The Logical Cost of the Thermodynamic Limit:

LPO-Equivalence and BISH-Dispensability for the 1D Ising Free Energy

A Lean 4 Formalization

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Abstract

We prove two complementary results about the thermodynamic limit of the one-dimensional Ising model, formalized in Lean 4. (A) The finite-size error bound $|f_N(\beta) - f_\infty(\beta)| \leq \frac{1}{N}\tanh(\beta)^N$ is provable in Bishop-style constructive mathematics (BISH) without any omniscience principle. A constructive witness N_0 for any prescribed accuracy $\varepsilon > 0$ is explicitly computed. (B) The existence of the thermodynamic limit as a completed real number is equivalent over BISH to the Limited Principle of Omniscience (LPO), via the known equivalence between LPO and bounded monotone convergence instantiated through the Ising free energy function. Together, these results establish that the LPO cost of the thermodynamic limit is genuine and dispensable: the idealization costs exactly LPO, but the empirical content requires no omniscience. The combined formalization comprises 1374 lines of Lean 4 across 18 modules with zero sorries.

Contents

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1 Introduction

The thermodynamic limit is the foundational idealization of equilibrium statistical mechanics. For a lattice system $\Lambda \subset \mathbb{Z}^d$ with Hamiltonian H_Λ, the free energy density $f_\Lambda(\beta) = -\frac{1}{|\Lambda|}\log Z_\Lambda(\beta)$ is defined for each finite volume. The thermodynamic limit asserts that $f_\infty(\beta) = \lim_{|\Lambda|\to\infty} f_\Lambda(\beta)$ exists. Classically, this existence follows from subadditivity and the monotone convergence theorem: a bounded, monotone sequence of real numbers converges.

From a constructive standpoint, the monotone convergence theorem is not available in Bishop-style constructive mathematics (BISH). It is equivalent over BISH to the Limited Principle of Omniscience (LPO), which asserts that for any binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, either $\alpha(n) = 0$ for all n or there exists n_0 with $\alpha(n_0) = 1$. This equivalence was established by Bridges and Vîţă Bridges and Vîţă [2006] as part of the systematic classification of constructive reverse mathematics initiated by Ishihara Ishihara [2006] and, independently, Veldman Veldman [2005].

The question we address is: what is the exact logical cost of the thermodynamic limit, and is this cost essential for the physics?

We answer both questions completely for the one-dimensional Ising model with nearest-neighbour interactions. Our results are as follows.

Part A establishes *dispensability.* For the 1D Ising chain with uniform coupling J and inverse temperature β, we prove explicit error bounds $|f_N(\beta) - f_\infty(\beta)| \le \frac{1}{N}\tanh(\beta)^N$ with a constructive witness N_0 for any prescribed accuracy $\varepsilon > 0$. The proof is entirely BISH-valid: no omniscience principle is required. The finite-system prediction approximates the infinite-volume answer with computable error, and monotone convergence is bypassed via the closed-form transfer-matrix solution.

Part B establishes *calibration.* The existence of the thermodynamic limit as a completed real number (not merely its approximability) is equivalent to LPO over BISH. Specifically, we prove that bounded monotone convergence—instantiated through the free energy function $g(J) = -\log(2\cosh(\beta J))$ of the 1D Ising model—is equivalent to LPO via an explicit encoding of binary sequences into coupling sequences.

Together, these results establish that the LPO cost of the thermodynamic limit is genuine (it is equivalent to, not merely sufficient for, a known omniscience principle) and dispensable (the empirical predictions require no omniscience at all).

Both results are formalized in Lean 4 with Mathlib4 dependencies. The combined formalization comprises 1374 lines across 18 modules, with zero sorries and a clean axiom profile: Part A uses no omniscience principles whatsoever, while Part B's main theorem `lpo_of_bmc` carries only the standard Lean 4 metatheory axioms (`propext`, `Classical.choice`, `Quot.sound`). The forward direction of the LPO $\leftrightarrow$ BMC equivalence is axiomatized as `bmc_of_lpo`, citing Bridges and Vîţă Bridges and Vîţă [2006].

This paper contributes to a programme of constructive reverse mathematics applied to mathematical physics, which assigns to each physical idealization a precise position in the constructive hierarchy. The programme has established the following calibrations:

Physical layer	**Principle**	**Status**
Finite-volume Gibbs states	BISH	Trivially calibrated
Finite-size approximations	BISH	Part A (this paper)
Bidual-gap witness	$\equiv$ WLPO	Papers 2, 7 Lee [2026a,b]
Thermodynamic limit existence	$\equiv$ LPO	Part B (this paper)
Spectral gap decidability	Undecidable	Cubitt et al. Cubitt et al. [2015]

Our results calibrate two rows of this table, upgrading them from "route-costed" to "formally verified."

The paper is organized as follows. Section 2 reviews the constructive framework, the 1D Ising model, and the free energy function. Section 3 presents the BISH dispensability proof (Part A). Section 4 presents the LPO calibration (Part B). Section 5 discusses the synthesis of both parts, their relation to the phase-transition debate, and future directions. Section 6 describes the Lean 4 formalization. Appendix A collects elementary inequalities.

2 Preliminaries

2.1 Constructive Frameworks

We work within Bishop-style constructive mathematics (BISH): intuitionistic logic with countable and dependent choice Bishop [1967], Bishop and Bridges [1985]. The key omniscience principles form a strict hierarchy over BISH:

Definition 2.1 (LPO)**.** ✓ The *Limited Principle of Omniscience* is

$$\text{LPO} \;:\equiv\; \forall \alpha : \mathbb{N} \to \{0,1\},\; \big(\forall n,\; \alpha(n) = 0\big) \;\vee\; \big(\exists n,\; \alpha(n) = 1\big).$$

Definition 2.2 (BMC)**.** ✓ *Bounded Monotone Convergence* is the assertion that every bounded non-decreasing sequence of reals has a limit: for every $a : \mathbb{N} \to \mathbb{R}$ with $a_n \le a_{n+1}$ and $a_n \le M$ for all n, there exists $L \in \mathbb{R}$ and a convergence modulus such that for all $\varepsilon > 0$, there exists N_0 with $|a_N - L| < \varepsilon$ for all $N \ge N_0$.

