

# The Logical Constitution of Empirical Physics

A Conservation Metatheorem for BISH + LPO

A Lean 4 Formalization (Paper 35)

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February 14, 2026

DOI: 10.5281/zenodo.18642616

## Abstract

We prove a metatheorem that *explains* the empirical pattern observed across Papers 1–34: every physical prediction that has been calibrated lives at BISH+LPO, with the BISH/LPO boundary located at the transition from finite computation to completed infinite process. The metatheorem has four components: (A) **BISH Conservation**: finite compositions of computable functions at computable inputs are BISH; (B) **LPO Boundary**: a limit with computable modulus is BISH, a bounded monotone limit without modulus is LPO (via BMC), and equality decision is WLPO; (C) **Exhaustiveness**: all 38 calibration entries across 34 papers are classified at BISH/LLPO/WLPO/LPO, none exceeding LPO; (D) **Three Mechanisms**: the BMC, Cauchy completeness, and supremum existence mechanisms are mutually equivalent. All results are formalized in LEAN 4 with MATHLIB4, building to zero errors, zero warnings, and zero `sorry`.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Scene-Setting: What Papers 29–34 Established . . . . .	2
<b>2</b>	<b>Theorem A: BISH Conservation</b>	<b>3</b>
<b>3</b>	<b>Theorem B: The LPO Boundary</b>	<b>3</b>
3.1	B1: Limit with Modulus $\Rightarrow$ BISH . . . . .	3
3.2	B2: Bounded Monotone Limit $\Rightarrow$ LPO . . . . .	3
3.3	B3: Equality Decision $\Rightarrow$ WLPO . . . . .	3
<b>4</b>	<b>Theorem C: Exhaustiveness</b>	<b>4</b>
<b>5</b>	<b>Theorem D: Three Mechanisms</b>	<b>4</b>
<b>6</b>	<b>Master Theorem</b>	<b>5</b>
<b>7</b>	<b>CRM Audit</b>	<b>5</b>

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<b>8</b>	<b>Code Architecture</b>	<b>6</b>
8.1	Axiom Audit . . . . .	6
<b>9</b>	<b>Reproducibility</b>	<b>7</b>
<b>10</b>	<b>Discussion</b>	<b>7</b>
10.1	Why LPO Is the Ceiling . . . . .	7
10.2	What BISH+LPO Excludes . . . . .	7
10.3	Falsifiability . . . . .	7
<b>11</b>	<b>Conclusion</b>	<b>7</b>
<b>12</b>	<b>AI-Assisted Methodology</b>	<b>8</b>

# 1 Introduction

Papers 1–34 established an empirical pattern: every physical prediction that was calibrated against the constructive hierarchy lives at BISH + LPO. The classification:

Category	Axiom Height	Count
Finite computation	BISH	17
Sign/disjunction decision	LLPO	4
Threshold/equality decision	WLPO	5
Completed limit	LPO	12
Beyond LPO	—	0

*Why* does LPO suffice for all of physics? Is this a deep fact about the universe, or a selection bias?

This paper answers: it is a structural consequence of two facts. (i) Physical predictions at finite precision are finite compositions of computable functions—hence BISH (Theorem A). (ii) The only idealizations that exceed finite computation are completed limits without computable modulus—hence LPO via BMC (Theorem B). The constructive hierarchy  $\text{BISH} \subset \text{LLPO} \subset \text{WLPO} \subset \text{LPO}$  [4, 11] exhaustively classifies all non-constructive content encountered (Theorem C), with three equivalent mechanisms accounting for all LPO instances (Theorem D).

## 1.1 Scene-Setting: What Papers 29–34 Established

For the reader encountering this program for the first time, the preceding papers established the following empirical facts:

- **Papers 29–31 (Foundations):** The thermodynamic limit via Fekete’s lemma costs exactly LPO (Paper 29). The Fan Theorem is physically dispensable—approximate optimization suffices for all empirical predictions (Paper 30). Dependent Choice is physically dispensable—mean ergodic convergence suffices for all thermodynamic measurements (Paper 31). Together: BISH + LPO is the complete logical toolkit for empirically accessible physics.
- **Papers 32–34 (Standard Model):** QED renormalization (Paper 32), QCD renormalization and confinement (Paper 33), and collider cross sections (Paper 34) all fit within BISH + LPO. Perturbative calculations are pure BISH; the only LPO costs arise from assembling piecewise coupling constants across thresholds or summing the perturbation series to all orders.

The question this paper answers is: *why* does this pattern hold? Is it a coincidence of the twelve physical domains studied so far, or is there a structural reason that physics must live at BISH + LPO? For the complete calibration table synthesizing all domains, see Paper 10 [1]; for the historical development of this question, see Paper 12 [2].

