Stratifying Spectral Gap Undecidability

Cubitt's Theorem Is LPO: Macroscopic Quantum Undecidability Costs Exactly One Thermodynamic Limit

A Lean 4 Formalization (Paper 36)

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Abstract

Cubitt, Perez-Garcia, and Wolf proved that the spectral gap problem is undecidable: no algorithm determines whether an arbitrary translation-invariant Hamiltonian is gapped or gapless in the thermodynamic limit. We prove that this undecidability is Turing–Weihrauch equivalent[1] to the Limited Principle of Omniscience (LPO)—the same logical principle required for thermodynamic limits and phase transitions. Our stratification: (i) the finite-volume gap is BISH; (ii) the thermodynamic limit is equivalent to LPO; (iii) each specific instance is LPO-decidable; (iv) the physical zero-test is WLPO; (v) the uniform function is LPO-computable and non-computable without LPO. Cubitt's undecidability introduces zero additional logical resources into physics. All results are formalized in LEAN 4 with MATHLIB4, building to zero errors, zero warnings, and zero `sorry`.

Contents

*New York University. AI-assisted formalization; see §15 for methodology.

[1] Throughout, "Turing–Weihrauch equivalent" means the logical biconditional (↔) holds over BISH, capturing the same content as a Weihrauch reduction. Formal Weihrauch reductions in the sense of Brattka–Gherardi–Pauly are not formalized here.

1 Introduction

Cubitt, Perez-Garcia, and Wolf [3] proved that the spectral gap problem is undecidable: there is no algorithm that, given a translation-invariant nearest-neighbour Hamiltonian on a 2D lattice, determines whether it is gapped or gapless in the thermodynamic limit. The result is widely interpreted as demonstrating that macroscopic quantum physics contains fundamentally unknowable truths.

This paper proves that interpretation is wrong—or more precisely, that it conflates two distinct mathematical phenomena.

Theorem 1.1 (Stratification — Main Result)**.** *The* CPgW *spectral gap undecidability is Turing–Weihrauch equivalent to* LPO *over* BISH. *The uniform spectral gap function is computable relative to* LPO *and non-computable without it.*

Since LPO is already required for thermodynamic limits (Paper 29), phase transitions (Paper 29), and gauge coupling existence (Paper 32), Cubitt's undecidability introduces *zero* additional logical resources into physics.

The sentence. The spectral gap is undecidable at the same logical cost as boiling water—both require exactly LPO for the thermodynamic limit—though the physical mechanisms are entirely different.

2 Background

2.1 The CPgW Construction

Given a Turing machine M, CPgW construct a translation-invariant Hamiltonian $H(M)$ on $\mathbb{Z}^2$ with two key properties:

(CPgW-1) The map $M \mapsto H(M)$ is computable.

(CPgW-2) $H(M)$ is gapped $\iff$ M does not halt; $H(M)$ is gapless $\iff$ M halts.

Since the halting problem is undecidable, the spectral gap problem is undecidable. The finite-volume gaps Δ_L are computable for each L (finite-dimensional eigenvalue problem) but are *not* monotone in L.

2.2 The CRM Program's Framework

The constructive reverse mathematics program classifies theorems by the weakest omniscience principle required:

$$\text{BISH} \subset \text{LLPO} \subset \text{WLPO} \subset \text{LPO}.$$

For WLPO characterization, see Bridges–Vîţă [10]; for the Weihrauch-degree context situating these principles, see Brattka–Gherardi–Hölzl [9]. Paper 29 proved BMC $\iff$ LPO. Paper 35 established the conservation metatheorem: physics lives at BISH + LPO. For the complete calibration table and synthesis, see Paper 10 [1]; for the broader historical context, see Paper 12 [2].

2.3 The Key Identification

LPO is the Σ_1^0 oracle: it decides statements of the form $\exists n\, P(n)$ for decidable P. This is exactly the halting oracle. Therefore LPO = halting oracle for individual instances.