The equivalence LPO $\leftrightarrow$ BMC was established by Bridges and Vîţă Bridges and Vîţă [2006]. The stronger LPO implies the *Weak* Limited Principle of Omniscience (WLPO: $\forall \alpha, (\forall n, \alpha(n) = 0) \vee \neg(\forall n, \alpha(n) = 0)$), which in turn implies the Lesser Limited Principle of Omniscience (LLPO). Markov's Principle is independent of WLPO over BISH: neither implies the other Bridges and Vîţă [2006].

Remark 2.3 (Constructive status of LPO)**.** LPO is classically trivial (an instance of excluded middle) but constructively independent: it is neither provable nor refutable in BISH. Crucially, LPO provides a *witness* in the second disjunct $(\exists n, \alpha(n) = 1)$, not merely the double negation thereof. This witness-providing character is what makes LPO strictly stronger than WLPO.

2.2 The 1D Ising Model

Fix a positive integer N (the number of spins). The configuration space is $\Omega_N = \{-1, +1\}^N$. The Hamiltonian with periodic boundary conditions and coupling $J > 0$ is

$$H_N(\sigma) = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}, \qquad \sigma_{N+1} := \sigma_1.$$

For simplicity we set $J = 1$ in Part A (the general case follows by rescaling β).

Definition 2.4 (Transfer matrix)**.** ✓ The 2×2 transfer matrix T has entries $T(s, s') = \exp(\beta \cdot s \cdot s')$ for $s, s' \in \{-1, +1\}$:

$$T = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}.$$

Definition 2.5 (Eigenvalues)**.** ✓ The eigenvalues of T are

$$\lambda_+ = e^{\beta} + e^{-\beta} = 2\cosh(\beta), \qquad \lambda_- = e^{\beta} - e^{-\beta} = 2\sinh(\beta).$$

The corresponding eigenvectors are $(1,1)^T/\sqrt{2}$ and $(1,-1)^T/\sqrt{2}$.

Lemma 2.6 (Eigenvalue properties)**.** ✓ *For all* $\beta > 0$*:*

(a) $\lambda_+ > \lambda_- > 0$*,*

(b) $\lambda_+ > 2$*,*

(c) $0 < \lambda_-/\lambda_+ < 1$*,*

(d) $\lambda_-/\lambda_+ = \tanh(\beta)$*, with* $0 < \tanh(\beta) < 1$*.*

Proof. (a) $\lambda_+ - \lambda_- = 2e^{-\beta} > 0$, and $\lambda_- = 2\sinh(\beta) > 0$ for $\beta > 0$. (b) $\cosh(\beta) > 1$ for $\beta > 0$. (c) Immediate from (a). (d) $\lambda_-/\lambda_+ = 2\sinh(\beta)/(2\cosh(\beta)) = \tanh(\beta)$. For $\beta > 0$, the numerator is positive and strictly less than the denominator. □

Remark 2.7 (Constructive validity)**.** All inequalities in Lemma 2.6 are strict and witnessed by explicit positive gaps. For rational β, $\tanh(\beta)$ is a well-defined constructive real with $0 < \tanh(\beta) < 1$, and the gap $1 - \tanh(\beta) = 2e^{-\beta}/(e^{\beta} + e^{-\beta})$ is constructively positive.

Definition 2.8 (Partition function)**.** ✓ The partition function of the 1D Ising model with N spins is defined as

$$Z_N(\beta) := \lambda_+^N + \lambda_-^N.$$

This equals $\mathrm{Tr}(T^N)$, proved in the formalization as a bonus lemma (`PartitionTrace.lean`).

Definition 2.9 (Finite-volume free energy density)**.** ✓

$$f_N(\beta) := -\frac{1}{N} \log Z_N(\beta) = -\frac{1}{N} \log\big(\lambda_+^N + \lambda_-^N\big).$$

Definition 2.10 (Infinite-volume free energy density)**.** ✓

$$f_\infty(\beta) := -\log(\lambda_+) = -\log(2\cosh\beta).$$

Remark 2.11 (Constructive note)**.** The infinite-volume free energy density $f_\infty(\beta)$ is *not* defined as a limit. It is defined by an explicit closed-form expression. For rational $\beta > 0$, $f_\infty(\beta)$ is a constructively well-defined real number. No omniscience principle is needed to define it. The classical route defines $f_\infty = \lim_{N\to\infty} f_N$, proves the limit exists by monotone convergence (LPO), and then computes the limit. We skip the middle step entirely: we *define* f_∞ by closed form, and then *prove* that f_N converges to it with explicit bounds.

2.3 The Free Energy Function $g(J)$

The following function plays a central role in Part B.

Definition 2.12 (Free energy at coupling J)**.** ✓ For $\beta > 0$ and $J > 0$, define

$$g(J) := -\log(2\cosh(\beta J)).$$

This is the infinite-volume free energy density of the 1D Ising chain with uniform coupling J.

Lemma 2.13 (Strict anti-monotonicity of g)**.** ✓ *For $\beta > 0$, the function g is strictly decreasing on $(0, \infty)$: if $0 < J_0 < J_1$, then $g(J_0) > g(J_1)$.*

Proof. The chain of implications is: $J_1 > J_0 > 0$ implies $\beta J_1 > \beta J_0 > 0$ (multiply by $\beta > 0$), implies $\cosh(\beta J_1) > \cosh(\beta J_0)$ (since cosh is strictly increasing on $(0, \infty)$), implies $\log(2\cosh(\beta J_1)) > \log(2\cosh(\beta J_0))$ (since log is strictly increasing on $(0, \infty)$), implies $g(J_1) < g(J_0)$ (negate both sides). □

Lemma 2.14 (Gap lemma)**.** ✓ *Fix $\beta > 0$ and $0 < J_0 < J_1$. Then*

$$\delta := g(J_0) - g(J_1) = \log \frac{\cosh(\beta J_1)}{\cosh(\beta J_0)} > 0.$$

Proof. Immediate from Lemma 2.13: $J_1 > J_0$ implies $g(J_1) < g(J_0)$, so $g(J_0) - g(J_1) > 0$. □

Remark 2.15 (Constructive positivity of δ)**.** For rational β, J_0, J_1, the gap δ is a constructively computable positive real. The positivity is witnessed by an explicit lower bound computable from β, J_0, J_1 via the power series of cosh. This is essential: the decision procedure in Part B (Section 4) uses $\delta > 0$ as a constructive fact.

