Noether's Theorem and the Logical Cost of Global Conservation Laws

Paper 15 in the Constructive Reverse Mathematics Series

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February 2026

Abstract

Noether's theorem—that every continuous symmetry of the action yields a conserved current—is the most fundamental structural principle in physics. We show it splits cleanly across the constructive hierarchy. The local conservation law ($\partial_\mu J^\mu = 0$) is an algebraic identity provable in Bishop's constructive mathematics (BISH). The finite-volume conserved charge is a computable real number (BISH). The global conserved charge—the assertion that the total energy of an infinite system exists as a definite real number—is equivalent to the Limited Principle of Omniscience (LPO) via Bounded Monotone Convergence (BMC), precisely when the conserved density is non-negative (as for energy, via the weak energy condition). All results are formalised in Lean 4 with machine-checked proofs (520 lines, zero `sorry`). This constitutes a fourth independent domain—after statistical mechanics, general relativity, and quantum decoherence—exhibiting the same BISH/LPO boundary at bounded monotone convergence, and the first to calibrate a *structural law* rather than a physical prediction.

Dedicated to Mimi, my wife.

Soli Deo gloria

Contents

*New York University. AI-assisted formalization; see §10 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 Physical Context

In 1918, Emmy Noether proved that every continuous symmetry of the action functional yields a conserved current Noether [1918]. The theorem generates conservation of energy (time translation), momentum (space translation), angular momentum (rotation), electric charge ($U(1)$ gauge), and every other conservation law in the Standard Model. It is, in the words of Lederman and Hill, among the most important results guiding the development of modern physics Byers [1998].

This paper addresses a precise question: **What is the logical cost of Noether's theorem?** More specifically: does the theorem, or its consequences, require non-constructive principles?

1.2 The Answer

The answer decomposes into two layers:

- The **local conservation law**—the algebraic identity $\partial_\mu J^\mu = 0$ following from the equations of motion—is BISH (Height 0). It is a finite algebraic manipulation of sums over finitely many lattice sites. No omniscience principle is needed.
- The **global conserved charge**—the assertion that the total energy $E = \lim_{N\to\infty} E_N$ exists as a definite real number for every bounded field configuration with non-negative energy density—is equivalent to LPO.

The mechanism is the same as in the three preceding domains: the partial energies $E_N = \sum_{i<N} \varepsilon_i$ form a bounded monotone sequence (bounded by hypothesis, monotone because each $\varepsilon_i \geq 0$). Asserting the completed limit is BMC, which is LPO.

1.3 Programme Context

This is the fourth independent domain in a programme of constructive calibration of mathematical physics. The previous three domains are:

- **Statistical mechanics** (Paper 8) Lee [2026c]: The 1D Ising free energy f_N is BISH; the thermodynamic limit $f_\infty = \lim f_N$ is LPO via BMC.
- **General relativity** (Paper 13) Lee [2026e]: Finite geodesic computations in the Schwarzschild interior are BISH; the assertion $r(\tau) \to 0$ (geodesic incompleteness) is LPO via BMC.
- **Quantum mechanics** (Paper 14) Lee [2026f]: Finite-step decoherence bounds are BISH; exact decoherence (the assertion that coherence converges to a definite limit) is LPO via BMC.

All four domains produce bounded monotone sequences whose completed limits cost exactly LPO. The physics differs completely—partition functions, geodesic equations, density matrices, energy densities—but the logical structure is identical.

1.4 What Makes Paper 15 Different

Papers 8, 13, and 14 calibrate *predictions*: quantities physicists compute and compare to experiment (free energy values, singularity formation, measurement outcomes). Paper 15 calibrates a *structural law*—the principle that symmetries generate conservation laws. The local form of this law (the algebraic content of Noether's theorem) is constructive. The global form (total charge exists) is the idealisation. This is the first result in the programme to calibrate a structural principle rather than a physical prediction.

2 Background

2.1 Constructive Reverse Mathematics

Constructive reverse mathematics (CRM) classifies mathematical theorems by the weakest omniscience principle needed to prove them Bishop [1967], Bridges and Vîţă [2006], Ishihara [2006]. Bishop's constructive mathematics (BISH) avoids all omniscience principles; every existential claim comes with a computable witness.

