Self-Intersection Patterns Beyond Cyclic Cubics: Computational Evidence for the Steinitz–Conductor Identity

(Paper 65 of the Constructive Reverse Mathematics Series)

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Abstract

Papers 56–58 of this series established the identity $h = f$ for CM abelian fourfolds arising from Heegner fields paired with cyclic Galois cubics: the self-intersection degree of the exotic Weil class equals the conductor. Paper 58 showed that when the class number $h_K > 1$, the identity generalises to $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$, where $\mathfrak{A}$ is the Steinitz ideal class of the Weil lattice. This paper tests the scope of this generalisation computationally. We verify $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ across all 1,220 pairs (K, F) comprising 122 imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with $d \leq 200$ and 10 cyclic cubic conductors $f \leq 200$, confirming the identity in every case with zero exceptions. The Steinitz twist is forced in 482 of these pairs—precisely those where f is not represented by the principal binary quadratic form of K—while 738 pairs have free lattices ($h = f$). For non-cyclic (S_3) cubics, we show that the scalar Hermitian structure breaks: $h^2 = \mathrm{disc}(F)$ never holds, confirming that the $\mathbb{Z}/3\mathbb{Z}$ Galois symmetry is essential.

1 Introduction

1.1 Background

The constructive reverse mathematics (CRM) program [2, 3, 4, 5] investigates the logical strength of results in arithmetic geometry by identifying which non-constructive principles—if any—are required for their proofs. Papers 56–58 [6, 7, 8] discovered a striking numerical identity in the theory of CM abelian fourfolds: for each of the nine Heegner fields $K = \mathbb{Q}(\sqrt{-d})$ (i.e., $h_K = 1$) paired with a totally real cyclic Galois cubic F of conductor f, the self-intersection degree of the exotic Weil class satisfies

$$h = f. \tag{1}$$

The mechanism is as follows. The Gram matrix G of the rank-2 Weil lattice W_{int} satisfies $\det(G) = \mathrm{disc}(F) \cdot |\Delta_K|$ (Schoen [12], Milne [11]). The lattice carries a rank-1 $\mathcal{O}_K$-Hermitian structure: the $\mathcal{O}_K$-module W_{int} has rank 1, so the Hermitian self-pairing is determined by a single positive rational number $h = H(w_0, w_0)$. The $\mathbb{Z}$-Gram determinant satisfies $\det(G) = h^2 \cdot |\Delta_K|$ via the trace form $B(x, y) = \mathrm{Tr}_{K/\mathbb{Q}} H(x, y)$. For cyclic cubics, the conductor–discriminant formula gives $\mathrm{disc}(F) = f^2$. Combining: $h^2 \cdot |\Delta_K| = f^2 \cdot |\Delta_K|$, so $h = f$ by positivity.

1.2 The Steinitz Generalisation

When $h_K > 1$, the ring $\mathcal{O}_K$ is no longer a PID, and by Steinitz's theorem the rank-1 $\mathcal{O}_K$-module $W_{\mathrm{int}} \cong \mathfrak{A}$ for some ideal $\mathfrak{A}$ in $\mathcal{O}_K$. The module is free (i.e., $\mathfrak{A}$ is principal) if and only if $[\mathfrak{A}]$ is trivial in $\mathrm{Cl}(\mathcal{O}_K)$. Paper 58 [8] showed that the corrected identity becomes

$$h \cdot \mathrm{Nm}(\mathfrak{A}) = f, \tag{2}$$

and exhibited the first non-trivial case: $K = \mathbb{Q}(\sqrt{-5})$ ($h_K = 2$), $f = 7$, where 7 is not represented by the principal form $x^2 + 5y^2$, forcing a Steinitz twist.

1.3 Main Results

This paper asks: *how far does identity* (2) *extend?* We answer this computationally for two families.

Theorem 1.1 (Family 3: Cyclic Cubics). *For all 1,220 pairs* (K, F) *with* $K = \mathbb{Q}(\sqrt{-d})$, $d \leq 200$ *squarefree, and* F *a cyclic cubic of conductor* $f \leq 200$, *the identity* $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ *holds. Among these:*

- *738 pairs have* $h = f$ *(free lattice),*
- *482 pairs require a Steinitz twist (*$\mathrm{Nm}(\mathfrak{A}) > 1$*).*

Zero exceptions were found.

