A BISH-Complete Domain: Yukawa Renormalization as a Finite Discrete Map

Technical Note 18 in the Constructive Reverse Mathematics Series

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Abstract

The Standard Model Yukawa beta functions, treated as a discrete finite-step map rather than a continuous ODE, constitute a computation entirely within Bishop's constructive mathematics (BISH). We verify numerically that the top quark quasi-fixed-point is visible at $N = 10$ discrete RK4 steps and that the full fermion mass hierarchy is *not* an attractor of the one-loop Standard Model RG. This is the first domain in the constructive reverse mathematics series where the entire physically relevant computation is BISH with no LPO boundary, confirming that the BMC $\leftrightarrow$ LPO boundary observed in five other domains is a property of specific physics—completed limits of bounded monotone sequences—not a universal feature of mathematical physics.

Dedicated to Mimi, my wife.

Soli Deo gloria

1 Introduction

The constructive reverse mathematics (CRM) programme calibrates the logical cost of physical theories by identifying which constructive principles are needed at each layer of the mathematical description. Bishop's constructive mathematics (BISH) requires every existential claim to come with a construction; the Limited Principle of Omniscience (LPO) asserts that for any binary sequence, either some term equals 1 or all terms equal 0. LPO is equivalent to Bounded Monotone Convergence (BMC): every bounded monotone sequence converges to a limit Bridges and Vîţă [2006].

Five independent physics domains exhibit the BMC $\leftrightarrow$ LPO boundary: statistical mechanics (Paper 8 Lee [2026a]), general relativity (Paper 13 Lee [2026c]), quantum decoherence (Paper 14 Lee [2026d]), conservation laws (Paper 15 Lee [2026e]), and quantum gravity (Paper 17 Lee [2026f]). In each case, finite computations are BISH, while the assertion that a bounded monotone sequence converges to a completed real number costs LPO. The working

*New York University. AI-assisted numerical investigation; see the methodology statement for details. The author is a medical professional, not a domain expert in constructive mathematics, particle physics, or renormalization group theory; the numerical investigation and analysis were developed with extensive AI assistance.

hypothesis Lee [2026b] is that empirical predictions are BISH-derivable and stronger principles enter only through idealizations.

This pattern generates a prediction: in domains where no completed limit is needed, the entire computation should be BISH, with no LPO boundary. The Standard Model renormalization group beta functions, evaluated as a discrete finite-step map, provide a natural test case. Every step is finite arithmetic—polynomial evaluation and addition. No sequence converges; no limit is taken. We ask whether this BISH computation contains physically relevant structure, and whether it might reveal mechanisms for the fermion mass hierarchy that are invisible in the conventional continuous-flow formalism.

2 The Computation

The one-loop beta functions for the third-generation Yukawa couplings in the Standard Model are Machacek and Vaughn [1984], Luo et al. [2003]:

$$16\pi^2 \frac{dy_t}{dt} = y_t \left[\tfrac{9}{2}y_t^2 + \tfrac{3}{2}y_b^2 + y_\tau^2 - 8g_3^2 - \tfrac{9}{4}g_2^2 - \tfrac{17}{12}g_1^2\right], \tag{1}$$

$$16\pi^2 \frac{dy_b}{dt} = y_b \left[\tfrac{9}{2}y_b^2 + \tfrac{3}{2}y_t^2 + y_\tau^2 - 8g_3^2 - \tfrac{9}{4}g_2^2 - \tfrac{5}{12}g_1^2\right], \tag{2}$$

$$16\pi^2 \frac{dy_\tau}{dt} = y_\tau \left[\tfrac{5}{2}y_\tau^2 + 3y_b^2 + 3y_t^2 - \tfrac{9}{4}g_2^2 - \tfrac{15}{4}g_1^2\right], \tag{3}$$

where $t = \ln(\mu/\mu_0)$ and g_1, g_2, g_3 are the $\mathrm{U}(1)_Y$, $\mathrm{SU}(2)_L$, $\mathrm{SU}(3)_C$ gauge couplings, running at one loop as

$$16\pi^2 \frac{dg_i}{dt} = b_i\, g_i^3, \qquad b_1 = \tfrac{41}{6}, \quad b_2 = -\tfrac{19}{6}, \quad b_3 = -7. \tag{4}$$

Lighter-generation Yukawa couplings satisfy analogous equations with the same gauge coefficients.

We use gauge couplings at $M_Z = 91.1876$ GeV in the $\overline{\mathrm{MS}}$ scheme ($g_1 = 0.3574$, $g_2 = 0.6518$, $g_3 = 1.221$) and tree-level Yukawa couplings $y_f = \sqrt{2}\, m_f/v$ with $v = 246.22$ GeV and PDG 2024 pole masses Particle Data Group [2024]. The scale range from M_Z to the Planck mass is $t_{\text{range}} \approx 39.4$.

