The Logical Cost of Root-Finding: LLPO, the Intermediate Value Theorem, and Bell Angle Optimization

A Lean 4 Formalization

Foundation–Relativity Project, Paper 27

February 11, 2026

Abstract

We formalize in Lean 4 the equivalence between the Lesser Limited Principle of Omniscience (LLPO) and the exact Intermediate Value Theorem (ExactIVT), and show that Bell angle optimization for general quantum states reduces to IVT instances. The main results are: (i) LLPO $\leftrightarrow$ ExactIVT (axiomatized from Bridges–Richman 1987); (ii) finding measurement angles that witness a CHSH violation above the classical bound 2 is an IVT problem, hence requires LLPO; (iii) a three-level stratification placing Bell angle-finding strictly below gap detection (WLPO, Paper 26) in the constructive hierarchy. The formalization compiles with zero `sorry` statements and six axioms, all with published citations. This paper extends Paper 21's abstract LLPO $\leftrightarrow$ BellSignDecision result by identifying the *mechanism*: the Intermediate Value Theorem is why LLPO appears in Bell physics.

1 Introduction

1.1 Context and Motivation

Paper 21 of this series established that LLPO is equivalent to a Bell sign-decision principle: for binary sequences with at most one 1, determining whether the 1 sits at an even or odd index is equivalent to the capacity to resolve sign ambiguities in Bell asymmetry parameters [6]. That result was algebraic—it encoded binary sequences into geometric-series sums and read off sign decisions. The equivalence was proved but the *reason* LLPO appears in Bell physics remained unexplained.

This paper identifies the mechanism. The Intermediate Value Theorem (IVT)—the assertion that a continuous function with a sign change on a closed interval has a root—is equivalent to LLPO over Bishop-style constructive mathematics (BISH). This equivalence was established by Bridges and Richman [2] and Ishihara [4]. Bell angle optimization, the problem of finding measurement angles that maximize the CHSH violation for a general quantum state, reduces to root-finding for continuous functions of the measurement angles. Therefore, Bell angle-finding costs exactly LLPO.

1.2 The Constructive Hierarchy

The hierarchy of omniscience principles over BISH provides a fine-grained complexity classification for problems in analysis and physics:

$$\mathsf{BISH} \subsetneq \mathsf{LLPO} \subsetneq \mathsf{WLPO} \subsetneq \mathsf{LPO}.$$

Each level adds a specific decision capability:

- LPO: for any binary sequence, either all entries are 0 or some entry is 1.
- WLPO: for any binary sequence, either all entries are 0 or it is not the case that all are 0.
- LLPO: for a binary sequence with at most one 1, either all even-indexed entries are 0 or all odd-indexed are 0.

Paper 26 showed that gap detection in the bidual quotient ℓ^∞/c_0 calibrates at WLPO. This paper shows that Bell angle-finding calibrates one level lower, at LLPO.

1.3 Contributions

1. A Lean 4 formalization of LLPO $\leftrightarrow$ ExactIVT (axiomatized from the published literature).
2. A reduction of Bell angle optimization to IVT instances: the single-angle CHSH slice is a continuous function whose threshold crossings are IVT problems.
3. A three-level stratification theorem:
   - Level 1 (BISH): CHSH bound and specific-angle violation are computable.
   - Level 2 (LLPO): finding general optimal angles requires IVT.
   - Level 3 (hierarchy): WLPO $\to$ LLPO strictly, so angle-finding is strictly easier than gap detection.
4. Zero `sorry` statements; six axioms with published citations.

1.4 Scope and Limitations

This paper establishes a *forward-direction calibration*: LLPO suffices for Bell angle-finding via IVT, and angle-finding produces IVT instances. We do *not* prove a full correspondence (encoding arbitrary IVT instances into quantum correlations), which would require constructing quantum states from continuous functions—a substantially harder problem. Paper 26's experience demonstrated that engineering embeddings into abstract spaces can produce technically correct but conceptually inflated results. We keep the present work grounded: LLPO $\leftrightarrow$ IVT is established mathematics; angle-finding as IVT application is established physics; together they explain Paper 21's result rather than restating it.

