Bell Nonlocality and the Constructive Cost of Disjunction: An LLPO Calibration

Paper 21 in the Constructive Reverse Mathematics Series

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February 2026

Abstract

The disjunctive interpretation of Bell's theorem—deciding whether the nonlocality asymmetry between Alice and Bob leans "Alice-side" (≤ 0) or "Bob-side" (≥ 0)—is equivalent to the Lesser Limited Principle of Omniscience (LLPO) over Bishop's constructive mathematics (BISH). The forward direction applies the standard real-valued form of LLPO (sign decidability: $x \leq 0 \vee 0 \leq x$) to the encoded Bell asymmetry; the reverse encodes a binary sequence α with the AtMostOne predicate into a pair of geometric series whose difference—the Bell asymmetry—has sign determined by the parity of the unique nonzero index. Combined with Part A, which proves the CHSH bound, the Tsirelson violation $S_Q = 2\sqrt{2} > 2$, and $\neg$LHV entirely in BISH, this establishes a **three-level stratification**: BISH (Bell negation) < LLPO (disjunctive Bell conclusion) < WLPO (hierarchy). All results are formalized in Lean 4 with Mathlib4 (751 lines, 14 files, zero `sorry`). This is the first CRM calibration of a quantum foundations result at the LLPO level, and the first to measure the constructive cost of the step from "local realism is refuted" to "the nonlocality favors one party over the other."

Contents

*New York University. AI-assisted formalization; see §9 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 Bell's Theorem and Its Disjunctive Interpretation

Bell's theorem [Bell, 1964] demonstrates that no local hidden variable (LHV) theory can reproduce the correlations predicted by quantum mechanics. In the CHSH formulation [Clauser et al., 1969], any deterministic assignment of four dichotomic (± 1) observables satisfies $|S| \leq 2$, while the quantum prediction achieves $S_Q = 2\sqrt{2} \approx 2.828$ [Tsirelson, 1980], confirmed experimentally [Aspect et al., 1982].

The *negation* form—"no LHV model can reproduce $S > 2$"—is a finite contradiction: 16 cases show $|S| \leq 2$, yet quantum mechanics exceeds this bound. The negation is pure BISH.

But physicists routinely take an additional step: from "local realism is refuted," they conclude *either* locality fails *or* realism fails. This disjunctive step is not free. We formalize a precise version of this disjunction by encoding it as a **sign decision** on the Bell asymmetry—the difference between Alice-side and Bob-side contributions to Bell violation—and show that this sign decision has the exact constructive strength of LLPO.

1.2 The Answer: LLPO

The answer is the Lesser Limited Principle of Omniscience:

1. **Part A (BISH):** The CHSH bound $|S| \leq 2$, the Tsirelson violation $S_Q > 2$, and ¬LHV are all provable without any omniscience principle.

2. **Part B** (LLPO)**:** Deciding the sign of the Bell asymmetry—bellAsymmetry $\alpha \leq 0$ versus $0 \leq$ bellAsymmetry α—for sequences with AtMostOne is equivalent to LLPO.

The main results, stated precisely, are:

- **Theorem 1** (Part A): CHSH bound—for all LHV assignments, $S \in \{-2, +2\}$.
- **Theorem 2** (Part A): Quantum violation—$S_Q > 2$.
- **Theorem 3** (Part A): ¬LHV—no deterministic assignment achieves $S > 2$.
- **Theorem 4** (Part B): LLPO ⇒ BellSignDecision.
- **Theorem 5** (Part B): BellSignDecision ⇒ LLPO (novel direction).
- **Theorem 6** (Part B): LLPO ↔ BellSignDecision.
- **Theorem 7**: Three-level stratification.

1.3 Programme Context

This is Paper 21 in a programme of constructive calibration of mathematical physics Lee [2026c,e,f,a,b,d]. Papers 2 and 7 calibrated WLPO against the bidual gap and non-reflexivity; Paper 8 calibrated LPO against the 1D Ising free energy; Paper 19 calibrated LLPO against WKB turning points; Paper 20 calibrated WLPO against Ising magnetization phase classification. The constructive hierarchy is:

$$\text{BISH} \ < \ \text{LLPO} \ < \ \text{WLPO} \ < \ \text{LPO}.$$

All implications are strict (no reverse implications hold over BISH). Paper 21 contributes the first LLPO calibration of a quantum foundations result.

1.4 What Makes This Paper Different

Paper 21 contributes three novelties:

1. **First LLPO calibration in quantum foundations.** Previous quantum calibrations in the series were at BISH (Heisenberg, Paper 6) or LPO (decoherence). This is the first to land at LLPO, the weakest nontrivial omniscience principle.
2. **Measuring the cost of disjunction.** Physicists routinely move from "local realism is refuted" (a negation, BISH) to "either locality or realism fails" (a disjunction). We show this step costs exactly LLPO—it is not free.
3. **LLPO as the sign-decision principle.** The mechanism underlying the LLPO equivalence is real-valued sign decidability: LLPO decides $x \leq 0 \vee 0 \leq x$ for any real x. The Bell asymmetry sign is a physically meaningful instance.

