Stabilization methods for Model Predictive Control

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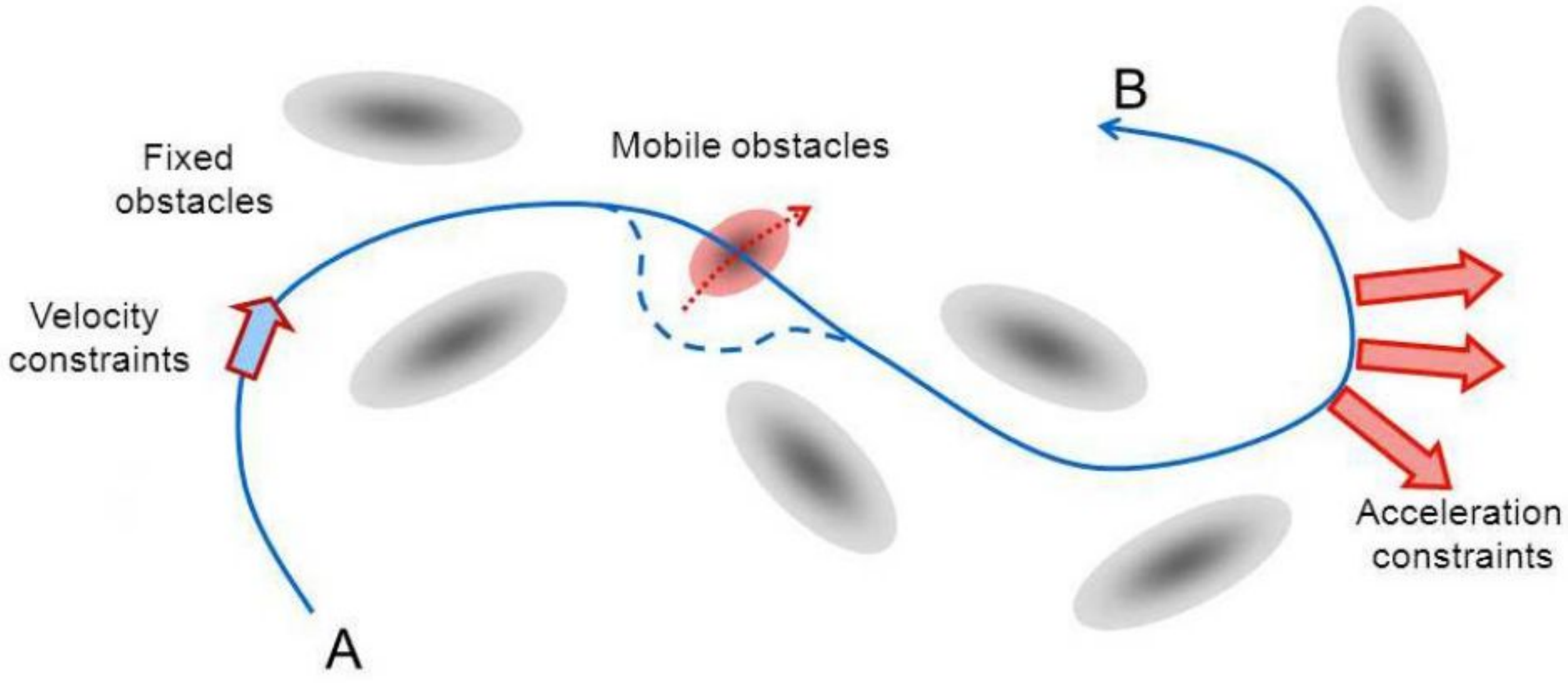
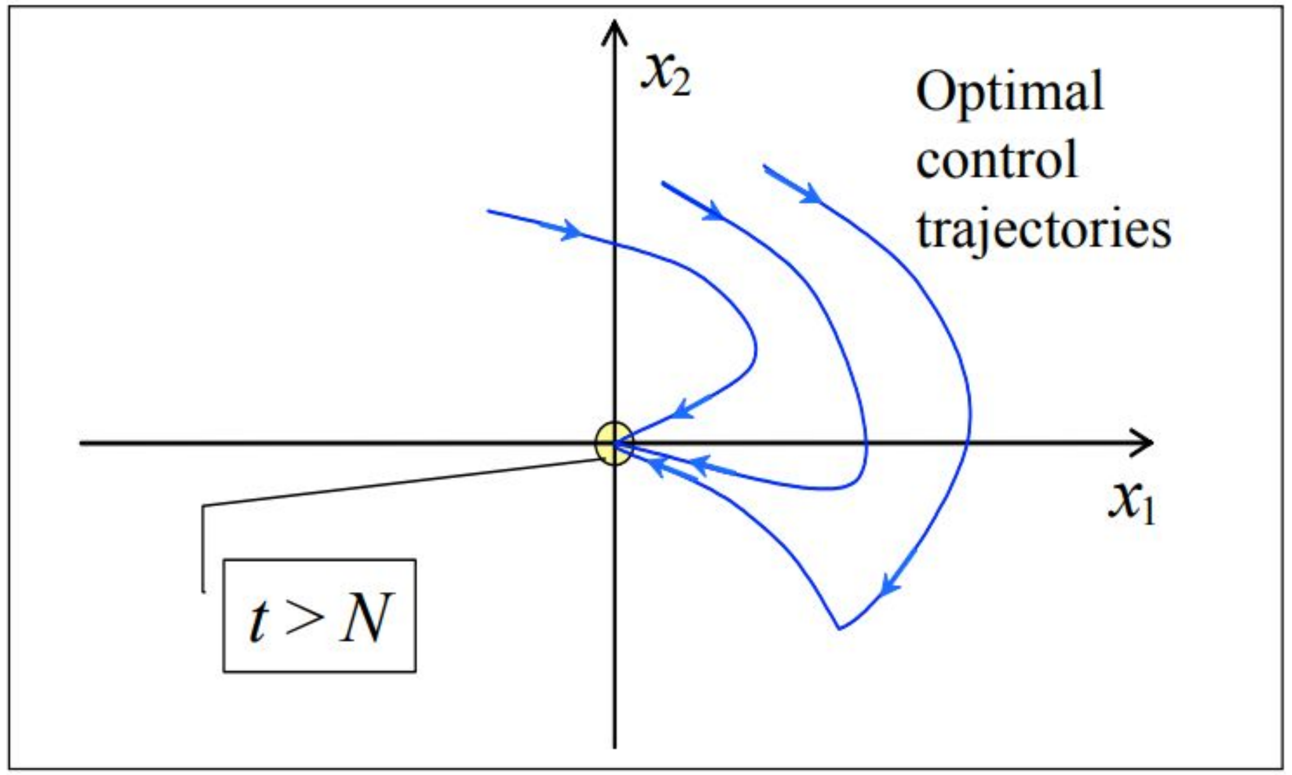
MPC task

