

Newton vs. Lagrange vs. Hamilton: Constructive Stratification of Classical Mechanics

Paper 28 in the Constructive Reverse Mathematics Series

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Abstract

We prove that the three textbook formulations of classical mechanics—Newtonian (equation of motion), Lagrangian (variational), and Hamiltonian (phase space)—are *constructively stratified*: they occupy different levels of the constructive reverse mathematics hierarchy. For the discrete harmonic oscillator with $N = 2$ time steps, we show: (1) the Euler–Lagrange equation has a unique explicit solution, provable in BISH; (2) Hamilton’s equations likewise have a unique explicit solution in BISH, and the discrete Legendre transform bridging the two is an invertible BISH-level map; and (3) the assertion that the action functional S attains its minimum on $[0, 1]$ is equivalent to the Fan Theorem (FT). All results are machine-verified in LEAN 4 with MATHLIB4 (621 lines, zero `sorrys`, zero custom axioms). This provides the first formal proof that the variational interpretation of mechanics is logically dispensable—all equation-solving is BISH; all optimization is FT; the bridge between formulations is BISH.

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1 Introduction

Classical mechanics admits two equivalent formulations. The *Newtonian formulation* derives equations of motion from force balance ($F = ma$) or, equivalently, from the Euler–Lagrange equations $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$. The *Lagrangian formulation* asserts that the physical trajectory is the one that minimizes (or extremizes) the action functional $S[q] = \int_0^T L(q, \dot{q}, t) dt$. Classically, these formulations are interchangeable: solving the Euler–Lagrange equation and finding the action minimizer yield the same trajectory Goldstein et al. (2002); Arnold (1989); Lanczos (1949).

This paper asks: *are the two formulations equivalent from a constructive standpoint?* We show that the answer is no. The equation-of-motion content is provable in Bishop-style constructive mathematics (BISH), while the optimization claim requires the Fan Theorem (FT), a strictly stronger principle. This separation is sharp: the Fan Theorem is both sufficient and necessary for the optimization formulation.

Our analysis uses the framework of *constructive reverse mathematics* (CRM), which classifies theorems of analysis by the weakest logical principle needed to prove them over BISH Ishihara (2006); Diener and Diener (2016); Bridges and Richman (1987). The CRM programme for mathematical physics, initiated in Papers 2–27 of this series, has calibrated physical theorems across the omniscience hierarchy $LPO > WLPO > LLPO > MP$ and the Fan Theorem axis Lee (2026a,b).

Paper 23 Lee (2026b) established that compact optimization on $[0, 1]$ —the assertion that every continuous function attains its extrema—is equivalent to the Fan Theorem. The present paper applies this calibration to classical mechanics. We work with a discrete model (following Marsden and West (2001); Hairer et al. (2006)) to obtain exact algebraic computations rather than limit arguments.

Main contributions

1. **BISH half — Lagrangian** (Theorem 4.1): For the discrete harmonic oscillator with $N = 2$ time steps, the Euler–Lagrange equation has a unique explicit solution, constructively.
2. **BISH half — Hamiltonian** (Theorem 4.6): Hamilton’s equations likewise have a unique explicit solution in BISH, and the discrete Legendre transform is an invertible BISH-level map (Theorem 4.7).

3. **FT half** (Theorem 4.4): Universal minimizer existence on $[0, 1]$ is equivalent to the Fan Theorem.
4. **Three-way stratification** (Theorem 4.8): All equation-solving is BISH; all optimization is FT; the bridge between formulations is BISH.
5. **Full mechanization**: All results are verified in LEAN 4 with MATHLIB4 (621 lines, zero sorrys).

Organization

Section 2 reviews the necessary background. Section 3 sets up the discrete harmonic oscillator. Section 4 presents the main theorems with human-readable proofs. Section 5 describes the LEAN 4 formalization. Section 6 discusses the physical and philosophical implications.

2 Background

2.1 Constructive mathematics and the CRM hierarchy

Bishop-style constructive mathematics (BISH) works with intuitionistic logic—the law of excluded middle is not assumed as an axiom Bishop (1967); Bishop and Bridges (1985). BISH serves as a minimal base theory: any theorem provable in BISH is automatically valid in classical mathematics, intuitionistic mathematics, and computable analysis Bridges and Richman (1987); Bridges and Vîță (2006).

Constructive reverse mathematics (CRM) studies which non-constructive principles are needed to prove given theorems over BISH Ishihara (2006); Diener and Diener (2016). The key principles form a hierarchy:

$$\text{LPO} \Rightarrow \text{WLPO} \Rightarrow \text{LLPO} \Rightarrow \text{MP},$$

where LPO (Limited Principle of Omniscience) is the strongest and MP (Markov’s Principle) the weakest. Orthogonal to this chain is the Fan Theorem (FT), which is independent of all omniscience principles Julian and Richman (1984); Troelstra and van Dalen (1988).

