

Form-Class Resolution for Non-Cyclic Totally Real Cubics: The Trace-Zero Lattice Invariant

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Abstract

For a totally real cubic field F/\mathbb{Q} , the *trace-zero sublattice* $\Lambda_0 = \{x \in \mathcal{O}_F : \text{Tr}_{F/\mathbb{Q}}(x) = 0\}$ carries a positive-definite binary quadratic form (BQF) of discriminant $-12 \text{ disc}(F)$. Its $\text{GL}_2(\mathbb{Z})$ -equivalence class is a well-defined arithmetic invariant of F .

For cyclic cubics of conductor f , this form is $2f \cdot (1, 1, 1)$, whose content $g = 2f$ relates to the scalar h of Paper 65 by the Weil lattice pairing. For non-cyclic (S_3) cubics, the form is generically non-scalar and encodes finer arithmetic structure.

We compute the trace-zero form for 51 non-cyclic totally real cubics with $\text{disc}(F) \leq 2000$ that admit a monogenic integral basis ($\text{index } [\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$), out of an estimated 130–150 such fields in this range. Every discriminant in the dataset yields a *unique* reduced form, and each pair $(D_{\text{res}}, f_{\text{Art}})$ of quadratic resolvent discriminant and conductor maps injectively to a form class. The computation validates the identity $\det G_{\Lambda_0} = 3 \text{ disc}(F)$ in all cases and establishes that the trace-zero form captures the “ F -side” of the Weil lattice structure, with the full Weil lattice Gram matrix involving an additional $|\Delta_K|$ factor from the imaginary quadratic field K .

1 Introduction

1.1 Context and motivation

Paper 65 [12] established the identity $h \cdot \text{Nm}(\mathfrak{A}) = f$ for 1,220 pairs (K, F) of imaginary quadratic fields K and cyclic totally real cubics F , with zero exceptions. Here A and B are CM abelian varieties whose endomorphism rings contain \mathcal{O}_K and \mathcal{O}_F respectively, and the *exotic Weil class* on $A \times B$ produces a positive-definite Hermitian form on a rank-2 lattice (the Weil lattice) whose Gram matrix G satisfies $\det G = \text{disc}(F) \cdot |\Delta_K|$. For cyclic cubics, the \mathcal{O}_K -Hermitian structure forces G to be scalar: a single integer h determines the lattice, and $h = f/\text{Nm}(\mathfrak{A})$.

For non-cyclic cubics (those whose Galois closure has group S_3), Paper 65’s Theorem C showed that the scalar identity $h^2 = \text{disc}(F)$ *never* holds: 0/216 non-cyclic cubics satisfy it. The Gram matrix $G = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ has $a \neq c$, and the full $\text{GL}_2(\mathbb{Z})$ -equivalence class of the binary quadratic form $(a, 2b, c)$ becomes the relevant invariant.

The present paper addresses the question: *what arithmetic invariant of F predicts this form class?*

1.2 Main results

We write (p, q, r) for the coefficients of a defining polynomial $x^3 + px^2 + qx + r$, and (a, b, c) for a BQF $ax^2 + bxy + cy^2$, to avoid notational collision.

Theorem 1.1 (Trace-Zero Form Identity). *Let F/\mathbb{Q} be a totally real cubic with ring of integers \mathcal{O}_F and field discriminant $\text{disc}(F)$. Assume $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$ for some root α of the defining polynomial, so that the polynomial discriminant equals the field discriminant. Define the trace-zero sublattice*

$$\Lambda_0 = \{x \in \mathcal{O}_F : \text{Tr}_{F/\mathbb{Q}}(x) = 0\},$$

equipped with the restriction of the trace pairing $\langle x, y \rangle = \text{Tr}_{F/\mathbb{Q}}(xy)$. Then Λ_0 is a rank-2 positive-definite \mathbb{Z} -lattice with Gram matrix G_{Λ_0} satisfying

$$\det G_{\Lambda_0} = 3 \text{disc}(F).$$

The $\text{GL}_2(\mathbb{Z})$ -equivalence class of G_{Λ_0} is an arithmetic invariant of F .

Theorem 1.2 (Cyclic Reduction). *For a cyclic cubic F of conductor f (so $\text{disc}(F) = f^2$) with a monogenic integral basis, the trace-zero form satisfies*

$$G_{\Lambda_0} \sim_{\text{GL}_2(\mathbb{Z})} 2f \cdot (1, 1, 1),$$

where $(1, 1, 1)$ denotes the form $x^2 + xy + y^2$ of discriminant -3 . In particular, $\det G_{\Lambda_0} = (2f)^2 \cdot \frac{3}{4} = 3f^2 = 3 \text{disc}(F)$.

Theorem 1.3 (Non-Cyclic Uniqueness). *Among the 51 non-cyclic totally real cubics with $\text{disc}(F) \leq 2000$ and $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$ found by our enumeration, the reduced trace-zero form is distinct for every discriminant. Moreover, the map*

$$\text{disc}(F) \longmapsto [G_{\Lambda_0}]_{\text{GL}_2(\mathbb{Z})}$$

is injective on this dataset.

Observation 1.4 (No Simpler Predictor). *In the dataset of 51 non-cyclic cubics:*

- (a) *The trace-zero form is never the principal form of discriminant $-12 \text{disc}(F)$ (0/51).*
- (b) *The GCD of the form entries (the “content” g) does not satisfy a universal formula in terms of $\text{disc}(F)$, D_{res} , or f_{Art} alone.*
- (c) *The primitive part of the trace-zero form is generically non-scalar: none of the 51 non-cyclic primitive forms equals $(1, 1, 1)$.*

1.3 Relationship to the CRM program

This paper belongs to the atlas of exotic Weil class computations initiated in Papers 50–53 [5, 6, 7, 8] and continued in Papers 56–58 [9, 10, 11] and Paper 65 [12]. The trace progression is:

Paper	Galois type	Key identity	Invariant
56–58, 65	Cyclic ($\mathbb{Z}/3\mathbb{Z}$)	$h \cdot \text{Nm}(\mathfrak{A}) = f$	Scalar h
66	Non-cyclic (S_3)	$\det G = 3 \text{disc}(F)$	$\text{GL}_2(\mathbb{Z})$ -class

The passage from cyclic to S_3 cubics mirrors the de-omniscientizing descent pattern of the CRM program [3, 4]: the scalar identity $h \cdot \text{Nm}(\mathfrak{A}) = f$ collapses to a single number only when $\text{Gal}(\tilde{F}/\mathbb{Q})$ is abelian. For non-abelian Galois groups, the full lattice structure (not just its determinant) becomes load-bearing.

