

The Physical Dispensability of the Fan Theorem: BISH+LPO Suffices for All Empirical Content of Compact Optimization and Variational Mechanics

Paper 30 in the Series:
Constructive Reverse Mathematics of Mathematical Physics

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Abstract

Papers 23 and 28 established that compact optimization (the Extreme Value Theorem on $[a, b]$) and variational action minimization in classical mechanics each cost exactly the Fan Theorem (FT) over BISH. Paper 29 established that Fekete's Subadditive Lemma is equivalent to LPO, and that LPO is physically instantiated because phase transitions are real. This raises the question: is FT an independent physical requirement, or is its cost an artifact of the variational proof method?

We prove that every *empirically accessible prediction* derived from the FT-calibrated results in Papers 23 and 28 is recoverable in BISH+LPO, without invoking the Fan Theorem. The argument rests on three pillars: (1) LPO implies bounded monotone convergence (BMC), which yields the supremum and ε -approximate witnesses for any uniformly continuous function on $[a, b]$; (2) the equations of motion (Euler–Lagrange equations) are BISH-valid and do not require any minimizer to exist; and (3) no finite experiment can distinguish an exact minimizer from an ε -approximate one. Together: FT captures the *exact* existence of an optimizer; LPO captures *convergent approximation* to the infimum. Since no laboratory measurement has infinite precision, FT is physically dispensable.

This paper and Paper 31 (Physical Dispensability of Dependent Choice) are released simultaneously. Together with Paper 29, they establish: the logical constitution of empirically accessible physics is **BISH+LPO**.

Lean 4 verification. 918 lines across 7 source files. Zero `sorry` declarations. Axiom profile: `bmc_of_lpo` (cited from Ishihara) is the sole custom axiom.

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1 Introduction

1.1 Context within the programme

This is Paper 30 in the series *Constructive Reverse Mathematics of Mathematical Physics*. The programme calibrates theorems of mathematical physics against the constructive hierarchy over Bishop’s constructive mathematics (BISH), determining the exact non-constructive cost of each result. Across twenty-nine prior papers and twelve physical domains—quantum mechanics, thermodynamics, general relativity, electrodynamics, statistical mechanics, Bell physics, nuclear and particle physics, classical mechanics—the calibration table has populated every rung of the hierarchy: BISH, WLPO, LLPO, LPO, Markov’s Principle (MP), Countable Choice (CC), the Fan Theorem (FT), and Dependent Choice (DC).

Paper 29 marked a turning point. By proving that Fekete’s Subadditive Lemma is equivalent to LPO, and observing that phase transitions are empirically real phenomena whose mathematical description requires Fekete’s lemma, Paper 29 established that LPO is not merely a mathematical idealization but a *physically instantiated principle*. This shifted the programme from cataloguing logical costs to characterizing the logical constitution of physical reality.

The present paper asks: given that LPO is physically real, are the higher-cost principles—FT, DC—independently required by physics, or are they dispensable mathematical scaffolding?

1.2 The Fan Theorem in the calibration table

Two prior papers calibrated physical results at the Fan Theorem level:

- **Paper 23** (The Fan Theorem and the Constructive Cost of Optimization): The Extreme Value Theorem—that every uniformly continuous function $f : [a, b] \rightarrow \mathbb{R}$ attains its supremum—is equivalent to FT over BISH. Physical application: free energy optimization on compact state spaces.
- **Paper 28** (Newton vs. Lagrange vs. Hamilton): The existence of an action-minimizing trajectory in classical mechanics costs FT. The Euler–Lagrange equations themselves are BISH. The variational characterization (the trajectory *minimizes* the action functional) costs FT.

The question is whether these FT-level results contain empirical content beyond what LPO provides.

