Axiom Calibration for General Relativity (Paper 5): Portals, Profiles, and a Hybrid Plan for EPS and Schwarzschild

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Abstract

We make *Axiom Calibration* (AxCal) the organizing principle for a foundations-first study of General Relativity (GR). The paper contributes three AxCal instruments for GR: (I) *witness families* pinned to a fixed Σ_0^{GR} signature; (II) *proof-route flags* and *portal theorems* that turn standard GR arguments into explicit frontier costs (Zorn$\Rightarrow$ $\{$AC$\}$, Limit-Curve/Ascoli$\Rightarrow$ $\{$FT/WKL$_0\}$, Serial-Chain$\Rightarrow$ $\{\mathrm{DC}_\omega\}$, Reductio$\Rightarrow$ $\{$LEM$\}$); and (III) *HeightCertificates* that compose costs across results. On this basis we calibrate five loci: G1 (explicit vacuum checks: mathematically Height 0), G2 (Cauchy/MGHD: PDE core vs. Zorn portal), G3 (singularity theorems: compactness and contradiction portals), G4 (maximal extensions: Zorn portal), and G5 (computable evolution: negative template after Pour–El–Richards).

Methodological stance. Our Lean 4 implementation uses `mathlib`, a classical library (ZFC with LEM and global choice mechanisms). Accordingly, we position the artifact as a form of *Structural Certification*: Lean checks the AxCalbookkeeping (portals, routes, and height composition), while deep theorems and constructive lower/upper bounds may be imported as named axioms. For G1 (Schwarzschild vacuum), we provide a complete symbolic verification (Sprint 3): all Christoffel symbols and Ricci tensor components are computed explicitly, proving $R_{\mu\nu} = 0$ for the Schwarzschild metric. The mathematics is constructive (Height 0), but the current Lean formalization verifies the computation in a classical environment; a formal constructive certificate is deferred to future work or to alternative libraries.

IMPORTANT DISCLAIMER

A Case Study: Using Multi-AI Agents to Tackle Formal Mathematics

This entire Lean 4 formalization project was produced by multi-AI agents working under human direction. All proofs, definitions, and mathematical structures in this repository were AI-generated. This represents a case study in using multi-AI agent systems to tackle complex formal mathematics problems with human guidance on project direction.

What is calibrated here (AxCal content). *For each GR target we (a) define a* witness family *at the pin, (b) mark* proof-route flags *indicating which portals are used, and (c) emit a* HeightCertificate *with axiswise heights* ($h_{\mathrm{Choice}}, h_{\mathrm{Comp}}, h_{\mathrm{Logic}}$). *Deep-dive deliverables (EPS; Schwarzschild) produce mathematically Height 0 certificates verified via structural certification in classical* `mathlib`*; imported heavy theorems are recorded as* named axioms *triggering portals and heights.*

Contents

1 AxCal instrumentation for GR

1.1 Pinned signature Σ_0^{GR}

We fix the smooth category (second-countable, Hausdorff manifolds), tensor fields, Lorentzian metrics, Levi–Civita connection, curvature and Einstein tensors, EFE, and pinned exemplars

(Minkowski; a Schwarzschild-type vacuum metric). Interpretations must fix Σ_0^{GR}.

1.2 Tokens and witness families

For foundations $F \in \mathsf{Found}$, we use tokens

$$[\mathrm{HasAC}\,F],\ [\mathrm{HasDC}_\omega\,F],\ [\mathrm{HasFT}\,F],\ [\mathrm{HasWKL}_0\,F],\ [\mathrm{HasLEM}\,F],\ [\mathrm{HasWLPO}\,F].$$

A *witness family* $\mathcal{W}$ assigns to F a groupoid of witnesses for the target statement over the pin.

1.3 Proof-route flags and portals

We make explicit route flags that, when present in a proof, trigger an *AxCal portal*:

- uses_zorn: applies Zorn on a Σ_0-definable poset of extensions $\Rightarrow$ *Zorn portal* ($\partial^+ \supseteq \{\mathrm{AC}\}$).
- uses_limit_curve: invokes Ascoli–Arzelà / compactness of causal curves $\Rightarrow$ *Compactness portal* ($\partial^+ \supseteq \{\mathrm{FT}/\mathrm{WKL}_0\}$).
- uses_serial_chain: builds an infinite dependent chain (e.g. curve prolongation) $\Rightarrow$ *Dependent-Choice portal* ($\partial^+ \supseteq \{\mathrm{DC}_\omega\}$).
- uses_reductio: essential proof by contradiction on Σ_0 data $\Rightarrow$ *LEM portal* ($\partial^+ \supseteq \{\mathrm{LEM}\}$).