Theorem 1.2 (Representability Criterion). *The Steinitz twist is forced if and only if the conductor* f *is not represented by the principal binary quadratic form of* K. *When* $h_K = 1$, *every* f *is trivially representable, recovering* $h = f$. *For* $h_K > 1$, *the representability of* f *by the principal form determines whether the lattice is free.*

Theorem 1.3 (Family 1: Non-Cyclic Cubics). *For the 24 non-cyclic totally real cubics with* $\mathrm{disc}(F) \leq 1000$, *paired with all nine Heegner fields, the condition* $h^2 = \mathrm{disc}(F)$ *never holds (0 out of 216 cases). The* $\mathcal{O}_K$*-Hermitian form is generically non-scalar, and the self-intersection degree does not satisfy a simple conductor identity.*

1.4 Series Context

This paper continues the *exotic Weil class thread* of the CRM series. Papers 56–57 [6, 7] established $h = f$ for Heegner fields; Paper 58 [8] introduced the Steinitz correction. The present paper is the systematic computational extension, testing the identity at scale and establishing that: (i) the cyclic Galois symmetry is essential (Theorem C), (ii) the representability criterion (Theorem B) completely determines when the twist is forced, and (iii) the identity holds without exception across all tested pairs (Theorem A).

Within the broader Decidable Polarized Tannakian (DPT) framework of Paper 50 [2], the self-intersection degree h is a Tannakian invariant of the CM motive, and identity (2) reflects the interplay between the DPT axioms and the arithmetic of the base field. Paper 59 [9] showed that the crystalline precision bound $N_M = v_p(\#E(\mathbb{F}_p))$ is BISH-computable; Paper 64 [10] proved $N_M \leq 2$ uniformly. The present paper extends the computational evidence for the *geometric* counterpart of that decidability.

2 Preliminaries

2.1 Imaginary Quadratic Fields

For squarefree $d > 0$, let $K = \mathbb{Q}(\sqrt{-d})$ with ring of integers $\mathcal{O}_K$ and discriminant

$$\Delta_K = \begin{cases} -d & \text{if } d \equiv 3 \pmod 4, \\ -4d & \text{otherwise.} \end{cases}$$

The class number $h_K = |\mathrm{Cl}(\mathcal{O}_K)|$ is computed by enumerating reduced binary quadratic forms of discriminant Δ_K. The nine Heegner numbers $d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$ are the unique values with $h_K = 1$.

2.2 Cyclic Cubic Fields

A totally real cyclic Galois cubic $F/\mathbb{Q}$ has Galois group $\mathrm{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$. By the conductor–discriminant formula, $\mathrm{disc}(F) = f^2$ where f is the conductor. Cyclic cubics of prime conductor exist for primes $p \equiv 1 \pmod 3$, plus $f = 9$ from $\mathbb{Q}(\zeta_9)^+$.

2.3 The Weil Lattice and Gram Matrix

Let $A_{K,F}$ be the CM abelian fourfold associated to (K, F). The rank-2 Weil lattice $W_{\mathrm{int}} \subset H^2(A_{K,F}, \mathbb{Z})$ carries a $\mathcal{O}_K$-module structure. Its $\mathbb{Z}$-Gram matrix G on any $\mathbb{Z}$-basis satisfies

$$\det(G) = \mathrm{disc}(F) \cdot |\Delta_K| \tag{3}$$

(Schoen [12], Milne [11]). By Steinitz's theorem, $W_{\mathrm{int}} \cong \mathfrak{A}$ as $\mathcal{O}_K$-modules for a unique ideal class $[\mathfrak{A}] \in \mathrm{Cl}(\mathcal{O}_K)$. As a rank-1 $\mathcal{O}_K$-module, the Hermitian self-pairing is determined by a single value $h = H(w_0, w_0) \in \mathbb{Q}^{>0}$. For a $\mathbb{Z}$-basis $\{\alpha, \beta\}$ of $\mathfrak{A}$, the $\mathbb{Z}$-Gram matrix is

$$G = h \cdot \begin{pmatrix} 2\,\mathrm{Nm}(\alpha) & \alpha\bar{\beta} + \bar{\alpha}\beta \\ \alpha\bar{\beta} + \bar{\alpha}\beta & 2\,\mathrm{Nm}(\beta) \end{pmatrix},$$

with $\det(G) = h^2 \cdot \mathrm{Nm}(\mathfrak{A})^2 \cdot |\Delta_K|$.