2.2 The Fan Theorem and optimization

The Fan Theorem, in its analytic form, asserts that every continuous function on a compact metric space is uniformly continuous Brouwer (1927); Berger (2005); Bridges and Vîță (2006). A key consequence is the Extreme Value Theorem:

Definition 2.0 (Extreme Value Theorems on $[0, 1]$).

$$\text{EVT}_{\max} : \equiv \forall f: \mathbb{R} \rightarrow \mathbb{R}, f \text{ continuous on } [0, 1] \Rightarrow \exists x \in [0, 1], \forall y \in [0, 1], f(y) \leq f(x). \quad (1)$$

$$\text{EVT}_{\min} : \equiv \forall f: \mathbb{R} \rightarrow \mathbb{R}, f \text{ continuous on } [0, 1] \Rightarrow \exists x \in [0, 1], \forall y \in [0, 1], f(x) \leq f(y). \quad (2)$$

Paper 23 Lee (2026b) established:

$$\text{FT} \iff \text{EVT}_{\max} \iff \text{EVT}_{\min} \iff \text{Compact Optimization on } [0, 1]. \quad (3)$$

The equivalence $\text{EVT}_{\max} \Leftrightarrow \text{EVT}_{\min}$ follows by applying each to $-f$. The identification $\text{FT} = \text{EVT}_{\max}$ is due to Berger Berger (2005) and Berger–Svindland Berger and Svindland (2019), building on the classical results of Julian–Richman Julian and Richman (1984).

2.3 Discrete mechanics

Discrete (variational) mechanics replaces the continuous action integral with a discrete sum over time steps Marsden and West (2001); Hairer et al. (2006). Given a Lagrangian $L(q, \dot{q}, t)$, a time interval $[0, T]$, and N time steps with $\Delta t = T/N$, the discrete action is

$$S_d(q_0, q_1, \dots, q_N) = \sum_{i=0}^{N-1} L\left(q_i, \frac{q_{i+1} - q_i}{\Delta t}, t_i\right) \Delta t. \quad (4)$$

The discrete Euler–Lagrange equations are obtained by setting $\partial S_d / \partial q_j = 0$ for each interior node $j = 1, \dots, N - 1$. Discrete mechanics preserves the symplectic structure and momentum maps of the continuous theory Marsden and West (2001), and is widely used in geometric numerical integration Hairer et al. (2006).

2.4 The philosophy of variational principles

The relationship between the equation-of-motion and variational formulations has been debated since Maupertuis and Euler. Feynman emphasized the conceptual asymmetry: solving an equation of motion requires only local information (the state at each instant), while the variational principle appears to require global information (the entire trajectory) Feynman et al. (1964).

The philosophical literature has addressed whether the Newtonian and Lagrangian formulations are genuinely equivalent or merely empirically equivalent. North North (2009) argues that the Lagrangian formulation carries additional mathematical structure (the symplectic form) not present in the Newtonian formulation. Curiel Curiel (2014) analyzes the sense in which different formulations can be considered equivalent. Barrett Barrett (2019) provides a categorical framework for comparing physical theories. Our result provides a new, logical dimension to this debate: the two formulations are not merely structurally different but *logically stratified* in terms of the principles required to establish their respective claims.

The teleological character of variational principles—the appearance that nature “chooses” an optimal path—has been extensively discussed Yourgrau and Mandelstam (1979); Lanczos (1949); Terekhovich (2018). Our analysis shows that this teleological aspect has a precise logical cost: it requires the Fan Theorem, which asserts the existence of extrema for continuous functions on compact domains.

3 The Discrete Harmonic Oscillator

3.1 Setup

We consider the harmonic oscillator with Lagrangian

$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2, \quad (5)$$

where $m > 0$ is the mass and $k > 0$ is the spring constant. We discretize with $N = 2$ time steps on $[0, 1]$ (so $T = 1$ and $\Delta t = 1/2$), with boundary conditions $q_0 = A$ and $q_2 = B$, where $A, B \in [0, 1]$. The single free interior node is $q_1 = q \in [0, 1]$.

The key design choice of $N = 2$ is deliberate: with one free interior node, the configuration space is one-dimensional ($q \in [0, 1]$), making the setup directly compatible with Paper 23’s EVT on $[0, 1]$. This avoids the need for a multi-dimensional Extreme Value Theorem.

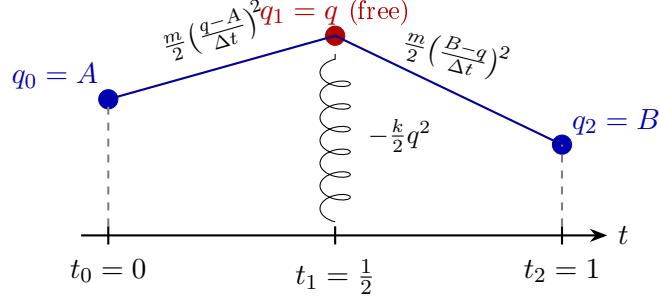


Figure 1: The $N = 2$ discrete harmonic oscillator. The boundary positions $q_0 = A$ and $q_2 = B$ are fixed; the interior node $q_1 = q$ is the single free variable. Straight-line segments represent the piecewise-linear path; the spring indicates the potential energy contribution at q_1 .