Proposition 1.1 (Portal soundness)**.** *If a proof of a Σ_0^{GR}-pinned statement uses a flagged route, the corresponding token is necessary along that route: Zorn $\Rightarrow$ HasAC; Limit-Curve $\Rightarrow$ HasFT/HasWKL$_0$ (depending on constructive/classical base); Serial-Chain $\Rightarrow$ HasDCω; Reductio $\Rightarrow$ HasLEM.*

Sketch. These are standard meta-implications: Zorn is equivalent to AC over ZF; Ascoli-type compactness aligns with FT constructively (and with WKL_0 classically); building infinite dependent sequences is a canonical use of DC_ω; essential reductio uses LEM. The novelty is the *proof-route* tagging that transports these meta-results to the Σ_0^{GR} pin. □

1.4 Height certificates and composition

Given portals triggered for a witness family $\mathcal{W}$, we record a HeightCertificate with $(h_{\mathrm{Choice}}, h_{\mathrm{Comp}}, h_{\mathrm{Logic}}) \in \{0, 1, \omega\}^3$. Products of claims compose componentwise by the AxCal product law (Paper 3A).

2 Literature anchors mapped to portals

Robb and Reichenbach provide axiomatic scaffolding for kinematics; EPS derives Lorentz classes from light and free-fall [1, 2, 3] (no Zorn; compactness may enter via curve families $\Rightarrow$ Compactness portal when maximizers are extracted). Pour–El–Richards show computable well-posed PDEs can yield non-computable evolutions [4] (Logic/Computability axis). Bishop–Bridges and Hellman/Bridges guide which analytic steps are Height 0 and which align with choice or LEM [5, 6, 7]. Wald, Hawking–Ellis, and Choquet–Bruhat are used to *locate* where standard GR proofs instantiate portals [8, 9, 10].

3 Calibration targets (G1–G5) with AxCal profiles

G1. Explicit vacuum checks (mathematically Height 0)

Definition 3.1 (G1 witness)**.** $\mathcal{C}^{\mathrm{G1}}$: the pinned Schwarzschild-type metric satisfies vacuum EFE at the pin.

Proposition 3.2 (G1 profile)**.** $(h_{\mathrm{Choice}}, h_{\mathrm{Comp}}, h_{\mathrm{Logic}}) = (0, 0, 0)$.

Sketch. Finite symbolic computation of $\Gamma^{\alpha}_{\mu\nu}$, $R_{\mu\nu}$, and $G_{\mu\nu}$ (cf. [8, §B.4]); no portals are triggered. In our Sprint 3 implementation, we compute:

- All non-zero Christoffel symbols: $\Gamma^{t}_{tr} = M/(r^2 f)$, $\Gamma^{r}_{tt} = Mf/r^2$, $\Gamma^{r}_{rr} = -M/(r^2 f)$, $\Gamma^{r}_{\theta\theta} = -r + 2M$, $\Gamma^{r}_{\varphi\varphi} = -(r-2M)\sin^2\theta$, $\Gamma^{\theta}_{r\theta} = 1/r$, $\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$, $\Gamma^{\varphi}_{r\varphi} = 1/r$, $\Gamma^{\varphi}_{\theta\varphi} = \cot\theta$
- Full Ricci tensor via $R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\mu\rho} + \Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho}$
- Verification that $R_{tt} = R_{rr} = R_{\theta\theta} = R_{\varphi\varphi} = 0$ for $f = 1 - 2M/r$

Note: While the mathematics is constructive, our Lean formalization uses classical `mathlib`, yielding structural certification rather than formal constructive proof. □

G2. Cauchy problem (local well-posedness and MGHD)

Definition 3.3 (G2 witness)**.** $\mathcal{C}^{\mathrm{G2}}$: local well-posedness for EFE and existence/uniqueness of MGHD from data (Σ, h, K).

Proposition 3.4 (G2 profile (route-separated))**.** *Local PDE (harmonic gauge, energy estimates) can be arranged with* $(0, 0, 0)$ *or low choice* (AC_ω) *in separable settings; the MGHD step has* uses_zorn*, so* $(h_{\mathrm{Choice}}, h_{\mathrm{Comp}}, h_{\mathrm{Logic}}) \geq (1, 0, 0)$ *for the global statement.*

Sketch. Local existence follows Choquet–Bruhat's PDE machinery [10]. MGHD standard proofs invoke Zorn on the poset of developments (Wald [8, Thm. 10.1.2]); by Prop. 1.1, the Zorn portal yields the AC frontier. □

G3. Singularity theorems (Penrose/Hawking)

Definition 3.5 (G3 witness)**.** $\mathcal{C}^{\mathrm{G3}}$: under trapped surface + energy conditions in a globally hyperbolic spacetime, geodesic incompleteness holds.

Proposition 3.6 (G3 profile)**.** *Standard proofs trigger* uses_limit_curve *and* uses_reductio*; hence* $(h_{\mathrm{Choice}}, h_{\mathrm{Comp}}, h_{\mathrm{Logic}}) \geq (0, 1, 1)$ *(the 1 on the second axis reads as* FT/WKL_0 *compactness; the 1 on the third as LEM).*

Sketch. Raychaudhuri focusing plus limit-curve compactness ([9, §8], [8, §14]) gives a maximizing geodesic; contradiction shows incompleteness. Portals: compactness and reductio. □

G4. Maximal extensions

Definition 3.7 (G4 witness)**.** $\mathcal{C}^{\mathrm{G4}}$: any local solution admits a maximal extension by isometric inclusion.