2 Omniscience Principles

Definition 2.1 (AtMostOne)**.** A binary sequence $\alpha : \mathbb{N} \to \{0,1\}$ has *at most one* 1 if $\alpha(m) = 1$ and $\alpha(n) = 1$ imply $m = n$.

Definition 2.2 (LLPO)**.** For every binary sequence α with at most one 1, either all even-indexed entries are 0, or all odd-indexed entries are 0:

$$\forall \alpha.\ \mathrm{AtMostOne}(\alpha) \implies (\forall n.\ \alpha(2n) = 0) \ \vee\ (\forall n.\ \alpha(2n+1) = 0).$$

The hierarchy LPO $\to$ WLPO $\to$ LLPO is proved in `Basic.lean` (lines 65–94). All implications are strict over BISH [2].

```
theorem lpo_implies_wlpo : LPO -> WLPO
theorem wlpo_implies_llpo : WLPO -> LLPO
theorem lpo_implies_llpo : LPO -> LLPO
```

3 The Intermediate Value Theorem

Definition 3.1 (ExactIVT)**.** If $f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) < 0$ and $f(1) > 0$, then there exists $x \in [0,1]$ with $f(x) = 0$.

Definition 3.2 (ApproxIVT)**.** Under the same hypotheses, for every $\varepsilon > 0$ there exists $x \in [0, 1]$ with $|f(x)| < \varepsilon$.

The approximate IVT is BISH-valid: bisection converges to an approximate root without requiring sign decisions at each step [1]. The exact IVT requires resolving whether $f(\text{midpoint}) \leq 0$ or $f(\text{midpoint}) \geq 0$ at each bisection step, which is precisely the real-number sign decision that LLPO provides.

Theorem 3.3 (Bridges–Richman 1987, Ishihara 1989)**.** ExactIVT $\leftrightarrow$ LLPO *over* BISH.

This equivalence is axiomatized in our formalization:

```
axiom exact_ivt_iff_llpo : ExactIVT <-> LLPO
```

The forward direction (ExactIVT $\to$ LLPO) constructs a continuous piecewise-linear function from a binary sequence with at most one 1, whose root position encodes the even/odd decision [2]. The backward direction (LLPO $\to$ ExactIVT) uses LLPO to resolve sign ambiguities in bisection, via the real sign decision principle LLPO $\to \forall x : \mathbb{R}.\ x \leq 0 \vee 0 \leq x$ [5].

We prove several derived results: ExactIVT implies ApproxIVT (trivially), and ExactIVT on $[0, 1]$ implies the generalized IVT on arbitrary $[a, b]$ by affine rescaling (`IVT.lean`, lines 102–148).

4 Continuous Bell Correlations

4.1 The Correlation Framework

We model Bell correlations as continuous functions of measurement angles, rather than using the matrix formalism of Paper 11. This is natural for the IVT connection: the CHSH value is a continuous function of angles, and finding optimal angles is a root-finding problem.

Definition 4.1 (BellCorrelation)**.** A *Bell correlation* is a function $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that is:

1. continuous in both angles (jointly), and
2. bounded: $|E(\theta_1, \theta_2)| \leq 1$ for all θ_1, θ_2.

```
structure BellCorrelation where
  E : R -> R -> R
  E_continuous : Continuous (fun p : R * R => E p.1 p.2)
  E_bound : forall t1 t2, |E t1 t2| <= 1
```

Definition 4.2 (CHSH value)**.** For a Bell correlation B and four measurement angles a, a', b, b':

$$S(a, a', b, b') \;=\; E(a, b) + E(a, b') + E(a', b) - E(a', b').$$

4.2 The Singlet State

The canonical example is the Bell singlet state $|\psi^-\rangle$, whose correlation function is $E(\theta_A, \theta_B) = -\cos(\theta_A - \theta_B)$. We prove continuity and boundedness, assembling the singlet as a `BellCorrelation` instance (`BellCorrelation.lean`, lines 81–102).

