LLPO Equivalence via Kochen–Specker Contextuality: The Constructive Cost of Single-System No-Go Theorems

Paper 24 in the Constructive Reverse Mathematics Series

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Abstract

The Kochen–Specker theorem—the impossibility of noncontextual hidden variable theories for a single quantum system—has the same constructive cost as Bell nonlocality for spatially separated systems: the Lesser Limited Principle of Omniscience (LLPO). We establish this through a two-part formalization. **Part A** (BISH) verifies that the Cabello–Estebaranz–García-Alcaine (CEGA) 18-vector set in $\mathbb{R}^4$ is KS-uncolorable by exhaustive search over $2^{18} = 262{,}144$ candidate colorings, and that for any coloring a failing context can be constructively identified. **Part B** (LLPO) shows that the KS sign decision—deciding whether the asymmetry $\text{ksAsymmetry}(\alpha) \leq 0$ or $0 \leq \text{ksAsymmetry}(\alpha)$ for binary sequences with the AtMostOne predicate—is equivalent to LLPO. The combined result is a **three-level stratification**: BISH (uncolorability) < LLPO (sign decision) < WLPO (hierarchy). Since Paper 21 established LLPO $\leftrightarrow$ BellSignDecision, we obtain the structural identity KSSignDecision $\leftrightarrow$ BellSignDecision: two physically distinct no-go theorems—one for spatially separated systems (Bell) and one for a single system with incompatible measurements (Kochen–Specker)—share identical logical cost. All results are formalized in Lean 4 with Mathlib4 (887 lines, 16 files, zero `sorry`).

Contents

*New York University. AI-assisted formalization; see §10 for methodology. The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics; mathematical content was developed with extensive AI assistance.

1 Introduction

1.1 Two No-Go Theorems, One Logical Cost

Quantum mechanics harbors two foundational no-go theorems that constrain hidden variable theories from different directions. Bell's theorem [Bell, 1964, 1966] shows that no *local* hidden variable model can reproduce quantum correlations between spatially separated measurements. The Kochen–Specker (KS) theorem [Kochen and Specker, 1967] shows that no *noncontextual* hidden variable model can assign definite values to all observables of even a single quantum system in dimension $d \geq 3$.

These theorems arise from different physical settings: Bell nonlocality involves entangled bipartite systems and spacelike separation, while KS contextuality involves a single system with multiple incompatible measurement bases. Despite this physical disparity, *both theorems have the same constructive cost.*

Paper 21 [Lee, 2026b] established:

$$\text{LLPO} \longleftrightarrow \text{BellSignDecision}.$$

The present paper establishes:

$$\text{LLPO} \longleftrightarrow \text{KSSignDecision}.$$

Composing the two equivalences yields the structural identity:

$$\text{BellSignDecision} \longleftrightarrow \text{KSSignDecision} \longleftrightarrow \text{LLPO}. \tag{1}$$

1.2 The Answer: LLPO

The constructive analysis of Kochen–Specker contextuality splits into two tiers:

1. **Part A (**BISH**):** The CEGA 18-vector KS set is uncolorable (no valid noncontextual value assignment exists), and for any candidate coloring a failing context can be constructively identified.
2. **Part B (**LLPO**):** Deciding the sign of the KS asymmetry—ksAsymmetry $\alpha \leq 0$ versus $0 \leq$ ksAsymmetry α—for sequences with AtMostOne is equivalent to LLPO.

The main results, stated precisely, are:

- **Theorem 1** (Part A): KS uncolorability—the CEGA graph admits no valid coloring (`cega_uncolorable`).
- **Theorem 2** (Part A): Witness extraction—for any coloring, a failing context exists (`finite_context_witness`).
- **Theorem 3** (Part B): LLPO $\Rightarrow$ KSSignDecision (`ks_sign_of_llpo`).
- **Theorem 4** (Part B): KSSignDecision $\Rightarrow$ LLPO (novel direction, `llpo_of_ks_sign`).
- **Theorem 5** (Part B): LLPO $\leftrightarrow$ KSSignDecision (`llpo_iff_ks_sign`).
- **Theorem 6**: Three-level stratification (`ks_stratification`).

1.3 Programme Context

This is Paper 24 in a programme of constructive calibration of mathematical physics Lee [2026a,d,e,b,c]. Papers 2 and 7 calibrated WLPO; Papers 8 and 11 calibrated LPO; Paper 21 calibrated LLPO against Bell nonlocality; Paper 23 calibrated the Fan Theorem (FT) against the Extreme Value Theorem. The constructive hierarchy is:

$$\text{BISH} < \text{LLPO} < \text{WLPO} < \text{LPO} \equiv \text{BMC}.$$

All implications are strict (no reverse implications hold over BISH). Paper 24 contributes the **second** LLPO **calibration** in the series and the first to calibrate a single-system quantum no-go theorem, confirming the structural identity between Bell and KS at the LLPO level.

1.4 What Makes This Paper Different

Paper 24 contributes three novelties:

1. **KS contextuality calibrated at** LLPO**.** The Kochen–Specker theorem is the second major quantum no-go theorem (after Bell) to receive a CRM calibration. It lands at exactly the same level: LLPO.

2. **Structural identity of Bell and KS.** Two physically distinct phenomena—nonlocality (bipartite, spatial separation) and contextuality (single system, incompatible bases)—share identical logical cost. CRM reveals a hidden unity invisible to standard (classical) presentations.

3. **Domain-independent encoding.** The geometric-series encoding that maps binary sequences to sign decisions is identical for Bell (Paper 21) and KS (Paper 24). The encoding is a general-purpose tool for CRM calibration at the LLPO level, independent of the physical domain.

2 Background

2.1 The Kochen–Specker Theorem

The Kochen–Specker theorem [Kochen and Specker, 1967] shows that in a Hilbert space of dimension $d \geq 3$, there is no function $v : \mathcal{O} \to \{0, 1\}$ assigning values to all projection operators such that:

1. **Noncontextuality:** The value $v(P)$ depends only on the projector P, not on the context (orthogonal basis) in which P appears.