The key principles for this paper are:

- **LPO** (Limited Principle of Omniscience): For every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, either all terms are zero or some term is one. Equivalent to: every bounded monotone sequence of reals converges (BMC). The equivalence is due to Bridges and Vîţă Bridges and Vîţă [2006] (Theorem 2.1.5).
- **BMC** (Bounded Monotone Convergence): Every monotone sequence bounded above has a limit.
- **NPSC** (Nonnegative Partial Sum Convergence): Every bounded sequence of partial sums of non-negative terms converges. A new abstraction introduced in this paper, equivalent to BMC and hence to LPO. This is the abstract framework connecting energy partial sums to the omniscience hierarchy.

2.2 The Diagnostic

When a physical theorem asserts the existence of a completed infinite limit (thermodynamic limit, singularity, decoherence, global charge), the diagnostic is: check whether the underlying sequence is bounded and monotone. If yes, the assertion costs exactly LPO. If the sequence has a computable modulus of convergence, the limit is BISH and LPO is not needed. The programme identifies physical theorems where the modulus is naturally absent—where the logical cost is an intrinsic feature of the physics, not an artefact of a particular proof strategy.

3 The Physics: Noether's Theorem on the Lattice

3.1 The Model

We work with a real scalar field on a 1D lattice with periodic boundary conditions. The lattice has N sites, with field values $\varphi_i \in \mathbb{R}$ and conjugate momenta $\pi_i \in \mathbb{R}$ at each site $i \in \mathrm{Fin}\ N$.

The lattice Lagrangian is:

$$L_N = \sum_{i=0}^{N-1} \left[\tfrac{1}{2}\pi_i^2 - \tfrac{1}{2}(\varphi_{i+1} - \varphi_i)^2 - V(\varphi_i)\right], \tag{1}$$

with $V(\varphi) \geq 0$ (non-negative potential) and periodic boundary conditions: $\varphi_N = \varphi_0$.

The equations of motion (discrete Euler–Lagrange) are:

$$\dot{\varphi}_i = \pi_i, \qquad \dot{\pi}_i = \Delta_d \varphi_i - V'(\varphi_i), \tag{2}$$

where $\Delta_d \varphi_i = \varphi_{i+1} - 2\varphi_i + \varphi_{i-1}$ is the discrete Laplacian.

The conserved energy (from time-translation invariance) is:

$$E_N = \sum_{i=0}^{N-1} \left[\tfrac{1}{2}\pi_i^2 + \tfrac{1}{2}(\varphi_{i+1} - \varphi_i)^2 + V(\varphi_i)\right]. \tag{3}$$

Every term in E_N is non-negative: $\frac{1}{2}\pi_i^2 \geq 0$ (kinetic), $\frac{1}{2}(\Delta\varphi_i)^2 \geq 0$ (gradient), $V(\varphi_i) \geq 0$ (potential). This positivity is the critical property.

3.2 Why the Lattice

We work on the lattice rather than in the continuum for three reasons:

1. **Physical honesty.** Lattice field theory is how physics is actually computed (lattice QCD, lattice gauge theory Creutz [1983]). The continuum limit is itself an idealisation whose logical cost could be separately calibrated.

2. **Lean practicality.** Everything is finite sums over Fin N. No Sobolev spaces, no distributions, no functional analysis infrastructure.

3. **Structural completeness.** The lattice model captures all three levels of the hierarchy: local conservation (algebraic identity), finite-volume energy (computable), and infinite-volume energy (BMC).

The mathematical theory of discrete Noether theorems is developed in Skopenkov Skopenkov [2023].

3.3 The Sign Observation

The logical cost of a global conservation law depends on whether the conserved density is positive definite:

- **Energy density** (T^{00}): Sum of squares plus non-negative potential. Always ≥ 0. Partial energies E_R are monotone increasing in R. $\to$ BMC $\to$ LPO.

- **Charge density** (J^0): Signed (positive for particles, negative for antiparticles). Partial charges Q_R oscillate as R grows. $\to$ Conditional convergence, does not fit BMC.

This parallels the weak energy condition in general relativity: $T_{ab}v^a v^b \geq 0$ for all timelike v^a Witten [1981], Kontou and Sanders [2020]. The positivity of energy density is a physical assumption. The programme shows this assumption is precisely what makes the infinite-volume limit cost LPO rather than something else.

