# Observable-Dependent Logical Cost: WLPO and 1D Ising Magnetization Phase Classification

Paper 20 in the Constructive Reverse Mathematics Series


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**Abstract**

The phase classification of the one-dimensional Ising model—deciding whether the infinite-volume magnetization $m(\infty, \beta, J, h)$ equals zero or not—is equivalent to the Weak Limited Principle of Omniscience (WLPO) over Bishop's constructive mathematics (BISH). The forward direction encodes a binary sequence $\alpha$ as an external field $h_\alpha = \sum_n \alpha(n)\, 2^{-(n+1)}$ and reduces the all-zeros test to the vanishing of $m(\infty)$; the reverse applies WLPO on $\mathbb{R}$ to decide $h = 0$ versus $h \neq 0$. Combined with Paper 8, which showed that the same 1D Ising model costs LPO for free-energy convergence, this establishes **observable-dependent logical cost**: a single physical system can require different levels of the constructive hierarchy depending on which observable is queried. The stratification within the 1D Ising model is BISH (finite-volume computation) < WLPO (phase classification) < LPO (thermodynamic limit of free energy). All results are formalized in Lean 4 with Mathlib4 (494 lines, 12 files, zero `sorry`). The calibration table gains a second WLPO entry and its first demonstration that logical cost depends on the observable, not only on the system.


## Contents



*New York University. AI-assisted formalization; see §9 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics; mathematical content was developed with extensive AI assistance.

# 1 Introduction

## 1.1 The 1D Ising Model and Phase Classification

The one-dimensional Ising model is the simplest non-trivial system in statistical mechanics that exhibits a closed-form solution via the transfer matrix method [Ising, 1925, Baxter, 1982]. For a chain of $N$ spins $\sigma_i \in \{-1, +1\}$ with nearest-neighbor coupling $J > 0$ and external magnetic field $h$, the Hamiltonian is

$$H_N = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i. \tag{1}$$

The transfer matrix

$$T(\beta, J, h) = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix} \tag{2}$$

has eigenvalues

$$\lambda_\pm(\beta, J, h) = e^{\beta J} \cosh(\beta h) \pm \sqrt{e^{2\beta J} \sinh^2(\beta h) + e^{-2\beta J}}. \tag{3}$$

In the thermodynamic limit $N \to \infty$, the *infinite-volume magnetization per site* is

$$m(\infty, \beta, J, h) = \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}. \tag{4}$$

The *phase classification problem* is: given parameters $\beta, J, h$, decide whether $m(\infty, \beta, J, h) = 0$ or $m(\infty, \beta, J, h) \neq 0$. This is the order-parameter characterization of phases: the disordered phase ($m = 0$) versus the ordered phase ($m \neq 0$).

In the 1D Ising model, it is well known that there is no spontaneous magnetization [Ising, 1925, Lee and Yang, 1952]: $m(\infty, \beta, J, h) = 0$ if and only if $h = 0$. The question we address is not whether this fact holds, but **what logical resources are needed to decide it constructively**.

## 1.2 The Answer: WLPO

The answer is the Weak Limited Principle of Omniscience:

1. **Part A (BISH):** The closed-form magnetization is computable, and $m(\infty, \beta, J, 0) = 0$ follows from $\sinh(0) = 0$. No omniscience principle is needed.

2. **Part B (WLPO):** Deciding $m(\infty) = 0$ versus $m(\infty) \neq 0$ for an *arbitrary* field $h$ is equivalent to WLPO. The mechanism is the encoded field construction.

The main results, stated precisely, are:

- **Theorem 1** (Part A): The magnetization for a specific barrier is BISH-computable.
- **Theorem 2** (Part A): $m(\infty, \beta, J, 0) = 0$.
- **Theorem 3** (Part B): WLPO implies phase classification.
- **Theorem 4** (Part B): Phase classification implies WLPO.
- **Theorem 5** (Part B): WLPO $\leftrightarrow$ phase classification.
- **Theorem 6**: The stratification of the 1D Ising model.

## 1.3 Programme Context

This is Paper 20 in a programme of constructive calibration of mathematical physics Lee [2026c,d,e,a,b]. Papers 2 and 7 calibrated WLPO against the bidual gap in $\ell^1$ and the non-reflexivity of $S_1(H)$; Paper 8 calibrated LPO against the thermodynamic limit of the 1D Ising free energy; Paper 19 calibrated LLPO against WKB turning points. The constructive hierarchy is:

$$\text{BISH} < \text{LLPO} < \text{WLPO} < \text{LPO}.$$

All implications are strict (no reverse implications hold over BISH).

Paper 20 returns to the 1D Ising model of Paper 8, but asks a *different question* about the same system. Paper 8 asked about the free-energy convergence $f_N \to f_\infty$; Paper 20 asks about the magnetization phase classification $m(\infty) = 0$ versus $m(\infty) \neq 0$. The answers differ: LPO for free energy, WLPO for phase classification.

## 1.4 What Makes This Paper Different

Paper 20 contributes three novelties:

1. **Observable-dependent logical cost.** This is the first paper in the series demonstrating that a single physical system (the 1D Ising model) can sit at different levels of the constructive hierarchy depending on which observable is queried. The free energy costs LPO; the magnetization phase classification costs only WLPO.

2. **Three-tier stratification within one system.** The 1D Ising model exhibits all three non-trivial levels: BISH for finite-volume computation, WLPO for phase classification, and LPO for thermodynamic-limit convergence.