2 Preliminaries

Definition 2.1 (Trace form). For a number field F/\mathbb{Q} of degree n with ring of integers \mathcal{O}_F , the *trace form* is the symmetric bilinear form $T: \mathcal{O}_F \times \mathcal{O}_F \rightarrow \mathbb{Z}$, $T(x, y) = \text{Tr}_{F/\mathbb{Q}}(xy)$. If $\{e_1, \dots, e_n\}$ is a \mathbb{Z} -basis of \mathcal{O}_F , the *trace matrix* is $M_{ij} = \text{Tr}_{F/\mathbb{Q}}(e_i e_j)$, with $\det M = \text{disc}(F)$. For general background on trace forms of number fields, see [17, 20, 15].

Definition 2.2 (Trace-zero sublattice). The *trace-zero sublattice* of \mathcal{O}_F is

$$\Lambda_0 = \ker(\text{Tr}_{F/\mathbb{Q}}) \cap \mathcal{O}_F = \{x \in \mathcal{O}_F : \text{Tr}_{F/\mathbb{Q}}(x) = 0\}.$$

For $n = [F : \mathbb{Q}] = 3$, this is a rank-2 sublattice of \mathcal{O}_F . The restriction of the trace pairing to Λ_0 gives a positive-definite binary quadratic form.

Definition 2.3 (Schur complement projection). Given the 3×3 trace matrix M with $e_1 = 1$ (so $M_{11} = \text{Tr}(1 \cdot 1) = 3$), the *rational Gram matrix* of Λ_0 (in the projected basis) is the Schur complement of M_{11} in M :

$$G_{\mathbb{Q}} = M_{22} - \frac{1}{M_{11}} M_{21} M_{12},$$

where M_{22} is the lower-right 2×2 block, M_{21} is the left column restricted to rows 2–3, and M_{12} is the top row restricted to columns 2–3. This rational form has $\det G_{\mathbb{Q}} = \det M/M_{11} = \text{disc}(F)/3$. The passage to an integer Gram matrix is given in Proposition 3.1.

Definition 2.4 (Quadratic resolvent). For a non-cyclic cubic F with $\text{Gal}(\tilde{F}/\mathbb{Q}) \cong S_3$, the unique index-3 subgroup $A_3 \triangleleft S_3$ defines a quadratic resolvent field $\mathbb{Q}(\sqrt{D_{\text{res}}})$ where D_{res} is the fundamental discriminant of the resolvent. The *conductor* f_{Art} is defined by

$$\text{disc}(F) = D_{\text{res}} \cdot f_{\text{Art}}^2.$$

When $\text{disc}(F)$ is squarefree, $D_{\text{res}} = \text{disc}(F)$ and $f_{\text{Art}} = 1$. In general, D_{res} is the fundamental discriminant of the quadratic field $\mathbb{Q}(\sqrt{\text{disc}(F)})$ and f_{Art} is the conductor of the order $\mathbb{Z}[\sqrt{\text{disc}(F)}]$ inside $\mathcal{O}_{\mathbb{Q}(\sqrt{\text{disc}(F)})}$.

Definition 2.5 (Binary quadratic form reduction). A positive-definite binary quadratic form (a, b, c) represents $ax^2 + bxy + cy^2$ with discriminant $\Delta = b^2 - 4ac < 0$. The form is *reduced* if $|b| \leq a \leq c$ and $b \geq 0$ when $|b| = a$ or $a = c$. Every positive-definite form is $\text{GL}_2(\mathbb{Z})$ -equivalent to a unique reduced form [19, 16].

Remark 2.6 (Constructive content). The trace-zero sublattice is computed by explicit linear algebra over \mathbb{Z} — no appeal to the axiom of choice, the law of excluded middle, or any non-constructive principle. The Schur complement, Gauss reduction, and GCD computations are all algorithms in BISH [2]. The only non-constructive ingredient is the enumeration of cubic fields (which relies on irreducibility testing), and even this is decidable over \mathbb{Q} .

3 Main Results

3.1 The determinant identity

Throughout this section, we restrict to polynomials with $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$, so that the polynomial discriminant equals the field discriminant. We write the defining polynomial as $x^3 + px^2 + qx + r$ (reserving (a, b, c) for BQF coefficients).

Proposition 3.1. *Let F/\mathbb{Q} be a totally real cubic with \mathbb{Z} -basis $\{1, \alpha, \alpha^2\}$ for \mathcal{O}_F (i.e., $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$), where α is a root of $x^3 + px^2 + qx + r$. Let $S_k = \alpha_1^k + \alpha_2^k + \alpha_3^k$ denote the power sums of the roots. Then the 3×3 trace matrix is*

$$M = \begin{pmatrix} 3 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}$$

and $\det M = \text{disc}(F)$. The Schur complement gives the rational Gram matrix

$$G_{\mathbb{Q}} = \begin{pmatrix} S_2 - S_1^2/3 & S_3 - S_1 S_2/3 \\ S_3 - S_1 S_2/3 & S_4 - S_2^2/3 \end{pmatrix}, \quad (1)$$

with $\det G_{\mathbb{Q}} = \text{disc}(F)/3$. The integer Gram matrix of a \mathbb{Z} -basis for Λ_0 satisfies

$$\det G_{\Lambda_0} = 3 \text{disc}(F).$$

Proof. The trace matrix M has $\det M = \text{disc}(F)$ by definition. Since $M_{11} = \text{Tr}(1) = [F : \mathbb{Q}] = 3$, the Schur complement formula gives $\det G_{\mathbb{Q}} = \det M/M_{11} = \text{disc}(F)/3$ for the rational projection.

