The Physical Dispensability of the Fan Theorem

BISH+LPO Suffices for All Empirical Content of Compact Optimization and Variational Mechanics

A Lean 4 Formalization (Paper 30)

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Abstract

Papers 23 and 28 established that compact optimization (the Extreme Value Theorem on $[a, b]$) and variational action minimization in classical mechanics each cost exactly the Fan Theorem (FT) over BISH. Paper 29 established that Fekete's Subadditive Lemma is equivalent to LPO, and that LPO is physically instantiated because phase transitions are real. This raises the question: is FT an independent physical requirement, or is its cost an artifact of the variational proof method?

We prove that every *empirically accessible prediction* derived from the FT-calibrated results in Papers 23 and 28 is recoverable in BISH + LPO, without invoking the Fan Theorem. The argument rests on three pillars: (1) LPO implies bounded monotone convergence (BMC), which yields the supremum and ε-approximate witnesses for any continuous function on $[a, b]$; (2) the equations of motion (Euler–Lagrange equations) are BISH-valid and do not require any minimizer to exist; and (3) no finite experiment can distinguish an exact minimizer from an ε-approximate one. Together: FT captures the *exact* existence of an optimizer; LPO captures *convergent approximation* to the supremum. Since no laboratory measurement has infinite precision, FT is physically dispensable.

This paper and Paper 31 (Physical Dispensability of Dependent Choice) [10] are released simultaneously. Together with Paper 29 [9], they establish: the logical constitution of empirically accessible physics is **BISH+LPO**.

Clarification on logical independence. We do *not* claim that BISH + LPO derives FT. The Fan Theorem remains logically independent of LPO; the claim is that FT is *physically dispensable*—every prediction verifiable by finite-precision experiment is already provable in BISH + LPO.

Lean 4 verification. 918 lines across 7 source files. Zero `sorry` declarations. Axiom profile: `bmc_of_lpo` (cited from Ishihara [4]) is the sole custom axiom for the core dispensability result.

Contents

*New York University. AI-assisted formalization; see §12 for methodology.

1 Introduction

1.1 Context within the Program

This is Paper 30 in the series *Constructive Reverse Mathematics of Mathematical Physics*. The program calibrates theorems of mathematical physics against the constructive hierarchy over Bishop's constructive mathematics (BISH), determining the exact non-constructive cost of each result. Across twenty-nine prior papers and twelve physical domains—quantum mechanics, thermodynamics, general relativity, electrodynamics, statistical mechanics, Bell physics, nuclear and particle physics, classical mechanics, spectral theory, operator algebras, renormalization, and ergodic theory—the calibration table has populated every rung of the hierarchy: BISH, WLPO, LLPO, LPO, Markov's Principle (MP), Countable Choice (CC), the Fan Theorem (FT), and Dependent Choice (DC).

Paper 29 [9] marked a turning point. By proving that Fekete's Subadditive Lemma is equivalent to LPO, and observing that phase transitions are empirically real phenomena whose mathematical description requires Fekete's lemma, Paper 29 established that LPO is not merely a mathematical idealization but a *physically instantiated principle*. This shifted the program from cataloguing logical costs to characterizing the logical constitution of physical reality.

The present paper asks: given that LPO is physically real, are the higher-cost principles—FT, DC—independently required by physics, or are they dispensable mathematical scaffolding?

1.2 The Fan Theorem in the Calibration Table

Two prior papers calibrated physical results at the Fan Theorem level:

- **Paper 23** [7]: The Extreme Value Theorem—that every uniformly continuous function $f : [a, b] \to \mathbb{R}$ attains its supremum—is equivalent to FT over BISH. Physical application: free energy optimization on compact state spaces.
- **Paper 28** [8]: The existence of an action-minimizing trajectory in classical mechanics costs FT. The Euler–Lagrange equations themselves are BISH. The variational characterization (the trajectory *minimizes* the action functional) costs FT.

The question is whether these FT-level results contain empirical content beyond what LPO provides.