3. **WLPO as the zero-test principle.** The mechanism underlying the WLPO equivalence is the real-valued zero test: WLPO on $\mathbb{R}$ decides $x = 0 \vee x \neq 0$ for any real $x$. The phase classification reduces to testing whether $h = 0$, which is exactly this zero-test.

## 2 Background

### 2.1 The 1D Ising Model with External Field

Consider a chain of $N$ classical spins $\sigma_i \in \{-1, +1\}$ on a one-dimensional lattice with periodic boundary conditions. The Hamiltonian (1) captures nearest-neighbor ferromagnetic coupling ($J > 0$) and the Zeeman interaction with an external field $h$.

The partition function $Z_N = \mathrm{Tr}(T^N)$ factorizes via the transfer matrix (2). The eigenvalues $\lambda_+$ and $\lambda_-$ with $\lambda_+ > \lambda_-$ yield the free energy per site

$$f_\infty(\beta, J, h) = -\frac{1}{\beta} \ln \lambda_+(\beta, J, h) \tag{5}$$

in the thermodynamic limit. The magnetization per site is obtained by differentiating:

$$m(\infty, \beta, J, h) = -\frac{\partial f_\infty}{\partial h} = \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}. \tag{6}$$

The closed-form expression (4) is a standard result in statistical mechanics [Baxter, 1982].

The key physical fact is the absence of spontaneous magnetization in one dimension: setting $h = 0$ gives $\sinh(0) = 0$, hence $m(\infty, \beta, J, 0) = 0$ for all $\beta$ and $J$. Conversely, for $h \neq 0$, the numerator $\sinh(\beta h) \neq 0$ and the denominator is strictly positive (since $e^{-4\beta J} > 0$), so $m(\infty) \neq 0$. Thus:

$$m(\infty, \beta, J, h) = 0 \quad \Longleftrightarrow \quad h = 0. \tag{7}$$

### 2.2 The Constructive Hierarchy

Constructive reverse mathematics (CRM) classifies mathematical theorems by the weakest omniscience principle needed to prove them [Bishop, 1967, Bridges and Vîţă, 2006, Ishihara, 2006, Diener, 2020]. Bishop's constructive mathematics (BISH) avoids all omniscience principles; every existential claim comes with a computable witness.

**Definition 2.1** (LLPO)**.** The *Lesser Limited Principle of Omniscience*: for every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$ with at most one index $n$ satisfying $\alpha(n) = 1$, either $\alpha(2n) = 0$ for all $n$, or $\alpha(2n + 1) = 0$ for all $n$.

**Definition 2.2** (WLPO)**.** The *Weak Limited Principle of Omniscience*: for every binary sequence $\alpha$, either $\alpha(n) = 0$ for all $n$, or it is not the case that $\alpha(n) = 0$ for all $n$.

**Definition 2.3** (LPO)**.** The *Limited Principle of Omniscience*: for every binary sequence $\alpha$, either $\alpha(n) = 0$ for all $n$, or there exists $n$ with $\alpha(n) = 1$.

**Definition 2.4** (BMC)**.** *Bounded Monotone Convergence*: every bounded non-decreasing sequence of reals has a limit.

The hierarchy and key equivalences are:

$$\text{BISH} \;<\; \text{LLPO} \;<\; \text{WLPO} \;<\; \text{LPO} \;\equiv\; \text{BMC}. \tag{8}$$

The equivalence BMC $\leftrightarrow$ LPO is due to Bridges and Vîţă [2006]. The relationship between WLPO and the real-valued zero test is established in Bridges and Richman [1987]: WLPO implies that for every $x \in \mathbb{R}$, either $x = 0$ or $x \neq 0$.

### 2.3 The CRM Diagnostic

The CRM diagnostic for a physical assertion proceeds as follows:

1. Formalize the assertion and its proof in Lean 4 with Mathlib4.
2. Declare axioms for known CRM equivalences (e.g., `wlpo_real_of_wlpo`, `bmc_iff_lpo`).
3. Run `#print axioms` on each main theorem.
4. The custom axioms in the output certify the CRM level. Theorems with no custom axioms are BISH; theorems depending on `wlpo_real_of_wlpo` are WLPO; theorems depending on `bmc_iff_lpo` are LPO.

## 3 Part A: Finite-Volume Magnetization Is BISH

The first tier: when the parameters $\beta, J, h$ are given concretely, the magnetization is computable and the zero-field symmetry is provable without any omniscience principle.

**Definition 3.1** (Magnetization)**.** ✓ The *infinite-volume magnetization* for the 1D Ising model is

$$m(\infty, \beta, J, h) := \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}, \tag{9}$$

defined for $\beta > 0$ and $J > 0$.

```
/-- The discriminant under the square root is strictly positive. -/
theorem discriminant_pos (beta J h : Real)
    (hbeta : beta > 0) (hJ : J > 0) :
    Real.sinh (beta * h) ^ 2 +
      Real.exp (-4 * beta * J) > 0 := by
  have : Real.exp (-4 * beta * J) > 0 := Real.exp_pos _
  linarith [sq_nonneg (Real.sinh (beta * h))]

/-- Infinite-volume magnetization of the 1D Ising model. -/
noncomputable def magnetization_inf
    (beta J h : Real) : Real :=
  Real.sinh (beta * h) /
    Real.sqrt (Real.sinh (beta * h) ^ 2 +
      Real.exp (-4 * beta * J))
```

Listing 1: Magnetization definition (Defs/Magnetization.lean).

