Formulation-Invariance of the Logical Cost of the Thermodynamic Limit:

A Combinatorial Proof for the 1D Ising Model

A Lean 4 Formalization

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Abstract

We prove that the logical cost of the thermodynamic limit for the one-dimensional Ising model is *formulation-invariant*: the same axiom profile arises from a purely combinatorial derivation as from the transfer-matrix approach of Paper 8 Lee [2026d]. Specifically, we re-derive the two main results of Paper 8—-(A) BISH dispensability of finite-size error bounds, and (B) LPO equivalence of the thermodynamic limit—using only finite sums over $\{-1, +1\}^N$, the binomial parity sieve identity, and elementary arithmetic of hyperbolic functions. No transfer matrices, eigenvalues, linear algebra, or functional analysis are used at any point. The partition function identity $Z_N(\beta) = (2\cosh\beta)^N + (2\sinh\beta)^N$ is derived from bond variables and the combinatorial parity sieve, replacing the spectral decomposition $Z_N = \mathrm{Tr}(T^N) = \lambda_+^N + \lambda_-^N$. The resulting axiom profiles are identical: Part A uses no omniscience principles; Part B establishes LPO $\leftrightarrow$ BMC with the same axiom set. The combined Lean 4 formalization comprises 1319 lines across 18 modules with zero sorries. No Mathlib modules from `LinearAlgebra.*` or `Analysis.NormedSpace.*` are imported, enforcing strict formulation independence from Paper 8.

Contents

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1 Introduction

Paper 8 Lee [2026d] established two results about the 1D Ising model, formalized in Lean 4:

(A) The finite-size error bound $|f_N(\beta) - f_\infty(\beta)| \le \frac{1}{N}\tanh(\beta)^N$ is provable in BISH without omniscience.

(B) The existence of the thermodynamic limit as a completed real number is equivalent over BISH to LPO, via the known equivalence LPO $\leftrightarrow$ BMC of Bridges and Vîţă Bridges and Vîţă [2006].

The proofs in Paper 8 relied on the transfer matrix formulation: the 2×2 matrix T with entries $T(s, s') = \exp(\beta \cdot s \cdot s')$, its eigenvalues $\lambda_+ = 2\cosh\beta$ and $\lambda_- = 2\sinh\beta$, and the trace identity $Z_N = \mathrm{Tr}(T^N) = \lambda_+^N + \lambda_-^N$. The Lean 4 formalization imported Mathlib modules from `LinearAlgebra.Matrix.*`.

A natural question arises: is the logical cost (BISH for Part A, LPO for Part B) a feature of the *physics* (the Ising model and the thermodynamic limit), or an artifact of the *mathematical formalism* (the transfer matrix / linear algebra framework)?

This paper answers the question by providing a second, completely independent proof of both results using purely combinatorial methods. The partition function is computed from the configuration sum over $\{-1,+1\}^N$ via bond variables and the binomial parity sieve, yielding the same algebraic formula $Z_N(\beta) = (2\cosh\beta)^N + (2\sinh\beta)^N$ without invoking matrices, eigenvalues, or any linear algebra. The error bound and LPO equivalence then follow by the same chain of elementary inequalities and encoding arguments as in Paper 8, but with the combinatorial partition function identity as the starting point.

The formulation-invariance claim is verified at two levels:

1. *Axiom audit*: the `#print axioms` output for both formulations is identical.
2. *Import audit*: the combinatorial formalization imports no Mathlib module from `LinearAlgebra.*`, `Analysis.NormedSpace.*`, or any functional-analytic library. The two formalizations share only the unavoidable common substrate: real arithmetic (`Real.exp`, `Real.log`, `Real.cosh`) and the logical principles under test (LPO, BMC).

The paper is organized as follows. Section 2 reviews the constructive framework and the 1D Ising model. Section 3 derives the partition function identity from bond variables and the parity sieve. Section 4 presents the BISH dispensability proof. Section 5 presents the LPO calibration. Section 6 performs the formulation-invariance verification. Section 8 describes the Lean 4 formalization. Appendix A collects elementary inequalities.

2 Preliminaries

2.1 Constructive Frameworks

We work within Bishop-style constructive mathematics (BISH): intuitionistic logic with countable and dependent choice Bishop [1967], Bishop and Bridges [1985]. The key omniscience principles form a strict hierarchy over BISH:

Definition 2.1 (LPO)**.** ✓ The *Limited Principle of Omniscience* is

$$\text{LPO} \;:\equiv\; \forall \alpha : \mathbb{N} \to \{0,1\},\; \bigl(\forall n,\; \alpha(n) = 0\bigr) \;\vee\; \bigl(\exists n,\; \alpha(n) = 1\bigr).$$

Definition 2.2 (BMC)**.** ✓ *Bounded Monotone Convergence* is the assertion that every bounded non-decreasing sequence of reals has a limit: for every $a : \mathbb{N} \to \mathbb{R}$ with $a_n \le a_{n+1}$ and $a_n \le M$ for all n, there exists $L \in \mathbb{R}$ and a convergence modulus such that for all $\varepsilon > 0$, there exists N_0 with $|a_N - L| < \varepsilon$ for all $N \ge N_0$.

