# Existence of solution of CT process

D.D.Kuznedelev (Skoltech)

September 10, 2021

# Description of CT process

Deterministic CT system is described by ODE of the form:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \tag{1}$$

Here $x \in \mathbb{R}^n, y \in \mathbb{R}^m, u \in \mathbb{R}^l$

Stochastic CT system is described by the SDE of the form:

$$\begin{aligned} dX_t &= f(X_t, U_t)dt + \sigma(X_t, U_t)dB_t \\ Y_t &= h(X_t, U_t) \end{aligned} \tag{2}$$

$u(t)$ $(U_t)$ is the policy in deterministic (stochastic case)

| Theorem | Conditions | Conclusion |
|---|---|---|
| Peano existence theorem | $f$ continuous | local existence (not uniqueness) |
| Picard theorem | $f$ Lipschitz continious | local existence and uniqueness |

Continuous function : $\|f(x) - f(y)\| \to 0$ if $\|x - y\| \to 0$

Lipschitz function (with constant $L$): $\|f(x) - f(y)\| < L\|x - y\|$

# Theorems (ODE)

## Peano theorem

Let $D$ be open subset of $\mathbb{R} \times \mathbb{R}^n$ and $f$ continious on $D$. Then every initial value problem $x(t_0) = x_0$ has a local solution:

$$x(t) : I_{x_0} \to \mathbb{R}^n$$

## Picard theorem

Let $D$ be open subset of $\mathbb{R} \times \mathbb{R}^n$ and $f$ uniformly Lipschitz on $D$ (constant $L$ doesn't depend on $t$). Then every initial value problem $x(t_0) = x_0$ has a unique local solution:

$$x(t) : I_{x_0} \to \mathbb{R}^n$$

# SDE DYNAMICS

The general form of the *stochastic differential equation* with drift and diffusion term is:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

Here $W_t$ is the standard *Wiener* process (Brownian motion).

For SDE there exist two notions of solutions:

- Strong solution
- Weak solution

# SDE dynamics

## Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space with an admissible filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$. *Strong solution* of SDE with initial condition $x \in \mathbb{R}$ is an adapted process with continuous paths:

$$X_t = x + \int_0^t \mu(X_t, s) ds + \int_0^t \sigma(X_s, s) dW_s$$

Here the last integral is the *Ito* integral

*Adapted* process, informally, cannot see into future.

Requirements for the solution to be well-defined:

- With probability 1:

$$\int_0^t |\mu(X_s)| ds < \infty \qquad \int_0^t \sigma^2(X_s) ds < \infty$$

- With probability 1 solutions exists $\forall t < \infty$

**Definition**

A *weak* solution of SDE with initial condition $x \in \mathbb{R}$ is a continious stochastic process $X_t$ defined on *some* probability space $(\Omega, \mathcal{F}, P)$ with *some* filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ and *some* realization of Wiener process $W_t$.

# Theorems for SDE

Here $\mu(X_t, t) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(X_t, t) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$.

## Ito's existence and uniqueness theorem

Let $\mu(X_t, t)$ and $\sigma(X_t, t)$ be *uniformly* Lipschitz on $\mathbb{R} \times \mathbb{R}^n$ with linear growth:

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq C|x-y|$$

$$|\mu(x,t)|^2 + |\sigma(x,t)|^2 \leq D^2(1 + |x|^2)$$

Where $C$ and $D$ constants (independent of $x, t$). Then there exists unique *strong* solution:

$$X(t) : I_{x_0} \to \mathbb{R}^n \quad \mathbb{E}\left[\int_0^T |X_t|^2 dt\right] < \infty$$

Lipschitz condition is important.

For $\dot{x} = 2\sqrt{|x|}$, r.h.s doesn't satisfy Lipschitz condition at $x = 0$.

For initial value $x(0) = 0$ both functions:

$$x_1(t) = 0 \quad x_2(t) = t^2$$

solve the initial value problem.

# Controlled SDE

Now we are going to consider the case of *controlled* diffusion with $U$ a control function, with $U$ taking values in $\mathbb{U}$.

$$dX_t = \mu(X_t, U_t, t)dt + \sigma(X_t, U_t, t)dW_t$$

Ito's theorem has to be modified a bit.

## Existence and uniqueness theorem for controlled process

If the conditions of Ito's theorem hold for any $t > 0$ and for any $\alpha \in \mathbb{U}$ *uniformly*:

$$|\mu(x, \alpha, t) - \mu(y, \alpha, t)| + |\sigma(x, \alpha, t) - \sigma(y, \alpha, t)| \leq C|x - y|$$

$$|\mu(x, \alpha, t)|^2 + |\sigma(x, \alpha, t)|^2 \leq D^2(1 + |x|^2)$$

Where $C$ and $D$ constants (independent of $x, t$). Then there exists unique *strong* solution:

$$X(t) : I_{x_0} \to \mathbb{R}^n \quad \mathbb{E}\left[\int_0^T |X_t|^2 dt\right] < \infty$$