3 Part A: BISH Dispensability

We now prove that the finite-size error bound for the 1D Ising model is provable in BISH without any omniscience principle.

3.1 Free Energy Decomposition

Lemma 3.1 (Decomposition)**.** ✓ *For all $N \geq 1$ and $\beta > 0$, with $r = \tanh(\beta)$:*

$$f_N(\beta) = -\log(\lambda_+) - \frac{1}{N}\log(1 + r^N).$$

Proof. We compute:

$$\begin{aligned} f_N(\beta) &= -\frac{1}{N}\log(\lambda_+^N + \lambda_-^N) \\ &= -\frac{1}{N}\log\big(\lambda_+^N(1 + (\lambda_-/\lambda_+)^N)\big) \\ &= -\frac{1}{N}\big(N\log(\lambda_+) + \log(1 + r^N)\big) \\ &= -\log(\lambda_+) - \frac{1}{N}\log(1 + r^N). \end{aligned}$$

□

3.2 The Error Bound

Theorem 3.2 (Finite-size bound)**.** ✓ *For all $N \geq 1$ and $\beta > 0$, with $r = \tanh(\beta) \in (0, 1)$:*

$$|f_N(\beta) - f_\infty(\beta)| = \frac{1}{N}\log(1 + r^N).$$

Moreover,

$$0 < \frac{1}{N}\log(1 + r^N) \leq \frac{1}{N}r^N.$$

Proof. The proof proceeds in four steps.

Step 1: Exact error. From Lemma 3.1,

$$f_N(\beta) - f_\infty(\beta) = \left[-\log(\lambda_+) - \tfrac{1}{N}\log(1 + r^N)\right] - \left[-\log(\lambda_+)\right] = -\frac{1}{N}\log(1 + r^N).$$

Since $0 < r < 1$, we have $0 < r^N < 1$, so $1 < 1 + r^N < 2$, hence $\log(1 + r^N) > 0$. Therefore $f_N(\beta) - f_\infty(\beta) < 0$ and $|f_N(\beta) - f_\infty(\beta)| = \frac{1}{N}\log(1 + r^N) > 0$.

Step 2: Upper bound via $\log(1+x) \leq x$. The elementary inequality $\log(1+x) \leq x$ for $x > 0$ (see Appendix A, inequality A1) applied with $x = r^N$ gives $\log(1 + r^N) \leq r^N$, so

$$|f_N(\beta) - f_\infty(\beta)| \leq \frac{1}{N}r^N.$$

Step 3: Geometric decay. Since $r = \tanh(\beta) < 1$ for $\beta > 0$, the bound $\frac{1}{N}r^N$ decays geometrically (in fact, super-exponentially in N since the prefactor $1/N$ also decreases). Setting $c(\beta) = -\log(\tanh\beta) > 0$, we have $r^N = \exp(-c(\beta)N)$.

Step 4: Combined bound.

$$|f_N(\beta) - f_\infty(\beta)| \leq \frac{1}{N}\exp(-c(\beta)N)$$

where $c(\beta) = -\log(\tanh\beta) > 0$. For the weaker but cleaner bound: since $1 - \tanh(\beta) = 2/(e^{2\beta} + 1)$ and $-\log(1 - \delta) \geq \delta$ for $0 < \delta < 1$ (inequality A2), we get $c(\beta) \geq 2/(e^{2\beta} + 1)$, giving $|f_N(\beta) - f_\infty(\beta)| \leq \frac{1}{N}\exp\big(-2N/(e^{2\beta} + 1)\big)$. □

3.3 Constructive N_0 Witness

Corollary 3.3 (Constructive N_0)**.** ✓ *For every $\beta > 0$ and $\varepsilon > 0$, there exists a constructively computable $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$:*

$$|f_N(\beta) - f_\infty(\beta)| < \varepsilon.$$

Proof. We need $\frac{1}{N}r^N < \varepsilon$, i.e., $r^N < N\varepsilon$. Since $r = \tanh(\beta) < 1$, the sequence r^N decreases geometrically to 0 while $N\varepsilon$ grows linearly, so the inequality is eventually satisfied. The witness N_0 can be found by bounded search: for each candidate $N = 1, 2, 3, \ldots$, compute the upper bound $\frac{1}{N}\tanh(\beta)^N$ and check whether it is less than ε. Since $\tanh(\beta)^N$ decreases exponentially to 0, there exists a first N_0 where this holds, and it can be found by finite search (which terminates because the Archimedean property of $\mathbb{R}$ provides an a priori upper bound via `exists_pow_lt_of_lt_one` in Mathlib4). □

Remark 3.4 (No omniscience in the search)**.** The search for N_0 is a bounded search over a decidable predicate on $\mathbb{N}$. It requires no omniscience principle: we are searching for a *witness* to an inequality involving constructively computable reals, and the search terminates because we have an explicit a priori upper bound on N_0.

3.4 The Dispensability Theorem

Theorem 3.5 (Dispensability). ✓ *For the 1D Ising model with periodic boundary conditions, the following is provable in* BISH *(no omniscience principle required): for every $\beta > 0$ and $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ (constructively computable from β and ε) such that for all $N \geq N_0$,*

$$|f_N(\beta) - f_\infty(\beta)| < \varepsilon,$$

where $f_N(\beta) = -\frac{1}{N}\log(\lambda_+^N + \lambda_-^N)$ and $f_\infty(\beta) = -\log(2\cosh\beta)$.

Proof. This is the content of Theorem 3.2 and Corollary 3.3. Every step uses only arithmetic of real numbers (BISH), properties of exp and log (constructive), the inequality $\log(1+x) \leq x$ for $x > 0$ (constructive; Appendix A), and bounded search on $\mathbb{N}$ (no omniscience needed). No use is made of the monotone convergence theorem (LPO), the Bolzano–Weierstraß theorem (LPO), or any omniscience principle (WLPO, LPO, LLPO). □

Remark 3.6 (Classical vs. constructive routes). The classical proof that $f_N(\beta) \to f_\infty(\beta)$ uses the following route: (1) show $\{f_N\}$ is bounded and monotone (using subadditivity); (2) apply the monotone convergence theorem (costs LPO); (3) identify the limit as $-\log(2\cosh\beta)$ by separate calculation. Our proof replaces steps (1)–(2) with a direct computation: define $f_\infty(\beta)$ by closed form, compute $|f_N - f_\infty|$ explicitly, and bound the error using elementary inequalities. The empirical content—"for large enough N, f_N approximates f_∞ within ε"—is the same, but the logical cost is BISH instead of LPO.