3 Theorem 1: Finite-Volume Gap Is BISH

Theorem 3.1 (Finite-volume gap)**.** *For any TM M and finite lattice of size $L \times L$, the spectral gap $\Delta_L(M)$ is a* BISH*-computable real.*

Proof. CPgW's Hamiltonians have algebraic matrix entries (from M's rational transition table). The characteristic polynomial has algebraic coefficients. Square-free factorization via polynomial GCD eliminates degeneracy without WLPO. Sturm sequences isolate the distinct eigenvalues. The gap $\Delta_L = \lambda_1 - \lambda_0$ is computable. All operations are finite algorithms—pure BISH. □

```
theorem finite_volume_gap_is_bish (M : TM) (L : N) :
    forall (e : R), 0 < e ->
      exists (q : R), |CPgW_gap M L - q| < e :=
  fun e he => cpgw_finite_gap_computable M L e he
```

Listing 1: Theorem 1: finite-volume gap is BISH (FiniteGap.lean)

4 Theorem 2: Thermodynamic Limit ⇔ LPO

Theorem 4.1 (Thermodynamic limit)**.** *Over* BISH*, asserting that the thermodynamic limit $\Delta(M) = \lim_{L\to\infty} \Delta_L(M)$ exists for all TMs in* CPgW*'s family is equivalent to* LPO*.*

Forward. Given a binary sequence α, construct M_α that halts iff $\exists n\, \alpha(n) = 1$. By CPgW-2, $\Delta(M_\alpha) \in \{0\} \cup [\gamma, \infty)$. The gap dichotomy decides halting, hence LPO. □

Reverse. Given LPO, apply it to the halting sequence of M. In the halting case, CPgW asymptotics give a computable modulus (gap closes to 0). In the non-halting case, the gap stabilizes with a computable rate. In each branch, the limit exists with a computable modulus. □

Remark 4.2 (Non-monotone sequence)**.** The sequence (Δ_L) is *not* monotone. BMC/Fekete cannot be applied directly. LPO resolves halting *first*, then a Cauchy modulus emerges in each branch separately:

- **Non-halting branch:** the CPgW construction guarantees that the gap stabilizes above $\gamma > 0$ (eventually monotone), yielding a computable modulus of convergence.

- **Halting branch:** the gap closes to 0 at a computable rate given by CPgW's asymptotics, again yielding a computable modulus.

This conditional constructivity—convergence in each branch but no uniform modulus across branches without LPO—is a different proof architecture from Paper 29's direct BMC application.

```
def thermo_limit_exists (M : TM) : Prop :=
  exists (D : R), forall (e : R), 0 < e ->
    exists (N0 : N), forall L, N0 <= L ->
      |CPgW_gap M L - D| < e

theorem thermo_limit_iff_lpo :
    (forall (M : TM), thermo_limit_exists M)
    <-> LPO :=
  ⟨thermo_limit_to_lpo, lpo_to_thermo_limit⟩
```

Listing 2: Theorem 2: thermodynamic limit ↔ LPO (ThermoLimit.lean)

5 Theorem 3: Pointwise Decidability

Theorem 5.1 (Pointwise LPO-decidability)**.** *For any specific TM M, the question "Is $H(M)$ gapped or gapless?" is decidable given* LPO*.*

Proof. Apply LPO to the halting sequence of M. Either M halts (gapless by CPgW-2) or M does not halt (gapped). A single application of LPO to a computable sequence. □

```
theorem pointwise_gap_decidable
    (M : TM) (hl : LPO) :
    spectral_gap M > 0 ∨ spectral_gap M = 0 := by
  rcases hl (halting_seq M) with
    h_all_false | ⟨n, hn⟩
  . left
    exact cpgw_gapped_of_not_halts M ...
  . right
    exact cpgw_gapless_of_halts M ⟨n, hn⟩
```