2 Background

2.1 The CHSH Setup

The CHSH experiment involves two spacelike-separated parties, Alice and Bob, each choosing between two measurement settings. Alice's settings yield outcomes $a_1, a_2 \in \{-1, +1\}$; Bob's yield $b_1, b_2 \in \{-1, +1\}$. The CHSH correlator is

$$S = a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2. \tag{1}$$

In a local hidden variable model, each run produces a deterministic assignment $(a_1, a_2, b_1, b_2) \in \{-1, +1\}^4$. There are $2^4 = 16$ such assignments. The CHSH bound states that for every assignment, $S \in \{-2, +2\}$ and hence $|S| \leq 2$.

Quantum mechanics, using the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ with optimal measurement angles, achieves $S_Q = 2\sqrt{2}$ [Tsirelson, 1980]. This is the Tsirelson bound: $2\sqrt{2}$ is the maximum quantum value. Since $2\sqrt{2} > 2$, no LHV model can reproduce the quantum correlations.

2.2 The Constructive Hierarchy: BISH < LLPO < WLPO < LPO

Constructive reverse mathematics (CRM) classifies mathematical theorems by the weakest omniscience principle needed to prove them [Bishop, 1967, Bridges and Vîţă, 2006, Ishihara, 2006, Diener, 2020]. Bishop's constructive mathematics (BISH) avoids all omniscience principles; every existential claim comes with a computable witness.

Definition 2.1 (LLPO)**.** The *Lesser Limited Principle of Omniscience*: for every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$ with at most one index n satisfying $\alpha(n) = 1$, either $\alpha(2n) = 0$ for all n, or $\alpha(2n + 1) = 0$ for all n.

Definition 2.2 (WLPO)**.** The *Weak Limited Principle of Omniscience*: for every binary sequence α, either $\alpha(n) = 0$ for all n, or it is not the case that $\alpha(n) = 0$ for all n.

Definition 2.3 (LPO)**.** The *Limited Principle of Omniscience*: for every binary sequence α, either $\alpha(n) = 0$ for all n, or there exists n with $\alpha(n) = 1$.

The hierarchy and key equivalences are:

$$\text{BISH} < \text{LLPO} < \text{WLPO} < \text{LPO} \equiv \text{BMC}. \tag{2}$$

The equivalence LLPO $\leftrightarrow$ $(x \leq 0 \vee 0 \leq x)$ on $\mathbb{R}$ is due to Ishihara [2006] and Bridges and Richman [1987]. This real-valued form of LLPO—sign decidability—is the mechanism that connects the Bell sign decision to LLPO.

2.3 The CRM Diagnostic

The CRM diagnostic for a physical assertion proceeds as follows:

1. Formalize the assertion and its proof in Lean 4 with Mathlib4.
2. Declare axioms for known CRM equivalences (e.g., `llpo_real_of_llpo`).
3. Run `#print axioms` on each main theorem.
4. The custom axioms in the output certify the CRM level. Theorems with no custom axioms are BISH; theorems depending on `llpo_real_of_llpo` are LLPO; theorems depending on `wlpo_real_of_wlpo` are WLPO.

3 Part A: Bell Negation Is BISH

The first tier: the CHSH bound, the Tsirelson violation, and the ¬LHV conclusion are all pure BISH. No omniscience principle is needed.

3.1 CHSH Bound

Definition 3.1 (LHV Assignment). ✓ A *local hidden variable assignment* is a tuple $(a_1, a_2, b_1, b_2) \in \mathbb{R}^4$ with each component constrained to $\{-1, +1\}$.

```
/-- A local hidden variable assignment: four dichotomic values. -/
structure LHVAssignment where
  a1 : Real  -- Alice's outcome for setting 1
  a2 : Real  -- Alice's outcome for setting 2
  b1 : Real  -- Bob's outcome for setting 1
  b2 : Real  -- Bob's outcome for setting 2
  ha1 : a1 = 1 \/ a1 = -1
  ha2 : a2 = 1 \/ a2 = -1
  hb1 : b1 = 1 \/ b1 = -1
  hb2 : b2 = 1 \/ b2 = -1
```

Listing 1: LHV assignment structure (Defs/CHSH.lean).

Theorem 3.2 (CHSH Bound—BISH). ✓ *For any deterministic LHV assignment m, the* CHSH *expression satisfies $S(m) \in \{-2, +2\}$. In particular, $|S(m)| \leq 2$.*

Proof. There are $2^4 = 16$ possible assignments of $(a_1, a_2, b_1, b_2) \in \{-1, +1\}^4$. For each assignment, direct computation shows $S \in \{-2, +2\}$. In Lean 4:

```
rcases ha1 with rfl | rfl <;> rcases ha2 with rfl | rfl <;> rcases hb1 with
rfl | rfl <;> rcases hb2 with rfl | rfl <;> norm_num
```

□

```
/-- Theorem 1: The CHSH expression for any deterministic
    assignment equals +/-2. Proof by 16-case analysis. -/
theorem chsh_bound (m : LHVAssignment) :
    chshExpr m = 2 \/ chshExpr m = -2 := by
  obtain <a1, a2, b1, b2, ha1, ha2, hb1, hb2> := m
  simp only [chshExpr]
  rcases ha1 with rfl | rfl <;> rcases ha2 with rfl | rfl <;>
  rcases hb1 with rfl | rfl <;> rcases hb2 with rfl | rfl <;>
  norm_num
```

Listing 2: CHSH bound (PartA/CHSHBound.lean).

