Analytic Rank Stratification of Mixed Motives

Completing the DPT Framework

Paper 60, Constructive Reverse Mathematics Series

Paul Chun-Kit Lee

New York University, Brooklyn, NY*

February 2026

Abstract

We prove that the DPT framework for numerical equivalence on pure motives is complete: Axioms 1–3 (decidable morphisms, integrality, Archimedean polarization) together with automatic de Rham decidability at finite primes (Paper 59) suffice. No mixed motive axiom is needed because numerical equivalence is a property of the quotient $\mathrm{CH}^*(X) \twoheadrightarrow \mathrm{NS}^*(X)$, and the extension groups Ext^1 governing the kernel $\mathrm{CH}^*(X)_{\mathrm{hom}}$ are invisible to this quotient.

We then initiate the extended framework for rational equivalence by proving an *analytic rank stratification theorem*: the logical complexity of computing $\mathrm{Ext}^1(\mathbb{Q}(0), M)$ is determined by the order of vanishing $r = \mathrm{ord}_{s=s_0} L(M, s)$.

r	Decidability	Mechanism
0	BISH	$\mathrm{Ext}^1 = 0$; verify $L(M, s_0) \neq 0$ to finite precision
1	BISH	1-dim regulator; Bloch–Kato + Northcott bound the search
≥ 2	MP	Covolume $\not\Rightarrow$ basis vector bound (Minkowski)

The rank ≥ 2 obstruction is structural: by Minkowski's geometry of numbers, lattice covolumes do not bound individual basis vectors in dimension ≥ 2. Removing this obstruction requires Lang's Height Lower Bound Conjecture (open).

CRM classification: $r = 0$: BISH; $r = 1$: BISH (conditional on Bloch–Kato); $r \geq 2$: MP.

Formalization: None. This paper is LaTeX and PDF only; the arguments are purely mathematical and do not admit formalization in the integer arithmetic framework of Papers 50–59.

1 Introduction

1.1 Main results

This paper has two results.

(A) DPT completeness for numerical equivalence (Theorem 2.1). The decidability of numerical equivalence on $\mathrm{CH}^*(X)$ for smooth projective varieties over $\mathbb{Q}$ requires only Axioms 1–3 of Paper 50 [1] plus de Rham decidability at finite primes (Paper 59 [6]). No mixed motive axiom is needed.

(B) Analytic rank stratification for Ext^1 (Theorem 4.1). For a motive M over $\mathbb{Q}$ possessing the Northcott property, the logical complexity of constructively computing a basis for $\mathrm{Ext}^1(\mathbb{Q}(0), M)$ is stratified by the analytic rank $r = \mathrm{ord}_{s=s_0} L(M, s)$: $r = 0$ and $r = 1$ are BISH-decidable; $r \geq 2$ requires MP.

*LaTeX source and PDF: `https://doi.org/10.5281/zenodo.18735873`. This paper has no Lean formalization; see §1.6.

1.2 CRM primer

Constructive reverse mathematics (CRM) [7, 8] calibrates the logical strength of mathematical theorems by identifying the weakest omniscience or choice principle needed for their proof. The hierarchy relevant to this paper:

- BISH (Bishop-style constructive mathematics): no omniscience principles; all existential claims come with explicit witnesses or bounded searches.
- MP (Markov's Principle): if a computation does not fail to terminate, then it terminates. Equivalently, $\forall f : \mathbb{N} \to \{0, 1\}$, $\neg\neg(\exists n. f(n) = 1) \to \exists n. f(n) = 1$. MP is strictly weaker than LPO but strictly stronger than BISH.
- LPO (Limited Principle of Omniscience), WLPO, LLPO: these appear in the pure motive programme (Papers 50–59) but are not needed here.

The CRM methodology asks: for a given theorem or computation, what is the *minimal* principle required? This paper identifies MP as a sufficient principle for Ext^1 decidability at rank ≥ 2, and argues that BISH alone does not suffice via the standard search method.

1.3 The DPT framework

Paper 50 [1] introduced three axioms for decidability of numerical equivalence in polarized Tannakian categories:

Ax 1. Decidable morphisms (linear algebra, unconditional).

Ax 2. Integrality (algebraic integers, unconditional).

Ax 3. Archimedean polarization (Rosati positive-definiteness, $u(\mathbb{R}) = 1$).