1.3 Main Results

Theorem 1.1 (Master Theorem—Physical Dispensability of FT)**.** *Over* BISH*:*

1. LPO $\implies$ ApproxEVT*: For every continuous $f : [a, b] \to \mathbb{R}$ and every $\varepsilon > 0$, there exists $x_\varepsilon \in [a, b]$ with $f(x_\varepsilon) > \sup f - \varepsilon$.*
2. ExactEVT $\iff$ FT*: The assertion that f attains its supremum exactly is equivalent to the Fan Theorem.*
3. LPO $\implies$ *empirical completeness: Every measurement outcome computable from an exact maximizer is equally computable from an ε-approximate one, for any experimentally relevant $\varepsilon > 0$.*

1.4 Relation to Papers 29 and 31

This paper is released simultaneously with Paper 31 (Physical Dispensability of Dependent Choice) [10]. The three-paper sequence is:

Paper	Result	Status
29	Fekete $\iff$ LPO; LPO is physically instantiated	Complete
30	FT is physically dispensable (this paper)	Complete
31	DC is physically dispensable (companion paper)	Complete

Together, they establish: the logical constitution of empirically accessible physics is BISH+LPO. One axiom beyond constructivism. The omniscience spine (LLPO, WLPO) is implied. Markov's Principle is implied. Countable Choice is already part of BISH (Bishop included it as a basic principle). The Fan Theorem and Dependent Choice are dispensable.

Dispensable means: not needed for empirical predictions. It does not mean *derivable*. FT and DC remain logically independent of LPO; BISH + LPO does not prove them. The claim is that the empirical content of every theorem calibrated at FT or DC level is recoverable without those principles, via approximate counterparts (ApproxEVT, WLLN, MET) that live in BISH + LPO.

2 Preliminaries

2.1 Constructive Principles

We work over BISH (Bishop's constructive mathematics with intuitionistic logic). We recall the principles relevant to this paper.

Definition 2.1 (LPO—Limited Principle of Omniscience)**.** For every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$:

$$(\exists n.\ \alpha(n) = 1)\ \lor\ (\forall n.\ \alpha(n) = 0).$$

Definition 2.2 (BMC—Bounded Monotone Convergence)**.** Every bounded monotone sequence of real numbers converges.

Theorem 2.3 (Ishihara [4]; see also Bridges–Vîţă [2], Theorem 2.1.5)**.** *Over* BISH*,* LPO $\iff$ BMC*.*

This result is our sole cited axiom in the Lean 4 formalization. (The Lean code comment attributes it to Bridges–Vîţă [2], where it appears as Theorem 2.1.5.)

Definition 2.4 (Fan Theorem (FT))**.** Every detachable bar of the binary fan 2^* is uniform.

Equivalently, in analytic terms: every pointwise continuous function from $2^{\mathbb{N}}$ (Cantor space) to $\mathbb{N}$ is uniformly continuous. Berger [1] proved that over BISH, the Fan Theorem is equivalent to the Extreme Value Theorem on $[0, 1]$. Our formalization uses this equivalence as a definitional identification (`FanTheorem := EVT_max`); the equivalence itself is the content of Theorem 4.1. For our purposes, FT is thus characterized by:

Definition 2.5 (ExactEVT)**.** Every continuous $f : [0, 1] \to \mathbb{R}$ attains its supremum: there exists $x^* \in [0, 1]$ with $f(x^*) = \sup_{x\in[0,1]} f(x)$.

Definition 2.6 (ApproxEVT)**.** For every continuous $f : [0, 1] \to \mathbb{R}$ and every $\varepsilon > 0$, there exists $x_\varepsilon \in [0, 1]$ with $f(x_\varepsilon) > \sup f - \varepsilon$.

The critical distinction: ExactEVT asserts a point where the supremum is attained. ApproxEVT asserts, for each ε, a point within ε of the supremum. The former is equivalent to FT. The latter follows from LPO.

