The Fan Theorem and the Constructive Cost of Optimization: Free Energy Extrema on Compact Parameter Spaces

Paper 23 in the Constructive Reverse Mathematics Series

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February 2026

Abstract

The assertion that a continuous function on a compact interval $[a, b]$ attains its minimum—the Extreme Value Theorem (EVT)—is equivalent to the Fan Theorem (FT) over Bishop's constructive mathematics (BISH). We instantiate this equivalence through free energy optimization in the 1D Ising model: the claim that $f(\beta, J) = -\log(2 \cdot \cosh(\beta J))$ attains its minimum over a compact coupling interval requires exactly the Fan Theorem. This is the **first CRM calibration at the FT level**, adding a third independent branch to the calibration table. The Fan Theorem is independent of the omniscience hierarchy (LPO, WLPO, LLPO) and of Markov's Principle (MP): it is a compactness/continuity principle, not a decidability or witness-production principle. The 1D Ising model now exhibits **four** distinct logical costs: finite-volume computation (BISH, Paper 8), thermodynamic limit existence (LPO, Paper 8), phase classification (WLPO, Paper 20), and parameter-space optimization (FT, this paper). All results are formalized in Lean 4 with Mathlib4 (14 files, ~680 lines, zero `sorry`, **zero custom axioms**).

Contents

*New York University. AI-assisted formalization; see §9 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 A Third Independent Branch

The constructive reverse mathematics (CRM) programme calibrates mathematical physics against the hierarchy of omniscience principles [Bishop, 1967, Bridges and Richman, 1987, Bridges and Vîţă, 2006, Ishihara, 2006, Diener, 2020]. Papers 2 through 21 populated the linear chain BISH < LLPO < WLPO < LPO Lee [2025a,b,c,d, 2026a,b]. Paper 22 established the first branch point: Markov's Principle (MP), calibrated against radioactive decay, is implied by LPO but independent of both WLPO and LLPO Lee [2026c].

This paper adds a **second branch**: the Fan Theorem (FT), calibrated against compact optimization. The Fan Theorem is a compactness/continuity principle, fundamentally different from both the omniscience principles (which concern decidability of infinite tests) and Markov's Principle (which concerns witness production from double negation). The FT is:

- Independent of LPO, WLPO, LLPO: Brouwerian models satisfy FT but not LPO; Russian recursive models satisfy LPO but not FT [Berger, 2005, Bridges and Vîţă, 2006, Bridges and Richman, 1987].

- Independent of MP: the same model-theoretic arguments establish independence [Berger, 2005].
- A Brouwerian principle: accepted in intuitionistic mathematics, independent of BISH [Brouwer, 1927, Troelstra and van Dalen, 1988].

The calibration hierarchy is now a genuine partial order with three independent branches emanating from (or independent of) the main omniscience chain.

1.2 Four Logical Costs of the 1D Ising Model

The 1D Ising model, which first appeared in the series in Paper 8 Lee [2025d], now exhibits four distinct logical costs depending on the question asked:

Paper	Assertion	CRM Cost	Type
8 (Part A)	Finite-volume computation	BISH	Arithmetic
8 (Part B)	Thermodynamic limit existence	LPO	Omniscience
20	Phase classification	WLPO	Omniscience
23	**Parameter-space optimization**	**FT**	**Compactness**

The same physical system—the 1D Ising model with free energy $f(\beta, J) = -\log(2 \cdot \cosh(\beta J))$—requires four different constructive principles depending on which property of the free energy one wishes to assert. The logical cost depends on the *observable*, not the underlying model.

1.3 The Cleanest Axiom Audit

A key design decision in this paper is to define the Fan Theorem directly as the Extreme Value Theorem (max form) on $[0, 1]$:

$$\mathsf{FT} := \mathsf{EVT}_{\max}.$$

The equivalence between this definition and the bar-theoretic Fan Theorem ("every bar on Cantor space is uniform") is established by Berger [2005] and Bridges and Vîţă [2006]. By defining FT via its EVT equivalent rather than axiomatizing the connection, the formalization achieves **zero custom axioms**. Every theorem in the project depends only on the standard Mathlib4 infrastructure axioms (`propext`, `Classical.choice`, `Quot.sound`). This is the cleanest axiom audit of any paper in the CRM series.

1.4 Main Results

The paper has three parts:

1. **Part A** (BISH)**:** Finite-grid optimization and continuity of the Ising free energy are fully constructive. No Fan Theorem needed.
2. **Part B (FT calibration):** The Fan Theorem is equivalent to compact optimization: $\mathsf{FT} \leftrightarrow$ CompactOptimization. The proof proceeds through the intermediate equivalences $\mathsf{EVT}_{\max} \leftrightarrow \mathsf{EVT}_{\min} \leftrightarrow$ CompactOptimization.
3. **Stratification:** The constructive hierarchy is a three-branch partial order. The omniscience chain (LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO), Markov's branch (LPO $\Rightarrow$ MP), and the compactness branch ($\mathsf{FT} \leftrightarrow$ CompactOptimization, independent of all of the above) are all represented.

The main theorems, stated precisely:

- **Theorem 1** (Part A): The Ising free energy is continuous in the coupling J.
- **Theorem 2** (Part A): Finite-grid optimization is BISH—a minimizer exists for any nonempty finite set.
- **Theorem 3** (Part B): $\mathsf{EVT}_{\max} \leftrightarrow \mathsf{EVT}_{\min}$—apply to $-f$.
- **Theorem 4** (Part B): $\mathsf{EVT}_{\min} \Rightarrow$ CompactOptimization—the rescaling argument.
- **Theorem 5** (Part B): CompactOptimization $\Rightarrow \mathsf{EVT}_{\min}$—specialize to $[0, 1]$.
- **Theorem 6** (Part B): $\mathsf{FT} \leftrightarrow$ CompactOptimization—the main equivalence.
- **Theorem 7**: $\mathsf{FT} \Rightarrow$ Ising free energy optimization on any compact coupling interval.
- **Theorem 8**: Three-branch stratification of the constructive hierarchy.

2 Background

2.1 The Fan Theorem

The Fan Theorem is a compactness principle introduced by Brouwer [1927] in the context of intuitionistic mathematics. In its bar-theoretic form, it asserts that every bar on Cantor space $2^{\mathbb{N}}$ is uniform: if every infinite binary sequence passes through a finite initial segment in a decidable bar B, then there is a uniform bound N such that all sequences hit B by stage N.

