The Tate Conjecture and LPO: Galois-Invariance, Cycle Verification, and Standard Conjecture D as a Decidability Axiom

(Paper 46, Constructive Reverse Mathematics Series)

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Abstract

We apply Constructive Reverse Mathematics to calibrate the logical strength of the Tate Conjecture for smooth projective varieties over finite fields. We establish four theorems (T1–T4) that constitute a *constructive calibration* of the conjecture's logical structure. Theorem T1 proves that deciding Galois-invariance (membership in $\ker(\mathrm{Frob} - I)$ on ℓ-adic cohomology) is equivalent to LPO($\mathbb{Q}_\ell$). Theorem T2 shows that numerical equivalence of algebraic cycles is decidable in BISH—requiring only integer arithmetic via the intersection pairing. Theorem T3 establishes that the Poincaré pairing on $V = H^{2r}_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell(r))$ cannot be anisotropic in dimension ≥ 5, permanently blocking the polarization strategy over $\mathbb{Q}_\ell$. Theorem T4 is the key new result: deciding homological equivalence requires LPO($\mathbb{Q}_\ell$), but Grothendieck's Standard Conjecture D converts it to BISH-decidable numerical equivalence. Standard Conjecture D is thus precisely the axiom that *de-omniscientizes* the morphism spaces of the motivic category. All results are formalized in Lean 4 over Mathlib; the bundle compiles with 0 errors, 0 warnings, and 0 `sorry`s. The formalization uses 21 custom axioms, all explicitly documented.

Contents

*Lean 4 formalization available at `https://doi.org/10.5281/zenodo.18682285`.

1 Introduction

1.1 Main results

Let X be a smooth projective variety over a finite field $\mathbb{F}_q$, and let $V = H^{2r}_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell(r))$ denote the ℓ-adic cohomology space carrying the action of the arithmetic Frobenius F. The Tate Conjecture (Tate, 1965 [19]) asserts that the cycle class map

$$\mathrm{cl} : \mathrm{CH}^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow V^{F=1} = \ker(\mathrm{Frob} - I)$$

is surjective: every Galois-invariant cohomology class comes from an algebraic cycle.

This paper applies Constructive Reverse Mathematics (CRM) to the logical structure of the Tate Conjecture, calibrating its components against the BISH–LPO hierarchy. We establish:

Theorem A (T1: Galois-Invariance $\leftrightarrow$ LPO). ✓ Deciding membership in $\ker(\mathrm{Frob} - I)$ over $\mathbb{Q}_\ell$ is equivalent to $\mathrm{LPO}(\mathbb{Q}_\ell)$:

$$(\forall\, x \in V,\; x \in V^{F=1} \vee x \notin V^{F=1}) \;\leftrightarrow\; \mathrm{LPO}(\mathbb{Q}_\ell).$$

Theorem B (T2: Cycle Verification in BISH). ✓ Given a finite complementary basis $\{W_1, \ldots, W_m\}$ for the Chow group $\mathrm{CH}^r(X) \otimes \mathbb{Q}$, numerical equivalence of algebraic cycles is decidable in BISH—requiring only finitely many integer equality tests.

Theorem C (T3: Polarization Obstruction). ✓ The Poincaré pairing on V cannot be anisotropic when $\dim_{\mathbb{Q}_\ell} V \geq 5$. Orthogonal projection onto $V^{F=1}$ is impossible over $\mathbb{Q}_\ell$.

Theorem D (T4: Standard Conjecture D as Decidability Axiom). ✓

- T4a: Deciding homological equivalence (cycle class equality in $\mathbb{Q}_\ell$-cohomology) requires $\mathrm{LPO}(\mathbb{Q}_\ell)$.
- T4b: Standard Conjecture D converts homological equivalence to numerical equivalence, making it BISH-decidable.

This is the key new result of Paper 46: SCD is precisely the axiom that *de-omniscientizes* the morphism spaces of the motivic category.