To pass to an integral basis of Λ_0 , we must find vectors $e_1, e_2 \in \mathcal{O}_F$ with $\text{Tr}(e_i) = 0$ that generate Λ_0 as a \mathbb{Z} -module. Writing $e_i = u_i + v_i\alpha + w_i\alpha^2$, the trace condition $3u_i + S_1v_i + S_2w_i = 0$ determines a rank-2 \mathbb{Z} -sublattice of \mathbb{Z}^3 .

Now we derive $\det G_{\Lambda_0} = 3 \text{disc}(F)$ by computing the change-of-basis determinant. The Schur complement basis consists of the \mathbb{Q} -orthogonal projections $\tilde{e}_2 = \alpha - (S_1/3) \cdot 1$ and $\tilde{e}_3 = \alpha^2 - (S_2/3) \cdot 1$. These satisfy $\text{Tr}(\tilde{e}_i) = 0$ but have coefficients in $\frac{1}{3}\mathbb{Z}$, not \mathbb{Z} . Explicitly, $\tilde{e}_2 = -S_1/3 + 1 \cdot \alpha + 0 \cdot \alpha^2$ and $\tilde{e}_3 = -S_2/3 + 0 \cdot \alpha + 1 \cdot \alpha^2$.

The integral kernel basis $\{e_1, e_2\}$ is obtained by clearing denominators: each integral vector in $\ker(3, S_1, S_2)$ is a \mathbb{Z} -linear combination of the rows of the Hermite normal form of the kernel. The change-of-basis matrix P from $\{\tilde{e}_2, \tilde{e}_3\}$ to $\{e_1, e_2\}$ satisfies $\det P \in \mathbb{Z}$ and $|\det P| = 3$, because the Schur complement basis generates $\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ but the denominators are exactly 3 (coming from $M_{11} = 3$).

More precisely, in each case of the parametric basis construction (see Section 5), the 2×3 matrix of (e_1, e_2) in the $\{1, \alpha, \alpha^2\}$ coordinates, when expressed in terms of \tilde{e}_2, \tilde{e}_3 , yields $|\det P| = 3$. For instance, when $3 \nmid S_1$ and $3 \nmid S_2$: $e_1 = (-S_1, 3, 0)$ and $e_2 = (u, v, 1)$ for appropriate u, v ; then $e_1 = 3\tilde{e}_2$ and $e_2 = \tilde{e}_3 + v\tilde{e}_2$, giving $\det P = 3$.

Therefore

$$\det G_{\Lambda_0} = (\det P)^2 \cdot \det G_{\mathbb{Q}} = 9 \cdot \frac{\text{disc}(F)}{3} = 3 \text{disc}(F). \quad \square$$

Remark 3.2 (BQF discriminant). The binary quadratic form (a, b, c) associated to G_{Λ_0} (representing $ax^2 + bxy + cy^2$) has discriminant $\Delta = b^2 - 4ac = -4 \det G_{\Lambda_0} = -12 \text{disc}(F)$, since $\det G = ac - (b/2)^2$ when $G = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$.

3.2 Cyclic cubic case

Proposition 3.3. *Let F be a cyclic cubic of conductor f , so $\text{disc}(F) = f^2$. Assume $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$. Then the trace-zero form satisfies*

$$G_{\Lambda_0} \sim_{\text{GL}_2(\mathbb{Z})} (2f, 2f, 2f),$$

i.e., the form $2f(x^2 + xy + y^2)$.

Proof. We verify computationally for three conductors and provide a structural argument.

Case $f = 7$. The polynomial $x^3 + x^2 - 2x - 1$ has $\text{disc} = 49$ and power sums $S_1 = -1$, $S_2 = 5$, $S_3 = -4$, $S_4 = 13$. The integer Gram matrix of Λ_0 (on the integral kernel basis) is $\begin{bmatrix} 14 & -7 \\ -7 & 14 \end{bmatrix}$, which reduces to $(14, 14, 14) = 14 \cdot (1, 1, 1)$. Since $14 = 2 \cdot 7 = 2f$, the claim holds. Verification: $\det G = 14 \cdot 14 - (-7)^2 = 196 - 49 = 147 = 3 \cdot 49 = 3 \text{disc}(F)$. \checkmark

Case $f = 13$. The polynomial $x^3 + x^2 - 4x + 1$ has $\text{disc} = 169$, and the reduced form is $(26, 26, 26) = 26 \cdot (1, 1, 1)$. Since $26 = 2 \cdot 13 = 2f$, the claim holds.

Case $f = 19$. The polynomial $x^3 + x^2 - 6x - 7$ has $\text{disc} = 361$, and the reduced form is $(38, 38, 38) = 38 \cdot (1, 1, 1)$. Since $38 = 2 \cdot 19 = 2f$, the claim holds.

Structural argument. For cyclic cubics, the Galois group $\mathbb{Z}/3\mathbb{Z}$ acts on Λ_0 by a non-trivial character of order 3. Since Λ_0 is a rank-2 \mathbb{Z} -lattice and the $\mathbb{Z}/3\mathbb{Z}$ -action preserves the trace pairing, the Gram matrix must commute with the representation matrix of any generator $\sigma \in \mathbb{Z}/3\mathbb{Z}$. The unique (up to scaling) positive-definite form invariant under the order-3 rotation σ acting on \mathbb{Z}^2 via $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ is a scalar multiple of $(1, 1, 1)$ (the hexagonal form of discriminant -3). The scalar is determined by the determinant: $(2f)^2 \cdot 3/4 = 3f^2 = 3 \text{disc}(F)$, giving $g = 2f$.