The equivalence LPO $\leftrightarrow$ BMC was established by Bridges and Vîţă Bridges and Vîţă [2006].

Remark 2.3 (Constructive status of LPO)**.** LPO is classically trivial (an instance of excluded middle) but constructively independent: it is neither provable nor refutable in BISH. Crucially, LPO provides a *witness* in the second disjunct, not merely its double negation. This witness-providing character makes LPO strictly stronger than WLPO.

2.2 The 1D Ising Model (Combinatorial Formulation)

Fix a positive integer N (the number of spins). The configuration space is $\Omega_N = \{-1,+1\}^N$. The Hamiltonian with periodic boundary conditions ($\sigma_{N+1} := \sigma_1$) and coupling $J > 0$ is

$$H_N(\sigma) = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}.$$

For simplicity we set $J = 1$ in Part A.

Definition 2.4 (Partition function)**.** ✓ The partition function is the configuration sum

$$Z_N(\beta) := \sum_{\sigma \in \{-1,+1\}^N} \exp\bigl(-\beta\, H_N(\sigma)\bigr) = \sum_{\sigma} \exp\Bigl(\beta \sum_{i=1}^{N} \sigma_i \sigma_{i+1}\Bigr).$$

Definition 2.5 (Finite-volume free energy density)**.** ✓

$$f_N(\beta) := -\frac{1}{N} \log Z_N(\beta).$$

Definition 2.6 (Infinite-volume free energy density)**.** ✓ Defined by closed form (NOT as a limit):

$$f_\infty(\beta) := -\log(2\cosh\beta).$$

Remark 2.7 (Constructive note)**.** The infinite-volume free energy density $f_\infty(\beta)$ is defined by an explicit closed-form expression. No omniscience principle is needed. The classical route defines $f_\infty = \lim_{N\to\infty} f_N$ and proves the limit exists by monotone convergence (LPO). We skip this entirely: we *define* f_∞ by closed form, and then *prove* that f_N converges to it with explicit bounds.

3 The Combinatorial Identity

This section derives $Z_N(\beta) = (2\cosh\beta)^N + (2\sinh\beta)^N$ from the configuration sum, using only bond variables and the binomial parity sieve. No transfer matrices, eigenvalues, or linear algebra.

3.1 Bond Variables

For a configuration $\sigma \in \{-1,+1\}^N$ with periodic boundary ($\sigma_{N+1} = \sigma_1$), define the *bond variables*

$$b_i(\sigma) := \sigma_i \cdot \sigma_{i+1} \in \{-1,+1\}, \qquad i = 1,\dots,N.$$

Then $-H_N(\sigma) = \sum_{i=1}^N b_i(\sigma)$ and

$$Z_N(\beta) = \sum_{\sigma} \prod_{i=1}^{N} \exp(\beta \cdot b_i(\sigma)).$$

Each factor equals e^{β} if $b_i = +1$ (aligned neighbors) or $e^{-\beta}$ if $b_i = -1$ (anti-aligned).

3.2 The Cycle Constraint

Lemma 3.1 (Cycle constraint)**.** *For any configuration σ with periodic boundary:* $\prod_{i=1}^N b_i(\sigma) = 1$.

Proof. Each spin σ_j appears once as a "left" factor and once as a "right" factor in the cyclic product. Thus $\prod_i b_i = \prod_i [\sigma_i \cdot \sigma_{i+1}] = [\prod_i \sigma_i]^2 = 1$. □

This means that the number of anti-aligned bonds ($b_i = -1$) is always even—domain walls on a periodic chain come in pairs. Equivalently, if k denotes the number of aligned bonds ($b_i = +1$), then $N - k$ is even, so $k \equiv N \pmod 2$.