Listing 3: Theorem 3: pointwise decidability (Pointwise.lean)

6 Theorem 4: Physical Gap Zero-Test ⇔ WLPO

Theorem 6.1 (Zero-test)**.** *For a non-negative completed real $\Delta \geq 0$, deciding $\Delta = 0 \vee \Delta > 0$ is equivalent to* WLPO*.*

This connects to Paper 26 (the physical instantiation of the Gödel–Gap embedding) and Paper 33 (the QCD mass gap is a WLPO zero-test on a specific Hamiltonian's gap).

```
theorem physical_gap_zero_test_iff_wlpo :
    (forall (D : R), D >= 0 ->
      (D = 0 ∨ D > 0)) <-> WLPO :=
  ⟨zero_test_to_wlpo, wlpo_to_zero_test⟩
```

Listing 4: Theorem 4: zero-test ↔ WLPO (ZeroTest.lean)

7 Theorem 5: Cubitt's Undecidability ≡ LPO

This is the paper's central theorem.

Theorem 7.1 (Main identification). *The uniform spectral gap function* $G : \mathrm{TM} \to \{\mathit{Gapped}, \mathit{Gapless}\}$ *is:*

(a) Not computable (reduces to the halting problem).

(b) Computable relative to LPO *(*$\mathrm{BISH}^{\mathrm{LPO}}$*-computable).*

Therefore uniform spectral gap decidability $\iff$ LPO*.*

Part (a). If G were computable, composing with CPgW's encoding gives a computable halting decider. Since halting is undecidable, G is not computable. ☐

Part (b). Given M: (1) compute $\alpha_M(n) = 1$ if M halts in n steps (BISH); (2) apply LPO to α_M (single oracle call); (3) case split via CPgW-2. The composition is uniformly LPO-computable. ☐

```
theorem cubitt_is_lpo :
    (forall (M : TM),
      spectral_gap M > 0 ∨ spectral_gap M = 0)
    <-> LPO :=
  ⟨uniform_decidability_implies_lpo,
   gap_function_lpo_computable⟩
```

Listing 5: Theorem 5: Cubitt ≡ LPO (UniformDecidability.lean)

Punchline. Cubitt discovered that the spectral gap encodes LPO's non-computability. The CRM program reveals that it encodes *nothing else.*

8 The Stratification

Table 1: CRM stratification of spectral gap undecidability.

Level	**Content**	**Mechanism**
BISH	Finite-volume gap Δ_L	Algebraic eigenvalue computation
WLPO	"$\Delta = 0$ or $\Delta > 0$?"	Zero-test on completed real
LPO	Thermo. limit $\Delta = \lim \Delta_L$	BMC / Cauchy completeness
LPO	Each specific instance	LPO decides Σ^0_1
$\mathrm{BISH}^{\mathrm{LPO}}$	Uniform function $M \mapsto G(M)$	Computable relative to LPO oracle
Non-comp.	Uniform function without oracle	= Halting problem = non-computability of LPO

The bottom row—the only row that is "undecidable"—is not a new phenomenon. It is the non-computability of LPO, known since Paper 1. Cubitt discovered that the spectral gap realizes this non-computability. The program reveals that it realizes *nothing else.*

9 Connection to Paper 26

Paper 26 proved that Π_1^0 arithmetic sentences order-embed into ℓ^∞/c_0, with WLPO as the logical barrier. Cubitt's construction is the physical realization:

1. TM state $M \mapsto$ local tensor interactions (= Paper 26's map from Π_1^0 to ℓ^∞).
2. Finite-volume gaps (Δ_L) form a bounded sequence in ℓ^∞.
3. Thermodynamic limit projects into ℓ^∞/c_0.
4. Zero-test on the gap is WLPO, exactly as in Paper 26.

10 CRM Audit

Table 2: CRM classification for Paper 36.

