The Event Horizon as a Logical Boundary:

Schwarzschild Interior Geodesic Incompleteness and LPO in Lean 4

Paper 13 in the Constructive Reverse Mathematics Series

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Abstract

We formalize in LEAN 4 a decomposition of Schwarzschild interior physics into constructive content layers. The explicit cycloid geodesic $r(\eta) = M(1+\cos\eta)$ reaches $r = 0$ at proper time $\tau^* = \pi M$ constructively; the Kretschmann scalar $K = 48M^2/r^6$ is constructively computable for any $r > 0$; and for any $\varepsilon > 0$, there exists an explicit η with $r(\eta) < \varepsilon$. All of this is BISH (Height 0). The abstract principle that *every* bounded monotone decreasing sequence in $[0, 2M)$ converges to a definite real limit—the completed-limit formulation of geodesic incompleteness—is equivalent to the Limited Principle of Omniscience (LPO). The event horizon at $r = 2M$ thus demarcates not only the causal boundary from which light cannot escape, but a logical boundary: the exterior geometry (Paper 1) and the interior's finite-time physics are BISH, while the singularity assertion as a completed-limit principle requires exactly LPO. The formalization comprises 1,021 lines across 8 modules with zero `sorry` statements. One interface assumption (`bmc_of_lpo`, the Bridges–Vîţă equivalence imported from Paper 8) is axiomatized with citation. The `Classical.choice` in the axiom profile arises from MATHLIB4 infrastructure; the constructive calibration is established by proof-content analysis (see §6 and Paper 10).

Contents

*New York University. AI-assisted formalization; see §7 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or general relativity; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 Physical Context

The Schwarzschild solution describes the geometry of a non-rotating, uncharged black hole of mass M. The event horizon at $r = 2M$ is the boundary of the region from which future-directed causal curves can reach infinity. Inside the horizon, the radial coordinate r becomes timelike: the metric component g_{rr} changes sign, and a freely falling observer necessarily moves toward decreasing r. The observer reaches the curvature singularity at $r = 0$ in finite proper time.

This "geodesic incompleteness"—the fact that timelike geodesics terminate at the singularity—is the physical content that the Penrose singularity theorem Penrose [1965] generalizes to arbitrary spacetimes satisfying energy conditions. For the Schwarzschild interior, the explicit solution is available: a radially infalling particle dropped from rest at the horizon follows the cycloid

$$r(\eta) = M(1 + \cos\eta), \qquad \tau(\eta) = M(\eta + \sin\eta), \qquad \eta \in [0, \pi], \tag{1}$$

reaching $r = 0$ at proper time $\tau^* = \pi M$.

1.2 The CRM Question

From the standpoint of constructive reverse mathematics (CRM), the question is: what is the logical cost of asserting geodesic incompleteness?

The answer turns out to be more subtle than a single classification. It decomposes into two layers:

- The **specific** cycloid geodesic reaching $r = 0$ is constructively computable (BISH). The endpoint is given by explicit trigonometric evaluation: $r(\pi) = M(1 + \cos\pi) = 0$.

- The **abstract principle** that every bounded monotone decreasing sequence in the interior converges to a definite real limit—the completed-limit formulation that underlies the general assertion of geodesic incompleteness—is equivalent to LPO.

This decomposition mirrors Paper 8's treatment of the 1D Ising model Lee [2026d], where finite-size bounds are BISH but the thermodynamic limit (a completed-limit assertion about the free energy sequence) costs LPO.

1.3 Contributions

1. Machine-verified proof that `SchwarzschildInteriorGeodesicIncompleteness` $\leftrightarrow$ LPO (1,021 lines of LEAN 4, zero `sorry`).
2. Explicit BISH content: cycloid computability, Kretschmann scalar divergence, constructive approaching of the singularity—all Height 0.
3. The event horizon as a logical boundary: the first result calibrating a general-relativistic singularity assertion in the constructive hierarchy.
4. Connection to Paper 8 via BMC $\leftrightarrow$ LPO, extending the dispensability–calibration pattern from statistical mechanics to gravitation.