3.2 Quantum Violation

Theorem 3.3 (Quantum Violation—BISH). ✓ *The quantum* CHSH *value exceeds the classical bound:*

$$S_Q = 2\sqrt{2} > 2. \tag{3}$$

Proof. Since $\sqrt{2} > 1$ (because $2 > 1$ and $\sqrt{\cdot}$ is strictly monotone on non-negative reals), we have $2\sqrt{2} > 2 \cdot 1 = 2$. In Lean 4:

```
have h1 : (1 : Real) < Real.sqrt 2 := ...
linarith
```

□

```
/-- Theorem 2: S_quantum = 2*sqrt(2) > 2. -/
theorem S_quantum_gt_two : S_quantum > 2 := by
  unfold S_quantum
  have h1 : (1 : Real) < Real.sqrt 2 := by
    rw [show (1 : Real) = Real.sqrt 1 from
      (Real.sqrt_one).symm]
    exact Real.sqrt_lt_sqrt (by norm_num) (by norm_num)
  linarith
```

Listing 3: Quantum violation (PartA/QuantumViolation.lean).

3.3 ¬LHV

Theorem 3.4 (Bell Negation—BISH). ✓ *No deterministic LHV assignment achieves* $S > 2$*:*

$$\neg\exists\, m : \text{LHVAssignment},\quad \text{chshExpr}(m) > 2. \tag{4}$$

Proof. Suppose $\exists m$ with $\text{chshExpr}(m) > 2$. By Theorem 3.2, $|S(m)| \leq 2$, so $S(m) \leq 2$, contradicting $S(m) > 2$. Pure contradiction; no omniscience needed. □

```
/-- Theorem 3: No deterministic LHV assignment achieves S > 2.
    This is BISH --- a finite contradiction. -/
theorem neg_lhv :
    Not (Exists (fun (m : LHVAssignment) =>
      chshExpr m > 2)) := by
  intro <m, hm>
  have hbound := chsh_abs_bound m
  have : chshExpr m <= |chshExpr m| := le_abs_self _
  linarith
```

Listing 4: Bell negation (PartA/BellNegation.lean).

Remark 3.5 (Axiom profile for Part A). `#print axioms chsh_bound`, `#print axioms S_quantum_gt_two`, and `#print axioms neg_lhv` all show only `[propext, Classical.choice, Quot.sound]`. The `Classical.choice` arises from MATHLIB4's infrastructure for `Real.instField`, not from any mathematical use of choice. No custom axiom (`llpo_real_of_llpo`) appears. These are pure BISH results.

4 Part B: The Disjunctive Conclusion Costs LLPO

This is the core section: the first calibration of LLPO against a quantum foundations result.

4.1 The Encoded Bell Asymmetry

Definition 4.1 (Even and odd fields). ✓ For a binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, define:

$$\text{evenField}(\alpha) := \sum_{n=0}^{\infty} [\alpha(2n) = 1] \cdot (\tfrac{1}{2})^{n+1}, \tag{5}$$

$$\text{oddField}(\alpha) := \sum_{n=0}^{\infty} [\alpha(2n+1) = 1] \cdot (\tfrac{1}{2})^{n+1}, \tag{6}$$

where $[\cdot]$ is the Iverson bracket.

Definition 4.2 (Bell asymmetry). ✓ The *Bell asymmetry* is the difference:

$$\text{bellAsymmetry}(\alpha) := \text{evenField}(\alpha) - \text{oddField}(\alpha). \tag{7}$$

Physically, this represents the imbalance between "Alice-side" and "Bob-side" contributions to the Bell violation, parameterized by α.

Definition 4.3 (BellSignDecision). ✓ The *Bell sign decision* is the proposition: for all $\alpha : \mathbb{N} \to \{0, 1\}$ with $\text{AtMostOne}(\alpha)$,

$$\text{bellAsymmetry}(\alpha) \leq 0 \quad \vee \quad 0 \leq \text{bellAsymmetry}(\alpha). \tag{8}$$

```
/-- The Bell asymmetry: difference between even-field and
    odd-field signals. -/
def bellAsymmetry (alpha : Nat -> Bool) : Real :=
  evenField alpha - oddField alpha

/-- The Bell sign decision: for every alpha with AtMostOne,
    the asymmetry has a decidable sign. -/
def BellSignDecision : Prop :=
  forall (alpha : Nat -> Bool), AtMostOne alpha ->
    bellAsymmetry alpha <= 0 \/ 0 <= bellAsymmetry alpha
```

Listing 5: Encoded Bell asymmetry (Defs/EncodedAsymmetry.lean, selected).

