Selected Answers to Exercise 3*

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1 P47-13(3)

$$
\begin{aligned}
&((x_1 * x_2)^{-1})_{\mathfrak{M}[\sigma]} \\
=&I(^{-1})((x_1 * x_2)_{\mathfrak{M}[\sigma]}) \\
=&(n - (x_1 * x_2)_{\mathfrak{M}[\sigma]})) \bmod n \\
=&(n - I(*)({x_1}_{\mathfrak{M}[\sigma]}, {x_2}_{\mathfrak{M}[\sigma]}) \bmod n \\
=&(n - ({x_1}_{\mathfrak{M}[\sigma]} +_n {x_2}_{\mathfrak{M}[\sigma]})) \bmod n \\
=&(n - (\sigma(x_1) +_n \sigma(x_2)) \bmod n \\
=&(n - (1 \bmod n) +_n (2 \bmod n)) \bmod n \\
=&-3 \bmod n
\end{aligned}
$$

$$
\begin{aligned}
&((x_1)^{-1} * (x_2)^{-1})_{\mathfrak{M}[\sigma]} \\
=&I(*)[(x_1^{-1})_{\mathfrak{M}[\sigma]}, (x_2^{-1})_{\mathfrak{M}[\sigma]}] \\
=&I(*)[I(^{-1})(\sigma(x_1)), I(^{-1})(\sigma(x_2))] \\
=&((n-1) \bmod n) +_n ((n-2) \bmod n) \\
=&-3 \bmod n
\end{aligned}
$$

Since $((x_1 * x_2)^{-1})_{\mathfrak{M}[\sigma]} = ((x_1)^{-1} * (x_2)^{-1})_{\mathfrak{M}[\sigma]}$, by Definition 3.12(2), p36, $A_{\mathfrak{M}[\sigma]} = T$.

2 P47-14(2)

Lemma 1: $\models A \Rightarrow \models \forall x.A$. ($\Leftrightarrow$ indeed).

*ver 2.2. This document is maintained on `https://github.com/sleepycoke/Mathematical_Logic_NJUCS`

Proof. By $\models A$, we have for any (M, σ), $A_{\mathfrak{M}[\sigma]} = T$. Therefore for any $a \in M$, $A_{\mathfrak{M}[\sigma[x:=a]]} = T$(note that $\sigma[x := a]$ is one specific assignment of all the σ), which implies $\forall x.A_{\mathfrak{M}[\sigma]} = T$, aka $\mathfrak{M} \models_\sigma \forall x.A$. As a result, since (M, σ) is arbitrary, $\models \forall x.A$. $\square$

Let $A \triangleq (x \doteq y \to y \doteq x)$,

$$\begin{aligned}
&A_{\mathfrak{M}[\sigma]} \\
=&B_{\to}[(x \doteq y)_{\mathfrak{M}[\sigma]}, (y \doteq x)_{\mathfrak{M}[\sigma]}] \\
=&\begin{cases} F & \text{if } (x \doteq y)_{\mathfrak{M}[\sigma]} = T \text{ and } (y \doteq x)_{\mathfrak{M}[\sigma]} = F \\ T & \text{otherwise} \end{cases}
\end{aligned}$$

However, $(x \doteq y)_{\mathfrak{M}[\sigma]} = T$ implies $\sigma(x) = \sigma(y)$, which implies $\sigma(y) = \sigma(x)$ and $(y \doteq x)_{\mathfrak{M}[\sigma]} = T$. Thus $A_{\mathfrak{M}[\sigma]} = T$. Applying Lemma 1 twice we get

$$\models \forall x \forall y.A.$$

3 P49-24

$\Rightarrow$

$$\begin{aligned}
&\mathfrak{M} \models_\sigma \forall x.A \\
\Rightarrow&\text{For all } a \in M, A_{\mathfrak{M}[\sigma[x:=a]]} = T \\
\Rightarrow&\text{For all } a \in M, A_{\mathfrak{M}[\rho[x:=a]]} = T, \text{ since } z \text{ is fresh in } A, \text{ where } \rho \triangleq \sigma[z := a]. \\
\Rightarrow&\text{For all } a \in M, A_{\mathfrak{M}[\rho[x:=z_{\mathfrak{M}[\rho]}]]} = T \\
\Rightarrow&\text{For all } a \in M, A[\frac{z}{x}]_{\mathfrak{M}[\rho]} = T, \text{ by Lemma 3.24, p40.} \\
\Rightarrow&\mathfrak{M} \models_\rho \forall z.A[\frac{z}{x}] \\
\Rightarrow&\mathfrak{M} \models_\sigma \forall z.A[\frac{z}{x}], \text{ since } z \notin FV(\forall z.A[\frac{z}{x}]).
\end{aligned}$$

$\Leftarrow$

$$\begin{aligned}
&\mathfrak{M} \models_\sigma \forall z.A[\frac{z}{x}] \\
\Rightarrow&\mathfrak{M} \models_\rho \forall z.A[\frac{z}{x}], \text{ since } z \notin FV(\forall z.A[\frac{z}{x}]), \text{ where } \rho \triangleq \sigma[z := a]. \\
\Rightarrow&\text{For all } a \in M, A[\frac{z}{x}]_{\mathfrak{M}[\rho]} = T \\
\Rightarrow&\text{For all } a \in M, A_{\mathfrak{M}[\rho[x:=z_{\mathfrak{M}[\rho]}]]} = T, \text{ by Lemma 3.24, p40.} \\
\Rightarrow&\text{For all } a \in M, A_{\mathfrak{M}[\rho[x:=a]]} = T \\
\Rightarrow&\text{For all } a \in M, A_{\mathfrak{M}[\sigma[x:=a]]} = T, \text{ since } z \text{ is fresh in } A. \\
\Rightarrow&\mathfrak{M} \models_\sigma \forall x.A
\end{aligned}$$