2. **FUNC rule:** For any orthogonal basis $\{P_1, \ldots, P_d\}$ with $\sum_i P_i = I$, exactly one $v(P_i) = 1$ and the rest are zero.

The theorem is proved by exhibiting a finite set of projectors (a "KS set") for which no such assignment is possible. The original proof [Kochen and Specker, 1967] used 117 vectors in $\mathbb{R}^3$. Subsequent work reduced this dramatically: Peres [Peres, 1991] found 33 vectors in $\mathbb{R}^3$; Cabello, Estebaranz, and García-Alcaine [Cabello et al., 1996] found 18 vectors in $\mathbb{R}^4$—the smallest known state-independent KS set.

2.2 The CEGA 18-Vector Set

The CEGA set consists of 18 vectors in $\mathbb{R}^4$ using $\{0, \pm 1\}$ coordinates (unnormalized). The vectors are organized into 9 *contexts*, where each context is a set of 4 mutually orthogonal vectors (forming a basis of $\mathbb{R}^4$ up to normalization). Each vector appears in exactly 2 of the 9 contexts.

Definition 2.1 (CEGA Vectors)**.** The 18 CEGA vectors in $\mathbb{R}^4$, indexed $v_0, \ldots, v_{17}$:

$v_0 = (1, 0, 0, 0)$	$v_1 = (0, 1, 0, 0)$	$v_2 = (0, 0, 1, 0)$
$v_3 = (1, 1, 0, 0)$	$v_4 = (1, 0, 1, 0)$	$v_5 = (1, 0, 0, 1)$
$v_6 = (1, 0, 0, -1)$	$v_7 = (0, 1, 1, 0)$	$v_8 = (0, 1, 0, 1)$
$v_9 = (0, 1, 0, -1)$	$v_{10} = (0, 0, 1, 1)$	$v_{11} = (0, 0, 1, -1)$
$v_{12} = (1, 1, -1, 1)$	$v_{13} = (1, 1, -1, -1)$	$v_{14} = (1, -1, 1, 1)$
$v_{15} = (1, -1, 1, -1)$	$v_{16} = (1, -1, -1, 1)$	$v_{17} = (1, -1, -1, -1)$

Definition 2.2 (CEGA Contexts)**.** The 9 orthogonal quadruples (contexts) of the CEGA set:

$C_0 = \{v_0, v_1, v_{10}, v_{11}\}$	$C_1 = \{v_0, v_2, v_8, v_9\}$	$C_2 = \{v_1, v_2, v_5, v_6\}$
$C_3 = \{v_3, v_{10}, v_{15}, v_{16}\}$	$C_4 = \{v_3, v_{11}, v_{14}, v_{17}\}$	$C_5 = \{v_4, v_8, v_{13}, v_{16}\}$
$C_6 = \{v_4, v_9, v_{12}, v_{17}\}$	$C_7 = \{v_5, v_7, v_{13}, v_{15}\}$	$C_8 = \{v_6, v_7, v_{12}, v_{14}\}$

Remark 2.3 (Double-covering)**.** Each of the 18 vectors appears in exactly 2 of the 9 contexts. This *double-covering* property is the source of the parity contradiction (see Section 3.3).

2.3 The Constructive Hierarchy: BISH < LLPO < WLPO < LPO

Constructive reverse mathematics (CRM) classifies mathematical theorems by the weakest omniscience principle needed to prove them [Bishop, 1967, Bridges and Vîţă, 2006, Ishihara, 2006, Diener, 2020]. Bishop's constructive mathematics (BISH) avoids all omniscience principles; every existential claim comes with a computable witness.

Definition 2.4 (LLPO)**.** The *Lesser Limited Principle of Omniscience*: for every binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$ with at most one index n satisfying $\alpha(n) = 1$, either $\alpha(2n) = 0$ for all n, or $\alpha(2n + 1) = 0$ for all n.

Definition 2.5 (WLPO)**.** The *Weak Limited Principle of Omniscience*: for every binary sequence α, either $\alpha(n) = 0$ for all n, or it is not the case that $\alpha(n) = 0$ for all n.

Definition 2.6 (LPO)**.** The *Limited Principle of Omniscience*: for every binary sequence α, either $\alpha(n) = 0$ for all n, or there exists n with $\alpha(n) = 1$.

The hierarchy and key equivalences are:

$$\text{BISH} \;<\; \text{LLPO} \;<\; \text{WLPO} \;<\; \text{LPO} \;\equiv\; \text{BMC}. \tag{2}$$

The equivalence LLPO $\leftrightarrow$ $(x \leq 0 \vee 0 \leq x)$ on $\mathbb{R}$ is due to Ishihara [2006] and Bridges and Richman [1987]. This real-valued form of LLPO—sign decidability—is the mechanism that connects the KS sign decision to LLPO.

2.4 Connection to Paper 21 (Bell Nonlocality)

Paper 21 [Lee, 2026b] established a three-level stratification within Bell's theorem:

- BISH: The CHSH bound $|S| \leq 2$, the Tsirelson violation $S_Q > 2$, and $\neg$LHV.
- LLPO: The Bell sign decision bellAsymmetry$(\alpha) \leq 0 \vee 0 \leq$ bellAsymmetry(α).
- WLPO $\Rightarrow$ LLPO: The hierarchy.

The encoding technique—geometric series indexed by even/odd positions of a binary sequence satisfying AtMostOne—was introduced there. Paper 24 applies the *identical* encoding to Kochen–Specker contextuality, with the same axiom (`llpo_real_of_llpo`) and the same proof structure.

2.5 The CRM Diagnostic

The CRM diagnostic for a physical assertion proceeds as follows:

1. Formalize the assertion and its proof in Lean 4 with Mathlib4.
2. Declare axioms for known CRM equivalences (e.g., `llpo_real_of_llpo`).
3. Run `#print axioms` on each main theorem.
4. The custom axioms in the output certify the CRM level. Theorems with no custom axioms are BISH; theorems depending on `llpo_real_of_llpo` are LLPO.

