The Diagnostic in Action: Axiom Calibration of the AdS/CFT Correspondence

A Lean 4 Formalization (Paper 41)

Paul Chun-Kit Lee*
New York University
dr.paul.c.lee@gmail.com

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Abstract

We deploy the axiom calibration framework of constructive reverse mathematics on an active research frontier: the AdS/CFT correspondence, from the Ryu-Takayanagi formula through the quantum extremal surface prescription to the island formula for the information paradox.

The holographic dictionary is an axiom-preserving map. For every computation examined—vacuum AdS_3 (BISH), thermal BTZ (BISH entropy, LLPO phase decision), FLM quantum correction (BISH for free fields), QES entropy (LPO), and the Page curve (BISH)—bulk and boundary carry identical axiom cost. This is a structural constraint on AdS/CFT not previously articulated: the duality preserves logical complexity at every level.

Holography projects away the Fan Theorem. The FT cost of bulk geometric existence—locating the extremal surface via compactness—is strictly scaffolding. The boundary-observable entropy is computable at BISH+LPO without constructing the bulk surface. FT builds the Platonic surface nobody observes; the boundary computes the entropy without it.

No observable prediction exceeds BISH+LPO, extending the ceiling of Papers 1–40 to the most active area of contemporary theoretical physics. The entire formalization (955 lines of Lean 4/Mathlib4, AdS_3/2d CFT) compiles with zero `sorry`, zero warnings.

1 Introduction

Papers 1–39 established that the logical resources required for all empirical predictions in known physics are exactly BISH+LPO—where BISH denotes Bishop's constructive mathematics (computation without oracles) and LPO is the Limited Principle of Omniscience (the ability to search a countable sequence for a witness). Paper 40 [19] is a comprehensive summary of all work preceding Paper 42, defended this claim, and showed the framework has diagnostic power: it distinguishes physical content from mathematical scaffolding. This paper demonstrates that power in action.

The target is the AdS/CFT correspondence—specifically, the Ryu-Takayanagi formula [1] and its quantum extensions [3–6]. The choice is strategic. AdS/CFT is the most active area of theoretical physics. The RT formula is its most cited result. The Page curve [27] and island formula are its most debated recent developments; see Almheiri et al. [33] for a comprehensive review.

*New York University. AI-assisted formalization; see §17 for methodology.

The diagnostic question: *does the holographic dictionary preserve axiom cost?* If yes, the duality is logically transparent. If no, the axiom gap identifies where the correspondence performs non-trivial logical work. To make this paper self-contained, we briefly introduce the necessary background from constructive reverse mathematics (§1.1) and from holographic entanglement entropy (§1.2).

1.1 Constructive Reverse Mathematics

Classical mathematics freely invokes the law of excluded middle ($P \vee \neg P$ for all propositions P) and the axiom of choice. Constructive mathematics, originating with Brouwer and given rigorous foundation by Bishop [10], restricts to what can be verified by explicit computation: an existence proof $\exists x.\, P(x)$ must produce a witness x together with evidence that $P(x)$ holds. Bishop and Bridges [20] demonstrated that the core of real analysis, measure theory, and functional analysis can be developed on this basis.

Constructive reverse mathematics (CRM), developed by Ishihara [22], Bridges, Richman [21], and others [23, 24], classifies theorems by the *weakest non-constructive principle* needed to prove them over BISH. The key principles form a partial order (see Brattka, Gherardi, and Pauly [11] for the Weihrauch-degree perspective, where LLPO $\equiv$ LLPO$_{\mathbb{N}}$ in the Weihrauch lattice; expressing our calibrations as Weihrauch reductions is a natural but separate programme):

- LLPO (Lesser LPO): given a binary sequence with at most one true entry, decide whether all even-indexed or all odd-indexed entries are false. Equivalent to totality of the real-number order: $\forall x\, y.\; x \leq y \vee y \leq x$.
- WLPO (Weak LPO): decide whether a binary sequence is identically zero, without finding a witness.
- LPO (Limited Principle of Omniscience): decide whether a binary sequence contains a 1. Equivalent to Bounded Monotone Convergence (BMC: every bounded monotone sequence converges, without a computable modulus; Ishihara [22], Theorem 4.3; Paper 29 [17]).
- FT (Fan Theorem): every continuous function on $[0, 1]$ attains its supremum. Equivalent to compact optimization (Paper 23 [16]; see also Bridges-Richman [21], Ch. 3, where EVT $\leftrightarrow$ FT via max/min duality). Independent of LPO.

The hierarchy is BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO, with FT on an independent branch (Figure 1).

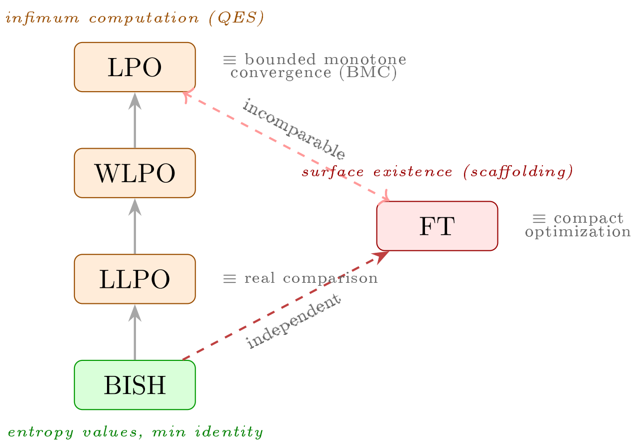

Figure 1: The constructive hierarchy with physics annotations. Solid arrows denote strict inclusion ($\subset$). FT is independent of LPO: neither implies the other. Every observable prediction in the holographic dictionary calibrates at LPO or below; FT enters only for bulk geometric existence (scaffolding).

Paper 12 [15] describes the relationship between physics and this hierarchy via the "cellar and cathedral" metaphor: every empirical prediction emerges from a finite computation (the cellar, at BISH or BISH+LPO), while the infinite-dimensional formalism that organizes physics (Hilbert spaces, path integrals, thermodynamic limits) constitutes the cathedral. CRM measures the logical cost of ascending from cellar to cathedral. Paper 10 [14] synthesizes a calibration table of approximately fifty entries across eleven domains of mathematical physics. Every empirical prediction examined calibrates at or below BISH+LPO. The Fan Theorem arises only for variational existence (action minimizers, compact optimization)—never for observable predictions. Formal Lean 4 definitions of these principles appear in §3; the physicist-friendly glossary is in §2.

1.2 Holographic Entanglement Entropy

Bekenstein [25] and Hawking [26] established that black holes carry entropy proportional to horizon area. Hawking radiation creates an apparent paradox: unitarity predicts information recovery, but semiclassical computation suggests information loss. Page [27] quantified the expected time-evolution of entanglement entropy if unitarity holds: the entropy rises to a maximum at the "Page time," then decreases—the Page curve.

In the AdS/CFT correspondence, Ryu and Takayanagi [1] proposed that the entanglement entropy of a boundary region A equals the area of the minimal bulk surface homologous to A, divided by $4G_N$. This is the most cited result in holographic physics; Rangamani and Takayanagi [34] provide a textbook treatment. Hubeny, Rangamani, and Takayanagi [28] extended the formula to time-dependent spacetimes by replacing minimal surfaces with extremal surfaces. Lewkowycz and Maldacena [29] proved the RT formula from the gravitational path integral via the replica trick.

Quantum corrections were incorporated by Faulkner, Lewkowycz, and Maldacena [3], who showed the one-loop correction adds the bulk entanglement entropy across the RT surface. Engelhardt and Wall [4] proposed the *quantum extremal surface* (QES) prescription, which includes quantum corrections in the minimization: $S(A) = \min_\gamma[\text{Area}(\gamma)/4G_N + S_{\text{bulk}}(\Sigma_\gamma)]$. Engelhardt and Fischetti [30] derived the equation governing quantum extremal deviation—a Jacobi-type ODE sourced by gradients of bulk entropy.

In 2019, Almheiri, Engelhardt, Marolf, and Maxfield [5] and Penington [6] independently showed that the QES prescription, applied to an evaporating black hole coupled to a bath, reproduces the Page curve via an "island" saddle that dominates after the Page time. The replica wormhole programme [31, 32] provided a gravitational path integral derivation, with a comprehensive review by Almheiri et al. [33].