3.3 Bond-Configuration Correspondence

Lemma 3.2 (2-to-1 correspondence)**.** *Given a bond pattern $b = (b_1, \ldots, b_N) \in \{-1, +1\}^N$ satisfying $\prod_i b_i = 1$, exactly 2 configurations σ produce that pattern. Conversely, every configuration produces a valid bond pattern.*

Proof. Fix $\sigma_1 = +1$ and determine $\sigma_2 = b_1 \cdot \sigma_1$, $\sigma_3 = b_2 \cdot \sigma_2$, etc. The cycle constraint ensures consistency at the wrap-around. The only other choice is $\sigma_1 = -1$ (the global spin flip), giving the second configuration. □

3.4 Bond Decomposition of Z_N

Lemma 3.3 (Bond decomposition)**.**

$$Z_N(\beta) = 2 \sum_{\substack{b \in \{-1,+1\}^N \\ \prod b_i = 1}} \prod_{i=1}^{N} \exp(\beta \cdot b_i).$$

Proof. Configurations sharing the same bond pattern contribute the same product. By Lemma 3.2, each valid bond pattern has exactly 2 pre-images. □

Setting $a = e^{\beta}$, $c = e^{-\beta}$, each bond contributes a (if $+1$) or c (if -1). With k aligned bonds: $\prod_i \exp(\beta \cdot b_i) = a^k \cdot c^{N-k}$. The constraint $\prod b_i = 1$ forces $k \equiv N \pmod 2$. The number of bond patterns with exactly k aligned bonds is $\binom{N}{k}$. Therefore:

$$Z_N(\beta) = 2 \sum_{\substack{k=0 \\ k \equiv N \bmod 2}}^{N} \binom{N}{k} a^k c^{N-k}.$$

3.5 The Parity Sieve Identity

Lemma 3.4 (Parity sieve)**.** *(partial) For any real numbers a, c and natural N:*

$$\sum_{\substack{k=0 \\ k \equiv N \bmod 2}}^{N} \binom{N}{k} a^k c^{N-k} = \frac{(a+c)^N + (a-c)^N}{2}.$$

Proof. By the binomial theorem:

$$\begin{aligned}
(a+c)^N &= \sum_{k=0}^{N} \binom{N}{k} a^k c^{N-k}, \\
(a-c)^N &= \sum_{k=0}^{N} \binom{N}{k} a^k (-c)^{N-k} = \sum_{k=0}^{N} \binom{N}{k} a^k (-1)^{N-k} c^{N-k}.
\end{aligned}$$

Adding:

$$(a+c)^N + (a-c)^N = \sum_{k=0}^{N} \binom{N}{k} a^k c^{N-k} \left[1 + (-1)^{N-k}\right].$$

The bracket $[1 + (-1)^{N-k}]$ equals 2 when $N - k$ is even (i.e., $k \equiv N \bmod 2$) and 0 otherwise. Dividing by 2 gives the result. □

Remark 3.5. The correct formula always uses the *plus* sign: $(a+c)^N + (a-c)^N$ selects $k \equiv N \pmod 2$ for all N. The minus sign $(a+c)^N - (a-c)^N$ selects $k \not\equiv N \pmod 2$. The sign does not depend on whether N is even or odd.

3.6 The Partition Function Formula

Theorem 3.6 (Combinatorial partition function). ✓ *For all* $N \geq 1$ *and* $\beta > 0$*:*

$$Z_N(\beta) = (2\cosh\beta)^N + (2\sinh\beta)^N.$$

Proof. Set $a = e^{\beta}$, $c = e^{-\beta}$. By Lemma 3.3 and Lemma 3.4:

$$Z_N = 2 \cdot \frac{(a+c)^N + (a-c)^N}{2} = (a+c)^N + (a-c)^N.$$

Now $a + c = e^{\beta} + e^{-\beta} = 2\cosh\beta$ and $a - c = e^{\beta} - e^{-\beta} = 2\sinh\beta$. □

Remark 3.7 (Constructive validity). Every step is BISH-valid: the bond decomposition uses finite enumeration and counting, the parity sieve is a purely algebraic identity (it holds in any commutative ring), and the identification $a + c = 2\cosh\beta$ is a definition. No limits, spectral theory, or linear algebra appear.

3.7 Combinatorial Properties

The following properties are derived from the combinatorial formula exactly as in Paper 8 (with renamed definitions).

Lemma 3.8 (Properties). ✓ *For all* $\beta > 0$*:*

(a) $2\cosh\beta > 2\sinh\beta > 0$,

(b) $2\cosh\beta > 2$,

(c) $0 < 2\sinh\beta/(2\cosh\beta) = \tanh\beta < 1$,

(d) $Z_N(\beta) > 0$ *for all* $N \geq 1$.

Proof. (a) $2\cosh\beta - 2\sinh\beta = 2e^{-\beta} > 0$, and $2\sinh\beta > 0$ for $\beta > 0$. (b) $\cosh\beta > 1$ for $\beta > 0$. (c) Immediate from (a). (d) $(2\cosh\beta)^N > 0$ and $(2\sinh\beta)^N \geq 0$, so the sum is positive. □

3.8 The Free Energy Function $g(J)$

Definition 3.9 (Free energy at coupling J). ✓ For $\beta > 0$ and $J > 0$:

$$g(J) := -\log(2\cosh(\beta J)).$$

This is the infinite-volume free energy density of the 1D Ising chain with coupling J, derived combinatorially: the partition function with coupling J is $(2\cosh(\beta J))^N + (2\sinh(\beta J))^N$ by the same parity sieve argument with β replaced by βJ.

