Algebraic Spectrum Is Necessary for Constructive Eigenvalue Decidability

(Paper 74, Constructive Reverse Mathematics Series)

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Abstract

We prove the reverse characterization of DPT Axiom 2: algebraic spectrum (geometric origin) is not merely sufficient but *necessary* for BISH-decidable eigenvalue comparison in a spectral category. Without geometric origin, the Langlands spectrum includes continuous parameters (Maass forms, unramified characters over $\mathbb{C}$); testing whether a continuous spectral parameter satisfies the Ramanujan bound is a real-number equality test, encoding WLPO. Combined with the forward direction (Paper 45, Deligne), this gives a biconditional: algebraic spectrum $\Leftrightarrow$ BISH eigenvalue decidability. This completes the DPT axiom trilogy: Paper 72 (Axiom 3 $\Leftrightarrow$ BISH cycle-search, cost without: LPO), Paper 73 (Axiom 1 $\Leftrightarrow$ BISH morphisms, cost without: LPO), Paper 74 (Axiom 2 $\Leftrightarrow$ BISH eigenvalues, cost without: WLPO). The WLPO asymmetry—not LPO—reflects the intrinsic computational difference between comparing a spectral value (equality test) and searching a geometric structure (existential search). Lean 4 formalization: $\sim$200 lines, zero `sorry`.

1 Introduction

Paper 72 of this series established three results about the DPT axiom system (Paper 50). First, each axiom is independently necessary (Theorem A, minimality). Second, Axiom 3 (Archimedean polarization) is both necessary and sufficient for BISH cycle-search (Theorem B, biconditional). Third, the Archimedean Principle is an equivalence, not merely a forward implication (Theorem C). Paper 73 carried out the analogous program for Axiom 1 (Standard Conjecture D $\Leftrightarrow$ BISH morphism decidability). The present paper completes the trilogy for Axiom 2.

Main results.

Theorem A (*Forward.*) With algebraic spectrum (geometric origin), eigenvalue comparison is BISH-decidable. Frobenius eigenvalues α are algebraic integers; testing $|\alpha| = q^{w/2}$ reduces to exact algebraic arithmetic. This is the content of Paper 45, reviewed here for completeness.

Theorem B (*Eigenvalue-Decidability Equivalence.*) For eigenvalue comparison in a spectral category:

$$\text{eigenvalue\_cost}(s) = \text{BISH} \iff s = \text{algebraic} \iff \text{Axiom 2 holds.}$$

Forward: algebraic spectrum ⇒ BISH. Reverse: BISH ⇒ algebraic spectrum (contrapositive: without Axiom 2, eigenvalue cost is WLPO).

Theorem C (*Axiom 2 Characterization.*) Algebraic spectrum (geometric origin) is the minimal and unique condition for BISH-decidable eigenvalue comparison. The Ramanujan resolution—proving the conjecture would make the analytic spectrum effectively algebraic—confirms the trade-off is real, not vacuous.

Why WLPO **and not** LPO**.** The cost of violating Axiom 2 is WLPO, not LPO. This is not an accident but reflects an intrinsic computational distinction. LPO decides an existential search: given a binary sequence (a_n), either $\exists n\,(a_n = 1)$ or $\forall n\,(a_n = 0)$. This is the structure behind cycle-search (Axiom 3) and radical detection (Axiom 1), where one searches through a finite-dimensional space for a witness. WLPO decides a single equality test: given a binary sequence (a_n), either $(a_n) = \bar{0}$ or $(a_n) \neq \bar{0}$, without producing a witness. Eigenvalue comparison—does this continuous spectral parameter equal this algebraic value?—is a single equality test, not a search.

In the CRM hierarchy, WLPO sits strictly below LPO: LPO implies WLPO but not conversely. The DPT axiom trio therefore stratifies the constructive cost across two distinct levels.

1.1 Three fatal flaws in the naive framing

One might attempt to frame Axiom 2's failure as "eigenvalues become transcendental." This framing suffers three fatal defects.

Flaw 1: Weil II (Deligne 1980). Deligne's Weil II theorem [4] proves that for every separated scheme of finite type over $\mathbb{F}_q$, the eigenvalues of the Frobenius endomorphism acting on ℓ-adic cohomology are algebraic numbers. More precisely, if X is a variety over $\mathbb{F}_q$ and α is an eigenvalue of Frob_q acting on $H^i_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, then α is an algebraic integer with $|\iota(\alpha)| = q^{w/2}$ for every embedding $\iota\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and some weight $w \leq i$. There are no transcendental Frobenius eigenvalues in the classical motivic setting. The "naive" framing contradicts established arithmetic geometry.