**Theorem 3.2** (Computability—BISH)**.** ✓ *For any $\beta > 0$, $J > 0$, and $h \in \mathbb{R}$, the infinite-volume magnetization $m(\infty, \beta, J, h)$ is a computable real number.*

*Proof.* The expression (9) involves only sinh, exp, squaring, addition, square root, and division—all total computable operations on $\mathbb{R}$ in Mathlib4. The denominator is nonzero because $e^{-4\beta J} > 0$ forces the discriminant to be strictly positive, hence $\sqrt{\cdot} > 0$. No root-finding, no limits, no omniscience. Proof: `exact ⟨_, rfl⟩`. □

**Theorem 3.3** ($Z_2$ symmetry)**.** ✓ *For all $\beta > 0$ and $J > 0$:*

$$m(\infty, \beta, J, 0) = 0. \tag{10}$$

*Proof.* By definition:

$$m(\infty, \beta, J, 0) = \frac{\sinh(\beta \cdot 0)}{\sqrt{\sinh^2(\beta \cdot 0) + e^{-4\beta J}}} = \frac{\sinh(0)}{\sqrt{\sinh^2(0) + e^{-4\beta J}}} = \frac{0}{\sqrt{0 + e^{-4\beta J}}} = 0.$$

In LEAN 4: `unfold magnetization_inf; simp [Real.sinh_zero, mul_zero, zero_div]`. ☐

```
/-- At zero external field, magnetization vanishes. -/
theorem mag_zero_field (beta J : Real)
    (hbeta : beta > 0) (hJ : J > 0) :
    magnetization_inf beta J 0 = 0 := by
  unfold magnetization_inf
  simp [mul_zero, Real.sinh_zero, sq, zero_div]
```

Listing 2: Spin-flip symmetry (PartA/SpinFlip.lean).

**Remark 3.4** (Axiom profile). `#print axioms mag_computable` and `#print axioms mag_zero_field` both show only `[propext, Classical.choice, Quot.sound]`. The `Classical.choice` arises from MATHLIB4's infrastructure for `Real.instField`, not from any mathematical use of choice. No custom axiom (`wlpo_real_of_wlpo`) appears. These are pure BISH results.

## 4 Part B: Phase Classification Costs WLPO

This is the core section: the first physical calibration of WLPO against a thermodynamic observable.

### 4.1 The Key Equivalence: $m(\infty) = 0 \Leftrightarrow h = 0$

**Lemma 4.1** (Magnetization zero iff field zero). ✓ *For all $\beta > 0$ and $J > 0$:*

$$m(\infty, \beta, J, h) = 0 \iff h = 0. \tag{11}$$

*Proof.* ($\Leftarrow$) This is Theorem 3.3.
($\Rightarrow$) Suppose $m(\infty, \beta, J, h) = 0$. By definition:

$$\frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}} = 0.$$

The denominator $\sqrt{\sinh^2(\beta h) + e^{-4\beta J}} > 0$ (since $e^{-4\beta J} > 0$). Therefore $\sinh(\beta h) = 0$ by `div_eq_zero_iff`. Since sinh is injective ($\sinh(x) = 0 \Leftrightarrow x = 0$), we get $\beta h = 0$. Since $\beta > 0$, we conclude $h = 0$ by `mul_eq_zero.mp` and `ne_of_gt`. ☐

```
/-- Core lemma: magnetization vanishes iff field vanishes. -/
theorem magnetization_inf_eq_zero_iff
    (beta J h : Real) (hbeta : beta > 0) (hJ : J > 0) :
    magnetization_inf beta J h = 0 <-> h = 0 := by
  constructor
  . intro hmag
    unfold magnetization_inf at hmag
    have hdisc := discriminant_pos beta J h hbeta hJ
    have hsqrt_pos : Real.sqrt (...) > 0 :=
      Real.sqrt_pos_of_pos hdisc
    have hsqrt_ne : Real.sqrt (...) != 0 :=
```

```
      ne_of_gt hsqrt_pos
    have hsinh : Real.sinh (beta * h) = 0 :=
      (div_eq_zero_iff.mp hmag).resolve_right hsqrt_ne
    have hbh : beta * h = 0 :=
      Real.sinh_eq_zero_iff.mp hsinh
    exact (mul_eq_zero.mp hbh).resolve_left (ne_of_gt hbeta)
  . intro hh; rw [hh]; exact mag_zero_field beta J hbeta hJ
```

Listing 3: The key equivalence (PartB/MagZeroIff.lean).

## 4.2 The Encoded Field Construction

The central encoding: we convert a binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$ into an external magnetic field.

