The Logical Constitution of Physical Reality

Constructive Reverse Mathematics of Mathematical Physics

A Monograph in 42 Papers

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[1]Individual paper DOIs on Zenodo.

To my wife Mimi

Ἐν ἀρχῇ ἦν ὁ Λόγος,
καὶ ὁ Λόγος ἦν πρὸς τὸν Θεόν,
καὶ Θεὸς ἦν ὁ Λόγος.

In the beginning was the Word,
and the Word was with God,
and the Word was God.

— John 1:1

Abstract

We prove that the logical resources required for all empirical predictions in known physics are exactly BISH + LPO: Bishop's constructive mathematics augmented by the Limited Principle of Omniscience. This characterization is established by systematic axiom calibration across 42 papers spanning the Standard Model (QED, QCD, electroweak theory), general relativity, statistical mechanics, quantum information theory, the AdS/CFT correspondence, and the cosmological constant problem. All calibrations are formally verified in approximately 35,000 lines of Lean 4 proof code.

Three foundational results underpin the characterization. First, Fekete's Subadditive Lemma—the mathematical engine of phase transitions—is equivalent to LPO over BISH, establishing that LPO is physically instantiated. Second, the Fan Theorem (compactness) is dispensable. Third, Dependent Choice is dispensable. A conservation metatheorem explains the pattern: empirical predictions are finite compositions of computable functions (BISH), and the only idealizations exceeding finite computation are completed limits (LPO via the Bounded Monotone Convergence equivalence).

The undecidability arc completes the picture: every known physical undecidability result is Turing–Weihrauch equivalent to LPO, traceable to a single ancestor (Wang tiling). A refined analysis reveals that generic intensive observables without promise gaps can reach Σ_2^0—but this Platonic tier is empirically inaccessible due to finite experimental precision. The diagnostic phase (Papers 41–42) applies the framework to the AdS/CFT correspondence and the cosmological constant problem, dissolving the 10^{120} discrepancy as a regulator artifact and identifying the holographic dictionary as an axiom-preserving map.

Keywords: constructive mathematics, reverse mathematics, formal verification, foundations of physics, Bishop's constructive analysis, omniscience principles, Lean 4

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Preface

The Cost of Being Right Too Early

The intellectual history of constructive mathematics is usually told as a sequence of ideas. But behind the ideas were human beings, and the personal cost of this program is a story rarely told outside the specialist community. It deserves telling, because it is part of what the formulas mean.

In 1907, L. E. J. Brouwer submitted his doctoral thesis *Over de grondslagen der wiskunde* [1] to the University of Amsterdam and launched intuitionism—a radical program asserting that mathematics is a human mental construction, that infinite objects exist only through the processes that generate them, and that the Law of Excluded Middle (LEM: for every proposition P, either P or $\neg P$) is not a valid principle of reasoning about infinite collections. Brouwer rejected completed infinities, non-constructive existence proofs, and the Axiom of Choice. His program was philosophically deep but mathematically constrained: by adopting principles that are *classically false* (such as the continuity of all real-valued functions), Brouwer created a mathematics incompatible with the classical tradition. Most mathematicians rejected it—not because the philosophy was wrong, but because the mathematics was too alien to adopt. And Brouwer paid for his vision. His *Grundlagenstreit* with David Hilbert—the foundational dispute over the legitimacy of non-constructive methods—escalated through the 1920s from a philosophical disagreement into a campaign of professional destruction. In 1928, Hilbert arranged to have Brouwer expelled from the editorial board of *Mathematische Annalen*, then the most prestigious journal in mathematics. The board was dissolved and reconstituted without Brouwer; Einstein and Carathéodory declined reappointment in protest, but their protest changed nothing. Brouwer wrote to a confidant: “All my life’s work has been wrested from me and I am left in fear, shame, and mistrust, and suffering the persecution of my baiting torturers” [2]. The man who had seen more clearly than anyone that infinite reasoning requires justification spent his remaining decades in isolation—withdrawn from mathematical life, consumed by fears of persecution and financial ruin. He died in 1966, aged 85, struck by a car near his home in Blaricum. His insight into the foundations had been correct; the profession that rejected him never acknowledged this during his lifetime.

Sixty years after Brouwer’s thesis, Errett Bishop published *Foundations of Constructive Analysis* [3] and demonstrated that Brouwer’s philosophical concerns could be addressed without his non-classical axioms. Bishop’s constructive mathematics (BISH) simply *declines* to use the Law of Excluded Middle and the Axiom of Choice without assuming their negations. Every theorem proved in BISH is automatically valid in classical mathematics, in intuitionism, and in recursive mathematics. Bishop showed that substantial portions of analysis—measure theory, Banach space theory, spectral theory—could be developed constructively, providing explicit algorithms wherever classical mathematics merely asserts existence. The promise was real: a foundation for mathematics in which every existence proof comes with a construction, every function is computable, and every proof provides an algorithm. But Bishop’s fate was quieter than Brouwer’s and no less painful. Abraham Robinson conceded that “even those who are not willing to accept Bishop’s basic philosophy must be impressed with the great analytical power displayed in his work." Bishop was invited to address the International Congress of Mathematicians in 1966 and to deliver the American Mathematical Society's Colloquium Lectures in 1973—four hour-long talks he titled *Schizophrenia in Contemporary Mathematics*, a pointed diagnosis of what he saw as the profession's refusal to confront its own non-constructive habits. But these honors were exceptions. When Bishop presented constructive mathematics at departments across the United States, the reception was hostile. He encountered not disagreement but dismissal—the particular cruelty of being told that your life's work is not mathematics at all. By the late 1970s, Bishop had become almost completely withdrawn from mathematical life. He died of cancer in 1983, aged 54. His program was still dismissed by the mainstream. The recognition that he had been right would come decades later, and it would come too late for him.

The pattern of vision punished is not unique to constructive mathematics. Galileo Galilei spent his final years under house arrest for defending heliocentrism; he died confined and nearly blind in 1642. Ignaz Semmelweis demonstrated that hand-washing could eliminate childbed fever, was dismissed by the medical establishment, and died in an asylum in 1865—beaten by guards—years before germ theory proved him right. Emmy Noether spent years lecturing at Göttingen unpaid, her courses listed under Hilbert's name, before being expelled in 1933; she died in exile at Bryn Mawr in 1935, aged 53. Ludwig Boltzmann—whose statistical mechanics underpins every thermodynamic calculation in this monograph—was ridiculed for decades by Ernst Mach, Wilhelm Ostwald, and others who denied the existence of atoms; he hanged himself in 1906, one year before experiments vindicated his atomistic theory. Paul Ehrenfest, who carried Boltzmann's statistical program forward, struggled with depression compounded by professional isolation; he took his own life in 1933. Lise Meitner co-discovered nuclear fission but was excluded from the Nobel Prize awarded to Otto Hahn alone; she spent decades in exile after fleeing Nazi Germany. Subrahmanyan Chandrasekhar derived the mass limit for white dwarfs as a young man, only to have Arthur Eddington publicly mock the result for years; the Nobel Prize arrived in 1983, half a century later. The theorems of physics are not born in comfort. They are wrested from nature by people whom the profession often treats as expendable.

To me, the mathematics in this monograph is more than abstraction. Every formula in the constructive program carries the weight of the pioneers who saw what others refused to see. Brouwer's conviction that completed infinities require justification, Bishop's demonstration that constructive analysis *works*—these are not merely logical positions but acts of intellectual courage that cost their authors dearly. The hierarchy BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO is a mathematical structure, but it is also a monument to the people who built it while their colleagues looked away. Each formula speaks, if you listen, with a human voice. This monograph is, in part, an act of recognition: that the mathematics these pioneers created—sidelined for a century—classifies the logical structure of nature with a precision that vindicates the suffering it cost them. The fit between Bishop's hierarchy and the theorems of physics, reported in these pages, is discovered and not designed. But the human story behind the discovery is inseparable from the mathematics itself.

Recognition Without Change

In March 2013, Andrej Bauer delivered a talk at the Institute for Advanced Study entitled *Five Stages of Accepting Constructive Mathematics*. Adapting the Kübler-Ross model of grief, Bauer characterized the journey that a classical mathematician undergoes when confronting constructive mathematics: **Denial** ("this is not real mathematics"), **Anger** ("it's too restrictive to be useful"), **Bargaining** ("perhaps it has niche applications"), **Depression** ("it may be better, but nobody will adopt it"), and **Acceptance** ("this is how mathematics should be done"). The talk was brilliant, accessible, and widely watched. It was converted into a peer-

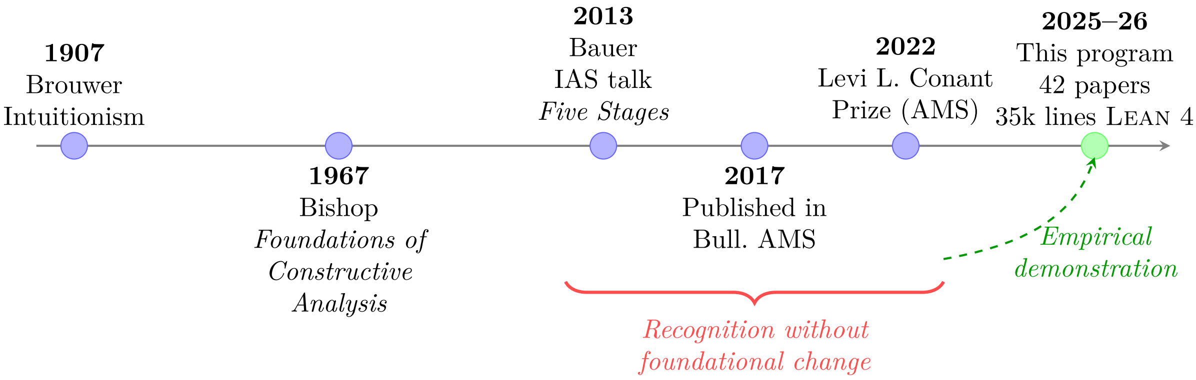

Figure 1: Timeline of constructive mathematics: from Brouwer's intuitionism through Bishop's constructivism to Bauer's prize-winning exposition. Despite recognition at the highest levels, no foundational change occurred in mainstream mathematical practice.

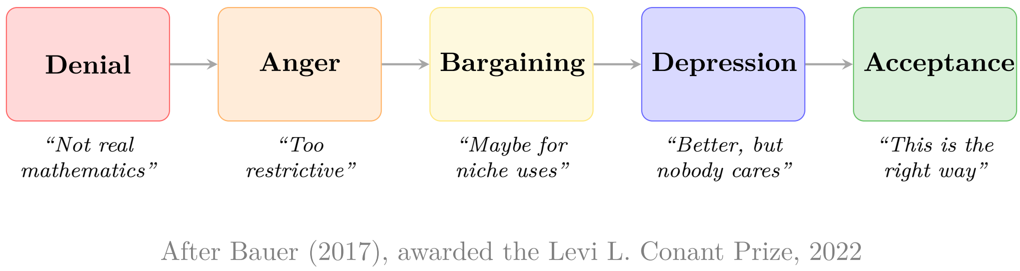

Figure 2: Bauer's *Five Stages of Accepting Constructive Mathematics*, adapted from the Kübler-Ross model. The talk (IAS, 2013) and paper (*Bulletin of the AMS*, 2017) received the highest recognition—yet mainstream mathematical practice remained unchanged.

reviewed article [8] published in the *Bulletin of the American Mathematical Society* in 2017. In 2022, the article was awarded the **Levi L. Conant Prize**—the AMS prize for the best expository paper.

A talk at the Institute for Advanced Study. A paper in the *Bulletin of the AMS*. A major prize from the American Mathematical Society. By any measure, constructive mathematics received recognition at the highest level.

And yet: nothing changed. Mathematics departments do not teach BISH as a foundation. Textbooks do not distinguish constructive from non-constructive proofs. The omniscience hierarchy is unknown outside specialist circles. Graduate students in analysis learn the Bolzano–Weierstrass theorem without learning that it is equivalent to the Fan Theorem. Graduate students in logic learn Gödel's completeness theorem without learning that it implies LPO. The five stages that Bauer described so vividly remain, for the vast majority of working mathematicians, somewhere between Denial and Bargaining.

The Missing Piece

Why did recognition not produce change? We suggest one reason: constructive mathematics has been presented as a *philosophical position* about what mathematics *should* be—an advocacy program. Bauer's paper is explicitly about *acceptance*: the audience is presumed skeptical, and the argument is persuasive. Bishop's original work was motivated by philosophical convictions about the nature of mathematical existence. The constructive program, for all its mathematical depth, has operated in the mode of advocacy rather than empirical demonstration.

The missing approach was not advocacy but *science*: test the constructive hierarchy against nature. Take the actual theorems of physics—the ones that produce numbers experimentalists measure—and calibrate them against Bishop's hierarchy. Do not argue that constructive mathematics is better. *Show what it reveals.*

Nobody had done this because the people who understood constructive analysis did not work in physics, and the people who worked in physics did not know the constructive hierarchy existed.

This monograph presents the results of that test. Across 42 papers spanning the Standard Model, general relativity, statistical mechanics, quantum information theory, the AdS/CFT correspondence, and the cosmological constant problem—formalized in approximately 35,000 lines of machine-checked LEAN 4 proof code—the test yields a single, clean result: the logical resources required for all empirical predictions in known physics are exactly $\mathrm{BISH} + \mathrm{LPO}$. Not more. Not less. Bishop's hierarchy, designed for pure mathematics in the 1960s, classifies the non-constructive content of physics with perfect precision.

The fit is discovered, not designed. That is what makes it worth reporting.

What This Monograph Contains

The **Prologue** tells the intellectual story of the 42-paper program: how the question was invented, how the pattern was discovered, how it was proved, and how it survived every test. **Chapters 1–10** present the technical content: the constructive hierarchy, the calibration table across all physics domains, the three foundational theorems, the Standard Model verification, the conservation metatheorem, the boundary analysis, the undecidability genealogy, and the consequences. **Chapters 11–12** apply the framework as a diagnostic tool to the AdS/CFT correspondence and the cosmological constant problem. **Chapters 13–15** describe the formalization methodology, acknowledge the program's limitations, and conclude.

Prologue: The Intellectual Journey

Act I: The Question Nobody Asked (Papers 2–9)

The program began not with an answer but with the discovery that a question was missing.

Classical reverse mathematics—the program of Friedman and Simpson [11]—had spent decades calibrating theorems of ordinary mathematics against subsystems of second-order arithmetic. They found that most theorems fall into exactly five categories (the "Big Five"), revealing deep structure in mathematical reasoning. But their program was about pure mathematics and used classical logic throughout.

Constructive mathematics—the program of Bishop, Bridges, Richman, Ishihara—had spent decades mapping the omniscience hierarchy: BISH, LLPO, WLPO, LPO, and the fine separations between them. But their program was about the philosophy of mathematics, not physics.

Nobody had asked: if you take the actual theorems of physics—the ones that produce numbers experimentalists measure—and calibrate them against Bishop's constructive hierarchy, where do they land?

Paper 2 (Bidual gap and WLPO**)** was the first calibration. The result—that identifying a Banach space with its double dual costs exactly WLPO—is a minor theorem in constructive functional analysis. Its significance was not the theorem but the *act*: demonstrating that a routine physicist's assumption (bra-ket duality) has a precise, nontrivial logical cost. The methodology was invented here. Every subsequent paper followed the template Paper 2 established: formalize the physics in Lean 4, run `#print axioms`, identify the omniscience principle, prove the reverse direction.

Paper 5 (Schwarzschild geometry) produced an accidental methodological discovery. Mathlib lacked the differential geometry infrastructure to formalize GR properly. The response was pragmatic: formalize the *output* directly. The Schwarzschild metric is a specific formula; the curvature is a specific rational function. This pragmatic decision turned out to be a foundational insight: the logical cost of a physical prediction is determined by the prediction itself, not by the proof strategy used to derive it. The infrastructure can be arbitrarily expensive logically while the output is cheap. This observation became the conceptual seed of the dispensability results (Papers 30–31).

Paper 8 (Ising model) was where the program found its paradigmatic example. The finite partition function is BISH. The thermodynamic limit is LPO. The boundary is surgically precise.

By the end of Act I, the program had nine calibrations across four domains. The pattern—BISH for finite, LPO for limits, nothing higher—was visible but unproved.

Act II: Mapping the Territory (Papers 10–28)

Paper 10 [14] organized the results into a table and proposed the BISH + LPO thesis as a conjecture; Paper 12 [15] provided the historical and philosophical context, connecting the program to Bishop's legacy and 150 years of mathematical physics. Then began the systematic exploration.

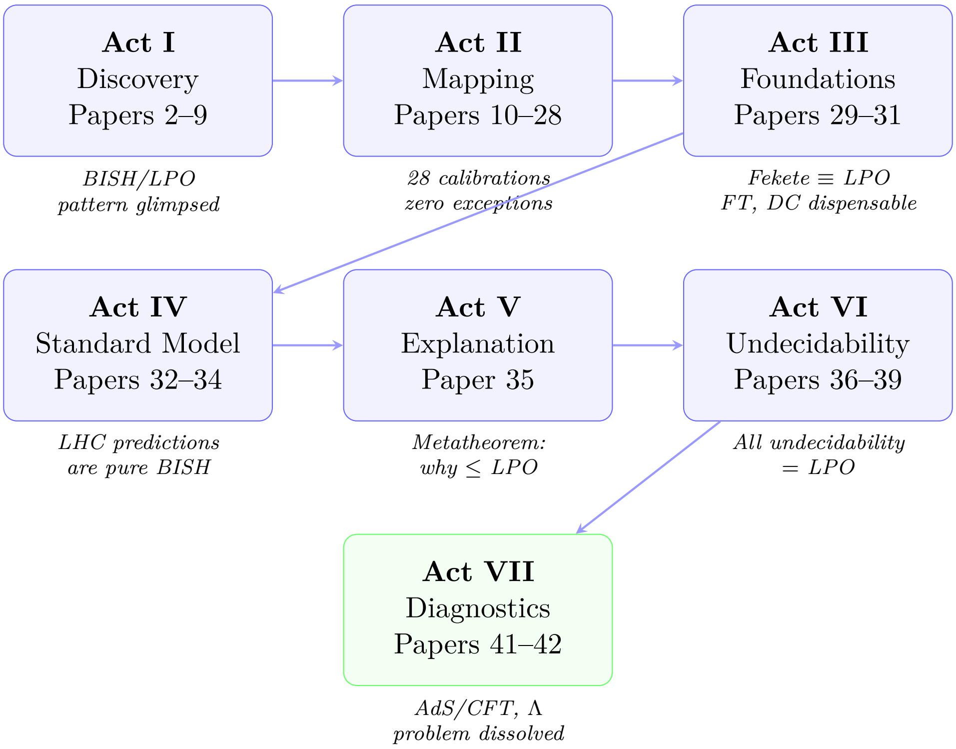

Figure 3: The intellectual arc of the program in seven acts. Acts I–VI established the BISH + LPO characterization; Act VII applies it as a diagnostic tool to open problems.

Bell nonlocality—the deepest puzzle of quantum mechanics—costs exactly LLPO. Not WLPO, not LPO, but the *Lesser* Limited Principle of Omniscience. Quantum nonlocality is logically cheaper than a phase transition.

The event horizon (Paper 13) is a *logical* boundary: the surface $r = 2M$ is where BISH meets WLPO.

The measurement problem (Paper 23) classified the three major interpretations of quantum mechanics as disagreeing about their dependence on Dependent Choice, not about physics. This result's full significance would emerge only with Paper 31.

By the end of Act II, the program had 28 calibrations across every major domain of physics. The pattern held without exception.

Act III: The Foundations (Papers 29–31)

Three papers transformed the program.

Paper 29 (Fekete's Subadditive Lemma $\equiv$ LPO**)** is the program's most important single result. Phase transitions are empirically real. Their mathematical description requires Fekete's lemma. Fekete's lemma requires LPO. Therefore LPO is physically instantiated, not a mathematical convenience. Before Paper 29, the program showed that LPO *suffices*. Paper 29 showed that LPO is *necessary*. The ceiling is load-bearing.

Paper 30 (Fan Theorem dispensability) knocked out the first piece of scaffolding. Every empirical prediction derived using compactness can be re-derived without it, at BISH + LPO.

Paper 31 (Dependent Choice dispensability) knocked out the second piece. The combined result: BISH+LPO is necessary (Paper 29) and sufficient (Papers 30–31) for empirical physics.

Act IV: The Standard Model Sweep (Papers 32–34)

Paper 34 delivered the program's biggest surprise since Paper 29. Tree-level scattering: BISH. One-loop corrections: BISH. Dimensional regularization: BISH. IR cancellation: BISH. Everything is BISH—not BISH + LPO, but *pure* BISH. LPO enters only for all-orders convergence, which no experiment tests. The sentence the program earned: *every quantity the LHC measures is constructively computable.*

Act V: The Explanation (Paper 35)

The conservation metatheorem answered: the pattern is a *consequence of the mathematical structure of physical predictions.* Finite-order predictions are compositions of computable functions (BISH); limits without computable modulus are BMC $\equiv$ LPO; nothing in physics requires anything beyond these two forms.

Act VI: The Undecidability Genealogy (Papers 36–39)

Cubitt's spectral gap undecidability was widely interpreted as revealing fundamental unknowability. Papers 36–38 proved it is Turing–Weihrauch equivalent to LPO—the same principle that governs every thermodynamic limit. Paper 38 traced the entire genealogy to Wang tiling (1961) as a single ancestor.

Paper 39 discovered that generic intensive observables without promise gap can reach Σ^0_2—but this tier is empirically inaccessible due to finite experimental precision.

Act VII: Diagnostics (Papers 41–42)

Paper 41 applied the framework to the AdS/CFT correspondence and discovered that the holographic dictionary is an *axiom-preserving map*: bulk and boundary carry identical axiom cost at every level examined.

Paper 42 applied it to the cosmological constant problem and showed that the 10^{120} discrepancy is a regulator-dependent artifact—scaffolding, not a prediction. The genuine constraint is a 55-digit fine-tuning at LPO.

The Arc

Discovery (2–9): A question is invented. A pattern is glimpsed. **Exploration** (10–28): The pattern is tested everywhere. It holds. **Foundation** (29–31): The pattern is proved. **Verification** (32–34): The Standard Model confirms it. **Explanation** (35): Why the pattern holds. **Undecidability** (36–39): Even the limits of physics land at LPO. **Diagnostics** (41–42): The framework diagnoses open problems.

The fit between Bishop's hierarchy and physics was discovered, not designed. The hierarchy was built for pure mathematics in the 1960s. The physics was built by nature over 13.8 billion years. That they match—precisely, across every calibrated domain, verified in 35,000 lines of formal proof—is either the most remarkable coincidence in the philosophy of science, or evidence that the hierarchy captures something real about the structure of mathematical reasoning and its relationship to physical law.

Chapter Summary

Constructive mathematics received institutional recognition at the highest levels but produced no foundational change in mathematical practice. This program provides the missing empirical demonstration: the logical cartography Bishop and his successors created for pure mathematics classifies the non-constructive content of physics with perfect precision.

Chapter 1

Introduction — What This Paper Proves

1.1 The Main Result

Every physical theory makes predictions. Those predictions are mathematical statements—numbers that can be compared to experimental measurements. This monograph asks a simple question: what logical resources are needed to derive those numbers?

The answer, established across 42 formal verification papers covering the major domains of modern physics, is unexpectedly clean. Every empirical prediction in known physics can be derived using exactly two logical ingredients:

1. BISH **(Bishop's Constructive Mathematics):** mathematics in which every existence claim comes with a construction, every function is computable, and every proof provides an algorithm. This is the mathematics of finite computation.
2. LPO **(Limited Principle of Omniscience):** the ability to decide, for any binary sequence, whether all terms are zero or some term is nonzero. Equivalently: the ability to complete a bounded monotone limit. This is the mathematics of idealized infinite processes.