Flaw 2: The linear algebra vacuum. Axiom 1 (Standard Conjecture D) provides that morphisms between motives are finite-dimensional $\mathbb{Q}$-vector spaces. Any endomorphism $\phi\colon M \to M$ of a motive therefore satisfies a minimal polynomial $p(x) \in \mathbb{Q}[x]$ of degree at most $\dim_{\mathbb{Q}} \mathrm{End}(M)$. The eigenvalues of ϕ are roots of $p(x)$, hence algebraic over $\mathbb{Q}$. Consequently, within the DPT framework where all three axioms are in play, algebraic spectrum is already forced by Axiom 1 alone. One cannot isolate the failure of Axiom 2 while keeping Axiom 1 intact—the axioms interact through the linear algebra of endomorphism rings.

Flaw 3: The ℓ-adic CLASS **trap.** Even granting the existence of non-algebraic elements in $\bar{\mathbb{Q}}_\ell$, comparing such elements with complex numbers requires an embedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. No canonical such embedding exists; constructing one requires the Axiom of Choice (specifically, extending a field isomorphism from $\bar{\mathbb{Q}}$ to a transcendence basis). This lands in CLASS, not WLPO. The naive framing would assign a cost of CLASS to Axiom 2 failure, which is both too expensive and mathematically misleading: it conflates the choice principle needed for ℓ-adic embeddings with the constructive content of eigenvalue comparison.

The correct framing. Axiom 2 fails when the category extends beyond geometric origin to the full analytic Langlands spectrum.[1] The spectral parameters of Maass forms and unramified characters live in $\mathbb{C}$ natively—they are archimedean analytic objects, not ℓ-adic algebraic objects. No embedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ is involved. The comparison $\mathrm{Re}(s) = \frac{1}{2}$ is a real-number equality test, which is exactly WLPO. This is why the correct cost is WLPO and not CLASS: the analytic Langlands spectrum avoids the ℓ-adic CLASS trap by working with archimedean parameters from the start.

Atlas position. Paper 74 sits between four earlier results: Paper 45 (Weil eigenvalue CRM: algebraic spectrum $\Rightarrow$ BISH eigenvalue verification), Paper 50 (the DPT axiom system with Axiom 2 as algebraic spectrum), Paper 72 (DPT minimality and Axiom 3 biconditional), and Paper 73 (Axiom 1 biconditional). The present paper extracts the Axiom 2 thread from Paper 72's minimality theorem and sharpens it from a one-directional necessity claim to a full biconditional.

Series context. The broader CRM series (Papers 1–73 [14, 15, 16, 17, 18, 19, 20, 21]) calibrates the logical cost of theorems across mathematics: arithmetic geometry, mathematical physics, number theory, and algebraic topology. The central finding (Paper 2 [15]): the logical cost of mathematics is the logical cost of $\mathbb{R}$. Papers 45–53 apply this to motivic conjectures; the present paper completes the DPT reverse-characterization program.

2 Preliminaries

2.1 CRM hierarchy

We work within Bishop's constructive mathematics (BISH) as the base. The CRM hierarchy [1, 7]:

$$\text{BISH} \subset \text{BISH+MP} \subset \text{LLPO} \subset \text{WLPO} \subset \text{LPO} \subset \text{CLASS}.$$

The principles relevant to this paper:

- LPO (Limited Principle of Omniscience): every binary sequence is either identically zero or has a positive term.
- WLPO (Weak LPO): every binary sequence is either identically zero or not identically zero, *without producing a witness.*
- WLPO is strictly weaker than LPO: knowing $(a_n) \neq \bar{0}$ does not provide the index n with $a_n = 1$.

See Papers 1–45 for extended treatment.

2.2 Frobenius eigenvalues and the Weil conjectures

For a smooth projective variety X of dimension d over $\mathbb{F}_q$, the Frobenius endomorphism acts on ℓ-adic cohomology $H^i(X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathbb{Q}_\ell)$. The Weil conjectures (proven by Deligne [3,

[1] Paper 72's descent table used the phrase "transcendental $|\alpha|$" for Axiom 2 failure. The present paper supersedes that terminology: the mechanism is continuous spectral parameters in the analytic Langlands spectrum, not transcendental algebraic numbers (which do not arise, per Weil II).

4]) assert:

1. *Rationality*: the zeta function $Z(X,t)$ is rational.
2. *Functional equation*: $Z(X, q^{-d}t^{-1})$ relates to $Z(X,t)$.
3. *Riemann hypothesis*: the eigenvalues $\alpha_{i,j}$ of Frobenius on H^i satisfy $|\alpha_{i,j}| = q^{i/2}$ under any embedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

Definition 2.1 (Algebraic spectrum)**.** A spectral category has *algebraic spectrum* if its Frobenius/Hecke eigenvalues are algebraic numbers. This is Axiom 2 of the DPT system (Paper 50).