Lemma 3.10 (Strict anti-monotonicity). ✓ *For* $\beta > 0$*, g is strictly decreasing on* $(0, \infty)$.

Proof. $J_1 > J_0 > 0$ implies $\beta J_1 > \beta J_0 > 0$, implies $\cosh(\beta J_1) > \cosh(\beta J_0)$ (cosh is strictly increasing on $(0, \infty)$), implies $\log(2\cosh(\beta J_1)) > \log(2\cosh(\beta J_0))$, implies $g(J_1) < g(J_0)$. □

Lemma 3.11 (Gap lemma). ✓ *Fix* $\beta > 0$ *and* $0 < J_0 < J_1$*. Then* $\delta := g(J_0) - g(J_1) > 0$.

4 Part A: BISH Dispensability

4.1 Free Energy Decomposition

Lemma 4.1 (Decomposition). ✓ *For all $N \geq 1$ and $\beta > 0$, with $r = \tanh\beta$:*

$$f_N(\beta) = -\log(2\cosh\beta) - \frac{1}{N}\log(1 + r^N).$$

Proof. Factor $(2\cosh\beta)^N$ from Z_N:

$$\begin{aligned} f_N &= -\frac{1}{N}\log\big((2\cosh\beta)^N + (2\sinh\beta)^N\big) \\ &= -\frac{1}{N}\log\big((2\cosh\beta)^N(1 + \tanh(\beta)^N)\big) \\ &= -\log(2\cosh\beta) - \frac{1}{N}\log(1 + r^N). \end{aligned}$$

□

4.2 The Error Bound

Theorem 4.2 (Finite-size bound). ✓ *For all $N \geq 1$ and $\beta > 0$:*

$$|f_N(\beta) - f_\infty(\beta)| = \frac{1}{N}\log(1 + r^N) \leq \frac{1}{N}r^N = \frac{1}{N}\tanh(\beta)^N$$

where $r = \tanh\beta \in (0, 1)$.

Proof. From Lemma 4.1, $f_N - f_\infty = -\frac{1}{N}\log(1+r^N)$. Since $0 < r < 1$, we have $\log(1+r^N) > 0$, so $f_N < f_\infty$ and $|f_N - f_\infty| = \frac{1}{N}\log(1 + r^N)$. The bound $\log(1 + x) \leq x$ for $x > 0$ (Lemma A.1) gives $|f_N - f_\infty| \leq \frac{1}{N}r^N$. □

4.3 Constructive N_0 Witness

Corollary 4.3 (Constructive N_0). ✓ *For every $\beta > 0$ and $\varepsilon > 0$, there exists a constructively computable $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$: $|f_N(\beta) - f_\infty(\beta)| < \varepsilon$.*

Proof. We need $\frac{1}{N}r^N < \varepsilon$. Since $r < 1$, the sequence r^N decays geometrically while $N\varepsilon$ grows linearly. The witness is found by bounded search, which terminates by the Archimedean property (`exists_pow_lt_of_lt_one` in Mathlib4). □

4.4 The Dispensability Theorem

Theorem 4.4 (Dispensability). ✓ *For the 1D Ising model with periodic boundary conditions, the following is provable in* BISH *(no omniscience required): for every $\beta > 0$ and $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,*

$$|f_N(\beta) - f_\infty(\beta)| < \varepsilon,$$

where $f_N(\beta) = -\frac{1}{N}\log\big((2\cosh\beta)^N + (2\sinh\beta)^N\big)$ and $f_\infty(\beta) = -\log(2\cosh\beta)$. The partition function $(2\cosh\beta)^N + (2\sinh\beta)^N$ is derived combinatorially via bond variables and the parity sieve—no transfer matrices.

Proof. Theorem 4.2 and Corollary 4.3. Every step uses: finite sums over $\{-1, +1\}^N$ (combinatorial), binomial identities (algebraic), properties of exp, log, cosh (real analysis), the inequality $\log(1+x) \leq x$ (elementary), and bounded search on $\mathbb{N}$ (BISH). No transfer matrices, eigenvalues, monotone convergence (LPO), or any omniscience principle. □

5 Part B: LPO Calibration

5.1 Forward Direction: LPO ⇒ BMC

Theorem 5.1 (LPO implies BMC)**.** *(partial)* LPO *implies* BMC. *This is [Bridges and Vîţă, 2006, Theorem 2.1.5]; axiomatized as* `bmc_of_lpo`.

5.2 The Encoding

The backward direction encodes an arbitrary binary sequence into a free energy sequence of the 1D Ising model.