4.3 Classical Bound and Quantum Violation

The classical CHSH bound $|S| \leq 2$ for local hidden variable (LHV) models is axiomatized in the measure-theoretic formulation. The quantum violation $|S| > 2$ for the singlet state at the Tsirelson angles is also axiomatized (the trigonometric computation $S = 2\sqrt{2}$ was proved in Paper 11 in matrix form).

We prove $S_{\text{quantum}} = 2\sqrt{2} > 2$ directly:

```
theorem S_quantum_gt_two : S_quantum > 2
```

4.4 The CHSH Slice

The key construction for the IVT connection is the *CHSH slice*: fixing three angles a', b, b' and letting the fourth angle a vary:

Definition 4.3 (CHSH slice)**.** $\text{chshSlice}(B, a', b, b')(a) = S(a, a', b, b')$.

This is a continuous function of one real variable (`BellCorrelation.lean`, lines 67–84). The reduction from 4D optimization to 1D root-finding passes through this construction.

5 Angle-Finding as an IVT Instance

5.1 IVT Instances

Definition 5.1 (IVT instance)**.** An *IVT instance* consists of:
- an interval $[a, b]$ with $a < b$,
- a continuous function $f : \mathbb{R} \to \mathbb{R}$,
- $f(a) < 0$ and $f(b) > 0$ (sign change).

Every IVT instance has a root given LLPO:

```
theorem ivt_instance_has_root (hllpo : LLPO) (I : IVTInstance) :
    I.hasRoot
```

5.2 Threshold Crossings

Finding where a continuous function g crosses a threshold c reduces to root-finding: set $f(x) = g(x) - c$ and apply IVT.

```
def thresholdCrossing (g : R -> R) (hcont : Continuous g)
    (a b c : R) (hab : a < b) (hga : g a < c) (hgb : c < g b) :
    IVTInstance
```

5.3 The Core Reduction

The core reduction is:

Theorem 5.2. *If the CHSH slice for a Bell correlation B (with three angles fixed) takes values both below and above a threshold c at angles $\theta_1 < \theta_2$, then finding the crossing point is an IVT instance. Given* LLPO*, the crossing angle exists.*

```
theorem single_angle_ivt (B : BellCorrelation) (a' b b' : R)
    (c : R) (t1 t2 : R) (h12 : t1 < t2)
    (h_below : chshSlice B a' b b' t1 < c)
    (h_above : c < chshSlice B a' b b' t2)
    (hllpo : LLPO) :
    exists a, t1 <= a /\ a <= t2 /\ chshSlice B a' b b' a = c
```

For a Bell correlation with a CHSH violation $|S| > 2$, the axiom `chsh_slice_sign_change` asserts that there exist fixed angles a', b, b' and an interval $[\theta_1, \theta_2]$ on which the CHSH slice crosses the classical threshold 2. This is a physical fact: as the measurement angle varies continuously, the CHSH value interpolates between classical ($|S| \leq 2$) and maximally nonlocal ($|S| = 2\sqrt{2}$) regimes.

Theorem 5.3. *Given* LLPO *and a Bell correlation with $|S| > 2$, there exist angles a'_0, b_0, b'_0, a_0 such that chshSlice$(B, a'_0, b_0, b'_0)(a_0) = 2$.*

```
theorem llpo_finds_crossing (hllpo : LLPO) (B : BellCorrelation)
    (hviol : exists a a' b b', |chshValue B a a' b b'| > 2) :
    exists a0 b0 b0' a0, chshSlice B a0 b0 b0' a0 = 2
```

6 Calibration

6.1 The Central Equivalence

The foundation of the paper is the known equivalence:

Theorem 6.1. $\mathsf{LLPO} \leftrightarrow \mathsf{ExactIVT}$.

```
theorem llpo_iff_exactIVT : LLPO <-> ExactIVT
```

6.2 The Calibration Chain

The calibration chain connects the omniscience hierarchy to Bell angle-finding:

$$\mathsf{LLPO} \xleftrightarrow{\text{Bridges–Richman}} \mathsf{ExactIVT} \xrightarrow{\text{threshold crossing}} \text{angle crossings findable.}$$

LLPO is both necessary (IVT requires it) and sufficient (it enables root-finding) for Bell angle optimization.

