Problem Set 2 for EE227C (Spring 2018): Convex Optimization and Approximation

Instructor: Moritz Hardt

Email: hardt+ee227c@berkeley.edu

Graduate Instructor: Max Simchowitz

Email: msimchow+ee227c@berkeley.edu

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Problem 1: Backtracking Line Search

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an m-strongly convex, M-smooth (and thus differentiable) function with global minimum x^*. Consider the following algorithm:

1 **Input:** Parameters $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$, $x_0 \in \mathbb{R}^n$;
2 **for** $t = 0, 1, 2, \ldots$ **do**
3   $g_t \leftarrow \nabla f(x_t)$;
4   **for** $k = 0, 1, 2, \ldots$ **do**
5   If "sufficient decrease" condition holds

$$f(x_t - \beta^k g_t) \leqslant f(x_t) - \alpha \beta^k \cdot \|g_t\|^2 \,, \tag{1}$$

    set $\eta_t = \beta^k$ and **break**
6   **end**
7   Set $x_t \leftarrow x_{t-1} - \eta_t g_t$
8 **end**

Algorithm 1: Backtracking Line Search

(A) Show that condition 1 holds for whenever $\beta^k \in (0, 1/M]$.

(B) Show that $\eta_t \geqslant \min\{1, \beta/M\}$. Conclude that the loop in Line 4 aways terminates.

(C) Using part b, show that

$$f(x_t - \eta_t g_t) \leqslant f(x_t) - \alpha \min\{1, \frac{\beta}{M}\} \|\nabla f(x_t)^2\| \tag{2}$$

(D) Show that there is a constant $C = C(\alpha, \beta, M, m) < 1$ such

$$f(x_t - \eta_t g_t) - f(x_*) \leqslant C(\alpha, \beta, M, m) \cdot (f(x_t) - f(x_*)) \tag{3}$$

Problem 2: Random Descent Directions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an m-strongly convex, M-smooth (and thus differentiable) function with global minimum x^*. Consider the following algorithm:

Input: Parameters $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$, $x_0 \in \mathbb{R}^n$;
for $t = 0, 1, 2, \ldots$ **do**
  Set $g_t \overset{\text{unif}}{\sim} \mathcal{S}^{n-1}$;
  Set $\eta_t := \min_{\eta \geqslant 0} f(x_t - \eta g_t)$;
  Set $x_{t+1} \leftarrow x_t - \eta g_t$;
end

Algorithm 2: Random Direction Line Search

$\mathcal{S}^{n-1} := \{v \in \mathbb{R}^n : \|v\|_2^2 = 1\}$ denotes the unit sphere in $\mathbb{R}^n$. $g_t \overset{\text{unif}}{\sim} \mathcal{S}^{n-1}$ denotes the unique rotation invariant distribution on the unit sphere. For example, if $h \sim \mathcal{N}(0, I)$, then $h/\|h\| \overset{\text{unif}}{\sim} \mathcal{S}^{n-1}$

(A) Prove that the above algorithm is a (non-strict) descent method; that is $f(x_t)$ is non-increasing in t. Also prove that unless $x_t = x_*$, $f(x_{t+1}) < f(x_t)$ with probability $1/2$.

(B) Prove that there exists a numerical constant C such that, if

$$t \geqslant T(\epsilon) := Cn \cdot \frac{M}{m} \log(\frac{f(x_0) - f(x^*)}{\epsilon}) \, , \tag{4}$$

then $\text{Exp}[f(x_t) - f(x^*)] \leqslant \epsilon$. *Hint:* Do not analyze the algorithm like you SGD, but more like coordinate descent. Since the method is a non-strict descent method, accept that on some rounds, you might not make any progress. Just ensure that, with constant probability on each round, you make some progress.

