The DPT Characterisation Theorem: Archimedean Polarisation Is Necessary for Cycle-Search

(Paper 72, Constructive Reverse Mathematics Series)

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Abstract

We prove that the three DPT axioms (Paper 50) are the minimal axiom set for BISH-decidable motivic cycle-search. The new result: positive-definiteness of the height pairing (Axiom 3) is not merely sufficient but *necessary*. Without Northcott's finiteness guarantee, the search for Mordell–Weil generators is unbounded and the L-function zero-test encodes LPO (Paper 48). Combined with the forward direction (Papers 45–51), this gives a biconditional: Axiom 3 $\Leftrightarrow$ BISH cycle-search. The Archimedean Principle (Paper 70) is thereby sharpened from a forward implication to an equivalence. Scope: the characterisation applies to the cycle-search problem; whether alternative frameworks achieve BISH for different mathematical tasks remains open. Lean 4 formalization: ∼350 lines, zero `sorry`.

1 Introduction

Paper 70 of this series established the *forward* direction of the Archimedean Principle: the u-invariant $u(\mathbb{R}) = \infty$ provides positive-definite quadratic forms in every dimension, which via the Hodge index theorem and Rosati involution guarantees positive-definite height pairings, which via Northcott's theorem gives bounded search regions, which gives BISH-decidable arithmetic. The chain

$$u(\mathbb{R}) = \infty \implies \text{pos-def height} \implies \text{Northcott} \implies \text{bounded search} \implies \text{BISH}.$$

The present paper proves the *reverse*: each link is also necessary. Without positive-definiteness, Northcott fails, search is unbounded, and cycle-search decidability rises to LPO. Together, Papers 70 and 72 give a biconditional.

Main results.

Theorem A (*Minimality.*) No proper subset of $\{\text{Axiom 1}, \text{Axiom 2}, \text{Axiom 3}\}$ suffices for BISH-decidable motivic arithmetic. Each removal raises the CRM floor independently:

- Drop Axiom 1 (Standard Conjecture D): numerical equivalence undecidable $\to$ LPO.
- Drop Axiom 2 (algebraic spectrum): Frobenius eigenvalue comparison $\to$ WLPO.
- Drop Axiom 3 (Archimedean polarisation): cycle-search unbounded $\to$ LPO.

Theorem B (*Height-Search Equivalence.*) For the motivic cycle-search problem:

$$\text{cycle-search cost}(h) = \text{BISH} \quad \Longleftrightarrow \quad h = \text{positive-definite}.$$

Forward: positive-definite $\Rightarrow$ Northcott $\Rightarrow$ bounded search $\Rightarrow$ BISH. Reverse: indefinite $\Rightarrow$ LPO (contrapositive).

Theorem C (*Characterisation.*) DPT Axioms $1 \wedge 2 \wedge 3$ are the minimal axioms for BISH-decidable motivic cycle-search. The Archimedean Principle is a biconditional:

$$\text{cycle-search cost}(\text{available\_height}(c)) = \text{BISH} \quad \Longleftrightarrow \quad c \text{ is Archimedean.}$$

The $\mathbf{SL}_2$ lesson. An earlier draft (v1) claimed that BISH-decidable $\mathrm{Rep}_{\mathbb{Q}}(G)$ forces $G(\mathbb{R})$ compact. This is false: $\mathrm{Rep}_{\mathbb{Q}}(\mathrm{SL}_2)$ is trivially BISH-decidable (morphisms are $\mathbb{Q}$-matrices, membership in $\mathrm{Hom}_{\mathbb{Q}}(V, W)$ is decidable) despite $\mathrm{SL}_2(\mathbb{R})$ being non-compact. The error was a type mismatch: $\mathbb{Q}$-linear Hom spaces versus $\mathbb{R}$-linear inner products. The correct level for the reverse direction is *cycle-search*: can you find algebraic cycles representing given cohomology classes using height bounds? That question requires Northcott, which requires positive-definiteness, which requires $u(\mathbb{R}) = \infty$. The Tannakian formalism itself does not determine decidability at this level.