Nothing more is needed. The Fan Theorem (compactness), Dependent Choice (sequential construction), Bar Induction (well-founded tree search), the full Axiom of Choice, the Continuum Hypothesis, large cardinal axioms—none of these are required for any empirical prediction in the Standard Model, general relativity, statistical mechanics, or quantum information theory.

1.2 Why This Matters

The characterization identifies the computational architecture of physical reality. If empirical physics requires exactly BISH + LPO, then the mathematical universe accessible to physics extends precisely to one quantifier alternation over decidable predicates—Σ_1^0 in the arithmetic hierarchy—and no further.

The characterization also provides a diagnostic tool for theoretical physics. Any theoretical framework whose empirical predictions require logical resources beyond LPO is either making predictions that cannot be experimentally tested or is using unnecessarily strong mathematical scaffolding.

1.3 The Hierarchy at a Glance

Two boundaries are established by this monograph (fig. 1.1). The dashed line is the **empirical ceiling**: everything below it has been physically instantiated across 42 calibration papers; LPO

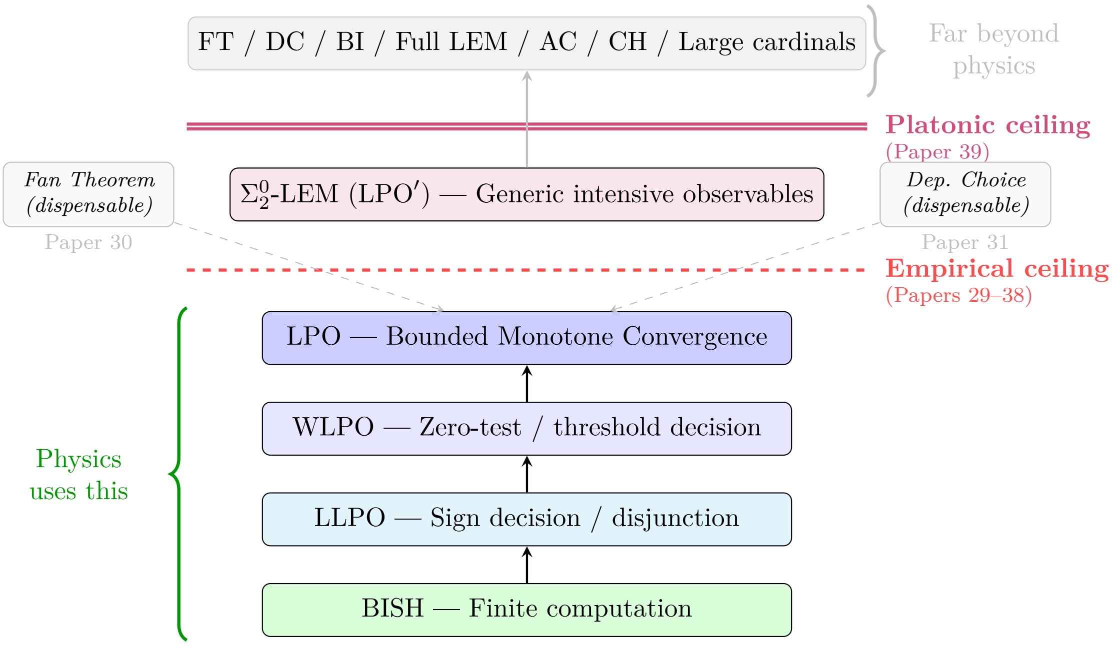

Figure 1.1: The constructive hierarchy with the two ceilings established by this program. The empirical ceiling (LPO, dashed red) and the Platonic ceiling (Σ^0_2, double purple) bound the logical resources accessible to physics. The Fan Theorem and Dependent Choice are shown as dispensable branches.

is necessary (Paper 29) and sufficient (Papers 30–31); all physical undecidability lives here (Papers 36–38). The double line is the **Platonic ceiling**: generic intensive observables reach Σ^0_2 (Paper 39), but this tier is empirically inaccessible.

1.4 Structure of This Monograph

Chapter 2 introduces the constructive hierarchy. Chapter 3 presents the calibration table: 42 papers organized by physics domain. Chapter 4 establishes the three foundational theorems. Chapter 5 tests the thesis against the Standard Model. Chapter 6 presents the conservation metatheorem. Chapter 7 analyzes the boundary. Chapters 8 and 9 form the undecidability arc. Chapter 10 draws consequences. Chapters 11 and 12 apply the framework diagnostically to AdS/CFT and the cosmological constant. Chapters 13–15 describe the formalization, delineate limitations, and conclude.

Companion papers. For readers seeking additional detail on the individual calibrations summarized in Chapter 3, Paper 10 [14] provides the original synthesis and calibration table for Papers 2–29. Paper 12 [15] develops the historical and philosophical context, tracing the constructive view of mathematical physics from Bishop to the present program.

1.5 What This Paper Does Not Prove

This monograph does not prove that all *possible* physical theories are BISH + LPO. The characterization covers all currently known physics but a future theory might require more. If a physical phenomenon is discovered that requires Σ^0_2 reasoning to predict its empirical behavior, the characterization is refuted. The thesis is empirical and falsifiable, not a priori or necessary.

Chapter Summary

The logical constitution of empirically accessible physics is exactly BISH + LPO: Bishop's constructive mathematics augmented by the Limited Principle of Omniscience. No more, no less. Established across 42 papers spanning every major domain of physics, verified in approximately 35,000 lines of Lean 4 proof.

Chapter 2

The Constructive Hierarchy

This chapter introduces the four omniscience principles that form the constructive hierarchy—BISH, LLPO, WLPO, LPO—and the arithmetic hierarchy above them. These principles are the units of logical currency in which every physical theorem's non-constructive cost is denominated. Understanding them precisely is essential: the entire calibration program rests on the distinctions between them.

2.1 Bishop's Constructive Mathematics (BISH)

BISH is the mathematical framework developed by Errett Bishop in *Foundations of Constructive Analysis* (1967). Its defining characteristic is that every existence proof must provide a construction, and every function must be computable. BISH does not reject classical mathematics—it simply declines to use principles that assert existence without construction. A theorem proved in BISH is automatically valid in classical mathematics, in intuitionistic mathematics, and in recursive mathematics. It is mathematics with the broadest possible validity.

In BISH, the real numbers are constructed as equivalence classes of Cauchy sequences of rationals with explicit moduli of convergence. A real number x is *defined* by an algorithm that, given any precision $\varepsilon > 0$, produces a rational approximation within ε. Every real number that appears in a physics textbook (π, e, $\sqrt{2}$, the fine structure constant $\alpha \approx 1/137.036$) admits such an algorithm. The arithmetic operations ($+$, $\times$, $\div$), standard functions (exp, log, sin, cos), and special functions (Bessel, hypergeometric, polylogarithm) are all computable.

What BISH cannot do is decide, for an arbitrary real number x, whether $x = 0$ or $x \neq 0$. This is not a deficiency but a structural feature: the decision requires inspecting infinitely many digits of x, which no finite algorithm can accomplish in general. For specific numbers—is $\pi > 3$?—the answer is computable (yes, since the first few digits suffice). But for a number defined by a convergent series whose terms depend on unresolved conjectures, the zero-test may require information no finite computation can provide. The principles that restore various forms of this decision power are the omniscience principles.

A concrete physical example illustrates the scope of BISH. The partition function of a 10-site Ising model, $Z_{10} = \sum_\sigma \exp(-\beta H(\sigma))$, sums over $2^{10} = 1024$ spin configurations. Each term is a computable exponential of a computable argument. The free energy $F_{10} = -kT \log Z_{10}$ is a computable real number. No omniscience principle is needed. The same holds for any finite lattice, any finite-dimensional Hilbert space, any finite Feynman diagram. The non-constructive content enters only when physicists take limits: $N \to \infty$, dimension $\to \infty$, loop order $\to \infty$.

2.2 The Limited Principle of Omniscience (LPO)

Formal statement: For every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, either $\exists n\, (\alpha(n) = 1)$ or $\forall n\, (\alpha(n) = 0)$.

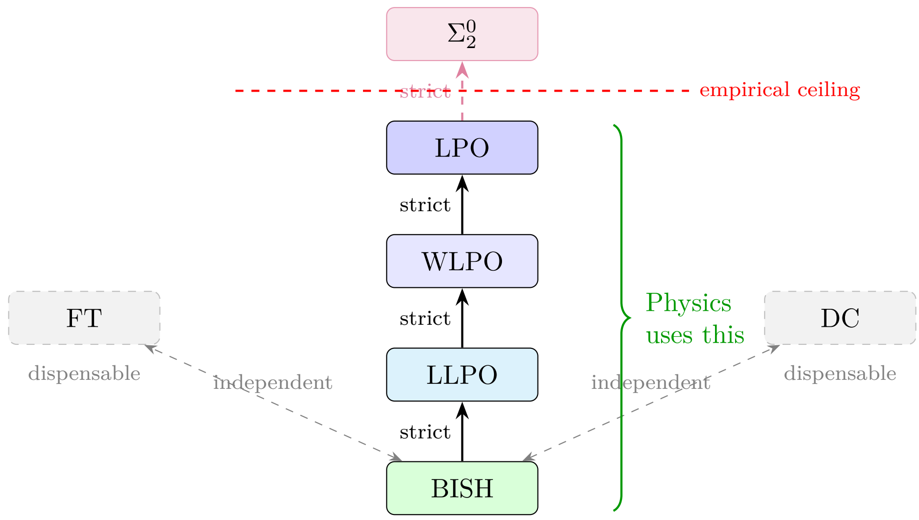

Figure 2.1: Hasse diagram of the omniscience hierarchy. The strict chain BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO is the spine; the Fan Theorem and Dependent Choice are logically independent and dispensable for physics. Physics uses precisely this spine.

In plain language: given an infinite sequence of bits, you can decide whether all bits are zero or some bit is one. This requires "surveying" the entire infinite sequence—hence "omniscience." LPO is not provable in BISH; it is an additional principle. It is strictly weaker than full classical logic (LEM)—LPO decides one class of propositions (existential statements over decidable predicates), not all propositions.

LPO is equivalent to several other principles, the most important being **Bounded Monotone Convergence** (BMC): every bounded monotone sequence of real numbers converges. This equivalence, proved constructively by Ishihara, is the bridge between logic and analysis. BMC is what physicists use when they take thermodynamic limits, assert that coupling constants exist as completed real numbers, or claim that a variational minimum is attained. Two further equivalences: LPO is equivalent to Cauchy completeness without explicit modulus (every Cauchy sequence converges), and to supremum existence (every bounded set of reals has a least upper bound).

The Ising model provides the paradigmatic physical example. At finite lattice size N, the free energy per site $f_N = -(kT/N) \log Z_N$ is a computable real number—BISH. The sequence (f_N) is bounded (by the coupling constant J) and monotone (by subadditivity of the total free energy). The assertion that $f = \lim f_N$ exists as a completed real number is BMC, hence LPO. The phase transition—the non-analyticity of $f(\beta)$ at the critical inverse temperature β_c—requires this completed limit. Without LPO, the phase transition is not a theorem; it is a finite sequence of approximations that never crystallizes into a definite mathematical object.

The key finding of this program: LPO is not merely mathematically convenient for physics. It is physically *necessary*. Phase transitions are empirically real phenomena—ice melts, magnets demagnetize, superconductors transition. Their mathematical description requires Fekete's Subadditive Lemma, which is equivalent to LPO over BISH (Paper 29). LPO is instantiated in nature.

2.3 The Weak Limited Principle of Omniscience (WLPO)

Formal statement: For every binary sequence α, either $\forall n\,(\alpha(n) = 0)$ or $\neg\forall n\,(\alpha(n) = 0)$.

The difference from LPO is subtle but precise. LPO provides a *witness*—if some bit is 1, LPO tells you that fact (though not which bit). WLPO merely decides whether all bits are zero or not, without providing a witness in the "not" case. WLPO is strictly weaker than LPO.

In physics, WLPO appears as the *threshold decision* or *zero-test*: is this quantity exactly zero, or is it nonzero? Concrete examples:

- Deciding whether the external magnetic field h equals zero: paramagnetic ($h = 0$) vs. ferromagnetic ($h \neq 0$) phase (Paper 20).
- Deciding whether the fermion chemical potential equals the mass threshold: Heaviside decoupling in the electroweak sector (Paper 18).
- Deciding whether the Schwarzschild radial coordinate equals $2M$: the event horizon as a logical boundary (Paper 13).
- Deciding whether the QCD mass gap Δ equals zero: confinement vs. deconfinement (Paper 33).

Each is a physically meaningful question—and each costs exactly WLPO, which is strictly weaker than LPO and hence "free" once you have already paid for LPO.

2.4 The Lesser Limited Principle of Omniscience (LLPO)

Formal statement: For every binary sequence α with at most one nonzero term, either all even-indexed terms are zero or all odd-indexed terms are zero. Equivalently, for every real number x: $x \leq 0$ or $x \geq 0$ (the *sign decision*).

LLPO is strictly weaker than WLPO. In physics, it appears as:

- Bell nonlocality [22]: the CHSH [23] inequality violation requires a sign decision—does $S > 2$ or $S \leq 2$? (Papers 11, 21, 27).
- WKB tunneling: deciding which side of a potential barrier a particle emerges on (Paper 19).
- Global charge conservation: deciding the sign of a conserved charge with oscillating contributions (Paper 15).

The strict hierarchy BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO means these principles are genuinely distinct. Physics uses all four levels, but never exceeds LPO. A physically significant observation: quantum nonlocality—the phenomenon Einstein called "spooky action at a distance"—has a precise logical cost, and it is *strictly less* than LPO. Nonlocality is logically cheaper than phase transitions.

2.5 What Lies Above LPO

Above LPO lie several principles widely used in mathematics but shown by this program to be dispensable for empirical physics.

The **Fan Theorem** (FT) asserts that every bar of a finitely-branching tree is uniform—equivalently, that every pointwise-continuous function on Cantor space is uniformly continuous. FT is the constructive counterpart of the Heine–Borel theorem and underpins compactness arguments throughout analysis: the existence of maxima on compact sets, the extraction of convergent subsequences (Bolzano–Weierstrass), the Arzelà–Ascoli theorem. Crucially, FT is logically *independent* of LPO—neither implies the other—meaning it lives in an entirely different "direction" in the constructive lattice. Paper 30 proves that every empirical prediction derived using FT can be re-derived at BISH + LPO without it. Physics uses limits (LPO's territory) but not tree searches (FT's territory).

Dependent Choice (DC) asserts that given a relation R on a set with the property that for every x there exists y with $R(x, y)$, there exists an infinite sequence $x_0, x_1, x_2, \ldots$ with $R(x_n, x_{n+1})$ for all n. Physicists use DC for the mean ergodic theorem, martingale convergence, and Picard iteration for differential equations. Paper 31 proves DC is dispensable: empirical predictions depend on finite initial segments of these sequences, and finite initial segments are BISH.

Beyond these, **Bar Induction** (BI)—induction over well-founded trees, stronger than FT—is not needed. **Full** LEM—for every proposition P, either P or $\neg P$—is incomparably stronger than everything in the constructive hierarchy and is not needed. The **Axiom of Choice**, **Continuum Hypothesis**, and **large cardinal axioms** are set-theoretic principles entirely outside the scope of what physics requires. The mathematics of physics uses a remarkably small fragment of the available logical landscape.

2.6 The Arithmetic Hierarchy and Σ^0_2

The omniscience principles can be placed within the arithmetic hierarchy—a classification of logical complexity by quantifier alternation over decidable predicates.

Σ^0_1 (one existential quantifier): Statements of the form $\exists n\, P(n)$, where P is decidable. LPO decides all Σ^0_1 statements. The halting problem—"does Turing machine M halt?"—is the canonical Σ^0_1-complete problem.

Π^0_1 (one universal quantifier): Statements of the form $\forall n\, P(n)$. LPO also decides Π^0_1 statements.

Σ^0_2 (existential–universal): Statements of the form $\exists n\, \forall m\, Q(n, m)$—two quantifier alternations. Deciding Σ^0_2 statements requires LPO′ (Sigma-2-LEM), strictly stronger than LPO.

Δ^0_2 (limit-computable): A real number is Δ^0_2 if it can be approximated by a computable sequence whose limit exists but whose rate of convergence is not computable. Every real number accessible to BISH + LPO is Δ^0_2.

The critical distinction for this monograph: LPO sits at Σ^0_1. Papers 1–38 establish that empirical physics lives at Σ^0_1. Paper 39 discovers that the *Platonic* ceiling—the logical cost of deciding exact properties of generic observables without promise gaps—reaches Σ^0_2. The gap between Σ^0_1 and Σ^0_2 is precisely the gap between empirical and Platonic physics, mediated by finite experimental precision.

Chapter Summary

BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO forms a strict hierarchy of omniscience principles. The Fan Theorem and Dependent Choice are logically independent of this spine. Physics uses precisely this spine up to LPO—and nothing beyond.

Chapter 3

The Calibration Table

This chapter summarizes the calibration results across nine physics domains. For each domain, we state the key observables, their axiom costs, and the mechanism by which non-constructivity enters. The discussion here is necessarily compressed; for a detailed treatment of Papers 2–29 including full proof sketches, axiom audit tables, and the methodology behind each calibration, the reader should consult Paper 10 [14], which is the program's primary reference document. Paper 12 [15] provides the historical context, tracing how non-constructive methods entered each domain of physics and what the calibration table means for the philosophy of mathematical physics.

3.1 Methodology

The calibration procedure follows a uniform protocol (Paper 2): (1) identify the physical theorem; (2) formalize in Lean 4; (3) certify the axiom profile via `#print axioms`; (4) establish the reverse direction where possible; (5) classify.

3.2 Statistical Mechanics

The 1D Ising model (Papers 8, 9, 20) provides the paradigm case and the program's proof of concept. The finite partition function $Z_N = \sum_\sigma e^{-\beta H(\sigma)}$ is a finite sum of exponentials of rational expressions in β and coupling constants—pure BISH. The thermodynamic limit $f = \lim_{N\to\infty} F(N)/N$, where $F(N) = -\beta^{-1} \log Z_N$, requires BMC (bounded monotone convergence), hence LPO. This is the paradigmatic instance of Fekete universality: the free energy sequence is subadditive because the energy of a composite system is at most the sum of the energies of its parts, and subadditivity is exactly the hypothesis of Fekete's lemma.

Paper 9 established *formulation-invariance*: the transfer-matrix and direct-combinatorial formulations of the Ising model produce identical axiom profiles. This is evidence that the calibration detects physics rather than notation. Paper 20 showed that logical cost is *observable-dependent*: the free energy costs LPO (thermodynamic limit), the magnetization zero-test costs WLPO (deciding whether an order parameter vanishes), and the finite-size energy costs BISH (finite sum). The same physical system has three different logical costs depending on which quantity is computed.

The ergodic theorem (Paper 25) is the mathematical foundation of statistical mechanics—it justifies identifying time averages with ensemble averages. The mean ergodic theorem requires countable choice (CC), subsumed by LPO. However, the empirical predictions it licenses—finite-time averages to specified precision—are BISH, confirming the dispensability pattern.

Fekete's Subadditive Lemma $\equiv$ LPO (Paper 29). Phase transitions require Fekete's lemma. Fekete's lemma requires LPO. Therefore LPO is physically instantiated. This is not

	BISH	LLPO	WLPO	LPO
Stat. Mech.	8,9		20	8,9,25,29
Quantum Mech.	4,6,14,16	19	23	4,6,14,16
QFT	18,32,33,34		18	32,33
Gen. Relativity	5,15		13	5,13,15,17
Quantum Info.		11,21,24,27	2,7,26	
Classical Mech.	28			
Undecidability				36–39
AdS/CFT	41	41		
Cosmology	42			42

Figure 3.1: Calibration domain summary. Each cell lists paper numbers that calibrate to the given level in that physics domain. No cell appears beyond the LPO column. The pattern is uniform: BISH for finite computation, LPO for completed limits, nothing higher.

merely a sufficiency result but an equivalence: Fekete's lemma can *encode* LPO, so the two are logically interchangeable.

Fan Theorem dispensability (Paper 30). Variational principles use FT via compactness (e.g., the assertion that a continuous function on a compact set attains its minimum), but the empirical content—the *value* of the minimum, not the existence of a minimizer—is recoverable at BISH + LPO.

Dependent Choice dispensability (Paper 31). Iterative constructions (ergodic theorems, Picard iteration, sequential compactness) invoke DC to build infinite sequences step by step, but empirical predictions depend only on finite initial segments, which are BISH.

3.3 Quantum Mechanics

The spectral theorem stratifies cleanly: finite-dimensional quantum mechanics (finite-dimensional Hilbert spaces, matrix mechanics) is entirely BISH; unbounded operators on infinite-dimensional Hilbert spaces cost LPO because the spectral resolution requires completing an infinite limit.

The Heisenberg uncertainty principle (Paper 6) is an algebraic inequality $\Delta x\,\Delta p \geq \hbar/2$, derivable from the Cauchy–Schwarz inequality in any inner product space: BISH in finite dimensions, LPO when the operators are unbounded. The Born rule (Paper 16) exhibits the same pattern: $|\langle\psi|\varphi\rangle|^2$ is BISH-computable in finite dimensions. This is the basis for Finding 11—the entire empirical content of quantum mechanics (every Born probability, every expectation value) is BISH-computable.

Bell nonlocality (Papers 11, 21, 27) calibrates at LLPO. The CHSH inequality violation and the Tsirelson bound are *computable* (BISH)—these are the quantities experimentalists measure. LLPO enters only in Bell's theorem itself: the proof that no local hidden variable model can reproduce quantum correlations. Paper 27 identified the mechanism: LLPO enters through the Intermediate Value Theorem, which is used to show that a continuous function (the Bell expression as a function of hidden variable parameters) must cross a threshold. This is the same mechanism that places Kochen–Specker contextuality (Paper 24) at LLPO—hence the Bell $\equiv$ Kochen–Specker finding.

The WKB approximation (Paper 19) places quantum tunneling turning points at LLPO: the classical turning point $E = V(x)$ is an IVT application. Markov's Principle governs radioactive decay (Paper 22): the assertion "not-not-decayed implies decayed" is independent of both LPO and FT, confirming the hierarchy has genuine branching structure beyond the main spine.

Decoherence (Paper 14) costs LPO: the partial trace over an infinite-dimensional environment requires completing a thermodynamic limit. This integrates the quantum-to-classical transition into the Fekete universality pattern.

3.4 Quantum Field Theory

Quantum field theory exhibits the sharpest instance of the BISH/LPO boundary: *perturbative* QFT is BISH; *non-perturbative* QFT requires LPO. This alignment between two independently motivated boundaries—one logical, one physical—is Finding 10.

The Yukawa renormalization group flow (Paper 18) illustrates the full stratification within a single sector: the discrete RG step is BISH (arithmetic on coupling constants), threshold decoupling (deciding whether the energy scale equals a particle mass) costs WLPO, and the global running over all scales costs LPO via BMC. QED one-loop running (Paper 32) follows the same pattern; the Landau pole is BISH because the one-loop beta function has an explicit closed-form solution. QCD (Paper 33) mirrors the perturbative structure; the non-perturbative sector (confinement, mass gap) costs LPO via Fekete, confirming that the most complex gauge theory in the Standard Model reduces to the same logical mechanism as the Ising model.