4 Finite Noether at BISH

4.1 The Conservation Identity

Theorem 4.1 (Energy conservation—Noether's theorem)**.** ✓ *For any field configuration* (φ, π) *on an* N*-site lattice (*$N \geq 2$*) with periodic boundary conditions, and any potential* V*, the total rate of energy change vanishes identically:*

$$\sum_{i=0}^{N-1} \Big[\pi_i \cdot (\Delta_d \varphi_i - V'(\varphi_i)) + (\varphi_{i+1} - \varphi_i)(\pi_{i+1} - \pi_i) + V'(\varphi_i) \cdot \pi_i\Big] = 0.$$

Proof. We compute $\frac{dE_N}{dt}$ by substituting the equations of motion (2) into the time derivative of E_N (3). Each summand receives three contributions: from the kinetic term $\pi_i \cdot \dot{\pi}_i = \pi_i(\Delta_d \varphi_i - V'(\varphi_i))$, from the gradient term $(\varphi_{i+1} - \varphi_i)(\dot{\varphi}_{i+1} - \dot{\varphi}_i) = (\varphi_{i+1} - \varphi_i)(\pi_{i+1} - \pi_i)$, and from the potential term $V'(\varphi_i)\dot{\varphi}_i = V'(\varphi_i)\pi_i$.

The first step is algebraic simplification. The $V'(\varphi_i)\pi_i$ terms from the kinetic and potential contributions cancel, leaving:

$$\text{summand}_i = \big(\varphi_{i+1}\pi_{i+1} - \varphi_i \pi_i\big) + \big(\pi_i \cdot \varphi_{i-1} - \varphi_i \cdot \pi_{i+1}\big).$$

This identity is verified by `ring` in Lean 4 after expanding the discrete Laplacian.

The sum of the first group is a telescoping sum: $\sum_i (\varphi_{i+1}\pi_{i+1} - \varphi_i\pi_i) = 0$ under periodic boundary conditions. This follows from **shift invariance of periodic sums** (Lemma 4.2): $\sum_i f(\text{next}(i)) = \sum_i f(i)$.

For the second group, we reindex the sum $\sum_i \pi_i \cdot \varphi_{i-1}$ using the substitution $i \mapsto \text{next}(j)$, giving $\sum_j \pi_{\text{next}(j)} \cdot \varphi_{\text{prev}(\text{next}(j))} = \sum_j \pi_{\text{next}(j)} \cdot \varphi_j$, where we used the inverse relationship prev $\circ$ next $=$ id. Then by commutativity: $\sum_j \pi_{\text{next}(j)} \cdot \varphi_j = \sum_j \varphi_j \cdot \pi_{\text{next}(j)}$, which equals $\sum_i \varphi_i \cdot \pi_{i+1}$. The two terms cancel, giving $\sum_i(\text{second group})_i = 0$.

Since both groups sum to zero, $dE_N/dt = 0$. □

Lemma 4.2 (Shift invariance of periodic sums). ✓ *For any $f : \mathrm{Fin}\ N \to \mathbb{R}$ and $N > 0$:*

$$\sum_{i \in \mathrm{Fin}\ N} f(\mathrm{next}(i)) = \sum_{i \in \mathrm{Fin}\ N} f(i).$$

Proof. The function next : Fin $N \to$ Fin N defined by $i \mapsto (i+1) \bmod N$ is a bijection. Its inverse is prev : $i \mapsto (i + N - 1) \bmod N$. Bijectivity is established by proving the inverse relationships next ∘ prev = id and prev ∘ next = id via case analysis on modular arithmetic. The result then follows from Mathlib4's `Fintype.sum_bijective`. □

Remark 4.3 (Logical status). Every step in the proof of Theorem 4.1 is finite arithmetic over Fin N. The telescoping uses periodic boundary conditions (a finite sum with wrapping indices). No limits, no compactness, no choice. The Lean 4 `#print axioms` output shows only `propext`, `Classical.choice` (from Mathlib4 infrastructure for `Fin.fintype` and `Real.instField`), and `Quot.sound`. No omniscience principles.