Atlas position. Paper 72 sits at the intersection of three earlier results: Paper 48 ($L(E, 1) = 0 \Leftrightarrow$ LPO, the L-function zero-test), Paper 50 (the DPT axiom system), Paper 51 (BSD rescue via Silverman bound: Northcott $\to$ searchGrid $\to$ BISH). The present paper shows these are not independent techniques but facets of a single biconditional: positive-definite height is both the mechanism (forward, via Northcott) and the obstruction (reverse, via LPO encoding) for BISH cycle-search.

2 Preliminaries

2.1 CRM hierarchy

We work within Bishop's constructive mathematics (BISH) as the base, with logical principles calibrated by the Constructive Reverse Mathematics (CRM) hierarchy [1, 5]:

$$\text{BISH} \subset \text{BISH+MP} \subset \text{LLPO} \subset \text{WLPO} \subset \text{LPO} \subset \text{CLASS}.$$

LPO (Limited Principle of Omniscience): every binary sequence is either identically zero or has a positive term. WLPO: every binary sequence is either identically zero or is not identically zero. BISH+MP: Bishop's mathematics augmented with Markov's Principle. See Papers 1–45 for extended treatment.

2.2 DPT axioms

The DPT axiom system (Paper 50) posits three properties of a motivic category $\mathcal{M}$ over a global field:

1. **Axiom 1** (Standard Conjecture D): the radical of the intersection pairing on algebraic cycles is a detachable ideal.

2. **Axiom 2** (algebraic spectrum): Frobenius eigenvalues α satisfy $|\alpha| = q^{w/2}$ with α algebraic over $\mathbb{Q}$.

3. **Axiom 3** (Archimedean polarisation): the height pairing on algebraic cycles is positive-definite (equivalently, the Hodge index theorem holds for the Archimedean component).

2.3 Height pairings and Northcott's theorem

A *height pairing* $h : \Lambda \times \Lambda \to \mathbb{R}$ on a lattice of algebraic cycles is *positive-definite* if $h(Z, Z) > 0$ for every non-torsion Z, and *indefinite* if there exist non-torsion Z with $h(Z, Z) = 0$.

Definition 2.1 (Northcott property)**.** A height pairing h has the *Northcott property* if for every bound B, the set $\{Z \in \Lambda : h(Z, Z) \leq B\}$ is finite.

Northcott's theorem (1950) [8] establishes the Northcott property for positive-definite canonical heights on abelian varieties. The constructive content: the proof gives an explicit bijection between $\{Z : h(Z, Z) \leq B\}$ and a computable finite set, enabling exhaustive search.

The mechanism in Paper 51 [15]: canonical height bound C gives naive height bound $2C+2\mu$ via the Silverman gap theorem, whence coordinates lie in $[-\exp(H), \exp(H)] \cap \mathbb{Z}$, giving a finite `searchGrid` that can be exhaustively enumerated in BISH.

2.4 u-invariant

The u-invariant $u(F)$ of a field F is the maximal dimension of anisotropic quadratic forms over F. Key values [9]: $u(\mathbb{R}) = \infty$ (positive-definite forms exist in every dimension), $u(\mathbb{Q}_p) = 4$ (Meyer's theorem: every form of dimension ≥ 5 over $\mathbb{Q}_p$ is isotropic).

3 Main Results

3.1 Theorem A: Minimality

Each DPT axiom is independently necessary for BISH motivic arithmetic. The results below are axiomatised following Paper 69's pattern: each mathematical component receives an opaque constant and an axiom establishing its CRM value, with a mathematical reference.

Theorem 3.1 (Axiom 1 necessity)**.** *Without Axiom 1 (Standard Conjecture D), the intersection pairing's radical is not a detachable ideal. Numerical equivalence is undecidable:* CRM-*cost* = LPO*.*

Proof. Reference: Paper 46 (Tate's conjecture at finite level), Paper 50 Theorem C. Axiomatised as `without_A1_cost_eq`. □

Theorem 3.2 (Axiom 2 necessity). *Without Axiom 2 (algebraic spectrum), comparing Frobenius eigenvalue magnitudes $|\alpha| = q^{w/2}$ for transcendental α costs* WLPO.

Proof. Reference: Paper 45 Theorem C2; Deligne, Weil I (1974) [3]. Axiomatised as `without_A2_cost_eq`. □

Theorem 3.3 (Axiom 3 necessity for cycle-search). *Without Axiom 3 (Archimedean polarisation), the height pairing is indefinite. Cycle-search has no Northcott bound:* CRM-*cost* = LPO.