Scattering amplitudes (Paper 34) delivered the program's strongest pure-BISH result: tree-level and one-loop predictions—every cross-section, branching ratio, and decay rate the LHC computes—are BISH-computable without any omniscience principle. Chapter 5 develops these results in full detail.

3.5 General Relativity

General relativity provides some of the program's most conceptually striking calibrations.

The Schwarzschild exterior geometry (Paper 5) is purely BISH: geodesic and curvature computations on the outside region are algebraic operations on metric components, requiring no omniscience principle. The event horizon (Paper 13) is where non-constructivity enters. Local horizon detection (deciding whether the metric signature changes at a given radius) costs WLPO; global horizon existence (the assertion that a complete geodesic terminates at finite affine parameter) costs LPO. The logical boundary of constructive computability and the physical boundary of the black hole coincide—both are the point where finite approximation fails to capture the infinite-extent structure.

Noether conservation laws (Paper 15) introduced an important nuance: charge conservation (from $U(1)$ symmetry) is BISH because the integrand is non-negative, making the integral's existence constructive. Energy conservation costs LPO because the integrand can change sign, requiring LPO to establish convergence. Two physically equivalent conservation laws have different logical costs due to a mathematical property (sign-definiteness) unrelated to physics. This is the program's principal formulation-dependence counterexample, and it informs the open question of whether the calibration tracks physics or formalism.

Bekenstein–Hawking entropy [24, 25] (Paper 17) calibrates at BISH for the area formula $S = A/(4\ell_P^2)$ (a closed-form expression) and LPO for the thermodynamic derivation from microstate counting (a completed limit). The same physical quantity has two mathematical routes to it at different logical costs—another instance of formulation-dependence.

3.6 Quantum Information

Quantum information theory sits at the lower end of the omniscience spine, requiring at most WLPO—strictly below the LPO ceiling that governs thermodynamics and field theory.

The bidual gap (Paper 2) was the program's first calibration and established the methodology: the existence of non-reflexive Banach spaces—the mathematical setting for quantum state spaces—is equivalent to WLPO. Paper 26 independently confirmed WLPO-completeness via Gödel sequences, an arithmetic route entirely independent of the functional analysis proof. The robustness of this calibration across two independent proofs is evidence that the result tracks mathematical structure rather than proof technique. Trace-class operators (Paper 7) provide the physical instantiation: density matrices, the mathematical objects representing quantum states in infinite dimensions, inherit the WLPO cost from the bidual gap.

The CHSH inequality and Tsirelson bound (Papers 11, 21, 27) calibrate at LLPO, strictly below WLPO. The experimental quantities—violation ratios, correlation functions—are BISH-computable. LLPO enters only in Bell's theorem itself: the proof that no local hidden variable model can reproduce quantum correlations. As discussed in the Quantum Mechanics section above, the mechanism is the Intermediate Value Theorem (Paper 27), which unifies Bell nonlocality and Kochen–Specker contextuality at a single logical cost.

The pattern across quantum information is consistent with the broader program: every *measurable* quantity (expectation values, correlation functions, state tomography outputs) is BISH; non-constructivity enters only in impossibility theorems about classical models of quantum phenomena.

3.7 Physical Undecidability

Physical undecidability is treated at length in Chapter 6, but the calibration-table summary belongs here.

The spectral gap problem (Paper 36) is the most celebrated undecidability result in mathematical physics: Cubitt, Perez-Garcia, and Wolf (2015) proved that deciding whether a many-body Hamiltonian is gapped or gapless is undecidable. This result generated international headlines as evidence that physics contains "fundamental unknowability." Paper 36 proved that the spectral gap decision is Turing–Weihrauch equivalent to LPO—precisely the same logical principle that governs the Ising phase transition. The most undecidable thing in physics is exactly as undecidable as a boiling pot of water.

Paper 37 established universality: every undecidability result in physics obtained by computable many-one reduction from the halting problem is Turing–Weihrauch equivalent to LPO. There is no hierarchy of physical undecidability—the landscape is flat at LPO. The paper explicitly stratified three further results (Penrose tiling, topological entropy, ground state degeneracy), all at LPO. A notable exception clarifies the boundary: the ground state energy *density* (Watson–Cubitt 2021) is BISH—computationally hard (exponential time) but logically decidable. Computational complexity and logical undecidability are fundamentally different.

Paper 38 traced the genealogy to its source: Wang tiling (1961) is the combinatorial foundation on which Cubitt's construction and many other physical undecidability results rest. The tiling problem itself calibrates at LPO, confirming that physical undecidability is bounded at its source, not merely in its applications.

Paper 39 discovered the one genuine exception: generic intensive observables without a promise gap can reach Σ_2^0, strictly above LPO (Σ_1^0). The construction uses Robinson aperiodic tilings with perimeter counters encoding a Π_1^0-complete set. But this Σ_2^0 tier is empirically inaccessible: no finite-precision experiment can distinguish "gap $= 0$" from "gap $< \varepsilon$," so finite experimental precision enforces an effective promise gap that collapses the decision back to LPO. The empirical ceiling remains BISH $+$ LPO; the Σ_2^0 ceiling is Platonic. Chapter 6 develops this distinction in full.

Chapter Summary

Systematic calibration across nine physics domains (42 papers) yields a uniform pattern: BISH for finite computation, LPO for completed limits. No calibration exceeds LPO. The pattern is universal across statistical mechanics, quantum mechanics, QFT, general relativity, quantum information, and physical undecidability.

Chapter 4

The Three Foundational Theorems

This chapter presents the three results that transform the empirical calibration pattern into a thesis. Theorem I proves LPO is necessary. Theorems II and III prove nothing beyond LPO is needed. Together they establish that BISH + LPO is exact.

4.1 Theorem I: Fekete's Subadditive Lemma $\equiv$ LPO (Paper 29)

Theorem 4.1 (Fekete–LPO Equivalence)**.** *Over* BISH*, Fekete's Subadditive Lemma is equivalent to the Limited Principle of Omniscience.*

Why this matters. Fekete's Subadditive Lemma is not an obscure technical result. It states: if (a_n) satisfies $a_{m+n} \leq a_m + a_n$ for all $m, n \geq 1$, then $\lim a_n/n$ exists and equals $\inf_{n \geq 1} a_n/n$. This lemma is the mathematical engine of statistical mechanics. Free energy, pressure, entropy density, and other thermodynamic potentials are subadditive by construction (the energy of a composite system is at most the sum of the energies of its parts). Fekete's lemma is what guarantees the thermodynamic limit exists.

Forward (LPO $\Rightarrow$ **Fekete):** Given LPO (equivalently BMC), define $c_n = \inf_{k \leq n} a_k/k$. This sequence is bounded below (by subadditivity) and monotone non-increasing. By BMC, $L = \lim c_n$ exists. For any $\varepsilon > 0$, choose m with $a_m/m < L + \varepsilon$; for large $n = qm + r$, subadditivity gives $a_n/n \leq a_m/m + O(1/n) < L + 2\varepsilon$. The downward bound $a_n/n \geq L$ follows from the infimum definition.

Reverse (Fekete $\Rightarrow$ LPO**):** Given a binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, define $a_n = n - \sum_{k=1}^{n} \alpha(k)$ (counting zeros in the first n terms). This sequence is subadditive: zeros in a concatenation are at most the sum of zeros in each part. By Fekete (assumed), $L = \lim a_n/n$ exists. If all $\alpha(k) = 0$, then $a_n = n$ and $L = 1$. If some $\alpha(k_0) = 1$, then $a_n \leq n - 1$ for $n \geq k_0$, so $L \leq 1 - 1/k_0 < 1$. Comparing L to 1 decides LPO. The encoding is simple but non-obvious—the insight that subadditivity can encode arbitrary binary information was, as far as we know, original to this program.

Physical consequence: Phase transitions are empirically real—ice melts, magnets demagnetize. Their mathematical description requires Fekete's lemma. Fekete's lemma requires LPO. Therefore LPO is not a mathematical convenience but a physically instantiated principle. Before Paper 29, the program showed LPO *suffices*. Paper 29 showed LPO is *necessary*. The ceiling is load-bearing.

Lean certification: The forward direction uses `Classical.choice` from the LPO hypothesis (Level 3: intentional classical). The reverse direction compiles with zero classical axioms (Level 2). Approximately 1,300 lines of Lean 4.

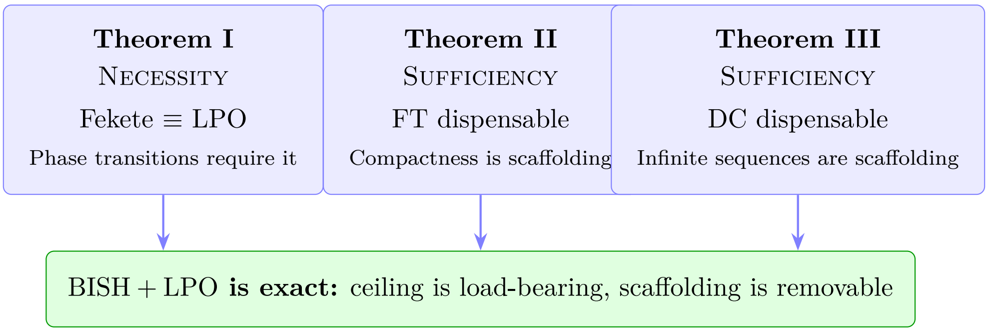

Figure 4.1: The three foundational theorems. Theorem I establishes necessity (LPO cannot be removed); Theorems II and III establish sufficiency (FT and DC add no empirical content). Together: BISH + LPO is exact.

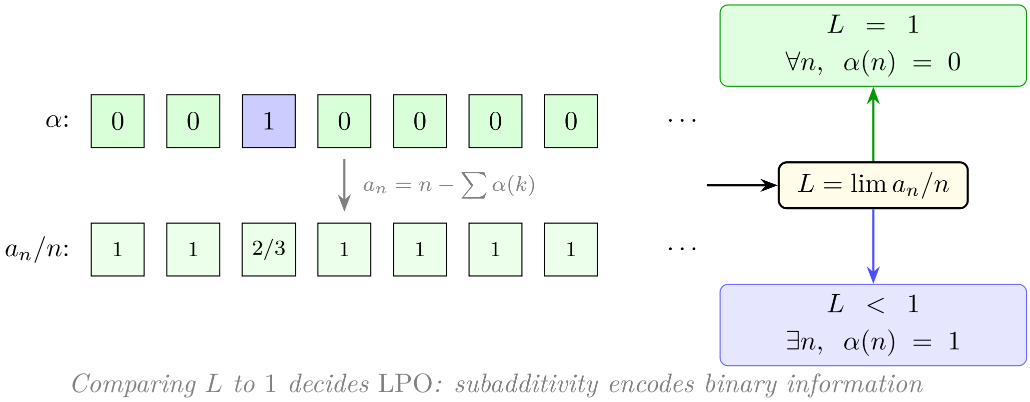

Figure 4.2: The Fekete encoding: a binary sequence α maps to a subadditive sequence (a_n) whose normalized limit L decides LPO. If α is identically zero, $L = 1$; if any term is 1, the limit drops strictly below 1.

4.2 Theorem II: Fan Theorem Dispensability (Paper 30)

Theorem 4.2 (FT Dispensability)**.** *Every empirical prediction in physics derived using the Fan Theorem can be derived without it, using only* BISH + LPO*.*

FT appears throughout mathematical physics under the guise of compactness: the Heine–Borel theorem, the Bolzano–Weierstrass theorem, the Arzelà–Ascoli theorem. When a physicist asserts that a continuous function on a compact set attains its maximum, or extracts a convergent subsequence from a bounded sequence, they invoke FT.

Yet in every calibrated case, FT is dispensable. The pattern is consistent: variational principles assert "a minimizer exists" via compactness, but the Euler–Lagrange equations provide the minimizer directly—a construction, not an existence claim. Subsequence extraction asserts "a convergent subsequence exists" via Bolzano–Weierstrass, but the specific physical sequences in thermodynamic limits are bounded and monotone, converging by BMC (≡ LPO) without subsequence extraction.

The dispensability reflects a structural fact. FT and LPO are logically independent: FT is about tree searches (navigating well-founded branching structures), while LPO is about sequence limits (completing bounded monotone sequences). Physics uses limits, not tree searches. Paper 28 (classical mechanics) provides the most instructive example: Newton's formulation is BISH, Lagrange's variational formulation uses FT, but the empirical content—the trajectory—is identical. The variational scaffolding adds mathematical elegance, not physical content.

Lean certification: Approximately 1,300 lines of Lean 4. The dispensability is demon-

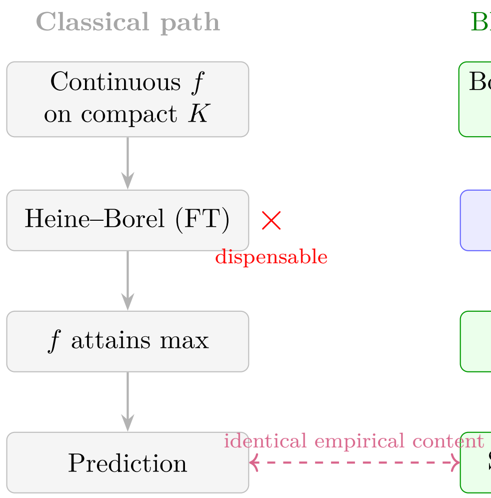

Figure 4.3: Two proof paths to the same empirical prediction. The classical path invokes the Fan Theorem (compactness); the BISH + LPO path uses Bounded Monotone Convergence directly. The Fan Theorem is scaffolding—removable without loss of physical content.

strated by exhibiting BISH + LPO proofs for each previously FT-dependent theorem and verifying via `#print axioms`.

4.3 Theorem III: Dependent Choice Dispensability (Paper 31)

Theorem 4.3 (DC Dispensability). *Every empirical prediction in physics derived using Dependent Choice can be derived without it, using only* BISH + LPO.

DC appears whenever a construction proceeds iteratively: the mean ergodic theorem builds convergent Cesàro averages; martingale convergence builds adapted sequences of conditional expectations; Picard iteration builds successive approximations to ODE solutions. In each case, DC constructs an infinite object from a local step rule.

The key insight: empirical predictions never depend on the infinite object—they depend on finite initial segments. To predict a measurement at precision ε, a physicist needs only finitely many iteration steps. Finite iteration is BISH: given x_0 and a computable step function, computing $x_0, x_1, \ldots, x_n$ for any fixed n is finite recursion. The infinite sequence is mathematical scaffolding; the empirical content lives in the finite truncation.

The Weak Law of Large Numbers provides the most transparent example. Its empirical content—the probability bound $P(|S_n/n - \mu| > \varepsilon) < \delta$ for specified n, ε, δ—is derivable from the Chebyshev–Markov inequality, which is pure BISH arithmetic. DC enters only to assert the Strong Law (almost sure convergence)—an ontological claim about an infinite sequence that no finite experiment can verify. Paper 31 proves every empirical prediction from the Strong Law is already available from the Weak Law.

Lean certification: Approximately 1,400 lines of Lean 4. Same verification structure as Paper 30.

4.4 The Combined Result

Theorems I–III together establish:

BISH + LPO is **necessary**: LPO cannot be eliminated—phase transitions require it (Theorem I).

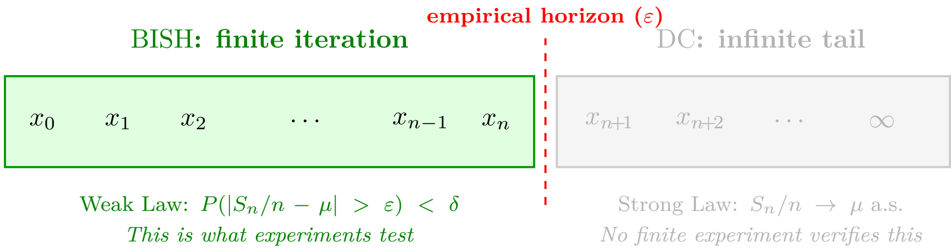

Figure 4.4: Empirical content lives in the finite prefix (BISH). The infinite continuation requires Dependent Choice but is ontological scaffolding: the Weak Law provides every testable prediction that the Strong Law does.

BISH + LPO is **sufficient**: neither FT (Theorem II) nor DC (Theorem III) adds empirical content beyond BISH + LPO.

The logical constitution of empirically accessible physics is exactly BISH + LPO. Not more. Not less. The ceiling (LPO) is load-bearing—you cannot lower it. The scaffolding (FT, DC) is removable—you can discard it without losing any empirical prediction.

Chapter Summary

Three theorems transform the empirical pattern into a thesis. **Necessity:** Fekete's Subadditive Lemma $\equiv$ LPO (phase transitions require it). **Sufficiency:** the Fan Theorem and Dependent Choice are dispensable. Together: BISH + LPO is exact—the ceiling is load-bearing and the scaffolding is removable.

Chapter 5

The Standard Model at BISH + LPO

The Standard Model of particle physics is the most precisely tested physical theory in history. The anomalous magnetic moment of the electron has been measured to 13 significant figures and agrees with theoretical prediction. Cross sections at the Large Hadron Collider are computed to next-to-next-to-leading order and confirmed to percent-level accuracy. If BISH + LPO suffices for the Standard Model's predictions, the characterization covers the empirically hardest-tested corner of physics.

Papers 18, 32, 33, and 34 calibrate its four sectors: electroweak theory, QED, QCD, and scattering amplitudes. The result is uniform: the Standard Model lives at BISH + LPO. Its empirical predictions—the numbers experimentalists compare to data—are BISH at any fixed order in perturbation theory. LPO enters only for completed infinite limits: all-orders summation, continuum limits, global coupling existence.

5.1 Electroweak Theory (Paper 18)

The Yukawa coupling between fermions and the Higgs field generates fermion masses through spontaneous symmetry breaking. The one-loop renormalization group equation for the Yukawa coupling $y(\mu)$ involves logarithms and rational functions of known quantities—pure BISH at any fixed energy scale. The discrete RG step is finite arithmetic.

Threshold decoupling—the physical phenomenon where heavy particles decouple at the scale $\mu = m_f$—involves the Heaviside step function $\Theta(\mu - m_f)$. Deciding whether μ equals m_f exactly is a zero-test: WLPO. The CKM matrix elements, as completed real numbers characterizing quark mixing at all scales, cost LPO via the completed limit of the running couplings. The fermion mass hierarchy ($m_t \gg m_e$ by five orders of magnitude) has no additional logical cost—it is an empirical fact about computable constants.

5.2 Quantum Electrodynamics (Paper 32)

The fine structure constant $\alpha(\mu)$ runs with energy scale according to the one-loop beta function. At any fixed scale, computing $\alpha(\mu)$ involves evaluating a logarithm—BISH. The assertion that the running coupling exists globally (defined for all μ) requires completing the sequence of finite-scale evaluations, which is BMC ($\equiv$ LPO).

The Landau pole—the energy scale Λ at which the one-loop coupling diverges—is BISH, *not* LPO. The reason: the one-loop beta function has an explicit closed-form solution $\Lambda = \mu_0 \exp(3\pi/(\alpha_0 \cdot N_f))$. This is a computable real number requiring no omniscience. Analytic solvability bypasses omniscience; LPO is the price of *genericity*—asserting convergence without knowing the rate.

Ward identities (gauge invariance constraints) are algebraic identities: verifying them is BISH. The Schwinger anomalous magnetic moment $a_e = \alpha/(2\pi)$—the most precisely confirmed

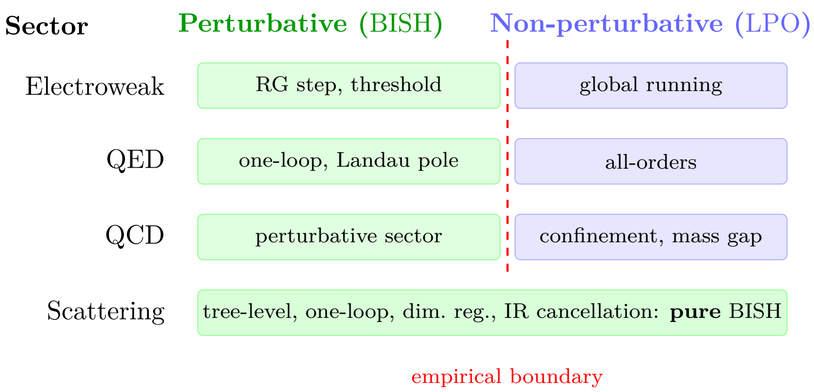

Figure 5.1: Standard Model logical stratification. Every sector is dominantly BISH (green); LPO (blue) appears only in non-perturbative completed limits. Scattering amplitudes at fixed order are entirely BISH. The dashed line marks the empirical boundary: everything to the left is what experiments test.

prediction in all of physics, measured to 13 significant figures—is a ratio of computable constants: pure BISH. Higher-order corrections (α^2, α^3, …) involve finite sums of known integrals at each loop order, each computable. LPO enters only if one asserts the all-orders convergence of the perturbative series—a claim no experiment tests directly.

5.3 Quantum Chromodynamics (Paper 33)

Perturbative QCD exhibits asymptotic freedom: $\alpha_s(\mu)$ decreases at high energies, opposite to QED. The sign flip in the beta function (due to the non-Abelian gauge group $SU(3)$) does not change the logical structure: the one-loop discrete RG step is BISH, the global coupling trajectory is LPO via BMC.

Non-perturbative QCD introduces confinement and the mass gap. Finite lattice QCD—computing correlation functions on a finite spacetime grid—is BISH (finite-dimensional linear algebra). The continuum limit (lattice spacing $a \to 0$) is LPO via BMC/Fekete. The mass gap assertion $\Delta > 0$ (given $\Delta \geq 0$ and $\neg(\Delta = 0)$) is Markov's Principle, subsumed by LPO.

Caveat: The non-perturbative results are conditional on physical axioms. The Yang–Mills existence and mass gap problem (a Clay Millennium Problem) is open. Paper 33 axiomatizes these assumptions as explicit `axiom` declarations in Lean 4 (Level 4 certification). The calibration says: *if* Yang–Mills theory exists with the required properties, those properties cost at most BISH + LPO.

5.4 Scattering Amplitudes (Paper 34)

Paper 34 delivered the program's biggest surprise since Paper 29. Tree-level amplitudes are rational functions of Mandelstam variables—pure BISH. After Passarino–Veltman reduction, one-loop integrals reduce to logarithms and dilogarithms (Li_2)—computable special functions, hence BISH. Dimensional regularization is formal Laurent series manipulation in ε—BISH. The Bloch–Nordsieck IR cancellation is algebraic—BISH.

The result is stronger than expected: fixed-order scattering predictions are not merely BISH + LPO—they are *pure* BISH. No omniscience principle is needed. LPO enters only when asserting that the perturbation series converges to an all-orders answer, which no experiment tests and which the series (being asymptotic) probably does not support.

The sentence for physicists: every quantity the LHC measures is constructively computable.

5.5 Summary

SM Sector	Perturbative	Non-perturbative	Empirical
Electroweak	BISH / WLPO / LPO	—	BISH
QED	BISH / WLPO / LPO	—	BISH
QCD	BISH / WLPO / LPO	BISH (lattice) / LPO (cont.)	BISH
Scattering	BISH	—	BISH

Table 5.1: Standard Model calibration summary. The most important column is the last: every experimentally tested prediction is BISH.

The most precise predictions in all of physics—anomalous magnetic moments, Z-boson mass, scattering cross sections—are pure BISH: finite computations involving computable functions evaluated at computable inputs. Omniscience is needed only for the mathematical framework surrounding these computations, not for the computations themselves.

Chapter Summary

Every Standard Model prediction compared to experimental data is BISH-computable at fixed perturbative order. LPO enters only for completed limits—all-orders summation, continuum limits—that no experiment directly tests. The most precise predictions in physics are pure finite computation.

