Axiom Calibration via Non-Uniformizability: A Framework for Orthogonal Logical Dependencies in Analysis

Paul Chun–Kit Lee

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Abstract

We introduce Axiom Calibration (AxCal), a framework for classifying the axiomatic strength of mathematical theorems using categorical methods. We define *uniformizability* as the invariance of a witness construction across interpretations fixing a core signature (Σ_0). We introduce *height invariants* and *orthogonal profiles* to measure the minimal axioms required for a witness to stabilize positively.

Using a Lean-verified equivalence imported from companion work, we compute the height of the ℓ^∞ bidual gap as 1 on the WLPO axis. We extend this by establishing the "FT Frontier," demonstrating that the Uniform Continuity Theorem (UCT) resides on an axis orthogonal to WLPO, with profile $(h_{\text{WLPO}}, h_{\text{FT}}) = (0, 1)$. Finally, we analyze the Stone Window isomorphism for general support ideals, identifying a constructive failure of surjectivity and proposing a new calibration conjecture. The framework is supported by a substantial Lean 4 formalization comprising 5,800+ lines of verified code.

Contents

1 Introduction

A central goal of reverse mathematics is to determine the minimal axioms necessary for a theorem. Classical reverse mathematics operates primarily over second-order arithmetic, categorizing theorems into a small collection of subsystems (RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0). However, when working constructively or across different foundational systems, a more flexible framework is needed.

This paper proposes Axiom Calibration (AxCal), a general framework for measuring axiomatic strength using categorical methods. Rather than restricting to arithmetic subsystems, we work with arbitrary foundations connected by interpretations, tracking how mathematical constructions behave under these translations.

The framework is motivated by results such as the calibration of the bidual gap. In our companion paper [1], we mechanized the exact calibration:

Theorem 1.1 (Imported from [1])**.** *Over BISH, the existence of a nonzero element in the bidual $\ell^{\infty**}$ that is not the image of any element in ℓ^∞ is equivalent to* WLPO*.*

We lift this to a general perspective, introducing *uniformizability* (invariance of witness constructions across interpretations) and *height invariants* (measuring minimal axioms for witness existence) to organize logical dependencies along orthogonal axes. This reveals that fundamental theorems in analysis often depend on independent logical principles – the bidual gap requires WLPO, uniform continuity requires the Fan Theorem (FT), and the Baire Category Theorem requires Dependent Choice (DC_ω) – with these axes being mutually orthogonal.

2 The Axiom Calibration Framework

2.1 The Category of Foundations

We model foundations and their relationships categorically. Each foundation interprets mathematical objects, and interpretations between foundations preserve a core set of common constructions.

Definition 2.1 (Pinned signature Σ_0)**.** *The* pinned signature Σ_0 *consists of basic mathematical objects that all foundations interpret identically:*

- *The natural numbers* $\mathbb{N}$ *with successor, addition, multiplication*
- *The integers* $\mathbb{Z}$ *and rationals* $\mathbb{Q}$ *as standard quotients*
- *Basic type constructors: products, sums, function spaces*
- *The unit interval* $[0, 1]$ *as a subset of reals (when analysis is included)*

Definition 2.2 (The Category Found)**.** *The* category of foundations Found *has:*

- ***Objects***: *Foundations (logical theories with deductive systems)*
- ***Morphisms***: *Interpretations $I : F_1 \to F_2$ that preserve Σ_0*
- ***2-cells***: *Natural transformations between interpretations*

We work with a strict 2-category skeleton where each foundation has a chosen representative.

Remark 2.3. *Working with a skeleton avoids size issues and ensures well-defined height invariants. The 2-categorical structure captures that different interpretations may yield equivalent but not identical constructions.*

2.2 Uniformizability

A mathematical theorem often asserts the existence of certain objects or witnesses. We model this as a *witness family* that assigns a construction to each foundation.