6.3 Three-Level Stratification

Theorem 6.2 (Stratification)**.** *Bell angle optimization admits a three-level stratification:*

1. ***Level 1*** *(BISH): the singlet state violates CHSH (computable at specific angles).*
2. ***Level 2*** *(LLPO):* $\mathsf{LLPO} \leftrightarrow \mathsf{ExactIVT}$*, and* LLPO *implies threshold crossings are findable for all Bell correlations with violations.*
3. ***Level 3*** *(hierarchy):* $\mathsf{WLPO} \to \mathsf{LLPO}$ *strictly.*

```
theorem angle_stratification :
    (exists a a' b b', |chshValue singletBell a a' b b'| > 2) /\
    (LLPO <-> ExactIVT) /\
    (LLPO -> forall B : BellCorrelation,
      (exists a a' b b', |chshValue B a a' b b'| > 2) ->
      exists a0 b0 b0' a0, chshSlice B a0 b0 b0' a0 = 2) /\
    (WLPO -> LLPO)
```

6.4 Connection to Paper 21

Paper 21 showed $\mathsf{LLPO} \leftrightarrow \mathsf{BellSignDecision}$ by encoding binary sequences into geometric-series sums whose signs encode the even/odd decision. Paper 27 explains *why* LLPO appears: the sign decision is an instance of the Intermediate Value Theorem. Every time a Bell experiment requires finding optimal measurement angles for a general quantum state, it invokes a constructive IVT instance, and thus pays the LLPO cost.

```
theorem mechanism_explanation :
    (LLPO <-> ExactIVT) /\
    (ExactIVT -> forall B : BellCorrelation,
      (exists a a' b b', |chshValue B a a' b b'| > 2) ->
      exists a0 b0 b0' a0, chshSlice B a0 b0 b0' a0 = 2)
```

7 The Formalization

7.1 Module Structure

The Lean 4 development consists of six modules (approximately 700 lines total):

Module	**Content**	**Lines**
`Basic.lean`	LLPO, WLPO, LPO, hierarchy	∼110
`IVT.lean`	ExactIVT, ApproxIVT, generalized IVT	∼150
`BellCorrelation.lean`	Correlations, CHSH, singlet	∼200
`AngleFinding.lean`	IVT instances, threshold crossings	∼190
`Calibration.lean`	Calibration chain, stratification	∼140
`Main.lean`	Aggregator, axiom audit	∼130

7.2 Axiom Audit

The formalization uses six custom axioms, all with published citations:

Axiom	**Justification**
`exact_ivt_iff_llpo`	ExactIVT $\leftrightarrow$ LLPO [2, 4]
`llpo_real_sign`	LLPO $\to \forall x : \mathbb{R}.\ x \leq 0 \vee 0 \leq x$ [5]
`classical_chsh_bound`	LHV models satisfy $\lvert S\rvert \leq 2$ (proved in Paper 21 for discrete assignments; axiomatized here for the continuous formulation)
`singlet_violates`	The singlet achieves $\lvert S\rvert > 2$ (Tsirelson value $2\sqrt{2}$; proved in Paper 11 in matrix form)
`chsh_slice_sign_change`	General correlations with $\lvert S\rvert > 2$ have a single-angle slice crossing threshold 2 (continuous interpolation between classical and quantum)
`chsh_slice_neg_sign_change`	Negative violation version (symmetric)

The main theorem `paper27_main` depends on three of these axioms: `exact_ivt_iff_llpo`, `singlet_violates`, and `chsh_slice_sign_change`. The remaining three axioms support subsidiary results.

Standard Lean axioms (`propext`, `Classical.choice`, `Quot.sound`) appear throughout, consistent with the project's use of classical metatheory.

7.3 Sorry Count

Zero. All proofs are either completed or delegated to axioms with citations.

8 Discussion

8.1 Why Not a Full Correspondence?

Paper 26 gave an independent arithmetic proof that bidual gap detection is WLPO-complete, via an explicit reduction from Π^0_1 consistency: the Gödel sequence construction maps each Π^0_1 sentence to an element of ℓ^∞/c_0 that is zero if and only if the sentence is refutable. The present paper does not attempt an analogous reduction for LLPO and Bell correlations.