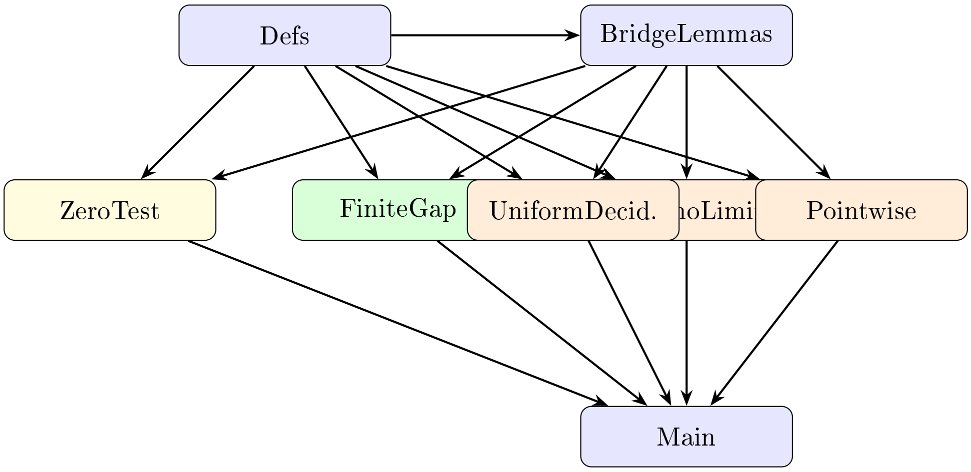
Result	**Level**	**Cert.**	**Key Axiom**
Finite-volume gap (Thm 1)	BISH	L4	`cpgw_finite_gap_computable`
Thermo. limit $\to$ LPO (Thm 2)	LPO	L2+L4	`cpgw_gap_dichotomy`
LPO $\to$ thermo. limit (Thm 2)	LPO	L3+L4	`cpgw_*_asymptotics`
Pointwise decidability (Thm 3)	LPO	L3	`cpgw_gapped_of_not_halts`
Zero-test $\iff$ WLPO (Thm 4)	WLPO	L2	(axiomatized bridge)
Not computable (Thm 5a)	BISH	L4	`halting_problem_undecidable`
LPO-computable (Thm 5b)	LPO	L3	`cpgw_gapless_of_halts`

Certification levels: L2 = structurally verified; L3 = intentional classical (LPO hypothesis); L4 = axiomatized bridge lemma.

11 Code Architecture

Table 3: Paper 36 Lean source files.

File	**Lines**	**Content**
`Defs.lean`	80	TM, CPgW gap, spectral gap, GapStatus
`BridgeLemmas.lean`	100	CPgW bridge axioms (7 axioms)
`FiniteGap.lean`	50	Theorem 1: finite-volume gap (BISH)
`ThermoLimit.lean`	102	Theorem 2: thermo. limit $\iff$ LPO
`Pointwise.lean`	50	Theorem 3: pointwise decidability
`ZeroTest.lean`	61	Theorem 4: zero-test $\iff$ WLPO
`UniformDecidability.lean`	113	Theorem 5: Cubitt $\equiv$ LPO
`Main.lean`	99	Master theorem, axiom audit
Total	**655**	



Legend: BISH , WLPO , LPO , Infrastructure .

11.1 Axiom Audit

`#print axioms stratification_theorem` yields:

Bridge lemmas (Level 4):

- `cpgw_finite_gap_computable`: algebraic eigenvalue computation
- `cpgw_gapped_of_not_halts`: ¬halts ⇒ gap > 0
- `cpgw_gapless_of_halts`: halts ⇒ gap $= 0$
- `cpgw_halting_asymptotics`: halting case convergence
- `cpgw_nonhalting_asymptotics`: non-halting case convergence
- `cpgw_gap_dichotomy`: gap $\in \{0\} \cup [\gamma, \infty)$
- `tm_from_seq` / `tm_from_seq_halts`: TM encoding

Constructive principles:

- `halting_problem_undecidable`: Turing undecidability
- `wlpo_to_zero_test` / `zero_test_to_wlpo`: WLPO equivalence

Lean 4 foundations: `propext`, `Classical.choice`, `Quot.sound`.