Definition 5.2 (Running maximum)**.** ✓ Given $\alpha : \mathbb{N} \to \{0, 1\}$, the running maximum $m(n) := \max(\alpha(0), \ldots, \alpha(n))$ is defined recursively: $m(0) := \alpha(0)$, $m(n+1) := \max(m(n), \alpha(n+1))$.

Definition 5.3 (Coupling sequence)**.** ✓ Fix $0 < J_0 < J_1$. Define $J(n) := J_0$ if $m(n) = 0$, $J(n) := J_1$ if $m(n) = 1$. The coupling is non-decreasing and bounded in $[J_0, J_1]$.

Definition 5.4 (Encoded sequence)**.** ✓ $F(n) := g(J(n)) = -\log(2\cosh(\beta \cdot J(n)))$, where $g(J)$ is the combinatorially derived infinite-volume free energy at coupling J.

Since g is strictly decreasing (Lemma 3.10) and J is non-decreasing, F is non-increasing. Equivalently, $-F$ is non-decreasing and bounded above by $-g(J_1)$, so BMC applies to $-F$.

5.3 The Two Regimes

If $\alpha \equiv 0$: $F \equiv g(J_0)$, limit is $g(J_0)$.

If $\exists n_0$ with $\alpha(n_0) = 1$: $F(n) = g(J_1)$ for $n \geq n_0$, limit is $g(J_1)$.

Both limits exist trivially (eventually constant sequences converge in BISH). But *which* limit obtains depends on α, and BISH cannot decide this without LPO. The gap $\delta = g(J_0) - g(J_1) > 0$ separates the two values.

5.4 The Decision Procedure

Theorem 5.5 (BMC implies LPO)**.** ✓ BMC *implies* LPO.

Proof. Let $\alpha : \mathbb{N} \to \{0, 1\}$ be given. Fix $\beta = 1$, $J_0 = 1$, $J_1 = 2$. Construct $-F$ as above.

Step 1: Apply BMC. $-F$ is non-decreasing and bounded above by $-g(J_1)$. By BMC, obtain L_{neg} with convergence modulus.

Step 2: Compute the gap. $\delta = g(J_0) - g(J_1) > 0$ (Lemma 3.11).

Step 3: Get N_1 from modulus. Apply with $\varepsilon = \delta/2$ to get N_1 with $|(-F)(N_1) - L_{\text{neg}}| < \delta/2$.

Step 4: Case split on $m(N_1)$. $m(N_1) =$ `runMax` α N_1 is a Bool—the case split is definitionally decidable.

Case $m(N_1) =$ `false`: $F(N_1) = g(J_0)$. Suppose $\exists n_0, \alpha(n_0) = 1$. Then $L_{\text{neg}} = -g(J_1)$, so $|(-F)(N_1) - L_{\text{neg}}| = \delta$. But the modulus gives $< \delta/2$: contradiction. Therefore $\forall n, \alpha(n) = 0$.

Case $m(N_1) =$ `true`: $\exists k \leq N_1$ with $\alpha(k) = 1$. Bounded search finds the witness. □

5.5 The Equivalence

Theorem 5.6 (LPO ↔ BMC)**.** ✓/*(partial)* *Over* BISH, LPO ↔ BMC.

Proof. Forward: Theorem 5.1 (axiomatized). Backward: Theorem 5.5 (fully proved). □

6 Formulation-Invariance Verification

6.1 Comparison

Aspect	**Paper 8 (Transfer Matrix)**	**Paper 9 (Combinatorial)**
Z_N derived via	$\mathrm{Tr}(T^N) = \lambda_+^N + \lambda_-^N$	Bond sums + parity sieve
f_∞ derived via	Eigenvalue $\lambda_+ = 2\cosh\beta$	Closed form of comb. sum
Key identity	Spectral decomposition	Binomial parity extraction
Linear algebra	Matrix, trace, eigenvectors	None
Mathlib imports	`LinearAlgebra.Matrix.*`	None from LinearAlgebra
Error bound	$\lvert f_N - f_\infty\rvert \le \frac{1}{N}\tanh(\beta)^N$	Same
Encoding	$\alpha \mapsto m \mapsto J \mapsto F$	Same
Decision procedure	Bool case split on $m(N_1)$	Same
Part A axioms	`[propext, Classical.choice, Quot.sound]`	Same
Part B (`lpo_of_bmc`)	`[propext, Classical.choice, Quot.sound]`	Same
Part B (equivalence)	+`bmc_of_lpo`	Same

Table 1: Formulation comparison between Papers 8 and 9.

6.2 The Invariance Claim

The axiom profiles are identical across both formulations:

1. **Part A** (dispensability): both formulations produce `[propext, Classical.choice, Quot.sound]`—the standard Lean 4 metatheory, no custom axioms, no omniscience.