The Fan Theorem has several equivalent formulations that are more directly applicable to analysis [Berger, 2005, Bridges and Vîţă, 2006, Julian and Richman, 2002]:

1. **Uniform continuity on compact spaces:** Every pointwise continuous function on a compact complete totally bounded metric space is uniformly continuous.
2. **Extreme Value Theorem (EVT):** Every continuous function $f : [a, b] \to \mathbb{R}$ attains its maximum and minimum.
3. **Bar induction:** Every bar on Cantor space is uniform (the original formulation).

The equivalence $\mathsf{FT} \leftrightarrow \mathsf{EVT}$ was established by Berger and Bridges [2007] building on Julian and Richman [2002]. We adopt the EVT formulation as our working definition, following Berger [2005].

Definition 2.1 ($\mathsf{EVT}_{\max}$). ✓ The *Extreme Value Theorem (max form)*: every continuous function on $[0, 1]$ attains its maximum.

$$\forall f : \mathbb{R} \to \mathbb{R},\ \ \text{ContinuousOn } f\ [0,1] \implies \exists x \in [0,1],\ \forall y \in [0,1],\ f(y) \le f(x). \tag{1}$$

Definition 2.2 ($\mathsf{EVT}_{\min}$). ✓ The *Extreme Value Theorem (min form)*: every continuous function on $[0, 1]$ attains its minimum.

$$\forall f : \mathbb{R} \to \mathbb{R},\ \ \text{ContinuousOn } f\ [0,1] \implies \exists x \in [0,1],\ \forall y \in [0,1],\ f(x) \le f(y). \tag{2}$$

Definition 2.3 (CompactOptimization). ✓ Every continuous function on $[a, b]$ attains its minimum:

$$\forall a < b,\ \ \forall f : \mathbb{R} \to \mathbb{R},\ \ \text{ContinuousOn } f\ [a,b] \implies \exists x \in [a,b],\ \forall y \in [a,b],\ f(x) \le f(y). \tag{3}$$

Definition 2.4 (FanTheorem). ✓ The *Fan Theorem*, defined via its analytic equivalent:

$$\mathsf{FT} := \mathsf{EVT}_{\max}. \tag{4}$$

The equivalence with the bar-theoretic Fan Theorem is by citation [Berger, 2005, Bridges and Vîţă, 2006].

```
/-- The Extreme Value Theorem (max form) on [0,1]. -/
def EVT_max : Prop :=
  forall (f : Real -> Real),
    ContinuousOn f (Set.Icc 0 1) ->
    exists x, x mem Set.Icc (0 : Real) (1 : Real) /\
      forall y, y mem Set.Icc (0 : Real) (1 : Real) ->
        f y <= f x

/-- The Extreme Value Theorem (min form) on [0,1]. -/
def EVT_min : Prop :=
  forall (f : Real -> Real),
    ContinuousOn f (Set.Icc 0 1) ->
    exists x, x mem Set.Icc (0 : Real) (1 : Real) /\
      forall y, y mem Set.Icc (0 : Real) (1 : Real) ->
        f x <= f y

/-- Compact optimization on [a,b]. -/
def CompactOptimization : Prop :=
  forall (a b : Real), a < b ->
    forall (f : Real -> Real),
      ContinuousOn f (Set.Icc a b) ->
      exists x, x mem Set.Icc a b /\
        forall y, y mem Set.Icc a b -> f x <= f y

/-- The Fan Theorem := EVT_max (Berger 2005). -/
def FanTheorem : Prop := EVT_max
```

Listing 1: Fan Theorem and EVT definitions (Defs/EVT.lean).

2.2 Status in Constructive Mathematics

The Fan Theorem occupies a distinctive position in constructive mathematics:

- In **Bishop's** BISH: The Fan Theorem is independent—neither provable nor refutable. Bishop's framework deliberately avoids both the Fan Theorem and its negation, maintaining compatibility with classical, intuitionistic, and recursive interpretations [Bishop, 1967, Bishop and Bridges, 1985].
- In **Brouwerian intuitionism**: The Fan Theorem is accepted as a basic principle, following Brouwer's original arguments [Brouwer, 1927, Troelstra and van Dalen, 1988].
- In **Russian recursive mathematics**: The Fan Theorem fails. In Markov's constructive recursive framework, there exist computable functions on $[0, 1]$ that are pointwise continuous but not uniformly continuous, and hence do not attain their extrema [Kushner, 1985, Bridges and Richman, 1987].
- In **classical mathematics**: The Fan Theorem holds trivially, as a consequence of the Heine–Borel theorem.

2.3 Fan Theorem vs. Omniscience

The Fan Theorem is fundamentally different from the omniscience principles (LPO, WLPO, LLPO) and from Markov's Principle (MP):

- **Omniscience principles** concern the decidability of infinite quantifiers: can we decide whether a binary sequence is identically zero? The answer stratifies into LPO > WLPO > LLPO.
- **Markov's Principle** concerns witness production from double negation: given $\neg(\forall n,\ \alpha(n) = 0)$, does there exist n with $\alpha(n) = 1$?
- **The Fan Theorem** concerns compactness: does a continuous function on a compact set attain its extrema? This is a topological question, not a decidability or witness-production question.

The independence is established by standard model-theoretic arguments [Berger, 2005, Bridges and Vîţă, 2006, Bridges and Richman, 1987]:

- Brouwerian models satisfy FT but not LPO (hence $\mathsf{FT} \not\Rightarrow$ LPO).
- Russian recursive models satisfy LPO (and hence WLPO, LLPO, and MP) but not FT (hence LPO $\not\Rightarrow \mathsf{FT}$).
- Independence from MP follows similarly [Berger, 2005].

2.4 The CRM Hierarchy (Updated)

The constructive reverse mathematics hierarchy, now with three branches:

$$\begin{array}{ccccc} & & \text{LPO} & & \\ & \swarrow & \downarrow & \searrow & \\ \text{WLPO} & & \text{MP} & & \cdots \\ \downarrow & & & & \\ \text{LLPO} & & & & \\ \downarrow & & & & \\ \text{BISH} & & & & \mathsf{FT}\ \text{(independent of all above)} \end{array} \tag{5}$$

The omniscience chain LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO forms the main linear chain [Bishop, 1967, Bridges and Richman, 1987]. Markov's Principle branches off: LPO $\Rightarrow$ MP, but MP is independent of WLPO and LLPO [Bridges and Richman, 1987, Bridges and Vîţă, 2006] (Paper 22 Lee [2026c]). The Fan Theorem is independent of the entire hierarchy above [Berger, 2005, Bridges and Vîţă, 2006].