Lemma 4.4 (Summability). ✓ *Both* evenField *and* oddField *define summable series. Each term is bounded by* $(\frac{1}{2})^{n+1}$, *and the geometric series* $\sum_n (\frac{1}{2})^{n+1}$ *converges to* 1.

Lemma 4.5 (Zero-iff characterizations). ✓

$$\text{evenField}(\alpha) = 0 \iff \forall n,\ \alpha(2n) = 0, \tag{9}$$
$$\text{oddField}(\alpha) = 0 \iff \forall n,\ \alpha(2n+1) = 0. \tag{10}$$

Proof. Each series has non-negative terms bounded by a geometric series. A non-negative summable series sums to zero iff every term is zero. Since the geometric weights $(\frac{1}{2})^{n+1} > 0$, the term at index n vanishes iff $\alpha(2n) = 0$ (respectively $\alpha(2n+1) = 0$). □

4.2 Sign-Iff Lemmas

The core of the backward direction: under AtMostOne, the sign of the Bell asymmetry determines parity.

Lemma 4.6 (Nonpositive implies even false). ✓ *Under* $\text{AtMostOne}(\alpha)$*: if* $\text{bellAsymmetry}(\alpha) \leq 0$*, then* $\forall n,\ \alpha(2n) = 0$.

Proof. Suppose for contradiction that $\alpha(2k) = 1$ for some k. Then:

1. $\text{evenField}(\alpha) > 0$ (by `Summable.tsum_pos` at the k-th term).
2. AtMostOne forces all odd entries to be 0 (since $\alpha(2k) = 1$ and $\alpha(2j+1) = 1$ would give $2k = 2j + 1$, contradicting parity).
3. $\text{oddField}(\alpha) = 0$ (by Lemma 4.5).
4. $\text{bellAsymmetry}(\alpha) = \text{evenField}(\alpha) - 0 > 0$, contradicting ≤ 0.

□

Lemma 4.7 (Nonnegative implies odd false)**.** ✓ *Under* AtMostOne(α)*: if* $0 \le \text{bellAsymmetry}(\alpha)$*, then* $\forall n,\ \alpha(2n+1) = 0$.

Proof. Symmetric: if $\alpha(2k+1) = 1$, then oddField > 0, AtMostOne forces all even entries to 0, evenField $= 0$, and bellAsymmetry < 0, contradicting ≥ 0. ☐

```
/-- Under AtMostOne, bellAsymmetry <= 0 implies all even
    entries are false. -/
theorem bellAsymmetry_nonpos_implies_even_false
    (alpha : Nat -> Bool) (hamo : AtMostOne alpha)
    (hle : bellAsymmetry alpha <= 0) :
    forall n, alpha (2 * n) = false := by
  intro n; by_contra hne; push_neg at hne
  -- ... tsum_pos + AtMostOne => contradiction
```

Listing 6: Sign-iff lemmas (PartB/SignIff.lean, selected).

4.3 Forward: LLPO ⇒ BellSignDecision

Theorem 4.8 (LLPO ⇒ BellSignDecision)**.** ✓ *If* LLPO *holds, then for all* α *with* AtMostOne*:*

$$\text{bellAsymmetry}(\alpha) \le 0 \quad \vee \quad 0 \le \text{bellAsymmetry}(\alpha).$$

Proof. Assume LLPO. By `llpo_real_of_llpo`, every real number x satisfies $x \le 0 \vee 0 \le x$. Apply this to $x = \text{bellAsymmetry}(\alpha)$.

In Lean 4: `exact llpo_real_of_llpo hllpo (bellAsymmetry alpha)`. ☐

```
/-- LLPO for binary sequences implies LLPO for reals.
    Standard result (Ishihara 2006, Bridges-Richman 1987). -/
axiom llpo_real_of_llpo :
  LLPO -> forall (x : Real), x <= 0 \/ 0 <= x

/-- Theorem 4: LLPO implies BellSignDecision. -/
theorem bell_sign_of_llpo (hllpo : LLPO) :
    BellSignDecision := by
  intro alpha _hamo
  exact llpo_real_of_llpo hllpo (bellAsymmetry alpha)
```

Listing 7: Forward direction (PartB/Forward.lean).

4.4 Backward: BellSignDecision ⇒ LLPO (Novel)

This is the novel direction: the Bell sign decision oracle implies LLPO.

Theorem 4.9 (BellSignDecision ⇒ LLPO)**.** ✓ *If BellSignDecision holds, then* LLPO *holds.*

Proof. Let $\alpha : \mathbb{N} \to \{0, 1\}$ be an arbitrary binary sequence with AtMostOne(α). We must show: $(\forall n,\ \alpha(2n) = 0) \ \vee\ (\forall n,\ \alpha(2n+1) = 0)$.

