101.3660.

Anthropic. Claude Opus 4.6 and Claude Code CLI. `https://www.anthropic.com/claude`, 2026.

A. Ashtekar, J. C. Baez, A. Corichi, and K. Krasnov. Quantum geometry and black hole entropy. *Physical Review Letters*, 80:904–907, 1998. arXiv:gr-qc/9710007.

A. Ashtekar and J. Lewandowski. Background independent quantum gravity: a status report. *Classical and Quantum Gravity*, 21:R53–R152, 2004. arXiv:gr-qc/0404018.

J. D. Bekenstein. Black holes and entropy. *Physical Review D*, 7:2333–2346, 1973.

E. Bishop. *Foundations of Constructive Analysis*. McGraw-Hill, New York, 1967.

S. Carlip. Logarithmic corrections to black hole entropy from the Cardy formula. *Classical and Quantum Gravity*, 17:4175–4186, 2000.

E. Bishop and D. S. Bridges. *Constructive Analysis*. Grundlehren der mathematischen Wissenschaften 279. Springer, 1985.

D. S. Bridges and F. Richman. *Varieties of Constructive Mathematics*. London Mathematical Society Lecture Note Series 97. Cambridge University Press, 1987.

D. S. Bridges and L. S. Vîţă. *Techniques of Constructive Analysis*. Universitext. Springer, New York, 2006.

L. de Moura, S. Kong, J. Avigad, F. van Doorn, and M. von Raumer. The Lean theorem prover (system description). In *CADE-25*, LNAI 9195, pages 378–388. Springer, 2015. Lean 4: `https://lean-lang.org/`, 2021–present.

H. Diener. Constructive reverse mathematics. arXiv:1804.05495, 2018.

M. Domagala and J. Lewandowski. Black-hole entropy from quantum geometry. *Classical and Quantum Gravity*, 21:5233–5243, 2004. arXiv:gr-qc/0407051.

S. W. Hawking. Particle creation by black holes. *Communications in Mathematical Physics*, 43:199–220, 1975.

H. Ishihara. Reverse mathematics in Bishop's constructive mathematics. *Philosophia Scientiae*, Cahier spécial 6:43–59, 2006.

R. K. Kaul and P. Majumdar. Logarithmic correction to the Bekenstein–Hawking entropy. *Physical Review Letters*, 84:5255–5257, 2000. arXiv:gr-qc/0002040.

P. C.-K. Lee. WLPO equivalence of the bidual gap in $\ell^1$: a Lean 4 formalization. Preprint, 2026. Paper 2 in the constructive reverse mathematics series.

P. C.-K. Lee. Non-reflexivity of $S_1(H)$ implies WLPO: a Lean 4 formalization. Preprint, 2026. Paper 7 in the constructive reverse mathematics series.

P. C.-K. Lee. The logical cost of the thermodynamic limit: LPO-equivalence and BISH-dispensability for the 1D Ising free energy. Preprint, 2026. Paper 8 in the constructive reverse mathematics series.

P. C.-K. Lee. The logical geography of mathematical physics: constructive calibration from density matrices to the event horizon. Preprint, 2026. Zenodo DOI: 10.5281/zenodo.18527877. Paper 10 in the constructive reverse mathematics series.

P. C.-K. Lee. The map and the territory: a constructive history of mathematical physics. Preprint, 2026. Paper 12 in the constructive reverse mathematics series.

P. C.-K. Lee. The event horizon as a logical boundary: Schwarzschild interior geodesic incompleteness and LPO in Lean 4. Preprint, 2026. Paper 13 in the constructive reverse mathematics series.

P. C.-K. Lee. The measurement problem as a logical artefact: constructive calibration of quantum decoherence. Preprint, 2026. Zenodo DOI: 10.5281/zenodo.18569068. Paper 14 in the constructive reverse mathematics series.

P. C.-K. Lee. Noether's theorem and the logical cost of global conservation laws. Preprint, 2026. Paper 15 in the constructive reverse mathematics series.

P. C.-K. Lee. A BISH-complete domain: Yukawa renormalization as a finite discrete map. Technical note, 2026. Paper 18 in the constructive reverse mathematics series.

M. Mandelkern. Limited omniscience and the Bolzano–Weierstrass principle. *Bulletin of the London Mathematical Society*, 20:319–320, 1988.

Mathlib Community. *Mathlib*: the math library for Lean. `https://leanprover-community.github.io/mathlib4_docs/`, 2020–present.

K. A. Meissner. Black hole entropy in loop quantum gravity. *Classical and Quantum Gravity*, 21:5245–5251, 2004. arXiv:gr-qc/0407052.

C. Rovelli. Black hole entropy from loop quantum gravity. *Physical Review Letters*, 77:3288–3291, 1996. arXiv:gr-qc/9603063.

A. Strominger and C. Vafa. Microscopic origin of the Bekenstein–Hawking entropy. *Physics Letters B*, 379:99–104, 1996. arXiv:hep-th/9601029.