Proposition 3.8 (G4 profile)**.** uses_zorn *holds;* $(h_{\mathrm{Choice}}, h_{\mathrm{Comp}}, h_{\mathrm{Logic}}) \geq (1, 0, 0)$.

Proof. Chains of extensions admit upper bounds; Zorn yields a maximal element (portal to AC). □

G5. Computable evolution

Definition 3.9 (G5 witness)**.** $\mathcal{C}^{\mathrm{G5}}$: computable initial data yield computable evolved fields (in a fixed representation) on a pinned globally hyperbolic class.

Proposition 3.10 (G5 negative template)**.** *Without added uniformity, $\mathcal{C}^{\mathrm{G5}}$ can fail by a Pour–El–Richards template [4]. The failure calibrates on the Logic/Computability axis; any attempt to extract definite infinite data sequences invokes* uses_serial_chain *(portal to* $\{\mathrm{DC}_\omega\}$*).*

Sketch. Linear prototypes show computable→non-computable evolution; for quasi-linear systems, similar non-uniformity can occur. If one insists on classical infinite sample paths from measurement-like procedures, the DC_ω portal is triggered. □

4 Auxiliary lemmas (EPS; limit-curve) with sketches

Lemma 4.1 (EPS kinematics)**.** *Under the Ehlers–Pirani–Schild axioms, light rays define a conformal structure and free fall a projective structure; compatibility yields a Weyl structure whose scale integrability produces a Lorentz metric class.*

Sketch. EPS axioms isolate the null cones and (unparameterized) timelike geodesics; the compatibility condition yields a torsion-free Weyl connection preserving the conformal class; vanishing length curvature selects a Levi–Civita metric representative [3]. No Zorn; compactness may enter only if one extracts maximizing curves. □

Lemma 4.2 (Limit-curve compactness)**.** *In globally hyperbolic spacetimes, causal curves between compact sets form a compact set in the C^0 topology; maximizing causal geodesics exist.*

Sketch. Global hyperbolicity yields compact diamond sets; equicontinuity gives Ascoli–Arzelà; upper semicontinuity of length gives maximizers (cf. [8, §14]). This triggers the Compactness portal. □

5 Hybrid plan: structured framework + selective deep dives

5.1 Sprint 3 Milestone: Schwarzschild Vacuum Solution

Achievement: Complete symbolic verification of the Schwarzschild vacuum solution, demonstrating that the Schwarzschild metric is a solution to Einstein's vacuum field equations $R_{\mu\nu} = 0$. The implementation includes:

1. **Metric components:** Diagonal metric with $g_{tt} = -f(r)$, $g_{rr} = f(r)^{-1}$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2 \sin^2\theta$, where $f(r) = 1 - 2M/r$.
2. **Christoffel symbols:** All 40 potentially non-zero symbols computed, with 9 non-vanishing components identified and verified.
3. **Ricci tensor:** Complete computation using the Ricci formula with systematic treatment of radial derivatives and angular components.
4. **Vacuum verification:** Explicit proof that all diagonal Ricci components vanish: $R_{tt} = R_{rr} = R_{\theta\theta} = R_{\varphi\varphi} = 0$.

This represents a Height 0 achievement as no axiom portals are triggered—the entire computation proceeds through finite symbolic algebra.

Structured framework (implemented). We have registered G1–G5 witness families as abstract propositions over the pinned signature Σ_0^{GR}. Route flags are attached per standard proofs, and HeightCertificates are emitted using portal soundness (Prop. 1.1). The framework maintains a verification ledger of *named axioms* (MGHD existence, Penrose/Hawking singularity theorems, limit-curve compactness) with bibliographic citations. This structured approach provides machine-checkable height profiles while using abstract placeholders (`True`) for the detailed mathematical content—a deliberate design choice that prioritizes axiomatic calibration over full formalization.

Deep dive anchors (Height 0 demonstrations).

- **D1 (EPS Core).** We implement the EPS kinematics framework showing how light rays and free fall determine metric structure. The implementation provides a mathematically Height 0 certificate by avoiding all portals, demonstrating that the EPS reconstruction can be done constructively. The Lean artifact verifies this in classical `mathlib`, providing structural certification of the portal-free nature.

- **D2 (Schwarzschild Engine).** We provide a complete symbolic verification of the Schwarzschild vacuum solution. The implementation computes all non-zero Christoffel symbols, Ricci tensor components, and verifies $R_{\mu\nu} = 0$ for the Schwarzschild metric in (t, r, θ, φ) coordinates. Key achievements include: (i) systematic computation of all connection coefficients; (ii) full Ricci tensor derivation using the contracted Bianchi identity; (iii) explicit verification that all Ricci components vanish for $f(r) = 1 - 2M/r$. The framework demonstrates mathematically Height 0 structure (no axiom portals) while verifying in classical `mathlib`.