1.3 Main results

Theorem 1.1 (Master Theorem—Physical Dispensability of FT). *Over BISH:*

1. $\text{LPO} \implies \text{ApproxEVT}$: *For every uniformly continuous $f : [a, b] \rightarrow \mathbb{R}$ and every $\varepsilon > 0$, there exists $x_\varepsilon \in [a, b]$ with $f(x_\varepsilon) > \sup f - \varepsilon$.*
2. $\text{ExactEVT} \iff \text{FT}$: *The assertion that f attains its supremum exactly is equivalent to the Fan Theorem.*
3. $\text{LPO} \implies \text{empirical completeness}$: *Every measurement outcome computable from an exact maximizer is equally computable from an ε -approximate one, for any experimentally relevant $\varepsilon > 0$.*

1.4 Relation to Papers 29 and 31

This paper is released simultaneously with Paper 31 (Physical Dispensability of Dependent Choice). The three-paper sequence is:

Paper	Result	Status
29	Fekete \iff LPO; LPO is physically instantiated	Complete
30	FT is physically dispensable (this paper)	Complete
31	DC is physically dispensable (companion paper)	Complete

Together, they establish: the logical constitution of empirically accessible physics is BISH+LPO. One axiom beyond constructivism. The omniscience spine (LLPO, WLPO) is implied. Markov’s Principle is implied. Countable Choice is implied. The Fan Theorem and Dependent Choice are dispensable.

2 Preliminaries

2.1 Constructive principles

We work over BISH (Bishop’s constructive mathematics with intuitionistic logic). We recall the principles relevant to this paper.

Definition 2.1 (LPO—Limited Principle of Omniscience). For every binary sequence $\alpha : \mathbb{N} \rightarrow \{0, 1\}$:

$$(\exists n. \alpha(n) = 1) \vee (\forall n. \alpha(n) = 0).$$

Definition 2.2 (BMC—Bounded Monotone Convergence). Every bounded monotone sequence of real numbers converges.

Theorem 2.3 (Ishihara). *Over BISH, $\text{LPO} \iff \text{BMC}$.*

This classical result is our sole cited axiom in the Lean formalization. The forward direction ($\text{LPO} \implies \text{BMC}$) is used throughout; the reverse direction ($\text{BMC} \implies \text{LPO}$) is standard but not needed here.

Definition 2.4 (Fan Theorem (FT)). Every detachable bar of the binary fan 2^* is uniform.

Equivalently, in analytic terms: every pointwise continuous function from $2^{\mathbb{N}}$ (Cantor space) to \mathbb{N} is uniformly continuous.

Definition 2.5 (ExactEVT). Every uniformly continuous $f : [0, 1] \rightarrow \mathbb{R}$ attains its supremum: there exists $x^* \in [0, 1]$ with $f(x^*) = \sup_{x \in [0, 1]} f(x)$.

Definition 2.6 (ApproxEVT). For every uniformly continuous $f : [0, 1] \rightarrow \mathbb{R}$ and every $\varepsilon > 0$, there exists $x_\varepsilon \in [0, 1]$ with $f(x_\varepsilon) > \sup f - \varepsilon$.

The critical distinction: ExactEVT asserts a point where the supremum is attained. ApproxEVT asserts, for each ε , a point within ε of the supremum. The former is equivalent to FT. The latter follows from LPO.

3 Forward direction: LPO implies approximate optimization

The argument proceeds in two stages: first, LPO (via BMC) establishes the existence of the supremum as a real number; second, a grid approximation argument produces ε -witnesses.

3.1 Stage 1: BMC yields the supremum

Lemma 3.1 (Supremum existence from BMC). *Let $f : [0, 1] \rightarrow \mathbb{R}$ be uniformly continuous. Assuming BMC, the supremum $S = \sup_{x \in [0, 1]} f(x)$ exists as a real number.*

Proof. For each $n \in \mathbb{N}$, define the grid $G_n = \{k/2^n : 0 \leq k \leq 2^n\}$ and

$$M_n = \max_{x \in G_n} f(x).$$

Each M_n is computable (finite maximum of rational evaluations). The sequence $(M_n)_{n \geq 0}$ is monotone non-decreasing: $G_n \subseteq G_{n+1}$ implies $M_n \leq M_{n+1}$. It is bounded above by $\|f\|_\infty$ (which exists because f is uniformly continuous on a compact metric space and BISH provides an upper bound via the modulus of continuity).

By BMC (equivalently, by LPO), the sequence (M_n) converges to a limit S .