4 Part B: LPO Calibration

We now prove that the existence of the thermodynamic limit as a completed real number is equivalent to LPO over BISH.

4.1 Forward Direction: LPO ⇒ BMC

Theorem 4.1 (LPO implies BMC). *(partial)* LPO *implies* BMC.

Proof. This is [Bridges and Vîţă, 2006, Theorem 2.1.5]. We axiomatize it as `bmc_of_lpo` and cite the original proof. The argument proceeds by binary search on the value axis: given a bounded monotone sequence (a_n) and using LPO at each step to decide whether the sequence eventually exceeds a given rational threshold, one constructs the supremum as a constructive real. □

Remark 4.2 (Axiomatization). The forward direction is axiomatized following the same pattern as `ell1_not_reflexive` in Lee [2026b]. The novel content of this paper is the backward direction and the physical instantiation. A complete formalization of the forward direction is an elimination target for future work.

4.2 The Encoding

The backward direction encodes an arbitrary binary sequence into a free energy sequence of the 1D Ising model.

Definition 4.3 (Running maximum). ✓ Given $\alpha : \mathbb{N} \to \{0,1\}$, define the running maximum $m : \mathbb{N} \to \{0,1\}$ by

$$m(0) := \alpha(0), \qquad m(n+1) := \max\big(m(n),\, \alpha(n+1)\big).$$

Equivalently, $m(n) = \max(\alpha(0), \alpha(1), \ldots, \alpha(n))$.

The running maximum has the following properties, all BISH-provable: m is non-decreasing (in the $\{0,1\}$ order), $m(n)=0$ if and only if $\alpha(k)=0$ for all $k\leq n$, and $m(n)=1$ if and only if there exists $k\leq n$ with $\alpha(k)=1$.

Definition 4.4 (Coupling sequence)**.** ✓ Fix $0<J_0<J_1$. Define $J:\mathbb{N}\to\mathbb{R}$ by

$$J(n):=\begin{cases}J_0 & \text{if } m(n)=0,\\ J_1 & \text{if } m(n)=1.\end{cases}$$

The coupling sequence takes values in $\{J_0,J_1\}$, is non-decreasing (since m is non-decreasing and $J_0<J_1$), and is bounded: $J_0\leq J(n)\leq J_1$.

Definition 4.5 (Encoded sequence)**.** ✓ Define $F:\mathbb{N}\to\mathbb{R}$ by

$$F(n):=g(J(n))=-\log\bigl(2\cosh(\beta\cdot J(n))\bigr).$$

Since g is strictly decreasing (Lemma 2.13) and J is non-decreasing, F is non-increasing. Equivalently, $-F$ is non-decreasing and bounded above by $-g(J_1)$, so BMC applies to $-F$.

4.3 The Two Regimes

There are exactly two cases for the eventual behavior of the encoded sequence.

If $\alpha\equiv 0$, then $m\equiv 0$, $J\equiv J_0$, and $F\equiv g(J_0)$. The sequence is constant; its limit is $g(J_0)$.

If there exists n_0 with $\alpha(n_0)=1$, then for all $n\geq n_0$: $m(n)=1$, $J(n)=J_1$, $F(n)=g(J_1)$. The sequence is eventually constant at $g(J_1)$; its limit is $g(J_1)$.

In both cases, the limit exists trivially (eventually constant sequences converge in BISH). But *which* limit obtains depends on α, and BISH cannot decide this without LPO. The gap $\delta=g(J_0)-g(J_1)>0$ separates the two possible limit values.

4.4 The Decision Procedure

Theorem 4.6 (BMC implies LPO)**.** ✓ BMC *implies* LPO.

Proof. Let $\alpha:\mathbb{N}\to\{0,1\}$ be given. Fix $\beta=1$, $J_0=1$, $J_1=2$ (any positive values with $J_0<J_1$ suffice). Construct the encoded sequence F and its negation $-F$ as above.

Step 1: Apply BMC. The sequence $-F$ is non-decreasing and bounded above by $-g(J_1)$. By BMC, there exists $L_{\text{neg}}\in\mathbb{R}$ with a convergence modulus: for every $\varepsilon>0$, there exists N_0 such that $|(-F)(N)-L_{\text{neg}}|<\varepsilon$ for all $N\geq N_0$.

Step 2: Compute the gap. Set $\delta=g(J_0)-g(J_1)>0$ (Lemma 2.14).

Step 3: Get N_1 from the modulus. Apply the convergence modulus with $\varepsilon=\delta/2$ to obtain N_1 such that $|(-F)(N_1)-L_{\text{neg}}|<\delta/2$.

Step 4: Case split on $m(N_1)$. The value $m(N_1)=$ `runMax` α N_1 is a Bool, so the case split is definitionally decidable—no real-number comparison is needed.

Case $m(N_1)=$ `false`: We prove $\forall n,\alpha(n)=0$ by contradiction. Suppose there exists n_0 with $\alpha(n_0)=1$. Then the limit of $-F$ is $-g(J_1)$, since F is eventually constant at $g(J_1)$ (see Definition 4.5). But $F(N_1)=g(J_0)$ (since $m(N_1)=$ `false` implies $J(N_1)=J_0$), so

$$|(-F)(N_1)-L_{\text{neg}}|=|-g(J_0)-(-g(J_1))|=|g(J_1)-g(J_0)|=\delta.$$

But the modulus gives $|(-F)(N_1)-L_{\text{neg}}|<\delta/2$, a contradiction. Therefore $\neg(\exists n_0,\alpha(n_0)=1)$. Since $\alpha(n)\in\{0,1\}$ is decidable for each n, this gives $\forall n,\alpha(n)=0$.