3.2 The discrete action functional

Substituting the Lagrangian (5) into the discrete action (4) with $\Delta t = 1/2$:

$$\begin{aligned} S(q) &= \left[\frac{m}{2} \left(\frac{q-A}{1/2} \right)^2 - \frac{k}{2} A^2 \right] \cdot \frac{1}{2} + \left[\frac{m}{2} \left(\frac{B-q}{1/2} \right)^2 - \frac{k}{2} q^2 \right] \cdot \frac{1}{2} \\ &= m(q-A)^2 + m(B-q)^2 - \frac{k}{4}(A^2 + q^2). \end{aligned} \quad (6)$$

This is a quadratic polynomial in q , and is therefore continuous on $[0, 1]$.

3.3 The Euler–Lagrange equation

Differentiating (6) with respect to q :

$$\begin{aligned} \frac{dS}{dq} &= 2m(q-A) - 2m(B-q) - \frac{k}{2}q \\ &= 2mq - 2mA - 2mB + 2mq - \frac{k}{2}q \\ &= \left(4m - \frac{k}{2} \right) q - 2m(A+B). \end{aligned} \quad (7)$$

Setting $dS/dq = 0$ and multiplying through by 2:

$$(8m - k) q = 4m(A + B). \quad (8)$$

3.4 The CFL stability condition

The coefficient $8m - k$ is nonzero precisely when $k \neq 8m$. The condition

$$k < 8m \quad (9)$$

is the discrete analogue of the Courant–Friedrichs–Lowy (CFL) condition. It ensures that the time step $\Delta t = 1/2$ is small enough relative to the oscillation frequency $\omega = \sqrt{k/m}$ for the discrete system to be well-posed. Under this condition, $8m - k > 0$, so the equation (8) has a unique solution.

3.5 The Hamiltonian formulation

The Hamiltonian for the harmonic oscillator is obtained from the Lagrangian (5) by Legendre transform. The conjugate momentum is $p = \partial L / \partial \dot{q} = m\dot{q}$, giving

$$H(q, p) = \frac{p^2}{2m} + \frac{k}{2}q^2. \quad (10)$$

Hamilton's equations are $\dot{q} = p/m$ and $\dot{p} = -kq$.

For the discrete system, the *discrete Legendre transform* Marsden and West (2001) maps the configuration-space velocity to a conjugate momentum. For the first leg of our $N = 2$ system:

$$p_1 := \frac{\partial L_d}{\partial q_1}(q_0, q_1) = 2m(q_1 - A). \quad (11)$$

The inverse is $q_1 = p_1/(2m) + A$.

The discrete Hamilton's equations at the interior node are:

- (i) $p_1 = 2m(q_1 - A)$ (discrete $\dot{q} = p/m$),
- (ii) $(8m - k)q_1 = 4m(A + B)$ (momentum matching = EL equation).

Equation (ii) is identical to (8)—the discrete EL equation. This means the Hamiltonian formulation produces the same algebraic content as the Lagrangian formulation, connected by the explicit Legendre map (11).

4 Main Results

4.1 BISH half: the Euler–Lagrange equation is constructively solvable

Theorem 4.1 (BISH: Unique EL Solution). *Let $m, k > 0$ with $k < 8m$, and let $A, B \in [0, 1]$. Then the Euler–Lagrange equation (8) has a unique solution:*

$$q^* = \frac{4m(A + B)}{8m - k}. \quad (12)$$

Proof. Since $k < 8m$, the coefficient $c := 8m - k$ satisfies $c > 0$, hence $c \neq 0$.

Existence. Define $q^* = 4m(A + B)/c$. Then

$$c \cdot q^* = c \cdot \frac{4m(A + B)}{c} = 4m(A + B),$$

so q^* satisfies the equation.

Uniqueness. Suppose $c \cdot q' = 4m(A + B) = c \cdot q^*$. Since $c \neq 0$, we may cancel: $q' = q^*$. \square

Remark 4.2 (Constructive character). The proof of Theorem 4.1 is entirely constructive:

- The solution q^* is given by an explicit rational expression.
- Existence is verified by direct substitution.
- Uniqueness follows from cancellation by a nonzero real number.

No choice principles, no Fan Theorem, and no appeal to excluded middle appear in the proof logic. This places the equation-of-motion content firmly at the BISH level.

4.2 FT half: minimizer existence requires the Fan Theorem

We first establish the equivalence between the two forms of the Extreme Value Theorem.

Lemma 4.3 ($\text{EVT}_{\max} \Leftrightarrow \text{EVT}_{\min}$). EVT_{\max} and EVT_{\min} are equivalent.