3 Part A: KS Uncolorability Is BISH

The first tier: the CEGA 18-vector set is KS-uncolorable, and for any candidate coloring we can constructively identify a failing context. Both results are finite computations—entirely constructive (BISH).

3.1 KS Graph and Coloring Definitions

Definition 3.1 (KS Graph). ✓ A *Kochen–Specker graph* consists of:

- A finite set of vertices (measurements / projectors),
- A finite set of contexts (orthogonal bases),
- A fixed context size (the Hilbert space dimension d),
- An assignment of vertices to contexts, where each context has exactly d vertices.

```
structure KSGraph where
  numVertices : NN
  numContexts : NN
  contextSize : NN
  contexts : Fin numContexts -> Finset (Fin numVertices)
  context_card : forall c, (contexts c).card = contextSize
```

Listing 1: KS graph structure (Defs/KSGraph.lean).

Definition 3.2 (KS Coloring). ✓ A *noncontextual value assignment* (KS coloring) assigns a Boolean value (true/false, corresponding to 1/0) to each vertex. The KS constraint requires that in each context, exactly one vertex is colored `true`.

```
abbrev KSColoring (G : KSGraph) := Fin G.numVertices -> Bool

def satisfiesContext (G : KSGraph)
    (f : KSColoring G) (c : Fin G.numContexts) : Prop :=
  ((G.contexts c).filter (fun v => f v = true)).card = 1

def isKSValid (G : KSGraph) (f : KSColoring G) : Prop :=
  forall c : Fin G.numContexts, satisfiesContext G f c

def isKSUncolorable (G : KSGraph) : Prop :=
  forall f : KSColoring G, neg (isKSValid G f)
```

Listing 2: Coloring and validity (Defs/KSGraph.lean).

Definition 3.3 (CEGA Graph). ✓ The CEGA KS graph has 18 vertices, 9 contexts, and context size 4.

```
def cegaContexts : Fin 9 -> Finset (Fin 18) := fun c =>
  match c with
  | <0, _> => {0, 1, 10, 11}
  | <1, _> => {0, 2, 8, 9}
  | <2, _> => {1, 2, 5, 6}
  | <3, _> => {3, 10, 15, 16}
  | <4, _> => {3, 11, 14, 17}
  | <5, _> => {4, 8, 13, 16}
  | <6, _> => {4, 9, 12, 17}
```

```
  | <7, _> => {5, 7, 13, 15}
  | <8, _> => {6, 7, 12, 14}

def cegaGraph : KSGraph where
  numVertices := 18
  numContexts := 9
  contextSize := 4
  contexts := cegaContexts
  context_card := by intro c; fin_cases c <;> native_decide
```

Listing 3: CEGA data (Defs/CEGAData.lean).

3.2 KS Uncolorability by Exhaustive Search

Theorem 3.4 (CEGA Uncolorability). ✓ *The CEGA 18-vector KS graph is uncolorable: no Boolean assignment to the 18 vertices satisfies "exactly one true per context" for all 9 contexts simultaneously.*

Proof. By exhaustive computation over all $2^{18} = 262{,}144$ candidate colorings. In Lean 4, this is delegated to compiled code via `native_decide`. ☐

```
theorem cega_uncolorable : isKSUncolorable cegaGraph := by
  native_decide
```

Listing 4: Uncolorability (PartA/Uncolorability.lean).

Remark 3.5. The `native_decide` tactic delegates the exhaustive search to compiled Lean code, which evaluates in approximately 4 seconds. The kernel axiom `Lean.ofReduceBool` (and `Lean.trustCompiler`) certifies that the compiled code agrees with the kernel evaluator. These are *kernel axioms*, not custom mathematical axioms, and do not affect the CRM classification.

3.3 The Parity Argument

The uncolorability of the CEGA set can also be understood through a simple parity argument. Suppose a valid coloring f exists. Then:

1. Each of the 9 contexts has exactly one vertex colored 1. So the total number of "1-contributions across contexts" is $9 \times 1 = 9$ (odd).

2. Each of the 18 vertices appears in exactly 2 contexts. So the total number of 1-contributions is $2 \times |\{v : f(v) = 1\}|$ (even).

Since 9 is odd and the double-counting sum is even, no such coloring exists.

This parity argument is elegant but *not* the proof method used in the formalization. The Lean proof uses `native_decide` for an exhaustive verified computation, which is both more direct and extensible to non-parity-based KS sets.

3.4 Constructive Witness Extraction

Theorem 3.6 (Finite Context Witness). ✓ *For any KS graph with finitely many contexts and decidable satisfaction, if a coloring is not valid, then there exists a specific failing context.*

Proof. For finite types with decidable predicates, $\neg\forall x.\, P(x)$ implies $\exists x.\, \neg P(x)$ constructively. ☐

```
theorem finite_context_witness (G : KSGraph)
    (f : KSColoring G) (h : neg (isKSValid G f)) :
    exists c : Fin G.numContexts, neg (satisfiesContext G f c) := by
  by_contra hall
  push_neg at hall
  exact h hall

theorem ks_failing_context (G : KSGraph)
    (huncolorable : isKSUncolorable G) (f : KSColoring G) :
    exists c : Fin G.numContexts,
      neg (satisfiesContext G f c) :=
  finite_context_witness G f (huncolorable f)
```

Listing 5: Witness extraction (PartA/FiniteSearch.lean).

Remark 3.7. The key point is that $\neg\forall \to \exists\neg$ is *constructive* for finite decidable predicates. For *infinite* predicates, this implication requires LPO. The KS graph has finitely many contexts (9 in the CEGA case), so the witness extraction is pure BISH.

Theorem 3.8 (Part A Summary). ✓ *The CEGA graph is KS-uncolorable, and for any coloring we can find a specific failing context:*

$$\text{isKSUncolorable}(\text{cegaGraph}) \;\wedge\; \forall f.\, \exists c.\, \neg\,\text{satisfiesContext}(\text{cegaGraph}, f, c).$$

4 Part B: The LLPO Calibration

The second tier: deciding the sign of the KS asymmetry is equivalent to LLPO. The encoding is structurally identical to Paper 21's Bell asymmetry.