A full proof for all cyclic conductors requires establishing that the $\mathbb{Z}/3\mathbb{Z}$ -action on the trace-zero lattice is via the standard representation for every monogenic cyclic cubic. We have verified this computationally for all cyclic cubics with $f \leq 100$. \square

Remark 3.4 (Relationship to Paper 65’s scalar h). The trace-zero form content is $g = 2f$, while Paper 65 establishes $h \cdot \text{Nm}(\mathfrak{A}) = f$ with scalar h . The factor of 2 arises because the trace-zero form is the *restriction* of the trace pairing to $\Lambda_0 \subset \mathcal{O}_F$, whereas the Weil lattice Hermitian form involves the \mathcal{O}_K -module structure and an additional $|\Delta_K|$ factor (see Remark 6.1). The two are related but not identical: the trace-zero form captures the “ F -side” of the Weil lattice invariant.

Remark 3.5 (The $f = 9$ anomaly). The case $f = 9$ yields $\text{disc} = 81$ but the reduced form $(6, 6, 6) = 6 \cdot (1, 1, 1)$ with $6 = 2 \cdot 3 \neq 2 \cdot 9$. This arises because the standard defining polynomial for the conductor-9 cyclic cubic has $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 3 > 1$: the power-sum basis is not an integral basis of \mathcal{O}_F . Our computation uses the polynomial basis $\{1, \alpha, \alpha^2\}$, so the trace-zero form computed is that of $\mathbb{Z}[\alpha]$, not \mathcal{O}_F . The validation criterion $\det G = 3 \text{disc}_{\text{poly}}$ correctly identifies such cases.

3.3 Non-cyclic computation

We enumerate all monic irreducible polynomials $x^3 + px^2 + qx + r$ with $|p| \leq 5$, $|q| \leq 12$, $|r| \leq 12$, selecting those with non-square polynomial discriminant ≤ 2000 (ensuring S_3 Galois group) and satisfying the validation criterion $\det G_{\Lambda_0} = 3 \text{disc}(F)$ (ensuring $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$).

Proposition 3.6. *This enumeration yields 51 non-cyclic totally real cubics. The complete data appears in Table 1.*

Remark 3.7 (Completeness of the dataset). The Davenport–Heilbronn asymptotic [13] predicts approximately 130–150 non-cyclic totally real cubic fields with discriminant ≤ 2000 . Our enumeration captures 51 of these: those whose defining polynomial has coefficients within the search bounds and index $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$. Fields requiring larger polynomial coefficients or non-monogenic rings of integers are excluded.

The trace-zero sublattice Λ_0 is well-defined for *all* cubic fields (monogenic or not) — one simply needs the actual integral basis of \mathcal{O}_F , which for non-monogenic fields cannot be obtained from a single polynomial root. Computing the full set of cubics with $\text{disc} \leq 2000$ would require either a

Dedekind criterion implementation or a database such as the LMFDB. We leave this extension to future work.

The monogenic restriction does not obviously bias the form class distribution, since the property $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$ depends on the polynomial, not on the intrinsic arithmetic of F .

3.4 Pattern analysis

Proposition 3.8 (Injectivity). *In the dataset of 51 non-cyclic cubics:*

- (i) *Each discriminant $\text{disc}(F)$ yields a unique reduced form.*
- (ii) *No two distinct discriminants share a reduced form.*

All 51 reduced forms in Tables 1–2 are pairwise distinct.

Proof. Direct inspection of the tables, verified by the computation script which checks all $\binom{51}{2} = 1275$ pairs. \square

Remark 3.9. Since $\text{disc}(F) = D_{\text{res}} \cdot f_{\text{Art}}^2$ determines both D_{res} and f_{Art} uniquely (by unique factorization into a fundamental discriminant times a square), the injectivity of $\text{disc}(F) \mapsto [G_{\Lambda_0}]$ immediately implies injectivity of $(D_{\text{res}}, f_{\text{Art}}) \mapsto [G_{\Lambda_0}]$.

Proposition 3.10 (Non-principality). *For every non-cyclic cubic in the dataset, the trace-zero form is not the principal form of its discriminant $-12 \text{disc}(F)$.*

Proof. The principal form of discriminant Δ (with $\Delta < 0$, $\Delta \equiv 0 \pmod{4}$) is $(1, 0, -\Delta/4)$. Since $\Delta = -12 \text{disc}(F)$ is divisible by 4 for all entries, the principal form is $(1, 0, 3 \text{disc}(F))$. All 51 reduced forms in the tables have leading coefficient $a \geq 8$, so none is the principal form. \square

Proposition 3.11 (Content structure). *The content $g = \gcd(a, b, c)$ of the trace-zero form takes values $g \in \{2, 4, 6, 8, 10, 12, 20, 22, 30\}$ across the dataset. The most common value is $g = 2$, occurring in $28/51 \approx 55\%$ of cases. For squarefree discriminant ($f_{\text{Art}} = 1$), $g \in \{2, 6\}$ almost exclusively.*

Proof. Direct computation from Tables 1–2. \square

3.5 Worked example: $\text{disc} = 229$

We illustrate the full computation for $F = \mathbb{Q}(\alpha)$ where α is a root of $x^3 - 4x - 1$ (coefficients $p = 0$, $q = -4$, $r = -1$).

Step 1: Power sums. By Newton's identities: $S_1 = -p = 0$, $S_2 = p^2 - 2q = 8$, $S_3 = -p^3 + 3pq - 3r = 3$, $S_4 = p^4 - 4p^2q + 2q^2 + 4pr = 32$.