1.2 Constructive Reverse Mathematics: a brief primer

CRM calibrates mathematical statements against logical principles of increasing strength within Bishop-style constructive mathematics (BISH). The hierarchy relevant to this paper is:

$$\mathrm{BISH} \subset \mathrm{BISH} + \mathrm{MP} \subset \mathrm{BISH} + \mathrm{LLPO} \subset \mathrm{BISH} + \mathrm{LPO} \subset \mathrm{CLASS}.$$

Here LPO (Limited Principle of Omniscience) states that every binary sequence is identically zero or contains a 1. In field-theoretic form, $\mathrm{LPO}(K)$ states $\forall x \in K,\ x = 0 \vee x \neq 0$. For a thorough treatment of CRM, see Bridges–Richman [2] and Bridges–Vîţă [3]; for reverse mathematics in Bishop's framework, see Ishihara [7]; for the broader program of which this paper is part, see Papers 1–45 of this series and the atlas survey [15]. Bishop–Bridges [1] provides the foundational reference for constructive analysis.

1.3 Current state of the art

The Tate Conjecture was formulated by Tate [19] in 1965. It is known for abelian varieties over finite fields (Tate [20]; Faltings [5] over number fields), for K3 surfaces over finite fields of odd characteristic (various authors; see Madapusi Pera [12]), and for divisors on abelian varieties. It remains open in general. Recent progress includes Scholze's perfectoid methods [17]. The conjecture is intimately related to the Birch–Swinnerton-Dyer conjecture [21] and to Grothendieck's Standard Conjectures [6]. For background on abelian varieties, see Milne [13].

Standard Conjecture D (Grothendieck [6]) asserts that homological and numerical equivalence coincide on algebraic cycles. This is a major open problem in algebraic geometry, implied by the Hodge Conjecture in characteristic zero and by the Tate Conjecture in positive characteristic. The constructive calibration we perform here is novel: no prior work has applied CRM to the logical structure of the Tate Conjecture or identified SCD as a decidability axiom.

1.4 Position in the atlas

This is Paper 46 of a series applying constructive reverse mathematics to five conjectures in arithmetic geometry (Papers 45–49, with atlas survey in Paper 50 [15]). Paper 45 [14] calibrates the Weight-Monodromy Conjecture and establishes the *de-omniscientizing descent* pattern. The present paper reuses the LPO definition, encoding pattern, and u-invariant obstruction from Paper 45, adapting them to the Tate Conjecture setting.

The novel contribution is Theorem D (T4): Standard Conjecture D as a decidability axiom. This result is unique to the Tate setting—the equivalence relation descent from homological to numerical equivalence has no analogue in the Weight-Monodromy Conjecture—and generalizes across the five-conjecture program.

2 Preliminaries

Definition 2.1 (Limited Principle of Omniscience)**.** LPO is the assertion that for every binary sequence $a : \mathbb{N} \to \{0,1\}$, either $\forall n,\ a(n) = 0$ or $\exists n,\ a(n) = 1$.

Definition 2.2 (LPO for a field)**.** LPO(K) is the assertion $\forall x \in K,\ x = 0 \vee x \neq 0$.

Definition 2.3 (Tate Conjecture)**.** For a smooth projective variety X over $\mathbb{F}_q$ and $V = H^{2r}_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell(r))$, the Tate Conjecture TC(X, r) asserts that the cycle class map $\text{cl} : \text{CH}^r(X) \otimes \mathbb{Q}_\ell \to V^{F=1}$ is surjective.

Definition 2.4 (Galois-fixed subspace)**.** The Galois-fixed subspace is $V^{F=1} = \ker(\text{Frob} - I)$, where $\text{Frob} : V \to V$ is the arithmetic Frobenius endomorphism. In the formalization: `galois_fixed := LinearMap.ker (Frob - LinearMap.id)`.

Definition 2.5 (Chow group and intersection pairing)**.** $\text{CH}^r(X) \otimes \mathbb{Q}$ is the group of algebraic r-cycles modulo rational equivalence, tensored with $\mathbb{Q}$. The intersection pairing $\cdot : \text{CH}^r(X) \times \text{CH}^r(X) \to \mathbb{Z}$ assigns to each pair of cycles an integer intersection number.

Definition 2.6 (Numerical equivalence)**.** Two cycles Z_1, Z_2 are *numerically equivalent*, written $Z_1 \sim_{\text{num}} Z_2$, if $Z_1 \cdot W = Z_2 \cdot W$ for all cycles W. In the formalization: `num_equiv Z₁ Z₂ := ∀ W, intersection Z₁ W = intersection Z₂ W`.

Definition 2.7 (Homological equivalence)**.** Two cycles Z_1, Z_2 are *homologically equivalent*, written $Z_1 \sim_{\text{hom}} Z_2$, if $\text{cl}(Z_1) = \text{cl}(Z_2)$ in V. In the formalization: `hom_equiv Z₁ Z₂ := cycle_class Z₁ = cycle_class Z₂`.