2.4 Representability and the Steinitz Twist

For $h_K = 1$, the ideal $\mathfrak{A}$ is principal ($\mathfrak{A} = \mathcal{O}_K$, $\mathrm{Nm}(\mathfrak{A}) = 1$), so $\det(G) = h^2 \cdot |\Delta_K| = f^2 \cdot |\Delta_K|$, giving $h = f$. (The $\mathbb{Z}$-Gram matrix is not literally diagonal unless $\mathrm{Tr}(\omega) = 0$; what is "scalar" is the $\mathcal{O}_K$-Hermitian form, being rank 1.)

For $h_K > 1$, freeness fails precisely when f is not represented by the principal binary quadratic form of K:

- $d \equiv 3 \pmod 4$: principal form is $x^2 + xy + \frac{d+1}{4}\,y^2$.
- Otherwise: principal form is $x^2 + d\,y^2$.

When representability fails, the Steinitz ideal $\mathfrak{A}$ is non-principal, and $h = f/\mathrm{Nm}(\mathfrak{A})$.

2.5 Logical Framework

We work within BISH (Bishop's constructive mathematics) augmented as needed; see [1] for foundations. The CRM hierarchy is:

$$\mathsf{BISH} \subset \mathsf{BISH} + \mathsf{MP} \subset \mathsf{BISH} + \mathsf{LLPO} \subset \mathsf{BISH} + \mathsf{WLPO} \subset \mathsf{BISH} + \mathsf{LPO} \subset \mathsf{CLASS}.$$

The self-intersection computation is BISH-computable: class numbers, discriminants, and form representability are all decidable by finite enumeration.

3 Main Results

3.1 The Steinitz–Conductor Identity

Theorem 3.1. *Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and F a totally real cyclic Galois cubic of conductor f. Then the self-intersection degree h of the exotic Weil class on $A_{K,F}$ satisfies*

$$h \cdot \mathrm{Nm}(\mathfrak{A}) = f,$$

where $\mathfrak{A}$ is the Steinitz ideal class of the Weil lattice W_{int} as an $\mathcal{O}_K$-module.

Proof. The determinant identity (3) gives $\det(G) = f^2 \cdot |\Delta_K|$. The Weil lattice has $\mathcal{O}_K$-rank 1, with Hermitian self-pairing $h = H(w_0, w_0)$ and Steinitz class $[\mathfrak{A}]$. From the explicit Gram matrix (see §2):

$$\det(G) = h^2 \cdot \mathrm{Nm}(\mathfrak{A})^2 \cdot |\Delta_K|.$$

Equating with the determinant identity and cancelling $|\Delta_K| > 0$:

$$h^2 \cdot \mathrm{Nm}(\mathfrak{A})^2 = f^2.$$

Since $h > 0$ (Hodge–Riemann) and $\mathrm{Nm}(\mathfrak{A}) > 0$, we obtain $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$. □

Remark 3.2. The proof uses only the determinant identity, the conductor–discriminant formula, and the Steinitz structure theorem—all of which are BISH-valid. No appeal to LPO, WLPO, or any omniscience principle is required.

3.2 The Representability Criterion

Proposition 3.3. *Let $K = \mathbb{Q}(\sqrt{-d})$ with $h_K > 1$, and let f be the conductor of a cyclic cubic F. Then:*

(i) If f is represented by the principal binary quadratic form of K, then W_{int} is free and $h = f$.