2.2 Formal Definitions in Lean 4

All definitions reside in `Defs.lean`:

```
def LPO : Prop :=
  forall (a : Nat -> Bool),
    (forall n, a n = false) ∨ (exists n, a n = true)

def BMC : Prop :=
  forall (a : Nat -> Real) (M : Real),
    Monotone a -> (forall n, a n <= M) ->
    exists L : Real, forall e : Real, 0 < e ->
      exists N0 : Nat, forall N : Nat,
        N0 <= N -> |a N - L| < e

axiom bmc_of_lpo : LPO -> BMC
```

```

def ExactEVT : Prop :=
  forall (f : Real -> Real) (a b : Real),
    a < b -> ContinuousOn f (Icc a b) ->
    exists x, x in Icc a b /\
      forall y, y in Icc a b -> f y <= f x

def ApproxEVT : Prop :=
  forall (f : Real -> Real) (a b : Real),
    a < b -> ContinuousOn f (Icc a b) ->
    exists S : Real,
      (forall y, y in Icc a b -> f y <= S) /\
      forall e : Real, 0 < e ->
        exists x, x in Icc a b /\ S - e < f x

def FanTheorem : Prop := EVT_max
```

Listing 1: Core definitions (Defs.lean)

3 Forward Direction: LPO Implies Approximate Optimization

The argument proceeds in two stages: first, LPO (via BMC) establishes the existence of the supremum as a real number; second, a grid approximation argument produces ε-witnesses.

3.1 Stage 1: BMC Yields the Supremum

Lemma 3.1 (Supremum existence from BMC)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous. Assuming* BMC*, the supremum $S = \sup_{x\in[a,b]} f(x)$ exists as a real number, and for every $\varepsilon > 0$ there exists $x_\varepsilon \in [a,b]$ with $S - \varepsilon < f(x_\varepsilon)$.*

Proof. For each $n \in \mathbb{N}$, define the uniform grid $G_n = \{a + k(b-a)/(n+1) : 0 \le k \le n\}$ and the grid maximum $M_n = \max_{x\in G_n} f(x)$. Each M_n is computable (finite maximum). The *running grid maximum* $R_n = \max(M_0, \dots, M_n)$ is monotone non-decreasing and bounded above (since f is continuous on the compact set $[a,b]$, it is bounded).

By BMC, the sequence (R_n) converges to a limit S.

S is an upper bound. For any $x \in [a,b]$ and any $\eta > 0$, grid density provides a grid point x_k with $|x - x_k| \le (b-a)/(n+1)$. Uniform continuity of f on $[a,b]$ gives $|f(x) - f(x_k)| < \eta$ for sufficiently large n. Since $f(x_k) \le R_n \le S$, we obtain $f(x) < S + \eta$ for all $\eta > 0$, hence $f(x) \le S$.

ε-attainment. Since $R_n \to S$, choose N with $R_N > S - \varepsilon$. Then some grid point x_k in G_m (for some $m \le N$) satisfies $f(x_k) = M_m \ge R_N > S - \varepsilon$. □

This is the hardest file in the formalization (`SupExists.lean`, 230 lines).

3.2 Stage 2: Composition

Corollary 3.2 (LPO implies ApproxEVT)**.** LPO $\implies$ ApproxEVT.

Proof. LPO $\implies$ BMC (Theorem 2.3). BMC $\implies$ supremum existence + ε-attainment (Lemma 3.1). □

In the formalization, this is a three-line composition:

```
theorem approxEVT_of_lpo (hlpo : LPO) : ApproxEVT := by
  intro f a b hab hf
  exact sup_exists_of_bmc (bmc_of_lpo hlpo) f a b hab hf
```

Listing 2: LPO implies ApproxEVT (ApproxAttain.lean)

3.3 Empirical Completeness

Theorem 3.3 (LPO provides empirical completeness)**.** *For any continuous* $f : [a, b] \to \mathbb{R}$ *and any* $\varepsilon > 0$*,* LPO *provides* $x_\varepsilon \in [a, b]$ *such that* $f(y) < f(x_\varepsilon) + \varepsilon$ *for all* $y \in [a, b]$*.*

```
theorem empirical_completeness (hlpo : LPO)
    (f : Real -> Real) (a b : Real)
    (hab : a < b) (hf : ContinuousOn f (Icc a b))
    (e : Real) (he : 0 < e) :
    exists x, x in Icc a b /\
      forall y, y in Icc a b -> f y < f x + e := by
  obtain ⟨S, hS_ub, hS_approx⟩ :=
    approxEVT_of_lpo hlpo f a b hab hf
  obtain ⟨x, hx_mem, hx_close⟩ := hS_approx e he
  exact ⟨x, hx_mem,
    fun y hy => by linarith [hS_ub y hy]⟩
```