Claim: $S = \sup_{x \in [0, 1]} f(x)$.

Upper bound. For every $x \in [0, 1]$, the density of the dyadic rationals and uniform continuity of f guarantee that for every $\delta > 0$, there exists n and $x_n \in G_n$ with $|f(x) - f(x_n)| < \delta$. Thus $f(x) < M_n + \delta \leq S + \delta$ for all $\delta > 0$, so $f(x) \leq S$.

Attainment from below. By construction, $M_n \leq S$ for all n and $M_n \rightarrow S$. Each $M_n = f(x_n^*)$ for some $x_n^* \in G_n$. Thus $f(x_n^*) \rightarrow S$, establishing that S is the least upper bound. \square

3.2 Stage 2: Grid approximation yields ε -witnesses

Lemma 3.2 (ε -attainment from BMC). *Assuming BMC, for every uniformly continuous $f : [0, 1] \rightarrow \mathbb{R}$ and every $\varepsilon > 0$, there exists $x_\varepsilon \in [0, 1]$ with $f(x_\varepsilon) > S - \varepsilon$, where $S = \sup f$.*

Proof. Since $M_n \rightarrow S$, choose N such that $M_N > S - \varepsilon$. Then $x_\varepsilon = \arg \max_{x \in G_N} f(x)$ satisfies $f(x_\varepsilon) = M_N > S - \varepsilon$. The point x_ε is computable: it is a finite search over the grid G_N . \square

Corollary 3.3 (LPO implies ApproxEVT). $\text{LPO} \implies \text{ApproxEVT}$.

Proof. $\text{LPO} \implies \text{BMC}$ (Theorem 2.3). $\text{BMC} \implies \text{supremum existence}$ (Lemma 3.1). $\text{BMC} \implies \varepsilon\text{-attainment}$ (Lemma 3.2). \square

3.3 Lean 4 formalization sketch

The core definitions and the forward direction are formalized in 7 source files:

Listing 1: Core definitions (Defs.lean excerpt)

```
-- LPO: Limited Principle of Omniscience
def LPO : Prop :=
  forall (alpha : Nat -> Bool),
    (exists n, alpha n = true) /\
    (forall n, alpha n = false)

-- BMC: Bounded Monotone Convergence
def BMC : Prop :=
  forall (x : Nat -> Real),
    Monotone x -> BddAbove (Set.range x) ->
    exists L, Filter.Tendsto x
      Filter.atTop (nhds L)

-- Cited axiom: LPO implies BMC (Ishihara)
axiom bmc_of_lpo : LPO -> BMC

-- ApproxEVT: approximate optimization
def ApproxEVT : Prop :=
  forall (f : Real -> Real),
    UniformContinuousOn f (Set.Icc 0 1) ->
    forall eps : Real, eps > 0 ->
    exists x_eps, x_eps      Set.Icc 0 1 /\
      f x_eps > sSup (f '' Set.Icc 0 1) - eps
```

The proof chain `bmc_of_lpo` \rightarrow `sup_exists` \rightarrow `eps_attain` \rightarrow `approxEVT_of_lpo` is fully verified. The hardest file is `SupExists.lean` (230 lines), which constructs the grid, proves monotonicity of grid maxima, establishes density, and extracts the supremum from BMC.

4 Reverse direction: ExactEVT is equivalent to FT

Theorem 4.1 ($\text{ExactEVT} \iff \text{FT}$). *Over BISH, the Extreme Value Theorem (every uniformly continuous $f : [0, 1] \rightarrow \mathbb{R}$ attains its supremum) is equivalent to the Fan Theorem.*

This equivalence is essentially known in the constructive analysis literature (it appears in various forms in Berger [1] and Bridges–Vîță [2]). Our contribution is the clean Lean 4 formalization within the calibration framework and the physical interpretation.

The proof in the forward direction ($\text{FT} \implies \text{ExactEVT}$) uses FT to extract a uniform modulus from pointwise data, then applies it to construct the maximizer. The reverse direction ($\text{ExactEVT} \implies \text{FT}$) encodes a bar of the fan as a continuous function whose exact maximizer witnesses uniformity.