1.4 Related Work

Paper 1 Lee [2026a] established that the Schwarzschild *exterior* geometry—metric components, Christoffel symbols, Riemann tensor, Ricci flatness, Kretschmann scalar—is BISH (Height 0). Paper 8 Lee [2026d] proved that bounded monotone convergence, instantiated through the 1D Ising free energy, is equivalent to LPO. The equivalence BMC $\leftrightarrow$ LPO itself is due to Bridges and Vîţă Bridges and Vîţă [2006] (Theorem 2.1.5).

To our knowledge, no prior work applies constructive reverse mathematics to general-relativistic singularities. Echenim and Mhalla Echenim and Mhalla [2024] formalized the CHSH inequality in Isabelle/HOL, but in a classical setting without constructive analysis.

2 Mathematical Content

2.1 The Schwarzschild Interior

The Schwarzschild metric in standard coordinates is

$$ds^2 = -f(M,r)\,dt^2 + f(M,r)^{-1}\,dr^2 + r^2\,d\Omega^2, \qquad f(M,r) = 1 - \frac{2M}{r}.$$

For $r > 2M$ (exterior), $f > 0$ and t is timelike. For $0 < r < 2M$ (interior), $f < 0$: the roles of t and r swap, making r a timelike coordinate and t a spacelike coordinate. This signature flip means that a freely falling observer in the interior cannot remain at constant r; the radial coordinate necessarily decreases.

Definition 2.1 (Interior domain). ✓ The *interior domain* is the set of pairs (M, r) satisfying $M > 0$, $r > 0$, and $r < 2M$. Equivalently, $f(M,r) < 0$.

2.2 The Cycloid Geodesic (BISH)

For a particle dropped from rest at the event horizon with specific energy $E = 1$, the radial geodesic equation has the closed-form cycloid solution (1). This explicit parametrization avoids formalizing ODE existence or uniqueness theory.

Theorem 2.2 (Cycloid properties). ✓ *For $M > 0$ and $\eta \in [0, \pi]$:*

(a) $r(0) = 2M$ (starts at the horizon) and $r(\pi) = 0$ (reaches the singularity).

(b) $\tau(0) = 0$ and $\tau(\pi) = \pi M$ (finite proper time).

(c) r is strictly decreasing on $[0, \pi]$: $r'(\eta) = -M \sin \eta < 0$ for $\eta \in (0, \pi)$.

(d) τ is strictly increasing on $[0, \pi]$: $\tau'(\eta) = M(1 + \cos \eta) > 0$ for $\eta \in (0, \pi)$.

(e) For $\eta \in (0, \pi)$: $0 < r(\eta) < 2M$ (lies in the interior).

Proof. All properties follow from direct computation using standard trigonometric identities. Part (a): $r(0) = M(1 + 1) = 2M$ and $r(\pi) = M(1 + (-1)) = 0$. Part (b): $\tau(0) = M(0 + 0) = 0$ and $\tau(\pi) = M(\pi + 0) = \pi M$. Part (c): the derivative $r'(\eta) = -M \sin \eta$ is negative for $\eta \in (0, \pi)$ since $\sin \eta > 0$ on this interval. Part (d): $\tau'(\eta) = M(1 + \cos \eta) > 0$ since $\cos \eta > -1$ for $\eta \in (0, \pi)$. Part (e): $\cos \eta \in (-1, 1)$ for $\eta \in (0, \pi)$, so $r(\eta) = M(1 + \cos \eta) \in (0, 2M)$. □

Theorem 2.3 (Constructive approaching). ✓ *For any $M > 0$ and $\varepsilon > 0$, there exists $\eta \in (0, \pi)$ such that $r(\eta) < \varepsilon$.*

Proof. Since r is continuous and $r(\pi) = 0$, by continuity at π, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|r(\eta) - 0| < \varepsilon$ whenever $|\eta - \pi| < \delta$. Choosing $\eta_0 = \pi - \min(\delta/2, \pi/2) \in (0, \pi)$ gives $r(\eta_0) < \varepsilon$. □

Remark 2.4 (BISH status). All results in this subsection are BISH (Height 0). The cycloid is an explicit computable function; no omniscience principle is needed. In particular, *reaching the singularity along the specific cycloid is constructive.* The LPO cost arises elsewhere.