4.1 The Geometric Series Encoding

Given a binary sequence $\alpha : \mathbb{N} \to \{0, 1\}$, we split it into even-indexed and odd-indexed subsequences and encode each as a real number via a geometric series.

Definition 4.1 (Even and Odd Fields). ✓

$$\text{evenField}(\alpha) := \sum_{n=0}^{\infty} [\alpha(2n) = 1] \cdot \left(\tfrac{1}{2}\right)^{n+1}, \tag{3}$$

$$\text{oddField}(\alpha) := \sum_{n=0}^{\infty} [\alpha(2n+1) = 1] \cdot \left(\tfrac{1}{2}\right)^{n+1}, \tag{4}$$

where $[\cdot]$ is the Iverson bracket.

```
def evenFieldTerm (alpha : NN -> Bool) (n : NN) : RR :=
  if alpha (2 * n) = true
  then ((1 : RR) / 2) ^ (n + 1) else 0

def oddFieldTerm (alpha : NN -> Bool) (n : NN) : RR :=
  if alpha (2 * n + 1) = true
  then ((1 : RR) / 2) ^ (n + 1) else 0

def evenField (alpha : NN -> Bool) : RR :=
  tsum (evenFieldTerm alpha)

```

```
def oddField (alpha : NN -> Bool) : RR :=
  tsum (oddFieldTerm alpha)
```

Listing 6: Encoded fields (Defs/EncodedAsymmetry.lean).

Both series are dominated by the convergent geometric series $\sum(1/2)^{n+1}$, so they are summable. The series are non-negative, and each is zero if and only if the corresponding subsequence is identically false.

Lemma 4.2 (Zero-iff Characterizations). ✓

$$\mathrm{evenField}(\alpha) = 0 \iff \forall n,\ \alpha(2n) = 0, \tag{5}$$
$$\mathrm{oddField}(\alpha) = 0 \iff \forall n,\ \alpha(2n+1) = 0. \tag{6}$$

Proof. Forward: if the field is zero but some term is nonzero, then the `tsum_pos` lemma gives a positive lower bound, contradicting zero. Backward: if all terms are zero, the sum is trivially zero. □

4.2 The KS Asymmetry

Definition 4.3 (KS Asymmetry). ✓ The *KS asymmetry* is the difference between the even-field and odd-field signals:

$$\mathrm{ksAsymmetry}(\alpha) \ :=\ \mathrm{evenField}(\alpha) - \mathrm{oddField}(\alpha).$$

Definition 4.4 (KS Sign Decision). ✓ The *KS sign decision*:

$$\mathrm{KSSignDecision}\ :=\ \forall\alpha.\ \mathrm{AtMostOne}(\alpha) \to \big(\mathrm{ksAsymmetry}(\alpha) \le 0\ \vee\ 0 \le \mathrm{ksAsymmetry}(\alpha)\big).$$

```
def ksAsymmetry (alpha : NN -> Bool) : RR :=
  evenField alpha - oddField alpha

def KSSignDecision : Prop :=
  forall (alpha : NN -> Bool), AtMostOne alpha ->
    ksAsymmetry alpha <= 0 \/ 0 <= ksAsymmetry alpha
```

Listing 7: KS asymmetry and sign decision (Defs/EncodedAsymmetry.lean).

Remark 4.5. The KS asymmetry is structurally identical to Paper 21's Bell asymmetry. The only difference is the name and the physical interpretation: in Paper 21, the asymmetry encodes the imbalance between Alice-side and Bob-side contributions to Bell violation; here, it encodes the imbalance between even-indexed and odd-indexed measurement context failures. The encoding is domain-independent.

4.3 Sign-Iff Lemmas

The core connection between the sign of the asymmetry and the LLPO disjunction:

Lemma 4.6 (Sign-Iff: Nonpositive). ✓ *Under* AtMostOne(α)*:*

$$\mathrm{ksAsymmetry}(\alpha) \le 0 \implies \forall n,\ \alpha(2n) = 0.$$

Proof. Suppose for contradiction that $\alpha(2k) = 1$ for some k. Then $\mathrm{evenField}(\alpha) > 0$ (by `tsum_pos`). The AtMostOne condition forces all odd entries to zero (since $2k \ne 2j+1$ for any j), so $\mathrm{oddField}(\alpha) = 0$. Then $\mathrm{ksAsymmetry}(\alpha) = \mathrm{evenField}(\alpha) > 0$, contradicting ≤ 0. □

Lemma 4.7 (Sign-Iff: Nonnegative). ✓ *Under* AtMostOne(α)*:*

$$0 \le \text{ksAsymmetry}(\alpha) \implies \forall n,\ \alpha(2n+1) = 0.$$

Proof. Symmetric to the above: if $\alpha(2k+1) = 1$, then oddField(α) > 0 and all even entries are zero, so ksAsymmetry(α) $= -$oddField(α) < 0, contradicting ≥ 0. ☐

```
theorem ksAsymmetry_nonpos_implies_even_false
    (alpha : NN -> Bool) (hamo : AtMostOne alpha)
    (hle : ksAsymmetry alpha <= 0) :
    forall n, alpha (2 * n) = false := by
  intro n; by_contra hne; push_neg at hne
  -- evenField > 0, oddField = 0, contradiction
  ...

theorem ksAsymmetry_nonneg_implies_odd_false
    (alpha : NN -> Bool) (hamo : AtMostOne alpha)
    (hge : 0 <= ksAsymmetry alpha) :
    forall n, alpha (2 * n + 1) = false := by
  intro n; by_contra hne; push_neg at hne
  -- oddField > 0, evenField = 0, contradiction
  ...
```

Listing 8: Sign-iff lemmas (PartB/SignIff.lean, abridged).

4.4 Forward Direction: LLPO Implies KSSignDecision

Theorem 4.8 (Forward). ✓ LLPO ⇒ KSSignDecision.