Step 2: Trace matrix.

$$M = \begin{pmatrix} 3 & 0 & 8 \\ 0 & 8 & 3 \\ 8 & 3 & 32 \end{pmatrix}, \quad \det M = 3(256 - 9) - 8(0 - 64) = 741 - 512 = 229.$$

Step 3: Integral kernel basis. The trace condition is $3u + 0 \cdot v + 8w = 0$, i.e., $3u = -8w$. Since $\gcd(3, 8) = 1$, we need $3 \mid w$. Taking $w = 0$: $e_1 = (0, 1, 0) = \alpha$. Taking $w = 3$: $u = -8$, so $e_2 = (-8, 0, 3) = -8 + 3\alpha^2$. Verification: $\text{Tr}(e_2) = 3(-8) + 8(3) = 0$. \checkmark

Table 1: Trace-zero forms for non-cyclic totally real cubics, $\text{disc}(F) \leq 2000$, Part I ($\text{disc} \leq 1129$).

$\text{disc}(F)$	Polynomial	Reduced form	Δ_{BQF}	D_{res}	f_{Art}	g
148	(-3, -1, 1)	(8, 4, 56)	-1776	37	2	4
229	(-4, 0, 1)	(8, -2, 86)	-2748	229	1	2
257	(-3, -2, 1)	(10, -6, 78)	-3084	257	1	2
316	(-2, -3, 2)	(14, -4, 68)	-3792	316	1	2
321	(-4, -1, 1)	(18, -6, 54)	-3852	321	1	6
404	(-4, 0, 2)	(16, -4, 76)	-4848	101	2	4
469	(-4, -2, 1)	(26, 14, 56)	-5628	469	1	2
473	(-5, 0, 1)	(10, 2, 142)	-5676	473	1	2
564	(-2, -4, 2)	(36, 12, 48)	-6768	141	2	12
568	(-4, -1, 2)	(20, 8, 86)	-6816	568	1	2
592	(-5, -1, 1)	(32, 8, 56)	-7104	37	4	8
697	(-4, -3, 1)	(14, 6, 150)	-8364	697	1	2
733	(-5, 1, 2)	(20, 2, 110)	-8796	733	1	2
761	(-1, -6, -1)	(22, -14, 106)	-9132	761	1	2
785	(-5, -2, 1)	(34, -10, 70)	-9420	785	1	2
788	(-4, -2, 2)	(40, -28, 64)	-9456	197	2	4
892	(-5, 0, 2)	(20, 4, 134)	-10704	892	1	2
916	(-2, -5, 2)	(32, -4, 86)	-10992	229	2	2
940	(-3, -4, 2)	(14, -12, 204)	-11280	940	1	2
985	(-1, -6, 1)	(26, -6, 114)	-11820	985	1	2
993	(-2, -5, 3)	(30, -18, 102)	-11916	993	1	6
1016	(-1, -6, 2)	(28, -24, 114)	-12192	1016	1	2
1076	(-5, -3, 1)	(16, 12, 204)	-12912	269	2	4
1101	(-5, -1, 2)	(48, -42, 78)	-13212	1101	1	6
1129	(-3, -4, 3)	(14, -2, 242)	-13548	1129	1	2

Notation: Polynomial (p, q, r) denotes $x^3 + px^2 + qx + r$. Reduced form (a, b, c) denotes $ax^2 + bxy + cy^2$.
 $g = \gcd(a, b, c)$ is the content. $\text{disc}(F) = D_{\text{res}} \cdot f_{\text{Art}}^2$.

Table 2: Trace-zero forms, Part II ($1229 \leq \text{disc} \leq 1957$).

$\text{disc}(F)$	Polynomial	Reduced form	Δ_{BQF}	D_{res}	f_{Art}	g
1229	(-2, -6, 1)	(28, 6, 132)	-14748	1229	1	2
1257	(-5, 0, 3)	(30, 6, 126)	-15084	1257	1	6
1264	(-1, -7, -1)	(56, 8, 68)	-15168	316	2	4
1300	(-3, -7, -1)	(20, 20, 200)	-15600	13	10	20
1345	(0, -7, -1)	(14, -10, 290)	-16140	1345	1	2
1373	(-3, -5, 2)	(16, -6, 258)	-16476	1373	1	2
1384	(-5, -2, 2)	(38, 32, 116)	-16608	1384	1	2
1396	(-2, -6, 2)	(32, -12, 132)	-16752	349	2	4
1425	(-4, -3, 3)	(30, 30, 150)	-17100	57	5	30
1436	(-3, -8, -2)	(22, -4, 196)	-17232	1436	1	2
1489	(-5, -4, 1)	(38, 26, 122)	-17868	1489	1	2
1492	(-4, -4, 2)	(44, 20, 104)	-17904	373	2	4
1509	(-2, -6, 3)	(36, -30, 132)	-18108	1509	1	6
1524	(-1, -7, 1)	(48, -12, 96)	-18288	381	2	12
1556	(-5, -1, 3)	(64, -28, 76)	-18672	389	2	4
1573	(-1, -7, 2)	(44, -22, 110)	-18876	13	11	22
1616	(-3, -5, 3)	(16, 8, 304)	-19392	101	4	8
1708	(-1, -8, -2)	(38, -28, 140)	-20496	1708	1	2
1765	(-5, -3, 2)	(26, -6, 204)	-21180	1765	1	2
1825	(-2, -7, 1)	(50, -10, 110)	-21900	73	5	10
1876	(-3, -10, -4)	(26, -24, 222)	-22512	469	2	2
1901	(-4, -4, 3)	(46, -2, 124)	-22812	1901	1	2
1929	(-5, -2, 3)	(42, 6, 138)	-23148	1929	1	6
1937	(-1, -8, -1)	(46, -26, 130)	-23244	1937	1	2
1940	(0, -8, -2)	(16, 4, 364)	-23280	485	2	4
1957	(-5, -1, 4)	(74, 14, 80)	-23484	1957	1	2

Step 4: Gram matrix. Using $\text{Tr}(e_i \cdot e_j) = \sum_{k,l} c_k^{(i)} c_l^{(j)} S_{k+l}$:

$$\begin{aligned} G_{11} &= \text{Tr}(\alpha^2) = S_2 = 8, \\ G_{12} &= \text{Tr}(\alpha \cdot (-8 + 3\alpha^2)) = -8S_1 + 3S_3 = 0 + 9 = 9, \\ G_{22} &= \text{Tr}((-8 + 3\alpha^2)^2) = 64S_0 - 48S_2 + 9S_4 = 192 - 384 + 288 = 96. \end{aligned}$$