Definition 2.8 (Standard Conjecture D)**.** Standard Conjecture D (Grothendieck [6]) asserts that homological and numerical equivalence coincide: $Z_1 \sim_{\text{hom}} Z_2 \iff Z_1 \sim_{\text{num}} Z_2$ for all cycles Z_1, Z_2.

Definition 2.9 (Anisotropic pairing)**.** A symmetric bilinear pairing $B : V \times V \to K$ is *anisotropic* if $B(v, v) = 0$ implies $v = 0$.

Remark 2.10. All axiomatized objects (the ℓ-adic field, cohomology space, Frobenius endomorphism, Chow group, cycle class map, intersection pairing, Poincaré pairing) are documented in the Lean files with explicit docstrings. See Section 5 for the full axiom inventory.

3 Main Results

3.1 Theorem A (T1): Galois-invariance ↔ LPO

Theorem 3.1 (T1)**.** $(\forall x \in V,\ x \in \ker(\text{Frob} - I) \vee x \notin \ker(\text{Frob} - I)) \quad \leftrightarrow \quad \text{LPO}(\mathbb{Q}_\ell)$.

Proof. ($\Rightarrow$) Given $a \in \mathbb{Q}_\ell$, the encoding axiom (`encode_scalar_to_galois`) provides $x \in V$ satisfying $x \in \ker(\text{Frob} - I) \iff a = 0$. The mathematical construction: choose $v \in V$ with $(\text{Frob} - I)(v) = w \neq 0$ and set $x = a \cdot v$. Then $(\text{Frob} - I)(x) = a \cdot w$, which equals 0 iff $a = 0$ (since $w \neq 0$ and $\mathbb{Q}_\ell$ is a field). The decidability oracle on x yields $a = 0 \vee a \neq 0$.

In the Lean formalization:

```
theorem galois_invariance_requires_LPO :
    (∀ (x : V), x ∈ galois_fixed ∨ x ∉ galois_fixed) → LPO_Q_ell := by
  intro h_dec a
  obtain ⟨x, hx⟩ := encode_scalar_to_galois a
  rcases h_dec x with h_in | h_not_in
  · left; exact hx.mp h_in
  · right; exact fun ha => h_not_in (hx.mpr ha)
```

($\Leftarrow$) Given LPO($\mathbb{Q}_\ell$), compute $y = (\mathrm{Frob} - I)(x)$ and express y in coordinates $(y_1, \dots, y_n)$ with respect to a basis of V. Apply LPO($\mathbb{Q}_\ell$) to each coordinate: $y_i = 0 \lor y_i \neq 0$. Since V is finite-dimensional, this is a finite conjunction. If all coordinates are zero, $x \in \ker(\mathrm{Frob} - I)$; if any is nonzero, $x \notin \ker(\mathrm{Frob} - I)$. This is axiomatized as `LPO_decides_ker_membership`.

Parallel to Paper 45. This is the Tate Conjecture analogue of Paper 45 Theorem C2, with $\mathrm{Frob} - I$ replacing the spectral sequence differential and $V^{F=1}$ replacing the degeneration locus. The encoding pattern is identical. □

3.2 Theorem B (T2): Cycle verification in BISH

Theorem 3.2 (T2). *Given a finite complementary basis $\{W_1, \dots, W_m\}$ for $\mathrm{CH}^r(X) \otimes \mathbb{Q}$, numerical equivalence is decidable: for any Z_1, Z_2,*

$$Z_1 \sim_{\mathrm{num}}^{\mathrm{fin}} Z_2 \;\lor\; Z_1 \not\sim_{\mathrm{num}}^{\mathrm{fin}} Z_2$$

where $Z_1 \sim_{\mathrm{num}}^{\mathrm{fin}} Z_2 := \forall j \in \{1, \dots, m\},\ Z_1 \cdot W_j = Z_2 \cdot W_j$.

Proof. The proof is a standard decidability argument over finite-dimensional modules with integer pairing.

Step 1. Each intersection number $Z \cdot W_j$ is an integer. Integer equality is decidable: $\forall a, b \in \mathbb{Z},\ a = b \lor a \neq b$ (Lean: `Int.decEq`).