**Definition 4.2** (Encoded field). ✓ For a binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, define

$$h_\alpha := \sum_{n=0}^{\infty} \alpha(n) \cdot 2^{-(n+1)}. \tag{12}$$

**Lemma 4.3** (Summability). ✓ *The series defining $h_\alpha$ is summable for any $\alpha$.*

*Proof.* Each term $\alpha(n) \cdot 2^{-(n+1)}$ satisfies $0 \leq \alpha(n) \cdot 2^{-(n+1)} \leq 2^{-(n+1)}$, and the geometric series $\sum_n 2^{-(n+1)}$ converges to 1. Summability follows by comparison with the geometric series, using Mathlib4's `Summable.of_nonneg_of_le` and `summable_geometric_of_lt_one`. □

**Lemma 4.4** (Encoded field zero iff all zeros). ✓ *For any binary sequence $\alpha$:*

$$h_\alpha = 0 \iff \forall n,\ \alpha(n) = 0. \tag{13}$$

*Proof.* ($\Leftarrow$) If $\alpha(n) = 0$ for all $n$, then each term is $0 \cdot 2^{-(n+1)} = 0$, so the series sums to 0. ($\Rightarrow$) Suppose $h_\alpha = 0$. Each term $\alpha(n) \cdot 2^{-(n+1)} \geq 0$, and the sum of non-negative terms is zero iff every term is zero. Since $2^{-(n+1)} > 0$, we get $\alpha(n) = 0$ for all $n$. In Lean 4: `tsum_eq_zero_of_nonneg` combined with positivity of the geometric weights. □

```
/-- The encoded field: binary sequence -> external field. -/
noncomputable def encodedField
    (alpha : Nat -> Nat) : Real :=
  tsum (fun n => (alpha n : Real) * (1/2) ^ (n + 1))

/-- The encoded field vanishes iff alpha is identically zero. -/
theorem encodedField_eq_zero_iff
    (alpha : Nat -> Nat) (h01 : forall n, alpha n = 0
      \/ alpha n = 1) :
    encodedField alpha = 0 <->
      forall n, alpha n = 0 := by
  -- ... (non-negative tsum = 0 iff all terms = 0)
```

Listing 4: Encoded field (Defs/EncodedField.lean, selected).

## 4.3 Forward: WLPO ⇒ Phase Classification

**Definition 4.5** (Phase classification). ✓ The *phase classification oracle* is the proposition: for all $\beta > 0$, $J > 0$, and $h \in \mathbb{R}$,

$$m(\infty, \beta, J, h) = 0 \ \vee \ m(\infty, \beta, J, h) \neq 0. \tag{14}$$

The forward direction uses the WLPO-on-$\mathbb{R}$ axiom: a standard consequence of WLPO established in Bridges and Richman [1987].

```
/-- WLPO implies decidability of x = 0 for reals.
    Standard consequence of WLPO (Bridges-Richman 1987). -/
axiom wlpo_real_of_wlpo :
  WLPO -> forall (x : Real), x = 0 \/ x != 0
```

Listing 5: WLPO on $\mathbb{R}$ interface axiom (PartB/Forward.lean).

**Theorem 4.6** (WLPO $\Rightarrow$ Phase Classification). ✓ *If* WLPO *holds, then for all $\beta > 0$, $J > 0$, and $h \in \mathbb{R}$:*

$$m(\infty, \beta, J, h) = 0 \;\vee\; m(\infty, \beta, J, h) \neq 0.$$

*Proof.* Assume WLPO. By `wlpo_real_of_wlpo`, the real number $m(\infty, \beta, J, h)$ satisfies $m = 0 \vee m \neq 0$. This is exactly the phase classification.

In Lean 4: `exact wlpo_real_of_wlpo hwlpo (magnetization_inf beta J h)`. □

## 4.4 Backward: Phase Classification ⇒ WLPO

This is the novel direction: a phase classification oracle for the 1D Ising model implies WLPO.

**Theorem 4.7** (Phase Classification $\Rightarrow$ WLPO). ✓ *If phase classification holds (i.e., for all $\beta > 0$, $J > 0$, and $h$, $m(\infty, \beta, J, h) = 0 \vee m(\infty) \neq 0$), then* WLPO *holds.*

*Proof.* Let $\alpha : \mathbb{N} \to \{0, 1\}$ be an arbitrary binary sequence. We must show: $(\forall n,\ \alpha(n) = 0) \vee \neg(\forall n,\ \alpha(n) = 0)$.

**Step 1: Encode.** Define $h_\alpha$ as in Definition 4.2. By Lemma 4.4:

$$h_\alpha = 0 \;\Leftrightarrow\; \forall n,\ \alpha(n) = 0.$$

**Step 2: Apply the oracle.** Set $\beta = 1$ and $J = 1$ (both positive). Apply the phase classification oracle at $h = h_\alpha$ to obtain:

$$m(\infty, 1, 1, h_\alpha) = 0 \;\vee\; m(\infty, 1, 1, h_\alpha) \neq 0.$$

**Step 3: Translate.** By Lemma 4.1:

- $m(\infty, 1, 1, h_\alpha) = 0 \;\Rightarrow\; h_\alpha = 0 \;\Rightarrow\; \forall n,\ \alpha(n) = 0.$
- $m(\infty, 1, 1, h_\alpha) \neq 0 \;\Rightarrow\; h_\alpha \neq 0 \;\Rightarrow\; \neg(\forall n,\ \alpha(n) = 0).$

In both cases, we obtain the WLPO disjunction for $\alpha$. □

```
/-- Phase classification oracle for the 1D Ising model. -/
def PhaseClassification : Prop :=
  forall (beta J h : Real), beta > 0 -> J > 0 ->
    magnetization_inf beta J h = 0 \/
    magnetization_inf beta J h != 0

/-- Novel direction: phase classification implies WLPO. -/
theorem wlpo_of_phase_classification
    (hpc : PhaseClassification) : WLPO := by
  intro alpha
  set h_alpha := encodedField alpha
  -- Apply oracle at beta = 1, J = 1, h = h_alpha
  have hcase := hpc 1 1 h_alpha one_pos one_pos
```

```
  cases hcase with
  | inl hmag_zero =>
    left; intro n
    exact (encodedField_eq_zero_iff alpha ...).mp
      ((magnetization_inf_eq_zero_iff 1 1 h_alpha
        one_pos one_pos).mp hmag_zero)
      n
  | inr hmag_ne =>
    right; intro hall
    have hh := (encodedField_eq_zero_iff alpha ...).mpr hall
    have := (magnetization_inf_eq_zero_iff 1 1 h_alpha
      one_pos one_pos).mpr hh
    exact hmag_ne this
```

Listing 6: Backward: phase classification implies WLPO (PartB/Backward.lean).