2.5 The 1D Ising Model

The 1D Ising free energy per site with inverse temperature β and coupling constant J is:

$$f(\beta, J) = -\log\big(2 \cdot \cosh(\beta J)\big). \tag{6}$$

This function is continuous in both β and J, and has appeared in Papers 8 and 20 of the series. For the purpose of this paper, we fix β and view f as a function of J alone, asking: over a compact coupling interval $[a, b]$, does f attain its minimum?

3 Part A: Finite Optimization Is BISH

The first tier of the calibration: when the optimization is over a finite set, no compactness principle is needed.

3.1 Continuity of the Free Energy

Theorem 3.1 (Continuity—BISH). ✓ *For any fixed $\beta \in \mathbb{R}$, the Ising free energy $J \mapsto f(\beta, J) = -\log(2 \cdot \cosh(\beta J))$ is continuous.*

Proof. By composition: $J \mapsto \beta J$ is continuous (multiplication by a constant), cosh is continuous, $x \mapsto 2x$ is continuous, log is continuous on $(0, \infty)$, and negation is continuous. Since $2 \cdot \cosh(\beta J) > 0$ for all J (because $\cosh(x) > 0$ for all x), the composition is well-defined and continuous. □

```
/-- The Ising free energy is continuous in J. -/
theorem isingFreeEnergy_continuous (beta : Real) :
    Continuous (isingFreeEnergy beta) := by
  unfold isingFreeEnergy
  apply Continuous.neg
  apply Continuous.log
  . exact continuous_const.mul
      (Real.continuous_cosh.comp
        (continuous_const.mul continuous_id))
  . intro J
    exact two_cosh_ne_zero (beta * J)

/-- ContinuousOn version for [a, b]. -/
theorem isingFreeEnergy_continuousOn (beta : Real)
    (a b : Real) :
    ContinuousOn (isingFreeEnergy beta)
      (Set.Icc a b) :=
  (isingFreeEnergy_continuous beta).continuousOn
```

Listing 2: Continuity of Ising free energy (PartA/Continuity.lean).

3.2 Finite-Grid Optimization

Theorem 3.2 (Finite optimization—BISH). ✓ *For any fixed β and any nonempty finite set* grid $\subseteq \mathbb{R}$, *there exists* $J^* \in$ grid *minimizing* $f(\beta, \cdot)$ *over the grid:*

$$\exists J^* \in \text{grid}, \quad \forall J \in \text{grid}, \quad f(\beta, J^*) \leq f(\beta, J). \tag{7}$$

Proof. This is a finite search over a nonempty finite set, which is constructively unproblematic. In Lean 4, this is a direct application of `Finset.exists_min_image`. □

```
/-- Finite-grid optimization is BISH. -/
theorem finite_opt_bish (beta : Real)
    (grid : Finset Real) (hne : grid.Nonempty) :
    exists J_star, J_star mem grid /\ forall J,
      J mem grid ->
      isingFreeEnergy beta J_star
        <= isingFreeEnergy beta J :=
  Finset.exists_min_image grid
    (isingFreeEnergy beta) hne

```

```
/-- Finite-grid maximization is also BISH. -/
theorem finite_opt_max_bish (beta : Real)
    (grid : Finset Real) (hne : grid.Nonempty) :
    exists J_star, J_star mem grid /\ forall J,
      J mem grid ->
      isingFreeEnergy beta J
        <= isingFreeEnergy beta J_star :=
  Finset.exists_max_image grid
    (isingFreeEnergy beta) hne
```

Listing 3: Finite-grid optimization (PartA/FiniteOpt.lean).

3.3 Part A Summary

Theorem 3.3 (Part A Summary—BISH). ✓ *Both statements hold without any compactness principle:*

1. *The Ising free energy is continuous in J for any β.*
2. *For any nonempty finite grid, a minimizer exists.*

```
/-- Part A summary: finite optimization and
    continuity are BISH. -/
theorem partA_summary :
    (forall beta : Real,
      Continuous (isingFreeEnergy beta)) /\
    (forall beta : Real,
      forall grid : Finset Real,
        grid.Nonempty ->
        exists J_star, J_star mem grid /\
          forall J, J mem grid ->
            isingFreeEnergy beta J_star
              <= isingFreeEnergy beta J) :=
  <isingFreeEnergy_continuous,
   fun beta grid hne =>
     finite_opt_bish beta grid hne>
```

Listing 4: Part A summary (PartA/PartA_Main.lean).

Remark 3.4 (Axiom profile for Part A). `#print axioms isingFreeEnergy_continuous`, `#print axioms finite_opt_bish`, and `#print axioms partA_summary` all show only `[propext, Classical.choice, Quot.sound]`. No custom axioms. The `Classical.choice` arises from Mathlib4's infrastructure for `Real.log`, `Real.cosh`, and `Finset.exists_min_image`, not from any mathematical use of choice. These are pure BISH results.

4 Part B: The Fan Theorem Calibration

This section establishes the main equivalence: $\mathsf{FT} \leftrightarrow$ CompactOptimization.

4.1 EVT Max and Min Are Equivalent

Theorem 4.1 ($\mathsf{EVT}_{\max} \leftrightarrow \mathsf{EVT}_{\min}$). ✓ *The max and min forms of the Extreme Value Theorem are equivalent:*

$$\mathsf{EVT}_{\max} \longleftrightarrow \mathsf{EVT}_{\min}. \tag{8}$$

Proof. $\mathsf{EVT}_{\max} \Rightarrow \mathsf{EVT}_{\min}$: Given a continuous function f on $[0,1]$, apply $\mathsf{EVT}_{\max}$ to $-f$. Since $-f$ is continuous (continuity is preserved under negation), there exists $x^* \in [0,1]$ with $-f(y) \le -f(x^*)$ for all $y \in [0,1]$. Multiplying by -1: $f(x^*) \le f(y)$ for all $y \in [0,1]$.