4.2 Non-negativity of Energy Density

Theorem 4.4 (Energy density is non-negative). ✓ *If $V(\varphi) \geq 0$ for all φ, then each term in E_N is non-negative:*

$$\varepsilon_i = \tfrac{1}{2}\pi_i^2 + \tfrac{1}{2}(\varphi_{i+1} - \varphi_i)^2 + V(\varphi_i) \geq 0.$$

Proof. The kinetic term $\frac{1}{2}\pi_i^2 \geq 0$ is a square divided by a positive constant. The gradient term $\frac{1}{2}(\varphi_{i+1} - \varphi_i)^2 \geq 0$ likewise. The potential term $V(\varphi_i) \geq 0$ by hypothesis. The sum of non-negative reals is non-negative. In Lean 4, the proof uses `positivity` for the squared terms and the hypothesis `hV` for the potential. □

4.3 Monotonicity of Partial Energies

Theorem 4.5 (Partial energy is monotone). ✓ *For a field on the infinite lattice $\mathbb{N}$, the partial energy $E_N = \sum_{i=0}^{N-1} \varepsilon_i$ is monotone increasing in N when $V \geq 0$.*

Proof. $E_{N+1} = E_N + \varepsilon_N \geq E_N$ since $\varepsilon_N \geq 0$ by Theorem 4.4. The Lean 4 proof applies `monotone_nat_of_le_succ` with the recurrence `partialEnergy_succ`. □

5 Global Energy and LPO

5.1 The Abstract Framework: NPSC

Definition 5.1 (NPSC). ✓ *Nonnegative Partial Sum Convergence* (NPSC): for every sequence $d : \mathbb{N} \to \mathbb{R}$ with $d_n \geq 0$ for all n, if the partial sums $\sum_{i<N} d_i$ are bounded, then they converge to a definite limit.

Theorem 5.2 (NPSC ↔ BMC). ✓ *Nonnegative Partial Sum Convergence is equivalent to Bounded Monotone Convergence.*

Proof. **(NPSC ⇒ BMC):** Given a monotone sequence $a : \mathbb{N} \to \mathbb{R}$ bounded above by M, define $d_n = a_{n+1} - a_n \geq 0$ (by monotonicity). The partial sums telescope:

$$\sum_{i<N} d_i = a_N - a_0,$$

which is bounded above by $M - a_0$. By NPSC, the partial sums converge to some L_d. Therefore $a_N = (\sum_{i<N} d_i) + a_0$ converges to $L_d + a_0$.

The telescoping identity is proved by induction on N. The base case is trivial ($\sum_{i<0} d_i = 0 = a_0 - a_0$). The inductive step uses `Finset.sum_range_succ`.

(BMC $\Rightarrow$ NPSC): Given $d : \mathbb{N} \to \mathbb{R}$ with $d_n \geq 0$ and bounded partial sums, the sequence $S_N = \sum_{i<N} d_i$ is monotone (adding a non-negative term cannot decrease the sum) and bounded. BMC gives convergence directly. □

Remark 5.3 (Logical status)**.** The equivalence NPSC $\leftrightarrow$ BMC is fully proved in Lean 4 with *no custom axioms.* The `#print axioms npsc_iff_bmc` output shows only `[propext, Classical.choice, Quot.sound]`—standard Mathlib4 infrastructure. This is the same pattern as Paper 14's `abc_iff_bmc`: the abstract equivalence is pure BISH.

Theorem 5.4 (NPSC $\leftrightarrow$ LPO)**.** ✓

$$\text{NPSC} \quad \longleftrightarrow \quad \text{LPO}.$$

Proof. Compose NPSC $\leftrightarrow$ BMC (Theorem 5.2) with BMC $\leftrightarrow$ LPO (Bridges–Vîţă Bridges and Vîţă [2006]). In Lean 4: `npsc_iff_bmc` composed with `lpo_iff_bmc.symm`. □

5.2 The Headline Result

Theorem 5.5 (Global energy existence $\leftrightarrow$ LPO)**.** ✓ *The assertion "for every bounded field configuration with $V \geq 0$, the total energy $E = \lim_{N\to\infty} E_N$ exists" is equivalent to* LPO*:*

$$\Big(\forall V, \varphi, \pi, M.\ V \geq 0 \to (\forall N,\ E_N \leq M) \to \exists E.\ E_N \to E\Big) \quad \longleftrightarrow \quad \text{LPO}.$$

Proof. **(Forward: Global energy $\Rightarrow$ LPO.)** Suppose the global energy assertion holds. We show NPSC holds (and hence LPO, by Theorem 5.4).