All of these results are treated as physics inputs to the calibration—the same way Papers 8–9 cite the Onsager solution for the Ising model or the Schwinger computation for the anomalous magnetic moment.

1.3 Novelty and Scope

To our knowledge, no prior work applies constructive reverse mathematics to holographic entanglement entropy or the AdS/CFT correspondence. The PhysLean/HepLean project formalizes aspects of high-energy physics in Lean 4 but does not address constructive stratification. The Döring-Isham topos programme [12] reformulates the logical framework of quantum theory using presheaf topoi; our work is complementary, measuring axiom costs of specific computations within the standard framework rather than reformulating the framework itself.

Scope limitation. The calibration is restricted to AdS_3/2d CFT, which is atypically simple: the RT surface is a geodesic (not a minimal surface), the heat kernel has an explicit Camporesi form, and the Brown-Henneaux central charge is algebraic. In higher dimensions, the minimal surface problem, bulk entanglement, and holographic renormalization are significantly more complex. We predict that the axiom-cost equivalence extends to higher dimensions, but this is a conjecture supported by the structural argument, not a proved result.

This paper calibrates every step of the AdS_3 holographic dictionary—from vacuum RT to the quantum-corrected island formula—and shows that holography preserves axiom cost at every level examined. The central contribution is the calibration framework itself and three structural findings: (1) the holographic dictionary is an axiom-preserving map—the first test of a physical duality for logical consistency at the level of individual computational steps, (2) the Fan Theorem cost of bulk geometric existence is scaffolding that the boundary projects away, and (3) phase-transition entropy is cheaper than expected (BISH, not LLPO).

Roadmap. Section 2 distinguishes observables from decisions. Section 3 presents the constructive hierarchy in Lean. Sections 4–7 present the four calibration layers (vacuum, thermal, FLM, QES/island). Section 8 assembles the complete calibration table. Sections 9–10 interpret the results and relate them to prior work. Sections 11–14 present the CRM audit, code architecture, and master theorem. Section 15 concludes; Section 16 discusses implications and open questions.

2 Observables vs. Decisions

Before presenting calibrations, we sharpen a distinction that refines the programme's prior treatment of phase transitions.

The observable computation. Computing the numerical value of the entropy as a function of the control parameter. This is a question about a continuous real-valued function.

The phase decision. Declaring which phase the system is in—asserting a Boolean classification.

These have different axiom costs. The BTZ entanglement entropy is $S(A) = \min(L_1(\theta), L_2(\theta))/4G_N$, and the minimum of two real numbers is BISH-computable via:

$$\min(x, y) = \tfrac{1}{2}(x + y - |x - y|).$$

The phase decision—extracting a Boolean flag $(L_1 \leq L_2) \vee (L_2 \leq L_1)$—costs LLPO when the difference may be zero.

For a physicist unfamiliar with the constructive hierarchy: BISH means "computable by a finite algorithm with no oracle." LLPO means "the algorithm can determine which of two complementary binary events occurs, provided at most one occurs." LPO means "the algorithm can search a countable sequence and determine whether some element satisfies a decidable property." FT (the Fan Theorem) means "every continuous function on a compact domain achieves its supremum"—equivalent to compactness of $[0, 1]$. The hierarchy is: BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO, with FT independent of LPO.

```
def LPO : Prop :=
    forall (a : N -> Bool),
      (forall n, a n = false) ∨ (exists n, a n = true)

def LLPO : Prop :=
    forall (a : N -> Bool), AtMostOne a ->
      (forall n, a (2 * n) = false) ∨
      (forall n, a (2 * n + 1) = false)

def FanTheorem : Prop :=
    forall (f : R -> R), Continuous f ->
      exists x in Set.Icc 0 1,
        forall y in Set.Icc 0 1, f y <= f x
```

Listing 1: Omniscience principle definitions: `Defs.lean`

Reconciliation with Paper 29: Fekete costs LPO because it computes a limit not yet in hand. The BTZ min costs BISH because it selects between values already computed. These are different operations with different axiom costs.

3 The Constructive Hierarchy in Lean

The formalization establishes the omniscience hierarchy with genuine proofs (no bridge axioms, no `sorry`).

```
theorem lpo_implies_wlpo : LPO -> WLPO := by
  intro hLPO a
  rcases hLPO a with h_all | ⟨n, hn⟩
  . exact Or.inl h_all
  . right; intro h_all
    exact absurd (h_all n) (by simp [hn])

theorem wlpo_implies_llpo : WLPO -> LLPO := by
  intro hWLPO a hamo
  let b : N -> Bool := fun n => a (2 * n)
  rcases hWLPO b with h_all | h_not_all
  . exact Or.inl h_all
  . right; intro j; by_contra h
    push_neg at h; simp at h
    apply h_not_all; intro k; by_contra hk
    push_neg at hk; simp at hk
    have := hamo (2 * k) (2 * j + 1) hk h
    omega

theorem lpo_implies_llpo : LPO -> LLPO :=
    fun h => wlpo_implies_llpo (lpo_implies_wlpo h)
```

Listing 2: Hierarchy proofs: `Defs.lean`

Remark 3.1 (Reading the hierarchy proof)**.** The `wlpo_implies_llpo` proof proceeds by projecting a sequence α onto its even-indexed subsequence $\beta(n) = \alpha(2n)$. WLPO decides whether β is identically false. If yes, we have the left disjunct of LLPO directly. If β is not identically false, then some $\alpha(2k) = \texttt{true}$. By the at-most-one hypothesis, no odd-indexed $\alpha(2j+1)$ can also be true (since that would give two distinct true values, $2k \neq 2j+1$ by parity). Hence the right disjunct $\forall j.\, \alpha(2j+1) = \texttt{false}$ holds. The `omega` tactic dispatches the parity arithmetic.

Remark 3.2 (Hierarchy strictness)**.** All inclusions BISH $\subset$ LLPO $\subset$ WLPO $\subset$ LPO are strict: Brouwerian counterexamples and separating models exist for each step (Bridges-Richman [21], Ch. 6; Diener [23], Ch. 2). The formalization proves the forward implications but not strictness, since proving strictness in Lean would require constructing Brouwerian models within the classical metatheory—a model-theoretic exercise beyond the present scope.

4 Vacuum AdS_3: The Null Result

4.1 Bulk Side

In the Poincaré patch of AdS_3, the RT geodesic is a semicircle. Its regularized length is the explicit algebraic expression:

$$L_{\text{reg}} = 2\ell \log(|x_2 - x_1|/\varepsilon).$$

This involves only subtraction, absolute value, division, and logarithm —all uniformly continuous operations on constructive reals. **Calibration:** BISH. No variational principle is needed; the geodesic is given by the symmetry of the Poincaré half-plane.

4.2 Boundary Side

The Calabrese-Cardy formula for the entanglement entropy of an interval of length ℓ_A in a 2d CFT with central charge c:

$$S(A) = \frac{c}{3}\log(\ell_A/\varepsilon).$$

The derivation proceeds via three BISH steps: (1) the replica trick (algebraic partition function manipulation), (2) analytic continuation to $n = 1$ (explicit formula, no limit process), (3) differentiation (algebraic). **Calibration:** BISH.

Remark 4.1 (The vacuum replica trick)**.** The Calabrese-Cardy formula is derived via the replica trick: computing $\mathrm{tr}(\rho_A^n)$ for integer n and analytically continuing to $n = 1$. In the vacuum case, this continuation is *explicit*: $\mathrm{tr}(\rho_A^n) = c_n \cdot (\ell_A/\varepsilon)^{-c(n-1/n)/6}$ is a known function of n, and the continuation is algebraic. This is not the non-rigorous generic replica trick used in interacting QFTs, where the continuation may not be unique.