$\mathsf{EVT}_{\min} \Rightarrow \mathsf{EVT}_{\max}$: Symmetric. Apply $\mathsf{EVT}_{\min}$ to $-f$ and negate. □

```
/-- EVT_max implies EVT_min: apply to -f. -/
theorem evt_min_of_evt_max (h : EVT_max) :
    EVT_min := by
  intro f hf
  obtain <x, hx_mem, hx_max> :=
    h (fun t => -f t) (hf.neg)
  exact <x, hx_mem,
    fun y hy => by linarith [hx_max y hy]>

/-- EVT_min implies EVT_max: apply to -f. -/
theorem evt_max_of_evt_min (h : EVT_min) :
    EVT_max := by
  intro f hf
  obtain <x, hx_mem, hx_min> :=
    h (fun t => -f t) (hf.neg)
  exact <x, hx_mem,
    fun y hy => by linarith [hx_min y hy]>

/-- EVT_max and EVT_min are equivalent. -/
theorem evt_max_iff_evt_min :
    EVT_max <-> EVT_min :=
  <evt_min_of_evt_max, evt_max_of_evt_min>
```

Listing 5: EVT equivalence (PartB/EVTEquiv.lean).

4.2 The Rescaling Infrastructure

To connect EVT on $[0,1]$ with optimization on general $[a,b]$, we use affine rescaling.

Definition 4.2 (Rescaling)**.** ✓ The affine map sending $[0,1]$ to $[a,b]$:

$$\mathrm{rescale}(a,b,t) := a + t \cdot (b-a). \tag{9}$$

Its inverse, sending $[a,b]$ back to $[0,1]$:

$$\mathrm{unscale}(a,b,x) := \frac{x-a}{b-a}. \tag{10}$$

Lemma 4.3 (Rescaling properties)**.** ✓ *For $a \le b$:*

1. $\mathrm{rescale}(a,b,0) = a$ *and* $\mathrm{rescale}(a,b,1) = b$.
2. rescale *maps* $[0,1]$ *into* $[a,b]$.
3. rescale *is continuous.*
4. *For $a < b$:* unscale *maps* $[a,b]$ *into* $[0,1]$.
5. $\mathrm{rescale} \circ \mathrm{unscale} = \mathrm{id}$ *on* $[a,b]$ *(when $a < b$).*
6. $\mathrm{unscale} \circ \mathrm{rescale} = \mathrm{id}$ *on* $[0,1]$ *(when $a < b$).*

```
/-- Affine map [0,1] -> [a,b]. -/
def rescale (a b : Real) (t : Real) : Real :=
  a + t * (b - a)

/-- Inverse: [a,b] -> [0,1]. -/
def unscale (a b : Real) (x : Real) : Real :=
  (x - a) / (b - a)

/-- rescale maps [0,1] into [a,b]. -/
theorem rescale_maps_Icc (a b : Real) (hab : a <= b)
    (t : Real) (ht : t mem Set.Icc (0 : Real) 1) :
    rescale a b t mem Set.Icc a b := by
  constructor
  . unfold rescale; nlinarith [ht.1, ht.2]
  . unfold rescale; nlinarith [ht.1, ht.2]

/-- rescale . unscale = id on [a,b]. -/
theorem rescale_unscale (a b : Real) (hab : a < b)
    (x : Real) (_hx : x mem Set.Icc a b) :
    rescale a b (unscale a b x) = x := by
  unfold rescale unscale
  have hba : (b - a) != 0 :=
    ne_of_gt (sub_pos.mpr hab)
  field_simp; ring
```

Listing 6: Rescaling infrastructure (Defs/Rescaling.lean, selected).

4.3 EVT Min Implies Compact Optimization

Theorem 4.4 ($\mathsf{EVT}_{\min} \Rightarrow$ CompactOptimization)**.** ✓ *If every continuous function on* $[0, 1]$ *attains its minimum, then every continuous function on any* $[a, b]$ *attains its minimum.*

Proof. Given $a < b$ and $f : \mathbb{R} \to \mathbb{R}$ continuous on $[a, b]$:

Step 1: Define the rescaled function. Let $g(t) = f(\text{rescale}(a, b, t)) = f(a+t(b-a))$. Then g is the composition of f (continuous on $[a, b]$) with rescale (continuous, mapping $[0, 1]$ into $[a, b]$). Hence g is continuous on $[0, 1]$.

Step 2: Apply $\mathsf{EVT}_{\min}$ to g. Obtain $t^* \in [0, 1]$ with $g(t^*) \le g(t)$ for all $t \in [0, 1]$.

Step 3: Return the minimizer. Set $x^* = \text{rescale}(a, b, t^*) \in [a, b]$. For any $y \in [a, b]$, let $s = \text{unscale}(a, b, y) \in [0, 1]$. Then:

$$\begin{aligned} f(x^*) &= f(\text{rescale}(a, b, t^*)) = g(t^*) \\ &\le g(s) = f(\text{rescale}(a, b, s)) \\ &= f(\text{rescale}(a, b, \text{unscale}(a, b, y))) = f(y). \end{aligned}$$

The last step uses rescale ∘ unscale = id on $[a, b]$. □

```
/-- EVT on [0,1] implies compact optimization
    on arbitrary [a,b]. -/
theorem compact_opt_of_evt_min (h : EVT_min) :
    CompactOptimization := by
  intro a b hab f hf
  let g : Real -> Real :=
    fun t => f (rescale a b t)
  have hg_cont : ContinuousOn g (Set.Icc 0 1) := by
```

```
    apply ContinuousOn.comp hf
      (rescale_continuous a b).continuousOn
    exact rescale_mapsTo a b (le_of_lt hab)
  obtain <t_star, ht_mem, ht_min> := h g hg_cont
  refine <rescale a b t_star,
    rescale_maps_Icc a b (le_of_lt hab)
      t_star ht_mem, ?>
  intro y hy
  have hs_mem : unscale a b y mem
      Set.Icc (0 : Real) 1 :=
    unscale_maps_Icc a b hab y hy
  have h1 : f (rescale a b t_star) <=
      f (rescale a b (unscale a b y)) :=
    ht_min (unscale a b y) hs_mem
  rw [rescale_unscale a b hab y hy] at h1
  exact h1
```

Listing 7: $\mathsf{EVT}_{\text{min}}$ implies CompactOptimization (PartB/CompactOpt.lean).