Given $d : \mathbb{N} \to \mathbb{R}$ with $d_n \geq 0$ and bounded partial sums, we encode d into a lattice field configuration as follows. Set $V = 0$ (zero potential, which is ≥ 0), $\varphi = 0$ (zero field), and $\pi_i = \sqrt{2 \cdot d_i}$. Then the energy density at site i reduces to

$$\varepsilon_i = \tfrac{1}{2}\pi_i^2 + 0 + 0 = \tfrac{1}{2} \cdot 2d_i = d_i,$$

where we used $(\sqrt{x})^2 = x$ for $x \geq 0$. Therefore $E_N = \sum_{i<N} d_i$, and the boundedness hypothesis carries over. The global energy assertion gives convergence.

(Reverse: LPO $\Rightarrow$ Global energy.) LPO $\Rightarrow$ BMC (by the Bridges–Vîţă equivalence). The partial energy sequence is monotone (Theorem 4.5) and bounded by hypothesis. BMC gives convergence. □

5.3 Axiom Certificate

The Lean 4 `#print axioms` output for `global_energy_iff_LPO` shows: `[propext, Classical.choice, Quot.sound, bmc_of_lpo, lpo_of_bmc]`. The only custom axioms are the two interface assumptions for the Bridges–Vîţă equivalence, cited from Bridges and Vîţă [2006] and Paper 8 Lee [2026c].

The abstract framework (`npsc_iff_bmc`) is pure BISH—no custom axioms.

6 The Sign Trap: Why Charge Doesn't Work

The conservation identity (Theorem 4.1) holds for *any* conserved current, not just energy. But the LPO equivalence (Theorem 5.5) depends critically on the positivity of the conserved density. For signed densities, the argument fails.

For a complex scalar field with $U(1)$ symmetry, the conserved charge density is

$$J^0 = i(\varphi^* \dot{\varphi} - \dot{\varphi}^* \varphi),$$

which is signed: positive for particles, negative for antiparticles. The partial charge $Q_R = \sum_{i=0}^{R-1} J_i^0$ is *not* monotone. Adding more sites can increase or decrease Q_R depending on the charge distribution.

The convergence of Q_R as $R \to \infty$ is a conditional convergence problem, not a monotone convergence problem. It does not map to BMC. Asserting convergence requires either:

- Explicit decay rates (collapsing to BISH), or
- Stronger principles than LPO (e.g., Bolzano–Weierstrass for subsequences).

The lesson is that the constructive calibration depends not just on the theorem (Noether) but on the *physical content* of the conserved quantity. Positivity (the weak energy condition) is what makes the BMC pattern available. The sign structure of the density is logically significant.

This is analogous to the distinction in Paper 14 between uniform-angle decoherence (geometric decay, BISH) and variable-angle decoherence (general monotone, LPO). The specific model determines where in the hierarchy the result lands.

7 Domain Invariance

7.1 The Four-Domain Table

Paper 15 is the fourth physical domain producing the BMC $\leftrightarrow$ LPO pattern:

Domain	**Paper**	**Bounded Monotone Seq.**	**BISH Content**	**LPO Content**
Stat. Mech.	8	Free energy f_N	f_N computed	$f_\infty = \lim f_N$
Gen. Rel.	13	Radial coord. $r(\tau)$	$r(\tau)$ to ε	$r(\tau) \to 0$
Quantum Meas.	14	Coherence $c(N)$	$c(N) < \varepsilon$	$c(N) \to 0$ (collapse)
Conserv. Laws	**15**	**Partial energy** E_N	$dE/dt = 0$, E_N	$E = \lim E_N$

7.2 What Domain Invariance Means

Four independent physical domains. Different physics: Paper 8 uses partition functions and Boltzmann weights; Paper 13 uses geodesic equations and the Schwarzschild metric; Paper 14 uses density matrices and partial trace; Paper 15 uses Lagrangians, symmetry flows, and energy densities.

Same logical structure: bounded monotone sequence, finite truncations at BISH, completed limit at LPO. Same abstract equivalence (BMC $\leftrightarrow$ LPO) instantiated in each domain.

One instance is a result. Two is a pattern. Three is evidence. Four begins to look like a structural feature of how physical theories relate to mathematical frameworks.

7.3 What Paper 15 Adds

Paper 15 is qualitatively different from the other three. Papers 8, 13, and 14 calibrate *predictions*—quantities physicists compute and compare to experiment. Paper 15 calibrates a *principle*—the structural law that symmetries generate conservation laws.

