Course Notes for EE227C (Spring 2018): Convex Optimization and Approximation

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6 Lecture 6: Discovering acceleration

In this lecture, we seek to find methods that converge faster than those discussed in previous lectures. To derive this accelerated method, we start by considering the special case of optimizing quadratic functions.

6.1 Quadratics

Definition 6.1 (Quadratic function)**.** A quadratic function $f\colon \mathbb{R}^n \to \mathbb{R}$ takes the form:

$$f(x) = \frac{1}{2}x^T Ax - b^T x + c,$$

where $A \in S^n$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Note that substituting $n = 1$ into the above definition recovers the familiar univariate quadratic function $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, as expected. There is one subtlety in this definition: we restrict A to be symmetric. In fact, we could allow $A \in \mathbb{R}^{n\times n}$ and this would define the same class of functions, since for any $A \in \mathbb{R}^{n\times n}$ there is a symmetric matrix $\tilde{A} = \frac{1}{2}\left(A + A^T\right)$ for which:

$$x^T Ax = x^T \tilde{A}x \; \forall x \in \mathbb{R}^n.$$

Restricting $A \in S^n$ ensures each quadratic function has a *unique* representation.

The gradient and Hessian of a general quadratic function take the form:

$$\nabla f(x) = Ax - b$$
$$\nabla^2 f(x) = A.$$

Note provided A is non-singular, the quadratic has a unique critical point at:

$$x^* = A^{-1}b.$$

When $A \succ 0$, the quadratic is *strictly convex* and this point is the unique global minima.

6.2 Gradient descent on a quadratic

In this section we will consider a quadratic $f(x)$ where A is positive definite, and in particular that:

$$\alpha I \preceq A \preceq \beta I,$$

for some $0 < \alpha < \beta$. This implies that f is α-strongly convex and β-smooth.

From Theorem **??** we know that under these conditions, gradient descent with the appropriate step size converges linearly at the rate $\exp\left(-t\frac{\alpha}{\beta}\right)$. Clearly the size of $\frac{\alpha}{\beta}$ can dramatically affect the convergence guarantee. In fact, in the case of a quadratic, this is related to the *condition number* of the matrix A.

Definition 6.2 (Condition number)**.** Let A be a real matrix. Its *condition number* (with respect to the Euclidean norm) is:

$$\kappa(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)},$$

the ratio of its largest and smallest eigenvalues.

So in particular, we have that $\kappa(A) \leqslant \frac{\beta}{\alpha}$; henceforth, we will assume that α, β correspond to the minimal and maximal eigenvalues of A so that $\kappa(A) = \frac{\beta}{\alpha}$. It follows that gradient descent with a constant step size $\frac{1}{\beta}$ converges at:

$$||x_{t+1} - x^*\|^2 \leqslant \exp\left(-t\frac{1}{\kappa}\right)\|x_1 - x^*\|^2.$$

In many cases, f is ill-conditioned and κ can easily take values in the hundreds or thousands. In this case, case convergence could be very slow: note that at $t > \kappa$, the error may have been reduced by only a factor of $3\times$. Can we do better than this?

In the case of a quadratic, we can of course use the analytic solution $x^* = A^{-1}b$. However, it will prove instructive to consider applying gradient descent to quadratic functions, and derive the convergence bound that we previously proved for any strongly convex smooth functions. This exercise will show us where we are losing performance, and suggest a method that can attain better guarantees.

6.2.1 Convergence analysis

Theorem 6.3. *Assume $f\colon \mathbb{R}^n \to \mathbb{R}$ is a quadratic where the quadratic coefficient matrix has a condition number κ. Let x^* be an optimizer of f, and let x_t be the updated point at step t using gradient descent with a constant step size $\frac{1}{\beta}$, i.e. using the update rule $x_{t+1} = x_t - \frac{1}{\beta}\nabla f(x_t)$. Then:*

$$\|x_{t+1} - x^*\|^2 \leqslant \exp\left(-t\frac{1}{\kappa}\right)\|x_1 - x^*\|^2.$$