4.4 Compact Optimization Implies EVT Min

Theorem 4.5 (CompactOptimization $\Rightarrow$ $\mathsf{EVT}_{\text{min}}$). ✓ *If every continuous function on any* $[a, b]$ *attains its minimum, then every continuous function on* $[0, 1]$ *attains its minimum.*

Proof. Specialize CompactOptimization to $a = 0$, $b = 1$ (noting $0 < 1$). □

```
/-- Compact optimization implies EVT on [0,1]. -/
theorem evt_min_of_compact_opt
    (h : CompactOptimization) : EVT_min := by
  intro f hf
  exact h 0 1 (by norm_num) f hf
```

Listing 8: CompactOptimization implies $\mathsf{EVT}_{\text{min}}$ (PartB/CompactOpt.lean).

4.5 The Main Equivalence

Theorem 4.6 ($\mathsf{FT} \leftrightarrow$ CompactOptimization). ✓ *Over* BISH, *the Fan Theorem is equivalent to compact optimization:*

$$\mathsf{FT} \quad \longleftrightarrow \quad \text{CompactOptimization.} \tag{11}$$

Proof. Compose the three equivalences. Since $\mathsf{FT} = \mathsf{EVT}_{\text{max}}$:

$$\mathsf{FT} = \mathsf{EVT}_{\text{max}} \xrightarrow{\text{Thm 4.1}} \mathsf{EVT}_{\text{min}} \xrightarrow{\text{Thm 4.4}} \text{CompactOptimization} \xrightarrow{\text{Thm 4.5}} \mathsf{EVT}_{\text{min}} \xrightarrow{\text{Thm 4.1}} \mathsf{EVT}_{\text{max}} = \mathsf{FT}.$$

□

```
/-- The Fan Theorem <-> CompactOptimization. -/
theorem ft_iff_compact_opt :
    FanTheorem <-> CompactOptimization := by
  unfold FanTheorem
  constructor
  . intro hmax
    exact compact_opt_of_evt_min
      (evt_min_of_evt_max hmax)
  . intro hco
    exact evt_max_of_evt_min
```

```
        (evt_min_of_compact_opt hco)
```

Listing 9: Main equivalence (PartB/PartB_Main.lean).

Remark 4.7 (Zero custom axioms). `#print axioms ft_iff_compact_opt` shows only `[propext, Classical.choice, Quot.sound]`. In fact, many of the intermediate results (`evt_min_of_evt_max`, `compact_opt_of_evt_min`, `evt_min_of_compact_opt`) require only `[propext]`. No custom axiom appears anywhere. This is possible because FT is defined as $\mathsf{EVT}_{\max}$, not axiomatized separately. The connection to the bar-theoretic Fan Theorem is by citation [Berger, 2005, Bridges and Vîţă, 2006], not by axiom.

5 The Stratification Theorem

5.1 Three-Branch Partial Order

The constructive hierarchy now has three independent branches:

Branch	Content	Example	Papers
Omniscience	LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO	Bidual gap, Ising limit	2–21
Markov	LPO $\Rightarrow$ MP	Eventual decay	22
Compactness	$\mathsf{FT} \leftrightarrow$ CompactOpt	Free energy optimization	23

Theorem 5.1 (Three-branch stratification). ✓ *The constructive hierarchy is a partial order with three branches:*

1. ***Omniscience chain:*** LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO.
2. ***Markov branch:*** LPO $\Rightarrow$ MP.
3. ***Compactness branch:*** $\mathsf{FT} \leftrightarrow$ CompactOptimization *(independent of all above).*

Proof. Items (1) and (2) are proved from first principles in the formalization:

- LPO $\Rightarrow$ WLPO: Given LPO, the sequence either is all false (first disjunct of WLPO) or has a witness (contradicting the all-false hypothesis in the second disjunct).
- WLPO $\Rightarrow$ LLPO: Given a sequence with at most one true value, apply WLPO to the even subsequence. If the even subsequence is all false, we are done. Otherwise, the even subsequence has a true value, which by the at-most-one condition forces the odd subsequence to be all false.
- LPO $\Rightarrow$ MP: Given LPO, the sequence either is all false (contradicting the hypothesis $\neg(\forall n,\ \alpha(n) = 0)$) or has a witness.

Item (3) is Theorem 4.6. The independence of FT from LPO, WLPO, LLPO, and MP is standard model theory [Berger, 2005, Bridges and Vîţă, 2006, Bridges and Richman, 1987]:

- **$\mathsf{FT} \not\Rightarrow$ LPO:** Brouwerian models validate FT (by Brouwer's Fan Theorem) but fail LPO (omniscience of infinite tests is not available).
- **LPO $\not\Rightarrow$ FT:** Russian recursive models validate LPO (all functions are recursive, hence decidable) but fail FT (there exist computable but not uniformly continuous functions on $[0, 1]$) [Kushner, 1985].
- **Independence from MP:** Follows from the same model separation [Berger, 2005].

□

```
/-- Three-branch stratification. -/
theorem fan_stratification :
    -- Omniscience chain
    (LPO -> WLPO) /\
    (WLPO -> LLPO) /\
    -- Markov branch
    (LPO -> MarkovPrinciple) /\
    -- Compactness branch
    (FanTheorem <-> CompactOptimization) :=
  <lpo_implies_wlpo, wlpo_implies_llpo,
   lpo_implies_mp, ft_iff_compact_opt>
```

Listing 10: Three-branch stratification (Main/Stratification.lean).

5.2 The Physical Instance

Theorem 5.2 (Ising optimization from FT). ✓ *The Fan Theorem implies that the Ising free energy attains its minimum on any compact coupling interval* $[a, b]$*:*

$$\mathsf{FT} \implies \forall \beta,\ \ \forall a < b,\ \ \exists J^* \in [a, b],\ \ \forall J \in [a, b],\ \ f(\beta, J^*) \leq f(\beta, J). \tag{12}$$

Proof. Apply Theorem 4.6 ($\mathsf{FT} \Rightarrow$ CompactOptimization) to the function $J \mapsto f(\beta, J)$, which is continuous on $[a, b]$ by Theorem 3.1. □

```
/-- FT implies Ising free energy optimization. -/
theorem ising_opt_of_ft (hft : FanTheorem)
    (beta : Real) (a b : Real) (hab : a < b) :
    exists J_star, J_star mem Set.Icc a b /\
      forall J, J mem Set.Icc a b ->
        isingFreeEnergy beta J_star
          <= isingFreeEnergy beta J := by
  exact ft_iff_compact_opt.mp hft a b hab
    (isingFreeEnergy beta)
    (isingFreeEnergy_continuousOn beta a b)
```

Listing 11: Ising optimization from FT (Main/PhysicalInstance.lean).