The local form (Noether's algebraic identity) is constructive. The global form (total charge exists) is the idealisation. This means: **the architecture of physical law is constructive.** The idealisation enters not in the law itself but in the totality assertion—the claim that the conserved quantity exists as a definite number summed over all of space. This is a statement about formalism, not nature. No finite experiment distinguishes "total energy = E" from "partial energy E_N agrees with E to within ε for all N we can probe."

7.4 Full Calibration Table

The calibration table for the full series, updated with Paper 15:

Physical layer	**Principle**	**Status**	**Source**
Finite-volume Gibbs states	BISH	Calibrated	Trivial
Finite-size approximations (Ising)	BISH	Calibrated	Paper 8
Schwarzschild exterior	BISH	Calibrated	Paper 1
Interior finite-time physics	BISH	Calibrated	Paper 13
Tsirelson bound (CHSH $\leq 2\sqrt{2}$)	BISH	Calibrated	Paper 11
Bell state entropy ($\log 2$)	BISH	Calibrated	Paper 11
Finite-step decoherence	BISH	Calibrated	Paper 14
Local conservation ($dE/dt = 0$)	**BISH**	**Calibrated**	**Paper 15**
Bidual-gap witness ($S_1(H)$)	$\equiv$ WLPO	Calibrated	Papers 2, 7
Thermodynamic limit (Ising)	$\equiv$ LPO	Calibrated	Paper 8
Geodesic incompleteness	$\equiv$ LPO	Calibrated	Paper 13
Exact decoherence (collapse)	$\equiv$ LPO	Calibrated	Paper 14
Global energy existence	**$\equiv$ LPO**	**Calibrated**	**Paper 15**

The pattern is consistent: all LPO costs arise from completed infinite limits; all finite-time and finite-size physics is BISH.

8 Lean Formalisation

8.1 Module Structure

The formalisation is organized as a single Lean 4 project with 6 modules:

Module	**Lines**	**Content**
`Defs.lean`	157	Lattice field types, energy density, fnext/fprev, non-negativity
`LocalConservation.lean`	190	Periodic BC shift lemmas, Noether's theorem ($dE/dt = 0$)
`Monotonicity.lean`	68	Partial energy recurrence, monotonicity, non-negativity
`LPO_BMC.lean`	57	LPO, BMC definitions, axiomatised equivalence
`GlobalEnergy.lean`	200	NPSC framework, npsc_iff_bmc, encoding, headline LPO theorem
`Main.lean`	100	Assembly + `#print axioms` audit
Total	**∼520**	

Table 1: Module structure of Paper 15.

8.2 Key Design Decisions

Periodic boundary conditions via explicit index wrapping. The lattice uses Fin N indices with explicit wrapping functions `fnext` ($i \mapsto (i+1) \bmod N$) and `fprev` ($i \mapsto (i + N - 1) \bmod N$). These are proved to be mutual inverses via explicit modular arithmetic: `fnext_fprev` and `fprev_fnext`. Bijectivity then follows from the inverse relationships, which is cleaner than direct injectivity/surjectivity proofs involving nested modular arithmetic.

Dual lattice models. The formalisation uses two complementary lattice models: a finite lattice with periodic boundary conditions (Fin N) for the conservation identity, and an infinite lattice ($\mathbb{N}$) with open boundary conditions for the partial energy and LPO equivalence. The finite lattice captures the algebraic content of Noether's theorem; the infinite lattice captures the passage to the global limit.

Standalone package. Paper 15 is a self-contained Lake package. It cannot import Paper 8 or 14 as Lake dependencies, so LPO, BMC, and the equivalence are re-defined. The BMC $\leftrightarrow$ LPO equivalence is axiomatised with citation.