2.3 The Completed-Limit Formulation

The key insight is to formulate geodesic incompleteness not as a statement about the specific cycloid solution, but as a universal principle about all infalling trajectories.

Definition 2.5 (Schwarzschild Interior Geodesic Incompleteness). ✓ *SchwarzschildInterior-GeodesicIncompleteness* is the assertion: for every $M > 0$ and every sequence $a : \mathbb{N} \to \mathbb{R}$ satisfying

(i) a is antitone (non-increasing),

(ii) $a(n) \geq 0$ for all n,

(iii) $a(0) < 2M$,

there exists $L \in \mathbb{R}$ such that a converges to L: for every $\varepsilon > 0$, there exists N_0 with $|a(N) - L| < \varepsilon$ for all $N \geq N_0$.

This is the universally quantified completed-limit assertion. It says: any monotone bounded sequence in the interior has a definite limit. This captures the logical content of "every radially infalling trajectory has a definite endpoint."

Remark 2.6 (On the formulation). In classical general relativity, geodesic incompleteness *is* a completed-limit assertion about monotone sequences. The radial coordinate along any infalling timelike geodesic is a monotone decreasing function of proper time, bounded below by 0. The assertion "the spacetime is geodesically incomplete" means these sequences converge to definite limits.

Our formalization strips away the differential geometry—we do not formalize the Lorentzian metric, parallel transport, or the geodesic equation as an ODE—and isolates the exact logical content: bounded monotone convergence for decreasing sequences in $[0, 2M)$. This extraction of logical content from physical assertion is the standard methodology of constructive reverse mathematics.

Theorem 2.7 (Main Theorem). ✓

$$\textit{SchwarzschildInteriorGeodesicIncompleteness} \longleftrightarrow \text{LPO}.$$

2.4 Forward Direction: Incompleteness ⇒ LPO

Given an arbitrary binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, we construct a monotone bounded sequence whose limit encodes whether α is identically zero or has a true term.

Definition 2.8 (Geodesic coupling). ✓ Fix $M > 0$. Define

$$a(n) := \texttt{geodesicCoupling}\ \alpha\ M\ 0\ n = \begin{cases} M & \text{if } \texttt{runMax}\ \alpha\ n = \texttt{false}, \\ 0 & \text{if } \texttt{runMax}\ \alpha\ n = \texttt{true}, \end{cases}$$

where $\texttt{runMax}\ \alpha\ n = \max(\alpha(0), \dots, \alpha(n))$.

This sequence takes values in $\{M, 0\}$ and is antitone (it can only jump from M to 0, never back). It is non-negative, and $a(0) \le M < 2M$, so it satisfies the hypotheses of Definition 2.5.

Proof of the forward direction. Fix $M = 1$ (any $M > 0$ works). Apply *SchwarzschildInterior-GeodesicIncompleteness* to the coupling sequence a to obtain a limit L. Let N_1 be the convergence index for $\varepsilon = M/2$, so $|a(N_1) - L| < M/2$.

Case split on $\texttt{runMax}\ \alpha\ N_1$ (a Bool, so definitionally decidable—no real-number comparison needed):

Case $\texttt{runMax}\ \alpha\ N_1 = \texttt{true}$**:** There exists $k \le N_1$ with $\alpha(k) = \texttt{true}$. We have $\exists n, \alpha(n) = \texttt{true}$.

Case $\texttt{runMax}\ \alpha\ N_1 = \texttt{false}$**:** Then $a(N_1) = M$. Suppose for contradiction that $\exists n_0, \alpha(n_0) = \texttt{true}$. Then the sequence is eventually 0, so $L = 0$. But $|M - 0| = M$, while the convergence modulus gives $|a(N_1) - L| < M/2$, yielding $M < M/2$, a contradiction. Therefore $\forall n, \alpha(n) = \texttt{false}$.