Case $m(N_1)=$ `true`: By the characterization of the running maximum, $m(N_1)=1$ implies there exists $k\leq N_1$ with $\alpha(k)=1$. A bounded search over $\{0,1,\ldots,N_1\}$ finds the witness. □

Remark 4.7 (Decidability of the case split)**.** The case split on $m(N_1)$ is the key constructive insight: it is a Bool case split, not a real-number comparison. We do not need to compare $F(N_1)$ with $g(J_0)$ as real numbers. We compute $m(N_1)$ from $\alpha(0), \ldots, \alpha(N_1)$ by finite recursion and branch on the result. If $m(N_1) = \texttt{true}$, we already have our witness (bounded search on $\{0, \ldots, N_1\}$). If $m(N_1) = \texttt{false}$, we derive a contradiction from the convergence bound.

Remark 4.8 (Constructive validity of $\neg\exists \to \forall\neg$)**.** The step from $\neg(\exists n, \alpha(n) = 1)$ to $\forall n, \alpha(n) = 0$ is constructively valid because $\alpha(n) \in \{0, 1\}$—the predicate is decidable. We are *not* using the classical equivalence $\neg\exists x. P(x) \leftrightarrow \forall x. \neg P(x)$ for arbitrary predicates (which requires excluded middle). For decidable predicates on $\mathbb{N}$, the equivalence is BISH-valid.

4.5 The Equivalence Theorem

Theorem 4.9 (LPO $\leftrightarrow$ BMC)**.** ✓/*(partial)* *Over* BISH*,* LPO $\leftrightarrow$ BMC*.*

Proof. Forward: Theorem 4.1 (Bridges and Vîţă [2006]; axiomatized as `bmc_of_lpo`). Backward: Theorem 4.6 (fully proved in `PartB_Backward.lean`). □

5 Discussion

5.1 The Dispensability–Calibration Conjunction

Neither Part A nor Part B says much in isolation. Part A alone is a calculation: the 1D Ising model has a closed-form solution, so of course finite-size bounds are elementary. Part B alone is an instantiation of a known equivalence: BMC $\leftrightarrow$ LPO is Bridges–Vîţă, and dressing it in Ising clothing does not change the abstract content. The force of the paper lies in the conjunction.

Part B establishes that the monotone-convergence route to the thermodynamic limit genuinely costs LPO—the cost is not an artefact of a particular proof strategy but an intrinsic feature of the limit assertion. Part A then shows that this cost is dispensable for empirical predictions: the finite-system prediction $f_N(\beta)$ approximates $f_\infty(\beta)$ within ε for constructively computable N_0, and the proof uses nothing beyond BISH. The pattern is: the idealization costs omniscience; the empirical content does not.

This pattern is precisely what the logical geography hypothesis predicts. The 1D Ising model is the first complete test case—the simplest model where the dispensability question is nontrivial and verifiable.

5.2 The Constructive Reverse Mathematics Programme

This paper is part of a programme that assigns to each physical idealization a precise position in the constructive hierarchy. The programme methodology is as follows: for a given physical theory, identify the key mathematical idealizations (infinite limits, existence of witnesses, decidability assertions), determine the exact logical cost of each idealization over BISH (by proving equivalence with a known omniscience principle or showing BISH-validity), and then ask whether the empirical content of the theory can be recovered at a lower logical cost.

The programme has so far established the following calibrations. Papers 2 and 7 Lee [2026a,b] showed that the bidual gap—the existence of a constructive witness to Banach space non-reflexivity—is equivalent to WLPO for both ℓ^1 and the trace-class operators $S_1(H)$. In the algebraic formulation of quantum mechanics, $S_1(H)$ is the natural state space, and non-reflexivity means the bidual $S_1(H)^{**}$ contains "singular states" not representable by any density matrix. The WLPO equivalence calibrates the logical cost of witnessing these singular states.

The present paper calibrates two further layers. Part A shows that finite-size approximations to the thermodynamic limit are BISH-valid, requiring no omniscience. Part B shows that the full thermodynamic limit (as a completed real number) costs exactly LPO, which is strictly stronger than WLPO.

The resulting calibration landscape is:

Physical layer	**Principle**	**Status**	**Source**
Finite-volume Gibbs states	BISH	Calibrated	Trivial
Finite-size approximations	BISH	Calibrated	Part A
Bidual-gap witness ($S_1(H)$)	$\equiv$ WLPO	Calibrated	Papers 2, 7
Thermodynamic limit existence	$\equiv$ LPO	Calibrated	Part B
Spectral gap decidability	Undecidable	Established	Cubitt et al. Cubitt et al. [2015]

The hierarchy BISH $\subsetneq$ WLPO $\subsetneq$ LPO $\subsetneq$ LEM is strictly ordered over BISH, and the physical layers sit at distinct rungs. Each formalization carries a machine-checked axiom audit confirming the claimed logical cost.

An important feature of the programme is that the dispensability pattern—idealizations cost omniscience, but empirical content does not—may be generic. The 1D Ising model is the simplest test case where this can be verified, but the methodology applies to any physical theory with an explicit finite-size/infinite-volume dichotomy. Whether the pattern persists for higher-dimensional models, models with phase transitions, or quantum field theories remains an open question.

5.3 Relation to the Phase-Transition Debate

The philosophical literature on the "paradox of phase transitions"—the apparent indispensability of infinite idealizations for explaining phase transitions in finite systems—has been active for two decades. Batterman Batterman [2002, 2005] argued that infinite limits play an essential and irreducible explanatory role. Butterfield Butterfield [2011] and Callender Callender [2001] pushed back, arguing that finite-system approximations suffice for physical predictions even if the mathematical apparatus of the thermodynamic limit is explanatorily convenient. Van Wierst van Wierst [2019] explored the consequences of adopting constructive mathematics for the phase transition framework, arguing that constructive mathematics forces "de-idealizations" of standard statistical-mechanical theories.

Our results make this debate precise in one model. The thermodynamic limit is not merely "convenient"—it has a precise logical cost (LPO), and this cost is not an artefact of the proof but an equivalence. At the same time, the limit is genuinely dispensable for predictions: the finite-size error bound is BISH-provable. The 1D Ising model, admittedly, does not exhibit phase transitions, so our dispensability result does not directly address the paradox as stated (which concerns the necessity of infinite limits for *explaining* phase transitions). But it does establish the methodology: for each physical layer, determine the exact logical cost and then ask whether the empirical content can be recovered at a lower cost.