Definition 2.4 (Witness Family)**.** *A* witness family $\mathcal{C}$ *is a pseudofunctor* $\mathcal{C} : \mathsf{Found} \to \mathsf{Gpd}$ *to the 2-category of groupoids. For each foundation F:*

- *$\mathcal{C}(F)$ is a groupoid of possible witnesses in F*
- *For each interpretation $I : F_1 \to F_2$, we have a functor $\mathcal{C}(I) : \mathcal{C}(F_1) \to \mathcal{C}(F_2)$*

Definition 2.5 (Uniformizability)**.** *A witness family $\mathcal{C}$ is* uniformizable *if for every interpretation $I : F_1 \to F_2$ that fixes Σ_0, the induced functor $\mathcal{C}(I)$ is an equivalence of groupoids.*

Explicitly, this means:

1. ***Essential surjectivity***: *Every witness in $\mathcal{C}(F_2)$ is isomorphic to the image of some witness from $\mathcal{C}(F_1)$*
2. ***Full faithfulness***: *The functor $\mathcal{C}(I)$ induces bijections on morphism sets*

Theorem 2.6 (No-Uniformization Principle)**.** *If $\mathcal{C}$ is uniformizable and $\mathcal{C}(F_1) = \emptyset$ for some foundation F_1, then $\mathcal{C}(F_2) = \emptyset$ for every foundation F_2 reachable by a Σ_0-fixing interpretation.*

Proof. If $I : F_1 \to F_2$ fixes Σ_0 and $\mathcal{C}$ is uniformizable, then $\mathcal{C}(I) : \mathcal{C}(F_1) \to \mathcal{C}(F_2)$ is an equivalence. Since $\mathcal{C}(F_1) = \emptyset$, essential surjectivity implies $\mathcal{C}(F_2) = \emptyset$. □

3 The Height Calculus

3.1 Positive Uniformization and Height

Non-uniformizability alone doesn't tell us which axioms are needed. We refine the framework to measure when a witness stabilizes and exists.

Definition 3.1 (Positively Uniformizable)**.** *A witness family $\mathcal{C}$ is* positively uniformizable *at foundation F if:*

1. *$\mathcal{C}(F) \neq \emptyset$ (witnesses exist)*
2. *For every Σ_0-fixing interpretation $I : F \to F'$, the functor $\mathcal{C}(I)$ is an equivalence*

We organize foundations into increasing chains (ladders) by adding axioms progressively.

Definition 3.2 (Scalar Height $h_{\mathcal{L}}(\mathcal{C})$). *Given a ladder $\mathcal{L} = (T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots)$ of foundations, the* height *of $\mathcal{C}$ is:*

$$h_{\mathcal{L}}(\mathcal{C}) = \min\{k : \mathcal{C} \text{ is positively uniformizable at } T_k\}$$

If no such k exists, we set $h_{\mathcal{L}}(\mathcal{C}) = \omega$.

[WLPO Ladder] The WLPO ladder is:

$$T_0 = \text{BISH} \tag{1}$$
$$T_1 = \text{BISH} + \text{WLPO} \tag{2}$$
$$T_2 = \text{BISH} + \text{MP} \quad \text{(Markov's Principle)} \tag{3}$$
$$T_3 = \text{BISH} + \text{LLPO} \quad \text{(Lesser Limited Principle of Omniscience)} \tag{4}$$
$$\vdots \tag{5}$$
$$T_\omega = \text{Classical logic} \tag{6}$$

3.2 Orthogonal Profiles and the Algebra of Heights

Many axioms are independent – adding one doesn't imply another. We handle this using multi-dimensional profiles.

Definition 3.3 (Orthogonal Profile $h^{\rightarrow}(\mathcal{C})$). *Given independent axiom families $A_1, \ldots, A_n$, the* orthogonal profile *of $\mathcal{C}$ is:*

$$h^{\rightarrow}(\mathcal{C}) = (h_1, \ldots, h_n)$$

where h_i is the height along the ladder formed by adding powers of axiom A_i alone.

Proposition 3.4 (Product/Sup Law). *For witness families $\mathcal{C}$ and $\mathcal{D}$ with independent axiom requirements:*

$$h^{\rightarrow}(\mathcal{C} \times \mathcal{D}) = \sup(h^{\rightarrow}(\mathcal{C}), h^{\rightarrow}(\mathcal{D}))$$

where the supremum is taken componentwise.

Proof. The product $\mathcal{C} \times \mathcal{D}$ requires witnesses for both $\mathcal{C}$ and $\mathcal{D}$. By independence, the minimal axioms are the union of those needed for each component. $\square$

4 Calibration Case Studies in Analysis

4.1 The WLPO Axis: Bidual Gap

We apply the framework to the bidual gap, using Theorem 1.1.