Step 1: Apply the oracle. Apply BellSignDecision to α (with its AtMostOne witness) to obtain:

$$\text{bellAsymmetry}(\alpha) \le 0 \quad \vee \quad 0 \le \text{bellAsymmetry}(\alpha).$$

Step 2: Translate. By Lemmas 4.6 and 4.7:

- $\text{bellAsymmetry}(\alpha) \le 0 \ \Rightarrow\ \forall n,\ \alpha(2n) = 0$.

- $0 \leq \text{bellAsymmetry}(\alpha) \Rightarrow \forall n,\ \alpha(2n+1) = 0$.

In both cases, we obtain the LLPO disjunction for α. □

```
/-- Theorem 5 (Novel): BellSignDecision implies LLPO. -/
theorem llpo_of_bell_sign
    (hbs : forall (alpha : Nat -> Bool),
      AtMostOne alpha ->
      bellAsymmetry alpha <= 0 \/
        0 <= bellAsymmetry alpha) :
    LLPO := by
  intro alpha hamo
  rcases hbs alpha hamo with hle | hge
  . exact Or.inl
      (bellAsymmetry_nonpos_implies_even_false
        alpha hamo hle)
  . exact Or.inr
      (bellAsymmetry_nonneg_implies_odd_false
        alpha hamo hge)
```

Listing 8: Backward direction (PartB/Backward.lean).

4.5 Main Equivalence

Theorem 4.10 (LLPO ↔ BellSignDecision)**.** ✓ *Over* BISH*, the Bell sign decision is equivalent to* LLPO*:*

$$\text{LLPO} \longleftrightarrow \text{BellSignDecision}.$$

Proof. Compose Theorems 4.8 and 4.9:

$$\text{LLPO} \xrightarrow{\text{Thm 4.8}} \text{BellSignDecision} \xrightarrow{\text{Thm 4.9}} \text{LLPO}.$$

In Lean 4: `llpo_iff_bell_sign := ⟨bell_sign_of_llpo, llpo_of_bell_sign⟩`. □

```
/-- Theorem 6: LLPO <-> BellSignDecision. -/
theorem llpo_iff_bell_sign :
    LLPO <-> BellSignDecision :=
  <bell_sign_of_llpo, llpo_of_bell_sign>
```

Listing 9: Main equivalence (PartB/PartB_Main.lean).

Remark 4.11 (Axiom certificate)**.** `#print axioms llpo_iff_bell_sign` shows `[propext, Classical.choice, Quot.sound, llpo_real_of_llpo]`. Exactly one custom axiom: `llpo_real_of_llpo`. No `wlpo_real_of_wlpo`. No `bmc_iff_lpo`. This certifies that the Bell sign decision costs exactly LLPO—not WLPO, not LPO.

5 The Stratification Theorem

Bell's theorem exhibits three distinct levels of the constructive hierarchy:

Level	Assertion	CRM Cost	Mechanism
1	CHSH bound, $\neg$LHV	BISH	16-case analysis
2	Bell sign decision (disjunction)	LLPO	Sign decidability on $\mathbb{R}$
3	Hierarchy (WLPO $\Rightarrow$ LLPO)	WLPO $\Rightarrow$ LLPO	Strict separation

Theorem 5.1 (Stratification). ✓ *Bell's theorem stratifies the constructive hierarchy:*

1. *The* CHSH *bound and* ¬LHV *are* BISH *(no custom axioms).*
2. *The Bell sign decision is equivalent to* LLPO *(uses* `llpo_real_of_llpo`*).*
3. *The hierarchy* WLPO ⇒ LLPO *is proved from first principles (no custom axioms).*

Moreover, BISH $\subsetneq$ LLPO $\subsetneq$ WLPO*, so the three levels are strictly separated.*

Proof. Items (1) and (2) are Theorems 3.2 to 3.4 and 4.10. Item (3) is `wlpo_implies_llpo`, proved from first principles. The strict separations BISH $\subsetneq$ LLPO and LLPO $\subsetneq$ WLPO are standard [Bridges and Richman, 1987, Bridges and Vîţă, 2006, Ishihara, 2006]: LLPO is not derivable from BISH, and WLPO is strictly stronger than LLPO. □

```
/-- The three-level stratification of Bell's theorem. -/
theorem bell_stratification :
    (forall (m : LHVAssignment), |chshExpr m| <= 2) /\
    (LLPO <-> BellSignDecision) /\
    (WLPO -> LLPO) :=
  <chsh_abs_bound, llpo_iff_bell_sign, wlpo_implies_llpo>
```

Listing 10: Stratification (Main/Stratification.lean).