Step 2. The proposition $\forall j \in \mathrm{Fin}\ m,\ Z_1 \cdot W_j = Z_2 \cdot W_j$ is a finite conjunction of decidable propositions. A finite conjunction of decidable propositions is decidable (Lean: `Fintype.decidableForallFintype`).

Step 3. Therefore $Z_1 \sim_{\mathrm{num}}^{\mathrm{fin}} Z_2$ is decidable. Extracting `.em` from the `Decidable` instance gives the disjunction.

No LPO, no Markov's Principle, no omniscience of any kind is required. The proof uses only integer arithmetic and the decidability of finite conjunctions—both available in BISH. This is the constructive heart of the Tate Conjecture: the geometric side (cycle verification via intersection numbers) uses only decidable integer arithmetic. □

3.3 Theorem C (T3): Polarization obstruction

Theorem 3.3 (T3). *If $\dim_{\mathbb{Q}_\ell} V \geq 5$, the Poincaré pairing on V cannot be anisotropic. That is:*

$$5 \leq \dim_{\mathbb{Q}_\ell} V \implies \neg(\forall v \in V,\ B(v, v) = 0 \implies v = 0)$$

where B is the Poincaré (cup product) pairing on V.

Proof. The u-invariant of $\mathbb{Q}_\ell$ (a local field) is 4 (Hasse–Minkowski; Lam [10]; Serre [18]). The Poincaré pairing is a nondegenerate symmetric bilinear form on V of dimension $\geq 5 > 4 = u(\mathbb{Q}_\ell)$. By the definition of u-invariant, every quadratic form of dimension $> u(K)$ over K has a nontrivial zero. Therefore there exists $v \neq 0$ with $B(v, v) = 0$. But anisotropy requires $B(v, v) = 0 \implies v = 0$, giving $v = 0$—a contradiction.

In the Lean formalization:

```
theorem poincare_not_anisotropic
    (hdim : 5 ≤ Module.finrank Q_ell V) :
    ¬ (∀ v : V, poincare_pairing v v = 0 → v = 0) := by
  intro h_aniso
  obtain ⟨v, hv_ne, hv_iso⟩ := poincare_isotropic_high_dim hdim
  exact hv_ne (h_aniso v hv_iso)
```

Comparison with Paper 45 C3. Paper 45 uses $\dim \geq 3$ for Hermitian forms over a quadratic extension L/K, because the trace form $\mathrm{Tr}_{L/K} \circ H$ doubles the dimension: $2 \times 3 = 6 > 4$. Here we work directly with the symmetric bilinear Poincaré pairing on V, with no trace form doubling: the threshold is $\dim \geq 5$ rather than $\dim \geq 3$.

Consequence. The Hodge-theoretic strategy—splitting $V = \ker(\mathrm{Frob} - I) \oplus \ldots$ via orthogonal projection using a positive-definite metric—works over $\mathbb{C}$ (where positive-definite inner products exist in all dimensions) but is algebraically impossible over $\mathbb{Q}_\ell$ in dimension ≥ 5. Any proof of the Tate Conjecture must use a non-metric strategy. □

3.4 Theorem D (T4): Standard Conjecture D as decidability axiom

Theorem 3.4 (T4a)**.** *If homological equivalence is decidable for all cycle pairs, then* LPO($\mathbb{Q}_\ell$) *holds:*

$$(\forall\, Z_1, Z_2,\ \mathrm{Decidable}(Z_1 \sim_{\mathrm{hom}} Z_2)) \implies \mathrm{LPO}(\mathbb{Q}_\ell).$$

Proof. Given $a \in \mathbb{Q}_\ell$, the encoding axiom (`encode_scalar_to_hom_equiv`) provides cycles Z_1, Z_2 with $Z_1 \sim_{\mathrm{hom}} Z_2 \iff a = 0$. The mathematical construction: the cycle class map has nonzero image; fix a nonzero v in the image and construct Z_a mapping to $a \cdot v$. Then $\mathrm{cl}(Z_a) = \mathrm{cl}(Z_0)$ iff $a \cdot v = 0$ iff $a = 0$. The decidability oracle on (Z_1, Z_2) decides $a = 0 \vee a \neq 0$.

```
theorem hom_equiv_requires_LPO :
    (∀ (Z₁ Z₂ : ChowGroup), Decidable (hom_equiv Z₁ Z₂)) → LPO_Q_ell := by
  intro h_dec a
  obtain ⟨Z₁, Z₂, hZ⟩ := encode_scalar_to_hom_equiv a
  cases h_dec Z₁ Z₂ with
  | isTrue h => left; exact hZ.mp h
  | isFalse h => right; exact fun ha => h (hZ.mpr ha)
```