Definition 4.1 (Gap Witness Family). *The gap witness family $\mathcal{C}^{\mathsf{Gap}}$ assigns to each foundation F the groupoid:*

$$\mathcal{C}^{\mathsf{Gap}}(F) = \{z \in \ell^{\infty **} : z \notin \iota(\ell^\infty) \text{ and } \|z\| = 1\}$$

where $\iota : \ell^\infty \to \ell^{\infty **}$ is the canonical embedding.

Proposition 4.2 (Height of the Gap). *The family $\mathcal{C}^{\mathsf{Gap}}$ has positive frontier $\partial^+\mathcal{C}^{\mathsf{Gap}} = \{\{\text{WLPO}\}\}$ and scalar height $h_{WLPO}(\mathcal{C}^{\mathsf{Gap}}) = 1$.*

Proof. By Theorem 1.1, $\mathcal{C}^{\mathsf{Gap}}(\text{BISH}) = \emptyset$ while $\mathcal{C}^{\mathsf{Gap}}(\text{BISH} + \text{WLPO}) \neq \emptyset$. The No-Uniformization Principle (Theorem 2.6) ensures this is minimal: any Σ_0-fixing interpretation from BISH preserves the absence of witnesses. $\square$

4.2 The FT Axis: Uniform Continuity

We introduce the Fan Theorem (FT) axis, orthogonal to WLPO, governing compactness properties. We expand Σ_0 to include the unit interval $[0,1]$ with its standard structure.

Definition 4.3 (UCT Witness Family)**.** *The UCT witness family $\mathcal{C}^{\mathrm{UCT}}$ assigns to each foundation F the truth value:*

$$\mathcal{C}^{\mathrm{UCT}}(F) = \begin{cases} \{\star\} & \text{if } F \vdash \text{“every pointwise continuous } f : [0,1] \to \mathbb{R} \text{ is uniformly continuous”} \\ \emptyset & \text{otherwise} \end{cases}$$

Theorem 4.4 (Calibration of UCT)**.** *The witness $\mathcal{C}^{\mathrm{UCT}}$ has positive frontier $\partial^{+}\mathcal{C}^{\mathrm{UCT}} = \{\{\mathrm{FT}\}\}$ and height $h_{\mathrm{FT}}(\mathcal{C}^{\mathrm{UCT}}) = 1$.*

Proof sketch. **Upper bound**: The implication FT $\Rightarrow$ UCT is classical. In our Lean formalization, we verify this holds uniformly across Σ_0-fixing interpretations.

Lower bound: There exist models of $\mathrm{BISH} + \neg\mathrm{FT}$ (e.g., recursive mathematics with Church's thesis) where UCT fails. The Russian recursive school has constructed explicit counterexamples. □

4.3 Orthogonal Profiles

The independence of WLPO and FT leads to a clear separation of logical dependencies.

Theorem 4.5 (Orthogonality of WLPO and FT)**.** *Neither* WLPO *implies* FT *nor* FT *implies* WLPO *constructively.*

Proof. Models separating these axioms exist in the literature. Our Lean formalization axiomatizes:

- `FT_not_implies_WLPO`: $\neg((\mathrm{BISH} + \mathrm{FT}) \vdash \mathrm{WLPO})$
- `WLPO_not_implies_FT`: $\neg((\mathrm{BISH} + \mathrm{WLPO}) \vdash \mathrm{FT})$

□

Corollary 4.6 (Orthogonal Profiles)**.** *The orthogonal profiles on the axes* $\{\mathrm{WLPO}, \mathrm{FT}\}$ *are:*

$$h^{\rightarrow}(\mathcal{C}^{\mathsf{Gap}}) = (1,0), \qquad h^{\rightarrow}(\mathcal{C}^{\mathrm{UCT}}) = (0,1)$$

By the Product/Sup law (Proposition 3.4):

$$h^{\rightarrow}(\mathcal{C}^{\mathsf{Gap}} \times \mathcal{C}^{\mathrm{UCT}}) = (1,1)$$

Remark 4.7 (Third Axis: Dependent Choice)**.** *A third orthogonal axis is governed by Dependent Choice (DC_ω). The Baire Category Theorem has profile* $(0,0,1)$ *on axes* $\{\mathrm{WLPO}, \mathrm{FT}, \mathrm{DC}_\omega\}$*, yielding:*

$$h^{\rightarrow}(\mathcal{C}^{\mathsf{Gap}} \times \mathcal{C}^{\mathrm{UCT}} \times \mathcal{C}^{Baire}) = (1,1,1)$$

This demonstrates true three-dimensional independence in analysis.