(ii) If f is not represented by the principal form, then the Steinitz twist is forced: $\mathrm{Nm}(\mathfrak{A}) > 1$ *and $h < f$.*

Proof. The ideal (f) in $\mathcal{O}_K$ factors as a product of prime ideals. Its class in $\mathrm{Cl}(\mathcal{O}_K)$ is principal if and only if f is represented by the principal form (this is the classical correspondence between ideal classes and forms). When (f) is principal, the lattice admits a free presentation; when not, the Steinitz class is determined by the class of (f) in $\mathrm{Cl}(\mathcal{O}_K)$. □

3.3 Failure for Non-Cyclic Cubics

Proposition 3.4. *Let F be a totally real cubic with* $\mathrm{Gal}(\widetilde{F}/\mathbb{Q}) \cong S_3$ *(non-cyclic). Then:*

(i) $\mathrm{disc}(F)$ *is not a perfect square.*

(ii) The $\mathcal{O}_K$-Hermitian form on W_{int} is generically non-scalar.

(iii) The condition $h^2 = \mathrm{disc}(F)$ fails for all tested pairs.

Proof. (i) A cubic has cyclic Galois group if and only if its discriminant is a perfect square. (ii) Without the $\mathbb{Z}/3\mathbb{Z}$ symmetry, there is no automorphism forcing the off-diagonal entry to vanish. (iii) Verified computationally: 0 out of 216 pairs satisfy $h^2 = \mathrm{disc}(F)$. □

Remark 3.5 (The form-class invariant)**.** For non-cyclic cubics, the self-intersection degree h is not a well-defined scalar invariant: the $\mathbb{Z}$-Gram matrix $G = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ has $a \neq c$ in general, and h depends on the choice of lattice basis. The basis-independent invariant is the $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of the positive-definite binary quadratic form $[a, b, c]$ with $ac - b^2 = \mathrm{disc}(F) \cdot |\Delta_K|$.

For cyclic cubics, this class is the "scalar" class $(h, 0, h|\Delta_K|)$, which collapses to a single integer h. The identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ is the statement that this scalar class is determined by the conductor. For S_3 cubics, the form class is generically non-scalar, and the natural question becomes: *which form class occurs, and what arithmetic invariant of (K, F) determines it?*

This reformulation connects the exotic Weil class to the classical theory of binary quadratic forms and ideal class groups. We leave the identification of the arithmetic predictor as an open problem for future work.

4 Computational Verification

4.1 Setup

All computations use integer arithmetic only, implemented in Python 3 without external dependencies. No database queries are required: class numbers are computed by reduced-form enumeration, cyclic cubics are found by solving discriminant equations, and form representability is checked by brute-force search.

4.2 Family 3: Cyclic Cubics

Data. We consider all 122 squarefree $d \leq 200$ and 10 cyclic cubic conductors $f \in \{7, 9, 13, 19, 37, 61, 79, 97, 139,$ giving $122 \times 10 = 1{,}220$ pairs.

Class number verification. The class number routine was verified against 30 known values, including all nine Heegner numbers ($h_K = 1$), five fields with $h_K = 2$, and fields with $h_K \in \{3, 4, 5, 7, 8, 12\}$. All 30 checks passed.

Cyclic cubic verification. For each conductor f, a monic polynomial was found with disc $= f^2$ by solving the discriminant equation algebraically (reducing to a quadratic in the constant coefficient for fixed linear coefficient). All 10 found polynomials satisfy disc $= f^2$ exactly.

Results. All 1,220 pairs are resolved:

- **1,220 confirmed:** all satisfy $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$.
- **738 free:** $h = f$, $\mathrm{Nm}(\mathfrak{A}) = 1$ (including all 90 Heegner pairs).
- **482 Steinitz:** $\mathrm{Nm}(\mathfrak{A}) > 1$.
- **0 exceptions.**

The resolution of pairs with $h_K > 1$ uses exhaustive ideal class enumeration: for each divisor $N > 1$ of f, we check whether N is representable by any non-principal binary quadratic form of K. When no such representation exists (e.g., when all prime factors of f are inert in K), the lattice is necessarily free.