In both cases, we have decided: either $\forall n, \alpha(n) = \texttt{false}$ or $\exists n, \alpha(n) = \texttt{true}$. This is LPO. □

The gap $\delta = M - 0 = M > 0$ between the two possible limit values (M when $\alpha \equiv \texttt{false}$, and 0 when $\exists n, \alpha(n) = \texttt{true}$) serves as the "decision amplifier."

2.5 Reverse Direction: LPO ⇒ Incompleteness

Proof of the reverse direction. LPO implies BMC by the Bridges–Vîţă equivalence Bridges and Vîţă [2006] (axiomatized as `bmc_of_lpo` from Paper 8).

Given an antitone sequence a with $a(n) \ge 0$ and $a(0) < 2M$, define $b(n) = -a(n)$. Then b is monotone (non-decreasing) and bounded above by 0. By BMC, b converges to some limit L_{neg}. Then a converges to $-L_{\text{neg}}$. □

2.6 The Honest Decomposition

The paper does *not* claim "computing the cycloid endpoint costs LPO." The cycloid is BISH. The paper claims:

Content	Principle	Certification
Cycloid $r(\pi) = 0$	BISH	Height 0 (machine-verified)
Cycloid approaching: $\forall\varepsilon > 0,\ \exists\eta,\ r(\eta) < \varepsilon$	BISH	Height 0 (machine-verified)
Kretschmann $K = 48M^2/r^6$ computable	BISH	Height 0 (machine-verified)
"Every antitone bounded sequence in $[0, 2M)$ has a limit"	LPO	$\leftrightarrow$ equivalence (machine-verified)

The event horizon separates the BISH exterior (Paper 1) from the interior where the *general completed-limit principle* costs LPO. The specific solution is constructive; the universal assertion is not.

3 Lean Formalization

3.1 Architecture

The formalization is organized as a single Lean 4 project with 8 modules:

Module	Lines	Content
`Basic.lean`	168	LPO, BMC, Interior, `runMax` + lemmas
`RadialGeodesic.lean`	250	Cycloid parametrization, monotonicity, approaching
`Incompleteness.lean`	167	`SchwarzschildInteriorGeodesicIncompleteness` + coupling
`LPO_Forward.lean`	91	$\rightarrow$ direction via gap encoding
`LPO_Reverse.lean`	57	$\leftarrow$ direction via BMC
`BISH_Content.lean`	128	Kretschmann scalar, cycloid computability
`Certificates.lean`	85	`#print axioms` audit
`Main.lean`	75	Assembly
Total	**1,021**	

Table 1: Module structure of Paper 13.

3.2 Key Design Decisions

Cycloid-first approach. Rather than formalizing the geodesic equation as an ODE (which would require Lean ODE infrastructure that does not yet exist in Mathlib4), we work with the closed-form cycloid solution directly. Monotonicity, boundedness, and convergence properties follow from explicit trigonometric identities. This is a pragmatic choice: the cycloid is the physically relevant solution for $E = 1$, and its properties are all that the forward direction requires.

Abstract incompleteness. `SchwarzschildInteriorGeodesicIncompleteness` quantifies over *all* antitone non-negative sequences starting below $2M$, not over solutions of the geodesic equation. This is the correct level of abstraction for CRM: the logical content is the completed-limit principle, and the geodesic equation is the physical motivation for why such sequences arise. The Brouwerian counterexample (`geodesicCoupling`) is not a geodesic—it is a valid element of the domain of the universally quantified proposition. This is standard CRM methodology.

BMC import. The `bmc_of_lpo` axiom imports the Bridges–Vîţă equivalence Bridges and Vîţă [2006] from Paper 8. This is the same interface assumption used in Paper 8 and documented in the series methodology (Paper 10).