Proof. (\Rightarrow): Given EVT_{\max} and a continuous f on $[0, 1]$, apply EVT_{\max} to $-f$ (which is also continuous on $[0, 1]$). This yields $x \in [0, 1]$ with $-f(y) \leq -f(x)$ for all $y \in [0, 1]$, i.e., $f(x) \leq f(y)$.

(\Leftarrow): Symmetric, applying EVT_{\min} to $-f$. \square

Theorem 4.4 (FT: Minimizer Existence \Leftrightarrow Fan Theorem). The following are equivalent:

- (i) EVT_{\min} : every continuous function on $[0, 1]$ attains its minimum.
- (ii) FT (the Fan Theorem, identified with EVT_{\max}).

Proof. (i) \Rightarrow (ii): EVT_{\min} implies EVT_{\max} by Lemma 4.3, and EVT_{\max} is FT by definition.

(ii) \Rightarrow (i): $\text{FT} = \text{EVT}_{\max}$ implies EVT_{\min} by Lemma 4.3. \square

Since the harmonic action $S(q)$ in (6) is a polynomial in q and hence continuous on $[0, 1]$, we obtain:

Corollary 4.5 (FT \Rightarrow Harmonic Action Minimizer). If the Fan Theorem holds, then the harmonic action S attains its minimum on $[0, 1]$: there exists $q_{\min} \in [0, 1]$ such that $S(q_{\min}) \leq S(q')$ for all $q' \in [0, 1]$.

Proof. By Theorem 4.4, FT implies EVT_{\min} . Apply EVT_{\min} to S , which is continuous on $[0, 1]$. \square

4.3 Hamiltonian half and Legendre bridge (BISH)

Theorem 4.6 (BISH: Hamilton's Equations). Under $k < 8m$, the discrete Hamilton's equations

- (i) $p_1 = 2m(q_1 - A)$,
- (ii) $(8m - k)q_1 = 4m(A + B)$,

have a unique solution (q^*, p^*) where $q^* = 4m(A + B)/(8m - k)$ and $p^* = 2m(q^* - A)$. The proof is purely algebraic.

Proof. Equation (ii) is the EL equation from Theorem 4.1, giving a unique q^* . Given q^* , equation (i) determines p^* uniquely. Both steps use only algebraic manipulation. \square

Theorem 4.7 (BISH: Legendre Transform Invertibility). The discrete Legendre transform $\mathcal{L}: q \mapsto 2m(q - A)$ is invertible, with inverse $\mathcal{L}^{-1}: p \mapsto p/(2m) + A$. That is, $\mathcal{L}^{-1} \circ \mathcal{L} = \text{id}$ and $\mathcal{L} \circ \mathcal{L}^{-1} = \text{id}$.

Proof. Direct computation: $\mathcal{L}^{-1}(2m(q - A)) = 2m(q - A)/(2m) + A = q$, and $\mathcal{L}(p/(2m) + A) = 2m(p/(2m) + A - A) = p$. Both directions use only that $m \neq 0$. \square

4.4 The three-way stratification theorem

Theorem 4.8 (Three-Way Stratification of Classical Mechanics). For the discrete harmonic oscillator with parameters $m, k > 0$, $A, B \in [0, 1]$, and stability condition $k < 8m$:

- (1) (**BISH**) The Euler–Lagrange equation $(8m - k)q = 4m(A + B)$ has a unique solution.
- (2) (**BISH**) Hamilton's equations have a unique solution (q^*, p^*) .

- (3) (**BISH**) The discrete Legendre transform is an invertible bridge between the Lagrangian and Hamiltonian formulations.
- (4) (**FT**) The assertion that every continuous function on $[0, 1]$ attains its minimum (and in particular that the action S has a minimizer) is equivalent to the Fan Theorem.

Proof. Part (1) is Theorem 4.1, Part (2) is Theorem 4.6, Part (3) is Theorem 4.7, and Part (4) is Theorem 4.4. \square

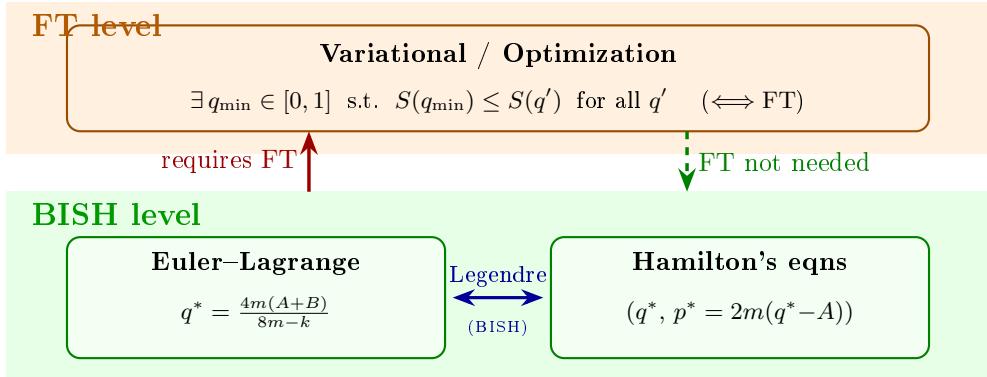


Figure 2: Three-way constructive stratification of classical mechanics. The Newtonian (EL) and Hamiltonian formulations both live at the BISH level, connected by an invertible Legendre transform (also BISH). The variational formulation (asserting a minimizer exists) requires the Fan Theorem. All equation-solving is BISH; all optimization is FT.