6 Updated Calibration Table (Papers 2–23)

The complete calibration table for the constructive reverse mathematics series, updated with Paper 23:

Paper	Physical System	Observable / Assertion	CRM Level	Position
2	Bidual gap (ℓ^1)	Gap witness $J - \kappa$	$\equiv$ WLPO	Chain
3	Brouwer fixed point	Fixed-point existence	$\equiv$ LLPO	Chain
4	Diagonal dominance	Dominance decision	$\equiv$ WLPO	Chain
5	Spectral threshold	Spectral gap	$\equiv$ LPO	Chain
6	Heisenberg uncertainty	Uncertainty bound	$\equiv$ WLPO	Chain
7	Reflexive dispensability	Non-reflexivity witness	$\equiv$ WLPO	Chain
8	1D Ising convergence	Thermodynamic limit f_∞	$\equiv$ LPO	Chain
9	Uniform convexity	Modulus of convexity	$\equiv$ WLPO	Chain
10	Compact embedding	Embedding decision	$\equiv$ WLPO	Chain
11	Hahn–Banach norm	Extension norm	$\equiv$ LLPO	Chain
12	Toeplitz index	Index computation	$\equiv$ WLPO	Chain
13	Minimax duality	Saddle point	$\equiv$ WLPO	Chain
14	Floquet discriminant	Stability boundary	$\equiv$ LLPO	Chain
15	Lyapunov exponent	Conservation law	$\equiv$ WLPO	Chain
16	Berry phase	Phase classification	$\equiv$ WLPO	Chain
17	Variational eigenvalue	Ground state energy	$\equiv$ WLPO	Chain
18	Fredholm alternative	Solvability decision	$\equiv$ WLPO	Chain
19	Hölder interpolation	Interpolation bound	$\equiv$ WLPO	Chain
20	1D Ising phase	Phase classification	$\equiv$ WLPO	Chain
21	Perturbation bound	Stability threshold	$\equiv$ LPO	Chain
22	Radioactive decay	Eventual decay	$\equiv$ MP	Branch 1
23	**Free energy optimization**	**Compact extremum**	$\equiv$ FT	**Branch 2**

The calibration table is now populated at every level of the constructive hierarchy and has branching structure:

- BISH: Finite-volume Ising computation (Paper 8, Part A), Heisenberg uncertainty bound (Paper 6), finite-grid optimization (Paper 23, Part A).
- LLPO: Brouwer fixed point (Paper 3), Hahn–Banach norm (Paper 11), Floquet discriminant (Paper 14).
- WLPO: Bidual gap (Paper 2), reflexive dispensability (Paper 7), Ising phase (Paper 20), and many others.
- LPO: Spectral threshold (Paper 5), Ising convergence (Paper 8), perturbation bound (Paper 21).
- MP (**Branch 1**): Eventual decay (Paper 22).
- FT (**Branch 2**): Free energy optimization (Paper 23).

Papers 22 and 23 demonstrate that the physics itself has a partially ordered logical structure: there exist physical assertions whose constructive costs are incomparable.

6.1 The Four-Level Ising Stratification

The 1D Ising model provides the most fine-grained stratification of any single physical system in the programme:

Level	Question	CRM Cost	Paper	Branch
1	Finite-volume f_N	BISH	8A	—
2	Limit $f_\infty = \lim f_N$	LPO	8B	Chain
3	Phase classification	WLPO	20	Chain
4	Parameter optimization	FT	23	Independent

The same physical system—the 1D Ising model—requires four different constructive principles depending on the question asked. Level 4 (FT) is particularly striking because it is *independent* of Levels 2 and 3: the compactness cost of parameter optimization is logically orthogonal to the omniscience cost of taking the thermodynamic limit or classifying phases.

7 Lean 4 Formalization

7.1 Module Structure

The formalization consists of 14 files organized in four directories:

Module	**Content**	**Lines**
`Defs/Principles.lean`	LPO, WLPO, LLPO, MP, hierarchy	88
`Defs/IsingFreeEnergy.lean`	$f(\beta, J) = -\log(2\cosh(\beta J))$	34
`Defs/EVT.lean`	$\mathrm{EVT}_{\max}$, $\mathrm{EVT}_{\min}$, CompactOpt, FanTheorem	57
`Defs/Rescaling.lean`	rescale, unscale, inverses	85
`PartA/Continuity.lean`	$f(\beta, \cdot)$ is continuous	33
`PartA/FiniteOpt.lean`	Finite-grid optimization	33
`PartA/PartA_Main.lean`	Part A summary and audit	30
`PartB/EVTEquiv.lean`	$\mathsf{EVT}_{\max} \leftrightarrow \mathsf{EVT}_{\min}$	43
`PartB/CompactOpt.lean`	$\mathsf{EVT}_{\min} \leftrightarrow$ CompactOpt	61
`PartB/PartB_Main.lean`	$\mathsf{FT} \leftrightarrow$ CompactOpt	37
`Main/PhysicalInstance.lean`	$\mathsf{FT} \Rightarrow$ Ising optimization	31
`Main/Stratification.lean`	Three-branch theorem	49
`Main/AxiomAudit.lean`	Comprehensive audit	99
`Main.lean`	Root imports	4
Total		$\sim$ **684**

Dependency graph:

```
Principles <-- IsingFreeEnergy
  |                |
  |           Rescaling <-- EVT
  |                |         |
  |          Continuity   FiniteOpt
  |                |         |
  |          PartA_Main -----+
  |
  +-- EVTEquiv
  |         |
  +-- CompactOpt
  |         |
  +-- PartB_Main
  |
  +-- PhysicalInstance
  |
  +-- Stratification
  |
  +-- AxiomAudit <-- Main
```

7.2 Design Decisions

FanTheorem := EVT_max. The central design decision is to define the Fan Theorem as $\mathsf{EVT}_{\max}$ rather than axiomatizing the bar-theoretic equivalence:

```
/-- The Fan Theorem, defined via EVT (Berger 2005). -/
def FanTheorem : Prop := EVT_max
```

Listing 12: The key design decision.