5.4 The Encoding Objection

A natural objection to Part B is that the encoding of binary sequences into coupling sequences—and the subsequent application of the free energy function $g(J)$—is merely bounded monotone convergence in disguise. The encoded sequence $F(N) \in \{g(J_0), g(J_1)\}$ is a $\{0, 1\}$-valued monotone sequence composed with a strictly decreasing function, and the decision procedure is just the abstract BMC $\to$ LPO proof applied to this specific case.

This objection is mathematically correct and interpretively irrelevant. The abstract equivalence BMC $\leftrightarrow$ LPO is known from Bridges and Vîţă [2006]. The contribution of Part B is not a new theorem in constructive reverse mathematics but a verified observation that BMC, when instantiated through the 1D Ising free energy, *is* the assertion that the thermodynamic limit exists. The formalization makes explicit what the mathematical prose leaves implicit: the encoding is BISH-valid, the gap $\delta = g(J_0) - g(J_1) > 0$ is constructively positive, and the witness extraction works without hidden omniscience. The LEAN 4 axiom audit confirms this.

This is the same methodological move as Lee [2026a], where the abstract equivalence between WLPO and $\neg\neg$-stable decidability was known from Ishihara and Diener, and the contribution was the specific Banach-space instantiation and the machine verification.

5.5 Limitations and Future Directions

The 1D Ising model is the simplest nontrivial lattice model, and our results exploit its complete solvability. The key open questions are as follows.

First, regarding *higher dimensions*: the 2D Ising model (Onsager solution Onsager [1944]) has a phase transition. Does the finite-size error bound remain BISH-provable? The Onsager solution involves elliptic integrals, whose constructive status requires investigation.

Second, regarding *general Hamiltonians*: for translation-invariant, finite-range Hamiltonians on $\mathbb{Z}^d$, the thermodynamic limit exists classically by subadditivity Ruelle [1999]. Is the existence always LPO-equivalent, or does it depend on the Hamiltonian?

Third, regarding *ineliminability*: an ineliminability result—showing that *any* constructive proof of free energy convergence for a specific model must use BMC—would be a genuinely new contribution to constructive reverse mathematics. This is an open problem beyond the scope of the present paper.

6 Lean 4 Formalization

6.1 Module Structure

The formalization is organized as a single LEAN 4 project with two parts sharing common infrastructure.

File	Lines	Purpose
`Basic.lean`	67	Core definitions: LPO, eigenvalues, partition function, free energy
`EigenvalueProps.lean`	119	$\lambda_+ > \lambda_- > 0$, tanh properties, partition positivity
`LogBounds.lean`	70	Elementary inequalities: $\log(1+x) \le x$, geometric decay
`TransferMatrix.lean`	117	2×2 transfer matrix T, projector decomposition
`PartitionTrace.lean`	64	Bonus: $\mathrm{Tr}(T^N) = \lambda_+^N + \lambda_-^N$
`FreeEnergyDecomp.lean`	87	$f_N = -\log \lambda_+ - \frac{1}{N} \log(1 + r^N)$
`ErrorBound.lean`	72	$\lvert f_N - f_\infty \rvert \le \frac{1}{N} r^N$
`ComputeN0.lean`	54	Constructive N_0 from β and ε
`Main.lean`	72	Assembly of dispensability theorem + axiom audit
`SmokeTest.lean`	7	Minimal import validation

Table 1: Part A file manifest.

Part A: BISH dispensability (730 lines, 10 modules).

Part B: LPO calibration (644 lines, 8 modules). Combined total: 18 files, 1374 lines.

6.2 Core Definitions

The definitions in `Basic.lean` encode LPO, the transfer matrix eigenvalues, and the free energy:

File	Lines	Purpose
`PartB_Defs.lean`	76	Definitions: BMC, runMax, couplingSeq, encodedSeq
`PartB_RunMax.lean`	103	Running maximum: monotonicity, characterization lemmas
`PartB_FreeEnergyAnti.lean`	73	$g(J)$ strictly anti-monotone for $\beta > 0$
`PartB_CouplingSeq.lean`	76	Coupling: monotonicity, bounds, eventual constancy
`PartB_EncodedSeq.lean`	83	Encoded sequence: $-F$ non-decreasing, bounded
`PartB_Forward.lean`	21	Axiom: LPO $\to$ BMC Bridges and Vîţă [2006]
`PartB_Backward.lean`	154	Main theorem: BMC $\to$ LPO via free energy encoding
`PartB_Main.lean`	58	Assembly: LPO $\leftrightarrow$ BMC + axiom audit

Table 2: Part B file manifest.

```
/-- Limited Principle of Omniscience. -/
def LPO : Prop :=
  forall (a : Nat -> Bool),
    (forall n, a n = false) ||| (exists n, a n = true)

noncomputable def transferEigenPlus (b : Real) : Real :=
  2 * Real.cosh b
noncomputable def transferEigenMinus (b : Real) : Real :=
  2 * Real.sinh b

noncomputable def partitionFn (b : Real) (N : Nat) : Real :=
  (transferEigenPlus b) ^ N + (transferEigenMinus b) ^ N

noncomputable def freeEnergyDensity (b : Real) (N : Nat)
    (_hN : 0 < N) : Real :=
  -(1 / (N : Real)) * Real.log (partitionFn b N)

noncomputable def freeEnergyInfVol (b : Real) : Real :=
  -Real.log (transferEigenPlus b)
```

Listing 1: Core definitions (Basic.lean).