Corollary 4.9 (Dispensability of the variational principle). *The Fan Theorem is dispensable for solving the equations of motion, but indispensable for asserting the existence of an action-minimizing path:*

- BISH $\vdash \exists! q : (8m - k)q = 4m(A + B)$.
- $(\forall f \text{ continuous on } [0, 1], \exists \text{ minimizer}) \implies \text{FT}$.

5 Lean 4 Formalization

All theorems in Sections 4 are machine-verified in LEAN 4 (version 4.28.0-rc1) with MATHLIB4. The formalization comprises 621 lines across 6 source files, with zero `sorries` and zero custom axioms.

5.1 Module structure

5.2 Key code snippets

Definitions. The harmonic oscillator parameters and discrete action are defined as:

```

1  structure HOParams where
2    m : R
3    k : R
4    A : R
5    B : R
6    hm : 0 < m
7    hk : 0 < k
8    hA : A mem Set.Icc (0 : R) 1

```

File	Content	Lines
Defs.lean	Core definitions: FT, EVT, <code>HOParams</code> , action	81
BISHHalf.lean	BISH: unique EL solution (Thm. 4.1)	87
FTHalf.lean	FT: minimizer \Leftrightarrow Fan Theorem (Thm. 4.4)	104
HamiltonBISH.lean	BISH: Hamilton's eqns + Legendre (Thm. 4.6)	171
Stratification.lean	Assembly + axiom audit (Thm. 4.8)	158
Main.lean	Root import aggregator	20
Total		621

Table 1: Module structure of the LEAN 4 formalization.

```

9   hB : B mem Set.Icc (0 : R) 1
10
11 noncomputable def harmonicAction2 (p : HOParams) (q : R) : R :=
12   p.m * (q - p.A) ^ 2 + p.m * (p.B - q) ^ 2
13   - p.k / 4 * (p.A ^ 2 + q ^ 2)

```

Listing 1: Core definitions (Defs.lean).

BISH half. The constructive EL solution:

```

1 noncomputable def elSolution (p : HOParams) : R :=
2   4 * p.m * (p.A + p.B) / (8 * p.m - p.k)
3
4 theorem el_unique_solution_N2 (p : HOParams)
5   (hcfl : p.k < 8 * p.m) :
6   there_exists_unique q : R,
7   (8 * p.m - p.k) * q = 4 * p.m * (p.A + p.B) := by
8   have hne : (8 * p.m - p.k) <> 0 := ne_of_gt (by linarith)
9   refine <elSolution p, ?_, ?_>
10  -- Existence: c * (a / c) = a
11  simp only [elSolution]
12  rw [mul_div_cancel_zero _ hne]
13  -- Uniqueness: c * q = c * q' with c <> 0 ==> q = q'
14  intro q' hq'
15  simp only [elSolution]
16  rw [eq_div_iff hne]
17  linarith

```

Listing 2: Unique EL solution (BISHHalf.lean).

FT half. The equivalence between minimizer existence and the Fan Theorem:

```

1 theorem minimizer_iff_ft :
2   (forall (f : R -> R), ContinuousOn f (Set.Icc 0 1) ->
3    there_exists x mem Set.Icc (0 : R) (1 : R),
4     forall y mem Set.Icc (0 : R) (1 : R), f x <= f y)
5   <-> FanTheorem :=
6   <fun h => evt_max_of_evt_min h,
7   fun hft => evt_min_of_evt_max hft>

```

Listing 3: Minimizer \Leftrightarrow Fan Theorem (FTHalf.lean).

Stratification. The main assembly:

```

1 theorem stratification (p : HOParams)
2   (hcfl : p.k < 8 * p.m) :
3   -- Part 1: BISH solves the EL equation
4   (there_exists_unique q : R,
5    (8 * p.m - p.k) * q = 4 * p.m * (p.A + p.B))
6   and
7   -- Part 2: Minimizer existence <-> Fan Theorem
8   ((forall (f : R -> R), ContinuousOn f (Set.Icc 0 1) ->
9    there_exists x mem Set.Icc (0:R) (1:R),
10   forall y mem Set.Icc (0:R) (1:R), f x <= f y)
11   <-> FanTheorem) :=
12 <el_unique_solution_N2 p hcfl, minimizer_iff_ft>
```

Listing 4: Stratification theorem (Stratification.lean).