This choice has a major payoff: **zero custom axioms**. Every theorem in the formalization depends only on MATHLIB4 infrastructure. The equivalence between $\mathsf{EVT}_{\max}$ and the bar-theoretic Fan Theorem is a well-established result in constructive reverse mathematics [Berger, 2005, Berger and Bridges, 2007, Bridges and Vîţă, 2006] and is handled by citation rather than formalization.

The alternative would be to define the Fan Theorem bar-theoretically and add an axiom `evt_of_fan_theorem`. This would give a correct formalization but would introduce a custom axiom that would appear in every theorem's axiom profile. The definitional approach is cleaner.

Self-contained bundle. Paper 23 is a standalone Lake package that re-declares LPO, WLPO, LLPO, and MP locally (in `Defs/Principles.lean`). The hierarchy proofs are proved from first principles, ensuring the formalization is fully self-contained.

Rescaling approach. The equivalence between EVT on $[0, 1]$ and CompactOptimization on general $[a, b]$ is proved via explicit affine rescaling rather than abstract compactness arguments. This makes the proof constructively transparent: the rescaling is an explicit, computable bijection.

7.3 Axiom Audit

Theorem	Custom Axioms	Infrastructure	Tier
`isingFreeEnergy_continuous`	None	propext, Classical.choice, Quot.sound	BISH
`finite_opt_bish`	None	propext, Classical.choice, Quot.sound	BISH
`partA_summary`	None	propext, Classical.choice, Quot.sound	BISH
`evt_min_of_evt_max`	None	propext	BISH
`evt_max_of_evt_min`	None	propext	BISH
`compact_opt_of_evt_min`	None	propext	BISH
`evt_min_of_compact_opt`	None	propext	BISH
`ft_iff_compact_opt`	None	propext	BISH
`ising_opt_of_ft`	None	propext, Classical.choice, Quot.sound	BISH
`fan_stratification`	None	propext, Classical.choice, Quot.sound	BISH
`lpo_implies_mp`	None	propext	BISH
`lpo_implies_wlpo`	None	propext	BISH
`wlpo_implies_llpo`	None	propext, Classical.choice, Quot.sound	BISH

All theorems: zero custom axioms. Only MATHLIB4 infrastructure appears. This is the cleanest axiom audit of any paper in the CRM series (Papers 2–23).

```
-- Part A (BISH):
#print axioms isingFreeEnergy_continuous
-- [propext, Classical.choice, Quot.sound]

#print axioms finite_opt_bish
-- [propext, Classical.choice, Quot.sound]

-- Part B (ZERO custom axioms!):
```

```
#print axioms evt_min_of_evt_max
-- [propext]

#print axioms compact_opt_of_evt_min
-- [propext]

#print axioms ft_iff_compact_opt
-- [propext]

-- Physical instance:
#print axioms ising_opt_of_ft
-- [propext, Classical.choice, Quot.sound]

-- Stratification:
#print axioms fan_stratification
-- [propext, Classical.choice, Quot.sound]

-- Hierarchy (pure logic):
#print axioms lpo_implies_mp
-- [propext]

#print axioms wlpo_implies_llpo
-- [propext, Classical.choice, Quot.sound]
```

Listing 13: Axiom audit (Main/AxiomAudit.lean, selected).

7.4 CRM Compliance Protocol

The axiom audit confirms:

- Part A theorems have **no custom axioms**—pure BISH.
- Part B theorems have **no custom axioms**—the equivalence $\mathsf{FT} \leftrightarrow$ CompactOptimization is proved without any axiomatic input beyond the definition $\mathsf{FT} := \mathsf{EVT}_{\max}$.
- The physical instance (Ising optimization from FT) has **no custom axioms**.
- The stratification theorem has **no custom axioms**.
- `Classical.choice` in results that use MATHLIB4 analysis (`Real.log`, `Real.cosh`, `ContinuousOn`) is a MATHLIB4 infrastructure artifact, not mathematical content. The mathematical content of all proofs is constructive.

The **contrast with Paper 22** is instructive. Paper 22 required one custom axiom (`mp_real_of_mp`) because Markov's Principle for reals is not derivable from the sequence-level definition without an equivalence proof that was cited rather than formalized. Paper 23 avoids this entirely by defining FT at the "right" level of abstraction ($\mathsf{EVT}_{\max}$), so that all equivalences are purely mathematical, requiring no interface axioms.

8 Discussion

8.1 Compactness vs. Omniscience

The Fan Theorem calibration reveals a fundamentally different logical dimension in mathematical physics. The omniscience principles (LPO, WLPO, LLPO) concern the decidability of infinite tests—can we decide whether a binary sequence is identically zero? Markov's Principle concerns witness production from double negation—given that a sequence is not all zeros, can we find a nonzero entry?

The Fan Theorem concerns *compactness*: does a continuous function on a compact set attain its extrema? This is a topological question whose logical content is orthogonal to both decidability and witness production. The independence of FT from LPO, WLPO, LLPO, and MP means that compactness is a *new logical dimension* in the constructive analysis of physics.

Concretely: knowing that the 1D Ising free energy has a thermodynamic limit (LPO, Paper 8) tells us nothing about whether the free energy attains its minimum over a compact parameter space (FT, Paper 23). Conversely, knowing that the free energy is optimizable over compact sets tells us nothing about whether the thermodynamic limit exists. These are logically independent questions about the same physical system.

8.2 The Fan Theorem in Physics

Free energy optimization is not an isolated example. The Fan Theorem—through its equivalence with compact optimization—is relevant wherever physics requires extremization over compact parameter spaces:

- **Variational principles:** Ground state energies in quantum mechanics are defined as infima of energy functionals over compact subsets of Hilbert space (via finite-dimensional approximation). The Fan Theorem provides the compactness needed to guarantee attainment.
- **Free energy minimization:** In statistical mechanics, equilibrium states minimize the free energy over the space of probability measures. On compact state spaces, this requires the Fan Theorem.
- **Optimal control:** Control theory frequently seeks minima of cost functionals over compact control sets. The existence of optimal controls depends on the Fan Theorem.
- **Phase diagrams:** Mapping out phase boundaries often involves optimizing order parameters over compact parameter ranges. The Fan Theorem guarantees the existence of the extremal parameter values.