□

Theorem 3.5 (T4b)**.** *Assuming Standard Conjecture D and a finite spanning basis, homological equivalence is decidable in* BISH*:*

$$\mathrm{SCD} \implies \forall\, Z_1, Z_2,\ Z_1 \sim_{\mathrm{hom}} Z_2 \vee Z_1 \not\sim_{\mathrm{hom}} Z_2.$$

Proof. The proof composes three equivalences:

1. SCD: $Z_1 \sim_{\mathrm{hom}} Z_2 \iff Z_1 \sim_{\mathrm{num}} Z_2$ (`standard_conjecture_D`).
2. Basis spanning: $Z_1 \sim_{\mathrm{num}} Z_2 \iff Z_1 \sim_{\mathrm{num}}^{\mathrm{fin}} Z_2$ (`basis_spans_num_equiv`).
3. Theorem T2: $Z_1 \sim_{\mathrm{num}}^{\mathrm{fin}} Z_2$ is decidable (`num_equiv_fin_decidable`).

Composing: $Z_1 \sim_{\mathrm{hom}} Z_2 \iff Z_1 \sim_{\mathrm{num}}^{\mathrm{fin}} Z_2$, and the right-hand side is decidable. Therefore homological equivalence is decidable.

```
def conjD_decidabilizes_morphisms {m : ℕ}
    (basis : Fin m → ChowGroup) (Z₁ Z₂ : ChowGroup) :
    Decidable (hom_equiv Z₁ Z₂) := by
  rw [show hom_equiv Z₁ Z₂ ↔ num_equiv_fin basis Z₁ Z₂ from
    (standard_conjecture_D Z₁ Z₂).trans (basis_spans_num_equiv basis Z₁ Z₂)]
  exact num_equiv_fin_decidable basis Z₁ Z₂
```

□

Remark 3.6 (Standard Conjecture D as the de-omniscientizing axiom). Theorems T4a and T4b together reveal the precise logical role of Standard Conjecture D:

- *Without* SCD: testing $\mathrm{cl}(Z_1) = \mathrm{cl}(Z_2)$ in $\mathbb{Q}_\ell$-cohomology requires $\mathrm{LPO}(\mathbb{Q}_\ell)$—exact zero-testing in an ℓ-adic field.
- *With* SCD: the same test reduces to $Z_1 \cdot W_j = Z_2 \cdot W_j$ for finitely many j—integer arithmetic in BISH.

In the language of motives, SCD asserts `DecidableEq` on Hom-spaces of the motivic category. It is the axiom that *de-omniscientizes* the morphism spaces: converting LPO-dependent equality testing to BISH-decidable computation.

Theorem 3.7 (Summary)**.** *The four results T1–T4 together yield:*

1. $(\forall\, x,\ x \in V^{F=1} \vee x \notin V^{F=1}) \leftrightarrow \mathrm{LPO}(\mathbb{Q}_\ell)$.
2. $\forall\, Z_1, Z_2,\ Z_1 \sim^{\mathrm{fin}}_{\mathrm{num}} Z_2 \vee Z_1 \not\sim^{\mathrm{fin}}_{\mathrm{num}} Z_2$.
3. $5 \le \dim V \implies \neg(\textit{Poincaré pairing anisotropic})$.
4. $\forall\, Z_1, Z_2,\ Z_1 \sim_{\mathrm{hom}} Z_2 \vee Z_1 \not\sim_{\mathrm{hom}} Z_2 \qquad$ *(assuming* SCD*).*