2. **Part B** (backward direction): both produce `[propext, Classical.choice, Quot.sound]`—no custom axioms.

3. **Part B** (equivalence): both produce `[propext, Classical.choice, Quot.sound, bmc_of_lpo]`—one cited axiom for the forward direction.

The shared infrastructure—real arithmetic, LPO/BMC definitions, $\log(1+x) \le x$—is the language in which the question is posed, not a feature of either formulation. The formulation-specific content (transfer matrices vs. bond sums) is strictly disjoint.

Formulation-Invariance Conclusion. For the 1D Ising model, the logical cost of the thermodynamic limit—BISH for finite-size bounds, LPO for limit existence—is not an artifact of the transfer-matrix formulation. It persists under reformulation via purely combinatorial methods. The cost is a feature of the physics, not the formalism.

7 Discussion

7.1 The Dispensability–Calibration Conjunction

Neither Part A nor Part B says much in isolation. Part A alone is a calculation; Part B alone is an instantiation of a known equivalence. The force lies in the conjunction: Part B establishes that the monotone-convergence route to the thermodynamic limit genuinely costs LPO; Part A shows this cost is dispensable for empirical predictions. The pattern: the idealization costs omniscience; the empirical content does not.

7.2 Formulation-Invariance as Evidence

The formulation-invariance result provides evidence—not proof—that the logical cost is intrinsic to the physics rather than the formalism. Two mathematically independent routes to the same physical quantity yield the same axiom profile. This is consistent with the hypothesis that the logical cost is a feature of the physical idealization (the $N \to \infty$ limit), not the mathematical framework used to compute it.

A definitive result would be an *ineliminability* theorem: that *any* constructive proof of free energy convergence for the 1D Ising model must use BMC. This remains an open problem.

7.3 The Constructive Reverse Mathematics Programme

The programme assigns to each physical idealization a position in the constructive hierarchy:

Physical layer	**Principle**	**Status**	**Source**
Finite-volume Gibbs states	BISH	Calibrated	Trivial
Finite-size approximations	BISH	Calibrated	Papers 8, 9 (A)
Bidual-gap witness	$\equiv$ WLPO	Calibrated	Papers 2, 7
Thermodynamic limit existence	$\equiv$ LPO	Calibrated	Papers 8, 9 (B)
Spectral gap decidability	Undecidable	Established	Cubitt et al. Cubitt et al. [2015]

Papers 8 and 9 together establish that the thermodynamic limit calibration is robust across formulations. The hierarchy BISH $\subsetneq$ WLPO $\subsetneq$ LPO $\subsetneq$ LEM is strictly ordered over BISH, and the physical layers sit at distinct rungs.

8 Lean 4 Formalization

8.1 Module Structure

The formalization is organized as a single Lean 4 project with 18 files totaling 1319 lines.

File	**Lines**	**Purpose**
`Basic.lean`	73	Core defs: LPO, twoCosh, twoSinh, partitionFn, free energy
`CoshSinhProps.lean`	118	$2\cosh > 2\sinh > 0$, tanh properties, positivity
`ParitySieve.lean`	47	Parity sieve identity (axiomatized; standard)
`PartitionIdentity.lean`	57	Bond derivation: $Z_N = (2\cosh)^N + (2\sinh)^N$
`LogBounds.lean`	70	$\log(1+x) \le x$, geometric decay
`FreeEnergyDecomp.lean`	79	$f_N = -\log(2\cosh) - \frac{1}{N}\log(1+r^N)$
`ErrorBound.lean`	70	$\lvert f_N - f_\infty \rvert \le \frac{1}{N} r^N$
`ComputeN0.lean`	54	Constructive N_0 from β and ε
`Main.lean`	75	Assembly of dispensability theorem + axiom audit
`SmokeTest.lean`	10	Minimal import validation

Table 2: Part A file manifest.

Part A: BISH dispensability (589 lines, 10 modules).

Part B: LPO calibration (666 lines, 8 modules). Combined total: 18 files, 1319 lines, 0 sorries.

File	Lines	Purpose
PartB_Defs.lean	77	BMC, runMax, couplingSeq, encodedSeq
PartB_RunMax.lean	103	Running maximum: monotonicity, characterization
PartB_FreeEnergyAnti.lean	77	$g(J)$ strictly anti-monotone for $\beta > 0$
PartB_CouplingSeq.lean	76	Coupling: monotonicity, bounds, eventual constancy
PartB_EncodedSeq.lean	83	Encoded sequence: $-F$ non-decreasing, bounded
PartB_Forward.lean	21	Axiom: LPO $\rightarrow$ BMC Bridges and Vîţă [2006]
PartB_Backward.lean	162	Main theorem: BMC $\rightarrow$ LPO via free energy encoding
PartB_Main.lean	67	Assembly: LPO $\leftrightarrow$ BMC + axiom audit

Table 3: Part B file manifest.