8.3 Core Definitions

```
/-- A state of the lattice scalar field. -/
structure LatticeState (N : Nat) where
  phi : Fin N -> Real
  pi  : Momenta N

/-- Next site index with periodic BC. -/
def fnext (hN : 0 < N) (i : Fin N) : Fin N :=
  <<(i.val + 1) % N, Nat.mod_lt _ hN>>

/-- Energy density at site i. -/
def energyDensity (V : Real -> Real) (hN : 0 < N)
    (s : LatticeState N) (i : Fin N) : Real :=
  kineticDensity s i + gradientDensity hN s i
    + potentialDensity V s i
```

Listing 1: Lattice state and energy density (Defs.lean, excerpt).

```
/-- The total rate of energy change expression. -/
def totalEnergyRate (hN : 0 < N)
    (phi pi : Fin N -> Real) (V' : Real -> Real) :=
  sum i : Fin N,
    (pi i * (discreteLaplacian hN phi i - V' (phi i))
     + (phi (fnext hN i) - phi i) *
       (pi (fnext hN i) - pi i)
     + V' (phi i) * pi i)

/-- Noether's theorem: energy is conserved. -/
theorem noether_conservation (hN : 2 <= N)
    (phi pi : Fin N -> Real) (V' : Real -> Real) :
    totalEnergyRate (by omega) phi pi V' = 0
```

Listing 2: Noether's theorem (LocalConservation.lean, excerpt).

```
/-- Nonnegative Partial Sum Convergence. -/
def NPSC : Prop :=
  forall (d : Nat -> Real), (forall n, 0 <= d n) ->
    (exists M, forall N,
      sum i in Finset.range N, d i <= M) ->
    exists L, forall e, 0 < e -> exists N0,
      forall N, N0 <= N ->
        |sum i in Finset.range N, d i - L| < e

/-- NPSC <-> BMC (pure BISH, no omniscience). -/
theorem npsc_iff_bmc : NPSC <-> BMC

```

```
/-- Global energy existence <-> LPO. -/
theorem global_energy_iff_LPO :
    (forall V, (forall x, 0 <= V x) ->
      forall phi pi M,
        (forall N, partialEnergy V phi pi N <= M) ->
      exists E, forall e, 0 < e -> exists N0,
        forall N, N0 <= N ->
          |partialEnergy V phi pi N - E| < e)
    <-> LPO
```

Listing 3: NPSC and headline theorem (GlobalEnergy.lean, excerpt).

8.4 Axiom Audit

The `Main.lean` module audits the axiom profile of each theorem:

```
-- Part A (BISH):
#print axioms noether_conservation
-- [propext, Classical.choice, Quot.sound]

#print axioms energyDensity_nonneg
-- [propext, Classical.choice, Quot.sound]

#print axioms partialEnergy_mono
-- [propext, Classical.choice, Quot.sound]

-- Part B (LPO equivalence):
#print axioms npsc_iff_bmc
-- [propext, Classical.choice, Quot.sound]
-- (no custom axioms!)

#print axioms global_energy_iff_LPO
-- [propext, Classical.choice, Quot.sound,
--  bmc_of_lpo, lpo_of_bmc]
```

Listing 4: Axiom audit (Main.lean, selected).

The `Classical.choice` appearing in the BISH results arises from MATHLIB4's real number infrastructure—specifically, `Fin.fintype` (for finite sums) and `Real.instField` (for real arithmetic). The mathematical content of these proofs is constructive: they involve only explicit finite-sum manipulation and algebraic identities. The constructive calibration is established by proof-content analysis, following the methodology described in Paper 10 Lee [2026d].

8.5 CRM Audit

The formalisation passes the CRM standard established in Papers 8, 13, and 14:

- Clean stratification: Part A never touches LPO axioms.
- Only declared axioms are `bmc_of_lpo` and `lpo_of_bmc`, properly cited.
- Abstract framework (`npsc_iff_bmc`) is pure BISH.
- `by_cases` in the inverse proofs uses decidable $\mathbb{N}$ equality (`instDecidableEqNat`, zero axioms), not `Classical.em`.
- `Nat.eq_or_lt_of_le` and `Nat.succ_le_of_lt` are axiom-free.
- No `Classical.em`, no `Classical.byContradiction`, no `decide` on propositions anywhere in the source code.

9 Discussion

9.1 The Local/Global Distinction

The result sharpens the local/global distinction in physics. Local conservation ($\partial_\mu J^\mu = 0$) is the empirical content—it determines what happens in any finite region. Global conservation (total charge exists) is the totalising assertion—it sums over all of space.

The constructive hierarchy reveals these are logically different claims. The local law is algebraic (BISH). The global assertion is infinitary (LPO). Physicists routinely treat them as equivalent ("conservation of energy"), but the programme shows they have different logical costs.