Proof. Direct assembly of Theorems 3.1–3.5. In Lean:

```
theorem tate_calibration_summary :
    ((∀ x : V, x ∈ galois_fixed ∨ x ∉ galois_fixed) ↔ LPO_Q_ell) ∧
    (∀ {m : ℕ} (basis : Fin m → ChowGroup) (Z₁ Z₂ : ChowGroup),
      num_equiv_fin basis Z₁ Z₂ ∨ ¬ num_equiv_fin basis Z₁ Z₂) ∧
    (5 ≤ Module.finrank Q_ell V →
      ¬ (∀ v : V, poincare_pairing v v = 0 → v = 0)) ∧
    (∀ {m : ℕ} (_basis : Fin m → ChowGroup) (Z₁ Z₂ : ChowGroup),
      hom_equiv Z₁ Z₂ ∨ ¬ hom_equiv Z₁ Z₂) := by
  exact ⟨galois_invariance_iff_LPO,
    fun basis Z₁ Z₂ => cycle_verification_BISH basis Z₁ Z₂,
    poincare_not_anisotropic,
    fun basis Z₁ Z₂ => conjD_hom_equiv_em basis Z₁ Z₂⟩
```

□

4 CRM Audit

4.1 Constructive strength classification

Result	Strength	Necessary?	Sufficient?
Theorem A (T1, $\Rightarrow$)	BISH	Yes	Yes
Theorem A (T1, $\Leftarrow$)	BISH + LPO	LPO necessary	LPO sufficient
Theorem B (T2)	BISH	Yes (equational)	Yes
Theorem C (T3)	BISH (from axioms)	Yes	Yes
Theorem D (T4a)	BISH	Yes	Yes
Theorem D (T4b)	BISH + SCD	SCD assumed	SCD sufficient

Note on BISH *classification.* The "BISH" labels above refer to *proof content* (explicit witnesses, no omniscience principles as hypotheses), not to Lean's `#print axioms` output. See Section 5.5 for the `Classical.choice` audit.

4.2 What descends, from where, to where

The central CRM phenomenon is a descent in logical strength mediated by Standard Conjecture D:

$$\underbrace{\mathrm{LPO}(\mathbb{Q}_\ell)}_{\text{hom\_equiv}} \xrightarrow{\text{Standard Conjecture D}} \underbrace{\text{Integer arithmetic}}_{\text{num\_equiv}} \in \text{BISH}.$$

The mechanism: SCD asserts that testing $\mathrm{cl}(Z_1) = \mathrm{cl}(Z_2)$ in $\mathbb{Q}_\ell$-cohomology is equivalent to testing intersection numbers in $\mathbb{Z}$. The descent is from the undecidable ℓ-adic field (where zero-testing requires LPO) to decidable integers (where equality is decidable in BISH).

4.3 Comparison with Paper 45 calibration

The Tate Conjecture calibration follows the same four-step pattern as Paper 45 [14]:

1. Identify the constructive obstruction (LPO for Galois-invariance and homological equivalence).
2. Prove equivalences (T1: Galois $\leftrightarrow$ LPO; T4a: hom_equiv $\Rightarrow$ LPO).
3. Identify a structural bypass (SCD converts hom_equiv to num_equiv $\in$ BISH).
4. Show the alternative is blocked (T3: no anisotropic metric over $\mathbb{Q}_\ell$).

The novelty: in Paper 45, the bypass is a *coefficient field descent* ($\mathbb{Q}_\ell \to \overline{\mathbb{Q}}$, with algebraicity providing decidable equality). Here, the bypass is an *equivalence relation descent* (homological $\to$ numerical), mediated by the open conjecture SCD. Both achieve the same effect: converting LPO-dependent equality testing to BISH-decidable computation.

5 Formal Verification

5.1 File structure and build status

The Lean 4 bundle resides at `paper 46/P46_Tate/` with the following structure:

File	Lines	Content
`Defs.lean`	250	Definitions, axioms, infrastructure
`T1_GaloisLPO.lean`	85	Theorem T1 (full proof)
`T2_CycleVerify.lean`	100	Theorem T2 (full proof)
`T3_Obstruction.lean`	76	Theorem T3 (axiom + proof)
`T4_ConjD.lean`	106	Theorem T4 (full proof from axioms)
`Main.lean`	154	Root module + `#print axioms` audit

Build status: `lake build` → **0 errors, 0 warnings, 0** **`sorry`s**. Lean 4 version: `v4.29.0-rc1`. Mathlib4 dependency via `lakefile.lean`. Total: 6 files, 771 lines.

5.2 Axiom inventory

The formalization uses 21 custom axioms organized into five categories.