MPC is for continuous-space optimal control problems. Task is to keep state close to:

- a goal state
- a given trajectory





- Optimal control trajectories converge to (0,0)
- If N is large, the part of the problem for $t > N$ can be neglected
- Infinite-horizon optimal control ≈ horizon-N optimal control

MPC algorithm

Let us describe the MPC algorithm for the deterministic problem just described. At the current state x_k:

(a) MPC solves an ℓ-step lookahead version of the problem, which requires that $x_{k+\ell} = 0$.

(b) If $\{\tilde{u}_k, \ldots, \tilde{u}_{k+\ell-1}\}$ is the optimal control sequence of this problem, MPC applies $\tilde{u}_k$ and discards the other controls $\tilde{u}_{k+1}, \ldots, \tilde{u}_{k+\ell-1}$.

(c) At the next stage, MPC repeats this process, once the next state x_{k+1} is revealed.

MPC objective

In particular, at the typical stage k and state $x_k \in X_k$, the MPC algorithm solves an ℓ-stage optimal control problem involving the same cost function and the requirement $x_{k+\ell} = 0$. This is the problem

$$\min_{u_i,\, i=k,\ldots,k+\ell-1} \sum_{i=k}^{k+\ell-1} g_i(x_i, u_i),$$

subject to the system equation constraints

$$x_{i+1} = f_i(x_i, u_i), \qquad i = k, \ldots, k+\ell-1,$$

the state and control constraints

$$x_i \in X_i, \qquad u_i \in U_i(x_i), \qquad i = k, \ldots, k+\ell-1,$$

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the state and control constraints

$$x_i \in X_i, \qquad u_i \in U_i(x_i), \qquad i = k, \ldots, k+\ell-1,$$

and the terminal state constraint

$$x_{k+\ell} = 0.$$

terminal constraint

$$f_k(0, \overline{u}_k) = 0, \quad g_k(0, \overline{u}_k) = 0$$

for some control $\overline{u}_k \in U_k(0)$.

Controllability condition

Constrained Controllability Condition

There exists an integer $\ell > 1$ such that for every initial state $x_k \in X_k$, we can find a sequence of controls $u_k, \ldots, u_{k+\ell-1}$ that drive to 0 the state $x_{k+\ell}$ of the system at time $k+\ell$, while satisfying all the intermediate state and control constraints

$$u_k \in U_k(x_k),\ x_{k+1} \in X_{k+1}, \ldots,$$
$$x_{k+\ell-1} \in X_{k+\ell-1},\ u_{k+\ell-1} \in U_{k+\ell-1}(x_{k+\ell-1}).$$

Stability

On the lecture: proved that stabilization is achieved for all sufficiently long horizons l

Stabilization condition:

$$\sum_{k=0}^{\infty} g_k(x_k, u_k) < \infty, \tag{2.39}$$

where $\{x_0, u_0, x_1, u_1, \ldots\}$ is the state and control sequence generated by MPC.

$$\hat{J}_k(x_k) = \min_{u_k \in U_k(x_k)} \left[g_k(x_k, u_k) + H_{k+1}\big(f_k(x_k.u_k)\big) \right].$$

$$\hat{J}_k(x_k) \leq H_k(x_k).$$

where $H_k(x_k)$ and $H_{k+1}(x_{k+1})$ are optimal costs for (l-1)-stage optimization problems that start at x_k and x_{k+1}, and drive the states x_{k+l-1} and x_{k+l}, respectively, to 0.

Stability

By combining previous relations, we obtain the sequential improvement condition

$$\min_{u_k \in U_k(x_k)} \left[g_k(x_k, u_k) + H_{k+1}\big(f_k(x_k.u_k)\big) \right] \le H_k(x_k),$$

$$g_k(x_k, u_k) + H_{k+1}(x_{k+1}) \le H_k(x_k), \qquad k = 0, 1, \ldots.$$

Adding this relation for all k in a range $[0, K]$, where $K = 0, 1, \ldots,$ we obtain

$$H_{K+1}(x_{K+1}) + \sum_{k=0}^{K} g_k(x_k, u_k) \le H_0(x_0).$$

Since $H_{K+1}(x_{K+1}) \ge 0$, it follows that

$$\sum_{k=0}^{K} g_k(x_k, u_k) \le H_0(x_0), \qquad K = 0, 1, \ldots, \tag{2.41}$$

and taking the limit as $K \to \infty$,

$$\sum_{k=0}^{\infty} g_k(x_k, u_k) \le H_0(x_0) < \infty,$$

[$H_0(x_0)$ is finite because the transfer from x_0 to $x_\ell = 0$ is feasible by the constrained controllability condition]. We have thus verified the stability condition (2.39).