3.3 Core Definitions

```
/-- Limited Principle of Omniscience. -/
def LPO : Prop :=
  forall (a : Nat -> Bool),
    (forall n, a n = false) ||| (exists n, a n = true)

/-- Bounded Monotone Convergence. -/
def BMC : Prop :=
  forall (a : Nat -> Real) (M : Real),
    Monotone a -> (forall n, a n <= M) ->
    exists L : Real, forall e : Real, 0 < e ->
      exists N0 : Nat, forall N : Nat,
        N0 <= N -> |a N - L| < e

/-- LPO implies BMC [Bridges-Vita 2006]. -/
axiom bmc_of_lpo : LPO -> BMC
```

Listing 1: Core definitions (Basic.lean, excerpts).

```
/-- Schwarzschild factor f(M, r) = 1 - 2M/r. -/
noncomputable def f (M r : Real) : Real := 1 - 2 * M / r

/-- Interior domain: 0 < r < 2M, M > 0. -/
structure Interior (M r : Real) : Prop where
  mass_pos : M > 0
  r_pos : r > 0
  r_inside : r < 2 * M
```

Listing 2: Interior domain and Schwarzschild factor (Basic.lean).

```
noncomputable def r_cycloid (M n : Real) : Real :=
  M * (1 + cos n)

noncomputable def t_cycloid (M n : Real) : Real :=
  M * (n + sin n)
```

Listing 3: Cycloid definitions (RadialGeodesic.lean).

```
def SchwarzschildInteriorGeodesicIncompleteness : Prop :=
  forall (M : Real), M > 0 ->
  forall (a : Nat -> Real),
    Antitone a ->
    (forall n, 0 <= a n) ->
    a 0 < 2 * M ->
    exists L : Real,
      forall e > 0, exists N0 : Nat,
        forall N >= N0, |a N - L| < e

noncomputable def geodesicCoupling
    (a : Nat -> Bool) (v0 v1 : Real) (n : Nat) : Real :=
  if runMax a n then v1 else v0
```

Listing 4: Geodesic incompleteness and coupling (Incompleteness.lean).

3.4 Main Theorems

```
theorem geodesic_incompleteness_implies_lpo
    (hGI : SchwarzschildInteriorGeodesicIncompleteness) :
    LPO := by
  intro a
  set M : Real := 1
  have hM : M > 0 := one_pos
  set a := geodesicCoupling a M 0 with ha_def
  have h_anti : Antitone a :=
    geodesicCoupling_antitone a (le_of_lt hM)
  have h_nn : forall n, 0 <= a n :=
    geodesicCoupling_nonneg a (le_of_lt hM) (le_refl 0)
  have h_a0 : a 0 < 2 * M := by
    have := geodesicCoupling_le a (le_of_lt hM) 0; linarith
  obtain <<L, hL>> := hGI M hM a h_anti h_nn h_a0
  obtain <<N1, hN1>> := hL (M / 2) (by linarith)
  have hN1_self := hN1 N1 (le_refl _)
  cases hm : runMax a N1
  . -- runMax = false: prove all n, a n = false
    left
    apply bool_not_exists_implies_all_false
    intro <<n0, hn0>>
    have hL_val := limit_of_exists_true a M 0 hL hn0
    have haN1 : a N1 = M := by
      simp only [ha_def, geodesicCoupling, hm,
        Bool.false_eq_true, reduce_ite]
    rw [haN1, hL_val] at hN1_self
    simp at hN1_self
    rw [abs_of_pos hM] at hN1_self
    linarith
  . -- runMax = true: extract witness
    right
    obtain <<k, _, hk>> :=
      runMax_witness a (show runMax a N1 = true from hm)
    exact <<k, hk>>
```

Listing 5: Forward direction (LPO_Forward.lean, complete).

```
theorem lpo_implies_geodesic_incompleteness (hLPO : LPO) :
    SchwarzschildInteriorGeodesicIncompleteness := by
  intro M _hM a ha hnn _ha0
  have hBMC := bmc_of_lpo hLPO
  have hMono : Monotone (fun n => -a n) :=
    fun m n hmn => by simp only [neg_le_neg_iff]; exact ha hmn
  have hBound : forall n, (fun n => -a n) n <= 0 :=
    fun n => by simp only [neg_nonpos]; exact hnn n
  obtain <<L_neg, hL>> :=
    hBMC (fun n => -a n) 0 hMono hBound
  refine <<-L_neg, fun e he => ?_>>
  obtain <<N0, hN0>> := hL e he
  exact <<N0, fun N hN => by
    have := hN0 N hN
    rwa [show a N - (-L_neg) = -((-a N) - L_neg)
      from by ring, abs_neg]>>
```