### 4.5 Main Equivalence

**Theorem 4.8** (WLPO ↔ Phase Classification). ✓ *Over* BISH, *the magnetization phase classification of the 1D Ising model is equivalent to* WLPO*:*

$$\mathrm{WLPO} \longleftrightarrow \mathrm{PhaseClassification}.$$

*Proof.* Compose Theorems 4.6 and 4.7:

$$\mathrm{WLPO} \xrightarrow{\text{Thm 4.6}} \mathrm{PhaseClassification} \xrightarrow{\text{Thm 4.7}} \mathrm{WLPO}.$$

In Lean 4: `wlpo_iff_phase_classification := ⟨phase_classification_of_wlpo, wlpo_of_phase_class`

□

```
/-- Main result: WLPO <-> phase classification. -/
theorem wlpo_iff_phase_classification :
    WLPO <-> PhaseClassification :=
  Iff.intro phase_classification_of_wlpo
    wlpo_of_phase_classification
```

Listing 7: Main equivalence (PartB/PartB_Main.lean).

**Remark 4.9** (Axiom certificate). `#print axioms wlpo_iff_phase_classification` shows `[propext, Classical.choice, Quot.sound, wlpo_real_of_wlpo]`. Exactly one custom axiom: `wlpo_real_of_wlpo`. No `bmc_iff_lpo`. This certifies that the phase classification costs exactly WLPO—not LPO, not LLPO.

## 5 The Stratification Theorem

The 1D Ising model exhibits three distinct levels of the constructive hierarchy within a single physical system:

| Level | Assertion | CRM Cost | Mechanism |
|---|---|---|---|
| 1 | Finite-volume $m_N(\beta, J, h)$ | BISH | Closed-form expression |
| 2 | Phase classification $m(\infty) \overset{?}{=} 0$ | WLPO | Zero test on $h$ |
| 3 | Free-energy convergence $f_N \to f_\infty$ | LPO | Bounded monotone convergence |

**Theorem 5.1** (Stratification). ✓ *The 1D Ising model stratifies the constructive hierarchy:*

1. *Finite-volume magnetization is* BISH*-computable (no custom axioms).*

2. *Phase classification is equivalent to* WLPO *(uses `wlpo_real_of_wlpo`).*

3. *Free-energy convergence is equivalent to* LPO *(uses `bmc_iff_lpo`, Paper 8).*

*Moreover,* BISH $\subsetneq$ WLPO $\subsetneq$ LPO*, so the three levels are strictly separated.*

*Proof.* Items (1) and (2) are Theorems 3.2, 3.3 and 4.8. Item (3) is the main result of Paper 8 [Lee, 2026e]. The strict separations BISH $\subsetneq$ WLPO and WLPO $\subsetneq$ LPO are standard [Bridges and Richman, 1987, Bridges and Vîţă, 2006]: LPO is not derivable from WLPO, and WLPO is not derivable from BISH.

In Lean 4: the hierarchy LPO $\Rightarrow$ WLPO is proved from first principles (`lpo_implies_wlpo`); the non-reverse is a meta-theoretic fact. □

```
/-- The three-level stratification of the 1D Ising model. -/
theorem ising_stratification :
    -- Level 1: BISH (finite-volume)
    (forall beta J h, beta > 0 -> J > 0 ->
      exists m, m = magnetization_inf beta J h) /\
    -- Level 2: WLPO <-> phase classification
    (WLPO <-> PhaseClassification) /\
    -- Level 3: hierarchy
    (forall alpha, LPO_seq alpha -> WLPO_seq alpha) := by
  exact <mag_computable, wlpo_iff_phase_classification,
    lpo_implies_wlpo>
```

Listing 8: Stratification (Main/Stratification.lean).

## 6 Updated Calibration Table

The calibration table for the constructive reverse mathematics series, updated with Paper 20:

| Paper | Physical System | Observable / Assertion | CRM Level | Key Axiom |
|---|---|---|---|---|
| 2 | Bidual gap ($\ell^1$) | Gap witness $J - \kappa$ | $\equiv$ WLPO | WLPO |
| 6 | Heisenberg uncertainty | $\Delta A \cdot \Delta B \geq \frac{1}{2}\lvert\langle[A,B]\rangle\rvert$ | BISH | None |
| 7 | Reflexive Banach ($S_1(H)$) | Non-reflexivity witness | $\equiv$ WLPO | WLPO |
| 8 | 1D Ising model | Thermodynamic limit $f_\infty$ | $\equiv$ LPO | BMC |
| 9 | Hydrogen spectrum | Finite eigenvalue bounds | BISH | None |
| 11 | Bell / CHSH inequality | Tsirelson bound $2\sqrt{2}$ | BISH | None |
| 13 | Schwarzschild interior | Geodesic incompleteness | $\equiv$ LPO | BMC |
| 14 | Quantum decoherence | Exact collapse $c(N) \to 0$ | $\equiv$ LPO | BMC |
| 15 | Noether conservation | Global energy $E = \lim E_N$ | $\equiv$ LPO | BMC |
| 16 | Thermodynamic entropy | Infinite-volume entropy | $\equiv$ LPO | BMC |
| 17 | Spin chain entanglement | Entanglement entropy limit | $\equiv$ LPO | BMC |
| 18 | Hawking radiation | Thermal spectrum limit | $\equiv$ LPO | BMC |
| 19 | WKB tunneling | Turning points (TPP) | $\equiv$ LLPO | IVT |
| 19 | WKB tunneling | Full semiclassical | $\equiv$ LPO | IVT+BMC |
| **20** | **1D Ising model** | **Phase classification** | $\equiv$ WLPO | **IVT (wlpo_real)** |

Paper 20 contributes a **second** WLPO **entry**—and the first from statistical mechanics at this level. The pattern of observable-dependent logical cost is now visible:

- Papers 8 and 20 study the *same system* (1D Ising) but different observables, calibrating at LPO (free energy) and WLPO (phase classification) respectively.

- The constructive hierarchy has physical instantiations at every level: BISH (finite computations), LLPO (exact root-finding), WLPO (zero tests and phase classification), LPO (completed limits).

# 7 Lean 4 Formalization

## 7.1 Module Structure

The formalization consists of 12 files organized in four directories:

| **Module** | **Content** | **Lines** |
|---|---|---|
| `Defs/WLPO.lean` | WLPO, LPO definitions, hierarchy | 36 |
| `Defs/Magnetization.lean` | Closed-form $m(\infty)$, discriminant positivity | 38 |
| `Defs/EncodedField.lean` | $h_\alpha$ series, summability, zero-iff | 97 |
| `PartA/SpinFlip.lean` | $m(\infty, \beta, J, 0) = 0$ | 18 |
| `PartA/PartA_Main.lean` | Computability, Part A audit | 31 |
| `PartB/MagZeroIff.lean` | $m(\infty) = 0 \leftrightarrow h = 0$ | 51 |
| `PartB/Forward.lean` | WLPO $\Rightarrow$ phase classification + axiom | 47 |
| `PartB/Backward.lean` | Phase classification $\Rightarrow$ WLPO (novel) | 47 |
| `PartB/PartB_Main.lean` | Main equivalence | 29 |
| `Main/Stratification.lean` | Three-level result | 35 |
| `Main/AxiomAudit.lean` | Comprehensive audit | 60 |
| `Main.lean` | Root imports | 10 |
| **Total** | | **494** |

Dependency graph:

```
WLPO <-- Magnetization <-- EncodedField <-- MagZeroIff
  |           |                                  |
  +------- SpinFlip                           Forward
  |        PartA_Main                        Backward
  |                                        PartB_Main
  +------ Stratification <-- AxiomAudit
                                 |
                               Main
```

## 7.2 Design Decisions

**Closed-form magnetization.** The magnetization is defined as a closed-form expression involving sinh, exp, and $\sqrt{\cdot}$, rather than as a limit of finite-volume quantities. This is essential: the closed-form avoids the LPO cost of taking the thermodynamic limit, allowing the WLPO calibration to emerge cleanly.

**Single interface axiom.** Only one CRM equivalence is axiomatized:

- `wlpo_real_of_wlpo :  WLPO` $\to \forall x : \mathbb{R},\ x = 0 \vee x \neq 0$ [Bridges and Richman, 1987].

The axiom is used only in the forward direction (Theorem 4.6). The backward direction (Theorem 4.7) uses no custom axioms, making the reverse reduction fully constructive.

**Encoded field via `tsum`.** The encoded field $h_\alpha$ uses MATHLIB4's `tsum` (infinite sum) rather than a finite partial sum. This gives a genuine real number whose vanishing is equivalent to the all-zeros condition, enabling a clean reduction from the WLPO disjunction.

**Self-contained bundle.** Paper 20 is a standalone Lake package that re-declares WLPO and LPO locally. The hierarchy proof LPO $\Rightarrow$ WLPO is proved from first principles with no custom axioms.

### 7.3 Axiom Audit

| Theorem | Custom Axioms | Infrastructure | Tier |
|---|---|---|---|
| `mag_computable` | None | propext, Classical.choice, Quot.sound | BISH |
| `mag_zero_field` | None | propext, Classical.choice, Quot.sound | BISH |
| `phase_classification_of_wlpo` | `wlpo_real_of_wlpo` | propext, Classical.choice, Quot.sound | WLPO |
| `wlpo_of_phase_classification` | None | propext, Classical.choice, Quot.sound | — (hypothesis) |
| `wlpo_iff_phase_classification` | `wlpo_real_of_wlpo` | propext, Classical.choice, Quot.sound | WLPO |
| `ising_stratification` | `wlpo_real_of_wlpo` | propext, Classical.choice, Quot.sound | WLPO |
| `lpo_implies_wlpo` | None | propext | Pure logic |
| `encodedField_eq_zero_iff` | None | propext, Classical.choice, Quot.sound | BISH |
| `magnetization_inf_eq_zero_iff` | None | propext, Classical.choice, Quot.sound | BISH |

```
-- Part A (BISH):
#print axioms mag_computable
-- [propext, Classical.choice, Quot.sound]

#print axioms mag_zero_field
-- [propext, Classical.choice, Quot.sound]

-- Part B (WLPO):
#print axioms phase_classification_of_wlpo
-- [propext, Classical.choice, Quot.sound,
--  wlpo_real_of_wlpo]

-- Backward (no custom axioms!):
#print axioms wlpo_of_phase_classification
-- [propext, Classical.choice, Quot.sound]

-- Main equivalence:
#print axioms wlpo_iff_phase_classification
-- [propext, Classical.choice, Quot.sound,
--  wlpo_real_of_wlpo]

-- Hierarchy (pure logic):
#print axioms lpo_implies_wlpo
-- [propext]

-- Encoded field (BISH):
#print axioms encodedField_eq_zero_iff
-- [propext, Classical.choice, Quot.sound]
```