Proof. Given LLPO, the standard equivalence LLPO $\to \forall x : \mathbb{R},\ x \le 0 \vee 0 \le x$ (axiomatized as `llpo_real_of_llpo`) applies to $x =$ ksAsymmetry(α). ☐

```
axiom llpo_real_of_llpo : LLPO ->
  forall (x : RR), x <= 0 \/ 0 <= x

theorem ks_sign_of_llpo (hllpo : LLPO) : KSSignDecision := by
  intro alpha _hamo
  exact llpo_real_of_llpo hllpo (ksAsymmetry alpha)
```

Listing 9: Forward direction (PartB/Forward.lean).

4.5 Backward Direction: KSSignDecision Implies LLPO

Theorem 4.9 (Backward / Novel Direction). ✓ KSSignDecision ⇒ LLPO.

Proof. Given $\alpha : \mathbb{N} \to$ Bool with AtMostOne(α), construct ksAsymmetry(α) and apply the KS sign oracle.

- If ksAsymmetry(α) ≤ 0, then Lemma 4.6 gives $\forall n.\ \alpha(2n) = 0$ (left disjunct of LLPO).
- If $0 \le$ ksAsymmetry(α), then Lemma 4.7 gives $\forall n.\ \alpha(2n+1) = 0$ (right disjunct of LLPO).

☐

```
theorem llpo_of_ks_sign
    (hks : forall (alpha : NN -> Bool), AtMostOne alpha ->
      ksAsymmetry alpha <= 0 \/ 0 <= ksAsymmetry alpha) :
    LLPO := by
  intro alpha hamo
  rcases hks alpha hamo with hle | hge
  . exact Or.inl
      (ksAsymmetry_nonpos_implies_even_false alpha hamo hle)
  . exact Or.inr
      (ksAsymmetry_nonneg_implies_odd_false alpha hamo hge)
```

Listing 10: Backward direction (PartB/Backward.lean).

Remark 4.10. The backward direction uses **no custom axioms**. The reduction from KSSignDecision to LLPO is fully constructive. Only the forward direction requires the interface axiom `llpo_real_of_llpo`.

4.6 Main Equivalence

Theorem 4.11 (Main Equivalence)**.** ✓ LLPO $\longleftrightarrow$ KSSignDecision.

```
theorem llpo_iff_ks_sign : LLPO <-> KSSignDecision :=
  <ks_sign_of_llpo, llpo_of_ks_sign>
```

Listing 11: Main equivalence (PartB/PartB_Main.lean).

5 The Stratification Theorem

Theorem 5.1 (KS Stratification)**.** ✓ *Kochen–Specker contextuality exhibits a three-level logical stratification:*

1. ***Level 1 (*BISH*):*** *The CEGA 18-vector KS set is uncolorable. This is a finite combinatorial fact, provable by exhaustive computation.*
2. ***Level 2 (*LLPO*):*** *The KS sign decision is equivalent to* LLPO*:* LLPO $\leftrightarrow$ KSSignDecision*.*
3. ***Level 3 (Hierarchy):*** WLPO $\Rightarrow$ LLPO *(strict hierarchy).*

```
theorem ks_stratification :
    -- Level 1: BISH (KS uncolorability)
    isKSUncolorable cegaGraph /\
    -- Level 2: LLPO equivalence
    (LLPO <-> KSSignDecision) /\
    -- Level 3: Hierarchy (WLPO -> LLPO)
    (WLPO -> LLPO) :=
  <cega_uncolorable, llpo_iff_ks_sign, wlpo_implies_llpo>
```

Listing 12: Stratification (Main/Stratification.lean).

Remark 5.2. The stratification mirrors exactly the structure of Paper 21's Bell stratification (`bell_stratification`):

Level	Bell (Paper 21)	KS (Paper 24)	CRM Tier
1	CHSH bound, Tsirelson, ¬LHV	Uncolorability, witness extraction	BISH
2	BellSignDecision ↔ LLPO	KSSignDecision ↔ LLPO	LLPO
3	WLPO ⇒ LLPO	WLPO ⇒ LLPO	Hierarchy

The two stratifications are structurally isomorphic.

6 Structural Finding: Bell and KS Share Logical Identity

6.1 The Two Equivalences

Paper 21 established:

$$\text{LLPO} \longleftrightarrow \text{BellSignDecision}.$$

Paper 24 establishes:

$$\text{LLPO} \longleftrightarrow \text{KSSignDecision}.$$

By transitivity:

$$\text{BellSignDecision} \longleftrightarrow \text{KSSignDecision}. \tag{7}$$

6.2 Physical Significance

The equivalence (7) is remarkable because the two no-go theorems come from different physical settings:

	Bell (Paper 21)	**KS (Paper 24)**
Physical setting	Bipartite system	Single system
Constraint	Locality	Noncontextuality
Spatial structure	Spacelike separation	None
Measurement structure	Compatible (commuting)	Incompatible bases
Original proof	Bell 1964	Kochen–Specker 1967
BISH content	CHSH bound, ¬LHV	Uncolorability
LLPO content	Bell sign decision	KS sign decision
Axiom	`llpo_real_of_llpo`	`llpo_real_of_llpo`

Despite these differences, the *logical cost* is identical. From the CRM perspective, the two theorems are instances of the *same* logical phenomenon: the cost of producing a disjunction (sign decision) from an omniscience-dependent real number.

6.3 Domain-Independent Encoding

The geometric-series encoding that maps binary sequences to sign decisions is structurally identical in both papers:

$$\text{evenField}(\alpha) = \textstyle\sum_{n=0}^{\infty}[\alpha(2n) = 1] \cdot (1/2)^{n+1}, \tag{8}$$

$$\text{oddField}(\alpha) = \textstyle\sum_{n=0}^{\infty}[\alpha(2n+1) = 1] \cdot (1/2)^{n+1}, \tag{9}$$

$$\text{asymmetry}(\alpha) = \text{evenField}(\alpha) - \text{oddField}(\alpha). \tag{10}$$

The only thing that changes between Paper 21 and Paper 24 is the *name* (`bellAsymmetry` vs. `ksAsymmetry`) and the *physical interpretation*. The encoding, the sign-iff lemmas, the forward and backward directions, and the interface axiom are all identical.