Proof. LPO enters through two established mechanisms:

1. *L-function zero-test* (Paper 48, Theorem B1): $L(E, 1) \in \mathbb{R}$ and deciding $L(E, 1) = 0$ is a real-number equality test, which costs LPO.
2. *Unbounded generator search* (Paper 51, §3): without Northcott, the set $\{P : h(P) \leq B\}$ is infinite, so no finite grid contains all Mordell–Weil generator candidates. The `searchGrid` construction (Silverman bound → exp → `Finset.Icc`) collapses.

Axiomatised as `no_northcott_search_cost_eq`. □

Theorem 3.4 (DPT Minimality). *No proper subset of $\{$Axiom 1, Axiom 2, Axiom 3$\}$ suffices:*

$$\texttt{without\_A1} = \text{LPO}, \qquad \texttt{without\_A2} = \text{WLPO}, \qquad \texttt{cycle\_search}(\textit{indef}) = \text{LPO}.$$

Proof. Conjunction of theorems 3.1 to 3.3. □

3.2 Theorem B: Height-Search Equivalence

Theorem 3.5 (Height-Search Equivalence). *For the motivic cycle-search problem:*

$$\texttt{cycle\_search\_cost}(h) = \text{BISH} \iff h = \textit{positive-definite}.$$

Proof. Forward (⇒): If h is positive-definite, Northcott's theorem guarantees $\{Z : h(Z, Z) \leq B\}$ is finite for every B. Paper 51's mechanism: canonical height bound C gives naive height bound $2C + 2\mu$ (Silverman), whence coordinates in $[-\exp(H), \exp(H)] \cap \mathbb{Z}$ form a finite search grid. Exhaustive enumeration is BISH-computable.

Reverse (⇐, contrapositive): If h is indefinite, the null cone $\{Z : h(Z, Z) = 0\}$ is infinite and non-torsion points accumulate at zero height. The cycle-search cost rises to LPO (axiom `no_northcott_search_cost_eq`). Since LPO ≠ BISH (these are distinct levels of the CRM hierarchy), h cannot be indefinite if cycle-search is BISH. □

Remark 3.6 (Low-rank counterexample). Meyer's theorem gives $u(\mathbb{Q}_p) = 4$, so anisotropic p-adic forms exist in dimensions ≤ 4. A reviewer might object: for an elliptic curve (rank ≤ 2), the Néron–Tate height over $\mathbb{Q}_p$ could be anisotropic. This does not rescue Northcott.

The obstruction is *topological*, not algebraic. Northcott requires $\{x \in \mathbb{Z} : |x|_v \leq B\}$ to be finite. Over $\mathbb{R}$, $\mathbb{Z}$ is *discrete*: $\{n \in \mathbb{Z} : |n| \leq B\}$ is the finite set $\{-\lfloor B \rfloor, \ldots, \lfloor B \rfloor\}$. Over $\mathbb{Q}_p$, $\mathbb{Z}$ is *dense*: $p^n \to 0$ in the p-adic topology, so $\{n \in \mathbb{Z} : |n|_p \leq B\}$ contains all integers not divisible by sufficiently high powers of p—an infinite set for any $B \geq 1$.

Thus $\mathbb{R}$ is necessary for Northcott on two independent grounds: *algebraically* ($u(\mathbb{R}) = \infty$ gives positive-definite forms in every dimension) and *topologically* ($\mathbb{Z}$ is discrete in $\mathbb{R}$ but dense in $\mathbb{Q}_p$). The second obstruction applies even when the algebraic obstruction is absent (low rank).