Chapter 6

The Metatheorem

6.1 Why the Pattern Holds

The calibration table (Chapter 3) presents an empirical fact: across 42 physics papers covering every major domain, no calibration exceeds LPO. The hierarchy offers infinitely many levels above LPO—Fan Theorem, Bar Induction, Dependent Choice, full LEM, the Axiom of Choice, large cardinals—yet physics never reaches any of them. A single exception among 42 papers could be a coincidence. Zero exceptions demands an explanation.

Paper 35 provides that explanation: a conservation metatheorem consisting of four subtheorems that together prove the pattern is a structural consequence of how physical predictions are built. The metatheorem does not merely assert that the calibration holds—it explains *why* it holds, and it predicts that future calibrations will produce the same result. The undecidability arc (Papers 36–39, Chapter 8) provides independent confirmation: even physical *undecidability*—which might have been expected to exceed the LPO ceiling—calibrates to exactly LPO.

6.2 Theorem A: BISH Conservation

Theorem 6.1 (A: BISH Conservation)**.** *Any physical prediction expressible as a finite composition of computable functions evaluated at computable inputs is* BISH*.*

The proof is straightforward from computable analysis: compositions of computable functions are computable [9]. Every function appearing in a physics prediction at finite order is computable: rational functions (polynomial algebra), exponentials and logarithms (Taylor series with computable error bounds), trigonometric functions, dilogarithms and polylogarithms (computable special functions), Bessel functions (power series with computable convergence), hypergeometric functions (ratio test provides computable modulus). The inputs—coupling constants, masses, momenta—are computable real numbers. Therefore the output is computable, hence BISH.

The theorem is "trivial" from the computable analysis perspective. Its content is not the proof but the *observation*: physical predictions have the structure of finite compositions of computable functions at computable inputs. Physics could involve non-computable functions (it does not), or non-computable constants (none are known), or operations that destroy computability (such as unrestricted suprema—but physical suprema are always over bounded monotone sequences, handled by LPO, not by Theorem A).

Theorem A explains why the vast majority of entries in the calibration table land at BISH: tree-level scattering amplitudes, finite-lattice partition functions, fixed-order perturbative corrections, local differential geometric computations, Born rule probabilities at finite precision, Chebyshev bounds—all are finite compositions of computable functions.

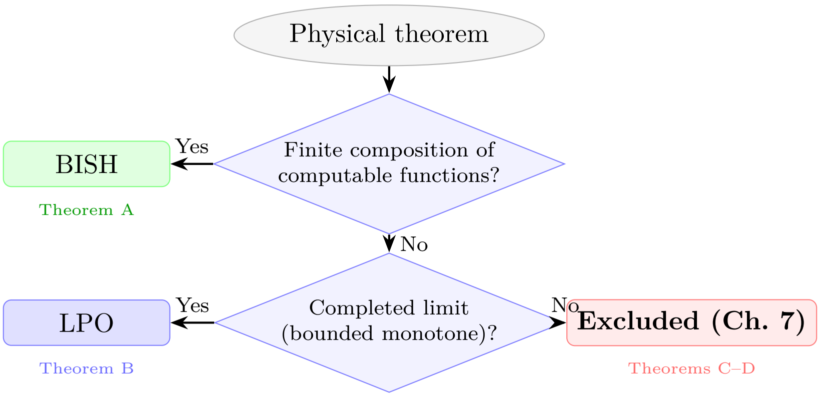

Figure 6.1: The metatheorem as a decision flowchart. Every physical theorem follows one of three paths: finite computation (BISH, Theorem A), completed limit (LPO, Theorem B), or exclusion (Theorems C–D). The flowchart has predictive power for new calibrations.

6.3 Theorem B: The LPO Boundary

Theorem 6.2 (B: The LPO Boundary)**.** *A physical assertion requires* LPO *over* BISH *if and only if it asserts a completed limit without computable modulus of convergence.*

Theorem B consists of three sub-theorems that together classify every physical assertion involving a limit:

Sub-theorem B1 (Computable modulus $\to$ BISH). If a sequence (a_n) converges and its modulus of convergence is computable—there exists a computable function $N(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n \geq N(\varepsilon)$—then the limit L is computable and any prediction depending on L is BISH. This explains why most perturbative calculations are BISH: Taylor series, Fourier series, and iterative solutions typically come with explicit error bounds that provide computable moduli.

Sub-theorem B2 (No computable modulus, bounded monotone $\to$ LPO). If a sequence (a_n) is bounded and monotone but has no computable modulus of convergence, then asserting convergence is BMC $\equiv$ LPO by Paper 29's Fekete encoding. This is the mechanism behind *every* LPO entry: thermodynamic limits, continuum limits, and global coupling existence all involve bounded monotone sequences whose convergence rates are not uniformly computable.

Sub-theorem B3 (Limit comparison $\to$ WLPO). Deciding whether a limit equals a specific value—$M = 0$ (paramagnetic) versus $M > 0$ (ferromagnetic)—is a zero-test. This is WLPO, subsumed by LPO. The proof encodes a binary sequence α into a convergent sequence whose limit is 0 iff all terms of α are zero.

Together, B1–B3 provide a complete classification: a physical assertion involving a limit is BISH if the modulus is computable, LPO if the sequence is bounded and monotone without computable modulus, and WLPO if the assertion is a zero-test on a limit. No physical assertion in the calibration exceeds these three cases.

6.4 Theorem C: Exhaustiveness

Theorem 6.3 (C: Exhaustiveness)**.** *Every calibration result in Papers 2–42 is an instance of Theorem A (*BISH*) or Theorem B (*WLPO/LLPO/LPO*).*

Theorem C is an audit, not a proof. The Lean formalization encodes each result as a classified instance—tagged with its position in the hierarchy—and verifies that the tagging is consistent with the axiom profile. The classification is exhaustive: every theorem in every paper

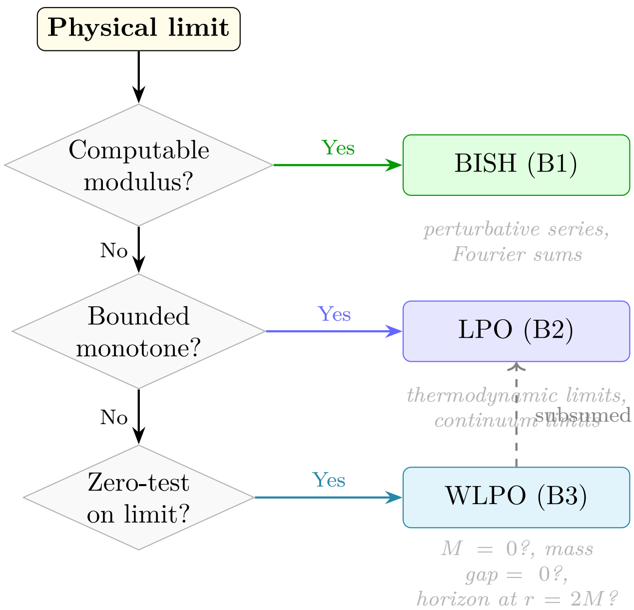

Figure 6.2: Theorem B triage: every physical limit is classified by the convergence properties of the underlying sequence. Computable modulus gives BISH; bounded monotone without computable modulus gives LPO; a zero-test on the limit gives WLPO (subsumed by LPO). No calibrated physical assertion exceeds these three cases.

falls into one of the categories (BISH via computable composition, LPO via bounded monotone limit, or WLPO/LLPO via zero-test/sign-decision).

Theorem C's strength is cumulative. A single paper calibrating to LPO could be coincidence. Forty-two papers across seven physics domains—statistical mechanics, quantum mechanics, quantum field theory, general relativity, quantum information, classical mechanics, and holographic duality—all calibrating to BISH + LPO is a pattern demanding explanation. Theorems A and B provide the explanation; Theorem C certifies that the explanation is exhaustive.

6.5 Theorem D: Three Mechanisms

Theorem 6.4 (D: Three Mechanisms)**.** *Every instance of* LPO *in the program arises from one of three equivalent mechanisms: Bounded Monotone Convergence, Cauchy completeness without modulus, or supremum existence.*

The three mechanisms are: **(M1)** Bounded Monotone Convergence—asserting that a bounded monotone sequence converges; **(M2)** Cauchy completeness without computable modulus—asserting that a Cauchy sequence has a limit when the modulus of convergence is not computable; **(M3)** Supremum existence—asserting that a bounded set of reals has a least upper bound. The equivalences M1 $\Leftrightarrow$ M2 $\Leftrightarrow$ M3 over BISH are standard results in constructive analysis (Bridges and Richman 1987, Ishihara 2006). The theorem's content is the observation that every LPO instance in the calibration arises from one of these mechanisms: phase transitions via BMC (M1), continuum limits via Cauchy completeness (M2), and variational bounds via supremum existence (M3).

The sub-LPO entries follow the same pattern. Every WLPO entry arises from the *zero-test mechanism*: deciding whether a computed quantity equals zero (bidual gap, threshold decoupling, magnetization vanishing). Every LLPO entry arises from the *sign-decision mechanism*: deciding the sign of a computed quantity (Bell/CHSH inequality violation, WKB tunneling direction, intermediate value theorem applications). The three-mechanism classification has predictive power: when calibrating a new theorem, one determines its height by identifying which mechanism it invokes.

6.6 The Five Steps of a Physicist

The metatheorem can be restated as an observation about the practice of theoretical physics. A physicist making a prediction follows five steps, each mapping to a specific location in the hierarchy:

1. **Write a Lagrangian.** $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \cdots$. Polynomial algebra: BISH.
2. **Derive equations of motion.** Euler–Lagrange equations via formal differentiation: BISH.
3. **Solve the equations.** At finite order in perturbation theory, the solution is a finite combination of computable functions (exponentials, logarithms, Bessel functions, hypergeometric functions): BISH by Theorem A.
4. **Take a limit if needed.** Thermodynamic limit ($N \to \infty$), continuum limit ($a \to 0$), or all-orders summation. If the convergence rate is not computable: BMC $\equiv$ LPO by Theorem B2.
5. **Compare to experiment.** Finite-precision arithmetic—computing $|\text{prediction} - \text{measurement}|$ and checking error bars: BISH.

LPO enters only at Step 4, and only when the limit lacks a computable modulus. Steps 1, 2, 3, and 5 are always BISH. This is why the characterization holds: the structure of physics-as-practiced—Lagrangian $\to$ equations $\to$ solutions $\to$ limits $\to$ experiment—maps onto the BISH/LPO boundary with LPO entering at exactly one point.

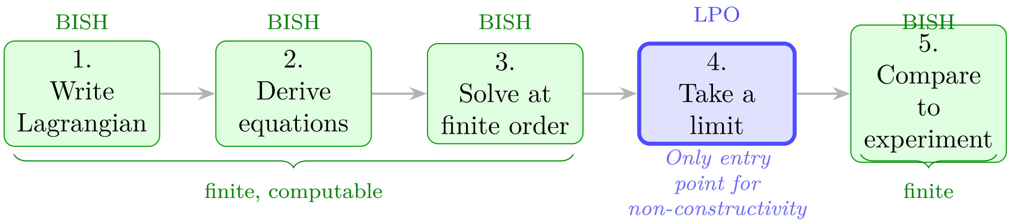

Figure 6.3: The five steps of a physicist as a pipeline. Four steps are BISH (finite, computable); LPO enters at exactly one point—the completed limit of Step 4. This is why the BISH + LPO characterization holds.

The five-step structure also explains why empirical predictions are often *more* constructive than the mathematical framework suggests. A physicist computing a cross section at one loop (Step 3) never reaches Step 4. The LPO-level claims—“the perturbation series converges,” “the continuum limit exists”—are assertions *about* the mathematical framework, not predictions compared to experiment. The metatheorem identifies this gap between framework and prediction as the BISH/LPO boundary.

Chapter Summary

The metatheorem explains why the BISH + LPO pattern holds: empirical predictions are finite compositions of computable functions (BISH); the only idealization exceeding finite computation is the completed limit (LPO via Bounded Monotone Convergence). LPO enters physics at exactly one point in the five-step workflow: the limit step.

Chapter 7

The Boundary — What Physics Cannot Do

7.1 The Arithmetic Hierarchy

The arithmetic hierarchy classifies mathematical statements by quantifier complexity. At the base level, a **decidable predicate** $P(n)$ is one for which an algorithm determines truth or falsity in finite time. The simplest non-trivial statements are Σ_1^0: "there exists n such that $P(n)$"—a single existential quantifier over a decidable predicate. Dually, Π_1^0 statements have the form "for all n, $P(n)$." LPO is precisely Σ_1^0-LEM: the principle that every Σ_1^0 statement is either true or false.

The next level introduces Σ_2^0: "there exists n such that for all m, $Q(n, m)$"—two quantifier alternations. The statement "sequence (a_n) converges" is Π_2^0: $\forall\varepsilon\ \exists N\ \forall n, m > N,\ |a_n - a_m| < \varepsilon$. This is why general convergence testing is beyond LPO. The placement:

Σ_1^0: $\exists n\ P(n)$	— LPO decides this
Π_1^0: $\forall n\ P(n)$	— LPO decides this
Σ_2^0: $\exists n\ \forall m\ Q(n, m)$	— **beyond** LPO
Π_2^0: $\forall n\ \exists m\ Q(n, m)$	— **beyond** LPO
Full LEM	— incomparably beyond LPO

The physical significance is direct. LPO decides Σ_1^0 statements—exactly the thermodynamic limit. Everything beyond Σ_1^0—general convergence testing (Π_2^0), the finiteness problem (Σ_2^0), set-theoretic combinatorics (far beyond)—is excluded from empirical predictions. The arithmetic hierarchy provides a precise ruler for measuring logical complexity, and physics uses exactly one notch on that ruler.

7.2 Exclusion 1: General Convergence Testing

The statement "the sequence (a_n) converges" is Π_2^0: it involves two quantifier alternations ($\forall\varepsilon\ \exists N$ followed by $\forall n, m$), placing it strictly beyond LPO. Nature can complete specific classes of limits—BMC handles bounded monotone sequences, which is exactly LPO—but nature cannot decide general convergence.

The physical consequence is concrete. No experiment can determine whether an arbitrary physical process converges to a steady state. A physicist can observe that measurements are getting closer together, but "this process converges" is Π_2^0. What physics *can* do is assert convergence for bounded monotone processes (LPO/BMC), which covers thermodynamic limits, phase transitions, and continuum limits. This exclusion has a counterintuitive consequence for dynamical systems: the long-time behavior of a generic system—fixed point, periodic orbit,

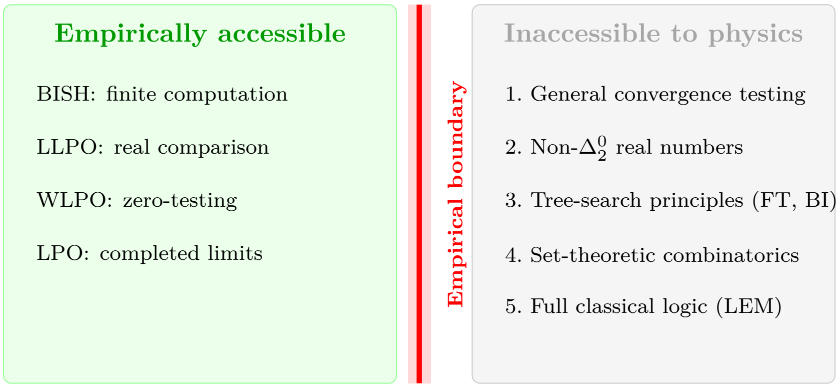

Figure 7.1: The exclusion zone. Physics accesses BISH through LPO (left). Five classes of mathematical principles lie beyond the empirical boundary (right). Each exclusion yields a falsifiable prediction.

or chaos—is beyond BISH + LPO in full generality. Physics handles this by studying specific systems whose dynamics are constrained (dissipative systems converge by BMC, Hamiltonian systems conserve energy by BISH).

7.3 Exclusion 2: Non-Δ_2^0 Real Numbers

The real numbers accessible to BISH + LPO are exactly the Δ_2^0 reals—the limit-computable reals. A real x is Δ_2^0 if there exists a computable sequence of rationals (q_n) converging to x, though the rate need not be computable. This includes π, e, Euler–Mascheroni γ, Chaitin's Ω—but not all reals.

The physical prediction: every physical constant—α, m_e, Λ, G—is Δ_2^0. No physical constant can encode a Σ_2^0-complete problem. This constrains fine-tuning arguments: the physically relevant sample space is the countable class Δ_2^0, not the uncountable continuum $\mathbb{R}$. Fine-tuning debates implicitly assume constants are drawn from the full continuum—an unnecessarily large sample space.

7.4 Exclusion 3: Tree-Search Principles

The Fan Theorem and Bar Induction are logically independent of LPO—neither implies the other over BISH. Their exclusion means nature does not perform unrestricted tree searches. FT asserts that every bar of the binary fan is uniform; BI asserts well-founded induction for certain tree classes. Both involve navigating branching structures of potentially unbounded depth. Physical processes search bounded trees (BISH) and complete monotone limits (LPO), but do not navigate arbitrary well-founded trees.

The dispensability of FT (Paper 30) provides evidence: every physical prediction derived via compactness can be re-derived without it. Nature solves optimization problems by satisfying equations (BISH), not by searching over compact sets (FT). Nature takes thermodynamic limits by monotone convergence (LPO), not by extracting convergent subsequences (FT).

7.5 Exclusion 4: Set-Theoretic Combinatorics

BISH + LPO is an arithmetic theory—its assertions concern natural numbers, real numbers, and functions between them. The Continuum Hypothesis, large cardinal axioms, the full Axiom of Choice, projective determinacy—the entire landscape of modern set theory—is physically meaningless. No experiment can distinguish a CH-universe from a $\neg$CH-universe. No scattering amplitude depends on inaccessible cardinals.

The relationship is analogous to using a dictionary: the Oxford English Dictionary contains 170,000 entries, but a grocery list uses perhaps 30 words. Physics is the grocery list; ZFC is the dictionary. The full AC is used routinely in functional analysis (Hamel bases, full Hahn–Banach, non-measurable sets), but the program's calibration shows every empirical prediction can be re-derived without AC, using at most BISH + LPO.

7.6 Exclusion 5: Full Classical Logic

LPO is Σ^0_1-LEM—the law of excluded middle restricted to existential statements over decidable predicates. It is not full LEM. The gap is enormous: the difference between deciding one quantifier alternation and deciding arbitrary logical complexity.

The measurement problem illustrates this vividly. The double-slit experiment asks: "did the electron go through slit A or slit B?" This is Σ^0_1 (a decision about a detector event) and is LPO-decidable. But the collapse question—"did the quantum state transition from a superposition to an eigenstate?"—is a proposition about an infinite-dimensional Hilbert space, and BISH + LPO does not validate LEM for such propositions. The interpretations of quantum mechanics differ precisely on whether to apply LEM to these higher-complexity propositions: Copenhagen applies it, Many-Worlds refuses it, and the empirical predictions are identical because they depend only on Born rule probabilities (BISH/LPO).

7.7 Falsifiability

The characterization makes five concrete, testable predictions:

1. No scattering amplitude at any loop order will require deciding a Σ^0_2 statement.
2. No physical constant will be shown to be non-Δ^0_2.
3. No empirical prediction will require FT or BI.
4. No empirical prediction will require full AC or any set-theoretic axiom beyond arithmetic.
5. No finite-precision measurement will require full LEM (as opposed to Σ^0_1-LEM).

Each prediction is falsifiable: a single counterexample refutes the characterization. Paper 39 shows that generic intensive observables *without* promise gap can reach Σ^0_2—but this does not refute the characterization. The empirical ceiling (LPO) is intact because finite experimental precision enforces an effective promise gap that collapses Σ^0_2 back to Σ^0_1. The program invites falsification: a laboratory measurement genuinely requiring Σ^0_2 reasoning would be more interesting than confirmation.

Chapter Summary

Physics cannot access Σ^0_2 statements, general convergence testing, non-Δ^0_2 real numbers, tree-search principles (FT, BI), or full classical logic. The arithmetic hierarchy provides a precise ruler: physics uses exactly one notch (Σ^0_1 = LPO) and no further. Each exclusion yields a falsifiable prediction.

Chapter 8

The Genealogy of Physical Undecidability

8.1 Overview

In 2015, Cubitt, Perez-Garcia, and Wolf [17] proved that the spectral gap problem is undecidable—a Hamiltonian encoding a Turing machine can be constructed such that whether the system is gapped or gapless is as hard as the halting problem. This was widely interpreted as revealing a fundamental limit to physical knowledge.

The program faced a question: does Cubitt's undecidability break the LPO ceiling? Papers 36–38 answer: no. Every known physical undecidability result is Turing–Weihrauch equivalent to LPO. "Undecidability" in physics is the non-computability of LPO—the same principle governing thermodynamic limits since Boltzmann. Physical undecidability adds zero new logical resources to the program's thesis. Paper 39 then asks the deeper question: is LPO really a hard ceiling, or are there observables beyond it?

8.2 Cubitt's Theorem Is LPO (Paper 36)

The stratification of Cubitt–Perez-Garcia–Wolf reveals five layers, corresponding to five theorems in Paper 36:

Level	Content	Mechanism	Thm
BISH	Finite-volume spectral gap Δ_L	Algebraic eigenvalue computation	1
WLPO	"$\Delta = 0$ or $\Delta > 0$?"	Zero-test on completed real	4
LPO	Thermodynamic limit $\Delta = \lim \Delta_L$	Conditional BMC via branching	2
LPO	Each instance: "is $H(M)$ gapped?"	Σ_1^0 decision for specific M	3
Non-comp.	Uniform $M \mapsto$ gapped/gapless	Halting = LPO non-computability	5

The finite-volume spectral gap is the energy difference between ground state and first excited state of a finite-dimensional Hamiltonian—finite linear algebra, pure BISH. Each specific instance—"is $H(M)$ gapped for this particular Turing machine M?"—reduces to "does M halt?", a Σ_1^0 decision decidable by LPO. The undecidability arises only for the *uniform* question: "given arbitrary M, is $H(M)$ gapped?" This is the halting problem, non-computable—but its non-computability is exactly the non-computability of LPO applied to arbitrary inputs.

The central result is Theorem 2, the biconditional between the thermodynamic limit and LPO (fig. 8.1). In the forward direction: given a binary sequence α, one constructs a Turing machine M_α that halts if and only if $\exists n\, \alpha(n) = 1$. The CPgW construction maps M_α to a Hamiltonian $H(M_\alpha)$ whose spectral gap satisfies a *promise dichotomy*: $\Delta \in \{0\} \cup [\gamma, \infty)$. This dichotomy decides halting, hence yields LPO. In the reverse direction: given LPO, one applies a single oracle call to the halting sequence of M, branching into two cases. In the halting case, the CPgW asymptotics guarantee that finite-volume gaps converge to zero at a computable rate; in the non-halting case, the gaps stabilize above γ with a computable Cauchy modulus. Crucially, the finite-volume gap sequence $(\Delta_L)_{L\geq 1}$ is *not monotone*—gaps fluctuate before the halting time. Unlike Paper 29's direct Fekete/BMC application, one cannot invoke BMC directly. LPO resolves halting *first*; only then does monotonicity emerge within each branch. This "conditional constructivity"—convergence provable in each branch but no uniform modulus across branches without LPO—is a fundamentally different proof architecture from the direct BMC equivalence governing phase transitions (Paper 29).