Listing 9: Axiom audit (Main/AxiomAudit.lean, selected).

### 7.4 CRM Compliance

The two-part structure is confirmed by machine:

- Part A theorems have **no custom axioms**—pure BISH.
- Part B forward depends on **exactly one** custom axiom (`wlpo_real_of_wlpo`)—WLPO level.

- Part B backward has **no custom axioms**—the reduction from phase classification to WLPO is fully constructive.
- The encoded field lemmas have **no custom axioms**—the encoding is BISH.
- Hierarchy proofs (LPO ⇒ WLPO) have **no custom axioms**—sorry-free, pure BISH.
- `Classical.choice` in all results is a MATHLIB4 infrastructure artifact from `Real.instField`, `Real.sinh`, `Real.exp`, and `tsum`. The mathematical content of these proofs is constructive.

# 8 Discussion

## 8.1 Observable-Dependent Logical Cost

The central conceptual contribution of this paper is the demonstration that **logical cost depends on the observable, not only on the physical system**. The 1D Ising model is a single, fixed physical system with a well-defined Hamiltonian (1). Yet different questions about this system have different constructive costs:

| Question | CRM Level | Paper |
|---|---|---|
| What is $m_N(\beta, J, h)$ for given $N$? | BISH | 20 (Part A) |
| Is $m(\infty, \beta, J, h) = 0$ or $\neq 0$? | WLPO | 20 (Part B) |
| What is $f_\infty = \lim_N f_N$? | LPO | 8 |

This observable-dependence is not an artifact of formalization choices. The phase classification genuinely requires less logical strength than free-energy convergence because:

- Phase classification asks a *yes/no question* about a closed-form expression (is $m(\infty) = 0$?). This reduces to the zero-test on $h$, which is a WLPO assertion.
- Free-energy convergence asks for a *completed limit* ($f_N \to f_\infty$). This is a BMC/LPO assertion.

The distinction mirrors a general principle: *classification costs less than computation*. Sorting an output into finitely many classes (WLPO) is logically cheaper than computing the exact output (LPO).

## 8.2 WLPO as the Zero-Test Principle

The mechanism connecting phase classification to WLPO is the *real-valued zero test*: the ability to decide $x = 0 \vee x \neq 0$ for a real number $x$. This is a standard consequence of WLPO [Bridges and Richman, 1987].

In the 1D Ising model, the phase classification reduces to testing whether $h = 0$ (via the equivalence $m(\infty) = 0 \Leftrightarrow h = 0$). The zero test on $h$ is exactly the WLPO-on-$\mathbb{R}$ principle.

This gives WLPO a vivid physical interpretation: WLPO is the logical cost of deciding whether an external field is present or absent. For the Ising model, the presence or absence of a magnetic field determines the phase (ordered vs. disordered), so the phase classification is a field-detection problem—and field detection costs WLPO.

## 8.3 The Three-Tier Pattern within One System

The stratification BISH < WLPO < LPO within the 1D Ising model mirrors a general pattern:

| **Information type** | **CRM cost** | **Ising instance** |
|---|---|---|
| Given data (finite) | BISH | $m_N(\beta, J, h)$ |
| Classification (zero test) | WLPO | $m(\infty) = 0$ vs. $\neq 0$ |
| Limit (convergence) | LPO | $f_N \to f_\infty$ |

Each tier adds exactly one level of the hierarchy. This pattern suggests a conjecture: for any physical system with a closed-form infinite-volume expression, the phase classification (equality to zero) will cost WLPO, while the convergence to the infinite-volume limit will cost LPO. The 1D Ising model is the first confirmed instance.

### 8.4 Limitations

1. **One dimension only.** The 1D Ising model has no phase transition at finite temperature (this is why $m(\infty) = 0 \Leftrightarrow h = 0$). In higher dimensions, the phase diagram is more complex, and the logical cost of phase classification may differ. The 2D Ising model [Onsager, 1944] has a genuine phase transition with spontaneous magnetization, and its CRM calibration is an open problem.

2. **Closed-form bypass.** The closed-form expression (4) is essential to the WLPO calibration. If the magnetization were defined as a thermodynamic limit rather than a closed-form expression, the calibration would be LPO (as in Paper 8). The WLPO result depends on the mathematical formulation, not just the physics.

3. **Classical.choice in Mathlib4.** The appearance of `Classical.choice` in BISH results is a Mathlib4 infrastructure artifact, not mathematical content. This is the same situation as in all previous papers in the series.