  Success metrics. *(i) D1 and D2 compiled without* `sorry` *in the classical* `mathlib` *environment; (ii) HeightCertificates present for all G1–G5 tracking axiom dependencies; (iii) explicit portal flags in ledger showing frontier costs; (iv) CI and "no-sorry" guards for deep-dive directories. The artifact provides* Structural Certification *of the AxCal bookkeeping, with mathematical Height 0 claims validated in a classical proof assistant.*

  Reproducibility Box: Building and Verifying Paper 5

  Prerequisites:

```
# Install elan (Lean version manager)
curl https://raw.githubusercontent.com/leanprover/elan/master/elan-init.sh -sSf | sh

# Clone repository
git clone https://github.com/AICardiologist/FoundationRelativity.git
cd FoundationRelativity
```

  Build Commands:

```
# Toolchain is pinned by the repository; no manual override needed
# (elan will read ./lean-toolchain automatically)

# Clean and update dependencies
lake clean && lake update
```

```
# Build Paper 5 AxCal framework
lake build Papers.P5_GeneralRelativity.Main

# Run smoke test (verifies all components)
lake build Papers.P5_GeneralRelativity.Smoke

# Verify no sorries in implementation
grep -r "sorry" Papers/P5_GeneralRelativity/*.lean | \
  grep -v "-- sorry" | wc -l  # Should output: 0

# Run schematic/axiom audits (match CI)
bash scripts/SchematicAudit.sh
bash scripts/AxiomDeclAudit.sh

# Optional (informational): print transitive axiom dependencies
# (will not fail; used for human-readable auditing in CI logs)
lake env lean scripts/AxiomAudit.lean || true
```

Verification Outputs:

- *All 7 height certificates compile: G1 (vacuum), G2 (local PDE + MGHD), G3 (Penrose), G4 (maximal extension), G5 (computable evolution + stream)*
- *Portal theorems correctly map flags to heights*
- ***Sprint 3 Complete**: Schwarzschild vacuum solution fully verified with explicit Christoffel symbols and Ricci tensor vanishing*
- *EPS and Schwarzschild Height 0 anchors verified (structurally)*
- *Profile computation: Zorn* $\to (1,0,0)$*, Limit-Curve* $\to (0,1,0)$*, Serial-Chain* $\to (1,0,0)$*, Reductio* $\to (0,0,1)$

6 Calibration table (profiles at a glance)

Target	**Profile** $(h_{\text{Choice}}, h_{\text{Comp}}, h_{\text{Logic}})$	**Flags/Portals used**	**Notes**
G1: explicit vacuum	$(0,0,0)$	none	symbol
G2: Cauchy/MGHD	$\geq (1,0,0)$ (global)	uses_zorn	local P
G3: singularities	$\geq (0,1,1)$	uses_limit_curve, uses_reductio	compac
G4: maximal extension	$\geq (1,0,0)$	uses_zorn	Zorn p
G5a: computability negative (PER)	$(0,0,0)$	none	negativ
G5b: measurement stream	$(1,0,0)$	uses_serial_chain	DC_ω p

7 Artifact Mapping: Paper Claims to Lean Implementation

7.1 Core AxCal Infrastructure

Paper Concept	Lean Symbol	Module
Height levels $(0, 1, \omega)$	`Height.zero/.one/.omega`	`AxCalCore.Axis`
Axis profile $(h_{\text{Choice}}, h_{\text{Comp}}, h_{\text{Logic}})$	`AxisProfile`	`AxCalCore.Axis`
Witness family type	`WitnessFamily`	`AxCalCore.Axis`
Height certificate structure	`HeightCertificate`	`GR.Certificates`
Portal flags (Zorn, Limit-Curve, etc.)	`PortalFlag`	`GR.Portals`
Route-to-profile mapping	`route_to_profile`	`GR.Portals`

7.2 GR-Specific Components

Paper Concept	Lean Symbol	Module
Pinned signature Σ_0^{GR}	`Spacetime`, `LorentzMetric`	`GR.Interfaces`
Einstein Field Equations	`EFE`, `VacuumEFE`	`GR.Interfaces`
Schwarzschild pinning	`IsPinnedSchwarzschild`	`GR.Interfaces`

7.3 Calibration Targets (G1–G5)

Target	Witness Family	Certificate	Verified Profile
G1: Vacuum	`GR.G1_Vacuum_W`	`G1_Vacuum_Cert`	$(0, 0, 0)$ ✓
G2: Local PDE	`GR.G2_LocalPDE_W`	`G2_LocalPDE_Cert`	$(0, 0, 0)$ ✓
G2: MGHD	`GR.G2_MGHD_W`	`G2_MGHD_Cert`	$(1, 0, 0)$ ✓
G3: Penrose	`GR.G3_Penrose_W`	`G3_Penrose_Cert`	$(0, 1, 1)$ ✓
G4: MaxExt	`GR.G4_MaxExt_W`	`G4_MaxExt_Cert`	$(1, 0, 0)$ ✓
G5: CompNeg	`GR.G5_CompNeg_W`	`G5_CompNeg_Cert`	$(0, 0, 0)$ ✓
G5: Stream	`GR.G5_MeasStream_W`	`G5_MeasStream_Cert`	$(1, 0, 0)$ ✓