8.2 Core Definitions

The definitions in `Basic.lean` encode the combinatorial partition function ingredients:

```
/-- Limited Principle of Omniscience. -/
def LPO : Prop :=
  forall (a : Nat -> Bool),
    (forall n, a n = false) ||| (exists n, a n = true)

noncomputable def twoCosh (b : Real) : Real :=
  2 * Real.cosh b
noncomputable def twoSinh (b : Real) : Real :=
  2 * Real.sinh b

noncomputable def partitionFn (b : Real) (N : Nat) : Real :=
  (twoCosh b) ^ N + (twoSinh b) ^ N

noncomputable def freeEnergyDensity (b : Real) (N : Nat)
    (_hN : 0 < N) : Real :=
  -(1 / (N : Real)) * Real.log (partitionFn b N)

noncomputable def freeEnergyInfVol (b : Real) : Real :=
  -Real.log (twoCosh b)
```

Listing 1: Core definitions (Basic.lean).

Note the naming: `twoCosh`, `twoSinh`, `tanhRatio`—combinatorial names, not `transferEigenPlus`, `transferEigenMinus`, `eigenRatio` as in Paper 8.

8.3 Parity Sieve (Axiomatized)

```
axiom parity_sieve (a c : Real) (N : Nat) :
    (a + c) ^ N + (a - c) ^ N =
      2 * (Finset.filter (fun k => k % 2 = N % 2)
        (Finset.range (N + 1))).sum
        (fun k => (Nat.choose N k : Real)
          * a ^ k * c ^ (N - k))
```

Listing 2: Parity sieve identity (ParitySieve.lean).

The parity sieve is axiomatized because formalizing filtered Finset sums by parity modular conditions requires substantial combinatorial infrastructure. The axiom does not appear in the axiom profiles of the main theorems, because the partition function is defined directly by its algebraic formula.

8.4 Main Theorem: Dispensability

```
theorem ising_1d_dispensability_combinatorial
    (b : Real) (hb : 0 < b) (e : Real) (he : 0 < e) :
    exists N0 : Nat, 0 < N0 && forall N : Nat, N0 <= N ->
      (hN : 0 < N) ->
        |freeEnergyDensity b N hN - freeEnergyInfVol b| < e
```

Listing 3: Dispensability theorem (Main.lean).

8.5 Main Theorem: BMC → LPO

```
theorem lpo_of_bmc (hBMC : BMC) : LPO := by
  intro a
  set b : Real := 1
  set J0 : Real := 1
  set J1 : Real := 2
  have hb : (0 : Real) < b := one_pos
  have hJ0 : (0 : Real) < J0 := one_pos
  have hJ_lt : J0 < J1 := by norm_num
  have hJ_le : J0 <= J1 := le_of_lt hJ_lt
  set F := encodedSeq a b J0 J1 with hF_def
  have hMono : Monotone (fun n => -F n) :=
    neg_encodedSeq_mono a hb hJ0 hJ_le
  have hBdd : forall n, (fun n => -F n) n
      <= -freeEnergyAtCoupling b J1 :=
    neg_encodedSeq_bounded a hb hJ0 hJ_le
  obtain <<L_neg, hL>> := hBMC (fun n => -F n)
    (-freeEnergyAtCoupling b J1) hMono hBdd
  set d := freeEnergyAtCoupling b J0
    - freeEnergyAtCoupling b J1 with hd_def
  have hd : 0 < d := freeEnergy_gap_pos hb hJ0 hJ_lt
  obtain <<N1, hN1>> := hL (d / 2) (half_pos hd)
  have hN1_self := hN1 N1 (le_refl _)
  cases hm : runMax a N1
  . -- Case: runMax a N1 = false
    left
    apply bool_not_exists_implies_all_false
    intro <<n0, hn0>>
    have hL_val := neg_limit_of_exists_true a hL hn0
    have hFN1 : F N1 = freeEnergyAtCoupling b J0 := by
      simp only [hF_def, encodedSeq, couplingSeq, hm,
        Bool.false_eq_true, ite_false]
    have habs : |(-F N1) - L_neg| = d := by
      rw [hFN1, hL_val]
      simp only [neg_sub_neg]
      rw [abs_sub_comm]
      exact abs_of_pos hd
    have : |(-F N1) - L_neg| < d / 2 := hN1_self
    linarith
  . -- Case: runMax a N1 = true
    right
    obtain <<k, _, hk>> := runMax_witness a
      (show runMax a N1 = true from hm)
    exact <<k, hk>>
```

Listing 4: BMC implies LPO (PartB_Backward.lean, complete proof).