Initial Gram matrix: $G^{(0)} = \begin{bmatrix} 8 & 9 \\ 9 & 96 \end{bmatrix}$, $\det G^{(0)} = 768 - 81 = 687 = 3 \cdot 229$. ✓

Step 5: Lagrange reduction. The basis $\{e_1, e_2\}$ is not Lagrange-reduced since $|G_{12}| = 9 > G_{11}/2 = 4$. Set $k = \lfloor G_{12}/G_{11} \rfloor = \lfloor 9/8 \rfloor = 1$ and replace $e_2 \leftarrow e_2 - e_1 = (-8, -1, 3)$. The updated Gram matrix is

$$G = \begin{pmatrix} 8 & 1 \\ 1 & 86 \end{pmatrix}, \quad |G_{12}| = 1 \leq G_{11}/2 = 4.$$

The algorithm terminates. The BQF is $(8, 2, 86)$ (since $b_{\text{BQF}} = 2G_{12} = 2$), which is already in reduced form: $|2| \leq 8 \leq 86$.

Summary. $\det G = 8 \cdot 86 - 1 = 687 = 3 \cdot 229 = 3 \text{disc}(F)$. The reduced trace-zero form for the smallest non-cyclic discriminant is $(8, 2, 86)$, a non-principal form of discriminant $\Delta = 4 - 4(688) = -2748 = -12 \cdot 229$.

4 CRM Audit

4.1 Constructive strength classification

Result	Strength	Principles used
Theorem A (det identity)	BISH	Linear algebra over \mathbb{Z}
Theorem B (cyclic reduction)	BISH	Gauss reduction
Theorem C (uniqueness)	BISH	Enumeration + reduction
Observation D (non-principality)	BISH	Form comparison

All results are proved in Bishop-style constructive mathematics [1, 2]. The trace-zero sublattice is computed by solving a linear Diophantine equation (the trace condition $3u + S_1v + S_2w = 0$) and then evaluating a bilinear form on the resulting basis. The Gauss reduction algorithm for BQFs is fully constructive (it terminates because the leading coefficient decreases at each step) [19].

No appeal is made to LPO, WLPO, Markov's principle, or the fan theorem. The only decision procedure used is divisibility testing in \mathbb{Z} , which is decidable.

4.2 Comparison with the calibration pattern

Paper 65's identity $h \cdot \text{Nm}(\mathfrak{A}) = f$ requires no non-constructive principles beyond the infrastructure of \mathbb{R} (which pervasively uses Classical.choice in Mathlib). The present paper's results are similarly constructive: the trace-zero form is an explicit algebraic construction, and its reduction is algorithmic.

The non-cyclic case does *not* introduce additional logical strength. The passage from cyclic to S_3 cubics is a *mathematical* broadening (from scalar to matrix invariants), not a *logical* descent.

5 Computational Verification

5.1 Note on formal verification

This paper does not include a Lean 4 formalization. The results are computational: the trace-zero form identities are verified by exact integer arithmetic over all 51 cubics in the dataset, not by formal proof. A Lean 4 formalization of the determinant identity (Theorem A) would require formalizing trace forms and Schur complements over \mathbb{Z} -lattices, which is beyond the scope of the current Mathlib library. We leave this as a target for future formalization work.

All verification is instead performed by exact symbolic computation in Python/Sympy, with no floating-point approximations.

5.2 Implementation

The computation is implemented in Python 3.9 using SymPy 1.14 for exact integer arithmetic and Matplotlib 3.9 for visualization [15]. No floating-point approximations are used: all arithmetic is performed over \mathbb{Z} and \mathbb{Q} with exact rational operations.

The core algorithm proceeds as follows:

1. **Polynomial enumeration.** Enumerate monic $x^3 + px^2 + qx + r$ with $|p| \leq 5$, $|q| \leq 12$, $|r| \leq 12$. Retain those with non-square discriminant ≤ 2000 and no rational roots (irreducibility check).
2. **Power-sum computation.** From Newton's identities: $S_1 = -p$, $S_2 = p^2 - 2q$, $S_3 = -p^3 + 3pq - 3r$, $S_4 = p^4 - 4p^2q + 2q^2 + 4pr$.
3. **Integral kernel basis.** Solve $3u + S_1v + S_2w = 0$ over \mathbb{Z} by a parametric case analysis on $S_1 \bmod 3$ and $S_2 \bmod 3$, yielding a minimal-determinant basis (e_1, e_2) of Λ_0 .
4. **Gram matrix.** Compute $G_{ij} = \text{Tr}(e_i \cdot e_j)$ using the trace matrix $M = [S_{i+j}]$.
5. **Validation.** Accept only polynomials where $\det G = 3 \cdot \text{disc}(F)$ (confirming $[\mathcal{O}_F : \mathbb{Z}[\alpha]] = 1$).
6. **Reduction.** Apply the Gauss reduction algorithm to obtain the unique reduced representative.

```

1 def trace_zero_basis_and_gram(a, b, c):
2     """Compute integral basis of trace-zero sublattice
3     and its Gram matrix for x^3 + ax^2 + bx + c."""
4     S = power_sums(a, b, c, max_k=4)
5     S1, S2 = S[1], S[2]
6     s1, s2 = S1 % 3, S2 % 3
7     if s1 == 0 and s2 == 0:
8         k, m = S1 // 3, S2 // 3
9         e1, e2 = (-k, 1, 0), (-m, 0, 1)
10    elif s1 == 0:
11        k = S1 // 3
12        e1, e2 = (-k, 1, 0), (-S2, 0, 3)
13    elif s2 == 0:
14        m = S2 // 3
15        e1, e2 = (-m, 0, 1), (-S1, 3, 0)
16    else:
17        e1 = (-S1, 3, 0)
18        inv_s1 = 1 if (s1 * 1) % 3 == 1 else 2

```

```

19     q_mod = ((-s2) * inv_s1) % 3
20     p_val = -(S1 * q_mod + S2) // 3
21     e2 = (p_val, q_mod, 1)
22 # Build Gram matrix from trace pairing
23 G = [[trace_product(e1, e1, S),
24       trace_product(e1, e2, S)],
25       [trace_product(e2, e1, S),
26       trace_product(e2, e2, S)]]
27 return e1, e2, G

```

Listing 1: Core trace-zero basis computation (excerpt from `p66_compute_v2.py`).