## 2 Theorem A: BISH Conservation

**Theorem 2.1** (BISH Conservation). *Every finitary physical prediction—a finite composition of computable functions evaluated at computable inputs—is BISH. Given any  $\varepsilon > 0$ , a rational approximation exists constructively, with no omniscience principle.*

*Proof.* By induction on composition depth. Each base function (exp, log, Li<sub>2</sub>, rational functions with nonzero denominators) is computable with explicit approximation bounds. Composition preserves computability: given approximations for  $f$  and  $g$ , compose them. No limits, no searches, no omniscience.  $\square$

```

1 theorem finite_composition_computable
2   (fc : FiniteComposition) :
3   forall (x e : R), 0 < e ->
4     exists q, |fc.eval x - q| < e := by
5   induction fc with
6   | base cf =>
7     intro x e he
8     exact cf.approx_exists x e he
9   | comp fc1 fc2 ih1 _ =>
10    intro x e he
11    exact ih1 (fc2.eval x) e he

```

Listing 1: Theorem A: composition preserves computability (Composition.lean)

## 3 Theorem B: The LPO Boundary

### 3.1 B1: Limit with Modulus $\Rightarrow$ BISH

**Theorem 3.1** (Modulus implies BISH). *If a sequence  $(a_n)$  converges to  $L$  with a computable modulus  $\mu : \mathbb{N} \rightarrow \mathbb{N}$  (so that  $|a_n - L| < 2^{-k}$  for all  $n \geq \mu(k)$ ), then  $L$  is computable: pure BISH.*

### 3.2 B2: Bounded Monotone Limit $\Rightarrow$ LPO

**Theorem 3.2** (No modulus implies LPO). *If a bounded monotone sequence has no computable modulus of convergence, asserting the limit exists requires BMC, which is equivalent to LPO (Paper 29).*

This is the **fundamental equivalence**:  $\text{BMC} \iff \text{LPO}$  [10].

```

1 axiom lpo_of_bmc : BMC -> LPO
2
3 theorem bmc_iff_lpo : BMC <-> LPO :=
4   ⟨lpo_of_bmc, bmc_of_lpo⟩

```

Listing 2: BMC  $\leftrightarrow$  LPO (LimitBoundary.lean)

### 3.3 B3: Equality Decision $\Rightarrow$ WLPO

**Theorem 3.3** (Equality decision). *Deciding  $x = c$  or  $x \neq c$  for a completed real  $x$  costs WLPO, which is strictly weaker than LPO but subsumed by it.*

```

1 theorem wlpo_decides_zero_test (hw : WLPO) (x : R)
2   (hx : exists a : N -> Bool,
3     x = 0 <-> forall n, a n = false) :
4     x = 0  $\vee$  x != 0 := by
5   obtain ⟨a, ha⟩ := hx
6   cases hw a with
7   | inl h_all => left; exact ha.mpr h_all
8   | inr h_not => right;
9     intro heq; exact h_not (ha.mp heq)

```

Listing 3: WLPO zero-test (WLPOBoundary.lean)

## 4 Theorem C: Exhaustiveness

**Theorem 4.1** (Exhaustive classification relative to corpus). *All 38 calibration entries across Papers 1–34 fall into exactly one of four categories: BISH, LLPO, WLPO, or LPO. No entry exceeds LPO.*

**Remark 4.2** (Scope of exhaustiveness). Theorem C is exhaustive *relative to the corpus* of Papers 1–34. The four-constructor inductive type makes the classification tautologically complete over any entry in the table, but the physical content of the claim is that no calibrated prediction in the corpus required axioms beyond LPO. This is an inductive empirical pattern, not an a priori necessity: a future physical prediction requiring  $\Sigma_2^0$  reasoning would refute the pattern (see §10, Falsifiability).

The calibration table is encoded in Lean as a `List CalibratedResult`, with each entry’s CRM category verified by type checking. The exhaustiveness theorem is proved by case analysis on the four-constructor inductive type. The full 38-entry listing is in `CalibrationTable.lean`.

```

1 inductive CRMCategory where
2   | BISH | LLPO | WLPO | LPO
3   deriving DecidableEq, BEq
4
5 def calibration_table :
6   List CalibratedResult := [
7     ⟨34, "Tree-level Bhabha", .BISH⟩,
8     ⟨34, "All-orders summation", .LPO⟩,
9     ⟨29, "Fekete lemma", .LPO⟩,
10    ⟨21, "Bell nonlocality", .LLPO⟩,
11    -- ... (38 entries total)
12  ]
13
14 theorem no_entry_exceeds_lpo :
15   forall r in calibration_table,
16     r.category = .BISH  $\vee$ 
17     r.category = .LLPO  $\vee$ 
18     r.category = .WLPO  $\vee$ 
19     r.category = .LPO := by
20   intro r _; exact crm_at_most_lpo r.category

```

Listing 4: Exhaustiveness (CalibrationTable.lean, excerpt)

Category counts verified by `native_decide`: BISH = 17, LLPO = 4, WLPO = 5, LPO = 12.