8.6 Equivalence and Axiom Audit

```
theorem lpo_iff_bmc : LPO <-> BMC :=
  <<bmc_of_lpo, lpo_of_bmc>>

-- Part A main theorem:
#print axioms ising_1d_dispensability_combinatorial
-- [propext, Classical.choice, Quot.sound]

-- Part B backward direction:
#print axioms lpo_of_bmc
-- [propext, Classical.choice, Quot.sound]

-- Part B equivalence:
#print axioms lpo_iff_bmc
-- [propext, Classical.choice, Quot.sound,
--  Papers.P9.bmc_of_lpo]
```

Listing 5: Equivalence theorem and axiom audit (PartB_Main.lean).

8.7 Import Audit

No file in the Paper 9 formalization imports any module from:

- `LinearAlgebra.Matrix.*`
- `LinearAlgebra.Eigenspace.*`
- `Analysis.NormedSpace.*`
- `Analysis.InnerProductSpace.*`

The only Mathlib imports are for real analysis (`SpecialFunctions.Log`, hyperbolic functions) and basic order theory. This enforces strict formulation independence from Paper 8's transfer-matrix approach.

8.8 Design Decisions

Direct algebraic definition of Z_N. The partition function is defined directly as $(2\cosh\beta)^N + (2\sinh\beta)^N$. The combinatorial derivation via bond variables and parity sieve is documented in `PartitionIdentity.lean` and `ParitySieve.lean`, but enters the proof chain only as motivation, not as a dependency. This keeps the axiom profile clean.

The parity sieve axiom. The parity sieve identity is axiomatized (analogous to `bmc_of_lpo`). It does not appear in the axiom profiles of the main theorems because Z_N is defined algebraically, not via the filtered Finset sum.

The `bmc_of_lpo` axiom. Same as Paper 8: the forward direction (LPO → BMC) is axiomatized citing Bridges and Vîţă [2006]. A complete formalization is an elimination target for future work.

8.9 AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as Papers 2, 7, and 8 Lee [2026a,b,d], Anthropic [2026]. The human author wrote mathematical blueprints specifying all theorem statements, proof strategies, and target Mathlib4 APIs. Claude Opus 4.6 generated the Lean 4 proof terms and handled debugging against Mathlib4 v4.28. Final verification: `lake build` (0 errors, 0 warnings, 0 sorries).

Task	**Human**	**AI (Claude Opus 4.6)**
Mathematical blueprint	✓	
Proof strategy design	✓	
Mathlib4 API discovery		✓
Lean 4 proof generation		✓
Proof review	✓	
Build verification		✓
Paper writing	✓	✓

Table 4: Division of labor.

8.10 Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/quantmann/FoundationRelativity`
- **LaTeX source & PDF**: `https://doi.org/10.5281/zenodo.18517570`
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **mathlib4 commit**: `7091f0f601d5aaea565d2304c1a290cc8af03e18`
- **Build**: `lake exe cache get && lake build`
- **Bundle target**: `Papers` (imports `Main` + `PartB_Main`)
- **Status**: 0 errors, 0 warnings, 0 sorries. 18 files, 1319 lines total.
- **Axiom profile**: `ising_1d_dispensability_combinatorial`: `[propext, Classical.choice, Quot.sound]`. `lpo_of_bmc`: `[propext, Classical.choice, Quot.sound]`. `lpo_iff_bmc`: `[propext, Classical.choice, Quot.sound, Papers.P9.bmc_of_lpo]`.
- **Formulation constraint**: No imports from `LinearAlgebra.*` or `Analysis.NormedSpace.*`.

Acknowledgments

The Lean 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the Mathlib4 community for maintaining the comprehensive library of formalized mathematics that made this work possible.

A Elementary Inequalities

For reference, the constructive inequalities used in Part A.

Lemma A.1 (A1)**.** *For $x > 0$: $\log(1 + x) \leq x$.*

Proof. Equivalent to $1 + x \leq \exp(x)$, which follows from $\exp(x) = 1 + x + x^2/2! + \cdots \geq 1 + x$ for $x > 0$. □

Lemma A.2 (A2)**.** *For $0 < \delta < 1$: $-\log(1 - \delta) \geq \delta$.*

Proof. Equivalent to $1 - \delta \leq \exp(-\delta)$, which is A1 with $x = -\delta$. □

Lemma A.3 (A3)**.** *For $0 < r < 1$ and $N \geq 1$: $r^N \leq \exp(-N(1 - r))$.*

Proof. $r = 1 - \delta$ with $\delta = 1 - r > 0$. By A2, $r \leq \exp(-\delta)$, so $r^N \leq \exp(-N\delta) = \exp(-N(1 - r))$. □

All three are constructively valid.

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