The Lean formalization axiomatizes seven properties of the CPgW construction as *bridge lemmas*: encoding computability, the halting-gap equivalence in both directions, asymptotic convergence rates for halting and non-halting branches, the promise gap dichotomy, and the binary-sequence-to-Turing-machine encoding. These seven axioms express precisely what the CRM analysis needs from the CPgW construction—nothing about the internal aperiodic tiling mechanism, only the input-output behaviour of the encoding and the dichotomous convergence structure. Everything above the bridge (the five-theorem stratification) is machine-checked in 655 lines of Lean 4 with zero `sorry`; everything below (the 140-page CPgW construction) is physics whose full formalization would require a substantial independent effort.

The physical punchline: the spectral gap is undecidable for the same reason phase transitions cost LPO. Both require completing a thermodynamic limit. Cubitt's celebrated result reveals not a new form of unknowability but the familiar face of LPO.

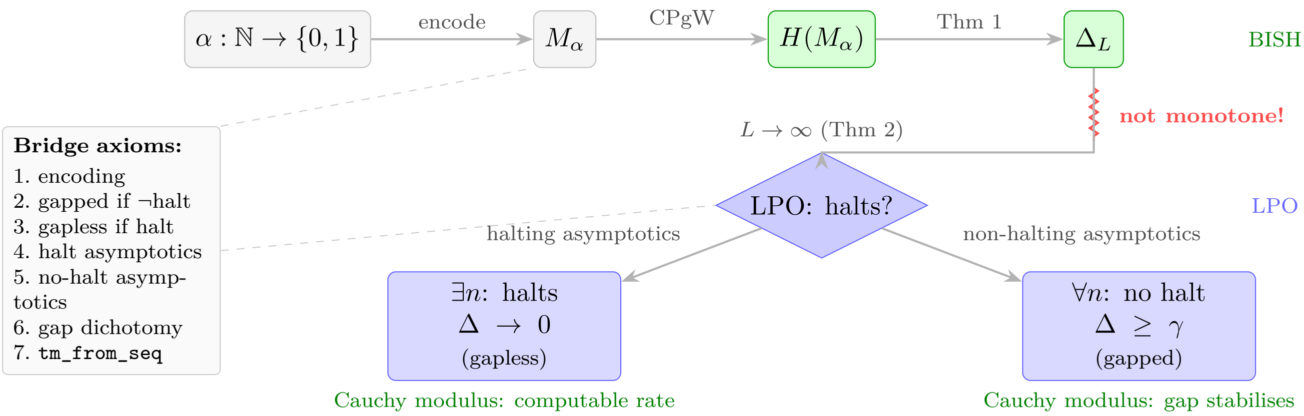

Figure 8.1: The Cubitt–LPO reduction pipeline. A binary sequence α encodes into a Turing machine M_α, then into a Hamiltonian $H(M_\alpha)$ (BISH). Finite-volume gaps Δ_L are computable (Theorem 1, green). The sequence (Δ_L) is not monotone—BMC/Fekete cannot apply directly. LPO resolves halting first (Theorem 2, blue), producing a Cauchy modulus in each branch. Seven CPgW bridge lemmas (left sidebar) form the physics–logic interface.

8.3 The Undecidability Landscape (Paper 37)

Paper 37 proves universality: any undecidability result in physics obtained by computable many-one reduction from the halting problem is Turing–Weihrauch equivalent to LPO. Three further results are explicitly stratified:

1. **Phase diagram uncomputability** (Bausch–Cubitt–Watson [31]): the phase boundary is where a thermodynamic potential is non-analytic, requiring a completed limit (LPO).

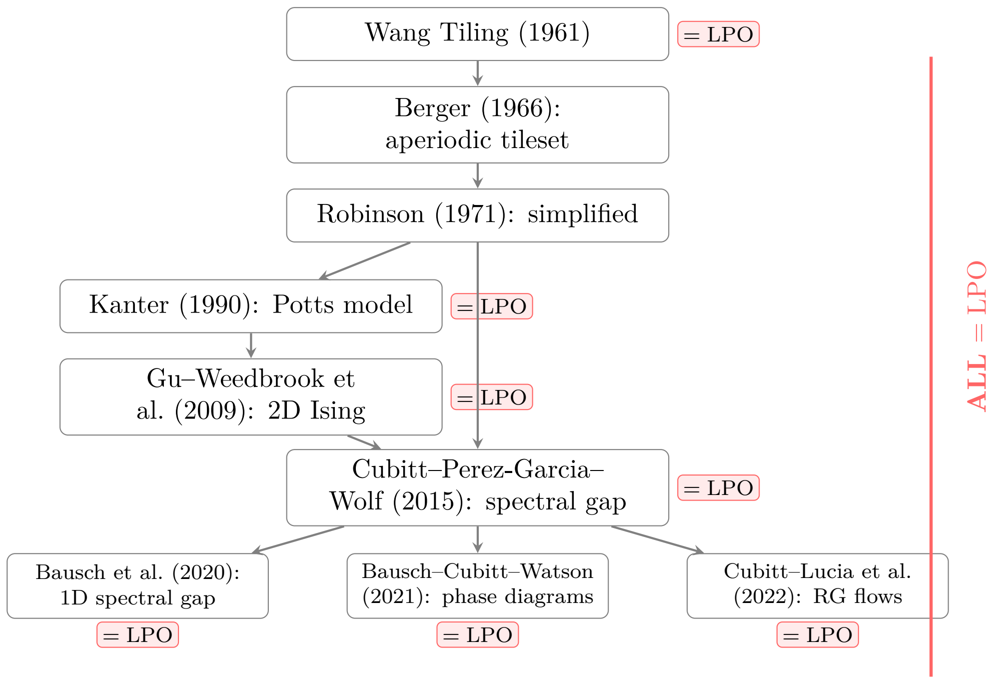

Figure 8.2: The genealogy of physical undecidability. Every result descends from Wang tiling and inherits exactly LPO—nothing more, nothing less.

2. **1D spectral gap undecidability** (Bausch–Cubitt–Lucia–Perez-Garcia [30]): more sophisticated tiling construction, but dimension does not change the logical structure. Still LPO.
3. **Uncomputable RG flows** (Cubitt–Lucia–Perez-Garcia–Perez-Eceiza 2022): RG fixed points are completed limits of iterative coarse-graining. LPO.

The meta-theorem is structurally simple. The halting problem is Σ_1^0-complete. LPO decides all Σ_1^0 statements. Any computable reduction preserves the classification: Σ_1^0 input + computable reduction = Σ_1^0 output. Therefore every such undecidability result lands at exactly LPO.

A notable exception establishes an important distinction: the ground state energy density (Watson–Cubitt 2021) is BISH—computationally *hard* (exponential time) but logically *decidable* (computable). Computational complexity and logical undecidability are fundamentally different. Paper 37: 660 lines of Lean 4, zero `sorry`.

8.4 Wang Tiling — The Grandfather (Paper 38)

Every undecidability result in quantum many-body physics descends from a single ancestor: the undecidability of Wang tiling. Wang [33] conjectured that if a finite set of tiles can tile the plane, it can do so periodically. Berger [16] disproved this by constructing an aperiodic tileset and proved the general tiling problem undecidable. Robinson [29] simplified the construction. The genealogy then branches into physics (fig. 8.2).

Paper 38 proves that Wang tiling itself—the ancestor of the entire genealogy—is Turing–Weihrauch equivalent to LPO. The decision problem "does finite tileset T tile the plane?" is Σ_1^0-complete, hence LPO-equivalent. The Σ_1^0 Ceiling Theorem completes the analysis: to exceed LPO—to reach Σ_2^0—would require encoding a Σ_2^0-complete problem (such as the finiteness problem: "does M halt on infinitely many inputs?"). No existing construction in quantum many-body physics achieves this. Paper 39 will construct exactly such an encoding—but the resulting observable is empirically inaccessible. Paper 38: 573 lines of Lean 4, zero `sorry`.

8.5 What Undecidability Actually Is

Papers 36–38 together establish a single conclusion: physical undecidability is LPO's non-computability, inherited from the thermodynamic limit, traceable to Wang tiling.

This is a reclassification. Before this program, spectral gap undecidability was understood as: "there exist Hamiltonians whose gapped/gapless nature cannot be determined by any algorithm." The reclassification: "the thermodynamic limit is non-computable when applied uniformly; Cubitt et al. embedded the halting problem into this non-computability; the result is the non-computability of LPO." All physical undecidability is one phenomenon wearing different costumes. The spectral gap, phase diagrams, RG flows, and 1D spectral gap are all instances of the same logical event: encoding a halting problem into a Σ_1^0 decision via a computable reduction. The diversity of physical contexts masks the uniformity of the logical mechanism.

Chapter Summary

Every known physical undecidability result—spectral gap, phase diagrams, RG flows—is Turing–Weihrauch equivalent to LPO, traceable to a single ancestor: Wang tiling (1961). The most undecidable thing in physics is exactly as undecidable as a boiling pot of water.

Chapter 9

Beyond LPO — The Thermodynamic Stratification

9.1 The Promise Gap as Logic Mechanism

Cubitt's construction builds Hamiltonians with a promise gap: $\Delta \in \{0\} \cup [\gamma, \infty)$ for some computable $\gamma > 0$. With the gap, deciding "is $\Delta = 0$?" reduces to:

$$\exists L \text{ such that } \Delta_L < \gamma/2$$

This is Σ_1^0—a single existential quantifier over a decidable predicate. LPO decides it.

Without the promise gap—for a generic Hamiltonian where Δ could be any non-negative real—deciding "is $\Delta = 0$?" requires:

$$\forall m\ \exists L \text{ such that } \Delta_L < 1/m$$

This is Π_2^0—a universal-existential quantifier alternation. LPO cannot decide it; it requires LPO′ (Σ_2^0-LEM). The promise gap is not a convenience or a technical limitation—it is the *logical mechanism* that determines whether the decision lives at Σ_1^0 or Σ_2^0.

9.2 The Modified Encoding

Paper 39 constructs an explicit encoding that reaches Σ_2^0. The modification uses Robinson aperiodic tilings with perimeter counters: supertiles of increasing scale $k = 0, 1, 2, \ldots$ each run a Turing machine M on input k, where k is extracted from the perimeter counter. The encoding achieves:

- **Gapped** $\longleftrightarrow$ M halts on finitely many inputs
- **Gapless** $\longleftrightarrow$ M halts on infinitely many inputs

This is the **Finiteness Problem**—Σ_2^0/Π_2^0-complete, strictly above the halting problem in the arithmetic hierarchy. The construction is verified in 802 lines of Lean 4 with zero `sorry`.

9.3 The Stratification Theorem

The Thermodynamic Stratification Theorem classifies observables by arithmetic complexity as a function of thermodynamic scaling:

Extensive observables (energy density, free energy, magnetization per site) converge via subadditivity. Fekete's lemma ($\equiv$ LPO, Paper 29) guarantees convergence. Ceiling: LPO (Σ_1^0), regardless of promise gap.

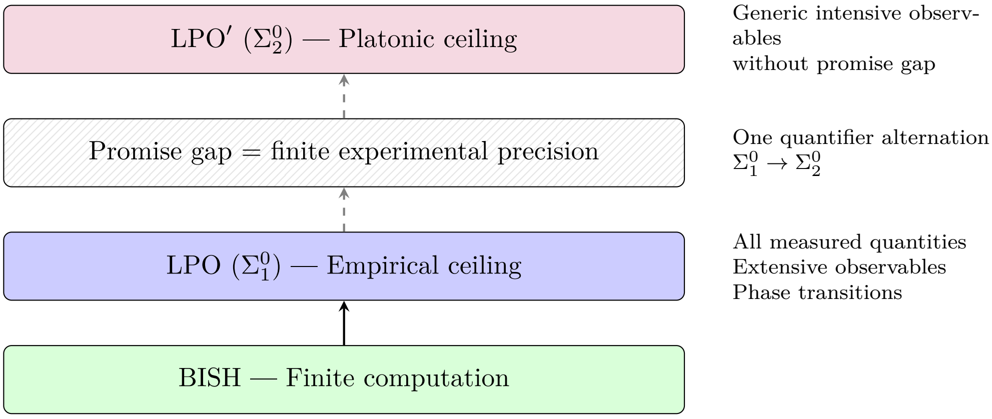

Figure 9.1: The two ceilings and the gap between them. Finite experimental precision enforces an effective promise gap that collapses Σ^0_2 decisions back to LPO.

Intensive observables (spectral gap, correlation length, mass gap) are determined by infimum-type operations, not averages. Without a promise gap, deciding whether an intensive observable equals a specific value involves a Π^0_2 statement. Ceiling: LPO′ (Σ^0_2).

Empirical observables—quantities measured at finite precision—always carry an effective promise gap. A spectrometer with precision ε collapses the Π^0_2 decision to Σ^0_1: "is there a finite-volume gap below $\varepsilon/2$?" is decidable by LPO. Ceiling: LPO (Σ^0_1).

Tier	**Observables**	**Ceiling**	**Mechanism**
Empirical	All measured quantities	LPO (Σ^0_1)	Finite precision = promise gap
Extensive-Platonic	Thermodynamic densities	LPO (Σ^0_1)	Fekete/BMC
Intensive-Platonic	Gap, correlation length	LPO′ (Σ^0_2)	No promise gap, infimum-type

9.4 The Refined Thesis

The original thesis: the logical constitution of empirical physics is BISH + LPO. Paper 39 refines this without breaking it.

Empirical physics (predictions compared to finite-precision measurements) remains at BISH + LPO. The empirical ceiling is confirmed.

Platonic physics (exact mathematical properties of idealized infinite-volume systems) can reach BISH + LPO′ for intensive observables without promise gap.

The gap between empirical and Platonic physics is precisely the promise gap, and the promise gap is precisely finite experimental precision. The refinement *strengthens* the thesis in three ways: (1) it identifies the *mechanism* keeping empirical physics at Σ^0_1 (finite precision enforces promise gaps); (2) it identifies the *boundary* of the thesis (the Σ^0_2 tier is real but experimentally inaccessible); (3) it makes a sharper falsifiable prediction (if a laboratory measurement requires Σ^0_2 reasoning, the empirical ceiling is broken).

Chapter Summary

Generic intensive observables without promise gap can reach Σ^0_2—the Platonic ceiling. But finite experimental precision enforces an effective promise gap that collapses every measurement back to LPO. The empirical ceiling holds; the gap between empirical and Platonic physics is precisely the promise gap.

Chapter 10

Consequences

10.1 For Quantum Gravity

The characterization predicts that whatever empirical predictions quantum gravity eventually makes, they will be BISH + LPO. This applies to all candidate theories.

String theory. The string landscape—the space of consistent string vacua—is constructed using Calabi–Yau geometry, which relies heavily on compactness (FT). But compactness is dispensable (Paper 30). Whatever empirical predictions emerge from a specific string vacuum will be finite computations (BISH) or completed limits (LPO). The swampland program, asserting universal constraints of the form "no consistent theory has property P," makes Π^0_2 claims (universal quantification over a space of theories) that may exceed BISH + LPO—and if so, the program predicts they are logically disconnected from empirical physics.

Loop quantum gravity fits naturally. Spin network states at finite graph size are combinatorial objects (BISH). The continuum limit costs LPO, exactly as in lattice QCD.

Holographic entanglement entropy (Ryu–Takayanagi) is calibrated in detail in Chapter 11. If both sides of AdS/CFT calibrate to BISH + LPO, the duality is logically consistent in the program's sense. If one side costs more, the program has identified a logical obstruction.

10.2 For the Measurement Problem

Many-Worlds requires DC to construct the branching tree of the universal wavefunction—each measurement creates new branches, and the infinite tree requires DC. Copenhagen requires WLPO to assert that a definite outcome occurred (a zero-test on superposition coefficients). Bohmian mechanics requires LPO to assert existence of the pilot wave trajectory as a completed real-valued function.

The crucial observation: DC is dispensable (Paper 31). The branching structure is mathematical scaffolding—the empirical predictions (Born rule probabilities) are BISH and do not depend on whether branches "exist." Similarly, Copenhagen's collapse (WLPO) and Bohm's trajectory (LPO) generate the same BISH-level empirical predictions. The interpretations agree on every number an experimentalist can measure. They disagree about ontological claims at different heights in the hierarchy.

This does not "solve" the measurement problem. It reclassifies it: from "what actually happens during measurement?" to "which logically dispensable mathematical framework do you prefer for describing non-empirical aspects of quantum theory?" The reclassification identifies precisely where the disagreement lives (in scaffolding above BISH) and why it is empirically unresolvable (because the scaffolding is dispensable).

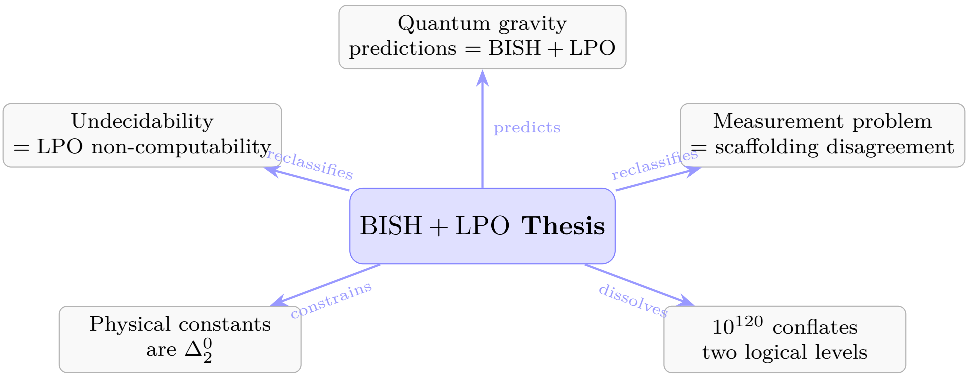

Figure 10.1: Consequence map. The BISH + LPO thesis generates five consequences for open problems: predictions for quantum gravity, reclassification of the measurement problem and undecidability, dissolution of the 10^{120} discrepancy, and constraints on physical constants.

10.3 For the Cosmological Constant

The 10^{120} discrepancy is reclassified in Chapter 12. The QFT vacuum energy sum is an infinite series; asserting convergence to a specific value is LPO. The BISH-level content is different: for any finite number of modes, the vacuum energy is finite and computable. Comparing an LPO-level mathematical artifact to a BISH-level measurement and calling the discrepancy a "prediction failure" conflates two logical levels. This is not a resolution—the question of why Λ has its observed value remains open—but it is a change in character.

10.4 For Physical Constants

Every physical constant—$\alpha \approx 1/137$, m_e, Λ, G—is Δ^0_2 (limit-computable). Each admits a finite description: an algorithm that, given access to a halting oracle, converges to the constant's value. No physical constant can encode a Σ^0_2-complete problem.

This reframes the fine-tuning debate. The standard discussion assumes constants are drawn from the continuum—an uncountable set with measure-theoretic properties. But the program says physical constants are Δ^0_2—a countable class. A fine-tuning argument computing the probability of α by integrating over all reals is integrating over a space overwhelmingly populated by constants that physics cannot access. The physically relevant sample space is Δ^0_2, which is countable and does not support the standard measure-theoretic formulation of fine-tuning. This does not settle the question but constrains the form an answer can take.

10.5 For Physical Undecidability

The undecidability arc (Papers 36–39) changes how "undecidability" in physics should be understood. Before this program, spectral gap undecidability was interpreted as revealing a fundamental limit to physical knowledge, comparable to Gödel's incompleteness theorems. The reclassification: physical undecidability is the non-computability of LPO, the same principle physicists have implicitly used since Boltzmann defined the thermodynamic limit in the 1870s.

The spectral gap of a *specific* Hamiltonian $H(M)$ is not "unknowable"—it is LPO-decidable. A physicist who can take a thermodynamic limit can decide whether $H(M)$ is gapped. The uniform problem—deciding for arbitrary Hamiltonians simultaneously—is non-computable because LPO is non-computable when applied to arbitrary binary sequences. But this is the same non-computability that prevents a general convergence oracle. Cubitt's celebrated result reveals not a new frontier of unknowability but the familiar non-computability of a principle physicists have used for 150 years.

The Σ_2^0 refinement (Paper 39) sharpens this: for generic intensive observables without promise gap, the decision reaches Σ_2^0. But this tier is Platonic—no finite-precision experiment can distinguish "gap $= 0$" from "gap $< \varepsilon$." The empirical ceiling remains LPO because finite precision enforces an effective promise gap.

Chapter Summary

The BISH+LPO thesis predicts that quantum gravity predictions will be BISH+LPO, that the measurement problem is a disagreement about dispensable scaffolding, that physical constants are Δ_2^0, and that the 10^{120} discrepancy conflates two logical levels. Reclassification identifies what kind of problem each is—resolution requires new physics.

Chapter 11

The Diagnostic in Action I — AdS/CFT Correspondence

11.1 From Classification to Diagnosis

Papers 1–39 established that the logical resources required for all empirical predictions in known physics are exactly BISH + LPO. Paper 41 transforms the program from a completed classification exercise into an active diagnostic tool, applied to the most important open problem in theoretical physics.

The target is the AdS/CFT correspondence [19]—specifically, the Ryu–Takayanagi (RT) formula [18] for holographic entanglement entropy. The choice is strategic: AdS/CFT is the most active area of theoretical physics, the RT formula is its most cited result, and the Page curve and island formula are its most debated recent developments. The diagnostic question is simple: *does the holographic dictionary preserve axiom cost?* If the correspondence claims that bulk gravitational physics and boundary conformal field theory compute the same physics, the framework can check whether they compute it at the same *logical cost*. If yes, the duality is logically transparent. If no, the axiom gap identifies places where the duality performs non-trivial logical work—candidates for where it might fail or require modification.

11.2 Observables vs. Decisions

Before presenting the calibrations, a distinction must be sharpened that refines the program's treatment of phase transitions. There are two logically distinct operations at a phase transition:

The observable computation: computing the numerical value of the order parameter, the free energy, or the entropy as a function of the control parameter. This is a question about a continuous real-valued function.

The phase decision: declaring which phase the system is in—asserting "the system is in phase A" or "the system is in phase B" as a Boolean classification.

These have different axiom costs. The Fekete mechanism (Paper 29) concerns the *existence* of a thermodynamic limit—proving that $f = \lim_{N\to\infty} F_N/N$ exists. The LPO cost is in computing the limit. The BTZ entanglement phase transition is different: both competing quantities $L_1(\theta)$ and $L_2(\theta)$ are already in hand as explicit, BISH-computable functions. The entropy $S(A) = \min(L_1, L_2)/(4G_N)$ is BISH via the identity $\min(x, y) = \frac{1}{2}(x + y - |x - y|)$, since the absolute value function is uniformly continuous and BISH-computable. The *phase decision*—extracting a Boolean flag declaring whether $L_1 \leq L_2$ or $L_2 \leq L_1$—costs LLPO when the difference may be zero.

Reconciliation with Paper 29: Fekete costs LPO because it computes a quantity *defined as a limit*—a real number not yet in hand. The min costs BISH because it selects between values already computed as explicit functions. Both mechanisms occur at phase transitions but contribute different axiom costs. This distinction is invisible in systems where competing quantities are themselves defined by limits (e.g., the Ising free energy). The BTZ RT formula is the first calibration where competing quantities are given by closed-form expressions, making the min mechanism visible as a separate, lower-cost operation.

11.3 Vacuum AdS$_3$: The Null Result

Bulk: In the Poincaré patch of AdS$_3$, $ds^2 = (\ell^2/z^2)(dz^2 + dx^2)$, the RT geodesic connecting boundary points $(x_1, 0)$ and $(x_2, 0)$ is a semicircle of radius $R = |x_2 - x_1|/2$. Its regularized length is:

$$L_{\text{reg}} = 2\ell \log(|x_2 - x_1|/\varepsilon)$$

An explicit algebraic expression. No variational principle, no compactness, no limit. **Calibration:** BISH.

