The Born Rule as a Logical Artefact

Technical Note 16 in the Constructive Calibration Programme

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February 2026

Abstract

We observe that the Born rule splits across the constructive hierarchy in a manner consistent with the general pattern of the calibration programme. The single-trial probability $p_i = \|P_i\psi\|^2$, the expectation value, variance, and Chebyshev bound (weak law of large numbers) are all provable in Bishop's constructive mathematics (BISH)—unsurprisingly, since they are finite-dimensional linear algebra and real arithmetic. The frequentist assertion—that relative frequencies converge almost surely to the Born probabilities—requires Dependent Choice over $\mathbb{N}$ (DC_ω), a fact well known in constructive probability theory.

Our contribution is not mathematical novelty but programme completeness: this note fills the DC_ω axis of the calibration table (Paper 10) with physical content from quantum measurement statistics, complementing the LPO entries from statistical mechanics (Paper 8), general relativity (Paper 13), decoherence (Paper 14), and conservation laws (Paper 15). All BISH results are formalised in Lean 4 with machine-checked proofs (564 lines, zero `sorry`). The DC_ω layer is axiomatised following established series methodology; a full constructive proof of the strong law with explicit DC_ω tracking remains open.

Contents

*New York University. AI-assisted formalization; see §8 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 Physical Context

The Born rule is the bridge between quantum geometry and laboratory data. Given a normalized state $\psi \in \mathbb{C}^d$ and an observable A with spectral decomposition $A = \sum_i \lambda_i P_i$, the rule asserts: the probability of measuring eigenvalue λ_i is

$$p_i \;=\; \|P_i\psi\|^2 \;=\; \operatorname{Re}\langle\psi, P_i\psi\rangle.$$

Every prediction of quantum mechanics flows from this postulate.

1.2 The Question

The Born rule admits two readings with different logical content. If "probability p_i" means "the squared norm of the projected state"—a single computation on one finite-dimensional vector—then the Born rule is a statement of finite-dimensional linear algebra, and its constructive status is immediate. If it means "the long-run relative frequency in infinitely many repeated measurements converges almost surely to p_i"—the frequentist interpretation—then it is a statement about infinite sequences, requiring countable product spaces and almost-sure convergence.

This note records the logical cost of each reading. The result is expected—constructive probabilists have understood the role of Dependent Choice in the strong law since at least Bishop and Bridges [Bishop and Bridges, 1985]—but its explicit placement in the calibration table has not previously been formalised.

1.3 Summary of Results

Layer	Physical content	Logical cost
Single-trial probability	$p_i = \|P_i\psi\|^2 \in [0,1]$	BISH
Expectation, variance	$\langle\psi, A\psi\rangle \in \mathbb{R}$, $\mathrm{Var} \geq 0$	BISH
Chebyshev bound (weak law)	$\Pr(\lvert\mathrm{freq} - p\rvert > \varepsilon) \leq 1/(4N\varepsilon^2)$	BISH
Relative frequency sums	$\sum_i \mathrm{freq}_N(i) = 1$	BISH
Strong law (SLLN)	$\mathrm{freq}_N \to p$ almost surely	DC_ω

The weak law row deserves emphasis: the Chebyshev bound gives an explicit, computable error estimate $\Pr(|\mathrm{freq}-p| > \varepsilon) \leq 1/(4N\varepsilon^2)$ that shrinks as $1/N$. This is what experimentalists actually use, and it is fully constructive.

1.4 Programme Context

This note extends the calibration table along the DC_ω axis, complementing existing entries:

- Paper 4: approximate spectral membership is BISH, exact membership costs MP.
- Paper 6: preparation uncertainty is BISH, measurement uncertainty costs DC_ω.
- Paper 14: finite-time decoherence bounds are BISH, exact collapse costs LPO.
- Paper 15: local energy conservation is BISH, global energy existence costs LPO.
- **Paper 16 (this):** finite-sample frequency bounds are BISH, exact frequentist convergence costs DC_ω.

Papers 6 and 16 together populate the DC_ω axis with two independent physical instances (measurement uncertainty and measurement statistics), paralleling the multiple LPO entries on the omniscience spine.

2 Background

2.1 Constructive Foundations

We work within Bishop's constructive mathematics (BISH): intuitionistic logic plus dependent choice restricted to finite operations. BISH admits all of classical analysis that can be carried out without appeal to excluded middle or uncountable choice.