The standard RG flow is a continuous ODE whose solution requires computing an integral—a limit of Riemann sums—which in general costs LPO. The discrete map replaces the ODE with a finite iteration: N applications of a fourth-order Runge–Kutta (RK4) step, each evaluating the beta functions (1)–(4) at intermediate points. Each step is finite arithmetic—BISH by definition. For $N = 10$, the entire computation involves approximately 480 floating-point multiplications and a comparable number of additions.

3 Results

Top Quasi-Fixed-Point

We scan y_t(Planck) over $[0.1, 10]$ with $N = 1{,}000$ RK4 steps. The Pendleton–Ross quasi-fixed-point (fig. 1) is confirmed: y_t(EW) converges to ≈ 1.29 for all $y_t(\text{Planck}) \gtrsim 0.7$, a basin encompassing 58% of scanned initial conditions. The 30% overshoot relative to the observed $y_t(M_Z) = 0.99$ is a known artifact of one-loop running without threshold corrections Pendleton and Ross [1981], Hill [1981].

Discrete Map vs. Continuous Flow

The QFP is genuinely finite-order structure. Table 1 shows that RK4 at $N = 50$ matches scipy's adaptive integrator to four decimal places. Figure 2 shows the basin is already visible at $N = 10$: the characteristic flattening for $y_t(\text{Planck}) > 0.7$ is present with spread less than 20% of the mean across the plateau.

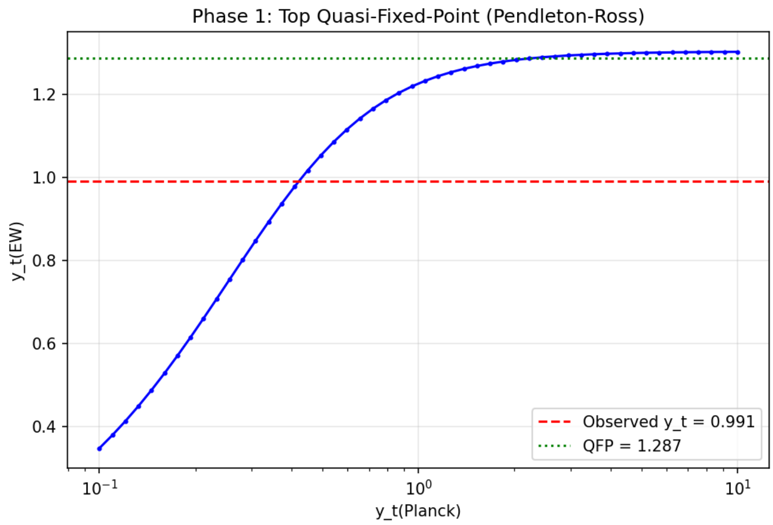

Figure 1: Top Yukawa at the EW scale versus initial value at the Planck scale ($N = 1{,}000$ RK4 steps). The curve flattens for $y_t(\text{Planck}) \gtrsim 0.7$, demonstrating the Pendleton–Ross quasi-fixed-point.

N	Euler	RK4
10	0.4293	1.2871
50	1.2390	1.2936
100	1.2644	1.2936
500	1.2874	1.2936
1,000	1.2904	1.2936
10,000	1.2932	1.2936
scipy (continuous)		1.2936

Table 1: $y_t(\text{EW})$ for various step counts N and integration methods. RK4 at $N = 50$ matches the continuous reference to four decimal places.

Mass Hierarchy

Of 3,000 randomly sampled initial conditions (all Yukawa couplings log-uniform on $[0.01, 10]$), none produce the observed mass ratios within one order of magnitude (fig. 3). The median RMS log-ratio error is 3.38 dex. The full fermion mass hierarchy is *not* an attractor of the one-loop SM RG: mass ratios are sensitive to initial conditions.

Secondary Observations

The bottom/tau mass ratio shows weak structure: median $y_b/y_\tau = 2.25$ at the EW scale (observed 2.35), but only 6% of initial conditions fall within 20% of the observed value. The Koide ratio Q yields a mean of 0.50 (observed 2/3), with only 3.2% of filtered samples within 1% of 2/3. Neither relation emerges generically from SM RG evolution. Two-loop corrections shift the QFP by -0.65% without producing qualitatively new structure.