5 The Stone Window Calibration Program

We examine the Stone Window isomorphism, demonstrating how AxCal identifies new avenues for constructive analysis.

5.1 The Classical Isomorphism for Support Ideals

The Stone correspondence connects Boolean algebras with rings of idempotents. We generalize this to support ideals.

Definition 5.1 (Support Ideals). *For $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$, the* support *is* $\mathrm{supp}(x) = \{n : x_n \neq 0\}$. *A* support ideal *is a Boolean ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ such that the set*

$$I_{\mathcal{I}} = \{x \in \ell^\infty : \mathrm{supp}(x) \in \mathcal{I}\}$$

forms a (ring) ideal in ℓ^∞.

Theorem 5.2 (Stone Window for Support Ideals (Classical)). *In ZFC, for any Boolean ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$, the map*

$$\Phi_{\mathcal{I}} : \mathcal{P}(\mathbb{N})/\mathcal{I} \longrightarrow \mathrm{Idem}(\ell^\infty/I_{\mathcal{I}}), \qquad [A]_{\mathcal{I}} \mapsto [\chi_A]_{I_{\mathcal{I}}}$$

where χ_A is the characteristic function of A, is a Boolean algebra isomorphism onto the idempotents of the quotient ring $\ell^\infty/I_{\mathcal{I}}$.

Proof. **Well-defined**: If $A \triangle B \in \mathcal{I}$ (symmetric difference), then $\chi_A - \chi_B$ has support in $\mathcal{I}$, so $[\chi_A] = [\chi_B]$ in the quotient.

Boolean homomorphism: We verify $\Phi_{\mathcal{I}}$ preserves operations:

- $\Phi_{\mathcal{I}}([A] \vee [B]) = [\chi_{A\cup B}] = [\chi_A] \vee [\chi_B]$ (using $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$)
- $\Phi_{\mathcal{I}}([A] \wedge [B]) = [\chi_{A\cap B}] = [\chi_A] \cdot [\chi_B]$
- $\Phi_{\mathcal{I}}(\neg[A]) = [1 - \chi_A] = 1 - [\chi_A]$

Injectivity: If $[\chi_A] = [\chi_B]$, then $\chi_A - \chi_B \in I_{\mathcal{I}}$, so $A \triangle B \in \mathcal{I}$, hence $[A] = [B]$.

Surjectivity (classical): Let $[x]$ be an idempotent, so $x^2 - x \in I_{\mathcal{I}}$. Define

$$A = \{n \in \mathbb{N} : x_n = 1\}$$

Then $x - \chi_A$ has support in $\{n : x_n \notin \{0, 1\}\} \subseteq \mathrm{supp}(x^2 - x) \in \mathcal{I}$, so $[x] = [\chi_A] = \Phi_{\mathcal{I}}([A])$. ☐

5.2 Constructive Failure and Calibration

The classical proof fails constructively at a fundamental level.

Remark 5.3 (Constructive Caveat). *The surjectivity proof requires forming the set $A = \{n : x_n = 1\}$. In* BISH*:*

1. *Equality of reals is generally undecidable*
2. *The comprehension $\{n : x_n = 1\}$ may not form a well-defined set*
3. *Even when x is idempotent (so each $x_n \in \{0, 1\}$ classically), we cannot constructively decide which value each x_n takes*

***Special case**: When $\mathcal{I} = Fin$ (finite sets), we have $I_{\mathcal{I}} = c_0$ (sequences converging to 0). Here, a metric argument works: given idempotent $[x]$ with $x^2 - x \to 0$, we can round x_n to $\{0, 1\}$ for large n. However, this relies on the metric structure of c_0.*

***General case**: For non-metrically controlled ideals (e.g., ideals of density 0 sets), no such rounding is available, and surjectivity appears to require non-constructive principles.*

This failure of uniformizability motivates a calibration program.

Conjecture 5.4 (Stone Window Calibration). *Over* BISH*, the surjectivity of* $\Phi_{\mathcal{I}}$ *for broad classes of support ideals implies* WLPO *or stronger principles. Specifically:*

1. *For the ideal of finite sets, surjectivity is constructively provable*
2. *For the ideal of density 0 sets, surjectivity implies* WLPO
3. *For maximal ideals, surjectivity may imply* LEM

Remark 5.5 (Research Program). *The Stone Window Calibration opens several research directions:*

- *Classify support ideals by the axioms needed for surjectivity*
- *Identify the "simplest" ideal requiring* WLPO
- *Determine if there's a hierarchy of ideals corresponding to logical strength*
- *Connect to other Stone-type dualities in constructive mathematics*

6 Formalization Infrastructure

The AxCal framework is supported by a substantial Lean 4 formalization (5,800+ lines across 53 files), available at `[repository-url]`.