Paper 59 [6] established de Rham decidability at finite primes: the precision bound $N_M = v_p(\#E(\mathbb{F}_p))$ is computable in BISH. Paper 54 [2] posed the question of whether a further "Axiom 5" is needed for p-adic decidability; Paper 59 showed it is not (for geometric representations).

This paper closes the pure motive programme by proving that Axioms 1–3 plus de Rham decidability are *complete*: no further axioms are needed for numerical equivalence. It then opens the mixed motive frontier.

1.4 Trajectory: Papers 50–59 to Paper 60

The pure motive programme followed two threads:

- **Thread 1** (Axiom 1 boundary): Papers 56–58 [3, 4, 5] investigated exotic Weil classes and the conductor formula at codimension ≥ 2.
- **Thread 2** (Axiom 3 boundary): Paper 59 [6] healed the p-adic fracture point (Paper 54 [2]) by computing the precision bound.

Paper 60 draws the conclusion: the pure motive programme is complete. It then initiates the mixed motive programme, where the relevant equivalence relation is *rational* rather than numerical, and the relevant invariant is Ext^1 rather than the intersection pairing.

1.5 State of the art

The Bloch–Kato conjecture [9] predicts that L-values encode the arithmetic of motives. For elliptic curves, the BSD conjecture is a special case. Kolyvagin [11] proved finiteness of $E(\mathbb{Q})$ when $\mathrm{ord}_{s=1}L(E,s) = 0$; Gross–Zagier [12] related $L'(E,1)$ to Heegner point heights when $\mathrm{ord}_{s=1}L(E,s) = 1$. No comparable result exists for rank ≥ 2: the regulator gives a lattice covolume, but no method is known to bound individual generators.

The CRM lens applied here is, to our knowledge, new: no prior work identifies the analytic rank as the parameter governing the logical complexity (BISH vs. MP) of arithmetic generator extraction.

1.6 Caveats

(i) The completeness theorem (Theorem A) is a structural observation: numerical equivalence factors through pure motives, so mixed motive data is invisible. The proof is short.

(ii) The rank stratification (Theorem B) is conditional on the Bloch–Kato conjecture and effective height bounds (Silverman). The rank ≥ 2 obstruction via Minkowski is unconditional.

(iii) This paper has no Lean formalization. The arguments involve real-valued L-functions, Arakelov heights, and lattice geometry—none of which admit the integer arithmetic treatment of Papers 50–59. A formalization would require substantial real analysis infrastructure beyond current Mathlib scope.

(iv) The contribution is framing: the CRM identification of MP as the precise logical cost of rank ≥ 2 generator extraction. The underlying arithmetic (Bloch–Kato formula, Minkowski bound) is standard material. The novelty, if any, lies in recognizing that the geometry-of-numbers obstruction is exactly Markov's Principle.

(v) The paper is self-contained in the sense that it does not depend on unproved conjectures beyond Bloch–Kato (which is needed only for $r = 1$). The $r = 0$ and $r \geq 2$ arguments are unconditional (given the L-function value and the Northcott property).

2 Completeness of the pure motive framework

Theorem 2.1 (DPT completeness for numerical equivalence)**.** *Let X be a smooth projective variety over $\mathbb{Q}$. The decidability of numerical equivalence on $\mathrm{CH}^*(X)$ is governed entirely by the pure motive $h^*(X)$ and requires only:*

Ax 1. *Decidable morphisms in the category of pure motives.*

Ax 2. *Integrality of Frobenius eigenvalues.*

Ax 3. *Archimedean polarization (Rosati positive-definiteness, $u(\mathbb{R}) = 1$).*

dR. *De Rham decidability at finite primes (automatic for geometric representations; Paper 59 [6]).*

No mixed motive data (Ext^1, intermediate Jacobians, regulators) is required.

Proof. Numerical equivalence on $\mathrm{CH}^k(X)$ is defined by the intersection pairing: $\alpha \sim_{\mathrm{num}} 0$ iff $\deg(\alpha \cdot \beta) = 0$ for all $\beta \in \mathrm{CH}^{\dim X - k}(X)$. This pairing factors through the cycle class map to cohomology and is computed by traces of endomorphisms in the pure motive $h^*(X)$. The kernel of the cycle class map—homologically trivial cycles—is annihilated by the intersection pairing and hence invisible to numerical equivalence.

The extension groups $\mathrm{Ext}^1(M, N)$ in the mixed motive category govern the structure of this kernel (Abel–Jacobi images, Griffiths groups, Mordell–Weil groups). Since numerical equivalence projects away from the kernel, no Ext^1 computation is needed.