7.4 Deep-Dive Deliverables (Height 0 Anchors)

Deliverable	Main Theorem	Implementation Status
D1: EPS Kinematics Core	`EPS_Height_Zero` `EPS_Kinematics_Height0`	Complete framework `GR.EPSCore`

D2: Schwarzschild Vacuum	`Ricci_vanishing`
	
	`Schwarzschild_is_vacuum`
		Sprint 3 Complete:
		
		Full symbolic verification
		`GR.Schwarzschild`

7.5 Portal Theorems

Portal	Lean Implementation	Effect on Profile
Zorn Portal	`Zorn_portal` axiom	$h_{\text{Choice}} \leftarrow 1$
Limit-Curve Portal	`LimitCurve_portal` axiom	$h_{\text{Comp}} \leftarrow 1$
Serial-Chain Portal	`SerialChain_portal` axiom	$h_{\text{Choice}} \leftarrow 1$ (via DC_ω)
Reductio Portal	`Reductio_portal` axiom	$h_{\text{Logic}} \leftarrow 1$

7.6 Verification Infrastructure

Component	Lean Symbol	Purpose
Main aggregator	`Paper5_Main`	Verifies framework completeness `Main`
Profile computation test	`Profile_Computation_Works`	Tests portal→height mapping `Main`
Smoke test	`Paper5_Smoke_Success`	CI aggregator, no-sorry guard `Smoke`
Certificate registry	`Certificates.all_certificates`	Lists all 7 height certificates `GR.Certificates`

7.7 Verification Ledger

The AxCal framework maintains a structured ledger that tracks the provenance of each height assignment. Each certificate in `GR/Certificates.lean` includes: (i) the witness family defining the mathematical claim, (ii) the list of portal flags triggered by the standard proof route, (iii) the resulting height profile computed via `route_to_profile`, (iv) bibliographic citations to the source literature, and (v) a constructive upper bound proof or axiom import. This ledger design ensures that height costs are auditable and that alternative proof routes (avoiding certain portals) can be systematically explored. The framework correctly computes that Zorn's lemma triggers $h_{\text{Choice}} = 1$, limit-curve compactness triggers $h_{\text{Comp}} = 1$, and proof by contradiction triggers $h_{\text{Logic}} = 1$. All seven certificates (G1 vacuum, G2 local PDE, G2 MGHD, G3 Penrose, G4 maximal extension, G5 computability negative, G5 measurement stream) compile without `sorry` in classical `mathlib` and produce the expected height profiles as verified by `Paper5_Main` and the smoke tests. This constitutes *Structural Certification* of the AxCal bookkeeping rather than formal constructive proof.

8 Conclusion

This paper is not a GR formalization for its own sake: it is an *AxCal map* of GR. Portals, route flags, and HeightCertificates turn the folklore "this uses choice/compactness/LEM" into machine-checkable artifacts that compose across the theory. The deep-dive tasks (EPS; Schwarzschild) supply mathematically Height 0 anchors verified via *Structural Certification* in classical `mathlib`, ensuring the project yields verifiable AxCal infrastructure while the schematic layer documents axiomatic cost with precision. Future work could provide formal constructive certificates using dedicated constructive libraries.

References

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A Portals and Proof–Route Flags: Soundness

We make explicit the mechanism that transports standard proof routes in GR into AxCal height costs.

A.1 Route flags and their semantics

Definition A.1 (Route flag)**.** A *route flag* is a marker indicating that a derivation of a witness explicitly invokes a standard device:

$$\text{Flag} \in \{\text{uses\_zorn},\ \text{uses\_limit\_curve},\ \text{uses\_serial\_chain},\ \text{uses\_reductio}\}.$$

Definition A.2 (Usage predicate)**.** For a given derivation $\mathcal{D}$, $\text{Uses}(\text{Flag}, \mathcal{D})$ is the proposition that the corresponding device is *actually used as a step* in $\mathcal{D}$ (not merely available or admissible). Formally, in Lean we record this as a Prop argument to the witness:

$$\mathsf{Uses}\ \text{Flag} \quad \text{is a hypothesis to the witness family for that result.}$$

Remark A.3 (Route sensitivity)*.* Height costs attach to *routes*, not just to statements. If the same theorem admits a proof avoiding a flagged device, then the certificate for that alternate route carries a lower profile.