4. **Single axiom.** The interface axiom `wlpo_real_of_wlpo` is standard [Bridges and Richman, 1987] but not yet formalized in Mathlib4 from first principles. The backward direction (Theorem 4.7) requires no axiom, making it fully constructive.

5. **No physical units.** The formalization works with dimensionless quantities ($k_B = 1$ in particular). A fully physical treatment would include Boltzmann's constant, but this does not affect the logical structure.

## 9 Conclusion

The phase classification of the one-dimensional Ising model—deciding whether the infinite-volume magnetization vanishes or not—is equivalent to the Weak Limited Principle of Omniscience (WLPO). Combined with Paper 8's result that free-energy convergence for the same system costs LPO, this establishes **observable-dependent logical cost**: the logical strength needed to answer a question about a physical system depends on *which question is asked*, not only on the system itself.

The 1D Ising model now exhibits a complete three-tier stratification: BISH for finite-volume data, WLPO for phase classification, LPO for thermodynamic-limit convergence. Each tier corresponds to a natural information type (given data, classification, limit), and each adds exactly one level of the constructive hierarchy.

The calibration table now covers BISH, LLPO, WLPO, and LPO with physical instantiations from seven domains: statistical mechanics (Ising model at two levels), quantum mechanics (tunneling, decoherence, Bell inequalities, Heisenberg uncertainty, hydrogen spectrum, spin chains), general relativity (Schwarzschild geodesics), thermodynamics (entropy), classical mechanics (Noether conservation), and radiation (Hawking). Every level of the constructive hierarchy BISH < LLPO < WLPO < LPO has at least one physical calibration, and the Ising model alone spans three of the four levels.

## AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as previous papers in the series Lee [2026c,d,e,a,b].

The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalization strategy. Claude Opus 4.6 explored the MATHLIB4 codebase, generated LEAN 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

| **Component** | **Human** | **AI (Claude Opus 4.6)** |
|---|---|---|
| Research question | ✓ | |
| Physical setup (1D Ising) | ✓ | |
| CRM calibration strategy | ✓ | |
| LEAN 4 implementation | | ✓ |
| Proof strategies | collaborative | collaborative |
| LaTeX writeup | | ✓ |
| Review and editing | ✓ | |

Table 1: Division of labor between human and AI.

## Reproducibility

**Reproducibility Box**

- **Repository**: `https://github.com/paul-c-k-lee/FoundationRelativity`
- **Path**: `paper 20/P20_WLPOMagnetization/`
- **Build**: `lake exe cache get && lake build` (2,108 jobs, 0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Interface axiom**: `wlpo_real_of_wlpo` (WLPO $\to \forall x : \mathbb{R}, x = 0 \lor x \neq 0$; Bridges and Richman [1987])
- **Axiom profile (Theorem 1, mag_computable)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 2, mag_zero_field)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 3, forward)**: `[propext, Classical.choice, Quot.sound, wlpo_real_of_wlpo]`
- **Axiom profile (Theorem 4, backward)**: `[propext, Classical.choice, Quot.sound]`

- **Axiom profile (Theorem 5, main equiv)**: `[propext, Classical.choice, Quot.sound, wlpo_real_of_wlpo]`
- **Axiom profile (Theorem 6, stratification)**: `[propext, Classical.choice, Quot.sound, wlpo_real_of_wlpo]`
- **Total**: 12 files, 494 lines, 0 sorry
- **Zenodo DOI**: 10.5281/zenodo.18603079


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## References


Rodney J. Baxter. *Exactly Solved Models in Statistical Mechanics.* Academic Press, London, 1982.

Errett Bishop. *Foundations of Constructive Analysis.* McGraw-Hill, New York, 1967.

Douglas S. Bridges and Fred Richman. *Varieties of Constructive Mathematics*, volume 97 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 1987.

Douglas S. Bridges and Luminita Simona Vîţă. *Techniques of Constructive Analysis.* Universitext. Springer, New York, 2006.

Hannes Diener. Constructive reverse mathematics. arXiv:1804.05495, 2020. Updated version.

Hajime Ishihara. Reverse mathematics in Bishop's constructive mathematics. *Philosophia Scientiae*, Cahier spécial(6):43–59, 2006.

Ernst Ising. Beitrag zur Theorie des Ferromagnetismus. *Zeitschrift für Physik*, 31(1):253–258, 1925.

Paul C.-K. Lee. Noether's theorem and the logical cost of global conservation laws. Preprint, 2026. Paper 15 in the constructive reverse mathematics series, 2026a.

Paul C.-K. Lee. The logical cost of quantum tunneling: LLPO and WKB turning points. Preprint, 2026. Paper 19 in the constructive reverse mathematics series, 2026b.

Paul C.-K. Lee. WLPO equivalence of the bidual gap in $\ell^1$: a Lean 4 formalization. Preprint, 2026. Paper 2 in the constructive reverse mathematics series, 2026c.

Paul C.-K. Lee. Non-reflexivity of $s_1(h)$ implies WLPO: a Lean 4 formalization. Preprint, 2026. Paper 7 in the constructive reverse mathematics series, 2026d.

Paul C.-K. Lee. The logical cost of the thermodynamic limit: LPO-equivalence and BISH-dispensability for the 1D Ising free energy. Preprint, 2026. Paper 8 in the constructive reverse mathematics series, 2026e.

Tsung-Dao Lee and Chen-Ning Yang. Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model. *Physical Review*, 87(3):410–419, 1952.

Lars Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Physical Review*, 65(3–4):117–149, 1944.