

Floating-Point Math and Accuracy

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Errors in Scientific Computing

- Before computations:
 - Modeling: neglecting certain properties
 - Empirical data: not every input is known perfectly
 - Previous computations: data may be taken from other (error-prone) numerical methods
 - Sloppy programming (e.g. inconsistent conversions)
- During computations:
 - Truncation: a numerical method approximates a continuous solution
 - Rounding: computers offer only finite precision in representing real numbers

Example

- Computing the surface of the earth using

$$A = 4\pi r^2$$

- This involves several approximations:
 - Modeling: the earth is not exactly a sphere
 - Measurement: earth's radius is an empirical number
 - Truncation: the value of π is truncated
 - Rounding: all numbers used are rounded due to arithmetic operations in the computer
- Total error is the sum of all errors, but one of them is often the dominant error

Representing Numbers (1)

- Real numbers have unlimited accuracy
- Yet computers “think” digital, i.e. in integer math
=> only a fixed **range** of numbers can be represented by a fixed number of bits
=> **distance** between two integers is 1
- We can reduce the distance through fractions (= fixed point), but that also reduces the range

	16-bit	32-bit	64-bit	28-bit / 4-bit	22-bit / 10-bit
Min.	-32768	-2147483648	$\sim -9.2233 \times 10^{-18}$	-16777216.0000	-2048.000000
Max.	32767	2147483647	$\sim 9.2233 \times 10^{-18}$	16777215.9375	~ 2047.999023
Dist.	1	1	1	0.0635	0.0009765625

Representing Numbers (2)

- Need a way to represent a wider range of numbers with a same number of bits
- Need a way to represent numbers with a reasonable amount of precision (distance)
- Same relative precision often sufficient:

=> Scientific notation:

$\pm(\text{mantissa}) * (\text{base})^{\pm(\text{exponent})}$

Mantissa -> integer fraction

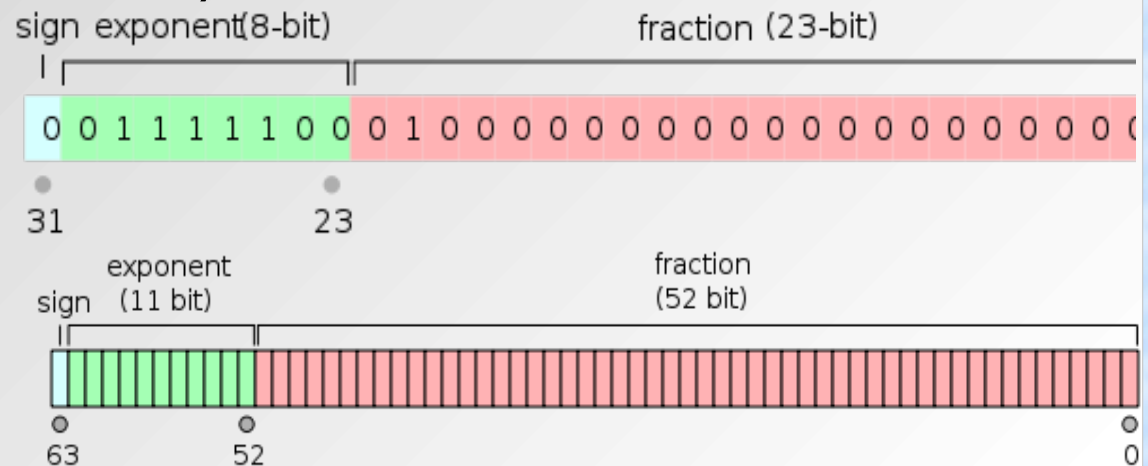
Base -> 2

Exponent -> a small integer

IEEE 754 Floating-point Numbers

- The IEEE 754 standard defines: storage format, result of operations, special values (infinity, overflow, invalid number), error handling
=> portability of compute kernels ensured
- Numbers are defined as bit patterns with a sign bit, an exponential field, and a fraction field

- Single precision:
8-bit exponent
23-bit fraction
- Double precision:
11-bit exponent
52-bit fraction