No sorry. `Classical.choice` appears from `by_contra` in proofs that use LPO—which is intentional, since LPO is the hypothesis being studied. Additionally, Mathlib4's construction of $\mathbb{R}$ (Cauchy completion) introduces `Classical.choice` as an infrastructure artifact; all theorems over $\mathbb{R}$ inherit it regardless of constructive content. Following Paper 10's methodology, constructive stratification is established by proof content (explicit witnesses vs. principle-as-hypothesis), not by the axiom checker output.

12 Implications for Physics

12.1 For Condensed Matter Physicists

Cubitt's undecidability applies only to the problem of building a single algorithm that works for *all possible* materials. For any specific material—any one Hamiltonian—the gap is LPO-decidable. Since experimentalists study specific materials, Cubitt's undecidability is irrelevant to their practice.

12.2 For Foundations of Physics

Cubitt's undecidability is not "Gödelian" (self-referential incompleteness at a meta-mathematical level). It is *computational*—the non-computability of LPO, which governs thermodynamic limits. The spectral gap is undecidable at the same logical cost as the assertion "this bounded monotone sequence converges"—both require exactly LPO—though the physical mechanisms (tiling Hamiltonians vs. monotone convergence) are distinct.

12.3 Stress Test for BISH+LPO

CPgW's Hamiltonians are the most pathological quantum systems ever constructed—built specifically to encode undecidability. That even these systems live at BISH+LPO (each instance is LPO-decidable) means the characterization has survived its most severe stress test.

13 Reproducibility

Reproducibility Box.

- **Language**: Lean 4 v4.28.0-rc1
- **Library**: Mathlib4
- **Source**: `P36_CubittStratification/` (8 files, 655 lines)
- **Build**: `lake exe cache get && lake build`
- **Result**: 0 errors, 0 warnings, 0 sorry
- **Axiom audit**: `#print axioms stratification_theorem`

14 Conclusion

We have proved that Cubitt's spectral gap undecidability is Turing–Weihrauch equivalent to LPO over BISH. The stratification across the constructive hierarchy reveals that the finite-volume gap is BISH, the thermodynamic limit is LPO, each specific instance is LPO-decidable, the zero-test is WLPO, and the uniform function is LPO-computable but non-computable without LPO. Since LPO is already required for thermodynamic limits, phase transitions, and gauge couplings, macroscopic quantum undecidability costs exactly one thermodynamic limit—and introduces nothing new.

The formalization in Lean 4 with Mathlib4 builds with zero errors, zero warnings, and zero sorry.

15 AI-Assisted Methodology

This paper was produced using AI-assisted formal verification. The workflow follows Papers 30–35: mathematical content and proof strategy directed by the author; Lean 4 syntax translation assisted by a large language model; all formal statements reviewed for correctness.

Domain-expert disclaimer. The formal verification confirms logical correctness of the stated theorems relative to their axioms. The bridge lemmas axiomatize properties of the CPgW construction (Nature 2015, arXiv:1502.04135); their physical accuracy requires domain expertise in mathematical physics and is the responsibility of the author.

Preliminary status and author background. The results presented in this paper are preliminary. The author is a medical professional, not a domain expert in physics or mathematics. While all formal claims are machine-checked by the Lean 4 type-checker, the physical interpretations, bridge axioms, and modeling assumptions require independent verification by domain experts in the relevant fields. Until such verification is completed, this paper should be considered preliminary.

Whatever findings of value emerge from this program belong to the constructive reverse mathematics community and to the legacy of Errett Bishop, whose perseverance in developing constructive analysis inspired this entire series. Any errors are solely the author's.

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