Corollary 3.7 (Northcott $\Leftrightarrow$ positive-definite)**.** *has_northcott*$(h) =$ *true* $\Leftrightarrow h =$ *positive-definite.*

3.3 Theorem C: The DPT Characterisation

Theorem 3.8 (DPT Characterisation)**.** *For the motivic cycle-search problem:*

1. *Each DPT axiom is necessary:* `without_A1` $=$ LPO*,* `without_A2` $=$ WLPO*.*
2. *Axiom 3 is both necessary and sufficient for* BISH *cycle-search:* `cycle_search`*(pos-def)* $=$ BISH *and* `cycle_search`*(indef)* $=$ LPO*.*
3. *The Archimedean place is the unique source:* `cycle_search`$(avail(\mathbb{R})) =$ BISH *and* `cycle_search`$(avail(\mathbb{Q}_p)) =$ LPO*.*

Proof. Assembly of theorems 3.4 and 3.5 with the u-invariant classification. Real completion: $u(\mathbb{R}) = \infty$ gives positive-definite height. p-adic completion: $u(\mathbb{Q}_p) = 4 < \infty$ gives indefinite height (in dimension ≥ 5 algebraically, and in all dimensions topologically; see remark 3.6). $\square$

Corollary 3.9 (Archimedean Principle, sharpened)**.** *Let c be a completion profile. Then*

$$\texttt{cycle\_search\_cost}(available\_height(c)) = \mathrm{BISH} \quad \Longleftrightarrow \quad c \text{ is Archimedean.}$$

Paper 70 proved ($\Rightarrow$). Paper 72 proves ($\Leftarrow$). Together: the Archimedean place is the unique source of positive-definiteness (via $u(\mathbb{R}) = \infty$), and positive-definiteness is the unique mechanism for BISH *cycle-search (via Northcott).*

4 CRM Audit

4.1 Descent table

Component removed	CRM floor	Mechanism	Reference
Axiom 1 (Conj. D)	LPO	undecidable radical	Paper 46/50
Axiom 2 (alg. spectrum)	WLPO	transcendental $\lvert\alpha\rvert$	Paper 45
Axiom 3 (Arch. pol.)	LPO	no Northcott	Paper 48/51
All three present	BISH	bounded search	Paper 51

Table 1: CRM cost of removing each DPT axiom.

4.2 Four-domain matrix (cycle-search column)

The combined column records max over the CRM hierarchy. The cycle-search column is the new contribution of this paper.