Definition 2.1 (Dependent Choice over $\mathbb{N}$)**.** DC_ω asserts: for any type α, any total relation $R : \alpha \to \alpha \to \mathrm{Prop}$, and any starting point $a_0 : \alpha$, if $\forall a, \exists b, R(a, b)$, then there exists a function $f : \mathbb{N} \to \alpha$ with $f(0) = a_0$ and $R(f(n), f(n+1))$ for all n.

DC_ω is strictly weaker than the full axiom of choice but strictly stronger than countable choice. It is the standard choice principle for constructing infinite sequences from local extension data—precisely what the strong law of large numbers requires.

2.2 Certification Methodology (Paper 10)

Our diagnostic for calibrating a physical theorem T against the constructive hierarchy is:

1. Formalise T in Lean 4 with Mathlib4.
2. Run `#print axioms T`.
3. If the axiom closure contains only `propext`, `Classical.choice`, `Quot.sound` (Mathlib infrastructure), classify T as BISH.
4. If the axiom closure additionally contains a custom axiom such as `dc_omega_holds`, classify T at the corresponding level (DC_ω).

The appearance of `Classical.choice` in BISH theorems is an artefact of Mathlib's type-class infrastructure (the `Fintype` instance for `Fin d`, the field structure on $\mathbb{R}$), not a mathematical use of excluded middle. This is the Paper 10 methodology, validated across all papers in the series.

3 Setup: Finite-Dimensional Quantum Measurement

We work in finite dimensions throughout. Fix:

- $H = \mathbb{C}^d$ (finite-dimensional Hilbert space, $d \geq 1$).
- $\psi \in H$ with $\|\psi\|^2 = 1$ (normalized state).
- $A : H \to H$ a Hermitian operator (observable).

- Spectral decomposition: $A = \sum_i \lambda_i P_i$ where each P_i is an orthogonal projection ($P_i^2 = P_i$, $P_i^\dagger = P_i$) and $\sum_i P_i = I$.

Definition 3.1 (Complex inner product)**.** For $\psi, \varphi : \mathrm{Fin}\, d \to \mathbb{C}$, define

$$\langle \psi, \varphi \rangle := \sum_{i=0}^{d-1} \overline{\psi_i}\, \varphi_i.$$

Definition 3.2 (Born probability)**.** $p_i := \mathrm{Re}\, \langle \psi, P_i \psi \rangle$. For a projection P_i with $P_i^2 = P_i$ and $P_i^\dagger = P_i$, this equals $\|P_i \psi\|^2$.

Definition 3.3 (Spectral decomposition)**.** In Lean 4:

```
structure SpectralDecomp (d : N) where
  eigenvalues    : Fin d -> R
  projections    : Fin d -> Matrix (Fin d) (Fin d) C
  is_projection : forall i, projections i * projections i =
     projections i
  is_hermitian   : forall i, (projections i).conjTranspose =
     projections i
  is_orthogonal : forall i j, i != j -> projections i * projections
     j = 0
  is_complete    : sum i, projections i = 1
```

Listing 1: SpectralDecomp structure (Defs.lean)

4 BISH Content: Theorems 1–5

4.1 Theorem 1: Born Probability Distribution

Theorem 4.1 (Born probability is BISH)**.** *For any normalized state ψ and spectral decomposition $\{P_i\}$:*

1. $0 \leq p_i$ *for each* i *(born_prob_nonneg)*
2. $\sum_i p_i = 1$ *(born_prob_sum_one)*

Proof. **Non-negativity.** Since $P_i^2 = P_i$ and $P_i^\dagger = P_i$:

$$\langle \psi, P_i \psi \rangle = \langle \psi, P_i(P_i \psi) \rangle = \langle P_i \psi, P_i \psi \rangle = \|P_i \psi\|^2 \geq 0.$$

The first equality uses $P_i^2 = P_i$. The second uses $P_i^\dagger = P_i$ (the Hermitian swap lemma: $\langle u, Pv \rangle = \langle Pu, v \rangle$ for $P^\dagger = P$). The inequality is BISH: a sum of $|\cdot|^2$ terms.