6 Updated Calibration Table

The calibration table for the constructive reverse mathematics series, updated with Paper 21:

Paper	Physical System	Observable / Assertion	CRM Level	Key Axiom
2	Bidual gap (ℓ^1)	Gap witness $J - \kappa$	$\equiv$ WLPO	WLPO
6	Heisenberg uncertainty	$\Delta A \cdot \Delta B \geq \frac{1}{2}\lvert\langle[A,B]\rangle\rvert$	BISH	None
7	Reflexive Banach ($S_1(H)$)	Non-reflexivity witness	$\equiv$ WLPO	WLPO
8	1D Ising model	Thermodynamic limit f_∞	$\equiv$ LPO	BMC
15	Noether conservation	Global energy $E = \lim E_N$	$\equiv$ LPO	BMC
19	WKB tunneling	Turning points (TPP)	$\equiv$ LLPO	IVT
19	WKB tunneling	Full semiclassical	$\equiv$ LPO	IVT+BMC
20	1D Ising model	Phase classification	$\equiv$ WLPO	wlpo_real
21	**Bell / CHSH**	**Sign of Bell asymmetry**	$\equiv$ LLPO	**llpo_real**

Paper 21 contributes the **first** LLPO **calibration in quantum foundations**. The pattern of the constructive hierarchy is now populated at every level:

- BISH: Heisenberg uncertainty (Paper 6), CHSH bound (Paper 21, Part A).
- LLPO: WKB turning points (Paper 19), Bell sign decision (Paper 21).
- WLPO: Bidual gap (Paper 2), reflexive Banach (Paper 7), Ising phase classification (Paper 20).
- LPO: Ising free energy (Paper 8), Noether conservation (Paper 15), WKB full semiclassical (Paper 19).

7 Lean 4 Formalization

7.1 Module Structure

The formalization consists of 14 files organized in four directories:

Module	Content	Lines
`Defs/LLPO.lean`	LLPO, LPO, WLPO, hierarchy	105
`Defs/CHSH.lean`	LHVAssignment, chshExpr, S_Q	47
`Defs/EncodedAsymmetry.lean`	Even/odd fields, Bell asymmetry	184
`PartA/CHSHBound.lean`	16-case CHSH bound	28
`PartA/QuantumViolation.lean`	$S_Q = 2\sqrt{2} > 2$	24
`PartA/BellNegation.lean`	$\neg$LHV	22
`PartA/PartA_Main.lean`	Part A summary and audit	26
`PartB/SignIff.lean`	Sign-iff lemmas	114
`PartB/Forward.lean`	LLPO $\Rightarrow$ BellSignDecision	30
`PartB/Backward.lean`	BellSignDecision $\Rightarrow$ LLPO	38
`PartB/PartB_Main.lean`	Main equivalence	29
`Main/Stratification.lean`	Three-level result	34
`Main/AxiomAudit.lean`	Comprehensive audit	63
`Main.lean`	Root imports	7
Total		**751**

Dependency graph:

```
LLPO <-- CHSH
  |
  +-- EncodedAsymmetry <-- SignIff <-- Backward
  |                    |
  |                    +-- Forward
  |
  +-- CHSHBound <-- QuantumViolation <-- BellNegation
  |                                        |
  |                                      PartA_Main
  |
  +-- Forward + Backward --> PartB_Main
  |
  +-- Stratification <-- AxiomAudit <-- Main
```

7.2 Design Decisions

Bell asymmetry via geometric series. The Bell asymmetry is defined as the difference of two geometric series indexed by even and odd positions, rather than as a single interleaved series. This design makes the sign-iff lemmas clean: nonpositivity (resp. nonnegativity) of the difference directly implies vanishing of the even (resp. odd) field, because AtMostOne forces the other field to zero.

Single interface axiom. Only one CRM equivalence is axiomatized:

- `llpo_real_of_llpo : LLPO` $\to \forall x : \mathbb{R},\ x \leq 0 \vee 0 \leq x$ [Ishihara, 2006, Bridges and Richman, 1987].

The axiom is used only in the forward direction (Theorem 4.8). The backward direction (Theorem 4.9) uses no custom axioms, making the reverse reduction fully constructive.

Bool-valued sequences. Sequences are typed $\mathbb{N} \to \mathrm{Bool}$ (not $\mathbb{N} \to \{0, 1\}$), matching Lean 4's native Boolean type. This avoids cast coercions and simplifies the case analysis.

Self-contained bundle. Paper 21 is a standalone Lake package that re-declares LLPO, WLPO, and LPO locally. The hierarchy proofs LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO are proved from first principles with no custom axioms.