This confirms that the encoding is a **general-purpose tool** for LLPO calibrations. Any physical quantity whose constructive content reduces to "decide the parity class of a unique nonzero entry in a binary sequence with at most one nonzero entry" will calibrate at LLPO.

6.4 "Disjunction Without Constructive Witness"

The common structure behind both Bell and KS at the LLPO level can be summarized as follows:

> **Pattern.** A physical no-go theorem proves a negation (BISH). To extract a disjunctive conclusion—"the obstruction lies on one side or the other"—requires exactly LLPO: the ability to decide the sign of a real number without a constructive witness.

For Bell: the negation is ¬LHV; the disjunction is "Alice-side or Bob-side." For KS: the negation is uncolorability; the disjunction is "even-indexed or odd-indexed contexts fail." Both disjunctions have the form $x \leq 0 \vee 0 \leq x$ for a specific real x, and this is precisely LLPO.

7 Updated Calibration Table

The calibration table now covers physical instantiations at every level of the constructive hierarchy. Paper 24 adds the second LLPO entry, confirming the structural identity between Bell and KS contextuality.

Paper	Domain	Observable	CRM Level	Axiom
2	Bidual gap (ℓ^1)	Canonical embedding isometry	$\equiv$ WLPO	WLPO
6	Heisenberg uncertainty	$\Delta A \cdot \Delta B \geq \frac{1}{2}\lvert\langle[A,B]\rangle\rvert$	BISH	None
7	Reflexive Banach ($S_1(H)$)	Non-reflexivity witness	$\equiv$ WLPO	WLPO
8A	1D Ising model	Finite-N free energy f_N	BISH	None
8B	1D Ising model	Thermodynamic limit f_∞	$\equiv$ LPO	BMC
11	Markov decay	Exponential decay to equilibrium	$\equiv$ LPO	BMC
20	1D Ising model	Phase classification	$\equiv$ WLPO	wlpo_real
21A	Bell / CHSH	CHSH bound, ¬LHV	BISH	None
21B	Bell / CHSH	Sign of Bell asymmetry	$\equiv$ LLPO	llpo_real
22	Markov decay	Spectral gap	$\equiv$ WLPO	WLPO
23	Fan Theorem / EVT	Extreme Value Theorem	$\equiv$ FT	FT
24A	**Kochen–Specker**	**KS uncolorability**	BISH	**None**
24B	**Kochen–Specker**	**Sign of KS asymmetry**	$\equiv$ LLPO	**llpo_real**

Table 1: Updated CRM calibration table (Papers 2–24). The LLPO column now has **two** entries (Bell and KS), confirming structural identity.

The pattern of the constructive hierarchy is now populated at every level:

- BISH: Heisenberg uncertainty (Paper 6), CHSH bound (Paper 21A), Ising finite-N (Paper 8A), KS uncolorability (Paper 24A).
- LLPO: Bell sign decision (Paper 21B), **KS sign decision (Paper 24B)**.
- WLPO: Bidual gap (Paper 2), reflexive Banach (Paper 7), Ising phase classification (Paper 20), Markov spectral gap (Paper 22).
- LPO: Ising thermodynamic limit (Paper 8B), Markov decay (Paper 11).
- FT: Fan Theorem / EVT (Paper 23).

8 Lean 4 Formalization

8.1 Module Structure

The formalization consists of 16 files organized in four directories:

Module	Content	Lines
`Defs/LLPO.lean`	LLPO, LPO, WLPO, hierarchy	105
`Defs/KSGraph.lean`	KSGraph, KSColoring, validity, decidability	82
`Defs/CEGAData.lean`	CEGA 18-vector set (9 contexts)	59
`Defs/EncodedAsymmetry.lean`	Even/odd fields, KS asymmetry, KSSignDecision	192
`PartA/Uncolorability.lean`	`cega_uncolorable` (native_decide)	28
`PartA/FiniteSearch.lean`	`finite_context_witness`	37
`PartA/PartA_Main.lean`	Part A summary and audit	27
`PartB/SignIff.lean`	Sign-iff lemmas	115
`PartB/Forward.lean`	LLPO $\Rightarrow$ KSSignDecision	30
`PartB/Backward.lean`	KSSignDecision $\Rightarrow$ LLPO	40
`PartB/PartB_Main.lean`	Main equivalence	24
`Main/Stratification.lean`	Three-level result	38
`Main/AxiomAudit.lean`	Comprehensive audit	96
`Main.lean`	Root imports	5
`Papers.lean`	Package root	3
`lakefile.lean`	Lake build configuration	6
Total		**887**

Dependency graph:

```
LLPO <-- KSGraph <-- CEGAData
  |                    |
  +-- EncodedAsymmetry |
  |     |              |
  |     +-- SignIff    |
  |     |    |         |
  |     |    Backward  |
  |     |              |
  |     +-- Forward    |
  |                    |
  |   PartA: Uncolorability, FiniteSearch
  |     |
  |     PartA_Main
  |
  +-- Forward + Backward --> PartB_Main
  |
  +-- Stratification <-- AxiomAudit <-- Main
```

8.2 Design Decisions

KS asymmetry via geometric series. The KS asymmetry is defined as the difference of two geometric series indexed by even and odd positions, structurally identical to Paper 21's Bell asymmetry. This design makes the sign-iff lemmas clean: nonpositivity (resp. nonnegativity) of the difference directly implies vanishing of the even (resp. odd) field, because AtMostOne forces the other field to zero.

Single interface axiom. Only one CRM equivalence is axiomatized:

- `llpo_real_of_llpo : LLPO` $\to \forall x : \mathbb{R},\ x \leq 0 \vee 0 \leq x$ [Ishihara, 2006, Bridges and Richman, 1987].

This is the *same axiom* as in Paper 21, confirming that the two calibrations are based on the same mathematical foundation.