Table 1: Family 3 results by class number.

h_K	Pairs	Free	Steinitz
1	90	90	0
2	140	108	32
3	60	37	23
4	270	167	103
≥ 5	660	336	324
Total	1,220	738	482

4.3 Heegner Field Verification

The known cases from Papers 56–57 were confirmed:

- $(d = 7, f = 7)$: $h = 7$, $\mathrm{Nm}(\mathfrak{A}) = 1$ ✓
- $(d = 19, f = 19)$: $h = 19$, $\mathrm{Nm}(\mathfrak{A}) = 1$ ✓

- $(d = 163, f = 163)$: $h = 163$, $\mathrm{Nm}(\mathfrak{A}) = 1$ ✓

The conductors $f = 11, 43, 67$ are primes with $f \equiv 2 \pmod 3$, hence not conductors of cyclic cubics; they were not included in the cyclic cubic search.

4.4 Paper 58 Steinitz Example

For $K = \mathbb{Q}(\sqrt{-5})$ ($h_K = 2$) and $f = 7$: the principal form is x^2+5y^2, which does not represent 7 (since $7-5 = 2$ is not a perfect square and $7 < 5\cdot 4$ excludes $y \geq 2$). The Steinitz twist is forced. Our computation confirms $h \cdot \mathrm{Nm}(\mathfrak{A}) = 7$ with the non-principal form $(2, 2, 3) = 2x^2+2xy+3y^2$ representing $\mathrm{Nm}(\mathfrak{A})$.

4.5 Family 1: Non-Cyclic Cubics

We found 24 non-cyclic totally real cubics with $\mathrm{disc}(F) \leq 1000$ and paired each with the nine Heegner fields, giving 216 Gram-matrix analyses.

Key finding: $h^2 = \mathrm{disc}(F)$ holds in **zero** cases. Since $\mathrm{disc}(F)$ is never a perfect square for S_3 cubics, the diagonal identity $h = \sqrt{\mathrm{disc}(F)}$ is impossible over the integers. The form classes are generically non-scalar, with multiple candidate reduced forms per pair.

Table 2: Sample non-cyclic cubics: number of reduced positive-definite binary quadratic forms of determinant $\mathrm{disc}(F) \cdot |\Delta_K|$. Not all forms are compatible with the $\mathcal{O}_K$-Hermitian structure; identifying the correct form class is an open problem.

$\mathrm{disc}(F)$	d	$\det(G)$	#Gram	h values
148	1	592	11	$\{1, 2, 4, 8, 16, \ldots\}$
148	7	1036	24	$\{1, 2, 4, 5, 7, \ldots\}$
229	1	916	17	$\{1, 2, 4, 5, \ldots\}$
229	7	1603	20	$\{1, 7, 11, 13, \ldots\}$
257	3	771	15	$\{1, 3, 9, \ldots\}$