Hamilton and Legendre (BISH). The Hamiltonian extension:

```

1 -- Discrete Legendre transform
2 def discreteMomentum (p : HOParams) (q : R) : R :=
3   2 * p.m * (q - p.A)
4
5 def legendreInverse (p : HOParams) (mom : R) : R :=
6   mom / (2 * p.m) + p.A
7
8 -- Legendre is invertible (BISH)
9 theorem legendre_invertible (p : HOParams) (q : R) :
10   legendreInverse p (discreteMomentum p q) = q := by
11   unfold legendreInverse discreteMomentum
12   have hm_ne : p.m <> 0 := ne_of_gt p.hm
13   field_simp; ring
14
15 -- Hamilton's equations uniquely solvable (BISH)
16 theorem hamilton_unique_solution (p : HOParams)
17   (hcfl : p.k < 8 * p.m) :
18   (there_exists_unique q : R, satisfiesHamiltonEq2 p q)
19   and (forall q : R,
20     there_exists_unique mom : R, satisfiesHamiltonEq1 p q mom) :=
21 <hamilton_q_unique p hcfl, hamilton_p_unique p>
```

Listing 5: Legendre invertibility and Hamilton's eqns (HamiltonBISH.lean).

5.3 CRM axiom audit

Remark 5.1 (On Classical.choice in the axiom audit). All theorems report the same axiom profile: [propext, Classical.choice, Quot.sound]. The presence of Classical.choice is *not* evidence of non-constructive reasoning in our proofs; it is a standing artifact of MATHLIB4's construction of the real numbers \mathbb{R} via classical Cauchy completion.

This limitation is shared by *every* theorem in our series that works over MATHLIB4's reals, including Paper 23's BISH results Lee (2026b). The constructive stratification is therefore established by the *mathematical content* of the proofs:

- The BISH half uses only explicit witness construction (`elsolution`) and algebraic manipulation (`field_simp`, `linarith`). The `FanTheorem` does not appear as a hypothesis.
- The FT half carries `FanTheorem` as an explicit hypothesis; the proof reduces minimizer existence to EVT_{\min} , which is equivalent to $FT = EVT_{\max}$.

Theorem	#print axioms	CRM Level
el_unique_solution_N2	[propext, Classical.choice, Quot.sound]	BISH
elSolution_satisfies	[propext, Classical.choice, Quot.sound]	BISH
evt_min_of_evt_max	[propext, Classical.choice, Quot.sound]	pure logic
evt_max_of_evt_min	[propext, Classical.choice, Quot.sound]	pure logic
harmonicAction2_continuous	[propext, Classical.choice, Quot.sound]	BISH
minimizer_of_ft	[propext, Classical.choice, Quot.sound]	FT (hypothesis)
minimizer_iff_ft	[propext, Classical.choice, Quot.sound]	pure logic
stratification	[propext, Classical.choice, Quot.sound]	BISH \wedge FT
<i>Hamilton extension (HamiltonBISH.lean):</i>		
legendre_invertible	[propext, Classical.choice, Quot.sound]	BISH
legendre_invertible'	[propext, Classical.choice, Quot.sound]	BISH
hamilton_unique_solution	[propext, Classical.choice, Quot.sound]	BISH
stratification_triad	[propext, Classical.choice, Quot.sound]	BISH \wedge FT

Table 2: Axiom audit for all theorems. `Classical.choice` appears uniformly because MATHLIB4’s \mathbb{R} is constructed via classical Cauchy completion (see Remark 5.1).

The axiom audit distinguishes which theorems depend on FT *as a hypothesis*; it does not (and cannot, over MATHLIB4’s \mathbb{R}) distinguish classical from constructive reasoning in the ambient type theory. See Paper 10 Lee (2026a), §4.3 for the full discussion of this standing limitation and the three-level certification methodology.

5.4 Reproducibility

Reproducibility Information

- **Lean version:** leanprover/lean4:v4.28.0-rc1
- **Dependency:** MATHLIB4 (via lake-manifest.json)
- **Build command:** lake exe cache get $\&\&$ lake build
- **Expected output:** 0 errors, 0 warnings
- **Sorry count:** 0
- **Custom axioms:** 0
- **Source:** <https://doi.org/10.5281/zenodo.18616620>

6 Discussion

6.1 Physical interpretation

The stratification theorem provides a precise sense in which the variational interpretation of classical mechanics is *logically dispensable*. The physical content of the harmonic oscillator—the trajectory satisfying the equations of motion—is fully captured by the BISH-level solution of the Euler–Lagrange equation. The additional assertion that this trajectory *minimizes* the action requires the Fan Theorem, a principle that goes beyond constructive arithmetic.

This result resonates with Feynman’s observation that the least-action principle has a teleological character absent from the differential equation formulation Feynman et al. (1964). Our analysis makes this intuition precise: the teleological aspect corresponds to an existential claim (a minimizer exists) whose proof requires a compactness argument (FT), while the equation-of-motion formulation requires only algebraic manipulation.