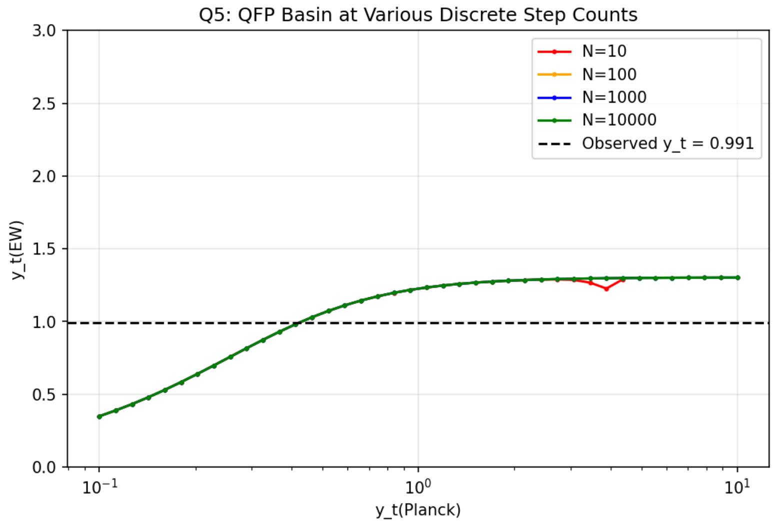

Figure 2: Top QFP basin at $N = 10$, 100, 1,000, and 10,000 discrete RK4 steps. The quasi-fixed-point structure is already visible at $N = 10$.

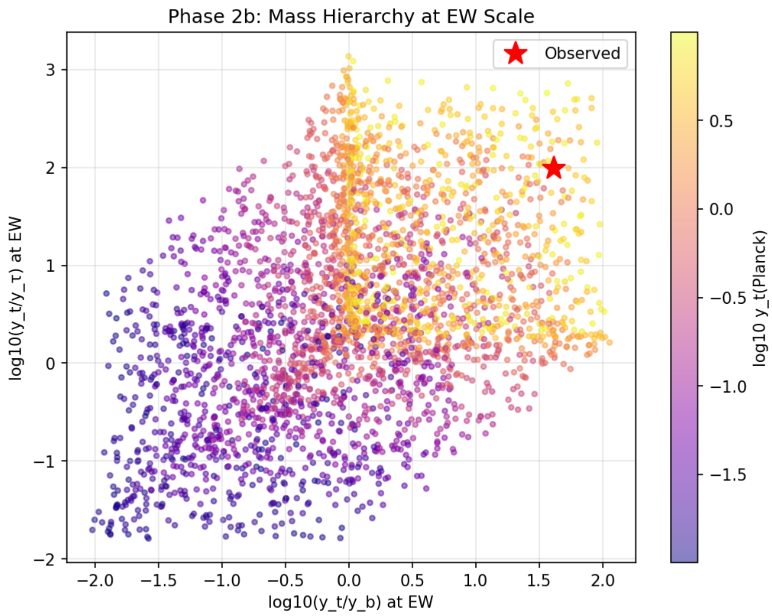

Figure 3: Mass hierarchy at the EW scale for 3,000 random Planck-scale initial conditions. The red star marks the observed values. No clustering near the observed point is visible.

4 Discussion

4.1 CRM as Diagnostic, Not Generative

The investigation tested a stronger hypothesis than the one confirmed: whether CRM thinking could be used *generatively*—not just to diagnose the logical cost of known physics, but to reveal mechanisms for open problems. The fermion mass hierarchy was the test case. The Standard Model has 13 free Yukawa parameters spanning six orders of magnitude; the specific hope was that stripping the LPO scaffolding from the RG (replacing the continuous flow with a finite discrete map) would expose algebraic structure—quasi-fixed-points for the full mass spectrum—invisible in the conventional formalism.

The hypothesis is refuted. The discrete map reveals the same structure as the continuous flow: a top-quark attractor and nothing else. The LPO content of the continuous formulation (the completed flow as a limit of discrete steps) is dispensable—the BISH content suffices—but the BISH content is not *different*. The mass hierarchy requires ultraviolet input that the SM's infrared dynamics do not determine, and no logical reorganization changes this. CRM is an excellent diagnostic tool; it is not a physics generator.

4.2 The BISH-Only Domain

What the investigation does provide is the first domain in the CRM series where everything is BISH and no LPO boundary appears. The five domains in Papers 8–17 all involve bounded monotone sequences whose completed limits cost LPO. The Yukawa RG involves no such sequence: evaluate the beta function a finite number of times, obtain the coupling at the electroweak scale. The "flow" need not *converge*; one only needs the output at the target scale.

The discrimination is structural (table 2): LPO appears when physicists assert the existence of a completed limit, and does not appear when the physical prediction requires only a finite computation. This confirms that the BMC $\leftrightarrow$ LPO boundary is not an artifact of the formalization method but a genuine property of specific physics—thermodynamic limits, geodesic incompleteness, exact decoherence, global conservation, and entropy density convergence all require completed limits; the RG discrete map does not.