Sum to one. By linearity of the inner product in the second argument:

$$\sum_i \langle \psi, P_i \psi \rangle = \langle \psi, \Big(\sum_i P_i\Big) \psi \rangle = \langle \psi, I \psi \rangle = \langle \psi, \psi \rangle = \|\psi\|^2 = 1.$$

Every step is a finite sum or matrix operation. BISH. □

```
theorem born_prob_nonneg {d : N} (psi : Fin d -> C)
    (spec : SpectralDecomp d) (i : Fin d) :
    0 <= bornProb psi spec i := by
  unfold bornProb
  suffices h : (cdot psi ((spec.projections i).mulVec psi)).re =
    cnorm_sq ((spec.projections i).mulVec psi) from
    h |> cnorm_sq_nonneg _
  unfold cnorm_sq; congr 1
```

```
  set Pi := spec.projections i
  have hP2 : Pi.mulVec (Pi.mulVec psi) = Pi.mulVec psi := by
    rw [Matrix.mulVec_mulVec, spec.is_projection i]
  conv_lhs => rw [show Pi.mulVec psi =
    Pi.mulVec (Pi.mulVec psi) from hP2.symm]
  exact hermitian_cdot_swap psi (Pi.mulVec psi) Pi
      (spec.is_hermitian i)
```

Listing 2: Born probability proofs (BornProbability.lean, excerpt)

4.2 Theorem 2: Expectation Value is Real

Theorem 4.2 (Expectation is BISH)**.** *For Hermitian A ($A^\dagger = A$), the expectation value $\langle\psi, A\psi\rangle$ is real:* $\operatorname{Im}\langle\psi, A\psi\rangle = 0$.

Proof. By the Hermitian swap lemma, $\langle\psi, A\psi\rangle = \langle A\psi, \psi\rangle$. By conjugate symmetry of the inner product, $\langle A\psi, \psi\rangle = \overline{\langle\psi, A\psi\rangle}$. So $z = \bar{z}$, which implies $\operatorname{Im}(z) = 0$. All operations are finite sums. BISH. □

```
theorem expectation_real {d : N} (psi : Fin d -> C)
    (A : Matrix (Fin d) (Fin d) C) (hA : A.conjTranspose = A) :
    (expectationValue psi A).im = 0 := by
  unfold expectationValue
  have h1 : cdot psi (A.mulVec psi) = cdot (A.mulVec psi) psi :=
    hermitian_cdot_swap psi psi A hA
  have h2 : cdot (A.mulVec psi) psi =
    starRingEnd C (cdot psi (A.mulVec psi)) :=
    cdot_hermitian (A.mulVec psi) psi
  have h3 := h1.trans h2  -- z = conj(z)
  have h4 := Complex.ext_iff.mp h3
  simp only [Complex.conj_re, Complex.conj_im] at h4
  linarith [h4.2]
```

Listing 3: Expectation reality (Expectation.lean)

4.3 Theorem 3: Variance is Non-negative

Theorem 4.3 (Variance is BISH)**.** *For any Hermitian A and complex μ,* $\|(A - \mu I)\psi\|^2 \geq 0$.

Proof. Immediate: the squared norm of any vector is non-negative. BISH. □

```
theorem variance_nonneg {d : N} (psi : Fin d -> C)
    (A : Matrix (Fin d) (Fin d) C) (mu : C) :
    0 <= cnorm_sq ((A - mu * (1 : Matrix (Fin d) (Fin d) C)).mulVec
        psi) :=
  cnorm_sq_nonneg _
```

Listing 4: Variance non-negativity (Variance.lean)

4.4 Theorem 4: Relative Frequency Bounds

Theorem 4.4 (Relative frequency is BISH)**.** *For N measurement outcomes $x_1, \ldots, x_N \in \{1, \ldots, d\}$, the relative frequency $\mathrm{freq}_N(i) = \#\{j : x_j = i\}/N$ satisfies:*

1. $0 \leq \mathrm{freq}_N(i)$ *(relative_freq_nonneg)*
2. $\mathrm{freq}_N(i) \leq 1$ *(relative_freq_le_one)*

3. $\sum_i \text{freq}_N(i) = 1$ *(relative_freq_sum)*

Proof. Non-negativity: the count is ≥ 0 and $N > 0$. Upper bound: the count $\leq N$. Sum to one: the counts partition $\{1, \ldots, N\}$, so $\sum \text{count}_i = N$. Dividing by N gives 1. All operations are finite counting. BISH. □

4.5 Theorem 5: Chebyshev Bound (Weak Law)

Theorem 4.5 (Chebyshev bound is BISH)**.** *For a Bernoulli process with parameter $p \in [0,1]$, N trials, and tolerance $\varepsilon > 0$:*

$$\frac{p(1-p)}{N\varepsilon^2} \leq \frac{1}{4N\varepsilon^2}.$$

Proof. **Step 1.** The Bernoulli variance bound: $p(1-p) \leq 1/4$ for $p \in [0,1]$. Proof: $(1/2-p)^2 \geq 0$ implies $1/4 - p + p^2 \geq 0$, so $p(1-p) \leq 1/4$.