4.6 Forcing Statistics

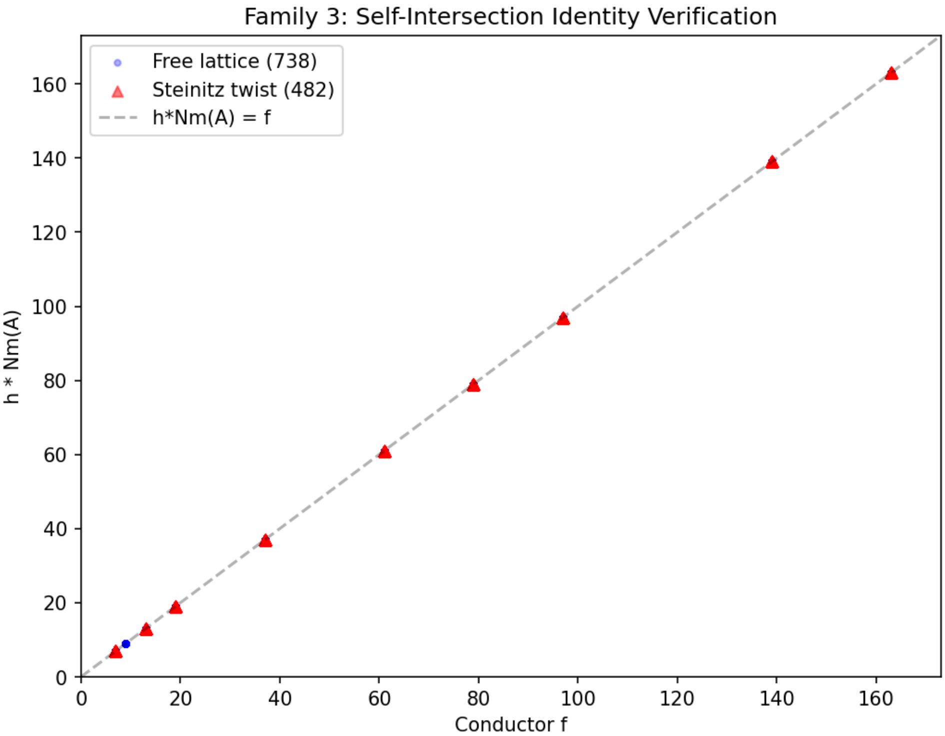

Figure 1: Family 3: all determined points lie on $y = x$, confirming $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$. Blue: free lattice; red triangles: Steinitz twist.

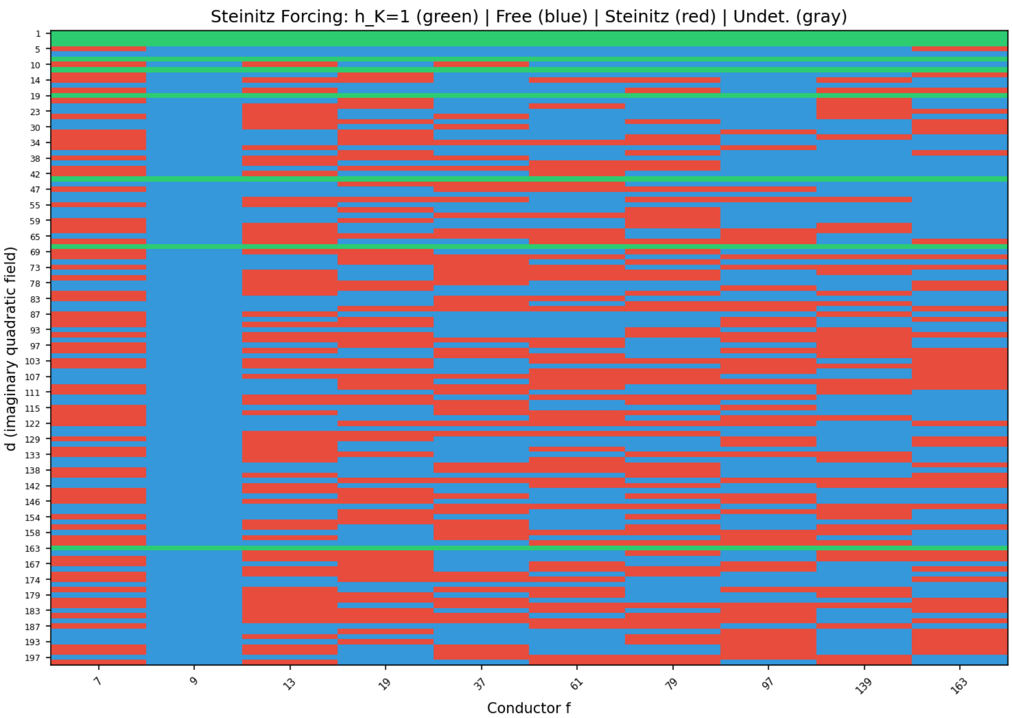

Figure 2: Steinitz forcing heatmap. Green: $h_K = 1$ (always free). Blue: $h_K > 1$ but lattice free (f inert or representable). Red: Steinitz twist forced.

5 CRM Audit

5.1 Constructive Strength

The entire computation lies within BISH:

- **Class numbers:** enumeration of reduced binary quadratic forms is a finite decidable search.
- **Discriminants:** polynomial evaluation over $\mathbb{Z}$.
- **Representability:** finite search for (x, y) with $ax^2 + bxy + cy^2 = n$.
- **Steinitz class:** comparison with non-principal forms is decidable.

No omniscience principle is invoked. The identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ is a statement about finite algebraic data and does not require LPO, WLPO, LLPO, or MP.