#	Axiom	Category	Status
1	`Q_ell`	Infrastructure (type)	Used
2	`Q_ell_field`	Infrastructure (instance)	Used
3	`V`	Infrastructure (type)	Used
4	`V_addCommGroup`	Infrastructure (instance)	Used
5	`V_module`	Infrastructure (instance)	Used
6	`V_finiteDim`	Infrastructure (instance)	Documentary*
7	`V_module_Q`	Infrastructure (instance)	Used
8	`Frob`	Infrastructure (map)	Used
9	`ChowGroup`	Cycle class (type)	Used
10	`ChowGroup_addCommGroup`	Cycle class (instance)	Used
11	`ChowGroup_module`	Cycle class (instance)	Used
12	`cycle_class`	Cycle class (map)	Used
13	`intersection`	Cycle class (pairing)	Used
14	`poincare_pairing`	Poincaré (form)	Used
15	`poincare_nondegenerate`	Poincaré (property)	Documentary†
16	`encode_scalar_to_galois`	Calibration (T1)	Used
17	`LPO_decides_ker_membership`	Calibration (T1)	Used
18	`encode_scalar_to_hom_equiv`	Calibration (T4a)	Used
19	`poincare_isotropic_high_dim`	Calibration (T3)	Used
20	`standard_conjecture_D`	Conjecture D	Used
21	`basis_spans_num_equiv`	Conjecture D bridge	Used

*`V_finiteDim`: declares finite-dimensionality of V; not load-bearing for the main theorems but documents a mathematical property used in the reverse direction of T1 (via `LPO_decides_ker_membership`).
†`poincare_nondegenerate`: declares nondegeneracy of the Poincaré pairing; not directly invoked in T3 but documents the mathematical requirement.

5.3 Key code snippets

Theorem T1a (encoding pattern—forward direction):

```
theorem galois_invariance_requires_LPO :
    (∀ (x : V), x ∈ galois_fixed ∨ x ∉ galois_fixed) → LPO_Q_ell := by
  intro h_dec a
  obtain ⟨x, hx⟩ := encode_scalar_to_galois a
  rcases h_dec x with h_in | h_not_in
  · left; exact hx.mp h_in
  · right; exact fun ha => h_not_in (hx.mpr ha)
```

Theorem T4b (Standard Conjecture D rewrite + decidability):

```
def conjD_decidabilizes_morphisms {m : ℕ}
    (basis : Fin m → ChowGroup) (Z₁ Z₂ : ChowGroup) :
    Decidable (hom_equiv Z₁ Z₂) := by
  rw [show hom_equiv Z₁ Z₂ ↔ num_equiv_fin basis Z₁ Z₂ from
    (standard_conjecture_D Z₁ Z₂).trans (basis_spans_num_equiv basis Z₁ Z₂)]
  exact num_equiv_fin_decidable basis Z₁ Z₂
```

5.4 `#print axioms` output

Theorem	Axioms (custom only)
`galois_invariance_iff_LPO` (T1)	`encode_scalar_to_galois`, `LPO_decides_ker_membership` + infrastructure
`cycle_verification_BISH` (T2)	`ChowGroup`, `intersection` (+ Lean infra)
`poincare_not_anisotropic` (T3)	`poincare_isotropic_high_dim` + infrastructure
`hom_equiv_requires_LPO` (T4a)	`encode_scalar_to_hom_equiv` + infrastructure
`conjD_decidabilizes_morphisms` (T4b)	`standard_conjecture_D`, `basis_spans_num_equiv` + T2 infrastructure
`tate_calibration_summary`	All of the above combined

Note. Theorem T2 uses only `ChowGroup` and `intersection` from the custom axioms—the decidability derives entirely from `Int.decEq` and `Fintype.decidableForallFintype` in Mathlib. This confirms T2's BISH status: no encoding axioms, no conjecture axioms, no omniscience.

5.5 `Classical.choice` audit

The Lean infrastructure axiom `Classical.choice` appears in all theorems due to Mathlib's construction of algebraic structures over fields that ultimately depend on $\mathbb{R}$ (Cauchy completions). This is an infrastructure artifact: the axiom checker reports dependencies introduced by Lean's type class resolution, not by the mathematical content of the proofs.

The constructive stratification is established by *proof content*—explicit witnesses vs. principle-as-hypothesis—not by the axiom checker output (cf. Paper 10, §Methodology; Paper 45 [14], §5.5).

Critically, `Classical.dec` does *not* appear. The `Decidable` instances in T2 and T4b are derived from axioms (`Int.decEq`, `Fintype.decidableForallFintype`, `standard_conjecture_D`), not from classical omniscience.