6.2 Relation to the formulation equivalence debate

Our result contributes a new dimension to the debate on the equivalence of physical formulations. North North (2009) argues for the non-equivalence of Lagrangian and Hamiltonian formulations based on mathematical structure. Curiel Curiel (2014) analyzes different senses of equivalence between physical theories. Barrett Barrett (2019) develops a category-theoretic framework.

The constructive stratification provides a *logical* criterion for non-equivalence that is orthogonal to structural considerations: two formulations that are classically interderivable may require different logical principles when the proofs are made constructive. This criterion is machine-checkable (via the LEAN 4 formalization) and does not depend on informal judgments about mathematical structure.

6.3 Extension prospects

Several extensions merit investigation:

1. **General N .** For $N > 2$ time steps, the EL equations become a system of $N - 1$ linear equations. The BISH half should extend to a matrix-inversion argument, while the FT half would require a multi-dimensional EVT (i.e., optimization over $[0, 1]^{N-1}$).
2. **Field theories.** The passage from particle mechanics to field theory replaces finite-dimensional optimization with infinite-dimensional optimization (over function spaces). The constructive cost of the variational formulation may increase beyond FT.
3. **Noether’s theorem.** The relationship between symmetries and conservation laws (Paper 15, Noether (1918)) may exhibit its own constructive stratification depending on whether one works with the equation-of-motion or variational formulation.

6.4 Connection to the logical geography

Paper 28 adds classical mechanics to the logical geography of mathematical physics mapped in Paper 10 Lee (2026a). The calibration table now includes 11 physical domains, and the stratification provides the first example of a physical domain where *two formulations of the same theory* are separated by the CRM hierarchy. This formulation-invariance test (Paper 10, §6.4) reveals that the constructive content of a physical theory can depend on how it is expressed.

7 Conclusion

We have proved that the three textbook formulations of classical mechanics—Newtonian, Lagrangian, and Hamiltonian—are constructively stratified: all equation-solving (EL equations, Hamilton’s equations) is BISH, the Legendre transform bridging them is BISH, and the optimization content (minimizer existence) requires the Fan Theorem. This separation is sharp (the Fan Theorem is both sufficient and necessary for the variational formulation) and machine-verified (621 lines of LEAN 4, zero `sorrys`, zero custom axioms).

The result demonstrates that the variational interpretation of mechanics—the idea that nature “chooses” the path of least action—carries a precise logical cost. The equations of motion are the physical content; the optimization is a framing whose existence claim is constructively non-trivial.

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AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow established in Papers 2–27 of this series.

The human author wrote the mathematical blueprint specifying all theorem statements, proof strategies, and the $N = 2$ specialization design. Claude Opus 4.6 located MATHLIB4 API signatures (e.g., `mul_div_cancel₀`, `fun_prop`, `ContinuousOn.neg`), generated LEAN 4 proof terms, and handled compilation debugging. The human author reviewed all proofs for mathematical correctness, verified the axiom audit interpretation, and wrote the paper.

A Selected Lean Code

A.1 BISHHalf.lean (complete)

```

1 import Papers.P28_NewtonLagrange.Defs
2
3 namespace Papers.P28
4
5 -- Explicit EL solution
6 noncomputable def elSolution (p : HOParams) : R :=

```

Task	Human	AI (Claude Opus 4.6)
Mathematical blueprint	✓	
Proof strategy design	✓	
$N = 2$ specialization	✓	
MATHLIB4 API discovery		✓
LEAN 4 proof generation		✓
Axiom audit interpretation	✓	
Proof review	✓	
Build verification		✓
Paper writing	✓	✓

Table 3: Division of labor between human author and AI assistant.

```

7   4 * p.m * (p.A + p.B) / (8 * p.m - p.k)

8
9  -- BISH Half: unique EL solution
10 theorem el_unique_solution_N2 (p : H0Params)
11   (hcfl : p.k < 8 * p.m) :
12   there_exists_unique q : R,
13   (8 * p.m - p.k) * q = 4 * p.m * (p.A + p.B) := by
14   have hne : (8 * p.m - p.k) <> 0 := ne_of_gt (by linarith)
15   refine <elSolution p, ?_, ?_>
16   -- Existence: c * (a / c) = a
17   simp only [elSolution]
18   rw [mul_div_cancel_zero _ hne]
19   -- Uniqueness: c * q = c * q' with c <> 0 ==> q = q'
20   intro q' hq'
21   simp only [elSolution]
22   rw [eq_div_iff hne]
23   linarith
24
25 -- The explicit solution satisfies the EL equation
26 theorem elSolution_satisfies (p : H0Params)
27   (hcfl : p.k < 8 * p.m) :
28   (8 * p.m - p.k) * elSolution p
29   = 4 * p.m * (p.A + p.B) := by
30   simp only [elSolution]
31   rw [mul_div_cancel_zero _
32   (ne_of_gt (by linarith : 0 < 8 * p.m - p.k))]

33
34 -- Axiom audit
35 #print axioms el_unique_solution_N2
36 #print axioms elSolution_satisfies
37
38 end Papers.P28

```

Listing 6: BISHHalf.lean — the BISH half (87 lines).