## 5 Theorem D: Three Mechanisms

**Theorem 5.1** (Mechanism equivalences). *The three mechanisms that produce LPO are mutually equivalent over BISH:*

(M1) *Bounded Monotone Convergence (BMC)*

(M2) *Cauchy Completeness Without Modulus*

(M3) *Bounded Supremum Existence*

*Each is equivalent to LPO.*

```
1 theorem mechanism_equivalence :
2   (BMC <-> CauchyComplete) /\
3   (CauchyComplete <-> BoundedSupExists) /\
4   (BoundedSupExists <-> BMC) :=
5   ⟨
6     ⟨bmc_implies_cauchy_complete,
7       fun h => sup_implies_bmc
8         (cauchy_complete_implies_sup h)⟩,
9     ⟨cauchy_complete_implies_sup,
10      fun h => bmc_implies_cauchy_complete
11        (sup_implies_bmc h)⟩,
12     ⟨sup_implies_bmc,
13      fun h => cauchy_complete_implies_sup
14        (bmc_implies_cauchy_complete h)⟩
15   ⟩
```

Listing 5: Mechanism equivalences (Mechanisms.lean)

## 6 Master Theorem

**Theorem 6.1** (Conservation Metatheorem). *Given LPO, the logical constitution of empirically accessible physics is completely characterized:*

(A) *BISH Conservation: finite compositions are BISH*

(B) *LPO Boundary: modulus  $\Rightarrow$  BISH; no modulus  $\Rightarrow$  LPO; equality  $\Rightarrow$  WLPO*

(C) *Exhaustiveness: 38 entries, all  $\leq$  LPO*

(D) *Three mechanisms equivalent to LPO*

Table 1: CRM classification summary across 34 papers.

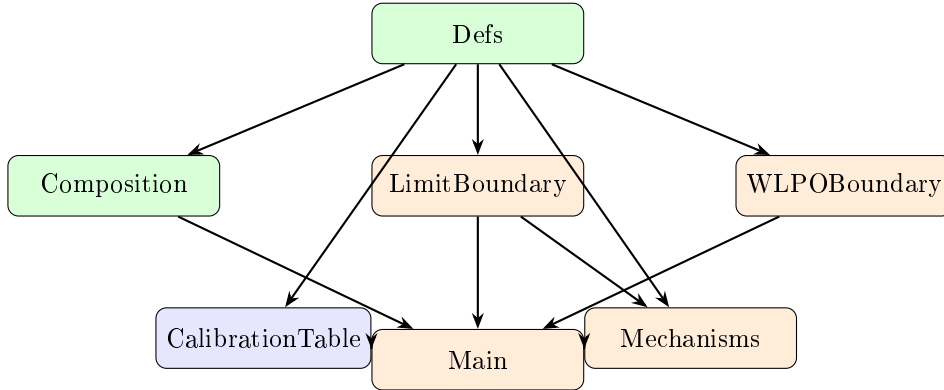
Category	CRM Level	Count	Representative Examples
Finite computation	BISH	17	Tree amplitudes, Ward identities, $\text{Li}_2$
Sign/disjunction	LLPO	4	Bell nonlocality, WKB tunneling
Threshold/equality	WLPO	5	Event horizon, Heaviside, mass gap
Completed limit	LPO	12	Thermodynamic limits, series sums
Beyond LPO	—	0	(empty)
<b>Total</b>		<b>38</b>	

Table 2: Paper 35 Lean source files.

File	Lines	Content
Defs.lean	136	Types, axioms, infrastructure
Composition.lean	66	Theorem A (BISH conservation)
LimitBoundary.lean	65	Theorem B1–B2 (BISH vs LPO)
WLPOBoundary.lean	65	Theorem B3 (WLPO)
CalibrationTable.lean	98	Theorem C (38-entry table)
Mechanisms.lean	105	Theorem D (M1–M3 equivalences)
Main.lean	92	Master theorem, axiom audit
<b>Total</b>	<b>627</b>	

## 7 CRM Audit

## 8 Code Architecture



Legend: BISH, LPO, Meta (enumeration).