A.2 Zorn's lemma portal (Choice axis)

Proposition A.4 (Zorn portal: uses_zorn $\Rightarrow$ {AC})**.** *Over* ZF*, any derivation* $\mathcal{D}$ *that uses Zorn's Lemma triggers a positive frontier on the Choice axis: if* $\text{Uses}(\text{uses\_zorn}, \mathcal{D})$*, then* AC *is required.*

Proof. In ZF, the following are equivalent: the Axiom of Choice (AC), Zorn's Lemma, and the Hausdorff Maximal Principle; see any standard reference (e.g. Jech, *Set Theory*, Thm. 8.1; Howard–Rubin, *Consequences of the Axiom of Choice*). Hence invoking Zorn in $\mathcal{D}$ imports a principle equivalent to AC. In AxCal, we reflect this by a portal axiom

$$\text{Uses}(\text{uses\_zorn}, \mathcal{D}) \ \Rightarrow\ \text{HasAC}(F),$$

which feeds the Choice coordinate of the height profile. $\square$

A.3 Limit–curve portal (Compactness axis)

Proposition A.5 (Limit–curve portal: uses_limit_curve $\Rightarrow$ {FT or WKL_0})**.** *Suppose a GR derivation $\mathcal{D}$ invokes a* limit–curve *argument: from a sequence of causal curves with uniform local bounds (e.g. equicontinuity and uniform speed control), $\mathcal{D}$ extracts a convergent subsequence (or a limit curve) without quantitative moduli of compactness. Then*

$$\mathrm{Uses}(\text{uses\_limit\_curve}, \mathcal{D}) \;\Rightarrow\; (\mathrm{HasFT}(F) \textit{ or } \mathrm{HasWKL}_0(F)).$$

Proof sketch. Limit–curve arguments in Lorentzian geometry are typically instances of Arzelà–Ascoli–type compactness on spaces of curves (or a diagonal Bolzano–Weierstraß selection). Over the classical base RCA_0, the necessary sequential compactness for $[0, 1]$ and the Bolzano–Weierstraß theorem are equivalent to WKL_0 (Simpson, *Subsystems of Second Order Arithmetic*, Chs. III–IV). Over constructive bases, Heine–Borel/compactness of Cantor space (hence $[0, 1]$ via coding) is calibrated by the Fan Theorem FT (Bishop–Bridges, *Constructive Analysis*; Troelstra–van Dalen, *Constructivism in Mathematics*). Thus a non-quantitative subsequence/limit extraction imports compactness strength: classically WKL_0; constructively FT. We record this disjunctively, to be resolved by the chosen base. □

Remark A.6. If one supplies explicit moduli (e.g. an effective Arzelà–Ascoli hypothesis), the portal can be avoided; the certificate for that route will then carry a lower compactness height.

A.4 Serial–chain portal (Dependent Choice axis)

Proposition A.7 (Serial–chain portal: uses_serial_chain $\Rightarrow$ {DC_ω})**.** *Let $R \subseteq X \times X$ be serial: $\forall x \in X\ \exists y \in X\ (xRy)$. If a derivation $\mathcal{D}$ requires the existence of an infinite R–chain $(x_n)_{n\in\mathbb{N}}$ with $x_n R x_{n+1}$, then*

$$\mathrm{Uses}(\text{uses\_serial\_chain}, \mathcal{D}) \;\Rightarrow\; \mathrm{HasDC}_\omega(F).$$

In the AxCal profile we record this on the Choice axis: $h_{\mathrm{Choice}} \leftarrow 1$.

Proof. This is exactly the axiom of Dependent Choice (for ω) specialized to a serial relation. In ZF, DC_ω is strictly weaker than AC and sufficient to construct such chains; in BISH, the same scheme expresses the iteration of countably many dependent selections. The portal records this as a foundation-scoped token. □

A.5 Reductio portal (Logic axis)

Proposition A.8 (Reductio portal: uses_reductio $\Rightarrow$ {LEM} (upper bound))**.** *If a derivation $\mathcal{D}$ obtains an* existential *or a* disjunctive *conclusion solely by contradiction (i.e. using $\neg\neg\varphi \Rightarrow \varphi$ at top level, not under stable predicates), then*

$$\mathrm{Uses}(\text{uses\_reductio}, \mathcal{D}) \;\Rightarrow\; \mathrm{HasLEM}(F),$$

giving an upper bound on the Logic axis.

Proof sketch. In intuitionistic/constructive settings, double–negation elimination is not generally valid; it becomes available under LEM (or specific semi–classical schemes for restricted formula classes). The standard proofs of the singularity theorems—e.g. Penrose—often conclude by contradiction from a global completeness hypothesis without providing a constructed witness; see Hawking–Ellis, *Large Scale Structure*, and Wald, *General Relativity*. We therefore mark the route with uses_reductio and import LEM as a conservative upper bound. If a route is reworked into a stable/existentially constructive form, the flag can be removed and the profile lowered. References: Troelstra–van Dalen; Bridges–Richman. □

A.6 Portal soundness summary

Combining Propositions A.4–A.8 yields the meta–level transport principle used throughout:

> *Portal Soundness.* If a certificate includes a set of route flags, then the corresponding tokens on the AxCal axes are admissible in the foundation, yielding the advertised height profile as an *upper bound*. When the route is replaced by one without a given flag, the corresponding coordinate can be lowered.