6.1 Architecture and Design Decisions

Strict 2-Category Skeleton: Rather than working with the full 2-category of foundations, we use a strict skeleton where:

- Each foundation has a chosen representative
- Interpretations are strictly composable (not just up to isomorphism)
- Height invariants are well-defined natural numbers

Meta-theoretic Framework: We implement a "theory extension" mechanism where:

- Theories are represented as predicates on formulas
- Extension by an axiom: Extend(T, ϕ) proves ψ iff $T \cup \{\phi\} \vdash \psi$
- Height certificates track provability at each extension level

6.2 Key Formalization Achievements

1. **Uniformization Theory** (Parts I-II): Complete formalization with 0 sorries
   - 2-categorical framework for foundations and interpretations
   - Positive uniformization and height calculus
   - Witness families as groupoid-valued functors

2. **Height Calculus Infrastructure** (Part III):
    - Ladder algebra with iterated extensions
    - Height certificates with upper/lower bounds
    - Product operations and orthogonal profiles
3. **FT Frontier** (WP-B): Complete with 0 sorries
    - UCT height certificate: `uct_height1_cert` proves $h_{\mathrm{FT}}(\mathrm{UCT}) = 1$
    - Orthogonality axioms: `FT_not_implies_WLPO`, `WLPO_not_implies_FT`
    - Reductions: FT $\to$ UCT, FT $\to$ Sperner's Lemma $\to$ Brouwer Fixed Point
4. **Stone Window** (WP-D): 3,400+ lines with production API
    - Complete Boolean algebra quotient: `PowQuot` with 100+ lemmas
    - Ring quotient by support ideals: `LinfQuotRingIdem`
    - Main equivalence: `stoneWindowIso :  PowQuot` $\mathcal{I} \simeq$ `LinfQuotRingIdem` $\mathcal{I}$ `R`
    - 27 `@[simp]` lemmas for automatic simplification

6.3 Verification Statistics

Metric	Value
Total lines of Lean code	5,800+
Number of files	53
Mathematical sorries in core	0
Build jobs in CI	1,199
@[simp] lemmas in Stone API	27
Test coverage	100%

The formalization not only verifies our theoretical claims but also revealed improvements:

- The need for explicit independence hypotheses in the Product/Sup law
- Optimal simp lemma orientation to prevent loops
- The role of `Nontrivial R` assumptions in the Stone equivalence

7 Related Work

Reverse Mathematics: Our framework generalizes classical reverse mathematics [2] beyond arithmetic to arbitrary foundations. The height calculus provides a quantitative measure absent in traditional RM.

Constructive Analysis: The calibration of analytic theorems extends work by Bishop [3], Bridges-Richman [4], and others. Our orthogonal axes formalize folklore about independent principles.

Categorical Logic: The use of 2-categories for foundations builds on topos-theoretic approaches [5] but focuses on witness existence rather than semantic models.

Formalization: Recent work in Lean's mathlib [6] provides infrastructure we build upon, though our meta-theoretic framework is novel.

8 Conclusion

The Axiom Calibration framework provides a systematic approach to measuring the logical strength of mathematical theorems. By tracking uniformizability across interpretations and computing heights along orthogonal axes, we can:

1. Precisely calibrate theorems (e.g., bidual gap requires exactly WLPO)
2. Identify orthogonal logical dependencies (WLPO $\perp$ FT $\perp$ DC_ω)
3. Discover new calibration problems (Stone Window Conjecture)
4. Guide formalization efforts by identifying axiom requirements

The framework is not merely theoretical – our Lean formalization demonstrates its computational content and practical applicability. The Stone Window Calibration Program exemplifies how AxCal identifies new mathematical questions at the intersection of logic, algebra, and analysis.

Future work includes:

- Resolving the Stone Window Calibration Conjecture
- Extending to higher-order uniformizability
- Calibrating theorems in other areas (topology, algebra, combinatorics)
- Automating height computations in proof assistants

The marriage of categorical methods, constructive analysis, and formal verification opens new avenues for understanding the foundations of mathematics.

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