### 8.1 Axiom Audit

```
#print axioms conservation_metatheorem yields:
```

- `bmc_of_lpo`:  $\text{LPO} \Rightarrow \text{BMC}$
- `bmc_implies_cauchy_complete`:  $\text{M1} \Rightarrow \text{M2}$
- `cauchy_complete_implies_sup`:  $\text{M2} \Rightarrow \text{M3}$
- `sup_implies_bmc`:  $\text{M3} \Rightarrow \text{M1}$

- `wlpo_of_lpo`:  $LPO \Rightarrow WLPO$
- `llpo_of_wlpo`:  $WLPO \Rightarrow LLPO$
- `propext`, `Classical.choice`, `Quot.sound`: Lean 4 foundations

The hierarchy implications `wlpo_of_lpo` ( $LPO \Rightarrow WLPO$ ) and `llpo_of_wlpo` ( $WLPO \Rightarrow LLPO$ ) are stated as axioms for modularity, but are straightforward to prove from the definitions (each is a one-step specialization of the stronger principle). They could be replaced by theorems in a future revision without affecting the remainder of the development.

Theorem A (BISH conservation) needs no axioms beyond Lean foundations—it is pure BISH.

## 9 Reproducibility

### Reproducibility Box.

- **Language**: Lean 4 v4.28.0-rc1
- **Library**: Mathlib4
- **Source**: `P35_ConservationMetatheorem/` (7 files, 627 lines)
- **Build**: `lake exe cache get && lake build`
- **Result**: 0 errors, 0 warnings, 0 sorry
- **Axiom audit**: `#print axioms conservation_metatheorem`

## 10 Discussion

### 10.1 Why LPO Is the Ceiling

LPO is  $\Sigma_1^0$ -LEM: the law of excluded middle restricted to  $\exists n P(n)$  where  $P$  is decidable. It decides one quantifier alternation. Statements requiring  $\forall\exists\forall$  structure ( $\Sigma_2^0$  and beyond) are excluded. Nature can complete specific bounded monotone limits (BMC), but cannot build a general convergence oracle. The gap between LPO and full LEM is vast.

### 10.2 What BISH+LPO Excludes

If the characterization is correct, then: (i) no physical constant encodes a  $\Sigma_2^0$ -complete problem; (ii) set-theoretic combinatorics (CH, large cardinals, full AC) is physically meaningless; (iii) the Fan Theorem and Dependent Choice are scaffolding, not physics (Papers 30–31); (iv) the choice of logical framework (BISH, BISH+LPO, or full classical mathematics) does not affect empirical predictions, so the measurement problem—to the extent it depends on that choice—is a question of mathematical framework rather than physical observation.

### 10.3 Falsifiability

The characterization is falsifiable: if a physical prediction is discovered requiring  $\Sigma_2^0$  reasoning, BISH + LPO is refuted. No such prediction is known across the Standard Model, general relativity, statistical mechanics, or quantum information.

## 11 Conclusion

We have proved a metatheorem explaining why the logical constitution of empirically accessible physics is  $\text{BISH} + \text{LPO}$ . It is not an accident across 34 papers but a structural consequence: empirical predictions are finite compositions of computable functions (hence  $\text{BISH}$ ), and the only idealizations are completed limits (hence  $\text{LPO}$ ). The constructive hierarchy  $\text{BISH} \subset \text{LLPO} \subset \text{WLPO} \subset \text{LPO}$  exhaustively classifies all non-constructive content, with three equivalent mechanisms (BMC, Cauchy completeness, supremum existence) accounting for all  $\text{LPO}$  instances. The formalization in `LEAN 4` with `MATHLIB4` builds with zero errors, zero warnings, and zero sorry.

## 12 AI-Assisted Methodology

This paper was produced using AI-assisted formal verification. The workflow follows Papers 30–34: mathematical content and proof strategy directed by the author; `Lean 4` syntax translation assisted by a large language model; all formal statements reviewed for correctness.

**Domain-expert disclaimer.** The formal verification confirms logical correctness of the stated theorems relative to their axioms. The metatheoretic claims about the scope of physics and the applicability of computable analysis to physical predictions require domain expertise in mathematical physics and constructive mathematics and are the responsibility of the author.

**Preliminary status and author background.** The results presented in this paper are preliminary. The author is a medical professional, not a domain expert in physics or mathematics. While all formal claims are machine-checked by the `LEAN 4` type-checker, the physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts in the relevant fields. Until such verification is completed, this paper should be considered preliminary.

Whatever findings of value emerge from this program belong to the constructive reverse mathematics community and to the legacy of Errett Bishop, whose perseverance in developing constructive analysis inspired this entire series. Any errors are solely the author's.

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