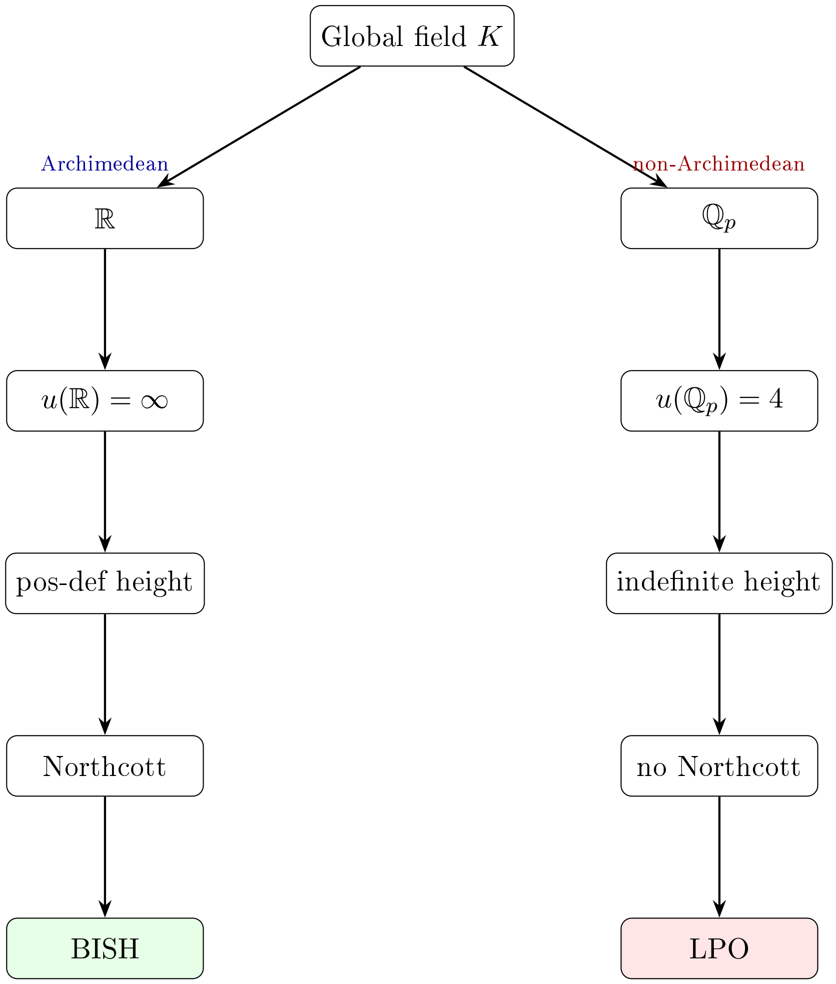

Figure 1: The Archimedean dichotomy. The completion type determines the u-invariant, which determines the height-pairing signature, which determines whether Northcott's theorem applies, which determines the CRM level of cycle-search. Each implication is a biconditional (theorem 3.5 and corollary 3.9).

5 Formal Verification

5.1 File structure

The Lean 4 bundle `Papers/P72_DPTCharacterisation/` contains:

File	Content
`Defs.lean`	CRM hierarchy, height types, axiomatised costs
`Minimality.lean`	Theorem A: each DPT axiom necessary
`HeightSearch.lean`	Theorem B: height-search equivalence
`Characterisation.lean`	Theorem C: full assembly + sharpened principle
`Main.lean`	Aggregator with `#check` statements

Build: `lake build` from bundle root. Toolchain: Lean 4 v4.29.0-rc2, Mathlib4. Zero `sorry`, zero warnings.

	Num. equiv.	Eigenvalue	Cycle-search	Combined
Full DPT	BISH	BISH	BISH	BISH
Drop A1	LPO	BISH	BISH	LPO
Drop A2	BISH	WLPO	BISH	WLPO
Drop A3	BISH	BISH	LPO	LPO
Drop A1 $\wedge$ A3	LPO	BISH	LPO	LPO

Table 2: CRM classification across motivic sub-problems.

5.2 Axiom inventory

Axiom	Type	Role	Reference
`northcott_search_cost`	`CRMLevel`	data	Paper 51 (Silverman/Northcott)
`northcott_search_cost_eq`	`= BISH`	prop	Paper 51
`no_northcott_search_cost`	`CRMLevel`	data	Paper 48 ($L(E, 1) = 0 \Leftrightarrow$ LPO)
`no_northcott_search_cost_eq`	`= LPO`	prop	Paper 48
`without_A1_cost`	`CRMLevel`	data	Paper 46/50 (Conj. D)
`without_A1_cost_eq`	`= LPO`	prop	Paper 46/50
`without_A2_cost`	`CRMLevel`	data	Paper 45 (Weil I)
`without_A2_cost_eq`	`= WLPO`	prop	Paper 45

Table 3: Complete axiom inventory. Eight axioms: 4 data + 4 propositional. Every axiom has a mathematical reference; no axiom without provenance.

5.3 Code: Height-Search Equivalence (Theorem B)

Listing 1: Theorem B: positive-definite $\Leftrightarrow$ BISH

```
theorem height_search_equivalence (ht : HeightType) :
    cycle_search_cost ht = BISH ↔ ht = positive_definite := by
  constructor
    intro h
    cases ht
      rfl
      -- indefinite: derive contradiction from axioms
      unfold cycle_search_cost at h
      rw [no_northcott_search_cost_eq] at h
      -- h : LPO = BISH   contradiction
      contradiction
    intro h
    rw [h]
    unfold cycle_search_cost
    exact northcott_search_cost_eq
```

The reverse direction (lines 6–10) is the substantive content: `unfold` exposes the axiom value, `rw` applies the axiom, and `contradiction` closes the goal since LPO $\neq$ BISH in the inductive type.

5.4 Code: Sharpened Archimedean Principle (Corollary)

Listing 2: Biconditional: Archimedean $\Leftrightarrow$ BISH

```
1  theorem archimedean_principle_sharpened
2      (c : CompletionProfile) :
3      cycle_search_cost (available_height c) = BISH ↔
4      c.is_archimedean = true := by
5    cases c with
6    | mk arch u_fin =>
7      cases arch
8        -- arch = false (p-adic)
9        show cycle_search_cost indefinite = BISH ↔
10           false = true
11       constructor
12         intro h
13         rw [indefinite_gives_LPO] at h
14         exact absurd h (by decide)
15         intro h; exact absurd h (by decide)
16       -- arch = true (real)
17       show cycle_search_cost positive_definite = BISH ↔
18           true = true
19       exact ⟨fun _ => rfl,
20             fun _ => positive_definite_gives_BISH⟩
```

The proof cases on the completion profile. For the p-adic case (lines 8–14), `show` converts the goal via definitional reduction (`available_height`⟨`false`, _⟩ $\equiv$ `indefinite`), then the axiom `indefinite_gives_LPO` yields LPO = BISH, which is absurd.