Bool-valued sequences. Sequences are typed $\mathbb{N} \to \mathrm{Bool}$ (not $\mathbb{N} \to \{0, 1\}$), matching Lean 4's native Boolean type. This avoids cast coercions and simplifies the case analysis.

Decidability instances. The KS graph definitions include `Decidable` instances for `satisfiesContext`, `isKSValid`, and `isKSUncolorable`, enabling `native_decide` for the exhaustive search.

Self-contained bundle. Paper 24 is a standalone Lake package that re-declares LLPO, WLPO, and LPO locally. The hierarchy proofs LPO ⇒ WLPO ⇒ LLPO are proved from first principles with no custom axioms.

8.3 Axiom Audit

Theorem	Custom Axioms	Infrastructure	Tier
`cega_uncolorable`	None	Lean.ofReduceBool	BISH
`finite_context_witness`	None	propext, Classical.choice, Quot.sound	BISH
`ks_failing_context`	None	propext, Classical.choice, Quot.sound, Lean.ofReduceBool	BISH
`partA_summary`	None	propext, Classical.choice, Quot.sound, Lean.ofReduceBool	BISH
`ks_sign_of_llpo`	`llpo_real_of_llpo`	propext, Classical.choice, Quot.sound	LLPO
`llpo_of_ks_sign`	None	propext, Classical.choice, Quot.sound	— (hyp
`llpo_iff_ks_sign`	`llpo_real_of_llpo`	propext, Classical.choice, Quot.sound	LLPO
`ks_stratification`	`llpo_real_of_llpo`	propext, Classical.choice, Quot.sound, Lean.ofReduceBool	LLPO
`lpo_implies_wlpo`	None	propext	Pure lo
`wlpo_implies_llpo`	None	propext, Quot.sound	Pure lo
`evenField_eq_zero_iff`	None	propext, Classical.choice, Quot.sound	BISH
`oddField_eq_zero_iff`	None	propext, Classical.choice, Quot.sound	BISH

```
-- Part A (BISH):
#print axioms cega_uncolorable
-- [Lean.ofReduceBool]

#print axioms finite_context_witness
-- [propext, Classical.choice, Quot.sound]

#print axioms partA_summary
-- [propext, Classical.choice, Quot.sound,
--  Lean.ofReduceBool]

-- Part B (LLPO):
#print axioms ks_sign_of_llpo
-- [propext, Classical.choice, Quot.sound,
--  llpo_real_of_llpo]

-- Backward (no custom axioms!):
#print axioms llpo_of_ks_sign
-- [propext, Classical.choice, Quot.sound]

-- Main equivalence:
#print axioms llpo_iff_ks_sign
-- [propext, Classical.choice, Quot.sound,
--  llpo_real_of_llpo]

-- Hierarchy (pure logic):
#print axioms lpo_implies_wlpo
-- [propext]

```

```
#print axioms wlpo_implies_llpo
-- [propext, Classical.choice, Quot.sound]
```

Listing 13: Axiom audit (Main/AxiomAudit.lean, selected).

8.4 CRM Compliance Protocol

The two-part structure is confirmed by machine:

- Part A theorems have **no custom axioms**—pure BISH. The only non-standard axiom is `Lean.ofReduceBool` from `native_decide`, which is a kernel axiom certifying that compiled evaluation agrees with the kernel.
- Part B forward depends on **exactly one** custom axiom (`llpo_real_of_llpo`)—LLPO level.
- Part B backward has **no custom axioms**—the reduction from KSSignDecision to LLPO is fully constructive.
- The encoded asymmetry lemmas have **no custom axioms**—the encoding is BISH.
- Hierarchy proofs (LPO $\Rightarrow$ WLPO $\Rightarrow$ LLPO) have **no custom axioms**—sorry-free, pure BISH.
- `Classical.choice` in all results is a MATHLIB4 infrastructure artifact from `Real.instField` and `tsum`. The mathematical content of these proofs is constructive.

9 Discussion

9.1 Bell and KS as the Same Logical Phenomenon

The central conceptual contribution of this paper is the demonstration that Bell nonlocality and KS contextuality represent **the same logical phenomenon at the** LLPO **level**. The logical structure of both theorems decomposes identically:

Step	**Bell (Paper 21)**	**KS (Paper 24)**
Negation (BISH)	No LHV model reproduces $S > 2$	No coloring satisfies all contexts
Disjunction (LLPO)	Bell asymmetry sign decision	KS asymmetry sign decision

In both cases:

- The negation is a finite computation, provable in BISH.
- The disjunction—deciding which "side" the obstruction lies on—requires exactly LLPO: the sign-decidability of a real number.

This identity is invisible from the classical perspective, where both theorems are simply proved by contradiction. CRM reveals the hidden unity by measuring the precise logical cost of each step.

9.2 The Encoding as a General-Purpose Tool

The geometric-series encoding used in both Paper 21 and Paper 24 is a general-purpose tool for LLPO calibrations. The recipe is:

1. Start with a binary sequence α satisfying AtMostOne.
2. Split into even-indexed and odd-indexed subsequences.
3. Encode each as a non-negative real via $\sum[\alpha(\cdot)] \cdot (1/2)^{n+1}$.
4. Define the asymmetry as the difference.
5. Show: the sign-iff lemmas connect sign to parity.
6. Conclude: sign decision $\leftrightarrow$ LLPO.

This recipe applies to *any* physical quantity whose constructive content reduces to a sign decision on an encoded asymmetry. The recipe is independent of the physical domain—it works equally well for Bell correlations, KS contextuality, and potentially for other quantum no-go theorems.

9.3 CRM as a Diagnostic for Structural Unity

Previous papers in the series used CRM as a *classification* tool: measuring the constructive cost of individual physical assertions. Papers 21 and 24 together demonstrate a new use of CRM: as a *diagnostic for structural unity.*

When two physically distinct theorems calibrate at the same CRM level with the same encoding mechanism, CRM is telling us that the theorems share a common logical structure. The fact that Bell and KS both calibrate at LLPO via the same sign-decision mechanism reveals a structural unity that is not apparent from the physical formulations alone.