Step 2. Since $N > 0$ and $\varepsilon > 0$, we have $N\varepsilon^2 > 0$. Dividing the inequality $p(1-p) \leq 1/4$ by $N\varepsilon^2$ preserves the direction:

$$\frac{p(1-p)}{N\varepsilon^2} \leq \frac{1/4}{N\varepsilon^2} = \frac{1}{4N\varepsilon^2}.$$

All operations are finite real arithmetic. BISH. □

```
theorem bernoulli_variance_bound (p : R) (_hp : 0 <= p) (_hp1 : p <=
   1) :
    p * (1 - p) <= 1 / 4 := by
  nlinarith [sq_nonneg (1 / 2 - p)]

theorem chebyshev_bernoulli_bound (p : R) (hp : 0 <= p) (hp1 : p <=
   1)
    (N : Nat) (hN : 0 < N) (eps : R) (heps : 0 < eps) :
    p * (1 - p) / (N * eps ^ 2) <= 1 / (4 * N * eps ^ 2) := by
  have hvar := bernoulli_variance_bound p hp hp1
  calc p * (1 - p) / (N * eps ^ 2)
      <= (1 / 4) / (N * eps ^ 2) := by
        apply div_le_div_of_nonneg_right hvar (by positivity)
    _ = 1 / (4 * N * eps ^ 2) := by ring
```

Listing 5: Chebyshev bound (WeakLaw.lean)

Physical interpretation. After $N = 10{,}000$ measurements, the probability that the observed frequency deviates from the Born probability by more than 0.01 is at most $1/(4 \times 10^4 \times 10^{-4}) = 0.25$. After $N = 10^6$ measurements with the same tolerance, the bound drops to 0.0025. The bound is explicit, computable, and shrinks as $1/N$. No physicist needs more than this.

5 DC Content: Strong Law of Large Numbers

5.1 Theorem 6: The Strong Law Requires DC_ω

Theorem 5.1 (Strong law requires DC_ω)**.** *The assertion "for almost every infinite sequence of independent Born-rule measurements, $\text{freq}_N(\lambda_i) \to p_i$ as $N \to \infty$" requires Dependent Choice over $\mathbb{N}$.*

The standard proof of the strong law uses DC_ω at three points:

1. **Product space construction.** The countable product probability space $(\Omega, \mathcal{F}, P) = \prod_n \text{Bernoulli}(p)$ requires DC at each extension step: given the finite product up to stage n, extend consistently to stage $n+1$.

2. **Borel–Cantelli.** The lemma $\sum P(E_k) < \infty \implies P(\limsup E_k) = 0$ manipulates the events $E_k = \{|\mathrm{freq}_k - p| > \varepsilon\}$ through a countable intersection of countable unions, requiring DC to select witnesses at each stage.

3. **Almost-sure convergence.** Extracting convergent behaviour from $\omega \in \Omega$ — proving the Cauchy criterion for a specific outcome sequence — requires dependent choices along the sequence.

In the formalisation, DC_ω is introduced as an axiom and the SLLN is axiomatized as a consequence:

```
def DC_omega : Prop :=
  forall (alpha : Type) (R : alpha -> alpha -> Prop) (a0 : alpha),
    (forall a, exists b, R a b) ->
    exists f : Nat -> alpha, f 0 = a0 /\ forall n, R (f n) (f (n +
       1))

axiom dc_omega_holds : DC_omega
axiom slln_of_dc : DC_omega -> SLLN

theorem frequentist_convergence : SLLN :=
  slln_of_dc dc_omega_holds
```

Listing 6: DC axiom and SLLN (DCAxiom.lean + StrongLaw.lean)

6 Pattern Consistency

The weak-law/strong-law split instantiates the programme's recurring pattern: finite approximations are BISH, completed limits have measurable cost. The specific mechanism differs from the LPO entries. In Papers 8 and 15, bounded monotone convergence drives the cost; the partial sums are non-negative and increasing, and asserting their limit exists is equivalent to LPO. Here, the cost arises from a different source: the product-space construction and almost-sure convergence machinery of the strong law, which require DC_ω for the sequential extension of probability measures.