Proof. Write:

$$f(x) = \frac{1}{2}x^T Ax - b^T x + c,$$

where $A \in S^n$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. A gradient descent update with step size η_t takes the form:

$$x_{t+1} = x_t - \eta_t \nabla f(x_t) = x_t - \eta_t\left(Ax_t - b\right)$$

Subtracting x^* from both sides of this equation and using the property that $Ax^* - b = \nabla f(x^*) = 0$:

$$\begin{aligned} x_{t+1} - x^* &= \left(x_t - \eta_t\left(Ax_t - b\right)\right) - \left(x^* - \eta_t\left(Ax^* - b\right)\right) \\ &= (I - \eta_t A)(x_t - x^*). \end{aligned}$$

S Thus:

$$\begin{aligned} \|x_{t+1} - x^*\|_2 &\leqslant \|(I - \eta_t A)\|_2 \|(x_t - x^*)\|_2 \\ &\leqslant \left(\prod_{k=1}^{t}\|I - \eta_k A\|_2\right)\|x_1 - x^*\|_2. \end{aligned}$$

Set $\eta_k = \frac{1}{\beta}$ for all k. Note that $\frac{\alpha}{\beta}I \preceq \frac{1}{\beta}A \preceq I$, so:

$$\max_{A\in\mathbb{R}^{n\times n}}\left\|I - \frac{1}{\beta}A\right\|_2 = 1 - \frac{\alpha}{\beta} = 1 - \frac{1}{\kappa}.$$

It follows that:

$$\begin{aligned} \|x_{t+1} - x^*\|_2 &\leqslant \left(1 - \frac{1}{\kappa}\right)^t \|x_1 - x^*\|_2 \\ &\leqslant \exp\left(-\frac{t}{\kappa}\right)\|x_1 - x^*\|_2. \end{aligned}$$

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6.3 Accelerated gradient descent

In the previous section, we proved a lower bound on the convergence rate. In this section, we would like to improve on this. Consider whether there was any point where we were careless in the proof above? One obvious candidate is that our choice of step size, $\eta_k = \frac{1}{\beta}$, was chosen rather arbitrarily. In fact, by choosing the sequence η_k we can select *any* degree-t polynomial of the form:

$$p(A) = \prod_{k=1}^{t} (I - \eta_K A).$$

Note that:

$$\|p(A)\|_2 \leqslant \max_{x \in \lambda(A)} |p(x)|$$

where $p(A)$ is a matrix polynomial, and $p(t)$ is the corresponding scalar polynomial. In general, we may not know the set of eigenvalues $\lambda(A)$, but we do know that all eigenvalues are in the range $[\alpha, \beta]$. Relaxing the inequality accordingly:

$$\|p(A)\| \leqslant \max_{x \in [\alpha, \beta]} |p(x)|.$$

We want a polynomial $p(t)$ that takes on small values in $[\alpha, \beta]$. We impose an additional normalization constraint that $p(0) = 1$ (otherwise we could 'cheat' by scaling the polynomial down everywhere on its domain), but allow it to take on arbitrary values elsewhere.

6.3.1 A naive polynomial solution

A naive solution chooses a uniform step size $\eta_t = \frac{2}{\alpha+\beta}$. Note that:

$$\max_{x \in [\alpha, \beta]} \left| 1 - \frac{2}{\alpha + \beta} x \right| = \frac{\beta - \alpha}{\alpha + \beta} \leqslant \frac{\beta}{\alpha} = \kappa,$$

recovering the same convergence rate we proved previously.

The resulting polynomial $p_t(x)$ is plotted in figure 1 for degrees $t = 3$ and $t = 6$, with $\alpha = 1$ and $\beta = 10$. Note that doubling the degree from three to six only halves the maximum absolute value the polynomial attains in $[\alpha, \beta]$, explaining why convergence is so slow.