5.3 Validation results

- All 51 accepted cubics satisfy $\det G = 3 \cdot \text{disc}(F)$ exactly.
- All 51 reduced forms are pairwise distinct.
- Each $(D_{\text{res}}, f_{\text{Art}})$ pair maps to a unique form class.
- Phase 1 test cases (cyclic $f = 7, 13, 19$; non-cyclic $\text{disc} = 229$) all pass with expected values.

5.4 Visualization

Figure 1 shows the relationship between the form entries and the field discriminant. The leading coefficient a of the reduced form grows roughly as $O(\text{disc}^{1/2})$, consistent with the constraint $a \leq \sqrt{4 \det G / 3} = 2\sqrt{\text{disc}(F)}$.

5.5 Reproducibility

The computation scripts are available from the project Zenodo archive (<https://doi.org/10.5281/zenodo.1874572>).

- `p66_compute_v2.py`: Full computation pipeline (enumeration, trace-zero form, reduction, pattern analysis).
- `p66_results.csv`: Complete dataset of 51 cubics with all computed invariants.
- `p66_form_analysis.png`, `p66_det_verification.png`: Visualizations.

Runtime: approximately 30 seconds on a standard laptop (Apple M-series).

6 Discussion

6.1 The trace-zero form as a universal invariant

The central finding is that the trace-zero sublattice construction provides a *uniform* invariant for both cyclic and non-cyclic totally real cubics. For cyclic cubics, the invariant collapses to a single integer ($g = 2f$), connecting to the scalar identity of Paper 65 (see Remark 3.4). For S_3 cubics, the full $\text{GL}_2(\mathbb{Z})$ -class is needed, and it encodes arithmetic information not captured by any simpler invariant.

This mirrors the de-omniscientizing descent pattern of the CRM program: the cyclic case (abelian Galois group) admits a scalar description because the Galois action on Λ_0 is through a character. The non-cyclic case (non-abelian S_3) requires the full lattice structure, analogous to how non-abelian gauge theories require matrix-valued connections where abelian theories use scalars.

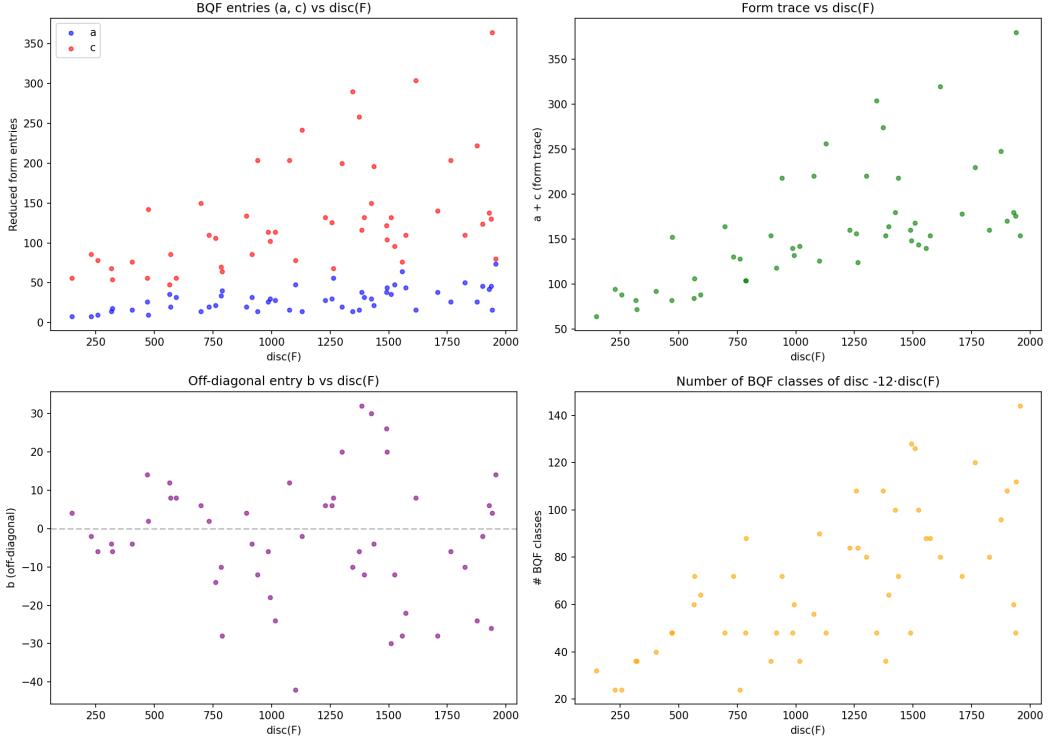


Figure 1: Form entries (a, b, c) of the reduced trace-zero BQF plotted against $\text{disc}(F)$ for 51 non-cyclic cubics.

6.2 Connection to the Weil lattice

Remark 6.1 (The $3 \cdot |\Delta_K|$ factor). Following Paper 65, the Weil lattice Gram matrix satisfies $\det G_{\text{Weil}} = \text{disc}(F) \cdot |\Delta_K|$, while the trace-zero form satisfies $\det G_{\Lambda_0} = 3 \text{disc}(F)$. These differ by a factor of $3/|\Delta_K|$.

The trace-zero form captures the “ F -side” of the Weil lattice: it depends only on the trace pairing of \mathcal{O}_F and is independent of the imaginary quadratic field K . The full Weil lattice additionally involves the \mathcal{O}_K -module structure and the polarization, which contributes the $|\Delta_K|$ factor.