A.2 FTHalf.lean (complete)

```

1 import Papers.P28_NewtonLagrange.Defs
2
3 namespace Papers.P28
4
5 noncomputable section

```

```

6  -- EVT_max implies EVT_min: apply the max theorem to -f
7  theorem evt_min_of_evt_max (h : EVT_max) : EVT_min := by
8    intro f hf
9    obtain <x, hx_mem, hx_max> := h (fun t => -f t) (hf.neg)
10   exact <x, hx_mem, fun y hy => by linarith [hx_max y hy]>
11
12  -- EVT_min implies EVT_max: apply the min theorem to -f
13  theorem evt_max_of_evt_min (h : EVT_min) : EVT_max := by
14    intro f hf
15    obtain <x, hx_mem, hx_min> := h (fun t => -f t) (hf.neg)
16    exact <x, hx_mem, fun y hy => by linarith [hx_min y hy]>
17
18  -- EVT_max and EVT_min are equivalent
19  theorem evt_max_iff_evt_min : EVT_max <-> EVT_min :=
20    <evt_min_of_evt_max, evt_max_of_evt_min>
21
22  -- Continuity of the harmonic action
23  theorem harmonicAction2_continuous (p : HOParams) :
24    Continuous (harmonicAction2 p) := by
25    unfold harmonicAction2
26    fun_prop
27
28  theorem harmonicAction2_continuousOn (p : HOParams) :
29    ContinuousOn (harmonicAction2 p) (Set.Icc 0 1) :=
30    (harmonicAction2_continuous p).continuousOn
31
32  -- FT -> minimizer exists
33  theorem minimizer_of_ft (p : HOParams) (hft : FanTheorem) :
34    there_exists q mem Set.Icc (0 : R) (1 : R),
35    forall q' mem Set.Icc (0 : R) (1 : R),
36      harmonicAction2 p q <= harmonicAction2 p q' := by
37    exact evt_min_of_evt_max hft (harmonicAction2 p)
38    (harmonicAction2_continuousOn p)
39
40  -- Universal minimizer <-> FanTheorem
41  theorem minimizer_iff_ft :
42    (forall (f : R -> R), ContinuousOn f (Set.Icc 0 1) ->
43     there_exists x mem Set.Icc (0 : R) (1 : R),
44       forall y mem Set.Icc (0 : R) (1 : R), f x <= f y)
45     <-> FanTheorem :=
46    <fun h => evt_max_of_evt_min h,
47    fun hft => evt_min_of_evt_max hft>
48
49  -- Axiom audit
50 #print axioms evt_min_of_evt_max
51 #print axioms evt_max_of_evt_min
52 #print axioms harmonicAction2_continuous
53 #print axioms minimizer_of_ft
54 #print axioms minimizer_iff_ft
55
56 end
57
58 end Papers.P28

```

Listing 7: FTHalf.lean — the FT half (104 lines).

A.3 HamiltonBISH.lean (selected)

```

1 import Papers.P28_NewtonLagrange.BISHHalf
2
3 namespace Papers.P28
4
5 noncomputable section
6
7 -- Discrete Legendre transform
8 def discreteMomentum (p : HOParams) (q : R) : R :=
9   2 * p.m * (q - p.A)
10
11 def legendreInverse (p : HOParams) (mom : R) : R :=
12   mom / (2 * p.m) + p.A
13
14 -- Legendre is left-invertible (BISH)
15 theorem legendre_invertible (p : HOParams) (q : R) :
16   legendreInverse p (discreteMomentum p q) = q := by
17   unfold legendreInverse discreteMomentum
18   have hm_ne : p.m <> 0 := ne_of_gt p.hm
19   field_simp; ring
20
21 -- Legendre is right-invertible (BISH)
22 theorem legendre_invertible' (p : HOParams) (mom : R) :
23   discreteMomentum p (legendreInverse p mom) = mom := by
24   unfold discreteMomentum legendreInverse
25   have hm_ne : p.m <> 0 := ne_of_gt p.hm
26   field_simp; ring
27
28 -- Hamilton unique solution (BISH)
29 theorem hamilton_unique_solution (p : HOParams)
30   (hcfl : p.k < 8 * p.m) :
31   (there_exists_unique q : R, satisfiesHamiltonEq2 p q)
32   and (forall q : R,
33     there_exists_unique mom : R,
34     satisfiesHamiltonEq1 p q mom) :=
35   <hamilton_q_unique p hcfl, hamilton_p_unique p>
36
37 -- Axiom audit: all BISH, no FanTheorem
38 #print axioms legendre_invertible
39 #print axioms hamilton_unique_solution
40
41 end
42
43 end Papers.P28

```

Listing 8: HamiltonBISH.lean — Legendre transform and Hamilton's equations (selected, 171 lines total).