B AxCal–Lean Ledger

This appendix records the Lean-facing names for tokens, portal axioms, witness families, and height certificates used in the calibration of G1–G5. It functions as a machine-checkable index aligning the paper's calibration table with repository artifacts.

B.1 Axis tokens and portal axioms

```
-- AxCal core tokens (foundation-scoped)
class HasAC   (F : Foundation) : Prop
class HasDCw  (F : Foundation) : Prop
class HasFT   (F : Foundation) : Prop
class HasWKL0 (F : Foundation) : Prop
class HasLEM  (F : Foundation) : Prop
class HasWLPO (F : Foundation) : Prop

/-- Proof-route flags (carried in certificates; see §\ref{app:certs}) -/
inductive PortalFlag
| uses_zorn
| uses_limit_curve
| uses_serial_chain
| uses_reductio

/-- Portal soundness axioms (paper Proposition~\ref{prop:portals}).
    They are registered once per foundation F. -/
axiom Zorn_portal     : forall {F}, Uses PortalFlag.uses_zorn         -> HasAC   F
axiom LimitCurve_portal : forall {F}, Uses PortalFlag.uses_limit_curve -> (HasFT F or HasWKL0
axiom SerialChain_portal : forall {F}, Uses PortalFlag.uses_serial_chain -> HasDCw F
axiom Reductio_portal  : forall {F}, Uses PortalFlag.uses_reductio     -> HasLEM F
```

Notes.

- The wrapper `Uses flag` is a Prop recording that the corresponding proof-route is actually used in the provided derivation (not merely available in the library). This is what ties the *route* to the *frontier cost*.
- The compactness portal is recorded disjunctively (`HasFT or HasWKL0`) to reflect constructive/classical bases; the certificate chooses the branch used in the imported argument.

B.2 Witness families for G1–G5

```
-- Pinned signature Sigma0^GR (interfaces only; no mathlib dependency)
structure Manifold := ...
structure LorentzMetric (M : Manifold) := ...
structure Spacetime := (M : Manifold) (g : LorentzMetric M)

-- Einstein tensor interface and EFE predicate
def EinsteinTensor (S : Spacetime) : Tensor := ...
def EFE (S : Spacetime) (T : Tensor) : Prop := ...

-- WitnessFamily type (from AxCal core)
--   WitnessFamily F  := Prop (witness existence over foundation F)

namespace GR

/-- G1: explicit vacuum check (Schwarzschild@pin) -/
def G1_Vacuum_W : WitnessFamily := fun F =>
  forall (Ssch : Spacetime), IsPinnedSchwarzschild Ssch -> EFE Ssch ZeroTensor

/-- G2: Cauchy problem split into local PDE and MGHD (global) -/
def G2_LocalPDE_W : WitnessFamily := fun F =>
  forall (ID : InitialData), LocalWellPosed ID         -- no portal flags
def G2_MGHD_W     : WitnessFamily := fun F =>
  forall (ID : InitialData), Uses PortalFlag.uses_zorn -> MGHD_Exists ID

/-- G3: Singularity theorem (schematic Penrose) -/
def G3_Penrose_W : WitnessFamily := fun F =>
  forall (S : Spacetime),
    (NullEnergyCondition S) →
    (HasTrappedSurface S)    →
    Uses PortalFlag.uses_limit_curve →
    Uses PortalFlag.uses_reductio     →
    ¬ GeodesicallyComplete S

/-- G4: Maximal extension existence -/
def G4_MaxExt_W : WitnessFamily := fun F =>
  forall (S : Spacetime),
    Uses PortalFlag.uses_zorn →
    exists Smax, IsMaximalExtension S Smax

/-- G5: Computable evolution (negative template and DC stream) -/
def G5_CompNeg_W : WitnessFamily := fun F =>
  exists (class : GHClass),
    ComputableInitialData class and
    NonComputableEvolution class  -- PER-style failure

def G5_MeasStream_W : WitnessFamily := fun F =>
```

```
  HasDCw F -> (forall proto : SerialProtocol, InfiniteHistory proto)

end GR
```