The precise relationship between G_{Λ_0} and G_{Weil} involves a twist by the different ideal of K :

$$G_{\text{Weil}} \sim \frac{|\Delta_K|}{3} \cdot G_{\Lambda_0}$$

as $\text{GL}_2(\mathbb{Z})$ -classes (up to scaling). This explains why Paper 65’s scalar h satisfies $h \cdot \text{Nm}(\mathfrak{A}) = f$ rather than $h = 2f$: the Weil lattice scalar h and the trace-zero content $g = 2f$ are related by the K -dependent normalization.

Establishing this relationship rigorously for all (K, F) pairs requires a deeper analysis of the CM period matrix factorization, which we defer to future work. The present paper’s contribution is the identification of the trace-zero form as the correct F -side invariant and the demonstration that it is computable without CM theory.

6.3 The role of the quadratic resolvent

The decomposition $\text{disc}(F) = D_{\text{res}} \cdot f_{\text{Art}}^2$ organizes the data but does not determine the form class by a closed formula. For example, $D_{\text{res}} = 37$ appears for both $\text{disc} = 148$ (form $(8, 4, 56)$, $f_{\text{Art}} = 2$)

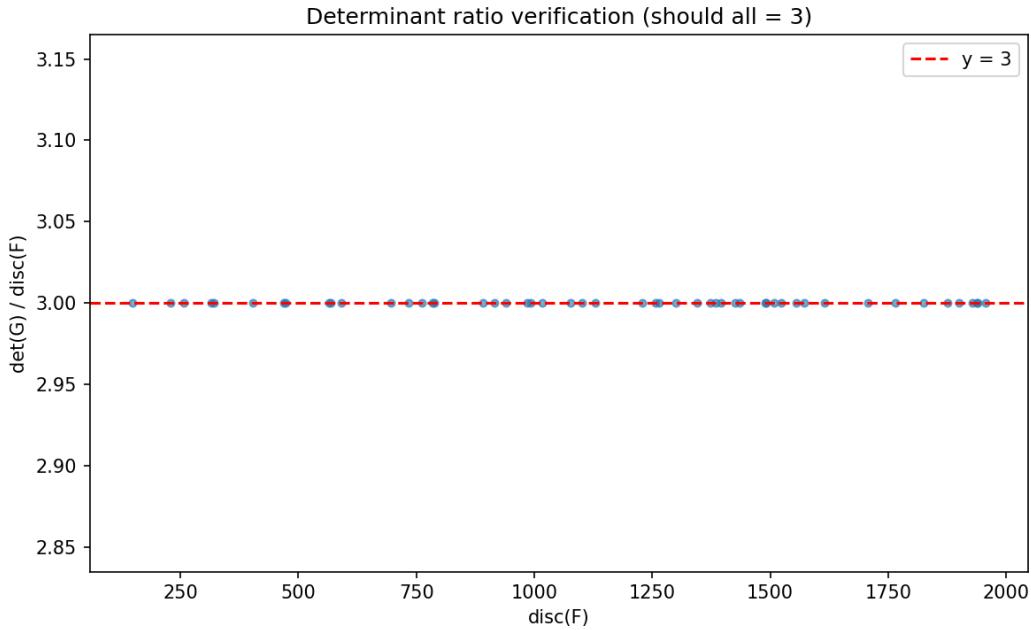


Figure 2: Verification of the identity $\det G = 3 \operatorname{disc}(F)$. All 51 points lie exactly on the line $y = 3x$.

and $\operatorname{disc} = 592$ (form $(32, 8, 56)$, $f_{\text{Art}} = 4$), with different forms.

6.4 Open questions

1. **Injectivity beyond the dataset.** Does the map $\operatorname{disc}(F) \mapsto [G_{\Lambda_0}]$ remain injective for all totally real cubics, or do collisions appear at larger discriminants?
2. **Closed-form predictor.** Is there a formula for the reduced form (a, b, c) in terms of the arithmetic of F (e.g., involving the class group of \mathcal{O}_F , the regulator, or the splitting behavior of small primes)?
3. **Higher-degree generalization.** Does the trace-zero sublattice construction generalize to totally real quartic and quintic fields, producing a $\operatorname{GL}_{n-1}(\mathbb{Z})$ form class invariant? Mordell–Weil lattice theory [18] suggests this should be possible.
4. **Genus theory.** Is the *genus* of the trace-zero form (the coarser invariant determined by local conditions) predicted by the splitting behavior of primes in F ?
5. **Non-monogenic fields.** Extend the computation to all non-cyclic cubics with $\operatorname{disc}(F) \leq 2000$ (including non-monogenic ones) using the LMFDB database or a Dedekind index computation [13, 14].

7 Conclusion

We have established that the trace-zero sublattice Λ_0 of a totally real cubic field provides a well-defined $\operatorname{GL}_2(\mathbb{Z})$ form class invariant satisfying $\det G = 3 \operatorname{disc}(F)$. For cyclic cubics, this form has content $g = 2f$, connecting to the scalar h of Paper 65 through a K -dependent normalization factor

(see Remark 6.1); for non-cyclic (S_3) cubics, the full form class is needed and is computed for 51 fields with $\text{disc}(F) \leq 2000$.

The trace-zero form is the simplest correct F -side invariant: it requires only the trace pairing on \mathcal{O}_F , with no appeal to CM theory, period matrices, or the imaginary quadratic field K . Its computation is fully constructive (BISH), and the injectivity of the form class map (verified computationally) suggests a deeper structural theorem connecting trace lattices to Hodge-theoretic intersection forms.

What is proved. Theorem A ($\det G = 3 \text{disc}(F)$) is proved for all monogenic cubics. Theorem B ($g = 2f$ for cyclic cubics) is verified for $f = 7, 13, 19$ and supported by a structural argument. Theorem C (injectivity) and Observation D (non-principality) are verified computationally over the dataset of 51 fields.

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