Listing 6: Reverse direction (LPO_Reverse.lean, complete).

```
theorem schwarzschild_interior_geodesic_incompleteness_iff_LPO :
    SchwarzschildInteriorGeodesicIncompleteness <-> LPO :=
```

```
<<geodesic_incompleteness_implies_lpo,
 lpo_implies_geodesic_incompleteness>>
```

Listing 7: Main theorem assembly (Main.lean).

3.5 Axiom Audit

The `Certificates.lean` module audits the axiom profile of each theorem via `#print axioms`:

```
#print axioms
  schwarzschild_interior_geodesic_incompleteness_iff_LPO
-- [propext, Classical.choice, Quot.sound, bmc_of_lpo]

#print axioms r_cycloid_at_pi
-- [propext, Classical.choice, Quot.sound]

#print axioms bish_content_complete
-- [propext, Classical.choice, Quot.sound]
```

Listing 8: Axiom audit (Certificates.lean, selected).

The `Classical.choice` appearing in the BISH results (cycloid, Kretschmann) arises from Mathlib4's real number infrastructure—specifically, the construction of $\mathbb{R}$ via Cauchy completion, which pervasively uses `Classical.choice` as a metatheoretic convenience. The mathematical content of these proofs is constructive: they involve only explicit trigonometric computation on computable real numbers. The constructive calibration is established by proof-content analysis, following the methodology described in Paper 10.

4 The Event Horizon as a Logical Boundary

4.1 The Causal Boundary

In classical GR, the event horizon at $r = 2M$ is the boundary of the causal past of future null infinity $\mathscr{I}^+$. Light signals emitted from $r < 2M$ cannot reach distant observers. This is a statement about the global causal structure of the Schwarzschild spacetime: future-directed null geodesics from the interior are trapped.

4.2 The Logical Boundary

Our result reveals a second boundary coinciding with the event horizon:

Exterior ($r > 2M$): Paper 1 establishes that the Schwarzschild geometry—metric components, Christoffel symbols, Riemann tensor, Ricci flatness, Kretschmann scalar—is BISH. No omniscience principle is needed.

Interior ($0 < r < 2M$): The finite-time physics remains BISH (cycloid computability, curvature divergence). But the *completed assertion* that every bounded monotone trajectory in the interior converges to a definite limit—the general singularity assertion—costs LPO.

The horizon thus demarcates not only what can communicate with infinity, but what can be asserted without surveying an infinite set.

4.3 Connection to Paper 8

The pattern is identical to the 1D Ising model:

	Paper 8 (Ising)	Paper 13 (Schwarzschild)
BISH content	Finite-size partition function	Cycloid geodesic, Kretschmann scalar
LPO content	Thermodynamic limit (free energy convergence)	Geodesic incompleteness (completed limit)
Physical interpretation	Phase transitions require idealization	Singularities require idealization
Encoding technique	Coupling constant modulation	Sequence coupling (`geodesicCoupling`)

Both cases instantiate the same principle: completed infinite limits in physics cost LPO, but the finite approximations that carry empirical content are BISH.

4.4 What This Does NOT Claim

This paper does not claim that "reaching the singularity costs LPO" in any operational sense. A freely falling observer following the cycloid geodesic computes $r(\tau)$ constructively at every finite proper time, and the limit $r \to 0$ is constructively approachable (for any ε, we can find η with $r(\eta) < \varepsilon$). The LPO cost attaches to the *universally quantified completeness principle*—the assertion that every bounded monotone trajectory in the interior converges to a definite limit—not to any particular trajectory.

The physical content is this: classical GR's assertion of geodesic incompleteness, when formulated as a completed-limit principle over all infalling trajectories, carries the same logical cost (LPO) as the thermodynamic limit in statistical mechanics. Both are instances of bounded monotone convergence applied to physically motivated monotone sequences.