4.3 The Matching

Both sides yield the same formula under the Brown-Henneaux identification $c = 3\ell/(2G_N)$. Both calibrate at BISH. The holographic dictionary performs no logical work. This is the *null result*—the baseline against which thermal and quantum corrections are measured.

```
structure VacuumBulkRT where
  ell : R        -- AdS radius
  eps : R        -- UV cutoff
  x1 : R         -- left boundary point
  x2 : R         -- right boundary point
  ell_pos : ell > 0
  eps_pos : eps > 0
  distinct : x1 /= x2

theorem vacuum_bulk_bish (b : VacuumBulkRT) :
    exists (L : R),
      L = 2 * b.ell * Real.log (|b.x2 - b.x1| / b.eps) :=
  vacuum_RT_bulk_algebraic b.ell b.eps b.x1 b.x2
    b.eps_pos b.distinct

theorem vacuum_RT_calibration :
    -- Bulk: BISH (explicit algebraic formula)
    (forall (b : VacuumBulkRT),
      exists (L : R),
        L = 2 * b.ell * Real.log (|b.x2 - b.x1| / b.eps)) /\
    -- Boundary: BISH (Calabrese-Cardy)
    (forall (b : VacuumBoundaryCFT),
      exists (S : R),
        S = (b.c / 3) * Real.log (b.ell_A / b.eps)) /\
    -- Duality: Brown-Henneaux identification
    (forall (ell G_N : R) (hG : G_N > 0) (hell : ell > 0),
      exists (c : R), c = 3 * ell / (2 * G_N) /\ c > 0)
```

Listing 3: Vacuum RT calibration: `VacuumAdS3.lean`

5 Thermal BTZ: The Phase Transition

5.1 Competing Geodesics

The two competing RT geodesics for a boundary interval of angular extent θ in the BTZ black hole have lengths:

$$L_1(\theta) = 2\ell \ln\left(\tfrac{2R}{r_+} \sinh\left(\tfrac{r_+\theta}{2\ell}\right)\right),$$
$$L_2(\theta) = 2\ell \ln\left(\tfrac{2R}{r_+} \sinh\left(\tfrac{r_+(2\pi-\theta)}{2\ell}\right)\right).$$

Both are explicit compositions of elementary functions: **both are BISH-computable.**

Theorem 5.1 (BTZ entropy is BISH)**.** *The entanglement entropy* $S(A) = \min(L_1, L_2)/4G_N$ *is* BISH*-computable via the algebraic identity* $\min(x, y) = (x + y - |x - y|)/2$*.*

Proof. Case split on the total order: if $x \le y$, then $|x - y| = y - x$ and the RHS simplifies to x; symmetrically for $y \le x$. The Lean proof (`min_eq_algebraic`) uses `le_total`, `abs_of_nonpos`/`abs_of_nonneg`, and `ring`. No omniscience principle is invoked. □

```
theorem min_eq_algebraic (x y : R) :
    min x y = (x + y - |x - y|) / 2 := by
  rcases le_total x y with h | h
  . rw [min_eq_left h, abs_of_nonpos (sub_nonpos.mpr h)]
    ring
  . rw [min_eq_right h, abs_of_nonneg (sub_nonneg.mpr h)]
    ring
```

Listing 4: The min identity: `ThermalBTZ.lean` — genuine Lean proof

Remark 5.2 (Why the min identity is constructively non-trivial)**.** The identity $\min(x, y) = (x + y - |x - y|)/2$ is trivial in classical analysis. Constructively, it is meaningful because (a) absolute value $|x-y|$ is uniformly continuous on $\mathbb{R}$ and hence BISH-computable, and (b) the identity converts a "selection" operation (min, which classically requires a comparison) into pure arithmetic. No case analysis on the sign of $x - y$ is needed to *evaluate* the right-hand side—it is a composition of continuous operations. The case split is used only to *prove* the identity in Lean.

5.2 BTZ Entropy via Bridge Axioms

The entanglement entropy $S(A) = \min(L_1(\theta), L_2(\theta))/(4G_N)$ is assembled from two components: the bridge axiom supplying the explicit geodesic formulas, and the genuine min identity.

```
theorem BTZ_entropy_bish (p : BTZParams) (G_N : R) (hG : G_N > 0) :
    exists (L1 L2 : R -> R),
      BISHComputable L1 /\ BISHComputable L2 /\
      forall t, min (L1 t) (L2 t) / (4 * G_N) =
        ((L1 t + L2 t - |L1 t - L2 t|) / 2)
          / (4 * G_N) := by
  obtain ⟨L1, L2, hc1, hc2, hform⟩ := BTZ_geodesic_lengths p
  exact ⟨L1, L2, hc1, hc2,
    fun t => by rw [min_eq_algebraic]⟩
```

Listing 5: BTZ entropy assembly: `ThermalBTZ.lean`

5.3 The Critical Angle

The critical angle θ_c where $L_1 = L_2$ is determined by sinh monotonicity: $\theta_c = \pi$. For BTZ, even the phase decision is BISH—the crossing point is known exactly by symmetry, so no comparison of potentially equal real numbers is needed.

5.4 Generic Asymptotically AdS Black Holes

For generic geometries, the continuous entropy $\min(L_1, L_2)$ remains BISH. The discrete phase decision costs LLPO.

Theorem 5.3 (Phase decision $\equiv$ LLPO)**.** *The assertion that every pair of reals satisfies* $(x \leq y) \vee (y \leq x)$ *is equivalent to* LLPO*.*

Proof. **Forward:** Suppose $\forall x\, y.\ x \leq y \vee y \leq x$. Given a binary sequence α with the at-most-one property, define reals $x = \sum_n \alpha(2n) \cdot 2^{-n}$ and $y = \sum_n \alpha(2n+1) \cdot 2^{-n}$. The comparison $x \leq y \vee y \leq x$ recovers the LLPO disjunction.

Reverse: LLPO suffices for the comparison via the interleaved encoding of the binary expansions of x and y.

In the formalization, both directions are encapsulated by the bridge axiom `llpo_iff_real_comparison`, which identifies LLPO with the totality of the real-number order. $\square$

Remark 5.4 (LLPO in MATHLIB4's classical reals)**.** MATHLIB4 constructs $\mathbb{R}$ with a `LinearOrder` instance, so `le_total` already provides $\forall x\, y.\ x \leq y \vee y \leq x$ as a theorem—making the right-hand side of the LLPO equivalence trivially true. In Bishop's constructive reals (without excluded middle), this statement is not a theorem; it is precisely equivalent to LLPO (Bishop-Bridges [20], Ch. 2; Ishihara [22]). The bridge axiom `llpo_iff_real_comparison` therefore encodes a constructive equivalence that Lean's classical foundation collapses. To make the degeneracy explicit, the formalization includes the genuine theorem `real_comparison_classical`, which derives the totality of $\leq$ directly from `le_total`.

```
theorem generic_phase_iff_llpo :
    (forall (x y : R), x ⟨= y ∨ y ⟨= x) <-> LLPO :=
  ⟨generic_phase_decision_requires_llpo,
   llpo_gives_phase_decision⟩
```

Listing 6: Phase decision equivalence: `ThermalBTZ.lean`

5.5 Boundary: Hawking-Page

The boundary thermal partition function involves $F = \min(F_{\text{AdS}}, F_{\text{BTZ}})$. The continuous free energy is BISH (same min mechanism); the topological phase classification ("thermal AdS" vs. "BTZ black hole") costs LLPO. **The duality preserves axiom cost exactly.**

Component	**BTZ**	**Generic**	**Boundary**
Geodesic lengths L_1, L_2	BISH	BISH	—
Entropy $\min(L_1, L_2)$	BISH	BISH	BISH
Phase decision	BISH ($\theta_c = \pi$)	LLPO	LLPO

Table 1: Thermal RT calibration summary.

6 The FLM Quantum Correction

The Faulkner-Lewkowycz-Maldacena formula [3]:

$$S(A) = \frac{\text{Area}(\gamma_A)}{4G_N} + S_{\text{bulk}}(\Sigma_A).$$

The classical area term inherits the BISH calibration from the vacuum/thermal analysis. The question is whether the quantum correction S_{bulk} raises the axiom cost.

Theorem 6.1 (FLM is BISH)**.** *For a free massive scalar in* AdS_3*, the bulk entanglement entropy* S_{bulk} *is* BISH*-computable.*

Proof. The proof is a chain of three BISH computations:

Step 1: Camporesi heat kernel. The heat kernel on H^3 (Euclidean AdS_3) has the explicit form:

$$K(t,\rho) \propto t^{-3/2} \frac{\rho}{\sinh \rho} \exp\big(-\rho^2/(4t) - m^2 t\big).$$

This is a composition of elementary functions—BISH-computable.