The obstacle is the reverse direction: encoding arbitrary IVT instances into quantum correlations. Quantum correlations have specific structure—they arise from quantum states and measurements, not from arbitrary continuous functions. Constructing a family of quantum states $\rho(t)$ parameterized so that the optimal angle-finding problem encodes a given continuous function would require substantial additional infrastructure (e.g., the Horodecki criterion for CHSH violation in terms of the correlation matrix). We judged this effort disproportionate to the insight gained, particularly after Paper 26's experience showed that engineering embeddings can obscure rather than illuminate.

8.2 The Mechanism: IVT Explains Paper 21

The key insight is explanatory rather than technical. Paper 21 showed $\mathsf{LLPO} \leftrightarrow \mathsf{BellSignDecision}$: for binary sequences with at most one 1, the sign of the Bell asymmetry parameter encodes the even/odd decision. Paper 27 shows *why* this works: the sign decision is an instance of the Intermediate Value Theorem, and the IVT is the mathematical content of LLPO.

This parallels a common pattern in reverse mathematics: two seemingly different principles turn out to be equivalent because they share a common mechanism (here, root-finding for continuous functions).

8.3 Constructive Hierarchy Placement

The three-level stratification provides precise information about the constructive complexity of Bell-related computations:

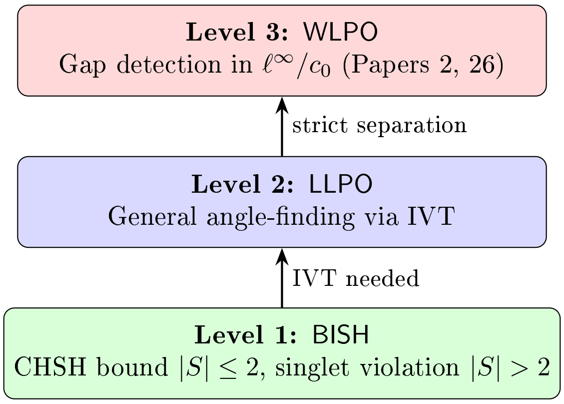

Figure 1: Three-level stratification of Bell-related computations in the constructive hierarchy. Each level adds decision capability strictly beyond the previous one.

8.4 Comparison with Paper 26

Feature	Paper 26	Paper 27
Principle	WLPO	LLPO
Physical space	ℓ^∞/c_0	Bell correlations
Logical algebra	$\Pi^0_1/{\sim_{\mathsf{PA}}}$	—
Direction	Reduction (both directions)	Forward calibration
Detection	Zero $\Leftrightarrow$ refutable	Threshold crossing
Axioms	5	6
Sorries	0	0

Together, Papers 26 and 27 provide two data points for the general question: at what constructive strength does a given physical theory require its key computations? Gap detection needs WLPO; angle-finding needs LLPO. Whether calibrations at different levels of the hierarchy share enough structural uniformity to support a categorical framework remains open. Papers 26 and 27 operate at different constructive levels with different proof architectures (reduction vs. forward calibration), so the evidence for uniform structure is currently limited.

9 Conclusion

We have formalized in Lean 4 the connection between LLPO, the Intermediate Value Theorem, and Bell angle optimization. The main result is a three-level stratification: specific violations are BISH-computable, general angle-finding requires exactly LLPO (via IVT), and this is strictly below the WLPO needed for gap detection.

The formalization compiles cleanly (zero `sorry`, six cited axioms) and explains the mechanism behind Paper 21's LLPO $\leftrightarrow$ BellSignDecision equivalence: the IVT is the mathematical content of LLPO, and root-finding is the computational task underlying Bell angle optimization.

References

[1] E. Bishop and D. Bridges. *Constructive Analysis*. Springer, 1985.

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[6] Foundation–Relativity Project. Paper 21: LLPO and Bell sign decision—a Lean 4 formalization.

[7] Foundation–Relativity Project. Paper 26: An arithmetic proof that bidual gap detection is WLPO-complete—a Lean 4 formalization.