Axioms 1–3 and de Rham decidability provide the complete set of certificates for evaluating the intersection pairing: Axiom 1 reduces the pairing to linear algebra, Axiom 2 ensures integrality, Axiom 3 provides the Archimedean bound, and de Rham decidability provides the p-adic bound $N_M = v_p(\det(1 - \varphi))$. □

Remark 2.2. The completeness theorem closes the pure motive programme as formulated in Paper 50. What follows is the first step of an extended framework governing rational equivalence and the mixed motive category.

3 The mixed motive frontier: Ext^1 decidability

3.1 The problem

For a pure motive M, the group $\mathrm{Ext}^1(\mathbb{Q}(0), M)$ in the (conjectural) category of mixed motives carries arithmetic information:

- $M = h^1(E)$, E an elliptic curve: $\mathrm{Ext}^1 \cong E(\mathbb{Q}) \otimes \mathbb{Q}$ (Mordell–Weil group).
- $M = h^1(A)$, A an abelian variety: $\mathrm{Ext}^1 \cong A(\mathbb{Q}) \otimes \mathbb{Q}$.
- $M = h^2(X)(1)$, X a surface: Ext^1 relates to the Griffiths group.
- $M = \mathbb{Q}(n)$: Ext^1 relates to algebraic K-theory [16].

The Bloch–Kato conjecture [9] predicts that the leading Taylor coefficient $L^*(M, 0)$ of the L-function encodes the "size" of Ext^1 via

$$L^*(M, 0) \sim \frac{\prod_v c_v(M) \cdot |\mathrm{III}(M)| \cdot R(M)}{|H^0(M)| \cdot |H^0(M^*(1))|}$$

where $R(M)$ is the regulator (covolume of the motivic lattice under the height pairing) and $\mathrm{III}(M)$ is the Tate–Shafarevich group.

The CRM question: given $L^*(M, 0)$, can one *constructively* extract the arithmetic generators of Ext^1?

3.2 The Northcott prerequisite

A motive M is said to possess the *Northcott property* if, for any $B > 0$, the set of motivic points of naïve height $\leq B$ is finite and effectively enumerable. Elliptic curves and abelian varieties satisfy this unconditionally (Northcott's theorem [14] for the Weil height). K3 surfaces and higher K-theory groups lack proven Northcott properties for the relevant cycle spaces.

The Northcott property is the structural prerequisite that converts a height bound into a finite search. Without it, even rank 1 computations become unbounded.

4 The analytic rank stratification

Theorem 4.1 (Analytic rank stratification)**.** *Let M be a motive over $\mathbb{Q}$ possessing the Northcott property, and let $r = \mathrm{ord}_{s=s_0} L(M, s)$. The logical complexity of constructively computing a basis for $\mathrm{Ext}^1(\mathbb{Q}(0), M)$ is stratified by r:*

r	*Decidability*	*Mechanism*
0	BISH	$\mathrm{Ext}^1 = 0$*; verify* $L(M, s_0) \neq 0$ *to finite precision*
1	BISH	$R(M) = \hat{h}(P)$*; BK bounds height; Northcott bounds search*
≥ 2	MP	$R(M) = \det(\langle P_i, P_j \rangle)$*; volume* $\not\Rightarrow$ *basis vector bound*

Proof. We treat each case.

Case $r = 0$. The Bloch–Kato conjecture predicts $\mathrm{Ext}^1(\mathbb{Q}(0), M) \otimes \mathbb{Q} = 0$ (the motivic rank is zero). Equivalently, the Selmer group is finite. The L-value $L(M, s_0)$ is a nonzero complex number, computable to arbitrary precision (for automorphic M). To verify $L(M, s_0) \neq 0$, compute it to precision $\varepsilon < |L(M, s_0)|$. This requires an *effective lower bound* on $|L(M, s_0)|$ to set the precision target; without such a bound, the computation becomes an unbounded search (and would require MP, not BISH). Effective lower bounds are available in many cases via subconvexity or non-vanishing results. BISH-decidable, conditional on effective lower bounds for $|L(M, s_0)|$.

Case $r = 1$. The regulator $R(M)$ is the canonical height $\hat{h}(P)$ of a single generator P. The Bloch–Kato formula determines $R(M) = L^*(M, s_0)/C$ where C is a product of local factors and $|\text{Ш}|$. (For elliptic curves of analytic rank 1, Kolyvagin's Euler system gives effective bounds on $|\text{Ш}|$, so C is computable.) The key steps:

(i) In rank 1, $R(M) = \hat{h}(P)$ (the 1×1 Gram determinant is the height itself), so $R(M)$ determines $\hat{h}(P)$ exactly.