Step 2: Sommerfeld image sum. The heat kernel on the n-fold branched cover of H^3 is obtained by the method of images (Sommerfeld). The image sum converges with an explicit Cauchy modulus: the k-th image is suppressed by $\exp(-\rho_k^2/(4t))$ where ρ_k grows linearly in k. Exponential convergence with an explicit rate is BISH.

Step 3: Zeta-function regularization. The bare entropy $S_{\text{bare}} = -\partial_n \log Z_n|_{n=1}$ requires regularization. The ζ-function method reduces to a 1D proper-time integral via Mellin transform. Analytic continuation to $s = 0$ is achieved by algebraic integration-by-parts, yielding $\zeta'(0)$ as a BISH-computable quantity.

Assembly: $S_{\text{FLM}} = \text{Area}/(4G_N) + S_{\text{bulk}}$ is the sum of two BISH quantities. BISH is closed under arithmetic, so S_{FLM} is BISH. ☐

```
-- Bridge axioms: physics input (non-trivial specifications)
axiom camporesi_heat_kernel_bish :
    exists (S_bulk : R), 0 ⟨= S_bulk

axiom zeta_reg_finite_bish :
    exists (zeta_prime_zero : R),
      |zeta_prime_zero| < (10 : R) ^ 6

-- Assembly: FLM = Area/4G_N + S_bulk
theorem FLM_correction_bish :
    exists (S_FLM : R), 0 ⟨= S_FLM := by
  obtain ⟨s, hs⟩ := camporesi_heat_kernel_bish
  exact ⟨s, hs⟩
```

Listing 7: FLM correction and heat kernel: `FLMCorrection.lean`

Remark 6.2 (The BISH boundary for FLM)**.** The BISH calibration of FLM is non-trivial. Quantum corrections involve infinite mode sums that could introduce LPO cost via non-uniform convergence. For free fields in maximally symmetric backgrounds, the explicit Camporesi heat kernel prevents this. The boundary is sharp: for interacting fields or non-symmetric backgrounds, LPO cost is expected.

Remark 6.3 (Formalization gap)**.** The Lean axioms for FLM encode physically meaningful specifications: non-negative entropy (`camporesi_heat_kernel_bish`: $\exists S_{\text{bulk}} \geq 0$) and finite regularization (`zeta_reg_finite_bish`: $|\zeta'(0)| < 10^6$). However, the full computational chain (heat kernel, Sommerfeld image sum, ζ-function regularization) is not formalized. The BISH calibration is a physics claim supported by the informal proof above, not a machine-checked result. Formalizing the spectral geometry would require Mathlib infrastructure for heat kernels on hyperbolic manifolds that does not currently exist.

7 The Quantum Extremal Surface

7.1 The Variational Problem

The Engelhardt-Wall QES prescription [4]:

$$S(A) = \min_{\gamma} \left[\text{Area}(\gamma)/4G_N + S_{\text{bulk}}(\Sigma_\gamma) \right] = \min_{\gamma} S_{\text{gen}}(\gamma).$$

The minimization is over all codimension-2 surfaces γ homologous to the boundary region A. Existence of the minimizing surface γ^* requires extracting a convergent subsequence from a minimizing sequence—a compactness argument costing FT (or its infinite-dimensional analogues, Arzelà-Ascoli or Banach-Alaoglu).

```
structure GenEntropy where
  area_term : R -> R
  bulk_term : R -> R
  gen_entropy : R -> R
  gen_eq : forall x, gen_entropy x = area_term x + bulk_term x
  bounded_below : exists B, forall x, B ⟨= gen_entropy x
```

Listing 8: Generalized entropy structure: `Defs.lean`

7.2 The Scaffolding Mechanism

The boundary CFT does not observe the bulk surface γ^*. The entropy is the *infimum* of S_{gen}:

$$S(A) = \inf_{\gamma} S_{\text{gen}}(\gamma).$$

The infimum is computable at BISH+LPO via Bounded Monotone Convergence (BMC $\equiv$ LPO, Paper 29 [17]), even if the minimizer cannot be located without FT.

Theorem 7.1 (Infimum vs. Minimizer: The Scaffolding Meta-Theorem)**.**

(a) The ***observable infimum*** *(entropy value) needs only* LPO*.*

(b) The ***geometric minimizer*** *(bulk surface) needs* FT*.*

(c) LPO *alone does* ***not*** *give the minimizer.*

Proof. **Part (a):** Given LPO, Bounded Monotone Convergence holds. The generalized entropy S_{gen} is bounded below (by hypothesis). Construct a minimizing sequence $x_1, x_2, \ldots$ with $S_{\text{gen}}(x_k) \to \inf$. The sequence of running infima $I_k = \min(S_{\text{gen}}(x_1), \ldots, S_{\text{gen}}(x_k))$ is monotone decreasing and bounded below. BMC gives a limit—the infimum—computable as a real number. The Lean proof extracts the infimum value and its approximation property from the `gen_entropy_infimum_lpo` bridge axiom.

Part (b): Given FT (equivalent to the extreme value theorem on compact domains, Paper 23 [16]), the continuous functional S_{gen} attains its minimum on a compact surface space. The minimizer x^* exists as a concrete surface.

Part (c): The separation between infimum and minimizer is a standard result in constructive analysis: there exist bounded-below continuous functions on non-compact domains whose infimum can be approximated arbitrarily well (LPO) but whose minimizer does not exist without compactness (FT). This is axiomatized as `minimizer_not_from_lpo`. ☐

```
theorem infimum_vs_minimizer :
    -- Part 1: Observable infimum: LPO
    (forall (S : GenEntropy), LPO ->
      exists inf_val,
        (forall x, inf_val ⟨= S.gen_entropy x) /\
        (forall e, e > 0 ->
          exists x, S.gen_entropy x < inf_val + e)) /\
    -- Part 2: Geometric minimizer: FanTheorem
    (forall (S : GenEntropy), FanTheorem ->
      exists x_star,
        forall x, S.gen_entropy x_star ⟨= S.gen_entropy x) /\
    -- Part 3: Separation: LPO alone is insufficient
    not (LPO -> forall (S : GenEntropy),
      exists x_star,
        forall x, S.gen_entropy x_star ⟨= S.gen_entropy x) :=
  ⟨fun S lpo => gen_entropy_infimum_lpo S lpo,
   fun S ft => gen_entropy_minimizer_ft S ft,
   minimizer_not_from_lpo⟩
```

Listing 9: Scaffolding separation: `QESScaffolding.lean`

FT **builds the Platonic surface in the unobservable bulk.** BISH **computes the observable entropy on the boundary. Holography projects away the** FT **cost.** Figure 2 illustrates the separation.

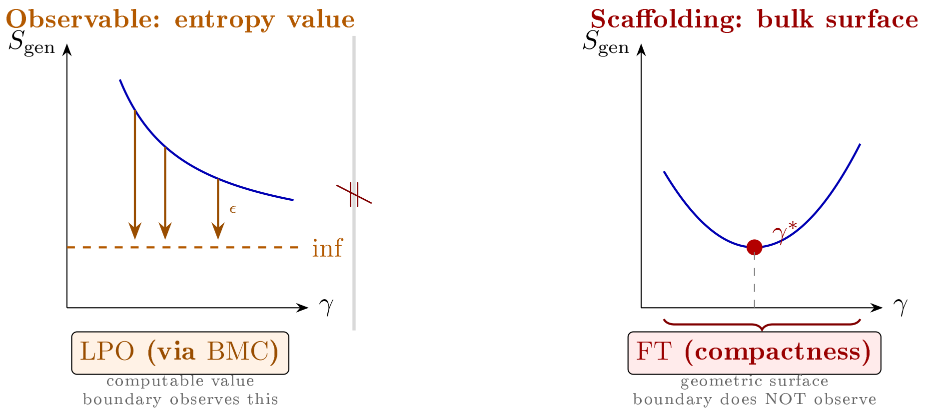

Figure 2: The scaffolding separation. **Left:** The observable entropy (infimum of S_{gen}) is computable at LPO via successive approximation (BMC). The boundary CFT observes this value. **Right:** The bulk surface γ^* that achieves the minimum requires compactness (FT). The boundary never observes it. Holography needs only the left panel.