This has implications for quantum gravity, where the definition of total energy is notoriously problematic (no local energy density for the gravitational field; see Witten Witten [1981] for the positive energy theorem). The programme suggests this difficulty is not an accident—total energy is an LPO-level assertion, and it becomes harder to formulate precisely in regimes where the positive-definiteness of T^{00} is not guaranteed.

9.2 Why Four Domains Matter

The accumulation of four independent domains all producing BMC $\leftrightarrow$ LPO at the boundary between finite physics and infinite formalism suggests this is a structural feature, not a coincidence. There is no obvious physical reason why Ising partition functions, Schwarzschild geodesics, decoherence coherence, and energy densities should share logical structure. They describe completely different phenomena.

The common feature is that all four involve sequences of finite computations (BISH) whose completed limits (LPO) are physically meaningful but experimentally indistinguishable from sufficiently close finite approximations. The programme has not proved this is universal—there may be physical theorems whose logical cost is different (e.g., dependent choice rather than LPO, as in the strong law of large numbers). But four independent data points sharing the same boundary is suggestive.

9.3 Limitations

1. **Lattice model.** We work on a lattice, not in the continuum. The continuum limit is itself an idealisation whose logical cost could be separately calibrated. We expect it to introduce additional non-constructive content (function spaces, compactness), but this is not formalised here.

2. **Encoding is abstract.** The reverse direction (global energy $\Rightarrow$ LPO) uses an encoding that maps arbitrary bounded monotone sequences to lattice field configurations. The resulting configurations (zero field, zero gradient, $\pi_i = \sqrt{2d_i}$) may not correspond to any realistic physical system. This is standard in reverse mathematics—the encoding in Paper 8 similarly produces non-physical Ising configurations.

3. **Energy only.** The result applies cleanly only to positive-definite conserved densities (energy, probability, number density). Signed densities (electric charge) do not fit the BMC pattern (Section 6). A complete calibration of all Standard Model conservation laws would require separate analysis for each type of charge.

4. **Classical.choice is a Mathlib4 infrastructure artifact.** The proof-content analysis methodology for handling this is described in Paper 10 Lee [2026d].

10 Conclusion

Noether's theorem splits across the constructive hierarchy. The local conservation law—the algebraic content of the theorem—is BISH. The global conserved charge—the assertion that total energy exists—is LPO, equivalent to bounded monotone convergence.

This is the fourth independent domain in which the BISH/LPO boundary falls at exactly the same place: the passage from finite computation to completed infinite limit. Four domains (statistical mechanics, general relativity, quantum measurement, conservation laws) with completely different physics, all producing bounded monotone sequences whose completed limits cost LPO.

The result calibrates not just a physical prediction but a structural principle. The architecture of physical law—the connection between symmetry and conservation—is constructive. The idealisation enters through the totality assertion, not through the law itself.

AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as Papers 2, 7, 8, 13, and 14 Lee [2026a,b,c,e,f], Anthropic [2026].

The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalisation strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Task	**Human**	**AI (Claude Opus 4.6)**
Research direction	✓	
Mathematical blueprint	✓	✓
Proof strategy design	✓	✓
Mathlib4 API discovery		✓
Lean 4 proof generation		✓
Proof review	✓	
Build verification		✓
Paper writing	✓	✓

Table 2: Division of labor between human and AI.

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/AICardiologist/FoundationRelativity`
- **Path**: `Papers/P15_Noether/`
- **Build**: `lake exe cache get && lake build` (1,955 jobs, 0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Mathlib version**: commit `7091f0f6`

- **Interface axioms**: `bmc_of_lpo` (Bridges–Vîţă Bridges and Vîţă [2006]), `lpo_of_bmc` (Paper 8 Lee [2026c])
- **Axiom audit**: `Main.lean`
- **Axiom profile (main theorem)**: `[propext, Classical.choice, Quot.sound, bmc_of_lpo, lpo_of_bmc]`
- **Axiom profile (BISH content)**: `[propext, Classical.choice, Quot.sound]` (Mathlib infra only)
- **Axiom profile (npsc_iff_bmc)**: `[propext, Classical.choice, Quot.sound]` (no custom axioms)
- **Total**: 6 files, ~520 lines, 0 sorry
- **Zenodo DOI**: 10.5281/zenodo.18572494

Acknowledgments

The LEAN 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the MATHLIB4 community for maintaining the comprehensive library of formalised mathematics that made this work possible.

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