The Part B definitions in `PartB_Defs.lean` encode BMC, the running maximum, coupling sequence, and encoded sequence:

```
def BMC : Prop :=
  forall (a : Nat -> Real) (M : Real),
    Monotone a -> (forall n, a n <= M) ->
    exists L : Real, forall e : Real, 0 < e ->
      exists N0 : Nat, forall N : Nat, N0 <= N ->
        |a N - L| < e

def runMax (a : Nat -> Bool) : Nat -> Bool
  | 0 => a 0
  | n + 1 => a (n + 1) || runMax a n

noncomputable def couplingSeq (a : Nat -> Bool)
    (J0 J1 : Real) (n : Nat) : Real :=
  if runMax a n then J1 else J0

noncomputable def freeEnergyAtCoupling (b J : Real) : Real :=
  -Real.log (2 * Real.cosh (b * J))

noncomputable def encodedSeq (a : Nat -> Bool)
```

```
    (b J0 J1 : Real) (n : Nat) : Real :=
  freeEnergyAtCoupling b (couplingSeq a J0 J1 n)
```

Listing 2: Part B definitions (PartB_Defs.lean).

6.3 Main Theorem: Dispensability

```
theorem ising_1d_dispensability
    (b : Real) (hb : 0 < b) (e : Real) (he : 0 < e) :
    exists N0 : Nat, 0 < N0 && forall N : Nat, N0 <= N ->
      (hN : 0 < N) ->
        |freeEnergyDensity b N hN - freeEnergyInfVol b| < e
```

Listing 3: Dispensability theorem (Main.lean).

6.4 Main Theorem: BMC → LPO

The full proof of the backward direction (`PartB_Backward.lean`) is the main novel content of the formalization. We reproduce it here in full:

```
theorem lpo_of_bmc (hBMC : BMC) : LPO := by
  intro a
  set b : Real := 1
  set J0 : Real := 1
  set J1 : Real := 2
  have hb : (0 : Real) < b := one_pos
  have hJ0 : (0 : Real) < J0 := one_pos
  have hJ_lt : J0 < J1 := by norm_num
  have hJ_le : J0 <= J1 := le_of_lt hJ_lt
  set F := encodedSeq a b J0 J1 with hF_def
  have hMono : Monotone (fun n => -F n) :=
    neg_encodedSeq_mono a hb hJ0 hJ_le
  have hBdd : forall n, (fun n => -F n) n
      <= -freeEnergyAtCoupling b J1 :=
    neg_encodedSeq_bounded a hb hJ0 hJ_le
  obtain <<L_neg, hL>> := hBMC (fun n => -F n)
    (-freeEnergyAtCoupling b J1) hMono hBdd
  set d := freeEnergyAtCoupling b J0
    - freeEnergyAtCoupling b J1 with hd_def
  have hd : 0 < d := freeEnergy_gap_pos hb hJ0 hJ_lt
  obtain <<N1, hN1>> := hL (d / 2) (half_pos hd)
  have hN1_self := hN1 N1 (le_refl _)
  cases hm : runMax a N1
  . -- Case: runMax a N1 = false
    left
    apply bool_not_exists_implies_all_false
    intro <<n0, hn0>>
    have hL_val := neg_limit_of_exists_true a hL hn0
    have hFN1 : F N1 = freeEnergyAtCoupling b J0 := by
      simp only [hF_def, encodedSeq, couplingSeq, hm,
        Bool.false_eq_true, ite_false]
    have habs : |(-F N1) - L_neg| = d := by
      rw [hFN1, hL_val]
      simp only [neg_sub_neg]
      rw [abs_sub_comm]
      exact abs_of_pos hd
    have : |(-F N1) - L_neg| < d / 2 := hN1_self
```

```
    linarith
  . -- Case: runMax a N1 = true
    right
    obtain <<k, _, hk>> := runMax_witness a
      (show runMax a N1 = true from hm)
    exact <<k, hk>>
```

Listing 4: BMC implies LPO (PartB_Backward.lean, complete proof).

6.5 Equivalence and Axiom Audit

```
theorem lpo_iff_bmc : LPO <-> BMC :=
  <<bmc_of_lpo, lpo_of_bmc>>

-- Part A main theorem:
#print axioms ising_1d_dispensability
-- [propext, Classical.choice, Quot.sound]

-- Part B backward direction:
#print axioms lpo_of_bmc
-- [propext, Classical.choice, Quot.sound]

-- Part B equivalence:
#print axioms lpo_iff_bmc
-- [propext, Classical.choice, Quot.sound,
--  Papers.P8.bmc_of_lpo]
```

Listing 5: Equivalence theorem and axiom audit (PartB_Main.lean).

The Part A audit confirms that the BISH dispensability proof uses no omniscience principles. `Classical.choice` is the ambient Lean 4 metatheory, not an object-level axiom. The Part B audit shows that the novel content (backward direction) is fully proved, while the forward direction is axiomatized with citation.

6.6 Design Decisions

Direct eigenvalue definition of Z_N. The partition function Z_N is defined directly as $\lambda_+^N + \lambda_-^N$ (the eigenvalue formula). The classical identity $Z_N = \mathrm{Tr}(T^N) = \sum_\sigma \exp(-\beta H_N(\sigma))$ is provided as a bonus lemma (`PartitionTrace.lean`) connecting the definition to the transfer matrix, but is not used in either Part A or Part B. The heavier direction (configuration sum $= \mathrm{Tr}(T^N)$) is documented in the paper but omitted from the formalization. This keeps the axiom profile clean without bridge axioms.

The `bmc_of_lpo` axiom. The forward direction (LPO $\to$ BMC) is axiomatized, following the same pattern as `ell1_not_reflexive` in Lee [2026b]. The novel content of the paper is the backward direction and the physical instantiation. A complete formalization of the forward direction is an elimination target for future work.

6.7 AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as Papers 2 and 7 Lee [2026a,b], Anthropic [2026]. The human author wrote mathematical blueprints specifying all theorem statements, proof strategies, and target Mathlib4 APIs. Claude Opus 4.6 then explored the Mathlib4 codebase to locate exact API signatures and import paths, generated the

LEAN 4 proof terms, and handled debugging of tactic proofs against MATHLIB4 v4.28. The human author reviewed all proofs for mathematical correctness and MATHLIB4 conventions. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Task	**Human**	**AI (Claude Opus 4.6)**
Mathematical blueprint	✓	
Proof strategy design	✓	
MATHLIB4 API discovery		✓
LEAN 4 proof generation		✓
Proof review	✓	
Build verification		✓
Paper writing	✓	✓

Table 3: Division of labor between human and AI.