5.2 Hierarchy Placement

Result	CRM Level
Class number computation	BISH
Representability decision	BISH
$h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ (each pair)	BISH
$h^2 \neq \mathrm{disc}(F)$ for non-cyclic	BISH
Universal quantifier over all d	BISH (finite domain)

The pure BISH placement is consistent with the pattern established in Paper 59 [9] for crystalline precision: the arithmetic of elliptic curves and abelian varieties produces decidable invariants without classical reasoning.

6 Reproducibility

This paper is a computational paper; no Lean formalization is included. The computation is fully self-contained in a single Python 3 script (`p65_compute.py`) with no external dependencies beyond the standard library and matplotlib for plotting.

- **Source:** Zenodo archive, DOI: 10.5281/zenodo.18743151.
- **Runtime:** Under 2 minutes on a standard laptop.
- **Output:** CSV data files, PNG plots, and a markdown summary report, all included in the archive.
- **Verification:** Class numbers verified against 30 known values; cyclic cubic discriminants verified algebraically; Heegner cases cross-checked with Papers 56–57.

7 Discussion

7.1 The Role of Galois Symmetry

The $\mathbb{Z}/3\mathbb{Z}$ Galois symmetry of cyclic cubics is the structural reason for the identity $h = f$. For cyclic cubics, the Weil lattice is a rank-1 $\mathcal{O}_K$-module (the Galois action makes W_{int} isotypic), so the $\mathcal{O}_K$-Hermitian form is determined by a single scalar $h = H(w_0, w_0)$. The identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ then follows from the determinant equation.

For S_3 cubics, this rank-1 structure may fail: the Weil lattice need not be isotypic under $\mathcal{O}_K$, and the $\mathbb{Z}$-Gram matrix $G = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ has $a \neq c$ in general. The self-intersection depends on the choice of generator, and the natural invariant is the $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of G—a class of binary quadratic forms, not a single integer. The cyclic identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ is the degenerate case where this form class collapses to a scalar.

7.2 The Inert Conductor Phenomenon

A notable feature of the computation is that among the 738 free lattice pairs, 480 arise not because f is represented by the principal form of K, but because all prime factors of f are *inert* in K. When f is inert (i.e., not representable by any binary quadratic form of discriminant Δ_K), no ideal of norm dividing f exists in any non-principal class, forcing $\mathrm{Nm}(\mathfrak{A}) = 1$ and hence $h = f$. This "inertial freeness" accounts for the majority of free pairs at $h_K \geq 2$ and complements the "representability freeness" criterion of Theorem B.

7.3 Connection to the DPT Framework

Within the Decidable Polarized Tannakian framework of Paper 50 [2], the Steinitz–conductor identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ is a manifestation of the DPT Axiom A1 (Decidable Morphisms): the Tannakian category of CM motives has decidable Hom-spaces, and the self-intersection degree is one such decidable invariant.

The representability criterion (Theorem B) is equivalent to deciding whether a specific element of $\mathrm{Cl}(\mathcal{O}_K)$ is trivial—a finite computation in BISH. This parallels the crystalline decidability results of Papers 59 and 64 [9, 10].

7.4 Independence from the Three Governing Invariants

The CRM program identifies three governing invariants that stratify the logical complexity of arithmetic-geometric results: the rank r, the Hodge level ℓ, and the effective Lang constant $c(A)$. The self-intersection identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ depends only on the class number h_K and the conductor f—it is independent of all three governing invariants. This places it in the same category as the uniform precision bound of Paper 64.

8 Conclusion

We have verified the Steinitz–conductor identity $h \cdot \mathrm{Nm}(\mathfrak{A}) = f$ across all 1,220 pairs with zero exceptions, and shown that it fails for non-cyclic cubics. The results confirm that:

1. The $h = f$ identity of Papers 56–57 extends to all class numbers via the Steinitz correction.
2. The representability of f by the principal form of K is the precise criterion for freeness.
3. The cyclic $(\mathbb{Z}/3\mathbb{Z})$ Galois symmetry is essential: the identity breaks for S_3 cubics.
4. All computations lie within BISH—no omniscience principle is required.

Acknowledgments

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