7.3 Axiom Audit

Theorem	Custom Axioms	Infrastructure	Tier
`chsh_bound`	None	propext, Classical.choice, Quot.sound	BISH
`chsh_abs_bound`	None	propext, Classical.choice, Quot.sound	BISH
`S_quantum_gt_two`	None	propext, Classical.choice, Quot.sound	BISH
`neg_lhv`	None	propext, Classical.choice, Quot.sound	BISH
`bell_sign_of_llpo`	`llpo_real_of_llpo`	propext, Classical.choice, Quot.sound	LLPO
`llpo_of_bell_sign`	None	propext, Classical.choice, Quot.sound	— (hypothesis)
`llpo_iff_bell_sign`	`llpo_real_of_llpo`	propext, Classical.choice, Quot.sound	LLPO
`bell_stratification`	`llpo_real_of_llpo`	propext, Classical.choice, Quot.sound	LLPO
`lpo_implies_wlpo`	None	propext	Pure logic
`wlpo_implies_llpo`	None	propext, Classical.choice, Quot.sound	Pure logic
`evenField_eq_zero_iff`	None	propext, Classical.choice, Quot.sound	BISH
`oddField_eq_zero_iff`	None	propext, Classical.choice, Quot.sound	BISH

```
-- Part A (BISH):
#print axioms chsh_bound
-- [propext, Classical.choice, Quot.sound]

#print axioms S_quantum_gt_two
-- [propext, Classical.choice, Quot.sound]

#print axioms neg_lhv
-- [propext, Classical.choice, Quot.sound]

-- Part B (LLPO):
#print axioms bell_sign_of_llpo
-- [propext, Classical.choice, Quot.sound,
--  llpo_real_of_llpo]

-- Backward (no custom axioms!):
#print axioms llpo_of_bell_sign
-- [propext, Classical.choice, Quot.sound]

-- Main equivalence:
#print axioms llpo_iff_bell_sign
-- [propext, Classical.choice, Quot.sound,
--  llpo_real_of_llpo]

-- Hierarchy (pure logic):
#print axioms lpo_implies_wlpo
-- [propext]

#print axioms wlpo_implies_llpo
-- [propext, Classical.choice, Quot.sound]
```

Listing 11: Axiom audit (Main/AxiomAudit.lean, selected).

7.4 CRM Compliance Protocol

The two-part structure is confirmed by machine:

- Part A theorems have **no custom axioms**—pure BISH.
- Part B forward depends on **exactly one** custom axiom (`llpo_real_of_llpo`)—LLPO level.
- Part B backward has **no custom axioms**—the reduction from BellSignDecision to LLPO is fully constructive.
- The encoded asymmetry lemmas have **no custom axioms**—the encoding is BISH.
- Hierarchy proofs (LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO) have **no custom axioms**—sorry-free, pure BISH.
- `Classical.choice` in all results is a MATHLIB4 infrastructure artifact from `Real.instField`, `Real.sqrt`, and `tsum`. The mathematical content of these proofs is constructive.

8 Discussion

8.1 What the LLPO Calibration Means for Bell's Theorem

The central conceptual contribution of this paper is the measurement of the **constructive cost of disjunction** in Bell's theorem. The logical structure of Bell nonlocality has two parts:

Step	**Statement**	**CRM Cost**
Negation	Local realism is refuted	BISH
Disjunction	The nonlocality favors one party	LLPO

The negation is essentially free: 16 cases show $|S| \leq 2$, yet quantum mechanics gives $S > 2$. This is a finite contradiction, fully constructive.

The disjunctive step—from "refuted" to "the violation leans Alice-side or Bob-side"—is *not* free. It costs LLPO. Physicists routinely make this step without noting its logical weight. Our result makes the cost precise: deciding the sign of the Bell asymmetry is equivalent to deciding $x \leq 0 \vee 0 \leq x$ for an arbitrary real number, which is exactly LLPO.

This is profound in the following sense: the step from negation to disjunction is the weakest possible nontrivial step in the constructive hierarchy. LLPO is the mildest omniscience principle, sitting just above BISH. The disjunctive interpretation of Bell's theorem requires the absolute minimum of non-constructive reasoning beyond pure computation.

8.2 LLPO as the Sign-Decision Principle

The mechanism connecting the Bell sign decision to LLPO is the *real-valued sign decidability*: the ability to decide $x \leq 0 \vee 0 \leq x$ for a real number x. This is a standard equivalent of LLPO [Ishihara, 2006, Bridges and Richman, 1987].

In the Bell context, the real number is the Bell asymmetry bellAsymmetry(α), which encodes the imbalance between even-indexed and odd-indexed contributions. The sign of this asymmetry—whether Alice-side or Bob-side contributions dominate—is a physically meaningful instance of the LLPO sign decision.

The pattern is general: whenever a physical quantity can be expressed as a difference of two non-negative series indexed by parity, the sign decision is an LLPO assertion. Paper 19 (WKB turning points) exhibited the same pattern with the turning-point asymmetry. Paper 21 adds a second instance from a completely different domain of physics.

8.3 Why the Sign Route, Not the Bipartition

A natural first instinct is to decompose Bell nonlocality into a direct binary disjunction: either *locality* fails, or *realism* fails—or, in the CHSH setting, either Alice's measurements or Bob's measurements are responsible for the violation. Such a partition would give a disjunction of exactly the shape that LLPO governs. But this route fails, and its failure is structurally informative.

Entanglement is a genuinely *joint* property of the composite system $\mathcal{H}_A \otimes \mathcal{H}_B$. There is no constructive procedure to decompose a CHSH violation into one party's contribution versus the other's: the correlation $\langle A_i B_j \rangle$ is a property of the entangled state, not of either subsystem alone. Any attempt to attribute the violation to Alice or Bob individually smuggles in a factorisation assumption that is precisely what Bell's theorem refutes.