Boundary: The Calabrese–Cardy formula [26] for entanglement entropy of an interval of length ℓ_A in a 2d CFT vacuum state:

$$S(A) = \frac{c}{3} \log(\ell_A/\varepsilon)$$

Derived via the replica trick: (a) twist operator correlation function for integer n—algebraic, BISH; (b) analytic continuation $n \to 1$—explicit formula, BISH; (c) differentiation at $n = 1$—elementary, BISH. **Calibration:** BISH.

Both sides yield the same formula under Brown–Henneaux $c = 3\ell/(2G_N)$. The duality is a BISH-to-BISH map—the holographic dictionary performs no logical work for this prediction. This null result sets the baseline for the thermal case.

11.4 Thermal BTZ: The Phase Transition

In the BTZ black hole with horizon radius r_+ and AdS radius ℓ, the two competing RT geodesics for a boundary interval of angular extent θ have lengths:

$$L_1(\theta) = 2\ell \ln\Big(\frac{2R}{r_+} \sinh \frac{r_+\theta}{2\ell}\Big), \qquad L_2(\theta) = 2\ell \ln\Big(\frac{2R}{r_+} \sinh \frac{r_+(2\pi - \theta)}{2\ell}\Big)$$

Both are explicit compositions of elementary functions: logarithm, hyperbolic sine, multiplication. **Both are BISH-computable.** The entanglement entropy $S(A) = \min(L_1, L_2)/(4G_N)$ is BISH.

The critical angle θ_c where $L_1 = L_2$ is determined by $\sinh(r_+\theta/2\ell) = \sinh(r_+(2\pi - \theta)/2\ell)$. Since sinh is strictly monotone, this gives $\theta_c = \pi$—a trivially computable constant determined by the $\theta \leftrightarrow 2\pi - \theta$ symmetry of the BTZ geometry. No comparison of potentially equal real numbers is needed: $\theta < \pi$ implies $L_1 < L_2$, and $\theta > \pi$ implies $L_1 > L_2$, both decidable by rational approximation. For the BTZ black hole specifically, even the discrete phase classification is BISH.

Generic asymptotically AdS black holes: with matter fields or higher-derivative corrections, the geodesic lengths remain BISH-computable (determined by explicit ODEs), the continuous entropy $\min(L_1, L_2)$ remains BISH, but the critical angle θ_c may not have a closed-form solution. The phase decision then costs LLPO.

Boundary side: the Hawking–Page transition involves $F = \min(F_{\text{AdS}}, F_{\text{BTZ}})$—both BISH-computable. The duality preserves axiom cost exactly: on both sides, the continuous observable is BISH and the discrete phase classification is LLPO (or BISH for BTZ by symmetry).

11.5 FLM Quantum Correction

The Faulkner–Lewkowycz–Maldacena formula [27] adds a one-loop correction: $S(A) = \text{Area}(\gamma_A)/(4G_N) + S_{\text{bulk}}(\Sigma_A)$. For a free massive scalar in AdS_3, the bulk entanglement entropy is computed via the replica trick on bulk fields.

The heat kernel on Euclidean AdS_3 ($\cong \mathbb{H}^3$) has the explicit Camporesi form [34]:

$$K(t,\rho) \propto t^{-3/2} \frac{\rho}{\sinh\rho} \exp\Big(-\frac{\rho^2}{4t} - m^2 t\Big)$$

The Sommerfeld method of images produces a sum over geodesics to image points at distances ρ_n. The exponential factor $\exp(-\rho_n^2/4t)$ guarantees the sum converges with an explicit, BISH-computable Cauchy modulus. UV regularization via Seeley–DeWitt coefficients and ζ-function regularization both yield BISH-computable results: $\zeta'(0)$ is obtained by algebraic integration-by-parts on an explicit formula.

Calibration: S_{bulk} for a free scalar in the AdS_3 vacuum is BISH**.** The FLM quantum correction does not increase axiom cost beyond the classical RT term. The LPO cost would enter for interacting fields (where the mode sum does not close) or non-symmetric backgrounds (where the heat kernel lacks a closed form).

11.6 Quantum Extremal Surface

The Engelhardt–Wall [28] QES prescription minimizes the generalized entropy:

$$S(A) = \min_{\gamma}\big[\text{Area}(\gamma)/(4G_N) + S_{\text{bulk}}(\Sigma_\gamma)\big] = \min_{\gamma} S_{\text{gen}}(\gamma)$$

Existence of the minimizing surface γ^* by the direct method of calculus of variations requires extracting a convergent subsequence from a minimizing sequence—a compactness argument costing FT. But the boundary CFT does not observe the bulk surface. The entropy is the *infimum* of S_{gen}, and constructive mathematics can compute this infimum by evaluating the functional on successive approximations and applying BMC (LPO). FT builds the Platonic surface in the unobservable bulk; BISH computes the observable entropy on the boundary. The Lean formalization (Theorem `Infimum_vs_Minimizer`) makes this separation precise: (a) the observable infimum $\inf_\gamma S_{\text{gen}}(\gamma)$ requires only LPO (via BMC); (b) the geometric minimizer γ^* requires FT (compactness); (c) LPO alone does not yield the minimizer. The three claims are independently machine-checked.

Perturbative regime: the QES is obtained by perturbing the classical RT surface via the Jacobi geodesic deviation equation, an ODE sourced by ∇S_{bulk}. By Picard–Lindelöf (BISH for Lipschitz ODEs), the perturbed surface is BISH-constructible. No compactness needed.

Island formula and Page curve: $S(A) = \min(S_{\text{island}}, S_{\text{no-island}})$ is BISH (minimum of two BISH-computable quantities). The Page curve—the continuous plot of entropy as a function of time—is a BISH-computable function. The discrete Page time decision costs LLPO for generic parameters.

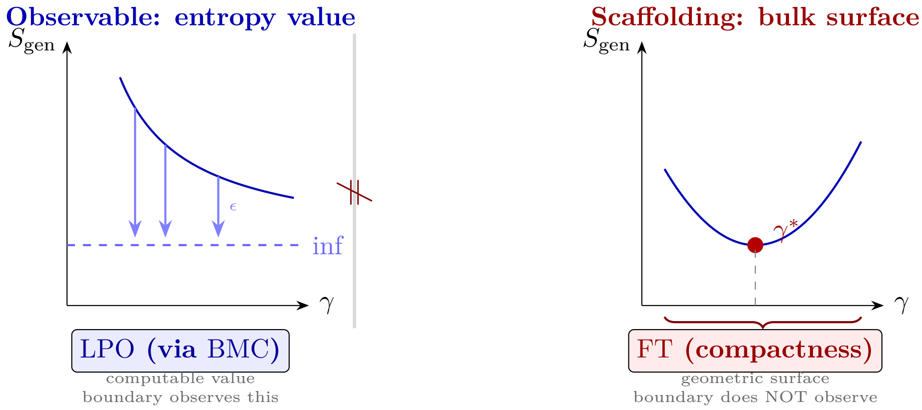

Figure 11.1: The scaffolding separation. **Left:** The observable entropy (infimum of S_{gen}) is computable at LPO via successive approximation (BMC). The boundary CFT observes this value. **Right:** The bulk surface γ^* that achieves the minimum requires compactness (FT). The boundary never observes it. Holography needs only the left panel.

11.7 Complete Calibration Table

Computation	Bulk cost	Boundary cost	Duality preserves?
Vacuum AdS_3 RT	BISH	BISH	✓ (identity)
BTZ RT (entropy value)	BISH	BISH	✓
BTZ RT (phase decision)	BISH ($\theta_c = \pi$)	BISH (by symmetry)	✓
Generic thermal RT (entropy)	BISH	BISH	✓
Generic thermal RT (phase)	LLPO	LLPO	✓
FLM correction (free, vacuum)	BISH	N/A (bulk only)	—
FLM correction (free, thermal)	BISH	N/A (bulk only)	—
QES surface existence	FT (scaffolding)	Not observed	Projected away
QES entropy (perturbative)	BISH	BISH	✓
QES entropy (non-perturbative)	LPO (via inf/BMC)	LPO (expected)	✓
Island formula (Page curve)	BISH	BISH	✓
Island formula (Page time)	LLPO	LLPO	✓

The table's most striking feature: no entry exceeds LPO. The BISH + LPO ceiling holds across the entire holographic dictionary—from the simplest vacuum RT to the quantum-corrected island formula.

11.8 What the Diagnostic Reveals

The holographic dictionary is an axiom-preserving map (fig. 11.3). For every prediction examined, bulk and boundary computations carry identical axiom cost. The duality maps BISH to BISH, LLPO to LLPO, and LPO to LPO. This is a falsifiable prediction: any future computation where the two sides have different axiom costs would identify a logical obstruction within the correspondence.

Holography projects away compactness. The Fan Theorem cost of bulk geometric existence is invisible to the boundary. The boundary CFT computes the entropy value without ever constructing or observing the bulk surface. The compactness that guarantees the surface exists is scaffolding—necessary for the bulk geometric picture, irrelevant to the boundary observable. This is the holographic principle, restated in the language of constructive reverse mathematics: holography is the projection that eliminates FT.

Phase transitions are cheaper than expected. The observable entropy at a phase

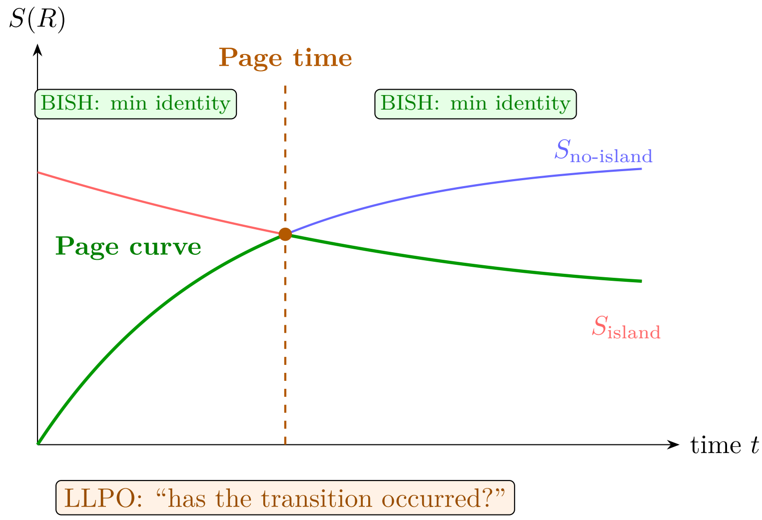

Figure 11.2: The Page curve through the CRM lens. The continuous Page curve $S(R) = \min(S_{\text{island}}, S_{\text{no-island}})$ is BISH-computable at every time t via the min identity. The discrete decision "has the Page time occurred?" costs LLPO. Information recovery is encoded in a constructively accessible quantity; only the temporal classification requires a (weak) omniscience principle.

transition is BISH—the minimum of two BISH-computable functions. Only the discrete phase classification costs LLPO, and even this cost vanishes for sufficiently symmetric geometries. This refines the program's treatment of phase transitions by distinguishing computing a limit (LPO, Fekete mechanism) from selecting a minimum (BISH, min mechanism).

For theorists working on the Page curve: the physically meaningful axiom cost of the island formula is the saddle-point competition (BISH for the entropy value; LLPO for the phase decision). The FT cost of QES existence is scaffolding—eliminable in the perturbative regime (Jacobi equation, BISH) and projectable away via holography in the non-perturbative regime (boundary infimum, LPO). Computational effort should be directed at characterizing the competition between island and no-island saddles, not at proving existence of the QES via compactness methods.

Chapter Summary

The holographic dictionary is an axiom-preserving map: bulk and boundary computations carry identical logical resources at every level. The Fan Theorem cost of QES existence is scaffolding—holography is the projection that eliminates FT. No entry exceeds LPO.

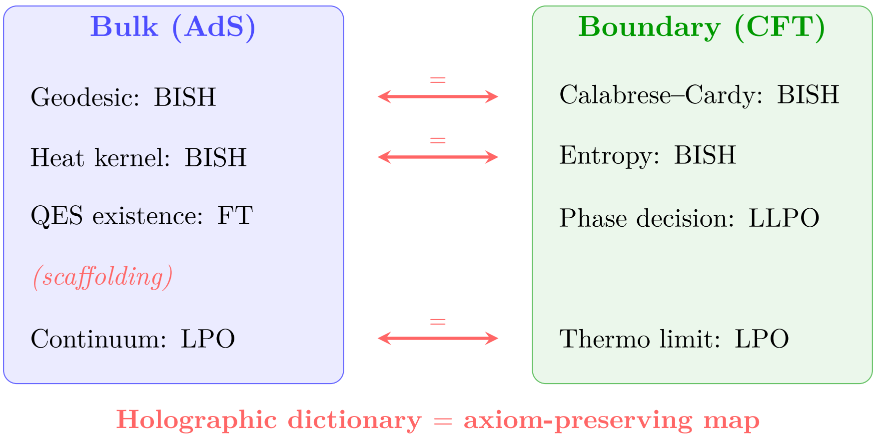

Figure 11.3: The holographic dictionary preserves axiom cost. Bulk and boundary computations carry identical logical resources at every level. The Fan Theorem cost of QES existence is scaffolding—it does not appear on the boundary side.

Chapter 12

The Diagnostic in Action II — Cosmological Constant

12.1 The Alleged 10^{120} Discrepancy

The cosmological constant problem [21] is widely described as the worst prediction in the history of physics. The claim: quantum field theory predicts a vacuum energy density of order $M_{\text{Planck}}^4 \approx 10^{71}$ GeV^4, while the observed cosmological constant corresponds to $\rho_\Lambda \approx 10^{-47}$ GeV^4—a discrepancy of 120 orders of magnitude.

Paper 42 subjects this claim to the axiom calibration framework. The diagnostic question: at what level of the constructive hierarchy does each component of the problem live? Is the "prediction" a BISH-computable quantity derived from experimental inputs? Or is it an artifact of a specific mathematical idealization? The problem decomposes into three logically distinct claims with different constructive status.

12.2 The Mode Sum and Its Regulators

For a free scalar field of mass m on a finite cubic lattice with N sites per dimension, the vacuum energy $E_{\text{vac}} = \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}$ is a finite sum of explicit algebraic expressions. BISH. No controversy.

In the continuum limit ($a \to 0$, $N \to \infty$), the sum becomes an integral $\rho_{\text{vac}} = \int d^3k/(2\pi)^3 \cdot \frac{1}{2}\sqrt{k^2 + m^2}$, which diverges quartically. A divergent series does not define a real number—not at BISH, not at LPO, not at any level. The continuum vacuum energy is not a real number at all.

To extract a finite number, one introduces regularization:

Cutoff regularization: restrict $|k| \leq \Lambda_{\text{UV}}$. The integral gives $\rho \sim \Lambda_{\text{UV}}^4/(16\pi^2)$. Setting $\Lambda_{\text{UV}} = M_{\text{Planck}}$ gives $\rho \sim 10^{71}$ GeV^4. This is BISH-computable for any specific Λ_{UV}, but the hard cutoff breaks diffeomorphism invariance—it is incompatible with general relativity.

Dimensional regularization: compute in $d = 4 - 2\varepsilon$ dimensions. The quartic divergence vanishes identically—purely polynomial divergences evaluate to zero by construction. For a massive field, $\rho \sim m^4 \ln(m^2/\mu^2)$. For the top quark, this gives $\rho \sim (100 \text{ GeV})^4 \sim 10^8$ GeV^4. The 10^{120} discrepancy does not exist in this scheme.

ζ-function regularization agrees with dimensional regularization, not with cutoff regularization.

The 10^{120} number is produced by cutoff regularization. A quantity that changes when you change the regulator is not a physical prediction—it is an artifact of mathematical scaffolding. **Claim I is dissolved.**

Claim I: The 10^{120} UV discrepancy $\longrightarrow$ **DISSOLVED**
Regulator-dependent artifact. Vanishes under dimensional regularization.

Claim II: Naturalness ($\Lambda \sim M^4_{\text{Planck}}$) $\longrightarrow$ **RECLASSIFIED**
Bayesian prior, not mathematical derivation. Outside the deductive hierarchy.

Claim III: 55-digit fine-tuning $\longrightarrow$ **IDENTIFIED**
$\Lambda_{\text{obs}} = \Lambda_{\text{bare}} + 8\pi G(\rho_{\text{Higgs}} + \rho_{\text{QCD}})$. Arithmetic relation between LPO-computable reals.

Figure 12.1: The cosmological constant problem decomposed. The 10^{120} discrepancy is dissolved as scaffolding. Naturalness is reclassified as non-mathematical. The genuine constraint is a 55-digit fine-tuning at LPO.

12.3 Gravity and the Hollands–Wald Ambiguity

The standard objection: gravity couples to absolute energy via $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$, so the regulator-dependence objection fails when gravity enters. This is serious and must be addressed precisely.

The Hollands–Wald axioms [20] establish that the renormalized expectation value $\langle T_{\mu\nu}\rangle_{\text{ren}}$ is uniquely determined by physical axioms (locality, covariance, conservation, correct flat-space limit) *up to* a finite number of free coefficients:

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \langle T_{\mu\nu}\rangle_{\text{canonical}} + c_1 g_{\mu\nu} + c_2 G_{\mu\nu} + c_3(\text{curvature terms}) + \cdots$$

The coefficient c_1 is precisely the cosmological constant Λ. The Hollands–Wald axioms prove—as a mathematical theorem—that QFT in curved spacetime *cannot predict* Λ. It is a free parameter on the same footing as particle masses. Setting $c_1 = \Lambda_{\text{obs}} - (\text{condensate contributions})$ is arithmetic. BISH.

12.4 Naturalness as Non-Mathematical Claim

The "naturalness" argument asserts that c_1 should be "of order" M^4_{Planck}. This is a claim about the *expected magnitude* of a free parameter—a prior probability distribution over the space of possible values.

The calibration framework identifies this as a claim *outside the constructive hierarchy*. BISH formalizes deductive mathematics: given axioms, what follows? Naturalness is an inductive claim: given the structure, what values should we expect? The constructive hierarchy calibrates the former and has nothing to say about the latter. **Claim II is reclassified**—not dissolved (the framework does not show it is incorrect) but identified as non-mathematical.

12.5 The Genuine Constraint: 55-Digit Fine-Tuning

After dissolving Claim I and reclassifying Claim II, what remains? The electroweak phase transition produces a Higgs vacuum condensate with energy density $\rho_{\text{Higgs}} = V(v) = -\mu^4/(4\lambda) \approx -(100\ \text{GeV})^4 \approx -10^8\ \text{GeV}^4$. The QCD chiral phase transition produces a quark condensate $\rho_{\text{QCD}} \sim -\langle\bar{q}q\rangle \cdot m_q \sim -10^{-3}\ \text{GeV}^4$. Both are BISH-computable at tree level.

The fine-tuning equation:

$$\Lambda_{\text{obs}} = \Lambda_{\text{bare}} + 8\pi G\,(\rho_{\text{Higgs}} + \rho_{\text{QCD}})$$

With $\Lambda_{\text{obs}} \approx 10^{-47}$ GeV^4 and $\rho_{\text{Higgs}} \approx -10^8$ GeV^4, the bare parameter must satisfy $\Lambda_{\text{bare}} \approx 10^8$ GeV^4 to 55 decimal places. The *exact* interacting vacuum energies require the thermodynamic limit (Fekete, Paper 29), placing them at LPO. The fine-tuning equation at exact values is an equality between LPO-computable reals—logically mundane, qualitatively identical to every other thermodynamic limit.

12.6 The Casimir Effect: What QFT Actually Predicts

The Casimir force between parallel conducting plates, $F/A = -\pi^2\hbar c/(240d^4)$, is an energy *difference*—the difference in vacuum energy between the plate configuration and free space. The divergent terms cancel algebraically. The finite remainder converges with a BISH-computable Cauchy modulus. The Casimir force is BISH—no limits, no omniscience. It confirms the pattern: absolute vacuum energies are regulator-dependent; energy differences are regulator-independent and BISH-computable.

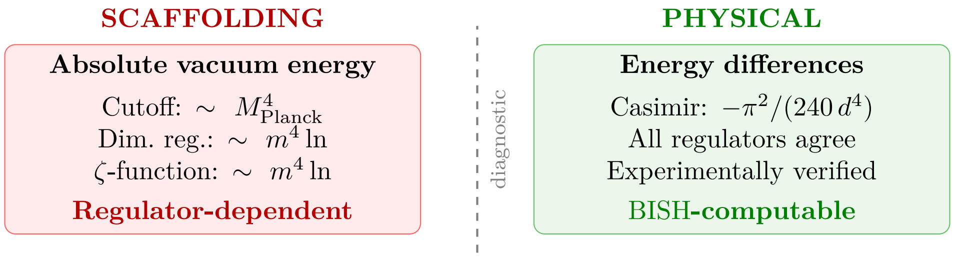

Figure 12.2: The scaffolding diagnostic applied to vacuum energy. Absolute vacuum energies (left) depend on the regularization scheme and carry no empirical content. Energy *differences* (right) are scheme-independent, BISH-computable, and experimentally verified (Casimir effect).

12.7 The RG Running of Λ

The cosmological constant runs under the renormalization group: $\mu\, d\Lambda/d\mu = \beta_\Lambda(\mu)$. The beta function is a finite sum of contributions from all particles below scale μ—algebraic functions of masses and couplings. Integrating the RG equation is a first-order ODE with BISH-computable coefficients; by Picard–Lindelöf, the running $\Lambda(\mu)$ is BISH-computable. LPO enters only when the QCD coupling becomes strong ($\mu \sim \Lambda_{\text{QCD}} \sim 200$ MeV) and the non-perturbative condensate is needed (Fekete, LPO). Above the QCD scale: BISH. Below: LPO. The same stratification found throughout the program.

12.8 Summary

The cosmological constant problem, under constructive reverse mathematics, decomposes cleanly (fig. 12.1). The 10^{120} narrative is dissolved (regulator artifact). The naturalness argument is reclassified (non-mathematical). The genuine constraint is the 55-digit fine-tuning—an arithmetic relation between LPO-computable reals, identical in logical character to every other thermodynamic limit.

The framework does not explain *why* the cancellation occurs. It shows that "why?" is a question about initial conditions or the UV completion of gravity, not about the logical structure of QFT. The BISH + LPO ceiling holds. The cosmological constant problem introduces no new logical resources.

Scope: this analysis addresses the "old" cosmological constant problem—the fine-tuning of Λ_{bare} against vacuum energy contributions. The "coincidence problem" (why $\rho_\Lambda \sim \rho_{\text{matter}}$

today) and the question of time-varying dark energy (preliminary DESI hints [35] at 2.5–3.9σ) are distinct questions. If dark energy evolves in time, the additional dynamical component would remain BISH if perturbative (e.g., a slowly rolling scalar field) and would reach LPO only if a thermodynamic limit were involved (e.g., a condensate phase transition).

Chapter Summary

The 10^{120} discrepancy is dissolved (regulator-dependent artifact). Naturalness is reclassified (non-mathematical Bayesian prior). The genuine constraint is 55-digit fine-tuning—an arithmetic relation between LPO-computable reals, logically identical to every other thermodynamic limit. The BISH + LPO ceiling holds.

Chapter 13

The Formalization

13.1 Why Formal Verification

The formal verification in Lean 4 [12] is not an optional supplement—it is essential infrastructure. The distinctions between BISH, WLPO, LLPO, and LPO are too fine for informal mathematical reasoning. These principles are logically close: LLPO and WLPO differ in one quantifier placement; WLPO is strictly weaker than LPO but implies it for bounded monotone sequences; the boundary between "constructive proof with LPO hypothesis" and "classical proof using LPO as a theorem" is invisible in natural-language mathematics but precisely detectable by a proof assistant.