(ii) The Silverman height difference bound [13] $|\hat{h}(P) - h_{\mathrm{naive}}(P)| \leq c(M)$ converts this to a bound on naïve height: $h_{\mathrm{naive}}(P) \leq R(M) + c(M)$.

(iii) By Northcott [14], there are finitely many points of bounded naïve height, and they are effectively enumerable.

(iv) Search this finite set for a non-torsion point. Termination is guaranteed because the set is finite.

This is a bounded computation. BISH-decidable (conditional on Bloch–Kato and effective Silverman bounds).

Case $r \geq 2$. The regulator $R(M) = \det(\langle P_i, P_j \rangle_{1 \leq i,j \leq r})$ is the Gram determinant of the Néron–Tate height pairing on the rank-r Mordell–Weil lattice. By Minkowski's theorem on successive minima: for a lattice $\Lambda \subset \mathbb{R}^r$ of covolume V,

$$\lambda_1 \cdots \lambda_r \leq \gamma_r^{r/2} V$$

where λ_i are the successive minima and γ_r is the Hermite constant. Critically, this bounds the *product* of the successive minima, not the *maximum*. In dimension $r \geq 2$, the shortest vector λ_1 can approach zero while $\lambda_r \to \infty$, maintaining fixed covolume. No finite bound on λ_r (hence on the naïve height of the r-th generator) follows from knowing V.

An algorithm that enumerates rational points by ascending naïve height will eventually find all r generators (assuming finiteness of Ш), but the termination time is unbounded. This is exactly Markov's Principle: the computation halts if the generators exist, but no a priori bound on the halting time is available.

Resolving rank ≥ 2 in BISH would require an effective lower bound on λ_1 (the shortest vector in the Mordell–Weil lattice), which is the content of Lang's Height Lower Bound Conjecture [15]—an open problem. $\square$

5 Examples

5.1 Rank 0: $E = X_0(11)$

$L(E, 1) = 0.2538\ldots \neq 0$. By Kolyvagin's theorem [11] (conditional on modularity, now proved), $E(\mathbb{Q})$ is finite. In fact $E(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z}$. The verification that $L(E, 1) \neq 0$ is a bounded real computation. BISH.

5.2 Rank 1: $E = \texttt{37a1}$

$L(E, 1) = 0$, $L'(E, 1) \neq 0$. Analytic rank 1. The generator is $P = (0, 0)$ with $\hat{h}(P) = 0.0511\ldots$ The Bloch–Kato / Gross–Zagier formula [12] determines $\hat{h}(P)$ from $L'(E, 1)$. The Silverman bound [13] gives $|h_{\text{naive}}(P) - \hat{h}(P)| \leq c$ for an explicit constant c. The naïve height search space is finite by Northcott. BISH.

5.3 Rank 2: $E = \texttt{389a1}$

$\mathrm{ord}_{s=1} L(E, s) = 2$. Generators: $P_1 = (0, 0)$, $P_2 = (-1, 1)$. The regulator $R = \det \begin{pmatrix} \hat{h}(P_1) & \langle P_1, P_2 \rangle \\ \langle P_1, P_2 \rangle & \hat{h}(P_2) \end{pmatrix}$ is known, but R alone provides no upper bound on $\max(\hat{h}(P_1), \hat{h}(P_2))$. Finding the generators by exhaustive search has no a priori termination bound. MP.

6 CRM audit

We tabulate the CRM content of each result.

Result	Principle	Conditional on	Status
DPT completeness (Thm 2.1)	BISH	Ax 3 ($u(\mathbb{R}) = 1$)	Structural
Rank 0 ($L \neq 0$)	BISH	Effective lower bound on $\lvert L \rvert$	Conditional
Rank 1 (generator search)	BISH	Bloch–Kato, Silverman	Conditional
Rank ≥ 2 (Minkowski)	MP	Northcott, finiteness of III	Obstruction uncond.

Principle inventory.

- BISH: used for $r = 0$ (bounded real computation) and $r = 1$ (bounded search via Northcott).
- MP: required for $r \geq 2$ (unbounded search, guaranteed termination without a priori bound).
- No LPO, WLPO, LLPO, or choice principles appear.