All of these applications share the same logical structure: a continuous function on a compact set must attain its extremum. The Fan Theorem is the constructive cost of this assertion.

8.3 Observable-Dependent Cost Revisited

The 1D Ising model now provides the most extensive example of observable-dependent cost in the programme. The same underlying model—the free energy $f(\beta, J) = -\log(2\cosh(\beta J))$—has four different logical costs depending on the question asked:

Question	**Principle**	**Type**	**Paper**
Compute f_N for finite N	BISH	Arithmetic	8A
Does $\lim f_N$ exist?	LPO	Decidability	8B
What phase is it in?	WLPO	Decidability	20
Does $\min_J f(\beta, J)$ exist?	FT	Compactness	23

Crucially, the last row (FT) is *independent* of the middle two (LPO, WLPO). This means the logical cost truly depends on the observable—the question asked—not merely on the difficulty of the question. The same encoding can yield costs on incomparable branches of the hierarchy.

8.4 The Architecture of the Hierarchy

With three independent branches, the CRM hierarchy has a richer structure than initially apparent:

$$\begin{array}{ccccc} & & \mathrm{LPO} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathrm{WLPO} & & \mathrm{MP} & & \cdots \\ \downarrow & & & & \\ \mathrm{LLPO} & & & & \\ \downarrow & & & & \\ \mathrm{BISH} & & & & \mathsf{FT} \end{array} \tag{13}$$

The hierarchy is a **partial order**, not a lattice: we do not know whether $\mathrm{WLPO} \wedge \mathsf{FT}$ or $\mathrm{MP} \wedge \mathsf{FT}$ have physical instantiations. Future work may discover assertions whose constructive cost is the conjunction of two branches—or demonstrate that no such conjunction arises naturally in physics.

8.5 Limitations

1. **Simple physical model.** The Ising free energy $f(\beta, J) = -\log(2\cosh(\beta J))$ is the simplest possible optimization target. More complex free energy landscapes (higher-dimensional, multi-phase) would provide richer physical content but the same CRM calibration.

2. **Bar-theoretic equivalence by citation.** The equivalence between $\mathsf{EVT}_{\max}$ and the bar-theoretic Fan Theorem is cited [Berger, 2005, Bridges and Vîţă, 2006], not formalized. Formalizing this equivalence in Lean 4 would require developing the theory of bars and fans in Cantor space, which is a substantial undertaking.

3. **Independence by citation.** The independence of FT from LPO, WLPO, LLPO, and MP relies on standard model-theoretic arguments [Berger, 2005, Bridges and Richman, 1987] that are not formalized. Formalizing these separations would require constructing Brouwerian and recursive models in Lean 4, which is beyond the current scope.

4. **Classical.choice in Mathlib4.** The appearance of `Classical.choice` in results using Mathlib4 analysis is a Mathlib4 infrastructure artifact. This is the same situation as in all previous papers.

5. **Optimization is for min, not argmin.** The formalization proves that a minimizer *exists*, but does not provide a computable procedure for finding it. This is inherent to the constructive content of the Fan Theorem: the theorem asserts existence but does not provide a uniform algorithm.

9 Conclusion

The assertion that a continuous function on a compact interval attains its extremum—the Extreme Value Theorem—is equivalent to the Fan Theorem over Bishop's constructive mathematics. We have instantiated this equivalence through free energy optimization in the 1D Ising model, establishing the **first CRM calibration at the FT level**.

This calibration adds a **third independent branch** to the constructive hierarchy. The omniscience chain (LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO), Markov's branch (LPO $\Rightarrow$ MP, Paper 22), and the compactness branch ($\mathsf{FT} \leftrightarrow$ CompactOptimization, this paper) are mutually independent.

The hierarchy is not a chain or even a tree—it is a genuine **partial order** with incomparable elements.

The 1D Ising model now exhibits four distinct logical costs:

- BISH: Finite-volume computation (Paper 8, Part A).
- LPO: Thermodynamic limit existence (Paper 8, Part B).
- WLPO: Phase classification (Paper 20).
- FT: Parameter-space optimization (this paper).

The fourth cost (FT) is independent of the second and third (LPO, WLPO), demonstrating that the logical cost of a physical assertion depends on the *observable* (the question asked), not on the underlying physical model.

The formalization achieves **zero custom axioms**—the cleanest axiom audit in the series. This is made possible by the design decision to define the Fan Theorem as $\mathsf{EVT}_{\max}$ directly, with the bar-theoretic equivalence handled by citation. All 14 files, ∼680 lines, compile with zero errors, zero warnings, and zero `sorry`.

Future work includes searching for physical assertions calibrated at other independent principles (e.g., the Weak Fan Theorem, the Anti-Specker property, or the Heine–Borel Theorem), discovering assertions whose cost is the conjunction of multiple branches, and extending the partial order to a more complete map of the constructive landscape of mathematical physics.

AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as previous papers in the series Lee [2025a,d, 2026a,c].

The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalization strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Component	**Human**	**AI (Claude Opus 4.6)**
Research question	✓	
Physical setup (Ising model)	✓	
CRM calibration strategy	✓	
Lean 4 implementation		✓
Proof strategies	collaborative	collaborative
LaTeX writeup		✓
Review and editing	✓	

Table 1: Division of labor between human and AI.

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/paul-c-k-lee/FoundationRelativity`

- **Path**: `paper 23/P23_FanTheorem/`
- **Build**: `lake exe cache get && lake build` (0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Custom axioms**: **NONE**
- **Axiom profile (Theorem 1, isingFreeEnergy_continuous)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 2, finite_opt_bish)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 3, evt_min_of_evt_max)**: `[propext]`
- **Axiom profile (Theorem 4, compact_opt_of_evt_min)**: `[propext]`
- **Axiom profile (Theorem 5, evt_min_of_compact_opt)**: `[propext]`
- **Axiom profile (Theorem 6, ft_iff_compact_opt)**: `[propext]`
- **Axiom profile (Theorem 7, ising_opt_of_ft)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 8, fan_stratification)**: `[propext, Classical.choice, Quot.sound]`
- **Total**: 14 files, ~680 lines, 0 sorry

Acknowledgments

The LEAN 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the MATHLIB4 community for maintaining the comprehensive library of formalized mathematics that made this work possible.

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