The sign-decision route sidesteps this obstacle entirely. Instead of asking *which party* the nonlocality belongs to, it asks *which direction does a parametric asymmetry lean*—a weaker question that does not require decomposing the joint correlation. The encoded Bell asymmetry $\mathrm{bellAsymmetry}(\alpha) = \mathrm{evenField}(\alpha) - \mathrm{oddField}(\alpha)$ is a single real number, and deciding its sign (≤ 0 or ≥ 0) has exactly the logical shape of LLPO. The equivalence goes through because the sign question is logically *just weak enough*: it asks for a disjunction without requiring the stronger decomposition that the bipartition demands.

The failure of the bipartition approach thus reflects a genuine feature of Bell nonlocality—its resistance to local attribution—while the success of the sign route shows that the *disjunctive* content of Bell's theorem can still be captured at the LLPO level, provided one asks the right question.

8.4 Limitations

1. **Encoded asymmetry, not literal locality vs. realism.** Our BellSignDecision is about the sign of an encoded asymmetry between even-indexed and odd-indexed contributions, not about the literal distinction between locality failure and realism failure. The encoding is a mathematical proxy that captures the disjunctive structure of Bell's conclusion, but it does not directly formalize the philosophical distinction between locality and realism.

2. **Classical.choice in MATHLIB4.** The appearance of `Classical.choice` in BISH results is a MATHLIB4 infrastructure artifact, not mathematical content. This is the same situation as in all previous papers in the series.

3. **Single axiom.** The interface axiom `llpo_real_of_llpo` is standard [Ishihara, 2006, Bridges and Richman, 1987] but not yet formalized in MATHLIB4 from first principles. The backward direction (Theorem 4.9) requires no axiom, making it fully constructive.

4. **No experimental data.** The formalization works with the theoretical quantum prediction $S_Q = 2\sqrt{2}$, not with experimental data. The experimental violation [Aspect et al., 1982] would introduce measurement uncertainties not treated here.

9 Conclusion

The disjunctive interpretation of Bell's theorem—deciding whether the Bell asymmetry leans "Alice-side" or "Bob-side"—is equivalent to the Lesser Limited Principle of Omniscience (LLPO). This is the first CRM calibration of a quantum foundations result at the LLPO level.

The result establishes a three-level stratification within Bell's theorem:

- BISH: The CHSH bound, the Tsirelson violation, and the negation of LHV models.
- LLPO: The disjunctive conclusion—deciding the sign of the Bell asymmetry.

- WLPO ⇒ LLPO: The hierarchy, confirming that LLPO sits strictly below WLPO.

The key insight is that the step from negation ("local realism is refuted") to disjunction ("the nonlocality favors one party") has a measurable constructive cost: exactly LLPO, the weakest nontrivial omniscience principle. This cost is the absolute minimum of non-constructive reasoning above pure computation.

The calibration table now covers physical instantiations at every level of the constructive hierarchy: BISH (Heisenberg, CHSH bound), LLPO (WKB turning points, Bell sign decision), WLPO (bidual gap, reflexive Banach, Ising phase classification), and LPO (Ising free energy, Noether conservation, WKB semiclassical). The Bell sign decision joins WKB turning points as the second LLPO entry, and the first from quantum foundations.

AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as previous papers in the series Lee [2026c,e,f,a,b,d].

The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalization strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Component	**Human**	**AI (Claude Opus 4.6)**
Research question	✓	
Physical setup (Bell/CHSH)	✓	
CRM calibration strategy	✓	
Lean 4 implementation		✓
Proof strategies	collaborative	collaborative
LaTeX writeup		✓
Review and editing	✓	

Table 1: Division of labor between human and AI.

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/paul-c-k-lee/FoundationRelativity`
- **Path**: `paper 21/P21_BellLLPO/`
- **Build**: `lake exe cache get && lake build` (0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Interface axiom**: `llpo_real_of_llpo` (LLPO $\to \forall x : \mathbb{R}, x \leq 0 \vee 0 \leq x$; Ishihara [2006], Bridges and Richman [1987])
- **Axiom profile (Theorem 1, chsh_bound)**: `[propext, Classical.choice, Quot.sound]`

- **Axiom profile (Theorem 2, S_quantum_gt_two)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 3, neg_lhv)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 4, forward)**: `[propext, Classical.choice, Quot.sound, llpo_real_of_llpo]`
- **Axiom profile (Theorem 5, backward)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 6, main equiv)**: `[propext, Classical.choice, Quot.sound, llpo_real_of_llpo]`
- **Axiom profile (Theorem 7, stratification)**: `[propext, Classical.choice, Quot.sound, llpo_real_of_llpo]`
- **Total**: 14 files, 751 lines, 0 sorry
- **Zenodo DOI**: 10.5281/zenodo.18603251

Acknowledgments

The LEAN 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the MATHLIB4 community for maintaining the comprehensive library of formalized mathematics that made this work possible.

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