Definition 2.2 (Geometric origin [5])**.** A Galois representation has *geometric origin* if it arises from the étale cohomology of an algebraic variety. The Fontaine–Mazur conjecture characterizes geometric representations as those that are de Rham at all primes above ℓ.

2.3 The Ramanujan conjecture and Maass forms

The Ramanujan conjecture for GL_2 over $\mathbb{Q}$ asserts that the Hecke eigenvalues of cusp forms satisfy the Ramanujan–Petersson bound. For holomorphic modular forms, this was proven by Deligne [3] as a consequence of the Weil conjectures. For Maass forms—eigenfunctions of the Laplacian on the upper half-plane that are not holomorphic—the Ramanujan conjecture remains open.

A Maass form has a spectral parameter $s \in \mathbb{C}$ with $\mathrm{Re}(s) \in [0,1]$. The Ramanujan conjecture asserts $\mathrm{Re}(s) = \frac{1}{2}$. Selberg [12] proved the weaker bound $\lambda_1 \geq \frac{3}{16}$ for the first eigenvalue of the Laplacian on congruence surfaces; the conjecture $\lambda_1 \geq \frac{1}{4}$ (equivalently, $\mathrm{Re}(s) = \frac{1}{2}$) remains open.

Definition 2.3 (Continuous spectrum)**.** A spectral category has *continuous spectrum* if its spectral parameters are continuous complex variables not constrained to be algebraic. This occurs when the category accommodates the full analytic Langlands spectrum: Maass forms, unramified characters of real groups, and automorphic representations without geometric origin.

2.4 WLPO vs LPO: the equality-search dichotomy

The distinction between WLPO and LPO can be stated precisely:

Principle	**Statement**	**Computational content**
LPO	$\forall (a_n),\ (\exists n,\, a_n = 1) \vee (\forall n,\, a_n = 0)$	search + witness
WLPO	$\forall (a_n),\ (a_n) = \bar{0} \vee (a_n) \neq \bar{0}$	equality test

LPO produces a witness index n; WLPO gives only a yes/no answer. Eigenvalue comparison ("does s equal $\frac{1}{2}$?") is a single equality test. Cycle-search ("is there Z with $h(Z) \leq B$?") and radical detection ("is there W with $\langle Z, W\rangle \neq 0$?") are existential searches.

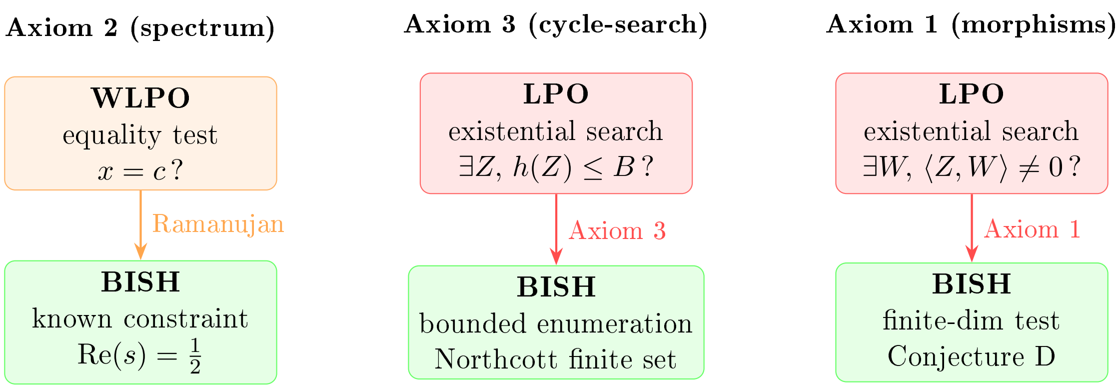

Figure 1: The equality-vs-search dichotomy across the DPT axiom trio. Axiom 2 (left, orange) involves a single equality test, costing WLPO when it fails. Axioms 1 and 3 (right, red) involve existential searches, costing LPO when they fail. Each axiom provides the descent arrow from omniscience to BISH.

3 Main Results

3.1 Theorem A: Forward direction

Theorem 3.1 (Algebraic spectrum $\Rightarrow$ BISH eigenvalues)**.** *With algebraic spectrum (geometric origin), eigenvalue comparison is* BISH*-decidable.*

Proof. Geometric origin (Deligne [3, 4]): Frobenius eigenvalues α are algebraic integers satisfying $|\alpha| = q^{w/2}$. Testing this condition reduces to: $|\alpha|^2 = \alpha\bar{\alpha}$ is a product of algebraic conjugates (algebraic), and q^w is an integer. Thus $|\alpha|^2 = q^w$ is an algebraic identity, decidable by computing the GCD of the minimal polynomials of $|\alpha|^2$ and q^w. This is a finite computation over $\mathbb{Z}[x]$, hence BISH. Axiomatized as `algebraic_gives_BISH`; mathematical reference: Paper 45 Theorem C1. □

