A Proof of Dual Replacement Lemma

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Let φ be a formula in basic modal and we denote φ' as the formula obtained by replacing a "$\Diamond$" inside φ with "$\neg\Box\neg$". We claim that

$$\vdash_K \varphi \leftrightarrow \varphi'.$$

Proof. We shall prove this by structural induction on φ.

Case 1: $\varphi = \Diamond\psi$ and we are replacing the first $\Diamond$, then $\varphi' = \neg\Box\neg\varphi$.

(1)	$\Diamond p \leftrightarrow \neg\Box\neg p$	DUAL
(2)	$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$	US:(1)

Case 2: $\varphi = \neg\psi$, $\varphi' = \neg\psi'$. We informally claim that the definition of ψ' is clear: For this case, if φ' replaces the ith $\Diamond$, ψ' does the ith.

(1)	$(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$	TAUT
(2)	$(\psi \leftrightarrow \psi') \leftrightarrow (\neg\psi \leftrightarrow \neg\psi')$	US:(1)
(3)	$\psi \leftrightarrow \psi'$	I.H.
(4)	$\neg\psi \leftrightarrow \neg\psi'$	MP(2)(3)

Case 3: $\varphi = \psi * \rho$, $\varphi' = \psi' * \rho$, where $* \in \{\wedge, \vee, \to, \leftrightarrow\}$.

(1)	$(p \leftrightarrow q) \to ((p * r) \leftrightarrow (q * r))$	TAUT
(2)	$(\psi \leftrightarrow \psi') \to ((\psi * \rho) \leftrightarrow (\psi' * \rho))$	US:(1)
(3)	$\psi \leftrightarrow \psi'$	I.H.
(4)	$(\psi * \rho) \leftrightarrow (\psi' * \rho)$	MP(2)(3)

Case 4: $\varphi = \psi * \rho$, $\varphi' = \psi * \rho'$. A similar proof as above holds.

Case 5: $\varphi = \Box\psi$, $\varphi' = \Box\psi'$.

(1)	$\psi \to \psi'$	PL:I.H.
(2)	$\Box(\psi \to \psi')$	N(1)
(3)	$\Box(\psi \to \psi') \to (\Box\psi \to \Box\psi')$	US:K
(4)	$\Box\psi \to \Box\psi'$	MP(3)(2)

Analogously, $\Box\psi' \to \Box\psi$. Therefore $\Box\psi \leftrightarrow \Box\psi'$.

Case 6: $\varphi = \Diamond\psi$ and we are replacing a $\Diamond$ inside ψ, then $\varphi' = \Diamond\psi'$.

(1)	$\Box\neg\psi \leftrightarrow \Box\neg\psi'$	I.H.(Case 5)
(2)	$\Diamond\psi \leftrightarrow \neg\Box\neg\psi$	I.H.(Case 1)
(3)	$\Diamond\psi' \leftrightarrow \neg\Box\neg\psi'$	I.H.(Case 1)
(4)	$\Diamond\psi \leftrightarrow \Diamond\psi'$	PL:(1),(2),(3)

Finally we have proved $\vdash_K \varphi \leftrightarrow \varphi'$ holds for any basic modal formula φ. $\Box$