6.4 Chebyshev polynomials

Fortunately, we can do better than this by speeding up gradient descent using Chebyshev polynomials. The Chebyshev polynomials are a sequence of orthogonal polynomials which are related to de Moivre's formula and which can be defined recursively. We will use Chebyshev polynomials of the first kind which are donated T_n.

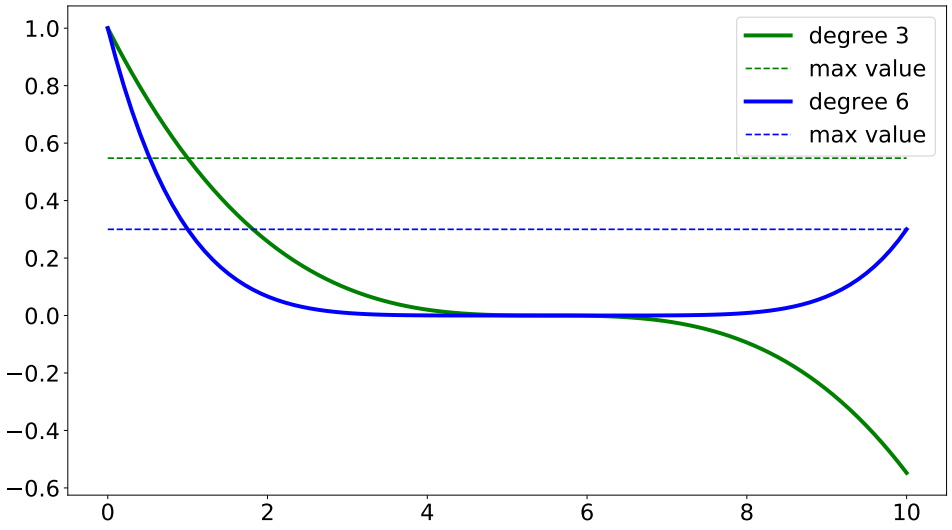

Figure 1: Naive Polynomial

The Chebyshev polynomials of the first kind are defined by the recurrence relation:

$$T_0(a) = 1, \quad T_1(a) = a$$
$$T_k(a) = 2aT_{k-1}(a) - T_{k-2}(a), \text{ for } k \geqslant 2.$$

And figure 2 is the plot of the Chebyshev polynomials for $i = 0, 1, 2, 3, 4$.

Why Chebyshev polynomials? Suitably rescaled, they minimize the absolute value in a desired interval $[\alpha, \beta]$ while satisfying the normalization constraint of having value 1 at the origin.

Recall that the eigenvalues of the matrix we consider are in the interval $[\alpha, \beta]$. We need to rescale the Chebyshev polynomials so that they're supported on this interval and still attain value 1 at the origin. This is accomplished by the polynomial

$$P_k(a) = \frac{T_k\left(\frac{\alpha+\beta-2a}{\beta-\alpha}\right)}{T_k\left(\frac{\alpha+\beta}{\beta-\alpha}\right)}.$$

We see on figure 3 that doubling the degree has a much more dramatic effect on the magnitude of the polynomial in the interval $[\alpha, \beta]$.

Let's compare on figure 4 this beautiful Chebyshev polynomial side by side with the naive polynomial we saw earlier. The Chebyshev polynomial does much better: at around 0.3 for degree 3 (needed degree 6 with naive polynomial), and below 0.1 for degree 6.