You might want to prove this lemma:

Lemma 0.1. *Let $g \overset{\text{unif}}{\sim} \mathcal{S}^{n-1}$, and let $v \in \mathbb{R}^n$. Then, there exist universal constants $c_1, c_2 > 0$ (independent of n and v) such that*

$$\mathbb{P}\text{r}[\langle v, g\rangle^2 \geqslant c_1 \text{Exp}[\langle v, g\rangle^2]] \geqslant c_2 \tag{5}$$

To prove the lemma, use the fact that if $g \overset{\text{unif}}{\sim} \mathcal{S}^{n-1}$, then g has the same distribution of $z/\|z\|$, where $z \sim \mathcal{N}(0, I/d)$.

(C) Ammend the stated algorithm to use line search instead of solving for the exactly-optimal step size. To be clear, you don't have access to $\nabla f(x_t)$, all you are allowed to do at round t is the following:

(C.1) Sample *one* direction $g_t \overset{\text{unif}}{\sim} \mathcal{S}^{n-1}$.

(C.2) Making (finite) function evaluations of the form $f(x_t - \eta g_t)$ for $\eta \in \mathbb{R}$. Ideally, this should be at most logarithmic in problem parameters.

State *both* the number of iterations and the number of function evaluations. Are the rates qualitatively similar?

Problem 3: Sh*t about Quadratics

In this problem, you are going to test the sharpness of our upper and lower bounds for quadratics on a randomly generated instance. Fix $n = 500$. We define the distribution over PSD matrix $\mathcal{D}(\epsilon)$:

Definition 0.2. $\mathcal{D}(\epsilon)$ is a distribution of matrix $\mathbf{M} = \mathbf{M}^\top$, defined as follows. Let $\mathbf{X} \in \mathbb{R}^{n \times n}$ denote a matrix with i.i.d $\mathcal{N}(0,1)$ entries. Generate a random vector $\mathbf{u}$ uniformly from the unit sphere. Define the matrix $\mathbf{M} = \frac{1}{\sqrt{2n}}(\mathbf{X} + \mathbf{X}^\top) + (1+\epsilon)\mathbf{u}\mathbf{u}^\top$.

Now, for each $\epsilon \in \mathcal{S} := \{1, .5, .2, .1, .05\}$, do the following

(A) Conduct trials $t = 1, 2, \ldots, 10$.

(A.1) Generate $\mathbf{M} \sim \mathcal{D}(\epsilon)$ as above, and a random vector $\mathbf{v}$ uniformly on the unit sphere.

(A.2) Set $\gamma = 2\lambda_{\max}(\mathbf{M}) - \lambda_2(\mathbf{M})$, and define the matrix $\mathbf{N} = \gamma I - \mathbf{M}$. Definally, define the function $\mathbf{f}(x) = \min_x x^T \mathbf{N} x - 2\langle \mathbf{v}, x \rangle$. What is the condition number of $\mathbf{N}$?

(A.3) Setting $x_0 = 0$, run gradient descent, a heavy-ball method or nesterov method to solve $\min_x \mathbf{f}(x)$ for a good number of iterations (use your discretion). You may compute the eigenvalues of $\mathbf{N}$ to tune your step parameters.

(A.4) For both gradient descent and heavy-ball, record for each trial iteration s, the difference between $\mathbf{f}(x_s) - \min_x \mathbf{f}(x)$ for each iteration.

(A.5) Using the step sizes, largest/smallest eigenvalues of $\mathbf{N}$, and the initial point $x_0 = 0$, compute a worst case upper bound for $\mathbf{f}(x_s) - \min_x \mathbf{f}(x)$ for each iteration s of gradient descent and the heavy ball method.

(A.6) Run gradient descent, but this time compute the optimality gap unising "best" iterate in the Krylov space. That is, compute

$$\min_{x \in \text{span}(x_1, \ldots, x_s)} \mathbf{f}(x) - \min_x \mathbf{f}(x) \tag{6}$$

(A.7) After each trial, you should have a list of 5 values for each iterate s: an upper bound for gradient descent, the rate actually attained by gradient descent, an upper bound for heavy ball/nesterov, the rate actualy attained by heavy ball/nesterov, and the "optimal" krylov algorith,

(B) For each of the lists above, average all 10 trials and plot them on the same plot. How sharp are the upper bounds?