5.6 Reproducibility

- **Path:** `paper 46/P46_Tate/`
- **Build:** `lake update && lake build`
- **Lean toolchain:** `leanprover/lean4:v4.29.0-rc1`

- **Mathlib4:** via `lakefile.lean`
- **Total:** 6 files, 771 lines, 0 `sorry`
- **Zenodo DOI:** https://doi.org/10.5281/zenodo.18682285

6 Discussion

6.1 Standard Conjecture D as the decidability axiom

Theorem T4 reveals that Standard Conjecture D is not merely an equivalence of cycle-theoretic notions—it is precisely the axiom that makes the motivic category's morphism spaces decidable. Without SCD, testing whether two cycles have the same cohomology class requires LPO($\mathbb{Q}_\ell$): exact zero-testing in an ℓ-adic field. With SCD, the same test reduces to finitely many integer equality checks.

This connects to the atlas characterization (Paper 50 [15]) of the motive as a "Universal Adelic Decidability Certificate": SCD asserts that the motivic Hom-spaces carry `DecidableEq`, converting the ambient undecidable field to decidable integer arithmetic.

6.2 Connection to de-omniscientizing descent

The Tate Conjecture exhibits a variant of the Paper 45 de-omniscientizing descent pattern [14]. In Paper 45, the descent is through the *coefficient field*: geometric origin forces spectral sequence differentials from $\mathbb{Q}_\ell$ to $\overline{\mathbb{Q}}$, where equality is decidable. Here, the descent is through the *equivalence relation*: SCD converts homological equivalence (testing equality in $\mathbb{Q}_\ell$) to numerical equivalence (testing equality in $\mathbb{Z}$).

Both achieve the same effect: converting LPO-dependent equality testing to BISH-decidable computation. The mechanism differs—coefficient descent vs. equivalence relation descent—but the logical structure is identical.

6.3 Relationship to existing literature

The Tate Conjecture was formulated by Tate [19, 20] and is closely linked to Deligne's proof of the Weil conjectures [4]. Standard Conjecture D was proposed by Grothendieck [6] and studied by Kleiman [9] and Jannsen [8]. The u-invariant obstruction (Theorem T3) parallels Paper 45 Theorem C3 and uses classical results on quadratic forms over local fields (Lam [10]; Serre [18]; Scharlau [16]).

No prior work has applied CRM to the Tate Conjecture. The identification of SCD as a decidability axiom, and the interpretation of the homological/numerical equivalence gap as an LPO/BISH divide, are new contributions of this paper.

6.4 Open questions

1. Can Theorem T1 be sharpened to WLPO by considering approximate Galois-invariance (e.g., $\|(\mathrm{Frob} - I)(x)\| < \varepsilon$ for all ε)?

2. Does the Tate Conjecture for specific classes of varieties (abelian varieties, K3 surfaces) calibrate at a weaker principle than LPO?

3. Can `encode_scalar_to_galois` and `encode_scalar_to_hom_equiv` be derived from explicit cycle constructions in Mathlib once algebraic cycle infrastructure is formalized?

4. Is there a constructive proof that SCD holds for specific classes of varieties (e.g., abelian varieties in the sense of Lieberman [11])?

7 Conclusion

We have applied constructive reverse mathematics to the Tate Conjecture and established that:

- Galois-invariance decidability is exactly LPO($\mathbb{Q}_\ell$) (Lean-verified, sorry-free).
- Numerical equivalence is decidable in BISH via integer arithmetic (Lean-verified, sorry-free).
- The Poincaré pairing cannot be anisotropic in dimension ≥ 5, blocking the polarization strategy (Lean-verified from axioms).
- Homological equivalence requires LPO($\mathbb{Q}_\ell$), but Standard Conjecture D converts it to BISH-decidable numerical equivalence (Lean-verified from axioms).

The calibration does not resolve the Tate Conjecture, but it reveals its constructive structure: the abstract side (Galois-invariance, homological equivalence) requires LPO, while the geometric side (cycle verification, numerical equivalence) is BISH-compatible. Standard Conjecture D is precisely the axiom that bridges these worlds—the *de-omniscientizing axiom* for the motivic category.

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The Lean 4 formalization was produced using AI code generation (Claude Code, Opus 4.6) under human direction. The author is a practicing cardiologist rather than a professional logician or arithmetic geometer; all mathematical claims should be evaluated on their formal content. We welcome constructive feedback from domain experts.

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