The pattern is uniform in structure (finite = BISH, infinite = non-BISH) but heterogeneous in mechanism (BMC for deterministic limits, DC_ω for probabilistic limits). This heterogeneity is itself informative: it shows that the calibration table's axes are genuinely independent, not artefacts of a single underlying phenomenon.

7 Programme Integration

Paper 16 extends the calibration table along the DC_ω axis:

Domain	**Paper**	**BISH Content**	**Non-BISH Content**
Stat. Mech.	P8	Finite-volume free energy	f_∞ exists (LPO)
Gen. Rel.	P13	Radial coordinate bounds	$r \to 0$ exactly (LPO)
Quantum Meas.	P14	Finite-time decoherence	Exact collapse (LPO)
Conservation	P15	Local conservation (Noether)	Global energy (LPO)
Born Rule	**P16**	**Probability, weak law**	**SLLN** (DC_ω)

Paper 16 is the first entry in the series where the non-constructive content lands at DC_ω rather than LPO. This is expected: the strong law of large numbers is a convergence theorem for *random* sequences (requiring product measure), not for *deterministic* bounded monotone sequences (which connect to LPO via BMC).

We note that the Born rule is not a new physical domain—it is quantum mechanics, already covered by Papers 4, 6, 11, and 14. What is new is the *aspect*: measurement statistics rather than spectral theory, entanglement, or decoherence. The DC_ω cost arises from the probabilistic infrastructure (product spaces, almost-sure convergence), not from quantum structure per se.

8 Lean Formalisation

8.1 Module Structure

The formalisation comprises 564 lines across 9 files:

File	Lines	Role
`Defs.lean`	86	Core definitions
`BornProbability.lean`	127	Theorems 1: Born probability (BISH)
`Expectation.lean`	37	Theorem 2: Expectation reality (BISH)
`Variance.lean`	29	Theorem 3: Variance non-negativity (BISH)
`RelativeFreq.lean`	67	Theorem 4: Frequency bounds (BISH)
`WeakLaw.lean`	47	Theorem 5: Chebyshev bound (BISH)
`DCAxiom.lean`	31	DC_ω definition + axiom
`StrongLaw.lean`	51	Theorem 6: SLLN requires DC_ω
`Main.lean`	89	Assembly + axiom audit

8.2 Design Decisions

1. **Custom inner product.** We define `cdot` as $\sum_i \overline{\psi_i}\,\varphi_i$ on `Fin d` $\to$ $\mathbb{C}$, bypassing Mathlib4's `InnerProductSpace` infrastructure. This avoids the `Classical.choice` contamination from `EuclideanSpace` instances while keeping all matrix operations available.

2. **Spectral decomposition as structure.** The `SpectralDecomp` structure bundles eigenvalues, projections, and their algebraic properties (idempotence, Hermiticity, orthogonality, completeness). The spectral theorem itself is not proved—it is a hypothesis of the Born rule.

3. DC_ω **as axiom.** Following the `bmc_of_lpo` pattern from Papers 14–15, we introduce `dc_omega_holds : DC_omega` as an axiom and `slln_of_dc : DC_omega → SLLN` as a second axiom. The `#print axioms` audit then cleanly separates BISH from DC_ω theorems.

4. **SLLN axiomatized.** The full proof of the strong law of large numbers is a substantial result in measure theory. We axiomatize it and verify that its axiom closure contains exactly `dc_omega_holds` + `slln_of_dc`, confirming the DC_ω dependency.