3.2 Theorem B: The reverse direction and biconditional

Theorem 3.2 (No algebraic spectrum $\Rightarrow$ WLPO eigenvalues)**.** *Without algebraic spectrum (continuous Langlands parameters), eigenvalue comparison costs* WLPO*.*

Proof. Without geometric origin, the spectral parameters are continuous complex variables. For a Maass form with spectral parameter $s \in \mathbb{C}$, testing whether $\mathrm{Re}(s) = \frac{1}{2}$ (the Ramanujan bound) is a real-number equality test. A real-number equality oracle $x = c$ for $x \in \mathbb{R}$ and algebraic c encodes WLPO (Paper 45 Theorem C2): given any binary sequence (a_n), the real number $x = \sum_n a_n 2^{-n}$ satisfies $x = 0$ iff $(a_n) = \bar{0}$. A spectral-comparison oracle therefore decides $(a_n) = \bar{0}$ for arbitrary sequences, yielding exactly WLPO.

Why WLPO and not LPO: the oracle answers "does s equal $\frac{1}{2}$?" without identifying which coefficient a_n is nonzero. This is a single equality test, not a search through an infinite structure. The stronger principle LPO would additionally provide a witness index n; eigenvalue comparison does not provide this. Axiomatized as `continuous_gives_WLPO`. □

Theorem 3.3 (Eigenvalue-Decidability Equivalence). *For eigenvalue comparison in a spectral category:*

$$\textit{eigenvalue\_cost}(s) = \mathrm{BISH} \iff s = \textit{algebraic} \iff \textit{Axiom 2 holds.}$$

Proof. *(⇐):* If the spectrum is algebraic, eigenvalue comparison is BISH (theorem 3.1).

(⇒, contrapositive): If the spectrum is continuous (Axiom 2 fails), eigenvalue comparison costs WLPO (theorem 3.2). Since WLPO $\neq$ BISH (these are distinct levels of the CRM hierarchy), the spectrum cannot be continuous if eigenvalue cost is BISH. □

Remark 3.4 (Weil II does not trivialize the reverse). Deligne's Weil II theorem [4] proves that all motives arising from algebraic geometry over $\mathbb{F}_q$ have algebraic Frobenius eigenvalues. One might object that this makes Axiom 2 a theorem, not a hypothesis, rendering the reverse characterization vacuous. The objection misses the point. Weil II proves Axiom 2 for *geometric motives*—motives with geometric origin. The reverse characterization identifies *where* the axiom is doing work: at the boundary between geometric and analytic. When the category extends to the full analytic Langlands spectrum (Maass forms, unramified characters), the eigenvalues are no longer constrained to be algebraic, and the axiom is genuinely needed. The reverse direction says: if eigenvalue comparison is BISH, the spectrum *must* be algebraic—geometric origin is not merely convenient but logically necessary.

3.3 The Deligne constraint

Theorem 3.5 (Deligne constraint). *Without geometric origin, one cannot simultaneously have:*

1. BISH-*decidable eigenvalue comparison, and*
2. *the full analytic Langlands spectrum.*

With geometric origin (Deligne), both hold. With the Ramanujan conjecture proven (but without geometric origin), the analytic spectrum becomes effectively algebraic.

Proof. The full analytic Langlands spectrum includes continuous parameters (e.g., Maass spectral parameters $s \in \mathbb{C}$). Eigenvalue comparison for continuous parameters costs WLPO (theorem 3.2), so (1) fails.

With geometric origin, Deligne's theorem restricts the spectrum to algebraic eigenvalues: both (1) and (2) hold (with "analytic spectrum" automatically algebraic).

If the Ramanujan conjecture is proven unconditionally, every Maass form satisfies $\mathrm{Re}(s) = \frac{1}{2}$; the spectral parameters satisfy a known constraint ($\mathrm{Re}(s) = \frac{1}{2}$), eliminating the equality test entirely and restoring BISH decidability without geometric origin. Axiomatized as `ramanujan_resolution`. □

3.4 The Selberg eigenvalue conjecture as a WLPO instance

The Selberg eigenvalue conjecture provides a concrete illustration of the WLPO cost identified in theorem 3.2.

For a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, the Laplacian Δ on $L^2(\Gamma\backslash\mathfrak{H})$ has a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ for cuspidal eigenfunctions. Each eigenvalue $\lambda = s(1-s)$ is parameterized by a spectral parameter s with $\mathrm{Re}(s) \in [0,1]$.