Values of Floating-Point Numbers

- Value: $(1 - (\text{mantissa}) / (2^{(\text{fraction bits})})) * 2^{(\text{exponent-bias})}$
 $1.0 \leq (\text{mantissa}) < 2.0, (\text{exponent}) \geq 0$
- Special case: 0.0 is all bits set to zero
Special case: -0.0 is like 0.0 but sign bit is set
More special cases: Inf, -Inf, NaN, -NaN
- Single precision: $\sim \pm 1.2 * 10^{-38} < x < \sim \pm 3.4 * 10^{38}$
actual precision: ~ 7 decimal digits
- Double precision: $\sim \pm 2.2 * 10^{-308} < x < \sim \pm 1.8 * 10^{308}$
actual precision: ~ 15 decimal digits

Density of Floating-point Numbers

- How can we represent so many more numbers in floating point than in integer? **We don't!**
- The number of unique bit patterns has to be the same as with integers of the same bitness
- There are 8,388,607 single precision numbers in $1.0 < x < 2.0$, but only 8191 in $1023.0 < x < 1024.0$
- \Rightarrow absolute precision depends on the magnitude
- \Rightarrow some numbers are not represented exactly
 \Rightarrow approximated using rounding mode (nearest)

Math with Floating Point Numbers

Addition:

- Right bitshift mantissa and increment exponent of smaller number until both exponents are the same
- Add mantissa of both numbers and bitshift until mantissa is between 1.0 and 2.0 again
- Only if both numbers have the same sign and the same exponent precision is preserved

Multiplication:

- Add exponents and multiply mantissa of both numbers
- Bitshift mantissa until its value is between 1.0 and 2.0
- No loss of precision; error is larger error of either number

Floating-Point Math Pitfalls

- Floating point math is commutative, but not associative! Example (single precision):
 $1.0 + (1.5 \times 10^{38} + (-1.5 \times 10^{38})) = 1.0$
 $(1.0 + 1.5 \times 10^{38}) + (-1.5 \times 10^{38}) = 0.0$
- \Rightarrow the result of a summation depends on the order of how the numbers are summed up
- \Rightarrow results may change significantly, if a compiler changes the order of operations for optimization
- \Rightarrow prefer adding numbers of same magnitude
 \Rightarrow avoid subtracting very similar numbers

How To Reduce Errors

- Use double precision unless you can be sure of error cancellation or using an imprecise model
=> collides with vectorization and GPU/MIC
- When summing numbers of different magnitude
 - Sort first and sum in ascending order
 - Sum in blocks (pairs) and then sum the sums
 - Use integer fraction, if range and precision allow it
- NOTE: summing numbers in parallel may give different results depending on parallelization

Floating Point Comparison

- Floating-point results are usually **inexact**
=> comparing for equality is dangerous
Example: don't use a floating point number for controlling a loop count. Integers are made for it
- It is OK to use exact comparison:
 - When results have to be bitwise identical
 - To prevent division by zero errors
- => compare against expected absolute error
- => don't expect higher accuracy than possible

Floating Point vs. Math Library

- libm is part of standard C, thus it is ubiquitous
- Provides a large variety of mathematical functions / operations on floating-point numbers but not many alternatives for x86/x86_64 exist
- Focus is typically put on standard compliance
- The x86 floating point unit contains most of the functionality internally, but most as firmware; SSE and AVX do not provide these
- The x86 FPU $\log()$ is slower than GNU libm

More Information on the Web

- <https://blog.reverberate.org/2014/09/what-every-computer-programmer-should.html>
- <https://blog.reverberate.org/2016/02/06/floating-point-demystified-part2.html>
- <https://floating-point-gui.de/>

Test Examples (1)

- **inverse**: computes $y=1/x$ and $z=x*y$ and checks if the result is exactly 1.0. Compare compilation using `gfortran -O2` and `gfortran -O2 -ffast-math`
- **loop**: advance x from 0.0 to 1.0 in increments of 0.01. Compare looping over integer and real
- **epsilon**: determine the floating-point precision through searching for the largest epsilon for which $1.0 + \epsilon == 1.0$. Start with $\epsilon = 1.0$ and repeatedly dividing by 2.0

Test Examples (2)

- **sum_number**: compare summing accuracy depending on ascending or descending order. Find the smallest N where the sums differ
- **paranoia**: IEEE-754 compliance test
=> use make to compile with different compiler flags for optimization and math accuracy
- **mathopt**: compute windowed average with a two and three numbers wide window.
=> speed of division by 2 vs division by 3
=> impact of compiler flags vs. code rewrite

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