8.3 Axiom Certificate

The build output confirms clean separation:

```
-- BISH theorems:
'born_prob_nonneg' depends on: [propext, Classical.choice, Quot.sound]
'born_prob_sum_one' depends on: [propext, Classical.choice, Quot.sound]
'expectation_real'  depends on: [propext, Classical.choice, Quot.sound]
'variance_nonneg'   depends on: [propext, Classical.choice, Quot.sound]
'bernoulli_variance_bound' depends on: [propext, Classical.choice, Quot.sound]
```

```
'chebyshev_bernoulli_bound' depends on: [propext, Classical.choice, Quot.sound]
'relative_freq_nonneg' depends on: [propext, Classical.choice, Quot.sound]
'relative_freq_le_one' depends on: [propext, Classical.choice, Quot.sound]
'relative_freq_sum'    depends on: [propext, Classical.choice, Quot.sound]
'cnorm_sq_nonneg'      depends on: [propext, Classical.choice, Quot.sound]

-- DC theorem:
'frequentist_convergence' depends on:
  [propext, Classical.choice, Quot.sound,
   Papers.P16.dc_omega_holds, Papers.P16.slln_of_dc]
```

All BISH theorems have axiom closure {`propext`, `Classical.choice`, `Quot.sound`}—standard Mathlib4 infrastructure with no custom axioms. The DC_ω theorem additionally contains `dc_omega_holds` and `slln_of_dc`.

8.4 AI-Assisted Methodology

The formalisation was developed with AI assistance (Claude, Anthropic). The author verified all mathematical content and the AI generated Lean 4 proof scripts. All proofs were machine-checked by the Lean 4 kernel—no trust is placed in the AI's correctness, only in the formal verification.

9 Discussion

9.1 Quantum Geometry vs. Quantum Statistics

The Born rule separates into two physical contents:

- **Quantum geometry (BISH):** the state ψ determines a probability distribution over outcomes. This is a fact about one vector in one finite-dimensional space. No measurements needed.
- **Quantum statistics (DC_ω):** repeated measurements of identically prepared systems yield frequencies converging to the geometric distribution. This requires constructing infinite product spaces and proving convergence.

The experimentalist who runs 10,000 measurements and checks that frequencies match Born probabilities within error bars is working in BISH. The textbook that asserts "in the limit of infinitely many measurements, the frequency equals the probability" is asserting DC_ω.

9.2 Comparison with Paper 14

Paper 14 calibrated the quantum measurement problem (decoherence) against LPO. Paper 16 calibrates the Born rule against DC_ω. The two results are complementary:

- Paper 14: the *mechanism* of measurement (decoherence) $\to$ LPO.
- Paper 16: the *statistics* of measurement (Born probabilities) $\to$ DC_ω.

Different aspects of quantum measurement land at different levels of the constructive hierarchy.

9.3 Limitations and Open Problems

1. **No new equivalence.** Unlike Papers 2, 8, and 13, which prove biconditional equivalences ($T \equiv P$ over BISH), this note establishes only that DC_ω is sufficient for the SLLN. The reverse direction (SLLN $\implies$ DC_ω) is not formalised. A full calibration would require proving that the strong law *cannot* be obtained from weaker principles (e.g., countable choice without dependence).

2. **SLLN axiomatised.** The strong law is introduced as an axiom (`slln_of_dc`), not derived from measure theory. A complete formalisation of the constructive SLLN with explicit DC_ω tracking in Lean 4 would be a substantial independent contribution.

3. **BISH content is unsurprising.** The constructive status of Theorems 1–5 follows immediately from their being finite-dimensional linear algebra and real arithmetic. The formalisation verifies correctness but does not reveal hidden non-constructive dependencies—there are none to reveal.

4. **Classical.choice artefact.** Per the Paper 10 methodology (§2.2), `Classical.choice` in the BISH theorems is an artefact of Mathlib4's typeclass infrastructure, not a mathematical use of excluded middle.

10 Conclusion

The Born rule decomposes along the constructive hierarchy: single-trial probability, expectation, variance, and finite-sample error bounds are BISH; exact frequentist convergence requires DC_ω. Neither half of this observation is mathematically surprising. The contribution is to the calibration programme: the DC_ω axis now has explicit physical content from quantum measurement statistics, and the pattern—physics as practised is constructive, physics as idealised has measurable logical cost—extends to yet another aspect of quantum mechanics.

The main open problem is sharpness: is DC_ω *necessary* for the strong law, or does a weaker principle suffice? A constructive proof of SLLN $\equiv$ DC_ω over BISH would convert this note's upper bound into an exact calibration.

Reproducibility. All source code, compiled PDF, and build instructions are archived at Zenodo DOI: 10.5281/zenodo.18575377. Lean 4 version: `leanprover/lean4:v4.28.0-rc1`. Mathlib4 commit: `7091f0f6`. Build: `lake exe cache get && lake build` (1662 jobs, 0 errors, 0 `sorry`).

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