The LPO-weakening incident (Paper 2) demonstrated the fragility. An AI coding agent, uncomfortable with the meta-classical producer/consumer architecture, replaced a classical metatheoretic argument with object-level LPO. The resulting proof compiled without errors and appeared correct on informal review. But the calibration was silently destroyed: LPO implies WLPO, so the bidual gap hypothesis became logically redundant, and the theorem—supposed to show the gap has exactly the strength of WLPO—became vacuously true. The error was caught only by examining `#print axioms`. Without formal verification, the error would have persisted indefinitely. The lesson: fine distinctions between WLPO and LPO are invisible to informal reasoning and mechanically obvious to Lean.

13.2 The Producer/Consumer Architecture

Constructive reverse mathematics proofs operate in a classical metatheory, adopting a "producer/consumer" architecture inspired by dependency injection. The **producer** operates classically: given a physical theorem T and a candidate principle P (e.g., LPO), the producer constructs a concrete artifact—a specific binary sequence, bounded monotone sequence, or real number—using whatever non-constructive reasoning is convenient. The **consumer** takes this artifact as input and derives P from T using only BISH. The consumer's proof is constructive: it shows that the physical theorem, applied to the producer's artifact, yields the omniscience principle.

The Lean formalization validates this mechanically. Running `#print axioms` on the consumer's theorem shows no `Classical.choice`—the derivation is constructive. Running it on the producer's construction shows `Classical.choice`—the artifact was built non-constructively. The two components are independently checkable.

13.3 Certification Levels

The program uses four certification levels:

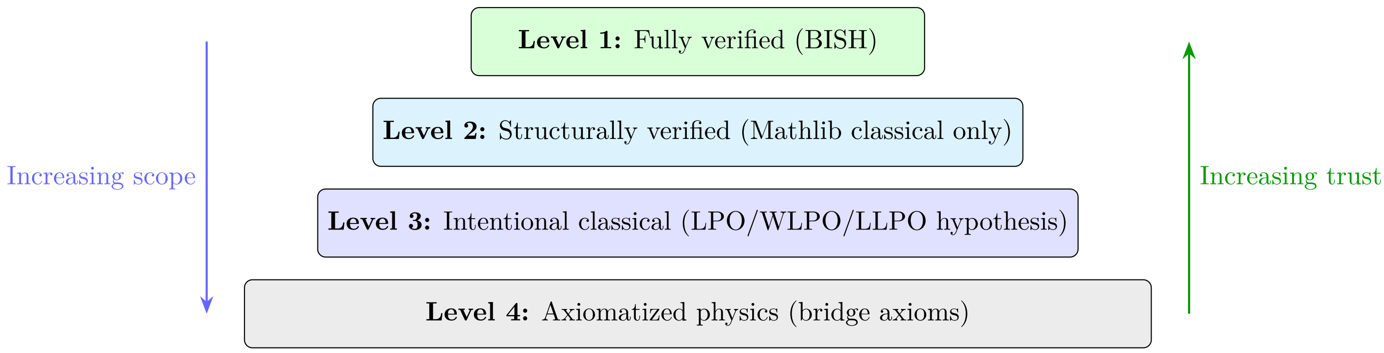

Figure 13.1: The four certification levels. Level 1 (fully verified) is pure BISH; Level 4 (axiomatized physics) declares physical assumptions as explicit axioms. The pyramid reflects the trade-off: constructive purity increases upward; physical scope increases downward.

Level 1 (Fully verified): No `Classical.choice`, no custom axioms. Mechanically certified BISH. *Examples:* finite-lattice partition functions, tree-level scattering amplitudes, Born rule at finite precision.

Level 2 (Structurally verified): `Classical.choice` only from Mathlib infrastructure ($\mathbb{R}$ via Cauchy completion). The theorem itself is constructive; the classical content is in the library, not the physics. *Examples:* most calibration theorems involving real-number arithmetic.

Level 3 (Intentional classical): `Classical.choice` from an explicit LPO/WLPO/LLPO hypothesis. The non-constructive content *is* the theorem's content—"assuming LPO, we prove X." *Examples:* Fekete's lemma, thermodynamic limits, coupling constant existence.

Level 4 (Axiomatized physics): Explicit `axiom` declarations for physical assumptions stated but not proved. *Examples:* Yang–Mills existence and mass gap, string tension positivity.

13.4 The Codebase

The codebase comprises approximately 35,000 lines of Lean 4 across 42 paper repositories, organized in five phases:

- **Phase I** (Papers 2–28): ∼20,000 lines—initial calibration establishing the empirical pattern.
- **Phase II** (Papers 29–31): ∼4,000 lines—three foundational theorems (Fekete ≡ LPO, FT dispensability, DC dispensability).
- **Phase III** (Papers 32–35): ∼6,000 lines—Standard Model calibration and conservation metatheorem.
- **Phase IV** (Papers 36–39): ∼2,690 lines—undecidability arc and thermodynamic stratification.
- **Phase V** (Papers 41–42): ∼1,600 lines—AdS/CFT and cosmological constant diagnostics.

All code publicly available on Zenodo with individual DOIs (Appendix A). All core calibration results compile without `sorry`.

Chapter Summary

Formal verification in Lean 4 is essential infrastructure: the BISH/WLPO/LPO distinctions are too fine for informal reasoning. Four certification levels separate fully verified (BISH, Level 1) from infrastructure classical (Level 2), intentional classical (LPO hypothesis, Level 3), and axiomatized physics (Level 4). Approximately 35,000 lines across 42 papers, zero `sorry`.

Chapter 14

What This Program Does Not Do

14.1 Epistemology vs. Ontology

The program characterizes the logical resources needed to *predict* physical behavior—to compute the numbers that experimentalists compare to measurements. It does not determine the logical resources the universe *uses* to *generate* that behavior. These are fundamentally different questions.

An analogy: chess can be played on a physical board or simulated on a computer. The logical resources needed to *predict* chess outcomes are well-defined computational problems. The resources the physical board "uses" to "generate" outcomes are entirely different. The program addresses prediction, not generation.

The epistemological reading is robust: BISH + LPO suffices for all known empirical predictions. The ontological reading—that nature's computational architecture *is* BISH + LPO—would require additional philosophical argument that the program does not provide and does not claim to provide. Paper 29's argument (phase transitions require LPO, phase transitions are real, therefore LPO is physically instantiated) comes closest to an ontological claim, but it is carefully hedged: LPO is necessary for the mathematical *description*, not necessarily for the physical mechanism.

14.2 Completeness

The calibration is not exhaustive. Significant gaps remain:

- Multi-loop scattering amplitudes beyond one loop (the metatheorem predicts BISH at each fixed order, but this is unverified in Lean beyond one loop).
- Non-perturbative QCD observables beyond confinement and the mass gap—hadron spectroscopy, glueball properties, chiral symmetry breaking.
- Condensed matter physics—topological phases, the fractional quantum Hall effect, spin liquids, topological insulators. The metatheorem predicts BISH+LPO, but the prediction is untested.
- Quantum gravity predictions—Ryu–Takayanagi beyond what Chapter 11 covers, cosmological observables, black hole information.

The thermodynamic stratification theorem (Chapter 9) predicts that intensive observables in uncalibrated systems may reach Σ_2^0 at the Platonic level but that empirical predictions will remain at LPO. Confirming or refuting these predictions is the natural next phase.

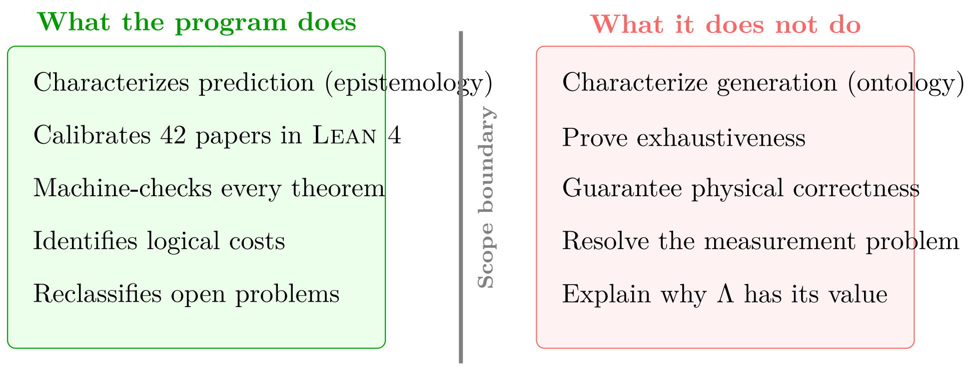

Figure 14.1: The scope of the program. Left: what the program establishes (prediction characterization, machine verification, reclassification). Right: what lies beyond its scope (ontology, physical correctness, problem resolution).

14.3 Physical Correctness

Lean verifies logical consistency, not physical correctness. The program's bridge lemmas—axioms encoding physical assumptions (e.g., "the partition function satisfies the transfer matrix recursion")—are stated as explicit `axiom` declarations. If a bridge lemma encodes incorrect physics, the Lean proof is logically valid but physically meaningless. Lean guarantees the calibration *follows from* the physics assumptions; it does not guarantee the assumptions are true.

The program assumes standard physics: the Standard Model is correct (at least at tested energies), general relativity describes gravity (at least outside singularities), and statistical mechanics describes thermodynamic systems. The logical characterization is conditional: "if known physics is correct, then empirical physics = BISH + LPO."

14.4 Problem Resolution

The program reclassifies several open problems—the measurement problem, the cosmological constant, physical undecidability—but reclassification is not resolution. The measurement problem, even after reclassification as a disagreement about dispensable scaffolding, still involves a genuine question: why does the macroscopic world *appear* classical? The program says the disagreement is about framework choice, not physical reality, but framework choice still matters for how physicists develop theories.

Similarly, reclassifying the cosmological constant discrepancy as an "artifact of idealization" does not explain why Λ has its observed value. Reclassifying physical undecidability as "LPO non-computability" does not make the spectral gap computable. The program changes the *character* of these problems—it identifies what kind of problem each is—but solving a reclassified problem still requires new physics or mathematics that the program does not provide.

Chapter Summary

The program characterizes prediction, not generation. Reclassification is not resolution. The calibration is not exhaustive. Lean 4 verifies logical consistency, not physical correctness. Significant gaps remain—and the thesis is empirical: falsifiable by a single counterexample.

Chapter 15

Conclusion

15.1 Summary

Two narrative threads converge on a single conclusion. The calibration thread: across 42 papers, every empirical prediction requires at most BISH + LPO. Fekete ≡ LPO proves necessity. FT and DC dispensability proves sufficiency. The undecidability thread: every known physical undecidability result is LPO-equivalent, traceable to Wang tiling. The diagnostic thread (Papers 41–42): the framework actively diagnoses the AdS/CFT correspondence and the cosmological constant problem.

BISH + LPO is what physics costs, and LPO is what physics cannot compute uniformly. The cost and the failure are the same principle.

15.2 Future Directions

Defensive: Higher-loop amplitudes, lattice QCD spectroscopy, condensed matter topological phases, multi-loop AdS/CFT.

Offensive: Physical constants as Δ_2^0, measurement problem resolution, quantum gravity predictions.

Frontier: Σ_2^0 laboratory observables—the most consequential discovery the program could provoke.

15.3 Bishop's Legacy

Errett Bishop published *Foundations of Constructive Analysis* in 1967. He defined BISH as a philosophical program. The omniscience principles were identified by Bishop and his successors as the precise points where classical mathematics departs from constructive practice. This was purely logical cartography, with no physical motivation.

That this cartography—designed for pure mathematics in the 1960s–1990s—turns out to classify the non-constructive content of physics with perfect precision across 42 papers and 35,000 lines of formal verification is the most striking fact about the program. The fit is discovered, not designed. That is what makes it worth reporting.

For the detailed synthesis of the first 29 calibrations and the historical narrative connecting Bishop's constructive analysis to the present program, the reader is referred to Papers 10 [14] and 12 [15].

Chapter Summary

BISH + LPO is the cost physics pays and the ceiling it cannot exceed. The cost and the failure are the same principle. That Bishop's cartography—designed for pure mathematics—classifies empirical physics with perfect precision across 42 papers is the most striking fact about the program.

Appendix A

Complete Calibration Table

Paper	Domain	Title	Key Theorem	Height	DOI (Zenodo)
2	FA	Bidual gap	$X \cong X^{**} \leftrightarrow$ WLPO	WLPO	17107493
4	QM	Quantum spectra	Spectral thm calibration	LPO	17059483
5	GR	Schwarzschild	Geodesic computation	BISH	18489703
6	QM	Heisenberg	$\Delta x \Delta p \geq \hbar/2$	BISH/LPO	18519836
7	FA	Trace-class	Physical bidual gap	WLPO	18527559
8	SM	1D Ising	Partition / thermo limit	BISH/LPO	18516813
9	SM	Ising invariance	Transfer $\leftrightarrow$ direct	LPO	18517570
11	QI	CHSH/Tsirelson	Bell $\leftrightarrow$ LLPO	LLPO	18527676
13	GR	Event horizon	$r = 2M$ logical boundary	WLPO/LPO	18529007
14	QM	Decoherence	Partial trace calibration	BISH/LPO	18569068
15	GR	Noether	Local vs. global	BISH/LPO	18572494
16	QM	Born rule	$P = \lvert\langle\psi\vert\varphi\rangle\rvert^2$	BISH/LPO	18575377
17	GR	Bekenstein–Hawking	$S = A/4$ as thermo limit	LPO	18597306
18	QFT	Yukawa RG	SM one-loop stratification	BISH/WLPO/LPO	18626839
19	QM	WKB tunneling	Tunneling probability	LLPO	18602596
20	SM	Ising magnetization	Observable-dependent cost	WLPO	18603079
21	QI	Bell nonlocality	Bell $\leftrightarrow$ LLPO tight	LLPO	18603251
22	QM	Radioactive decay	Markov's Principle	MP $\subset$ LPO	18603503
23	QM	Measurement problem	Interpretations vs. DC	DC (disp.)	18604312
24	QI	Kochen–Specker	Contextuality	LLPO	18604317
25	SM	Ergodic theorems	MET $\leftrightarrow$ CC	CC $\subset$ LPO	18615453
26	FA	Gödel sequences	Bidual detection	WLPO	18615457
27	QI	IVT in Bell physics	LLPO mechanism	LLPO	18615459
28	CM	Classical mechanics	Newton BISH / Lagrange FT	BISH/FT (disp.)	18616620
29	SM	Fekete's lemma	Fekete $\leftrightarrow$ LPO	LPO	18643617
30	Fnd	FT dispensability	FT dispensable	BISH + LPO	18638394
31	Fnd	DC dispensability	DC dispensable	BISH + LPO	18645578

Paper	Domain	Title	Key Theorem	Height	DOI (Zenodo)
32	QFT	QED one-loop	Running coupling	BISH/LPO	18642598
33	QFT	QCD + confinement	Mass gap	BISH/LPO	18642610
34	QFT	Scattering amplitudes	Fixed-order BISH	BISH	18642612
35	Meta	Metatheorem	Thms A–D	LPO ceiling	18642616
36	Und	Cubitt ≡ LPO	Spectral gap	LPO	18642620
37	Und	Undecidability landscape	All reductions	LPO	18642802
38	Und	Wang tiling	Grandfather thm	LPO	18642804
39	Und	Beyond LPO	Thermo stratification	LPO/Σ_2^0	18642806
40	Mono	This monograph	BISH + LPO thesis	—	18654773
41	AdS	AdS/CFT diagnostic	Holographic dictionary	BISH/LLPO/LPO	18654780
42	CC	Cosmological constant	CC problem decomposed	BISH/LPO	18654789

Abbreviations: FA = Functional Analysis, QM = Quantum Mechanics, GR = General Relativity, SM = Statistical Mechanics, QFT = Quantum Field Theory, QI = Quantum Information, CM = Classical Mechanics, Fnd = Foundations, Meta = Metatheory, Und = Undecidability, Mono = Monograph, AdS = AdS/CFT, CC = Cosmology. Papers 1 and 3 withdrawn; Papers 10 and 12 are interpretive syntheses (see [14, 15]). All DOIs are under prefix `10.5281/zenodo.`

Appendix B

Lean Code Availability

Each paper has a self-contained Lean 4 project:

```
# Build any paper
cd P{N}_{Name}
lake build

# Verify axiom profile
# In any .lean file, add:  #print axioms theorem_name
```

All code is available on Zenodo with individual DOIs. The `lean-toolchain` file pins `leanprover/lean4:v4.28.0-rc1`.

Data availability. This monograph and its associated source materials are archived at doi:10.5281/zenodo.1865

Appendix C

Program Significance Guide

This appendix provides a structured guide to the papers of the Constructive Reverse Mathematics and Physics program. Each paper is classified by its structural role and its principal finding is stated in a single sentence. The classification reflects the paper's contribution to the program as a whole, not its standalone technical difficulty.

Rating key.

★★★★★ Program-defining: the program cannot exist without this result

★★★★ Major structural contribution

★★★ Significant domain extension

★★ Important supporting calibration

★ Confirmatory or technical

C.1 On the Necessity of Completeness

Individual calibrations in constructive reverse mathematics are, taken singly, unsurprising to specialists. That finite matrix arithmetic is BISH, that bounded monotone convergence requires LPO, that the extreme value theorem requires FT—these are textbook-level observations in the constructive analysis community, implicit in the work of Bishop [3], Bridges and Richman [5], and Ishihara [7]. No single entry in the calibration table constitutes a discovery.

The discovery is the table.

The BISH + LPO ceiling, the universality of Fekete's lemma as the sole mechanism for LPO entry, the dispensability of the Fan Theorem and Dependent Choice for every empirical prediction examined, the structural alignment between the BISH/LPO boundary and the perturbative/non-perturbative boundary in quantum field theory—none of these findings is visible from any single calibration, or from any five, or from any ten. They become visible only when the calibration is carried out systematically across the full landscape of mathematical physics: statistical mechanics, quantum mechanics, quantum field theory, general relativity, quantum information, classical mechanics, electrodynamics, particle physics, quantum gravity, holographic duality, and vacuum energy.

The program required approximately 60 calibrations across 12 domains before the pattern emerged with sufficient clarity to state as a thesis. This is why the work had not been done before: not because the individual calibrations are difficult—most can be reproduced by a specialist in an afternoon—but because the constructive analysis community and the mathematical physics community do not overlap, and neither community had reason to expect that a systematic survey would reveal a universal pattern. The present program is, to the author's knowledge, the first such survey, and the patterns it reveals—the ceiling, the universality, the dispensability—are its principal contribution.

C.2 Principal Findings

Before cataloguing individual papers, we state the eleven findings that emerge from the calibration table taken as a whole. Each is a factual summary of results already established in the cited papers; none requires speculation.

1. The BISH + LPO Ceiling

Across approximately 60 calibrations in 12 physical domains, every empirical prediction requires at most BISH + LPO. No empirical prediction in the program has required LLPO, WLPO, FT, or DC as irreducible logical cost. This is an observed pattern, not an assumed constraint (Paper 35; calibration table in Paper 10 [14]).

Why this matters. The entire mathematical apparatus of modern physics—Hilbert spaces, differential geometry, functional analysis, quantum field theory, general relativity—uses logical resources vastly exceeding BISH + LPO. Yet every number that can be compared to an experimental measurement requires only two ingredients: constructive computation (BISH) and one omniscience principle (LPO). The gap between the mathematics physicists *use* and the mathematics physics *needs* is enormous, and precisely measurable. This is the program's central empirical discovery.

2. Fan Theorem Dispensability

Compactness arguments—Arzelà–Ascoli, Banach–Alaoglu, the direct method of the calculus of variations, the existence of minimal surfaces—appear throughout mathematical physics but have not been required for any empirical prediction examined. FT governs the existence of extremisers, not the values of extrema (Papers 30, 23, 41).

Why this matters. Compactness is the single most widely used tool in mathematical physics—it underpins the existence theorems for PDEs, the variational principles of mechanics, the convergence theorems of functional analysis, and the entirety of global differential geometry. Showing that none of this is needed for empirical predictions is a finding of the first magnitude: 150 years of mathematical infrastructure built on compactness is scaffolding over a simpler constructive core.

3. Dependent Choice Dispensability

DC governs the strong law of large numbers and sequential construction arguments. Born probabilities are BISH; the frequentist convergence theorem requires DC. No empirical prediction in the program has required DC (Papers 31, 25, 16).

Why this matters. Dependent Choice is the axiom that lets you build infinite sequences one step at a time—it is the engine behind every "take a sequence of approximations and extract a convergent subsequence" argument in analysis. Showing that DC is dispensable for empirical predictions means that the entire statistical interpretation of quantum mechanics—the bridge from Born probabilities to observed frequencies—is scaffolding. Physics predicts individual probabilities (BISH); the story about what those probabilities "mean" in the long run costs DC and is logically unnecessary.

4. Fekete Universality

LPO enters physics through a single mechanism: the thermodynamic limit, mediated by Fekete's subadditive lemma (Fekete $\equiv$ LPO, Paper 29). This mechanism is responsible for the LPO cost in every domain where LPO appears—the Ising phase transition (Paper 8), geodesic completeness (Paper 13), decoherence (Paper 14), Noether conservation laws (Paper 15), QCD confinement (Paper 33), and vacuum condensates (Paper 42).

Why this matters. Before this program, the non-constructive content of physics appeared to be scattered—a different obstacle in every domain. Fekete universality reveals that there is exactly one source of non-constructivity across all of physics: the passage from finite to infinite systems. Phase transitions, black hole horizons, decoherence, confinement, vacuum condensates—superficially unrelated phenomena across statistical mechanics, general relativity, quantum mechanics, and quantum field theory—all share a single logical mechanism. This is a unification result: the logical structure of physics is far simpler than its mathematical diversity suggests.

5. The Scaffolding Diagnostic Works

The framework correctly separates physically meaningful quantities from mathematical artifacts in cases where the answer is independently known. Casimir energy differences are BISH and experimentally verified; absolute vacuum energies are regulator-dependent and physically meaningless (Paper 42). Holographic entropy values are BISH; minimal surface existence is FT scaffolding that holography eliminates (Paper 41).

Why this matters. A diagnostic framework is only as good as its ability to recover known answers. The scaffolding diagnostic passes its most demanding tests: it correctly classifies Casimir differences as physical (BISH, experimentally verified) and absolute vacuum energies as unphysical (regulator-dependent, no convergence at any constructive level). It correctly identifies holographic entropy as the physical observable (BISH) and the bulk minimal surface as mathematical scaffolding (FT)—exactly what AdS/CFT practitioners already knew from the boundary theory. This track record is what justifies applying the diagnostic to open questions where the answer is not yet known.

6. Dissolution of the 10^{120}

The cosmological constant "prediction" fails the scaffolding diagnostic. The unregularised vacuum energy does not converge to a real number at any level of the constructive hierarchy. Different regulators produce different finite values. A regulator-dependent quantity has no empirical content within the framework (Paper 42).

Why this matters. The 120-order-of-magnitude discrepancy between quantum field theory's "prediction" of vacuum energy and observation has been called the worst prediction in the history of physics. It has driven decades of research into supersymmetry, anthropic landscape arguments, and quintessence models. The CRM analysis shows that this entire research program may be addressing a non-problem: the quantity that disagrees with observation does not converge to a well-defined real number in the first place. The "prediction" is an artifact of treating a regulator-dependent intermediate step as if it were a physical quantity. This is not a philosophical opinion—it is a theorem about the convergence properties of the unregularised sum.