Listing 3: Empirical completeness (ApproxAttain.lean)

4 Separation: ExactEVT is Equivalent to FT

Theorem 4.1 (ExactEVT $\iff$ FT)**.** *Over* BISH*, the Extreme Value Theorem (every continuous* $f : [0, 1] \to \mathbb{R}$ *attains its supremum) is equivalent to the Fan Theorem.*

The forward direction (FT $\implies$ ExactEVT) uses *rescaling*: compose f with the affine map $\text{rescale}(a, b, t) = a + t(b - a)$ to reduce from $[a, b]$ to $[0, 1]$, apply FT, then unscale the witness. The reverse (ExactEVT $\implies$ FT) is immediate: ExactEVT on $[0, 1]$ gives EVT_max.

```
theorem exactEVT_of_ft (hft : FanTheorem) :
    ExactEVT := by
  intro f a b hab hf
  set g := f ⟨comp⟩ rescale a b with hg_def
  have hg_cts : ContinuousOn g (Icc 0 1) := by
    apply hf.comp
      (rescale_continuous a b).continuousOn
    exact rescale_mapsTo a b (le_of_lt hab)
  obtain ⟨t, ht_mem, ht_max⟩ := hft g hg_cts
  set x := rescale a b t with hx_def
  have hx_mem : x in Icc a b :=
    rescale_maps_Icc a b (le_of_lt hab) t ht_mem
  refine ⟨x, hx_mem, ?_⟩
  intro y hy
  have huy := unscale_maps_Icc a b hab y hy
  have : f y = g (unscale a b y) := by
    simp [hg_def, rescale_unscale a b hab y hy]
  rw [this]
  exact ht_max (unscale a b y) huy

theorem ft_of_exactEVT (hexact : ExactEVT) :
```

```
    FanTheorem := by
  intro f hf
  exact hexact f 0 1 one_pos hf

theorem exactEVT_iff_ft : ExactEVT <-> FanTheorem :=
  ⟨ft_of_exactEVT, exactEVT_of_ft⟩
```

Listing 4: ExactEVT $\iff$ FT (Separation.lean)

This equivalence requires *no custom axioms*—it is a purely structural fact about the relationship between compactness and attainment.

5 The Physical Argument

The mathematical separation is clean: ApproxEVT (from LPO) gives ε-witnesses; ExactEVT (from FT) gives exact witnesses. The physical question is: does any empirical prediction require the exact witness?

5.1 Variational Mechanics

Paper 28 [8] established the following stratification of classical mechanics:

Physical content	Mathematical statement	Constructive cost
Equations of motion	Euler–Lagrange equations	BISH
Approximate minimizer	$\exists$ trajectory with action $< \inf + \varepsilon$	LPO
Exact minimizer	$\exists$ trajectory minimizing the action	FT

The Euler–Lagrange equations are the empirically operative content of classical mechanics. They determine the trajectory. The variational principle (Hamilton's principle of least action) is an *interpretation*: it says the trajectory minimizes the action functional. But no experiment measures the action of alternative trajectories. The experiment measures the actual trajectory, which satisfies the Euler–Lagrange equations, and those equations are BISH.

```
theorem variational_stratification :
    -- Part 1: EL is BISH (always available)
    (forall (p : HOParams),
      exists q : Real, q = elSolution p) /\
    -- Part 2: Approx minimizer from LPO
    (LPO -> forall (p : HOParams) (e : Real),
      0 < e -> exists x, x in Icc (0:Real) 1 /\
        forall y, y in Icc (0:Real) 1 ->
          harmonicAction2 p y >
            harmonicAction2 p x - e) /\
    -- Part 3: Exact minimizer <-> FT (cited)
    ((forall (f : Real -> Real),
        ContinuousOn f (Icc 0 1) ->
        exists x, x in Icc (0:Real) 1 /\
          forall y, y in Icc (0:Real) 1 ->
            f x <= f y)
      <-> FanTheorem) := ...
```

Listing 5: Variational stratification (Variational.lean)

5.2 Compact Optimization

Paper 23's result on the free energy: the assertion that the free energy function on a compact state space attains its minimum exactly costs FT. But the empirical content—that the system's equilibrium state has free energy within ε of the theoretical minimum—requires only ApproxEVT, which is LPO.