The Brouwerian counterexample (`geodesicCoupling`) is not a geodesic—it is a valid element of the domain of the universally quantified proposition. This is standard CRM methodology: the counterexample demonstrates that the universal principle carries LPO strength, while specific instances (the cycloid) are constructive.

5 Discussion

5.1 The Calibration Table

Paper 13 adds new rows to the programme's calibration landscape:

Physical layer	Principle	Status	Source
Finite-volume Gibbs states	BISH	Calibrated	Trivial
Finite-size approximations (Ising)	BISH	Calibrated	Paper 8
Schwarzschild exterior	BISH	Calibrated	Paper 1
Interior finite-time physics	BISH	Calibrated	Paper 13
Bidual-gap witness ($S_1(H)$)	$\equiv$ WLPO	Calibrated	Papers 2, 7
Tsirelson bound (CHSH $\leq 2\sqrt{2}$)	BISH	Calibrated	Paper 11
Bell state entropy ($\log 2$)	BISH	Calibrated	Paper 11
Partial trace (qubit systems)	BISH	Calibrated	Paper 11
Thermodynamic limit (Ising)	$\equiv$ LPO	Calibrated	Paper 8
Geodesic incompleteness	$\equiv$ LPO	Calibrated	Paper 13
Spectral gap decidability	Undecidable	Established	Cubitt et al. Cubitt et al. [2015]

The pattern strengthens: all LPO costs arise from completed infinite limits, all finite-time and finite-size physics is BISH.

5.2 The Encoding Objection

A natural objection is that the encoding of binary sequences into coupling sequences—and the subsequent application of the completeness principle—is merely bounded monotone convergence in disguise. The encoded sequence $a(n) \in \{M, 0\}$ is a $\{0,1\}$-valued monotone sequence composed with a scaling, and the decision procedure is the abstract BMC $\to$ LPO proof applied to this specific class.

This objection is mathematically correct and interpretively irrelevant. The abstract equivalence BMC $\leftrightarrow$ LPO is known from Bridges and Vîţă [2006]. The contribution of this paper is not a new theorem in abstract constructive reverse mathematics but a verified observation that BMC, when applied to bounded monotone sequences in $[0, 2M)$, *is* the completed-limit content of geodesic incompleteness. The formalization makes explicit what the mathematical prose leaves implicit: the encoding is BISH-valid, the gap $\delta = M > 0$ is constructively positive, and the witness extraction works without hidden omniscience. The Lean 4 axiom audit confirms this.

This is the same methodological move as Paper 8, where the abstract BMC $\leftrightarrow$ LPO equivalence was known and the contribution was the specific physical instantiation and machine verification.

5.3 Methodological Limitations

We are frank about the scope and limitations of this result:

1. **Classical.choice is a Mathlib infrastructure artifact.** The `Classical.choice` appearing in the axiom profile of BISH results (cycloid properties, Kretschmann scalar) arises from Mathlib4's pervasive use of classical logic in its real number library, not from the mathematical content of the proofs. The proof-content analysis methodology for handling this is described in Paper 10.

2. **The formalization abstracts from the geodesic ODE.** We do not formalize the Lorentzian metric, Christoffel symbols, parallel transport, or the geodesic equation as an ODE. The cycloid solution is used directly as an explicit function. This means our formalization does not verify that the cycloid actually solves the radial geodesic equation—only that it has the algebraic and analytic properties (monotonicity, boundedness, endpoint values) claimed. The physical interpretation relies on standard textbook results connecting the cycloid to the geodesic equation.

3. **The LPO cost is on the universal principle, not any specific geodesic.** The cycloid reaching $r = 0$ is constructive. The formalization proves this explicitly (`r_cycloid_at_pi`). The LPO cost attaches to the assertion that *every* bounded monotone sequence in the interior converges—a universal principle that subsumes but is not identical to any particular geodesic.

4. `SchwarzschildInteriorGeodesicIncompleteness` **is BMC for a specific class of sequences.** The "geodesic incompleteness" interpretation is physical motivation, not a formal derivation from the Einstein equations. The formalization establishes an equivalence between a specific instance of BMC (antitone sequences in $[0, 2M)$) and LPO. Whether this particular instance captures the full logical content of geodesic incompleteness in a more complete formalization of GR is an open question.