- **Selberg's proven bound** (1965) [12]: $\lambda_1 \geq \frac{3}{16}$ for all congruence subgroups. This is equivalent to $\mathrm{Re}(s) \geq \frac{1}{4}$ for the spectral parameter.
- **Selberg's conjecture**: $\lambda_1 \geq \frac{1}{4}$, equivalently $\mathrm{Re}(s) = \frac{1}{2}$ for all cuspidal Maass forms on congruence surfaces. This is equivalent to the Ramanujan conjecture for GL_2 over $\mathbb{Q}$ (Satake 1966).
- **Best known bound**: Kim and Sarnak (2003, appendix to Kim [2]) proved $\lambda_1 \geq \frac{1}{4} - (\frac{7}{64})^2 \approx 0.2380$, i.e., $\mathrm{Re}(s) \geq \frac{1}{2} - \frac{7}{64}$. The gap between this bound and $\frac{1}{4}$ remains open.

In the CRM framework, verifying the Selberg conjecture for a *specific* Maass form f amounts to testing the real-number equality $\mathrm{Re}(s_f) = \frac{1}{2}$. This is a single equality test on a continuous parameter, hence a WLPO instance. The proven bounds ($\frac{3}{16}$, Kim-Sarnak) narrow the range but do not decide equality.

Proving the full Selberg conjecture unconditionally would eliminate the WLPO cost for the $\mathrm{GL}_2(\mathbb{Q})$ case, paralleling how Deligne's theorem eliminates it for geometric motives. The conceptual parallel is precise:

Setting	**Constraint**	**Effect**
Geometric motives	Deligne (Weil II)	Algebraic eigenvalues $\to$ BISH
Maass forms ($\mathrm{GL}_2/\mathbb{Q}$)	Selberg conjecture	$\mathrm{Re}(s) = \frac{1}{2} \to$ BISH
General automorphic	Ramanujan conjecture	Eliminates all WLPO costs

3.5 Theorem C: The characterization

Theorem 3.6 (Axiom 2 Characterization)**.** *Algebraic spectrum (geometric origin) is the minimal and unique condition for* BISH*-decidable eigenvalue comparison:*

1. *eigenvalue_cost*(*algebraic*) $=$ BISH*;*
2. *eigenvalue_cost*(*continuous*) $=$ WLPO*;*
3. *without geometric origin, the full analytic spectrum costs* WLPO*;*
4. *the Ramanujan resolution (proving the conjecture makes the analytic spectrum effectively algebraic) confirms the trade-off is real.*

Proof. Assembly of theorems 3.1 to 3.3 and 3.5. Axiomatized as `axiom2_characterization`. ∎

Corollary 3.7 (Axiom 2 Principle, sharpened)**.**

$$\mathit{eigenvalue\_cost}(s) = \mathrm{BISH} \iff \mathit{is\_algebraic}(s) = \mathit{true}.$$

Paper 72 Theorem A asserted: without Axiom 2, cost = WLPO *(forward). Paper 74 proves the biconditional: algebraic spectrum is necessary and sufficient.*

Spectrum type	CRM cost	Mechanism	Reference
Algebraic (geometric origin)	BISH	algebraic number GCD	Paper 45 C1
Continuous (analytic Langlands)	WLPO	real-number equality test	Paper 45 C2

Table 1: CRM cost of eigenvalue decidability vs. spectrum type.

	Axiom 1 (Paper 73)	**Axiom 2 (Paper 74)**	**Axiom 3 (Paper 72)**
Conjecture	Conj. D	Algebraic spectrum	Archimedean pol.
Domain	Morphism equality	Eigenvalue comparison	Cycle-search
BISH case	detachable radical	algebraic (Deligne)	positive-definite
Failure cost	LPO	WLPO	LPO
Bridge	hom = num	geometric origin	Northcott property
Mechanism	integer tests	algebraic GCD	$u(\mathbb{R}) = \infty$
Escape clause	Jannsen escape	Ramanujan resolution	low-rank remark

Table 2: The DPT axiom trio: parallel structure across all three reverse characterizations. Note the WLPO asymmetry of Axiom 2.

4 CRM Audit

4.1 Descent table

4.2 The DPT axiom trio

The three characterizations are logically independent: Axiom 1 controls morphism *equality* (is $Z_1 \sim Z_2$?), Axiom 2 controls eigenvalue *comparison* (does $|\alpha| = q^{w/2}$?), and Axiom 3 controls cycle *search* (can you find Z with $h(Z) \leq B$?). Dropping one raises the CRM floor without affecting the others (Paper 72, table 2).

5 Formal Verification

5.1 File structure

The Lean 4 bundle `Papers/P74_Axiom2Reverse/` contains:

File	Content
`Defs.lean`	CRM hierarchy, spectrum type, axiomatized costs
`Forward.lean`	Theorem A: algebraic $\to$ BISH
`Reverse.lean`	Theorem B: biconditional + Deligne constraint
`Characterization.lean`	Theorem C: full assembly + sharpened principle
`Main.lean`	Aggregator with `#check` statements

Build: `lake build` from bundle root. Toolchain: Lean 4 v4.29.0-rc2, Mathlib4. Zero `sorry`, zero warnings.