Domain	**Paper**	BISH **Content**	LPO **Content**
Statistical Mechanics	8	Finite-volume free energy	Thermodynamic limit
General Relativity	13	Finite-time geodesic	Geodesic incompleteness
Quantum Measurement	14	Finite-step decoherence	Exact decoherence
Conservation Laws	15	Local energy conservation	Global energy
Quantum Gravity	17	Finite entropy count	Entropy density limit
Particle Physics (RG)	**18**	**Finite-step Yukawa evolution**	**None**

Table 2: Updated CRM calibration table. The sixth row—all BISH, no LPO—completes the diagnostic picture.

4.3 CRM Analysis of Mass-Problem Approaches

Applied to the fermion mass problem directly, CRM reveals a subtlety: every existing approach operates within BISH (table 3). Flavor symmetries replace 13 unexplained Yukawa couplings with 8–12 unexplained flavon parameters—BISH reorganization, not derivation. Randall–Sundrum models achieve better compression (mild $O(1)$ spread in bulk masses produces exponential hierarchy via warped geometry) but still require unexplained inputs. Radiative mass generation offers the best compression ratio: one universal Yukawa coupling plus perturbative loop suppression generates six orders of magnitude, though the required new particles have not been observed. The Koide formula $Q = 2/3$, satisfied to $\sim 10^{-5}$ precision, is empirically BISH (checking the relation is finite arithmetic), but Sumino's all-orders cancellation mechanism requires LPO Sumino [2009]—the only approach whose *explanation* costs more than BISH Koide [1983].

CRM does not discriminate these approaches by logical cost—they are all BISH. What it reveals is that they differ in *compression ratio*: the number of unexplained parameters needed to produce 13 observables. The mass problem is entirely a problem within BISH—a different kind of mystery from the ones CRM was designed to illuminate.

Limitations. The QFP value $y_t(\mathrm{EW}) \approx 1.29$ exceeds the observed 0.99 by $\sim$30%—a known limitation of one-loop running without threshold corrections. CKM mixing and complete two-loop corrections are neglected but do not affect qualitative conclusions. Unlike Papers 8–17, this investigation is a numerical experiment, not a formal verification; the statement that the discrete map is BISH is trivially true of any finite arithmetic.

Approach	**Logical Cost**	**Unexplained Inputs**
Standard Model (raw)	BISH	13 Yukawa couplings
Flavor symmetries	BISH	8–12 flavon parameters
Randall–Sundrum	BISH	$\sim$6 bulk masses + geometry
Radiative generation	BISH	$\sim$1 universal Yukawa
Koide (empirical)	BISH	2 parameters (μ, δ)
Koide (Sumino mechanism)	LPO	0 (if mechanism works)
Landscape	BISH (formally)	0 (but no prediction)

Table 3: CRM calibration of approaches to the fermion mass hierarchy. All approaches except Sumino's all-orders mechanism operate within BISH; they differ in compression ratio.

5 Conclusion

The Standard Model Yukawa renormalization group, treated as a discrete finite-step map, is entirely BISH. The top quasi-fixed-point—the only attractor structure in the system—is visible at $N = 10$ discrete RK4 steps. The full fermion mass hierarchy is not an attractor; mass ratios are sensitive to ultraviolet initial conditions. This is the first domain in the CRM series with no LPO boundary. Its existence confirms that LPO enters physics through completed limits of bounded monotone sequences, and nowhere else.

The programme archive is maintained at Zenodo (DOI: 10.5281/zenodo.18600243).

AI-Assisted Methodology

This investigation was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface Anthropic [2026], following the same human–AI workflow as Papers 2–17. The author specified the research direction and CRM framing; Claude Opus 4.6 implemented the beta functions, numerical scanning code, and plots. There is no formal verification component.

Task	**Human**	**AI (Claude Opus 4.6)**
Research direction	✓	
CRM framing	✓	
Beta function code		✓
Numerical scans		✓
Result interpretation	✓	✓
Paper writing	✓	✓

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/AICardiologist/FoundationRelativity`
- **Script**: `paper 18/rg_mass_hierarchy.py` ($\sim$600 lines)
- **Dependencies**: Python 3.9+, NumPy, SciPy, Matplotlib
- **Run**: `python3 rg_mass_hierarchy.py` ($\sim$16 min on a 2020 MacBook)
- **Output**: 10 plots in `plots/`, console summary

- **Zenodo DOI**: 10.5281/zenodo.18600243

Acknowledgments

The numerical investigation was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool.

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