5. **No Penrose theorem is formalized.** The Penrose singularity theorem Penrose [1965] generalizes geodesic incompleteness from Schwarzschild to arbitrary spacetimes with trapped surfaces and energy conditions. Formalizing this would require massive infrastructure (global hyperbolicity, trapped surfaces, energy conditions) far beyond our current scope.

6. **Scope is limited to Schwarzschild.** The result applies to the non-rotating, uncharged (Schwarzschild) case only. Extensions to Kerr (rotating) or Reissner–Nordström (charged) black holes would require additional analysis.

5.4 Open Problems

1. **Penrose theorem calibration.** Does the full Penrose singularity theorem, including trapped surface and energy condition hypotheses, calibrate above LPO? Our result suggests this is likely: the completed-limit content is the same, and additional hypotheses may introduce further costs.

2. **Cosmic censorship.** Weak cosmic censorship (singularities hidden behind horizons) involves a universal quantifier over all generic initial data sets. What is its logical cost?

3. **Hawking radiation.** The quantum process by which black holes evaporate involves the thermodynamic limit (infinite number of field modes). Does the Hawking temperature calculation inherit the LPO cost, or is it dispensable via finite-mode approximation?

6 Certification Methodology

6.1 Axiom Profile

All theorems carry `[propext, Classical.choice, Quot.sound]` from Mathlib4, plus `bmc_of_lpo` for the reverse direction. The BISH results (cycloid, Kretschmann) have `Classical.choice` from Mathlib4 infrastructure only.

6.2 Certification Level

Paper 13 contains two certification levels in the series terminology (Paper 10):

- The BISH content is finite-dimensional, explicit computation—constructively valid by inspection.
- The LPO equivalence uses `bmc_of_lpo` intentionally (this *is* the classical content).
- No minimal artifact is planned: the logical structure—BMC $\leftrightarrow$ LPO instantiated on $[0, 2M)$—is simple enough that the full artifact suffices.

6.3 The `bmc_of_lpo` Axiom

The equivalence BMC $\leftrightarrow$ LPO is due to Bridges and Vîţă Bridges and Vîţă [2006]. It is axiomatized in Paper 13 rather than re-proved because:

- Paper 8 already uses the same axiom.
- The Bridges–Vîţă proof is standard and uncontroversial.
- Re-proving it would add approximately 200 lines of Lean 4 for zero epistemic gain.

The axiom is conservative: it introduces no new logical content beyond what LPO already provides.

7 AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as Papers 2, 7, and 8 Lee [2026b,c,d], Anthropic [2026].

The author is a medical professional, not a domain expert in constructive mathematics or general relativity. The mathematical content of this paper—the connection between geodesic incompleteness and BMC, the choice of cycloid parametrization, the encoding strategy—was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalization strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging against Mathlib4 v4.28, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Task	**Human**	**AI (Claude Opus 4.6)**
Research direction	✓	
Mathematical blueprint	✓	✓
Proof strategy design	✓	✓
Mathlib4 API discovery		✓
Lean 4 proof generation		✓
Proof review	✓	
Build verification		✓
Paper writing	✓	✓

Table 2: Division of labor between human and AI.

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/AICardiologist/FoundationRelativity`
- **Path**: `Papers/P13_EventHorizon/`
- **Build**: `lake exe cache get && lake build` (3,100 jobs, 0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Interface axiom**: `bmc_of_lpo` (Bridges–Vîţă, imported from Paper 8)
- **Axiom audit**: `Certificates.lean`
- **Axiom profile (main theorem)**: `[propext, Classical.choice, Quot.sound, bmc_of_lpo]`
- **Axiom profile (BISH content)**: `[propext, Classical.choice, Quot.sound]` (Mathlib infra only)
- **Total**: 8 files, 1,021 lines, 0 sorry
- **Zenodo DOI**: 10.5281/zenodo.18529007

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