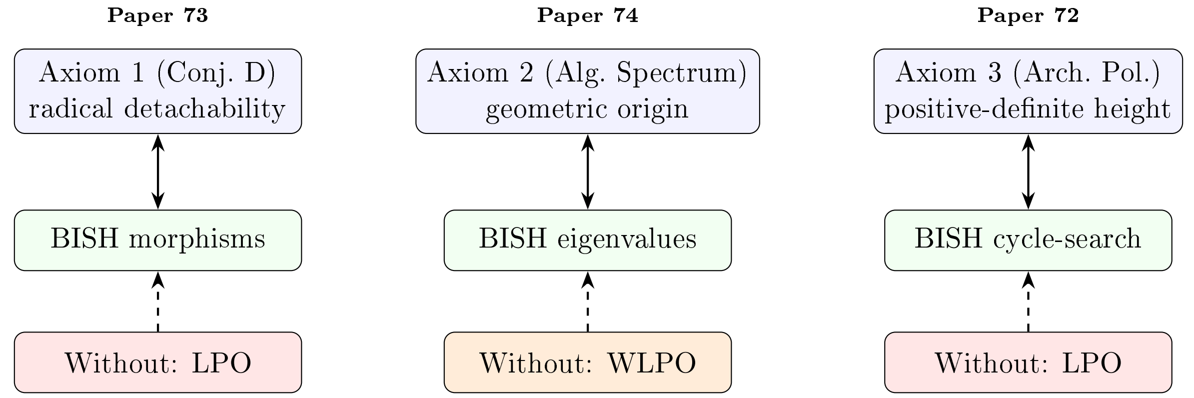

Figure 2: The DPT axiom trio. Each axiom is individually necessary and sufficient for BISH-decidable operations in its domain. Double arrows (↔) denote biconditionals. The WLPO asymmetry of Axiom 2 (orange) reflects the equality-test character of eigenvalue comparison, contrasting with the existential-search character (LPO, red) of morphism equality and cycle-search.

Axiom	**Type**	**Role**	**Reference**
`algebraic_eigenvalue_cost`	`CRMLevel`	data	Paper 45 C1, Deligne (1974)
`algebraic_eigenvalue_cost_eq`	`= BISH`	prop	Paper 45 C1
`continuous_eigenvalue_cost`	`CRMLevel`	data	Paper 45 C2, Bump (1997)
`continuous_eigenvalue_cost_eq`	`= WLPO`	prop	Paper 45 C2, Selberg (1965)

Table 3: Complete axiom inventory. Four axioms: 2 data + 2 propositional. Every axiom has a mathematical reference; no axiom without provenance.

5.2 Axiom inventory

5.3 Code: Eigenvalue-Decidability Equivalence (Theorem B)

Listing 1: Theorem B: algebraic spectrum ⇔ BISH

```
1  theorem eigenvalue_decidability_equivalence
2      (s : SpectrumType) :
3      eigenvalue_cost s = BISH ↔s = algebraic := by
4    constructor
5      intro h
6      cases s
7        rfl
8        -- continuous: derive contradiction
9        unfold eigenvalue_cost at h
10       rw [continuous_eigenvalue_cost_eq] at h
11       -- h : WLPO = BISH   contradiction
12       contradiction
13     intro h
14     rw [h]
15     exact algebraic_gives_BISH
```

The reverse direction (lines 6–11) mirrors Paper 72's height-search equivalence and Paper 73's morphism-decidability equivalence: `unfold` exposes the axiom value, `rw` applies the axiom, and `contradiction` closes the goal since WLPO $\neq$ BISH in the inductive type. The structural parallel across all three papers is exact—only the inductive type (`SpectrumType` vs. `HeightType` vs. `RadicalStatus`), the cost function, and the contradiction value (WLPO vs. LPO) differ.