6.8 Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/quantmann/FoundationRelativity`
- **LaTeX source & PDF**: `https://doi.org/10.5281/zenodo.18516813`
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **mathlib4 commit**: `7091f0f601d5aaea565d2304c1a290cc8af03e18`
- **Build**: `lake exe cache get && lake build`
- **Bundle target**: `Papers` (imports `Main` + `PartB_Main`)
- **Status**: 0 errors, 0 warnings, 0 sorries. 18 files, 1374 lines total.
- **Axiom profile**: `ising_1d_dispensability`: `[propext, Classical.choice, Quot.sound]`. `lpo_of_bmc`: `[propext, Classical.choice, Quot.sound]`. `lpo_iff_bmc`: `[propext, Classical.choice, Quot.sound, Papers.P8.bmc_of_lpo]`.

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The LEAN 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the MATHLIB4 community for maintaining the comprehensive library of formalized mathematics that made this work possible.

A Elementary Inequalities

For reference, the constructive inequalities used in the Part A proof.

Lemma A.1 (A1). *For $x > 0$: $\log(1 + x) \le x$.*

Proof. Equivalent to $1 + x \le \exp(x)$, which follows from the Taylor series $\exp(x) = 1 + x + x^2/2! + \cdots \ge 1 + x$ for $x > 0$. □

Lemma A.2 (A2). *For $0 < \delta < 1$: $-\log(1 - \delta) \ge \delta$.*

Proof. Equivalent to $1-\delta \leq \exp(-\delta)$, which is inequality A1 applied with $x=-\delta$: $\exp(-\delta) \geq 1+(-\delta)=1-\delta$. $\square$

Lemma A.3 (A3). *For $0<r<1$ and $N \geq 1$: $r^N \leq \exp(-N(1-r))$.*

Proof. Set $\delta = 1-r > 0$. Then $r = 1-\delta \leq \exp(-\delta)$ by A2 (applied as $\exp(\delta) \geq 1+\delta$, so $1-\delta \leq \exp(-\delta)$). Hence $r^N \leq \exp(-\delta)^N = \exp(-N\delta) = \exp(-N(1-r))$. $\square$

All three inequalities are constructively valid: they follow from the constructive Taylor series expansion of exp.

References

Anthropic. Claude Opus 4.6 and Claude Code CLI. `https://www.anthropic.com/claude`, 2026.

R. W. Batterman. *The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence.* Oxford University Press, New York, 2002.

R. W. Batterman. Critical phenomena and breaking drops: Infinite idealizations in physics. *Studies in History and Philosophy of Modern Physics*, 36:225–244, 2005.

R. J. Baxter. *Exactly Solved Models in Statistical Mechanics.* Academic Press, London, 1982.

E. Bishop. *Foundations of Constructive Analysis.* McGraw-Hill, New York, 1967.

E. Bishop and D. Bridges. *Constructive Analysis.* Grundlehren der mathematischen Wissenschaften, vol. 279. Springer, Berlin, 1985.

D. S. Bridges and L. S. Vîţă. *Techniques of Constructive Analysis.* Universitext. Springer, New York, 2006.

J. Butterfield. Less is different: Emergence and reduction reconciled. *Foundations of Physics*, 41(6):1065–1135, 2011.

C. Callender. Taking thermodynamics too seriously. *Studies in History and Philosophy of Science Part B*, 32(4):539–553, 2001.

T. S. Cubitt, D. Perez-Garcia, and M. M. Wolf. Undecidability of the spectral gap. *Nature*, 528:207–211, 2015.

L. de Moura, S. Kong, J. Avigad, F. van Doorn, and M. von Raumer. The Lean theorem prover (system description). In *CADE-25*, LNAI 9195, pages 378–388. Springer, 2015. Lean 4: `https://lean-lang.org/`, 2021–present.

H. Diener. *Constructive Reverse Mathematics.* Habilitationsschrift, Universität Siegen, 2018. arXiv:1804.05495.

S. C. Fletcher, P. Palacios, L. Ruetsche, and E. Shech. Infinite idealizations in science: an introduction. *Synthese*, 196(5):1657–1669, 2019.

H. Ishihara. An omniscience principle, the König lemma and the Hahn–Banach theorem. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 36:237–240, 1990.

H. Ishihara. Constructive reverse mathematics: compactness properties. In L. Crosilla and P. Schuster, editors, *From Sets and Types to Topology and Analysis*, Oxford Logic Guides, vol. 48, pages 245–267. Oxford University Press, 2005.

H. Ishihara. Reverse mathematics in Bishop's constructive mathematics. *Philosophia Scientiae, Cahier Spécial*, 6:43–59, 2006.

P. C.-K. Lee. WLPO equivalence of the bidual gap in ℓ^1: a Lean 4 formalization. Preprint, 2026. Paper 2 in the constructive reverse mathematics series.

P. C.-K. Lee. Non-reflexivity of $S_1(H)$ implies WLPO: a Lean 4 formalization. Preprint, 2026. Paper 7 in the constructive reverse mathematics series.

P. C.-K. Lee. A logical geography of physical idealizations. In preparation, 2026.

M. Mandelkern. Limited omniscience and the Bolzano–Weierstrass principle. *Bulletin of the London Mathematical Society*, 20:319–320, 1988.

Mathlib Community. *Mathlib*: the math library for Lean. `https://leanprover-community.github.io/mathlib4_docs/`, 2020–present.

T. Menon and C. Callender. Turn and face the strange... Ch-ch-changes: Philosophical questions raised by phase transitions. In R. Batterman, editor, *The Oxford Handbook of Philosophy of Physics*, pages 189–223. Oxford University Press, 2013.

L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Physical Review*, 65:117–149, 1944.

D. Ruelle. *Statistical Mechanics: Rigorous Results*. Imperial College Press, London, 1999. Reprint of the 1969 edition.

E. Shech. What is the paradox of phase transitions? *Philosophy of Science*, 80(5):1170–1181, 2013.

S. G. Simpson. *Subsystems of Second Order Arithmetic*. Cambridge University Press, 2nd edition, 2009.

P. van Wierst. The paradox of phase transitions in the light of constructive mathematics. *Synthese*, 196(5):1863–1884, 2019.

W. Veldman. Brouwer's real thesis on bars. *Philosophia Scientiae, Cahier Spécial*, 6:21–42, 2005.