B.3 Height certificates (profiles and routes)

```
-- Axis triple: (Choice, Compactness, Logic)
structure AxisProfile := (hChoice hComp hLogic : Height)   -- Height in {zero, one, omega}
structure HeightCertificate :=
{ W        : WitnessFamily
, profile  : AxisProfile
, flags    : List PortalFlag
, upper    : ProfileUpper profile W        -- constructive upper proof or portal imports
, cites    : List Citation                 -- paper-level references used
}

-- Concrete certificates (G1--G5)
def G1_Vacuum_Cert : HeightCertificate :=
{ W       := GR.G1_Vacuum_W
, profile := <zero, zero, zero>
, flags   := []
, upper   := by
    -- symbolic curvature computation at the pin (no portals)
    exact upper_height0_vacuum_check
, cites   := [cite "Wald §B.4"]
}

def G2_LocalPDE_Cert : HeightCertificate :=
{ W       := GR.G2_LocalPDE_W
, profile := <zero, zero, zero>  -- or ⟨one, zero, zero> if ACw is used in analysis
, flags   := []
, upper   := import_local_pde_result
, cites   := [cite "Choquet-Bruhat (2009)"]
}

def G2_MGHD_Cert : HeightCertificate :=
{ W       := GR.G2_MGHD_W
, profile := <one, zero, zero>
, flags   := [PortalFlag.uses_zorn]
, upper   := by
    intro F ID hzorn
    have hAC : HasAC F := Zorn_portal hzorn
    exact imported_mghd_existence hAC
, cites   := [cite "Wald Thm. 10.1.2"]
}

def G3_Penrose_Cert : HeightCertificate :=
{ W       := GR.G3_Penrose_W
```

```
, profile := <zero, one, one>
, flags   := [PortalFlag.uses_limit_curve, PortalFlag.uses_reductio]
, upper   := by
    intro F S nec trapped hlim hred
    have hComp : (HasFT F or HasWKL0 F) := LimitCurve_portal hlim
    have hLEM  : HasLEM F              := Reductio_portal hred
    exact imported_penrose hComp hLEM nec trapped
, cites   := [cite "Hawking-Ellis §8", cite "Wald §14"]
}

def G4_MaxExt_Cert : HeightCertificate :=
{ W       := GR.G4_MaxExt_W
, profile := <one, zero, zero>
, flags   := [PortalFlag.uses_zorn]
, upper   := by
    intro F S hz
    exact imported_maximal_extension (Zorn_portal hz)
, cites   := [cite "Wald §10.1"]
}

def G5_CompNeg_Cert : HeightCertificate :=
{ W       := GR.G5_CompNeg_W
, profile := <zero, zero, zero>  -- negative template; no portal cost tracked
, flags   := []
, upper   := imported_PER_negative_template
, cites   := [cite "Pour-El-Richards (1989)"]
}

def G5_MeasStream_Cert : HeightCertificate :=
{ W       := GR.G5_MeasStream_W
, profile := <one, zero, zero>  -- DC sits on the Choice axis
, flags   := [PortalFlag.uses_serial_chain]
, upper   := by
    intro F hDC proto
    exact SerialChain_portal_elim hDC proto
, cites   := [cite "AxCal DC eliminator"]
}
```

B.4 Verification table (names / profiles / portals)

Target	Witness (Lean)	Certificate (Lean)	Flags	Profile
G1	`GR.G1_Vacuum_W`	`G1_Vacuum_Cert`	–	$(0, 0, 0)$
G2 (local)	`GR.G2_LocalPDE_W`	`G2_LocalPDE_Cert`	–	$(0, 0, 0)$ or $(1, 0, 0)$
G2 (MGHD)	`GR.G2_MGHD_W`	`G2_MGHD_Cert`	Zorn	$(1, 0, 0)$
G3	`GR.G3_Penrose_W`	`G3_Penrose_Cert`	LimitCurve, Reductio	$(0, 1, 1)$
G4	`GR.G4_MaxExt_W`	`G4_MaxExt_Cert`	Zorn	$(1, 0, 0)$
G5 (neg.)	`GR.G5_CompNeg_W`	`G5_CompNeg_Cert`	–	$(0, 0, 0)$
G5 (stream)	`GR.G5_MeasStream_W`	`G5_MeasStream_Cert`	SerialChain	$(1, 0, 0)$

B.5 File map (proposed layout)

```
Papers/P5_GR/
  AxCalCore/Axis.lean              -- Height, AxisProfile, ProfileUpper
  AxCalCore/Tokens.lean            -- HasAC, HasDCw, HasFT, HasWKL0, HasLEM, HasWLPO
  GR/Interfaces.lean               -- Sigma0^GR: manifolds, Lorentz metrics, EFE predicate
  GR/Portals.lean                  -- PortalFlag, Zorn_portal, LimitCurve_portal, ...
  GR/Witnesses.lean                -- G1_Vacuum_W, G2_*, G3_*, G4_*, G5_*
  GR/Certificates.lean             -- G*_Cert definitions (HeightCertificate)
  GR/EPSCore.lean                  -- (deep-dive) EPS kinematics proofs (Height 0)
  GR/Schwarzschild.lean            -- (deep-dive) vacuum check engine (Height 0)
  Ledger/Citations.lean            -- structured bibliography handles for certificates
  Smoke.lean                       -- CI aggregator; no-sorry guard for deep-dive dirs
```

B.6 Ledger policy

- Every certificate includes *flags* (route evidence) and *cites* (bibliographic anchors).
- Replacing an imported theorem by an internal proof that avoids a flagged portal *automatically* lowers the certificate's profile (the AxCal algebra recomputes heights componentwise).
- Disjunctive compactness (`HasFT or HasWKL0`) must be resolved per foundation instance to produce a concrete (h_{Comp}) entry.