5.4 Code: Sharpened Axiom 2 Principle (Corollary)

Listing 2: Biconditional: algebraic spectrum $\Leftrightarrow$ BISH

```
theorem axiom2_principle_sharpened
    (s : SpectrumType) :
    eigenvalue_cost s = BISH ↔
    is_algebraic s = true :=
  ⟨fun h => (axiom2_iff_algebraic s).mpr
    ((eigenvalue_decidability_equivalence s).mp h),
   fun h => (eigenvalue_decidability_equivalence s).mpr
    ((axiom2_iff_algebraic s).mp h)⟩
```

5.5 `#print axioms` output

```
'axiom2_characterization' depends on axioms:
  [algebraic_eigenvalue_cost_eq, continuous_eigenvalue_cost_eq]

'axiom2_principle_sharpened' depends on axioms:
  [algebraic_eigenvalue_cost_eq, continuous_eigenvalue_cost_eq]

'eigenvalue_decidability_equivalence' depends on axioms:
  [algebraic_eigenvalue_cost_eq, continuous_eigenvalue_cost_eq]

'algebraic_gives_BISH' depends on axioms:
  [algebraic_eigenvalue_cost_eq]

'continuous_gives_WLPO' depends on axioms:
  [continuous_eigenvalue_cost_eq]

'axiom2_iff_algebraic' does not depend on any axioms

'deligne_constraint' depends on axioms:
  [continuous_eigenvalue_cost_eq]

'ramanujan_resolution' depends on axioms:
  [algebraic_eigenvalue_cost_eq]

'dpt_trio_costs' depends on axioms:
  [algebraic_eigenvalue_cost_eq, continuous_eigenvalue_cost_eq]
```

No theorem depends on `Classical.choice`, `propext`, or `Quot.sound`. The `opaque` data constants (`algebraic_eigenvalue_cost`, `continuous_eigenvalue_cost`) do not appear in the axiom trace because Lean 4 reports only propositional axioms; the `opaque` declarations contribute indirectly via their `_eq` axioms.

5.6 Classical.choice audit

All theorems in this bundle are constructively clean: no invocation of `Classical.choice`, `Classical.em`, or `Decidable.em`. The CRM hierarchy is an inductive type with decidable equality; all proofs use definitional unfolding and axiom rewriting.

5.7 Reproducibility

Lean 4 toolchain: `leanprover/lean4:v4.29.0-rc2`. Mathlib4 dependency resolved via `lake-manifest.json` (pinned commit). Build command: `lake build` from bundle root. Lean source and compiled PDF deposited on Zenodo: DOI: `https://doi.org/10.5281/zenodo.18773827`. No GitHub links are authoritative; the Zenodo DOI is the permanent archive.

6 Discussion

6.1 Why WLPO not LPO: the equality-vs-search dichotomy

The WLPO cost of Axiom 2 is the most distinctive feature of the DPT trilogy. Axioms 1 and 3 both cost LPO when violated, but Axiom 2 costs only WLPO. This is not an artifact of the formalization but a reflection of the underlying mathematics.

Axiom 1 (Conjecture D): testing morphism equality requires searching a finite-dimensional space for a cycle W witnessing $\langle Z, W\rangle \neq 0$. The search is finite but existential: one needs to *find* the witness. This is an LPO pattern.

Axiom 3 (Archimedean polarization): cycle-search requires finding Z with $h(Z) \leq B$ in a lattice. Again, an existential search for a concrete object: LPO.

Axiom 2 (algebraic spectrum): eigenvalue comparison asks whether a continuous parameter s equals an algebraic value $\frac{1}{2}$. No witness is produced; the answer is simply yes or no. This is a WLPO pattern.

The dichotomy is intrinsic. Morphism equality and cycle-search are *search problems*: they require finding objects in a space. Eigenvalue comparison is a *decision problem*: it requires testing equality of a value. The CRM hierarchy captures this distinction: LPO quantifies over searches, WLPO quantifies over equalities.

6.2 Completing the DPT trilogy

With Papers 72, 73, and 74, all three DPT axioms now have individual reverse characterizations:

- Axiom 1 $\Leftrightarrow$ BISH morphisms (Paper 73).
- Axiom 2 $\Leftrightarrow$ BISH eigenvalues (Paper 74).
- Axiom 3 $\Leftrightarrow$ BISH cycle-search (Paper 72).

The upgrade: Paper 72's minimality theorem (Theorem A) showed each axiom is necessary; Papers 73–74 show each is *uniquely necessary*, i.e., the biconditional holds. The DPT axiom system is not merely minimal (no axiom is redundant) but *canonical* (each axiom is the unique condition for its domain).

6.3 Connection to Paper 75 (conservation test)

Paper 75 (planned) will test the *conservation thesis*: that the CRM calibration of arithmetic results proved via condensed and perfectoid methods (Fargues–Scholze [13]) can be predicted from the DPT framework. The trilogy established in Papers 72–74 provides the tool: if one knows which spectral, morphism, or cycle-search operations a theorem employs, the DPT biconditionals predict the exact CRM cost.

The conservation question is stratified across three layers of the condensed mathematics programme:

1. *Algebraic layer* (solidification functor for condensed abelian groups): the relevant constructions involve countable inverse limits, where the Mittag-Leffler condition or Dependent Choice governs convergence. Preliminary analysis suggests this layer conserves: the arithmetic content descends to BISH + LPO or lower.