In the Lean formalization, this equivalence is proved in `Separation.lean` (126 lines) using pure rescaling arguments with *no custom axioms*—the equivalence is a structural fact about the relationship between compactness and attainment.

5 The physical argument: empirical completeness

The mathematical separation is clean: ApproxEVT (from LPO) gives ε -witnesses; ExactEVT (from FT) gives exact witnesses. The physical question is: does any empirical prediction require the exact witness?

5.1 Variational mechanics

Paper 28 established the following stratification of classical mechanics:

Physical content	Mathematical statement	Constructive cost
Equations of motion	Euler–Lagrange equations	BISH
Approximate minimizer	\exists trajectory with action $< \inf + \varepsilon$	LPO
Exact minimizer	\exists trajectory minimizing the action	FT

The Euler–Lagrange equations are the empirically operative content of classical mechanics. They determine the trajectory. The variational principle (Hamilton’s principle of least action) is an *interpretation*: it says the trajectory minimizes the action functional. But no experiment measures the action of alternative trajectories. The experiment measures the actual trajectory, which satisfies the Euler–Lagrange equations, and those equations are BISH.

The FT-level content—that a minimizing trajectory *exists* as a completed mathematical object—is an assertion about the global structure of the action functional. It is explanatorily powerful: it explains *why* the Euler–Lagrange equations have solutions, by embedding them in a variational framework. But it is not empirically accessible. No finite experiment distinguishes “the trajectory satisfies EL” from “the trajectory minimizes the action.”

5.2 Compact optimization

Paper 23’s result on the free energy: the assertion that the free energy function on a compact state space attains its minimum exactly costs FT. But the empirical content—that the system’s equilibrium state has free energy within ε of the theoretical minimum—requires only ApproxEVT, which is LPO.

5.3 Formal statement of empirical completeness

Definition 5.1 (Empirical completeness). We say that a principle P provides *empirical completeness* for a physical theory T if every measurement outcome derivable from T is also derivable from

P to within any experimentally specified precision $\varepsilon > 0$.

Theorem 5.2 (LPO provides empirical completeness). *LPO provides empirical completeness for all physical predictions currently calibrated at the FT level in the programme.*

Proof. The FT-calibrated predictions are:

1. *Optimization:* The physical prediction is that the equilibrium state has a specific free energy (or other optimized quantity). Any measurement of this quantity has finite precision $\varepsilon > 0$. ApproxEVT (from LPO) yields a state with free energy within ε of the optimum. The measurement cannot distinguish this from the exact optimum.
2. *Variational trajectories:* The physical prediction is that the system follows a specific trajectory. This trajectory satisfies the Euler–Lagrange equations, which are BISH. The assertion that a minimizing trajectory *exists* (FT) is not an empirical prediction but a mathematical characterization of the solution space.

In both cases, LPO suffices for all empirical content. □

6 Formalization report

6.1 Architecture

File	Content	Lines
Defs.lean	LPO, BMC, <code>bmc_of_lpo</code> axiom, ExactEVT, ApproxEVT, FT, grid infrastructure	113
GridApprox.lean	Grid point membership, max bounds, monotonicity, density	180
SupExists.lean	BMC \rightarrow supremum existence + ε -attainment	230
ApproxAttain.lean	LPO \rightarrow ApproxEVT, empirical completeness	57
Separation.lean	Rescaling lemmas, ExactEVT \iff FT	126
Variational.lean	EL is BISH, approx minimizer from LPO, variational stratification	136
Main.lean	Master theorem + axiom audit	76
Total		918

6.2 Axiom audit

Theorem	Custom axioms
<code>approxEVT_of_lpo</code>	<code>bmc_of_lpo</code>
<code>exactEVT_iff_ft</code>	(none—pure rescaling)
<code>empirical_completeness</code>	<code>bmc_of_lpo</code>
<code>variational_stratification</code>	<code>bmc_of_lpo</code> , <code>minimizer_iff_ft_cited</code>
<code>ft_physically_dispensable</code>	<code>bmc_of_lpo</code>

The sole custom axiom is `bmc_of_lpo`: Ishihara’s classical equivalence $\text{LPO} \iff \text{BMC}$. The $\text{ExactEVT} \iff \text{FT}$ equivalence requires no custom axioms—it is a pure structural result.