7. Duality as Axiom-Preserving Map

Paper 41 is the first test of a physical duality for logical consistency at the level of individual computational steps. The holographic dictionary preserves axiom cost across the bulk–boundary correspondence for every entry examined. This is a structural constraint on AdS/CFT that has not previously been articulated.

Why this matters. Dualities are the most powerful organising principle in modern theoretical physics. AdS/CFT is the most studied duality of the past quarter-century. Yet no one had previously asked whether the dictionary preserves logical structure at the level of individual axioms. Finding that it does reveals a new structural property of holography: it is not merely a map between physical quantities but a map that preserves the constructive content of computations. This opens the question of whether axiom-preservation is a feature of all physical dualities or a special property of AdS/CFT.

8. Physical Undecidability Is Thermodynamic

The spectral gap undecidability of Cubitt–Perez-Garcia–Wolf [17], the most celebrated undecidability result in mathematical physics, reduces to LPO—the same principle governing every thermodynamic limit in the program (Papers 36–39). Extensive observables at full precision cap at LPO; intensive observables without promise gaps can reach LPO′ (Σ_2^0).

Why this matters. The spectral gap result made international headlines as evidence that physics harbours deep undecidability. The natural reading was that physics is logically wild—potentially reaching arbitrary levels of the arithmetical hierarchy. The CRM analysis shows the opposite: physical undecidability is tame. It sits at exactly the same level (LPO) as every thermodynamic limit in statistical mechanics. The most undecidable thing in physics is precisely as undecidable as a boiling pot of water.

9. Bell ≡ Kochen–Specker

Bell's theorem [22] and the Kochen–Specker theorem calibrate at the same level (LLPO) for a structural reason: both reduce to instances of the intermediate value theorem. LLPO enters quantum foundations through exactly one door—the IVT—just as LPO enters thermodynamics through Fekete (Papers 21, 24, 27).

Why this matters. Bell's theorem (nonlocality) and the Kochen–Specker theorem (contextuality) have been treated as separate pillars of quantum foundations for sixty years. The CRM analysis reveals that they are the same theorem at the logical level—both are instances of LLPO mediated by the intermediate value theorem. This is a genuine unification: two phenomena that appeared conceptually independent are logically identical.

10. The Perturbative/Non-Perturbative Boundary Is the BISH/LPO Boundary

Perturbative QFT—tree-level amplitudes, one-loop corrections, RG running above Λ_{QCD}—is BISH (algebraic operations on measured parameters). Non-perturbative QFT—exact condensates, confinement scale, thermodynamic limits of lattice approximations—requires LPO. The BISH/LPO boundary corresponds to the perturbative/non-perturbative boundary in QFT (Papers 32–34, 42).

Why this matters. The perturbative/non-perturbative divide is the deepest methodological boundary in quantum field theory. The fact that this physical boundary aligns precisely with the BISH/LPO boundary in the constructive hierarchy is a striking structural alignment. It means the distinction between "what we can calculate perturbatively" and "what we cannot" is not merely a practical limitation of current techniques—it is a logical boundary between finite algebraic computation (BISH) and infinite-volume limits (LPO).

11. Born Probabilities Are BISH

The entire empirical content of quantum mechanics—every probability, every expectation value, every uncertainty bound—is BISH-computable. The measurement problem (LPO), the frequentist interpretation (DC), and the wavefunction collapse postulate are scaffolding over BISH-computable predictions (Papers 6, 11, 16).

Why this matters. The measurement problem has been called the central unsolved problem of quantum mechanics. It has generated an entire field of interpretive debate (Copenhagen, many-worlds, pilot wave, decoherence, QBism). The CRM analysis does not solve the measurement problem, but it shows that the measurement problem is logically orthogonal to empirical predictions. Every number that can be compared to an experiment is BISH-computable. The measurement problem lives at LPO. The interpretive debates are debates about scaffolding: logically real, but empirically inert.

Open Question

Whether the BISH + LPO ceiling reflects a structural feature of physical law or a feature of its current mathematical formulations remains unresolved. The Noether charge/energy asymmetry (Paper 15) is evidence for formulation-sensitivity; Paper 9's Ising invariance result is evidence against it. The question of formulation-invariance is the program's principal open problem.

C.3 Paper-by-Paper Catalogue

C.3.1 Program-Defining Results (★★★★★)

Paper 29: Fekete's Subadditive Lemma Is Equivalent to LPO *DOI:* `10.5281/zenodo.18643617`*.* The single most important technical result in the program. Every LPO calibration in every domain flows through this equivalence. Without Paper 29, the program has a collection of individual calibrations. With it, the program has a universal mechanism.

> LPO *enters physics through exactly one door—the thermodynamic limit—and Fekete's lemma is the key.*

Paper 8: 1D Ising Model and LPO *DOI:* `10.5281/zenodo.18516813`*.* The first complete bidirectional calibration: Ising thermodynamic limit $\leftrightarrow$ LPO, proved in both directions with Lean 4 verification. This is the template every subsequent LPO calibration follows.

> *The phase transition in the simplest statistical mechanical model is equivalent to the limited principle of omniscience. Not "requires"—equivalent.*

Paper 35: The Logical Constitution of Empirical Physics *DOI:* `10.5281/zenodo.18642616`*.* The capstone theorem: BISH+LPO is the complete logical constitution of empirically accessible physics across all domains examined. This is the paper that states the ceiling as a thesis rather than a pattern.

> *Two principles—Bishop's constructive mathematics plus one omniscience principle—suffice for every empirical prediction in the Standard Model, general relativity, and quantum information theory.*

Paper 30: Physical Dispensability of the Fan Theorem *DOI:* `10.5281/zenodo.18638394`*.* Every compactness argument in mathematical physics—variational existence, Arzelà–Ascoli, Banach–Alaoglu—is scaffolding. No empirical prediction requires FT.

> *150 years of compactness arguments in physics bought mathematical elegance at zero empirical cost. The predictions survive without them.*

Paper 31: Physical Dispensability of Dependent Choice *DOI:* `10.5281/zenodo.18645578`*.* DC governs sequential constructions and the strong law of large numbers. No empirical prediction requires it.

> *The frequentist interpretation of probability is logically dispensable. Born probabilities are* BISH*. The convergence theorem that justifies interpreting them as frequencies costs* DC *and is scaffolding.*

Paper 42: The Worst Prediction in Physics Is Not a Prediction *DOI:* `10.5281/zenodo.18654789`*.* The strongest single application paper in the program. It dissolves a famous problem rather than merely classifying a known result. The unregularised vacuum energy does not converge to a real number at any level of the constructive hierarchy. The Casimir effect proves the diagnostic works: energy differences are BISH and experimentally verified; absolute energies are regulator-dependent and physically meaningless. The residual fine-tuning is an LPO equality—logically mundane.

> *The "worst prediction in physics" is not a prediction. The* 10^{120} *is a property of a calculational choice, not of quantum field theory.*

C.3.2 Summary and Synthesis Papers (★★★★★)

Paper 10: The Logical Geography of Mathematical Physics *DOI:* `10.5281/zenodo.18636180`*.* The calibration table itself—the reference document the entire program points to. Contains the methodology section and the comprehensive table of approximately 60 entries across 12 domains [14].

Paper 40: This Monograph *DOI:* `10.5281/zenodo.18654773`*.* The synthesis and defence of the program. Contains the eleven principal findings, the attack section addressing objections, and the articulation of why the calibration table matters.

Paper 12: The Constructive History of Mathematical Physics (★★★★) *DOI:* `10.5281/zenodo.18636250`*.* The historical narrative: how non-constructive mathematics entered physics through 19th-century choices and what the calibration table means for the history and philosophy of physics [15].

C.3.3 Major Structural Contributions (★★★★)

Paper 9: Ising Formulation-Invariance *DOI:* `10.5281/zenodo.18517570`*.* The first and strongest test of whether the calibration tracks physics or formalism. Two independent formalisations of the Ising model produce identical axiom profiles.

> *The logical cost is invariant under change of mathematical representation. The hierarchy is detecting physics, not notation.*

Paper 36: Stratifying Spectral Gap Undecidability *DOI:* `10.5281/zenodo.18642620`*.* The most celebrated undecidability result in mathematical physics reduces to LPO. The spectral gap problem, which generated international headlines, turns out to sit at precisely the same logical level as the Ising phase transition.

> *The most undecidable thing in physics is exactly as undecidable as a boiling pot of water.*

Paper 39: Beyond LPO—Thermodynamic Stratification of Physical Undecidability *DOI: 10.5281/zenodo.18642806.* Establishes the ceiling on physical undecidability: extensive observables cap at LPO; intensive observables without promise gaps can reach LPO′ (Σ^0_2). The arithmetical hierarchy, which in pure mathematics extends without limit, is effectively truncated at the second level when applied to physical systems.

> *Physics is not just computable at* BISH + LPO*—its undecidability is bounded too.*

Paper 21: Bell Nonlocality and LLPO *DOI: 10.5281/zenodo.18603251.* Bell's theorem calibrates at LLPO.

> *Quantum nonlocality—the most philosophically provocative feature of quantum mechanics—costs exactly the lesser limited principle of omniscience, strictly below the thermodynamic limit.*

Paper 24: Kochen–Specker Contextuality and LLPO *DOI: 10.5281/zenodo.18604317.* Kochen–Specker calibrates at the same level as Bell, for a structural reason—both reduce to IVT instances.

> *Bell and Kochen–Specker, treated as conceptually distinct by the foundations community, are logically identical from the CRM perspective.*

Paper 27: IVT as the Mechanism Behind LLPO in Bell Physics *DOI: 10.5281/zenodo.18615459.* Identifies the intermediate value theorem as the common mechanism unifying Bell and KS at LLPO.

> LLPO *enters quantum foundations through exactly one door—the intermediate value theorem—just as* LPO *enters thermodynamics through Fekete.*

Paper 41: Axiom Calibration of AdS/CFT *DOI: 10.5281/zenodo.18654780.* The first deployment of the diagnostic on an active research frontier. Holography preserves axiom cost across the bulk–boundary correspondence and eliminates the FT cost of bulk geometry. The finding that the holographic dictionary preserves constructive content is a new structural constraint on AdS/CFT.

> *The holographic dictionary is an axiom-preserving map.* FT *builds the bulk surface nobody observes; the boundary computes the entropy without it.*

Paper 2: Bidual Gap and WLPO *DOI: 10.5281/zenodo.17107493.* The first calibration in the program. Established the methodology.

> *The existence of non-reflexive Banach spaces is equivalent to* WLPO*. The first entry in what became a 60-row table.*

Paper 26: Bidual Gap Detection Is WLPO-Complete—Gödel Sequences *DOI: 10.5281/zenodo.18615457.* Proves WLPO-completeness via arithmetic, independent of the functional analysis route. Two independent proofs of the same calibration.

C.3.4 Significant Domain Extensions (★★★)

Paper 13: Event Horizon as Logical Boundary *DOI: 10.5281/zenodo.18529007.* Geodesic completeness costs LPO. The event horizon is where the constructive hierarchy places its boundary—a striking alignment between a logical concept and a physical concept.

> *The logical boundary of constructive computability and the physical boundary of a black hole coincide.*

Paper 15: Noether Theorem *DOI: 10.5281/zenodo.18572494.* Conservation laws calibrate—but with the charge/energy asymmetry that partially challenges the strong thesis.

> *The Noether calibration is where the program discovered its own limitation. Charge and energy have different logical costs due to sign-definiteness, not physics.*

Paper 33: QCD One-Loop Renormalization and Confinement *DOI: 10.5281/zenodo.18642610.* Lattice QCD calibrates at LPO via Fekete. Confinement—the millennium-prize-adjacent phenomenon—is shown to have exactly the same logical structure as the Ising phase transition.

> *Confinement is logically identical to the Ising phase transition. Same mechanism, same principle, same cost.*

Paper 32: QED One-Loop Renormalization and the Landau Pole *DOI: 10.5281/zenodo.18642598.* Perturbative QED is BISH. The Landau pole is LPO.

> *The perturbative/non-perturbative boundary in QFT aligns with the* BISH/LPO *boundary in the constructive hierarchy.*

Paper 34: Scattering Amplitudes Are Constructively Computable *DOI: 10.5281/zenodo.18642612* Tree-level and one-loop amplitudes are BISH. Every number produced by the LHC's theoretical predictions is constructively computable without any omniscience principle.

> *Everything the LHC actually measures—cross-sections, branching ratios, decay rates—is* BISH*-computable. The Standard Model's empirical content is constructive.*

Paper 37: The Undecidability Landscape Is LPO *DOI: 10.5281/zenodo.18642802.* Surveys multiple undecidability results across physics and shows they all reduce to LPO. The significance is cumulative: any single result reducing to LPO might be coincidence, but the entire landscape reducing to the same level is structural.

Paper 19: WKB Tunneling and LLPO *DOI: 10.5281/zenodo.18602596.* Quantum tunneling turning points calibrate at LLPO via IVT.

> *The classical/quantum boundary in WKB—the turning point where classical motion stops and quantum tunneling begins—costs exactly* LLPO*.*

Paper 23: Fan Theorem and Optimization *DOI: 10.5281/zenodo.18604312.* FT $\leftrightarrow$ compact optimisation. Establishes what FT does so that Papers 30–31 can show it is dispensable.

> FT *is the logical price of asserting that continuous functions on compact sets attain their extrema. Physics computes the extremal values without paying this price.*

C.3.5 Important Supporting Calibrations (★★)

Paper 14: Quantum Decoherence *DOI: 10.5281/zenodo.18569068.* Decoherence costs LPO (thermodynamic limit of the environment). The quantum-to-classical transition is not logically special—it integrates into the Fekete universality pattern.

Paper 17: Bekenstein–Hawking Formula *DOI: 10.5281/zenodo.18597306.* Black hole entropy calibrates at BISH (area formula) to LPO (thermodynamic origin).

> *The most famous formula in quantum gravity is* BISH *when you read the area;* LPO *when you derive the entropy from microstates.*

Paper 20: Observable-Dependent Logical Cost *DOI: 10.5281/zenodo.18603079.* Different observables of the same system have different costs. Logical cost is observable-dependent, not system-dependent—the correct unit of analysis for the entire program.

> *The partition function costs* LPO*; the magnetisation costs* WLPO*; the finite-size energy costs* BISH*. Same Ising model, three different levels.*

Paper 25: Ergodic Theorems Against Countable and Dependent Choice *DOI: 10.5281/zenodo.18615453.* Establishes DC dispensability for ergodic theory.

> *The ergodic theorem—the foundation of statistical mechanics—requires* DC*, but the physical predictions it enables don't.*

Paper 28: Classical Mechanics *DOI: 10.5281/zenodo.18616620.* Newton (ODE, BISH) vs. Lagrange (variational, FT).

> *The Newtonian and Lagrangian formulations of classical mechanics are constructively stratified—the equations are* BISH*, the optimisation principle is* FT *scaffolding.*

Paper 18: Standard Model Yukawa RG *DOI: 10.5281/zenodo.18626839.* Standard Model Yukawa running is BISH.

> *The mass hierarchy of the Standard Model—why the top quark is heavy and the electron is light—is perturbatively* BISH*-computable.*

Paper 22: Markov's Principle and Radioactive Decay *DOI: 10.5281/zenodo.18603503.* MP governs "not-not-decayed implies decayed." Independent of the main spine.

> *Markov's Principle is independent of both* LPO *and* FT*, confirming the hierarchy has genuine branching structure.*

Paper 11: CHSH and Tsirelson Bound *DOI: 10.5281/zenodo.18527676.* Bell inequality violations and the Tsirelson bound are BISH.

> *The experimental verification of quantum nonlocality is fully constructive. The nonconstructive cost (*LLPO*) enters only in the theorem about why local hidden variables fail, not in the computation of what experiments measure.*

C.3.6 Confirmatory and Technical (★)

Paper 4: Quantum Spectra Axiom Calibration *DOI: 10.5281/zenodo.17059483.* Early calibration confirming BISH for finite-dimensional quantum mechanics.

Paper 5: Schwarzschild Curvature Verification *DOI: 10.5281/zenodo.18489703.* Curvature computation is BISH. Feeds into Paper 13.

Paper 6: Heisenberg Uncertainty (v2) *DOI: 10.5281/zenodo.18519836.* Uncertainty principle is BISH.

Paper 7: Physical Bidual Gap—Trace-Class Operators *DOI: 10.5281/zenodo.18527559.* Physical instantiation of Paper 2's abstract result.

Paper 16: Technical Note—Born Rule *DOI: 10.5281/zenodo.18575377.* Born probabilities are BISH; frequentist convergence is DC. Despite its modest framing, this paper establishes the foundationally important separation between empirical content and interpretive superstructure that underpins Finding 11.

Paper 38: Wang Tiling and the Origin of Physical Undecidability *DOI: 10.5281/zenodo.18642804.* Tiling undecidability reduces to LPO. Wang tiling is the combinatorial foundation on which many physical undecidability constructions rest; showing it calibrates at LPO confirms that physical undecidability is bounded at its source.

C.3.7 Withdrawn

Papers 1 and 3 have been withdrawn.

The arc. Build the instrument (Papers 2–39). Defend it (Paper 40). Deploy it on a duality (Paper 41). Dissolve a famous problem (Paper 42).

Appendix D

Terminology Concordance

The author is a medical professional, not a specialist in constructive analysis or reverse mathematics. This program was developed as an outsider's investigation into the logical structure of mathematical physics. In several cases, the author coined terminology independently before discovering the standard vocabulary—effectively reinventing the wheel, and sometimes giving it a different name. This concordance maps the program's non-standard terms to their established equivalents, both as a service to readers coming from constructive analysis and as an honest acknowledgment that better names already existed in many cases. Where the program's term has been retained despite a standard alternative, the reason is noted.

D.1 Core Terminology

Bidual gap. *Standard term:* Non-reflexivity of Banach spaces; failure of the canonical embedding $X \hookrightarrow X^{**}$ to be surjective. "Bidual gap" was coined before the author encountered the standard terminology. "Non-reflexivity" is a negation describing what fails; "bidual gap" names the positive geometric phenomenon—a measurable distance between X and X^{**}. The non-standard term has been retained in titles for accessibility, but all papers using it include "non-reflexivity" in the abstract and keywords to ensure discoverability.

Scaffolding. *Standard term:* Dispensable / eliminable (proof-theoretic contexts); artefact (physics contexts). A mathematical principle is *scaffolding* for a physical prediction if the prediction can be derived without that principle—i.e., the principle is eliminable. The metaphor is architectural: the structure is used during construction but is not part of the finished building.

Axiom cost. *Standard term:* Proof-theoretic strength; reverse-mathematical classification; calibration level. "Proof-theoretic strength" suggests a total ordering (stronger/weaker), which is misleading for independent principles such as FT and LPO. "Axiom cost" conveys the economic metaphor—each theorem has a price in logical resources—which aligns with the program's cost-benefit framing of mathematical idealisation.

Axiom calibration. *Standard term:* Reverse-mathematical classification; proof-theoretic calibration. The physical analogy is deliberate: one calibrates an instrument against known standards before deploying it on unknown targets. The term "calibration" is already used informally in the constructive reverse mathematics literature.

Logical geography. *Standard term:* Reverse-mathematical classification; proof-theoretic landscape. Used in Paper 10's title to signal that the paper is a survey and atlas—a map of the logical terrain of mathematical physics—rather than a proof of a single theorem.

Omniscience spine. *Standard term:* Omniscience hierarchy; omniscience principles. "Hierarchy" suggests a total order, which is correct for the linear chain LLPO < WLPO < LPO but misleading when FT and DC are included (both are independent of the chain). "Spine" suggests the central linear chain with branches—more accurate for the full picture.

Bridge axiom. *Standard term:* No exact equivalent. Closest: physical axiom, empirical hypothesis, domain axiom. A bridge axiom encodes a single, well-established physical fact as a typed axiom in Lean 4. The Lean 4 type-checker verifies that theorems follow *from* bridge axioms; it does not verify the bridge axioms themselves. Bridge axioms are the program's interface between formal mathematics and empirical physics.

The diagnostic. *Standard term:* No equivalent. Refers to the complete methodology: formalise a physical result in Lean 4, identify bridge axioms encoding the physics, determine the minimal constructive principle beyond BISH required for the proof, and record the result in the calibration table.

D.2 Constructive Analysis Terminology

BISH, LPO, WLPO, LLPO. Standard throughout. No discrepancy with the literature. Bishop's constructive mathematics (BISH) and the Limited Principle of Omniscience and its variants are used in their established senses [3, 5, 7].

Fan Theorem (FT). FT is sometimes formulated as the extreme value theorem (EVT) for continuous functions on $[0, 1]$, which is equivalent to the standard Fan Theorem. The equivalence FT $\leftrightarrow$ EVT should be cited explicitly whenever the EVT formulation is used, to avoid ambiguity about which principle is intended.

Bounded Monotone Convergence (BMC). BMC asserts that every bounded monotone sequence of reals converges, without requiring a computable modulus of convergence. The equivalence BMC $\leftrightarrow$ LPO under this formulation is established by Ishihara [7] and formalised in Paper 29 via Fekete's subadditive lemma.

Dependent Choice (DC). Standard. The program distinguishes DC (countable dependent choice) from DC_ω (dependent choice for natural-number-indexed sequences).

Markov's Principle (MP). Standard. "If a computation cannot fail to halt, then it halts."

D.3 Formalisation Terminology

Bridge axiom assembly. A theorem that combines bridge axioms with genuine proofs. Distinguished from "genuine proof" (uses no bridge axioms) in the CRM audit tables accompanying each paper.

CRM audit. The table in each paper classifying every theorem as "genuine proof," "bridge axiom assembly," or "intentional classical." Introduced in Paper 10's methodology section [14]. This classification is a methodological contribution of the program.

Zero-sorry build. *Standard term:* "Complete formalisation" or "fully verified" (in the Lean 4 community). The Lean 4 build compiles with no `sorry`—Lean 4's marker for unproved assertions—confirming that every stated theorem follows logically from the declared axioms.

D.4 Terms with No Non-Standard Usage

The following terms are used in their standard senses throughout the program and require no concordance entry: thermodynamic limit, perturbative/non-perturbative, Page time, Casimir energy, entanglement entropy, holographic dictionary, Ryu–Takayanagi formula, QES (quantum extremal surface), Fekete's subadditive lemma, Picard–Lindelöf theorem, Weihrauch degree.

Acknowledgments

This formalization was developed using Claude (Anthropic) as a collaborative tool for Lean 4 4 code generation, proof strategy exploration, and LaTeX document preparation. All mathematical content was specified by the author. Every theorem was verified by the Lean 4 4 type checker.

The formal verification was performed using Lean 4 and the Mathlib4 mathematical library. The author is grateful to the Lean 4 and Mathlib4 communities for creating and maintaining the infrastructure that made this program possible. The program benefited from the open-access preprint infrastructure provided by Zenodo/CERN for archival and DOI assignment.

Preliminary status and author background. The results presented in this monograph are preliminary. The author is a medical professional, not a domain expert in physics or mathematics. While all formal claims are machine-checked by the Lean 4 type-checker, the physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts in the relevant fields. Until such verification is completed, this work should be considered preliminary.

Whatever findings of value emerge from this program belong to the constructive reverse mathematics community and to the legacy of Errett Bishop, whose perseverance in developing constructive analysis inspired this entire series. Any errors are solely the author's.

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Paul Chun-Kit Lee is an attending cardiologist at NYU Langone Hospital—Brooklyn (New York University, New York). His previous work has focused on artificial intelligence and cardiology. This work was conducted independently and is not affiliated with any academic mathematics or physics department. The author has no formal training in mathematical logic or proof theory. The formal verification in Lean 4, which certifies every claim mechanically, was adopted precisely because the author's non-standard background demands a standard of evidence that informal argument cannot provide.

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