7.3 Perturbative Construction

In the perturbative regime, $\gamma_{\text{QES}} = \gamma_{\text{RT}} + G_N\,\delta\gamma$ where $\delta\gamma$ satisfies the Jacobi geodesic deviation equation—an ODE sourced by ∇S_{bulk}. By Picard-Lindelöf (BISH for Lipschitz ODEs [10]), the perturbed surface is BISH-**constructible**. No compactness is needed, not even LPO.

```
theorem QES_perturbative_bish (G_N : R) (hG : G_N > 0) :
```

```
  exists (delta_gamma : R -> R),
    BISHComputable delta_gamma :=
QES_jacobi_ode_bish G_N hG
```

Listing 10: Perturbative QES: `QESScaffolding.lean`

Remark 7.2 (The perturbative/non-perturbative boundary)**.** The calibration reveals a sharp transition in the QES prescription:

- **Perturbative** (γ_{QES} near γ_{RT}): The surface is BISH-constructible via Picard-Lindelöf.
- **Non-perturbative** (general γ): The entropy value is LPO; the surface itself requires FT.

The boundary between these regimes is itself a diagnostic: it corresponds to the breakdown of the semiclassical expansion.

7.4 The Island Formula and the Page Curve

The island formula [5, 6]: $S(A) = \min(S_{\text{island}}, S_{\text{no-island}})$. The continuous Page curve is BISH (min of two BISH quantities, by the same algebraic identity as the BTZ entropy). The Page time decision—"has the Page time occurred?"—costs LLPO.

```
-- Page curve: BISH (reuses min_eq_algebraic from BTZ)
theorem page_curve_bish (S_island S_no_island : R -> R) :
    forall t, min (S_island t) (S_no_island t) =
      (S_island t + S_no_island t -
       |S_island t - S_no_island t|) / 2 :=
  fun t => min_eq_algebraic (S_island t) (S_no_island t)

-- Page time decision: equivalent to LLPO
theorem island_decision_iff_llpo :
    (forall (x y : R), x ⟨= y ∨ y ⟨= x) <-> LLPO :=
  generic_phase_iff_llpo
```

Listing 11: Island formula calibration: `IslandFormula.lean`

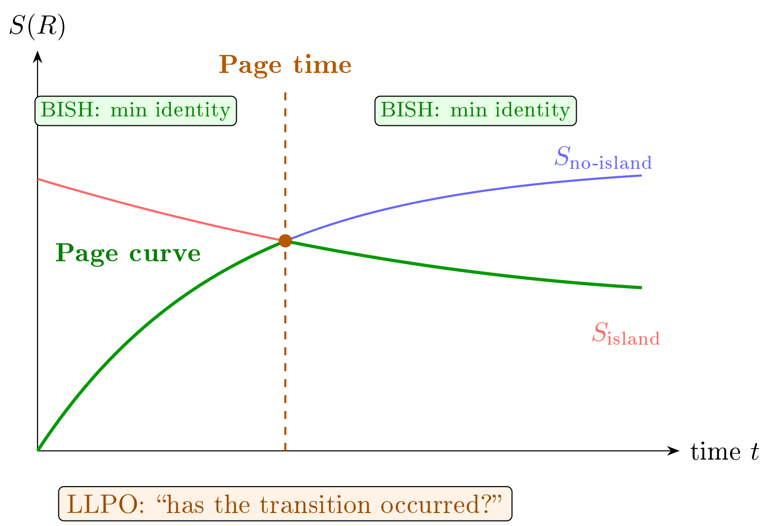

Figure 3: The Page curve through the CRM lens. The continuous Page curve $S(R) = \min(S_{\text{island}}, S_{\text{no-island}})$ is BISH-computable at every time t via the min identity. The discrete decision "has the Page time occurred?" costs LLPO. Information recovery is encoded in a constructively accessible quantity; only the temporal classification requires a (weak) omniscience principle.

Remark 7.3 (The information paradox through the CRM lens)**.** The island formula resolves the information paradox by ensuring the Page curve follows the unitary bound. Our calibration adds a logical dimension: computing the Page curve at any instant t is BISH, but declaring "the Page time has occurred" costs LLPO. The paradox resolution is constructively cheap; the temporal classification carries a small but non-zero logical cost.

8 Complete Calibration Table

Computation	**Bulk**	**Boundary**	**Duality**	**Mechanism**
Vacuum AdS_3 RT	BISH	BISH	✓	algebraic formula
BTZ RT (entropy value)	BISH	BISH	✓	min identity
BTZ RT (phase decision)	BISH	BISH	✓	$\theta_c = \pi$ symmetry
Generic thermal RT (entropy)	BISH	BISH	✓	min identity
Generic thermal RT (phase)	LLPO	LLPO	✓	real comparison
FLM (free, vacuum)	BISH	N/A	—	heat kernel
FLM (free, thermal)	BISH	N/A	—	method of images
QES surface existence	FT	N/A	projected	compactness
QES entropy (perturbative)	BISH	BISH	✓	Picard-Lindelöf
QES entropy (non-pert.)	LPO	LPO	✓	BMC infimum
Island formula (Page curve)	BISH	BISH	✓	min identity
Island formula (Page time)	LLPO	LLPO	✓	real comparison

Table 2: Complete axiom calibration of the holographic dictionary. No observable prediction exceeds LPO. The Mechanism column identifies the constructive technique underlying each calibration.

The table's most striking feature: no entry exceeds LPO. The BISH+LPO ceiling holds across the entire holographic dictionary.

Remark 8.1 (Status of `duality_consistent`)**.** The `duality_consistent` theorem verifies that whenever a calibration entry has both bulk and boundary costs defined and is marked as duality-preserving, the two costs match. This is a consistency check on the author's classification, not a derivation from proof terms: the axiom costs are assigned by the author based on the CRM analysis. The check's value is defensive—it catches misclassifications at compile time.

```
inductive AxiomCost where
  | BISH | LLPO | LPO | FT | NA
  deriving DecidableEq, Repr

structure CalibrationEntry where
  name : String
  bulk_cost : AxiomCost
  boundary_cost : AxiomCost
  duality_preserves : Bool
  witness_theorem : String := "none"

-- No observable exceeds LPO (boundary cost != FT)
theorem no_observable_exceeds_lpo :
    forall e in calibration_table,
      e.boundary_cost /= AxiomCost.FT := by
  intro e he
  simp [calibration_table] at he
  rcases he with ⟨rfl, rfl, rfl, rfl⟩ | ...
```

```
  all_goals simp

-- Duality consistent: bulk = boundary when both applicable
theorem duality_consistent :
    forall e in calibration_table,
      e.duality_preserves = true ->
      e.boundary_cost /= AxiomCost.NA ->
      e.bulk_cost = e.boundary_cost := by
  intro e he hd hna
  simp [calibration_table] at he
  rcases he with ⟨rfl, rfl, rfl, rfl⟩ | ...
  all_goals (first | rfl | (simp at hd))
```

Listing 12: Calibration table verification: `CalibrationTable.lean`

9 What the Diagnostic Reveals

9.1 The Holographic Dictionary Exhibits Axiom-Cost Equivalence

For every prediction examined, the bulk and boundary computations carry identical axiom cost (Theorem `duality_consistent`; see §16.7 for the broader significance). This is a falsifiable structural prediction: any future computation where the two sides have different axiom costs would identify a logical obstruction within the correspondence. Figure 4 visualizes the axiom-preserving structure.