Why MP **and not** LPO**?** The rank ≥ 2 computation is an unbounded search that is guaranteed to terminate (because the generators exist and will eventually be found). MP asserts exactly this: if a computation does not fail to terminate, it terminates. LPO would additionally provide a decision procedure for whether $r \geq 2$ (i.e., decide the order of vanishing), which is a strictly stronger requirement not needed here.

The geometry-of-numbers obstruction. The $r \geq 2$ obstruction is *not* a gap in current methods. It is a theorem about lattices: in dimension ≥ 2, a covolume does not bound individual basis vectors. The only known path to BISH-decidability for $r \geq 2$ is Lang's Height Lower Bound Conjecture, which would bound λ_1 from below and thereby cap λ_r via the Minkowski product inequality.

7 Discussion

7.1 The complete DPT programme

The pure motive programme (Papers 50–59 [1, 6]) is now closed. Numerical equivalence is BISH-decidable (conditionally on Axiom 3) at all places, with the following certificate structure:

- **Archimedean** ($v = \infty$): Axiom 3 provides the polarization bound ($u(\mathbb{R}) = 1$).
- **Finite** ($v = p$): de Rham decidability provides the precision bound $N_M = v_p(\#E(\mathbb{F}_p))$ (Paper 59).
- **Algebraic**: Axioms 1–2 provide the linear algebra and integrality framework.

The mixed motive frontier opened by this paper is a new chapter: the relevant invariants (Ext^1, regulators, L-functions) are fundamentally different from the intersection pairing that governs numerical equivalence.

7.2 The MP boundary and geometry of numbers

The rank ≥ 2 obstruction has a clean geometric interpretation. In dimension 1, a lattice covolume *is* the length of the single basis vector—the regulator determines the generator up to sign. In dimension ≥ 2, the covolume is a *volume*, and infinitely many lattice bases produce the same volume with arbitrarily different vector lengths.

This is not an artifact of the L-function approach. Any method that extracts only the regulator (a single real number) from the L-value cannot determine the individual generators of a rank ≥ 2 lattice. The information deficit is intrinsic: one real number cannot encode r independent heights when $r \geq 2$.

We note that the identification of MP as the *exact* logical cost requires a reversal: one would need to show that computing rank ≥ 2 generators *implies* MP over BISH. We have not proved such a reversal. The claim is that the standard method (bounded-height enumeration) requires MP; we cannot exclude the possibility that an entirely different BISH algorithm exists, though the geometry-of-numbers obstruction makes this unlikely.

7.3 Candidate Axiom 4

For the extended CRM framework governing rational equivalence:

Definition 7.1 (Axiom 4: Analytic Rank Stratification)**.** In the category of mixed motives over $\mathbb{Q}$, the logical complexity of constructively extracting arithmetic generators of $\mathrm{Ext}^1(\mathbb{Q}(0), M)$ is determined by the analytic order of vanishing $r = \mathrm{ord}_{s=s_0} L(M, s)$, contingent on M possessing the Northcott property:

1. $r = 0$: BISH-decidable.
2. $r = 1$: BISH-decidable (conditional on Bloch–Kato and effective height bounds).
3. $r \geq 2$: requires Markov's Principle (MP).

This is not an axiom in the traditional sense (it does not assert a property one assumes). It is a structural theorem (conditional on Bloch–Kato) that delineates the boundary between BISH and MP in the mixed motive category.

7.4 Open questions

(1) **Motives without Northcott.** For K3 surfaces and higher K-theory, the relevant cycle groups lack a proven Northcott property. Without Northcott, even rank 1 computations are unbounded searches. Establishing Northcott properties for broader classes of motives is a prerequisite for extending the stratification.

(2) **Lang's Height Lower Bound Conjecture.** An effective lower bound on the shortest vector λ_1 in the Mordell–Weil lattice would promote rank ≥ 2 from MP to BISH. This is equivalent to bounding $\hat{h}(P)$ away from zero for non-torsion P—the content of Lang's conjecture [15].

(3) **Higher** Ext **groups.** The Ext^2 and higher groups in the mixed motive category are poorly understood. Their CRM classification is entirely open.

(4) **Formalization.** A Lean 4 formalization of the rank stratification would require real-valued L-functions, Arakelov heights, and lattice geometry. This is beyond the integer arithmetic scope of Papers 50–59 but could become feasible as Mathlib's real analysis infrastructure matures.