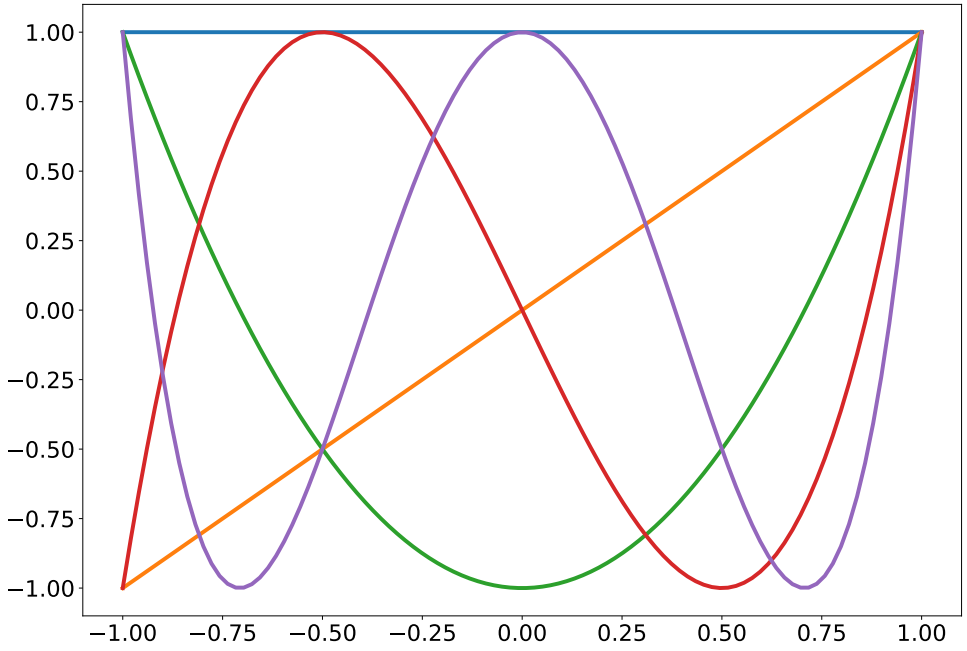

Figure 2: Chebychev polynomials

6.4.1 Accelerated gradient descent

The Chebyshev polynomial leads to an accelerated version of gradient descent. Before we describe the iterative process, let's first see what error bound comes out of the Chebyshev polynomial.

So, just how large is the polynomial in the interval $[\alpha, \beta]$? First, note that the maximum value is attained at α. Plugging this into the definition of the rescaled Chebyshev polynomial we get the upper bound for any $a \in [\alpha, \beta]$,

$$\begin{aligned} |P_k(a)| &\leqslant |P_k(\alpha)| \\ &= \frac{|T_k(1)|}{|T_K\left(\frac{\beta+\alpha}{\beta-\alpha}\right)|} \\ &= |T_K\left(\frac{\beta+\alpha}{\beta-\alpha}\right)^{-1}|. \end{aligned}$$

Recalling the condition number $\kappa = \beta/\alpha$, we have

$$\frac{\beta+\alpha}{\beta-\alpha} = \frac{\kappa+1}{\kappa-1}.$$

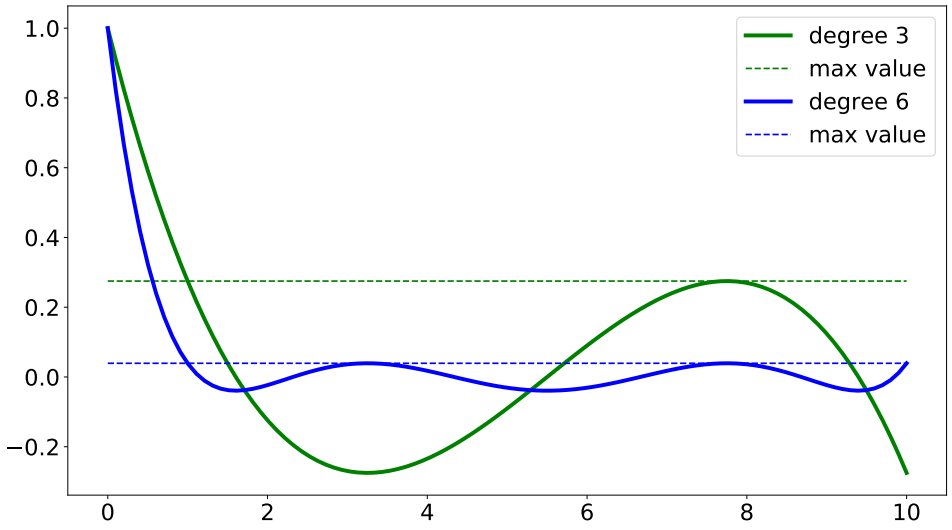

Figure 3: Rescaled Chebyshev

Typically κ is large, so this is $1+\epsilon$, $\epsilon \approx \frac{2}{\kappa}$. Therefore, we have