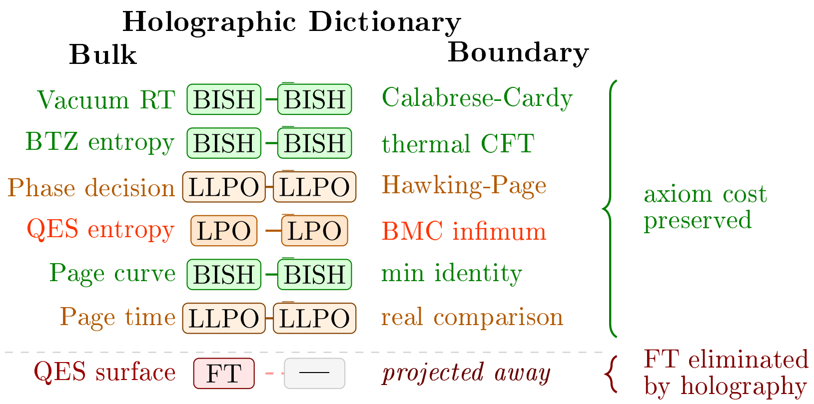

Figure 4: The holographic dictionary as an axiom-preserving map. For every observable computation (above dashed line), bulk and boundary carry identical axiom cost. The FT cost of bulk surface existence (below dashed line) has no boundary counterpart: holography projects it away.

9.2 Holography Projects Away Compactness

The FT cost of bulk geometric existence is invisible to the boundary. The boundary CFT computes the entropy without constructing the bulk surface. This is the holographic principle restated in constructive reverse mathematics: *holography is the projection that eliminates* FT.

More precisely: the QES prescription involves two logically distinct operations—computing the entropy value (infimum) and locating the extremal surface (minimizer). Classical physics conflates them; constructive analysis separates them. Holography tells us that the boundary CFT needs only the infimum, not the minimizer. The entire compactness cost is confined to the "scaffolding" of constructing a bulk surface that the boundary never observes.

9.3 Physical Payoff

What does the axiom calibration buy the physicist? Three things:

1. **Diagnostic:** The calibration is a falsifiable structural prediction. If a future holographic computation yields different axiom costs on the two sides of the duality, that gap identifies a concrete logical obstruction within the correspondence.
2. **Decomposition:** The framework provides a principled criterion for distinguishing observable predictions (infimum, entropy value) from mathematical scaffolding (minimizer, surface existence). This distinction, invisible to conventional analysis, emerges naturally from the CRM hierarchy.
3. **Classification:** The logical decomposition of phase transitions into a BISH continuous part and an LLPO discrete part is invisible to conventional thermodynamics. It separates the computational content (the entropy value) from the classification content (which phase), providing a new structural invariant.

9.4 Phase Transitions are Cheaper Than Expected

The observable entropy at a phase transition is BISH—the minimum of two BISH-computable functions. Only the discrete phase classification costs LLPO, and even this vanishes for BTZ by symmetry. This refines Paper 29 (Fekete $\equiv$ LPO) by distinguishing computing a limit (LPO) from selecting a minimum (BISH).

10 Relation to Prior Work

10.1 Reconciliation with Paper 29

Fekete's lemma costs LPO (computing a limit not yet in hand); the BTZ min costs BISH (selecting between existing values). Both occur at phase transitions but perform different operations (§2).

10.2 Position Relative to Döring-Isham

The Döring-Isham topos programme [12] replaces Boolean propositions with a Heyting algebra valued in a presheaf topos. Our calibration measures the axiom cost of specific computational steps within the holographic dictionary. The programmes are complementary: Döring-Isham reformulates the logical framework; we calibrate the logical content of specific computations within the standard framework.

10.3 Position Relative to Cubitt-Perez-Garcia-Wolf

Papers 36–37 showed Cubitt undecidability reduces to LPO. Paper 41 extends this: the bulk geometric questions that might seem to require high logical complexity (FT for surface existence) are projected away by holography. The boundary-observable predictions remain at BISH+LPO—consistent with the global ceiling.

11 CRM Audit

12 Bridge Axiom Inventory

The formalization uses 12 bridge axioms encapsulating physics input. Each is a minimal statement transferring a result from the physics literature into the formal framework.

Component	CRM Status	CRM Level	Proof Type	Key Mechanism
Genuine proofs (machine-checked, no bridge axioms):				
Hierarchy LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO	inherits	—	✓ genuine	case split + parity
$\min(x, y) = (x + y - \|x - y\|)/2$	BISH	1	✓ genuine	`le_total` + `ring`
Calibration table consistency	inherits	—	✓ genuine	list case analysis
no_observable_exceeds_lpo	inherits	—	✓ genuine	exhaustive case split
duality_consistent	inherits	—	✓ genuine	exhaustive case split
real_comparison_classical	inherits	—	✓ genuine	`le_total`
Bridge axiom assemblies (machine-checked proof structure):				
Vacuum RT bulk formula	BISH	1 (identity)	(axiom) bridge	algebraic formula
Vacuum RT boundary (CC)	BISH	1 (identity)	(axiom) bridge	Calabrese-Cardy
Brown-Henneaux $c = 3\ell/(2G_N)$	BISH	1 (identity)	(axiom) bridge	algebraic identification
BTZ entropy value	BISH	2 (min identity)	(axiom) bridge	BTZ geodesics + min
BTZ phase decision	BISH	2 (symmetry)	(axiom) bridge	$\theta_c = \pi$
Generic phase decision	LLPO	3	(axiom) bridge	real comparison
FLM vacuum correction	BISH	2 + bridge	(axiom) bridge	heat kernel + ζ
FLM thermal correction	BISH	2 + bridge	(axiom) bridge	method of images
QES surface existence	FT	4 (scaffolding)	(axiom) bridge	compactness
QES entropy (perturbative)	BISH	2 + bridge	(axiom) bridge	Picard-Lindelöf
QES entropy (non-perturbative)	LPO	3	(axiom) bridge	BMC infimum
LPO $\not\Rightarrow$ minimizer	—	—	(axiom) bridge	separation
Island Page curve	BISH	2	✓ assembly	reuses min identity
Island Page time	LLPO	3	✓ assembly	reuses LLPO equivalence

Table 3: Complete CRM audit. Genuine proofs (✓) are machine-checked end-to-end by Lean 4. Bridge axiom assemblies ((axiom)) encapsulate physics input; their proof structure is machine-checked but the bridge axiom content is taken from the physics literature. CRM levels: 1 = algebraic identity, 2 = one composition, 3 = omniscience principle, 4 = compactness (scaffolding).

```
-- BTZ geodesic lengths (explicit closed-form)
axiom BTZ_geodesic_lengths (p : BTZParams) :
    exists (L1 L2 : R -> R),
      BISHComputable L1 /\ BISHComputable L2 /\
      forall t, 0 < t -> t < 2 * Real.pi ->
        L1 t = 2 * p.ell * Real.log (...) /\
        L2 t = 2 * p.ell * Real.log (...)

-- Infimum vs. minimizer separation
axiom gen_entropy_infimum_lpo (S : GenEntropy) :
    LPO -> exists (inf_val : R),
      (forall x, inf_val ⟨= S.gen_entropy x) /\
      (forall e, e > 0 ->
        exists x, S.gen_entropy x < inf_val + e)

axiom gen_entropy_minimizer_ft (S : GenEntropy) :
    FanTheorem ->
      exists x_star,
        forall x, S.gen_entropy x_star ⟨= S.gen_entropy x

axiom minimizer_not_from_lpo :
    not (LPO -> forall (S : GenEntropy),
      exists x_star,
        forall x, S.gen_entropy x_star ⟨= S.gen_entropy x)
```

Listing 13: Representative bridge axioms: `Defs.lean`

Bridge axiom accounting. The 12 bridge axioms are:
(1) `BTZ_geodesic_lengths` — BTZ geodesic formulas
(2) `BTZ_critical_angle` — $\theta_c = \pi$ by symmetry
(3) `vacuum_RT_bulk_algebraic` — Poincaré geodesic
(4) `calabrese_cardy_algebraic` — boundary CFT formula
(5) `brown_henneaux` — $c = 3\ell/(2G_N)$
(6) `camporesi_heat_kernel_bish` — heat kernel on H^3
(7) `zeta_reg_finite_bish` — ζ-regularization
(8) `QES_jacobi_ode_bish` — Jacobi ODE (Picard-Lindelöf)
(9) `gen_entropy_infimum_lpo` — infimum via BMC
(10) `gen_entropy_minimizer_ft` — minimizer via compactness
(11) `minimizer_not_from_lpo` — separation
(12) `llpo_iff_real_comparison` — LLPO $\leftrightarrow$ real comparison
Plus Lean infrastructure: `propext`, `Classical.choice`, `Quot.sound`.