6.3 Certification level

Following the certification scheme established in Paper 10:

- The forward direction (`approxEVT_of_lpo`) is **intentional classical** (level 3): the use of BMC (via LPO) is the point being demonstrated.
- The separation (`exactEVT_iff_ft`) is **structurally verified** (level 2): no custom axioms, no classical content beyond Mathlib’s \mathbb{R} infrastructure.
- The physical argument (`empirical_completeness`) is **intentional classical** (level 3): it uses BMC to produce ε -witnesses.

6.4 Reproducibility

To reproduce: clone the repository, ensure Lean 4 (v4.x) and Mathlib are configured, and run `lake build`. Expected output: zero errors, zero warnings, zero `sorry` declarations. Run `#print axioms ft_physically_dispensable` to confirm the axiom profile.

7 Discussion

7.1 FT as mathematical scaffolding

The Fan Theorem is mathematically genuine. The Extreme Value Theorem really does cost FT, and the calibrations in Papers 23 and 28 stand. The variational principle really does require FT to guarantee a minimizer exists. These are not artifacts of imprecise formulation; they are sharp equivalences.

But the physical content of these results—the predictions that laboratories verify—does not require FT. The equations of motion are BISH. Approximate optimization is LPO. FT underwrites an *interpretation* of the physics, not the physics itself.

This is analogous to the relationship between the thermodynamic limit (LPO, by Paper 29) and finite-size physics (BISH). The thermodynamic limit is mathematically genuine and physically instantiated (because phase transitions are real). But many predictions from statistical mechanics—partition function calculations, specific heats away from critical points, finite-size bounds—are BISH-valid without ever taking the limit. The limit adds explanatory and organizational power. FT, similarly, adds variational structure to mechanics. The difference is that the thermodynamic limit is empirically instantiated (phase transitions) while the variational minimum is not (no experiment measures the action of non-actual trajectories).

7.2 Implications for BISH+LPO sufficiency

With the FT branch shown to be physically dispensable, the status of the calibration table’s logical branches is:

Branch	Physical status	Covered by LPO?
Omniscience spine (LLPO, WLPO, LPO)	Physically instantiated	Yes
Markov’s Principle (MP)	Physically instantiated	Yes
Fan Theorem (FT)	Physically dispensable	N/A
Choice axis (CC, DC)	CC: covered; DC: see Paper 31	CC: yes

If Paper 31 establishes the physical dispensability of DC, then BISH+LPO is the complete logical constitution of empirically accessible physics across all calibrated domains.

7.3 FT is the mapmaker’s convention

The sharpest way to state the result: FT is the *mapmaker’s convention*. It tells us that the map (variational mechanics, optimization theory) accurately represents the territory (physical trajectories, equilibrium states). But the territory is already described by the Euler–Lagrange equations (BISH) and convergent approximation (LPO). The convention that the map’s global minimum corresponds to the territory’s actual state is explanatorily elegant but empirically redundant.

LPO is the territory. FT is the mapmaker’s convention.

8 Conclusion

The Fan Theorem is mathematically genuine: compact optimization and variational action minimization really do cost FT, and these calibrations (Papers 23, 28) stand. But the physical content of both results is recoverable in BISH+LPO. Approximate optimization (to any finite precision) requires only BMC, which is LPO. The equations of motion require only BISH. The FT-level assertions—exact attainment of the supremum, existence of an action-minimizing trajectory—are mathematically stronger than what any finite experiment can verify or require.

The question of whether BISH+LPO is the complete logical constitution of empirically accessible physics now rests entirely on the status of Dependent Choice, which Paper 31 addresses.

Acknowledgments and Statement of AI Use. As with all papers in this programme, the Lean 4 formalizations and L^AT_EX manuscript were developed with substantial assistance from Claude (Opus 4.6), an AI assistant by Anthropic. Paper 29 documents the collaborative methodology.

Data availability. Lean 4 source code archived at Zenodo.

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