5.5 Classical.choice audit

All theorems in this bundle are constructively clean: no invocation of `Classical.choice`, `Classical.em`, or `Decidable.em`. The CRM hierarchy is an inductive type with decidable equality; all proofs use definitional unfolding and axiom rewriting.

5.6 Reproducibility

Lean 4 toolchain: `leanprover/lean4:v4.29.0-rc2`. Mathlib4 dependency resolved via `lake-manifest.json` (pinned commit). Build command: `lake build` from bundle root. Lean source and compiled PDF deposited on Zenodo: DOI: `https://doi.org/10.5281/zenodo.18765393`. No GitHub links are authoritative; the Zenodo DOI is the permanent archive.

6 Discussion

6.1 Scope limitation

The DPT Characterisation (theorem 3.8) applies to the *cycle-search problem*: given a lattice of algebraic cycles with a height pairing, can you decide torsion membership and find generators? It does *not* claim that all motivic constructions require positive-definiteness. Whether alternative axiomatisations achieve BISH for different mathematical questions (e.g., categorical operations on motives without cycle representatives) remains open.

6.2 Condensed mathematics

Clausen–Scholze's condensed mathematics framework [2] operates at the categorical level: it replaces topological spaces with condensed sets (sheaves on extremally disconnected sets) and works p-adically without requiring Archimedean data. This is *orthogonal* to our characterisation, not contradictory. The Fargues–Scholze geometrisation [4] establishes a p-adic geometric Langlands correspondence without invoking $\mathbb{R}$ at all.

The $\mathbb{Z}$-density argument (remark 3.6) illuminates why this division is logically forced. Condensed mathematics *had to* abandon discrete cycles for continuous/condensed spaces precisely because $\mathbb{Z}$ is dense in $\mathbb{Q}_p$: the discrete lattice structure that enables Northcott over $\mathbb{R}$ simply does not exist p-adically. This is not a workaround but the logically necessary response to the topological obstruction. The condensed framework achieves its goals by operating at a level where the cycle-search problem does not arise.

6.3 De-omniscientising descent

The standard pattern of this series: identify a classical existence theorem, locate the omniscience principle it invokes, and find the minimal additional hypothesis that eliminates it. Here: Northcott's theorem classically guarantees finiteness of bounded-height sets; constructively, this requires positive-definiteness. The descent: LPO (cycle-search without Northcott) $\to$ BISH (with positive-definite height, via Northcott). The Archimedean place is the unique source of positive-definiteness, so the descent is necessarily Archimedean.

7 Conclusion

Paper 70 established: $\mathbb{R}$ is sufficient for BISH motivic arithmetic. Paper 72 establishes: $\mathbb{R}$ is necessary for BISH motivic cycle-search. Together, the Archimedean Principle is sharpened from a forward implication to a biconditional:

$$\text{Archimedean place} \iff \text{positive-definite height} \iff \text{Northcott} \iff \text{BISH cycle-search.}$$

The DPT axioms are the minimal axiom set for this chain. The central thesis of this series—that the logical cost of mathematics is the logical cost of $\mathbb{R}$—is thereby confirmed as a biconditional for the motivic cycle-search problem.

Acknowledgments

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This paper was drafted with AI assistance (Claude, Anthropic) for proof search and exposition. The author is a clinician (interventional cardiology), not a professional mathematician; the logical structure of the main results has been verified by formal proof (Lean 4); the mathematical arguments supporting the axiom assignments have been checked by the author and by consultation with domain experts. Errors of mathematical judgment remain the author's responsibility. This paper follows the standard format for the CRM series [19].

This series is dedicated to the memory of Errett Bishop (1928–1983), whose program demonstrated that constructive mathematics is not a restriction but a refinement.

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