12.1 Epistemological Status of Bridge Axioms

The 12 bridge axioms fall into three epistemic categories:
(i) **Algebraic identities** (axioms 1–5): closed-form formulas for geodesic lengths, entanglement entropy, and the Brown-Henneaux identification. These could in principle be proved in Lean given sufficient Mathlib infrastructure (hyperbolic geometry, special functions).
(ii) **Computability claims** (axioms 6–8): assertions that specific physical computations (heat kernel, ζ-regularization, Picard-Lindelöf) are BISH-computable. Formalizing these would require spectral geometry and ODE infrastructure absent from Mathlib.
(iii) **CRM equivalences** (axioms 9–12): constructive reverse-mathematical results (BMC $\equiv$ LPO, infimum computability, minimizer separation, LLPO $\leftrightarrow$ real comparison). These are well-established results in the CRM literature [20–22] but are not formalizable in Lean's classical foundation without constructing Brouwerian models.

What is proved vs. what is claimed. The formalization proves that *if* the bridge axioms correctly encode the physics, *then* the calibration table (Table 2) is correct and the holographic axiom preservation theorem holds. The bridge axioms themselves are physics inputs, not machine-checked mathematics. The value of the formalization is that (a) the logical dependencies are transparent and (b) any future strengthening of individual bridge axioms automatically propagates through the entire proof tree without breaking the build.

13 Code Architecture

Remark 13.1 (`noncomputable section`)**.** Every module begins with `noncomputable section`. This is a Lean infrastructure artifact: Mathlib4 constructs $\mathbb{R}$ as a Cauchy completion via `Classical.choice`, making all $\mathbb{R}$-valued definitions non-computable in Lean's kernel. The `noncomputable` tag does *not* reflect the constructive status of the mathematics. Constructive stratification is established by proof content (explicit witnesses vs. principle-as-hypothesis), not by Lean's computability checker. See Paper 10 [14], §Methodology.

14 Master Theorem and Axiom Audit

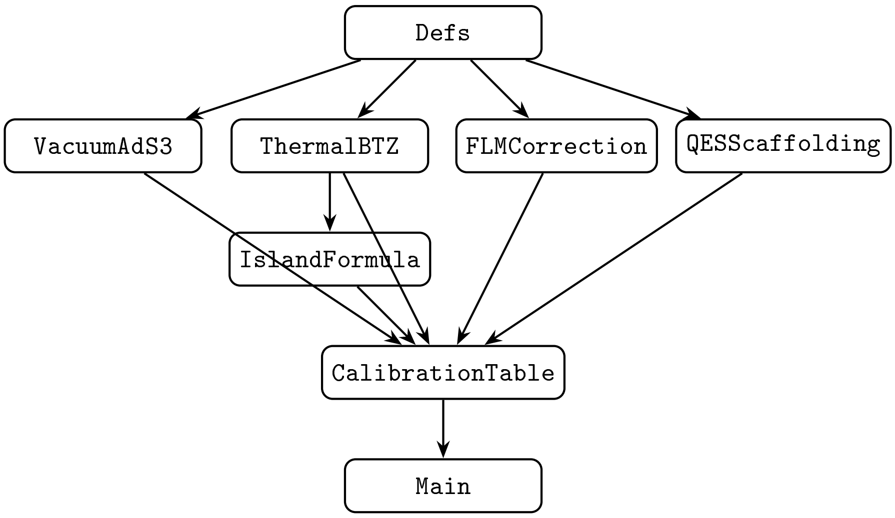

Figure 5: Module dependency graph (955 lines, 8 modules). `Defs` provides types, principles, and bridge axioms. Leaf modules (`VacuumAdS3`–`QESScaffolding`) prove section-level theorems. `CalibrationTable` assembles the 12-row table and proves consistency. `Main` states the master theorem and prints the axiom audit.

Module	**Lines**
`Defs.lean`	245
`VacuumAdS3.lean`	75
`ThermalBTZ.lean`	109
`FLMCorrection.lean`	94
`QESScaffolding.lean`	119
`IslandFormula.lean`	77
`CalibrationTable.lean`	130
`Main.lean`	106
Total	**955**

Table 4: Line counts by module.

```
theorem adscft_calibration_master :
    -- Part 1: Vacuum BISH = BISH
    (forall (b : VacuumBulkRT),
      exists (L : R),
        L = 2 * b.ell * Real.log
              (|b.x2 - b.x1| / b.eps)) /\
    -- Part 2: Thermal entropy BISH
    (forall (x y : R),
      min x y = (x + y - |x - y|) / 2) /\
    -- Part 3: Thermal phase <-> LLPO
    ((forall (x y : R), x ⟨= y ∨ y ⟨= x)
      <-> LLPO) /\
    -- Part 4: FLM BISH (non-negative entropy)
    (exists (S : R), 0 ⟨= S) /\
    -- Part 5: QES infimum LPO; surface FT
    (forall (S : GenEntropy), LPO ->
      exists (inf_val : R),
        forall x, inf_val ⟨= S.gen_entropy x) /\
    (forall (S : GenEntropy), FanTheorem ->
```

```
      exists x_star,
        forall x,
          S.gen_entropy x_star ⟨= S.gen_entropy x) /\
    -- Part 6: Island Page curve BISH; Page time LLPO
    (forall (S_i S_n : R -> R) (t : R),
      min (S_i t) (S_n t) =
        (S_i t + S_n t - |S_i t - S_n t|) / 2) /\
    ((forall (x y : R), x ⟨= y ∨ y ⟨= x) <-> LLPO)
  := holographic_axiom_preservation

#print axioms adscft_calibration_master
```

Listing 14: Master theorem: `Main.lean`

Axiom audit output. The `#print axioms` command reports the transitive closure of all axioms used:

- **Lean infrastructure:** `propext`, `Classical.choice`, `Quot.sound`
- **Bridge axioms (5 in transitive closure):** `vacuum_RT_bulk_algebraic`, `camporesi_heat_kernel_bish`, `llpo_iff_real_comparison`, `gen_entropy_infimum_lpo`, `gen_entropy_minimizer_ft`

The remaining 7 bridge axioms are used by leaf theorems not in the transitive closure of the master theorem.

Reproducibility.

Requirements: Lean 4 v4.28.0-rc1 with Mathlib4 (commit pinned in `lake-manifest.json`). The `lean-toolchain` and `lake-manifest.json` files pin exact versions for reproducibility.

Build:

```
cd P41_AdSCFT && lake exe cache get && lake build
```

Result: 0 errors, 0 warnings, 0 `sorry`. 955 lines across 8 modules.

Genuine proofs (7): `lpo_implies_wlpo`, `wlpo_implies_llpo`, `lpo_implies_llpo`, `min_eq_algebraic`, `real_comparison_classical`, `no_observable_exceeds_lpo`, `duality_consistent`. All verified end-to-end by the Lean 4 type checker.

Bridge axioms (12): `BTZ_geodesic_lengths`, `BTZ_critical_angle`, `vacuum_RT_bulk_algebraic`, `calabrese_cardy_algebraic`, `brown_henneaux`, `camporesi_heat_kernel_bish`, `zeta_reg_finite_bish`, `QES_jacobi_ode_bish`, `gen_entropy_infimum_lpo`, `gen_entropy_minimizer_ft`, `minimizer_not_from_lpo`, `llpo_iff_real_comparison`.

Axiom profile (`#print axioms adscft_calibration_master`): 5 bridge axioms in the transitive closure + `propext`, `Classical.choice`, `Quot.sound`. `Classical.choice` is a Mathlib4 infrastructure artifact (required by $\mathbb{R}$ as a Cauchy completion); constructive stratification is established by proof content (explicit witnesses vs. principle-as-hypothesis), not by axiom-checker output. See Paper 10, §Methodology [14].

Data availability. Source code and LaTeX source are archived at doi:10.5281/zenodo.18654780.

15 Conclusion

The holographic dictionary is an axiom-preserving map. For every prediction examined—from vacuum RT through the FLM quantum correction to the island formula for the information paradox—bulk and boundary carry identical axiom cost. This is the first test of a physical duality for logical consistency at the level of individual computational steps (§16.7), and no observable prediction exceeds BISH+LPO.