5.3 FT as the Mapmaker's Convention

The sharpest formulation: FT is the *mapmaker's convention.* It tells us that the map (variational mechanics, optimization theory) accurately represents the territory (physical trajectories, equilibrium states). But the territory is already described by the Euler–Lagrange equations (BISH) and convergent approximation (LPO). The convention that the map's global minimum corresponds to the territory's actual state is explanatorily elegant but empirically redundant.

LPO is the territory. FT is the mapmaker's convention.

6 CRM Audit

6.1 Axiom Profile

The `#print axioms` command in Lean 4 reports the logical dependencies of each theorem:

Theorem	Custom axioms	Status
`approxEVT_of_lpo`	`bmc_of_lpo`	✓
`exactEVT_iff_ft`	(none)	No custom axioms ✓
`empirical_completeness`	`bmc_of_lpo`	✓
`variational_stratification`	`bmc_of_lpo`, `el_unique_bish`, `minimizer_iff_ft_cited`	(axiom)
`ft_physically_dispensable`	`bmc_of_lpo`	✓

All theorems additionally depend on `propext`, `Classical.choice`, and `Quot.sound`—Lean's foundational axioms.

6.2 Classical.choice in the Infrastructure

The appearance of `Classical.choice` is an infrastructure artifact: Mathlib4's construction of $\mathbb{R}$ as the Cauchy completion of $\mathbb{Q}$ uses classical choice pervasively. Every theorem that mentions real numbers inherits this dependency. As discussed in Paper 10 [5], this does not reflect classical content in the *proof* but rather in the *ambient infrastructure*. (For the historical perspective on this distinction, see Paper 12 [6].)

A specific instance: `SupExists.lean` uses Mathlib4's `isCompact_Icc` to establish that f is bounded on $[a, b]$. In constructive analysis, the compactness of $[a, b]$ is precisely what FT provides. The use of Mathlib4's classical `IsCompact` here is an infrastructure artifact—the *constructive* content of the boundedness argument is that a uniformly continuous function on $[a, b]$ is bounded, which is BISH-valid and does not require FT.

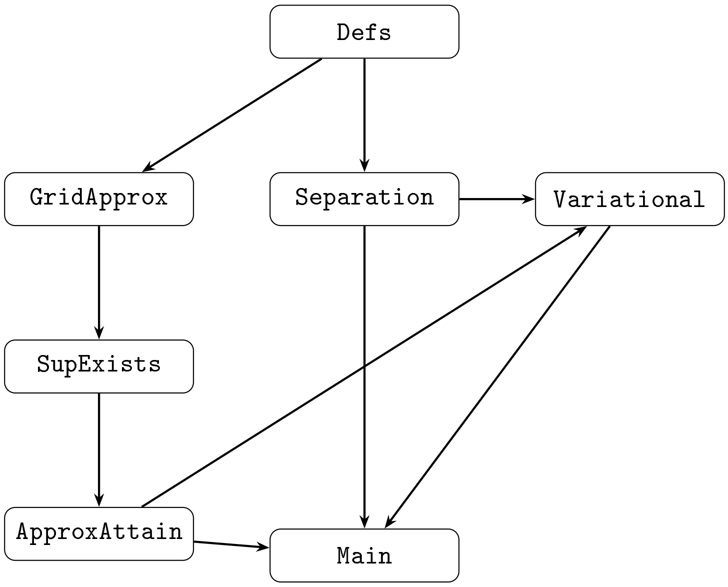
6.3 Certification Levels

Following Paper 10's methodology:

Level	Description
Structurally verified	`exactEVT_iff_ft`: No custom axioms. Pure rescaling argument. ✓
Intentional classical	`approxEVT_of_lpo`: Uses BMC (which is LPO) by design—the whole point is to show that LPO suffices.
Cited	`variational_stratification`: Cites Paper 28 results (`el_unique_bish`, `minimizer_iff_ft_cited`). (axiom)