2. *Homological layer* (injective resolutions in solid module categories): existence of enough injectives in general abelian categories requires Zorn's lemma (= CLASS). Conservation may fail here unless bounded derived categories can bypass injective resolutions via Čech complexes.

3. *Geometric layer* (the v-topology on perfectoid spaces): the existence of enough points for the v-site requires the Boolean Prime Ideal theorem (BPI, a consequence of the Axiom of Choice). Conservation fails at this layer in full generality.

The DPT biconditionals from Papers 72–74 predict exactly which layer an arithmetic conclusion depends on: if the conclusion involves only eigenvalue comparison (Axiom 2), the conservation question reduces to whether WLPO suffices; if it involves cycle-search (Axiom 3), LPO is the threshold.

6.4 The geometric-analytic boundary

The correct reading of Axiom 2 is not about transcendental numbers (which do not arise in the classical motivic setting, as Weil II proves) but about the boundary between geometric and analytic. Geometric motives (Deligne): spectrum algebraic, eigenvalue comparison BISH. Analytic representations (Langlands): spectrum continuous, eigenvalue comparison WLPO. The Ramanujan conjecture is the bridge: its proof would collapse the analytic side to the geometric side, eliminating the WLPO cost.

This parallels Conjecture D for Axiom 1: Conjecture D bridges numerical and homological equivalence, eliminating the LPO cost. And Archimedean polarization for Axiom 3: positivity bridges bounded and unbounded height, eliminating the LPO cost. In each case, the axiom is the logical bridge between a decidable world (BISH) and an omniscient one (WLPO or LPO).

6.5 De-omniscientizing descent

The standard pattern in the CRM programme is: (1) identify a classical theorem requiring omniscience, (2) locate the specific constructive principle, and (3) find the hypothesis that eliminates it. The DPT trilogy instantiates this pattern three times, with the same logical structure but different mathematical mechanisms:

Axiom	Classical operation	Omniscience	De-omniscientizing hypothesis
2 (Spectrum)	Eigenvalue comparison: $s = \frac{1}{2}$?	WLPO (equality test)	Algebraic spectrum: GCD replaces real equality
1 (Conj. D)	Radical detection: $\exists W, \langle Z, W\rangle \neq 0$?	LPO (search)	Conj. D: finite-dim $\mathbb{Q}$-space $\to$ bounded search
3 (Polar.)	Cycle search: $\exists Z, h(Z) \leq B$?	LPO (search)	Northcott: pos-def height $\to$ finite enumeration

Table 4: The de-omniscientizing descent across the DPT trilogy. Each axiom replaces an omniscient operation with a constructive computation, descending from WLPO or LPO to BISH.

The descent target is always BISH, but the omniscience cost varies. Axiom 2 costs WLPO because eigenvalue comparison is an equality test; Axioms 1 and 3 cost LPO because morphism and cycle operations are existential searches. This asymmetry reflects the intrinsic difference between testing a value and searching a structure—a distinction invisible to classical mathematics but sharp in the constructive hierarchy.

7 Conclusion

Paper 45 and the DPT framework (Paper 50) established: algebraic spectrum is sufficient for BISH-decidable eigenvalue comparison. Paper 74 establishes: algebraic spectrum is also *necessary*. Together:

$$\text{algebraic spectrum} \iff \text{geometric origin} \iff \text{BISH eigenvalue comparison.}$$

This completes the DPT axiom trilogy. All three axioms are now individually characterized by biconditionals, upgrading Paper 72's minimality theorem to a canonicity result. The WLPO asymmetry of Axiom 2—in contrast to the LPO cost of Axioms 1 and 3—is not a technicality but the CRM signature of the equality-vs-search dichotomy.

Status of claims. *Lean-verified* (zero `sorry`): Theorems A, B, C and the sharpened Axiom 2 Principle, conditional on four axioms with mathematical references (table 3). *Rigorous mathematical analysis* (not formalized): the Weil II non-triviality remark (remark 3.4), the Selberg eigenvalue conjecture as a WLPO instance (**??**), and the geometric-analytic boundary discussion. *Open*: whether the Ramanujan conjecture (or specific cases such as GL_2 over $\mathbb{Q}$) can be resolved, and whether the biconditional extends to categories intermediate between geometric motives and the full analytic spectrum.

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This paper was drafted with AI assistance (Claude, Anthropic) for proof search and exposition. The author is a clinician (interventional cardiology), not a professional mathematician; the logical structure of the main results has been verified by formal proof (Lean 4); the mathematical arguments supporting the axiom assignments have been checked by the author and by consultation with domain experts. Errors of mathematical judgment remain the author's responsibility.

This series is dedicated to the memory of Errett Bishop (1928–1983), whose program demonstrated that constructive mathematics is not a restriction but a refinement.

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