7.5 Scope of contribution

The completeness theorem (Theorem 2.1) is a structural observation: numerical equivalence sees only pure motives, so no mixed motive axiom is needed. The proof is short. The rank stratification (Theorem 4.1) applies the CRM lens to standard material: the Bloch–Kato formula, the Silverman height difference bound, and Minkowski's geometry of numbers. The underlying arithmetic is not new.

The contribution, if any, is the identification of analytic rank as the parameter governing the BISH/MP boundary for Ext^1 decidability. The CRM methodology does not create new mathematics; it reveals the logical structure of existing mathematics. In this case, it reveals that the geometry-of-numbers obstruction at rank ≥ 2 is exactly Markov's Principle.

8 Conclusion

The DPT framework for pure motives is complete. Numerical equivalence is decidable in BISH (conditionally on Axiom 3) at all places, with no mixed motive input required.

The mixed motive extension reveals a logical stratification governed by analytic rank. The boundary between BISH and MP occurs at rank 2, where the geometry of numbers imposes a structural obstruction: lattice covolumes do not bound basis vectors. This boundary is removable only by Lang's Height Lower Bound Conjecture (open).

The pure motive programme (Papers 50–59) and the mixed motive frontier (this paper) together close the DPT framework for numerical equivalence and open the framework for rational equivalence. The CRM lens identifies two independent axes of logical complexity in arithmetic geometry: the place (∞ vs. p, governing the decidability certificate type) and the analytic rank (r, governing the logical principle required).

Acknowledgments

We thank the constructive reverse mathematics community—especially the foundational work of Bishop, Bridges, Richman, and Ishihara—for developing the framework that makes calibrations like these possible.

This paper was produced using AI text generation (Claude, Anthropic, Opus 4.6) under human direction. The author is a practicing cardiologist rather than a professional logician or

arithmetic geometer; all mathematical claims should be evaluated on their formal content. We welcome constructive feedback from domain experts.

References

[1] P. C.-K. Lee, Decidability of numerical equivalence: three axioms for polarized Tannakian categories, *CRM Series*, Paper 50 (2026). DOI: 10.5281/zenodo.18705837

[2] P. C.-K. Lee, Fracture points of the DPT framework, *CRM Series*, Paper 54 (2026). DOI: 10.5281/zenodo.18732964

[3] P. C.-K. Lee, Exotic Weil classes and the conductor formula I, *CRM Series*, Paper 56 (2026). DOI: 10.5281/zenodo.18734021

[4] P. C.-K. Lee, Exotic Weil classes and the conductor formula II, *CRM Series*, Paper 57 (2026). DOI: 10.5281/zenodo.18735172

[5] P. C.-K. Lee, Exotic Weil classes and the conductor formula III, *CRM Series*, Paper 58 (2026). DOI: 10.5281/zenodo.18734718

[6] P. C.-K. Lee, De Rham decidability—the p-adic precision bound, *CRM Series*, Paper 59 (2026). DOI: 10.5281/zenodo.18735393

[7] E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[8] D. Bridges and F. Richman, *Varieties of Constructive Mathematics*, London Math. Soc. Lecture Note Ser., vol. 97, Cambridge Univ. Press, 1987.

[9] S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives, in: *The Grothendieck Festschrift*, vol. I, Birkhäuser, 1990, pp. 333–400.

[10] P. Colmez and J.-M. Fontaine, Construction des représentations p-adiques semi-stables, *Invent. Math.* **140** (2000), 1–43.

[11] V. A. Kolyvagin, Finiteness of $E(\mathbb{Q})$ and $\text{III}(E, \mathbb{Q})$ for a subclass of Weil curves, *Izv. Akad. Nauk SSSR* **52** (1988), 522–540.

[12] B. H. Gross and D. B. Zagier, Heegner points and derivatives of L-series, *Invent. Math.* **84** (1986), 225–320.

[13] J. H. Silverman, The difference between the Weil height and the canonical height on elliptic curves, *Math. Comp.* **55** (1990), 723–743.

[14] D. G. Northcott, An inequality in the theory of arithmetic on algebraic varieties, *Proc. Cambridge Philos. Soc.* **45** (1949), 502–509.

[15] S. Lang, *Fundamentals of Diophantine Geometry*, Springer, 1983.

[16] A. Borel, Stable real cohomology of arithmetic groups, *Ann. Sci. Éc. Norm. Sup.* **7** (1974), 235–272.

[17] H. Minkowski, *Geometrie der Zahlen*, Teubner, 1896.