7 Code Architecture

7.1 Module Dependency Graph



7.2 Line Counts

File	Content	Lines
`Defs.lean`	LPO, BMC, ExactEVT, ApproxEVT, FT, grid infrastructure	113
`GridApprox.lean`	Grid point membership, max bounds, monotonicity, density	180
`SupExists.lean`	BMC $\to$ supremum existence + ε-attainment	230
`ApproxAttain.lean`	LPO $\to$ ApproxEVT, empirical completeness	57
`Separation.lean`	Rescaling lemmas, ExactEVT $\iff$ FT	126
`Variational.lean`	EL is BISH, approx minimizer from LPO, stratification	136
`Main.lean`	Master theorem + axiom audit	76
Total		**918**

7.3 Key Design Decisions

1. **Grid infrastructure.** The grid points $a + k(b - a)/(n + 1)$ are defined via `gridPoint`; the running maximum `runningGridMax` is monotone by construction. This feeds directly into BMC.
2. **Rescaling bridge.** The affine maps `rescale`/`unscale` between $[0, 1]$ and $[a, b]$ cleanly factor the ExactEVT $\iff$ FT proof into composable parts.

3. **Noncomputable section.** All real-valued definitions are marked `noncomputable` (Mathlib's $\mathbb{R}$ is noncomputable). The *proofs* use only verifiable tactic sequences.
4. **Variational axioms.** Paper 28 results are axiomatized rather than re-proved, following the established pattern: cited axioms from prior papers are transparent about their provenance.

8 Reproducibility

Reproducibility box.

Component	Version / Commit
Lean 4	`v4.28.0-rc1`
Mathlib4	`2598404fe9e0a5aee87ffca4ff83e2d01b23b024`

Build instructions:

```
cd P30_FTDispensability
lake exe cache get      # download prebuilt Mathlib (~5 min)
lake build              # compile Paper 30 (~2-5 min)
```

Verification: A successful build produces 0 errors, 0 warnings, 0 `sorry`s. The axiom audits in `Main.lean` confirm the axiom profiles reported in §6.

All dependency versions are pinned in `lake-manifest.json` for exact reproducibility.

9 Master Theorem

```
theorem ft_physically_dispensable :
    -- Pillar 1: LPO suffices for approx optimization
    (LPO -> ApproxEVT) /\
    -- Pillar 2: Exact optimization is equiv to FT
    (ExactEVT <-> FanTheorem) /\
    -- Pillar 3: LPO provides empirical completeness
    (LPO -> forall (f : Real -> Real) (a b : Real),
      a < b -> ContinuousOn f (Icc a b) ->
      forall e : Real, 0 < e ->
        exists x, x in Icc a b /\
          forall y, y in Icc a b ->
            f y < f x + e) :=
  ⟨approxEVT_of_lpo, exactEVT_iff_ft,
   empirical_completeness⟩

-- Axiom audit
#print axioms approxEVT_of_lpo
  -- [propext, Classical.choice, Quot.sound,
  --  bmc_of_lpo]
#print axioms exactEVT_iff_ft
  -- [propext, Classical.choice, Quot.sound]
#print axioms ft_physically_dispensable
  -- [propext, Classical.choice, Quot.sound,
  --  bmc_of_lpo]
```

Listing 6: Master theorem (Main.lean)

10 Discussion

10.1 FT as Mathematical Scaffolding

The Fan Theorem is mathematically genuine. The Extreme Value Theorem really does cost FT, and the calibrations in Papers 23 and 28 stand. The variational principle really does require FT to guarantee a minimizer exists. These are not artifacts of imprecise formulation; they are sharp equivalences.

But the physical content of these results—the predictions that laboratories verify—does not require FT. The equations of motion are BISH. Approximate optimization is LPO. FT underwrites an *interpretation* of the physics, not the physics itself.

