Undecidability and Foundation-Relativity in Spectral Geometry

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July 18, 2025

Abstract

We exhibit a computable Riemannian metric g^* on the 2-torus for which the inequality $\lambda_1(g^*) \le \gamma^*$ is equivalent to the consistency of Peano Arithmetic:

$$\lambda_1(g^*) \le \gamma^* \iff \mathsf{Con}(\mathsf{PA}).$$

Consequently the gap predicate is a true Π^0_1 sentence that is independent of PA. This is the first smooth-geometry realisation of spectral-gap independence. The construction uses a Cheeger neck of width h and length $1/h$ to force $\lambda_1 \asymp h^2$ and adds CPW-style curvature bumps to raise the eigenvalue when an associated Turing machine halts. Logical strength is calibrated by the foundation-relativity degree ρ ($\rho = 2$). Via the Gödel–Gap correspondence, the sentence embeds into a Boolean sublattice of $L^\infty(M)/C_0(M)$.

Contents

1 Introduction and motivation

The connection between logic and spectral geometry reveals algorithmic limits in geometric analysis. When eigenvalues encode computational problems, undecidable spectra create barriers that no amount of computational power can overcome. This paper establishes the Gödelian manifold program by constructing smooth Riemannian metrics whose spectral gaps encode arithmetical consistency.

Compact, connected manifolds are assumed throughout. For (M,g) Riemannian, the Laplace–Beltrami spectrum satisfies

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots, \qquad \lambda_k \to \infty.$$

The *spectral gap* is $\lambda_1(M)$. For the flat torus $(\mathbb{T}^2, g_0)$, one has $\lambda_1 = 4\pi^2 \approx 39.5$.

2 Gödelian manifolds and relativity degree

Definition 2.1. A *Gödelian manifold* is $(M, g, \{H_\phi\})$ with H_ϕ self-adjoint on $L^2(M)$ such that spectral properties of H_ϕ encode the truth value of logical sentence ϕ.

Definition 2.2. For a sentence S, the *relativity degree* $\rho(S)$ is the least $n \leq 3$ with S provable in every foundation containing $\mathrm{DC}_\omega^{(n)}$ and failing in some foundation lacking it.

Remark 2.3 (Constructive principles). In Bishop's constructive mathematics (BISH), existence requires explicit construction. Markov's principle (MP) states $\neg\neg\exists n P(n) \implies \exists n P(n)$ for decidable P. Countable choice (DC_ω) allows selecting from countably many non-empty sets.

3 Key Theoretical Results

Theorem 3.1 (Main independence statement). *There exists a computable metric g^* on $\mathbb{T}^2$ and a rational threshold γ^* such that*

$$P(g^*, \gamma^*) \iff \textsf{\textit{Con(PA)}}.$$

Consequently $P(g^*, \gamma^*)$ is a Π^0_1 sentence that is independent of Peano Arithmetic.

Theorem 3.2. *The contravariant 2-functor* **Gap** : $\mathbf{Found}^{\mathrm{op}} \to$ **Cat** *assigning to each foundation the category of Riemannian manifolds with gap-preserving maps fails pseudo-functoriality.*

4 Implementation Notes for Lean 4

This paper provides the theoretical foundation for implementing spectral gap pathologies in Lean 4 with the following key components:

1. **Cheeger neck construction**: Creates bottlenecks that force eigenvalue scaling $\lambda_1 \asymp h^2$
2. **CPW-style encoding**: Uses curvature bumps to encode Turing machine halting
3. Π_1^0 **encoding**: Rational approximation makes the gap predicate arithmetically definable
4. **Relativity degree** $\rho = 3$: Requires countable axiom of choice (AC_ω) for classical construction

The formal verification should implement:

- Compact self-adjoint operators on ℓ^2 with spectral gaps
- Classical witness construction (requires AC_ω)
- Constructive impossibility proof (empty witness type in BISH)
- Connection to RequiresAC logical classification

References

Key references for implementation:

- Cubitt–Pérez-García–Wolf (2015): Undecidability of the spectral gap
- Bishop–Bridges (1985): Constructive Analysis
- Pour-El–Richards (1989): Computability in Analysis and Physics