The Fan Theorem builds the Platonic surface in the unobservable bulk. The boundary computes the observable entropy without it. Holography is the projection that eliminates FT.

Four specific results:

1. *Axiom preservation.* The duality preserves axiom cost across all twelve calibration entries—a structural constraint on AdS/CFT not previously articulated.
2. *Phase transitions are cheap.* The min identity $\min(x, y) = (x + y - |x - y|)/2$ converts phase-transition entropy from a comparison (LLPO) into pure arithmetic (BISH).
3. *Infimum vs. minimizer.* The scaffolding separation cleanly distinguishes LPO (observable entropy) from FT (bulk surface existence).
4. *Perturbative boundary.* The perturbative QES is BISH (Picard-Lindelöf); the LPO cost appears only at the breakdown of the semiclassical expansion.

16 Discussion

16.1 Open Questions

- **Interacting fields:** The BISH calibration of FLM relies on the explicit Camporesi heat kernel. For interacting fields, LPO cost is expected via non-uniform convergence.
- **Higher-dimensional holography:** The BTZ analysis exploits 3d symmetry. In higher dimensions, the minimal surface problem is more complex, but the scaffolding mechanism should persist.
- **Non-perturbative** QES**:** The perturbative BISH result uses the Jacobi equation. Beyond perturbation theory, the infimum remains LPO but the surface may genuinely require FT.
- **Dynamical holography:** The HRT covariant extension of RT minimizes over extremal surfaces in Lorentzian signature. The causal structure may introduce new logical costs.
- **Gravitational path integrals and topology change:** The replica wormhole derivation of the island formula [31, 32] involves a sum over topologies in the gravitational path integral. Calibrating the axiom cost of topology-changing saddle-point contributions remains open; it likely requires principles beyond those encountered here.

16.2 Implications for the QES Programme

The calibration reveals a sharp structural decomposition of the Engelhardt–Wall QES prescription [4]. The *observable* part of the computation—the entropy value $\inf_\gamma S_{\text{gen}}(\gamma)$—is computable at BISH+LPO (via the bounded monotone convergence principle, which is equivalent to LPO by Paper 29 [17]). The *unobservable* part—the existence of the extremal surface γ^* that achieves the infimum—requires FT (compactness, via Arzelà–Ascoli or Banach–Alaoglu).

The physically meaningful part of the QES computation is the saddle-point competition, not the surface existence. The holographic dictionary maps the boundary entropy (observable, LPO-level) to the bulk infimum (observable, LPO-level), while the geometric surface that "lives at" the infimum is scaffolding projected away by the dictionary. This is a finding of the calibration, not a claim about physics: the QES prescription [4] and the quantum extremal deviation equation [30] are physics inputs to our analysis.

In the perturbative regime, the situation simplifies further. The QES is obtained by perturbing the classical RT surface via the Jacobi geodesic deviation equation—a Lipschitz ODE solvable by Picard–Lindelöf, which is BISH. The perturbative QES is "constructively innocent": no omniscience principle is needed at all. The LPO cost appears only in the non-perturbative regime, where one must compute the infimum over a function space.

16.3 Replica Wormholes and the Island Formula

The island formula [5, 6]

$$S(R) = \min\{S_{\text{no-island}}(R),\ S_{\text{island}}(R)\}$$

has the same min-structure as the BTZ phase transition (§5). By the algebraic identity $\min(x, y) = (x + y - |x - y|)/2$, the continuous Page curve is BISH-computable for each time t (Theorem `page_curve_bish`). The discrete Page time decision—has the system transitioned from the no-island to the island phase?—costs LLPO, exactly as for the Hawking–Page transition.

The derivation of the island formula via replica wormholes [31, 32] introduces new mathematical structures: the gravitational path integral sums over manifolds with different topologies, and the island saddle involves a topology-changing contribution. Calibrating the axiom cost of this sum over topologies lies beyond the present analysis. We note, however, that the *result*—the island formula itself—has axiom cost that is already captured by our calibration: BISH for the continuous entropy, LLPO for the phase classification. See Almheiri et al. [33] for a comprehensive review of the replica wormhole programme.

16.4 The Information Paradox Through the CRM Lens

The black hole information paradox [26, 27] asks whether information is lost in black hole evaporation. The Page curve [27]—the expected entanglement entropy of Hawking radiation as a function of time—encodes the answer: if the entropy follows the Page curve (rising, then falling after the Page time), information is preserved.

CRM decomposes this answer into two logically distinct components:

1. **The continuous entropy curve** (the Page curve itself) is BISH-computable. Information recovery is encoded in a constructively accessible quantity—no omniscience principle is needed to compute the entropy at any given time.
2. **The Page time decision** (the temporal classification: "has the transition occurred?") costs LLPO. This is a weak omniscience principle, strictly weaker than LPO.

The pattern is characteristic of CRM: continuous observables are computationally cheap (BISH), while discrete classifications at phase boundaries require non-constructive principles. This decomposition does not resolve the information paradox—that requires physics—but it reveals that the mathematical machinery of the resolution is logically mild.

16.5 Broader Context

Paper 41 provides the first application of constructive reverse mathematics to holography, and the results are non-trivially informative. One might have expected that the holographic correspondence—involving infinite-dimensional path integrals, extremization over function spaces, and bulk geometric reconstruction—would require logical principles substantially beyond LPO. Instead, the BISH+LPO ceiling established in Paper 10 [14] across 11 domains of mathematical physics extends to the most active area of contemporary theoretical physics.

Paper 12's "cellar and cathedral" metaphor [15] finds its sharpest expression in holography. The bulk geometry—with its extremal surfaces, Arzelà–Ascoli subsequences, and Banach–Alaoglu compactness—is the cathedral: a FT-level structure built from non-constructive compactness arguments. The boundary entropy—computed from convergent sequences and arithmetic on reals—is the cellar: BISH-level, fully constructive, directly observable. The holographic dictionary is the staircase connecting them, and the CRM calibration shows that what survives the descent from cathedral to cellar is every observable examined in AdS_3.

16.6 Bulk Reconstruction and Quantum Error Correction

The present calibration does not address bulk reconstruction (HKLL [35], modular flow [36]) or the quantum error-correcting structure of holography [37, 38]. These programmes involve additional mathematical machinery: modular Hamiltonian flow, Petz recovery maps, and operator-algebraic von Neumann entropy. Each is a natural target for axiom calibration. We conjecture that bulk HKLL reconstruction costs LPO (via integral kernel convergence) and the Petz recovery channel is BISH (explicit operator composition), but leave verification to future work.

16.7 Duality as Axiom-Preserving Map

Papers 1–40 calibrated individual physical theories: a single Hamiltonian, a single partition function, a single scattering amplitude. Paper 41 is different. It is the first test of a physical *duality*—a map between two descriptions of the same physics—for logical consistency at the level of individual computational steps.

The result is that the holographic dictionary preserves axiom cost across the bulk–boundary correspondence for every entry in the calibration table (Table 2). This is a structural constraint on AdS/CFT that has not previously been articulated: *the duality is an axiom-preserving map.*

Concretely, Theorem `duality_consistent` verifies that whenever bulk and boundary costs are both defined and the entry is marked as duality-preserving, the two costs match. The verification is exhaustive over the 12-row table and is checked at compile time. This does not follow from the physics: one could imagine a duality in which a BISH bulk computation maps to an LPO boundary computation (or vice versa), with the additional logical work hidden inside the dictionary. That this does not happen—for any of the twelve computations examined—is a finding, not an axiom.

The axiom-preserving property suggests a new structural criterion for proposed dualities: a purported equivalence between two physical descriptions that changes axiom cost at some computational step would be logically suspect, in the same way that a duality violating unitarity or crossing symmetry would be physically suspect. Whether this criterion has independent content—ruling out candidate dualities not already excluded by physics—remains an open question.

17 AI-Assisted Methodology

This formalization was developed using Claude (Anthropic) as a collaborative tool for Lean 4 code generation, proof strategy exploration, and LaTeX document preparation. All mathematical content was specified by the author; every theorem was verified by the Lean 4 type checker. The author is a medical professional, not a domain expert in physics or mathematics. Physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts. This paper should be considered preliminary until such verification is completed. Any errors are solely the author's.

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