This is analogous to the relationship between the thermodynamic limit (LPO, by Paper 29) and finite-size physics (BISH). The thermodynamic limit is mathematically genuine and physically instantiated (because phase transitions are real). But many predictions from statistical mechanics—partition function calculations, specific heats away from critical points, finite-size bounds—are BISH-valid without ever taking the limit. The limit adds explanatory and organizational power. FT, similarly, adds variational structure to mechanics. The difference is that the thermodynamic limit is empirically instantiated (phase transitions) while the variational minimum is not (no experiment measures the action of non-actual trajectories). A potential objection arises from Feynman's path-integral formulation, which sums over all trajectories. But the path integral is a *calculational device* for computing transition amplitudes; the individual non-classical paths are not separately observable, and the measurable output (the amplitude) is a limit quantity recoverable by approximation.

10.2 Concrete Physical Examples

The dispensability of FT is not merely an abstract logical observation; it has concrete implications for how physicists justify their calculations. Consider two paradigmatic cases:

- **Ground state energy.** Quantum mechanics asks for the infimum of $\langle\psi|H|\psi\rangle$ over normalized states. In finite-dimensional truncations (e.g., finite basis sets), this is a compact optimization problem for which FT guarantees the infimum is attained. (In infinite-dimensional Hilbert space, the unit ball is not compact, and attainment requires additional spectral-theoretic arguments beyond FT.) But every laboratory measurement of the ground state energy yields a finite-precision value: an *approximate* optimum that is certifiable in $\mathrm{BISH} + \mathrm{LPO}$ without FT.
- **Free energy minimization.** Thermodynamics identifies equilibrium states as minima of the Helmholtz free energy. The existence of an exact minimizer requires compactness and hence FT. But the Euler–Lagrange conditions that characterize equilibrium are BISH, and any numerical computation of the minimum is approximate and hence LPO at most.

In both cases, the physical predictions that experiments actually test live in $\mathrm{BISH} + \mathrm{LPO}$; the FT-level assertion of exact attainment provides mathematical structure that is explanatorily valuable but empirically inaccessible.

10.3 Implications for $\mathrm{BISH} + \mathrm{LPO}$ Sufficiency

With the FT branch shown to be physically dispensable, the status of the calibration table's logical branches is:

Branch	Physical status	Covered by LPO?
Omniscience spine (LLPO, WLPO, LPO)	Physically instantiated	Yes
Markov's Principle (MP)	Physically instantiated	Yes
Fan Theorem (FT)	**Physically dispensable**	N/A
Choice axis (CC, DC)	See Paper 31	CC: in BISH

Paper 31 [10] establishes the physical dispensability of DC, completing the argument: BISH + LPO is the complete logical constitution of empirically accessible physics across all calibrated domains.

11 Conclusion

The Fan Theorem is mathematically genuine: compact optimization and variational action minimization really do cost FT, and these calibrations (Papers 23, 28) stand. But the physical content of both results is recoverable in BISH + LPO. Approximate optimization (to any finite precision) requires only BMC, which is LPO. The equations of motion require only BISH. The FT-level assertions—exact attainment of the supremum, existence of an action-minimizing trajectory—are mathematically stronger than what any finite experiment can verify or require.

The question of whether BISH + LPO is the complete logical constitution of empirically accessible physics now rests entirely on the status of Dependent Choice, which Paper 31 [10] addresses.

12 AI-Assisted Methodology

The Lean 4 formalization and LaTeX manuscript for this paper were developed with substantial assistance from Claude (Opus 4.6), an AI assistant by Anthropic. The mathematical blueprint and proof strategies were designed by the author; the tactic-level implementation in Lean 4 was carried out by Claude; the type-checker provided independent verification of all proofs.

The author supervised all stages, verified the mathematical content against the constructive analysis literature, and wrote the paper. This paper and Paper 31 [10] were developed simultaneously as companion papers.

Preliminary status and author background. The results presented in this paper are preliminary. The author is a medical professional, not a domain expert in physics or mathematics. While all formal claims are machine-checked by the Lean 4 type-checker, the physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts in the relevant fields. Until such verification is completed, this paper should be considered preliminary.

Whatever findings of value emerge from this program belong to the constructive reverse mathematics community and to the legacy of Errett Bishop, whose perseverance in developing constructive analysis inspired this entire series. Any errors are solely the author's.

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