Terminal costs

In particular, at the typical stage k and state $x_k \in X_k$, the MPC algorithm solves an ℓ-stage optimal control problem involving the same cost function and the requirement $x_{k+\ell} = 0$. This is the problem

$$\min_{u_i,\, i=k,\ldots,k+\ell-1} g_N(x_N) + \sum_{i=k}^{k+\ell-1} g_i(x_i, u_i),$$

penalizes the terminal constraint

subject to the system equation constraints

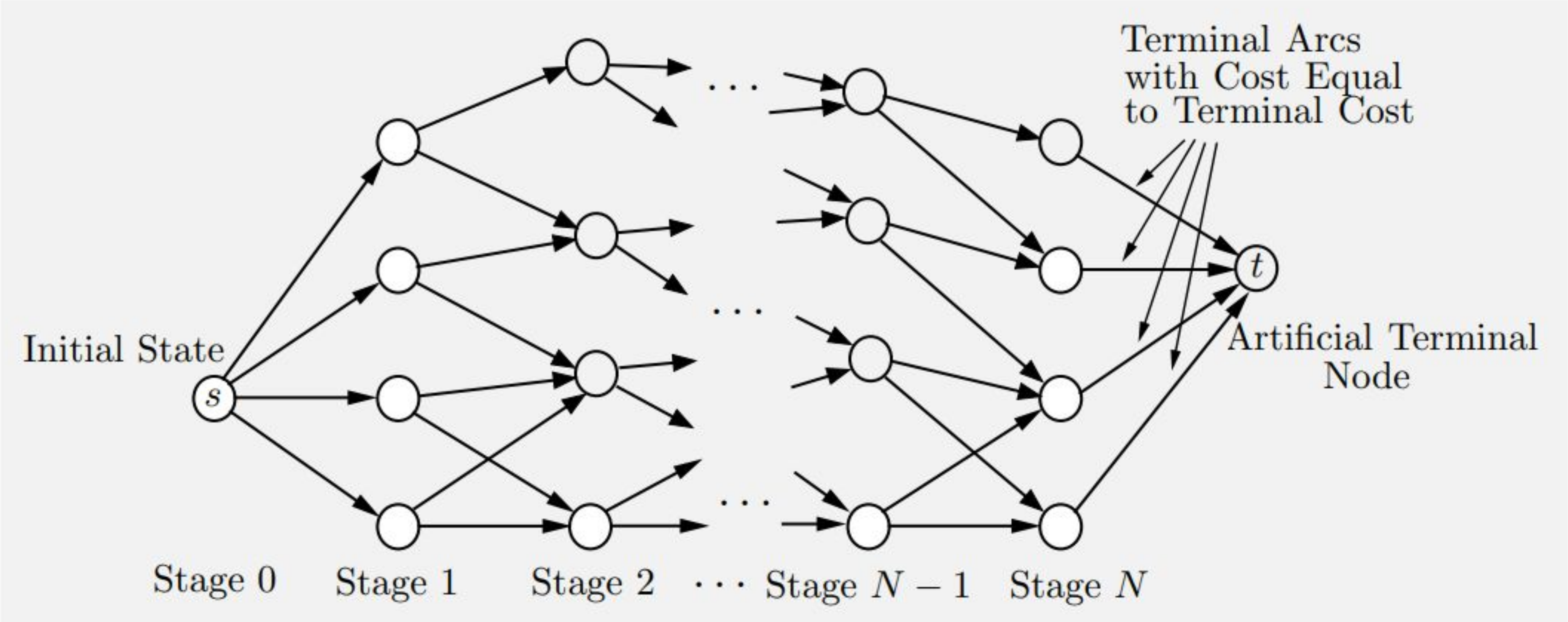
$$x_{i+1} = f_i(x_i, u_i), \qquad i = k, \ldots, k+\ell-1,$$

the state and control constraints

$$x_i \in X_i, \qquad u_i \in U_i(x_i), \qquad i = k, \ldots, k+\ell-1,$$

where $g_N(x_N)$ is a terminal cost incurred at the end of the process.

Terminal costs



References

[1] D. Bertsekas, Reinforcement Learning and Optimal Control, MIT, 2019.

[2] Lecture 14 of EE392m course (Control Engineering in Industry), Stanford University, 2004-2005. https://web.stanford.edu/class/archive/ee/ee392m/ee392m.1056/Lecture14_MPC.pdf

Thank you!

Any questions?