This diagnostic function of CRM goes beyond mere cataloguing. It generates *predictions*: any quantum no-go theorem whose constructive content reduces to a sign decision on an asymmetry between parity-indexed contributions should calibrate at LLPO. Future candidates include no-cloning theorems, quantum state discrimination bounds, and other contextuality inequalities.

9.4 Limitations

1. **Encoded asymmetry, not literal contextuality.** The KSSignDecision is about the sign of an encoded asymmetry between even-indexed and odd-indexed contributions, not about the literal distinction between "this context fails" and "that context fails." The encoding is a mathematical proxy that captures the disjunctive structure of the KS conclusion.

2. **Classical.choice in Mathlib4.** The appearance of `Classical.choice` in BISH results is a Mathlib4 infrastructure artifact, not mathematical content. This is the same situation as in all previous papers in the series.

3. **Single axiom.** The interface axiom `llpo_real_of_llpo` is standard [Ishihara, 2006, Bridges and Richman, 1987] but not yet formalized in Mathlib4 from first principles. The backward direction (Theorem 4.9) requires no axiom, making the reverse reduction fully constructive.

4. **State-independent KS only.** We consider the CEGA state-independent KS set. State-dependent contextuality proofs [Budroni et al., 2022] may have different constructive costs and are left for future work.

9.5 Future Directions

Several extensions are natural:

1. **Gleason's theorem.** Gleason's theorem generalizes both KS and Born's rule. A CRM calibration of Gleason's theorem would clarify whether the full theorem requires more than LLPO.

2. **Quantum state discrimination.** The Helstrom bound and other state discrimination bounds involve disjunctions (is the state ρ_0 or ρ_1?) that may calibrate at LLPO.

3. **No-cloning theorem.** The no-cloning theorem is a negation (BISH), but extracting which states are "close to cloneable" may require LLPO or higher.

4. **Other KS sets.** The parity argument for the CEGA set is specific to the double-covering structure. Other KS sets (Peres 33-vector in $\mathbb{R}^3$ [Peres, 1991], etc.) may exhibit different combinatorial obstructions but the same LLPO sign decision.

10 Conclusion

The Kochen–Specker theorem—the impossibility of noncontextual hidden variable theories for a single quantum system—has the same constructive cost as Bell nonlocality for spatially separated systems: the Lesser Limited Principle of Omniscience (LLPO).

The result establishes a three-level stratification within KS contextuality:

- BISH: The CEGA 18-vector set is KS-uncolorable (finite computation).
- LLPO: The KS sign decision is equivalent to LLPO.
- WLPO $\Rightarrow$ LLPO: The hierarchy.

Combined with Paper 21's LLPO $\leftrightarrow$ BellSignDecision, this yields the structural identity

$$\text{BellSignDecision} \longleftrightarrow \text{KSSignDecision} \longleftrightarrow \text{LLPO}.$$

Two physically distinct no-go theorems—one involving spatial separation and entanglement, the other involving a single system with incompatible measurements—share identical logical cost. CRM reveals this hidden structural unity by measuring the precise constructive cost of each theorem's disjunctive content.

The calibration table now includes physical instantiations at every level of the constructive hierarchy: BISH, LLPO, WLPO, LPO, and FT. The LLPO level is populated by both Bell (Paper 21) and Kochen–Specker (Paper 24), confirming that the geometric-series encoding is a domain-independent tool for CRM calibration at this level.

AI-Assisted Methodology

This formalization was developed using **Claude Opus 4.6** (Anthropic, 2026) via the **Claude Code** command-line interface, following the same human–AI workflow as previous papers in the series Lee [2026a,d,e,b,c].

The author is a medical professional, not a domain expert in constructive mathematics or mathematical physics. The mathematical content of this paper was developed with extensive AI assistance. The human author specified the research direction and high-level goals, reviewed all mathematical claims for plausibility, and directed the formalization strategy. Claude Opus 4.6 explored the Mathlib4 codebase, generated Lean 4 proof terms, handled debugging, and assisted with paper writing. Final verification was by `lake build` (0 errors, 0 warnings, 0 sorries).

Component	Human	AI (Claude Opus 4.6)
Research question	✓	
Physical setup (KS/CEGA)	✓	
CRM calibration strategy	✓	
Lean 4 implementation		✓
Proof strategies	collaborative	collaborative
LaTeX writeup		✓
Review and editing	✓	

Table 2: Division of labor between human and AI.

Reproducibility

Reproducibility Box

- **Repository**: `https://github.com/paul-c-k-lee/FoundationRelativity`
- **Path**: `paper 24/P24_KochenSpecker/`
- **Build**: `lake exe cache get && lake build` (0 errors, 0 sorry)
- **Lean toolchain**: `leanprover/lean4:v4.28.0-rc1`
- **Interface axiom**: `llpo_real_of_llpo` (LLPO $\to \forall x : \mathbb{R}, x \leq 0 \lor 0 \leq x$; Ishihara [2006], Bridges and Richman [1987])
- **Axiom profile (Theorem 1, cega_uncolorable)**: `[Lean.ofReduceBool]`
- **Axiom profile (Theorem 2, finite_context_witness)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 3, forward)**: `[propext, Classical.choice, Quot.sound, llpo_real_of_llpo]`
- **Axiom profile (Theorem 4, backward)**: `[propext, Classical.choice, Quot.sound]`
- **Axiom profile (Theorem 5, main equiv)**: `[propext, Classical.choice, Quot.sound, llpo_real_of_llpo]`
- **Axiom profile (Theorem 6, stratification)**: `[propext, Classical.choice, Quot.sound, Lean.ofReduceBool, llpo_real_of_llpo]`
- **Total**: 16 files, 887 lines, 0 sorry

Acknowledgments

The Lean 4 formalization was developed using Claude Opus 4.6 (Anthropic, 2026) via the Claude Code CLI tool. We thank the Mathlib4 community for maintaining the comprehensive library of formalized mathematics that made this work possible.

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