$$|P_k(a)| \leqslant |T_k(1+\epsilon)^{-1}|.$$

To upper bound $|P_k|$, we need to lower bound $|T_k(1+\epsilon)|$.
Fact: for $a > 1$, $T_k(a) = \cosh(k \cdot \operatorname{arccosh}(a))$ where:

$$\cosh(a) = \frac{e^a + e^{-a}}{2}, \quad \operatorname{arccosh}(a) = \ln\left(x + \sqrt{x^2 - 1}\right).$$

Now, letting $\phi = \operatorname{arccosh}(1+\epsilon)$:

$$e^{\phi} = 1 + \epsilon + \sqrt{2\epsilon + \epsilon^2} \geqslant 1 + \sqrt{\epsilon}.$$

So, we can lower bound $|T_k(1+\epsilon)|$:

$$\begin{aligned} |T_k(1+\epsilon)| &= \cosh(k\operatorname{arccosh}(1+\epsilon)) \\ &= \cosh(k\phi) \\ &= \frac{(e^{\phi})^k + (e^{-\phi})^k}{2} \\ &\geqslant \frac{(1+\sqrt{\epsilon})^k}{2}. \end{aligned}$$

Then, the reciprocal is what we needed to upper bound the error of our algorithm, so we have:

$$|P_k(a)| \leqslant |T_k(1+\epsilon)^{-1}| \leqslant 2(1+\sqrt{\epsilon})^{-k}.$$

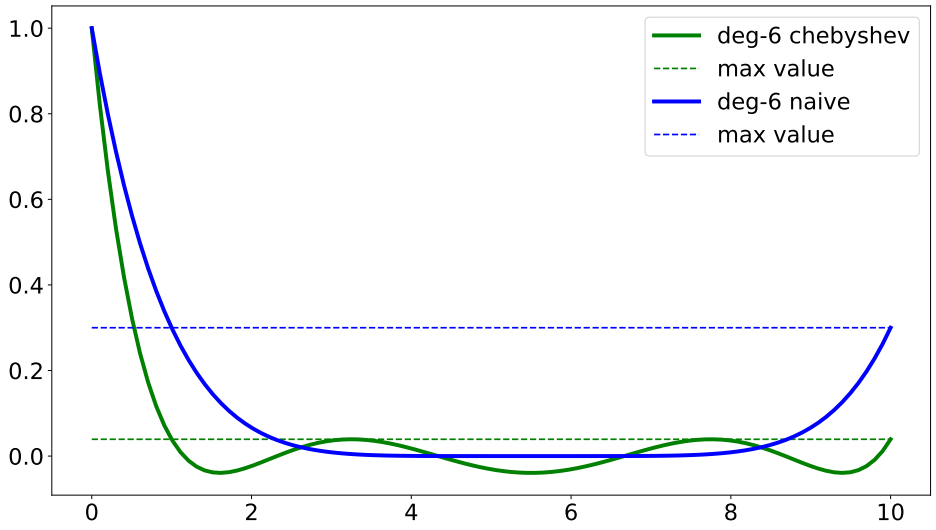

Figure 4: Rescaled Chebyshev VS Naive Polynomial

Thus, this establishes that the Chebyshev polynomial achieves the error bound:

$$\begin{aligned}\|X_{t+1} - X^*\| &\leqslant 2(1+\sqrt{\epsilon})^{-t} \|X_0 - X^*\| \\ &\approx 2(1+\sqrt{\frac{2}{\kappa}})^{-t} \|X_0 - X^*\| \\ &\leqslant 2\exp\left(-t\sqrt{\frac{2}{\kappa}}\right) \|X_0 - X^*\|.\end{aligned}$$

This means that for large κ, we get quadratic savings in the degree we need before the error drops off exponentially. Figure 5 shows the different rates of convergence, we clearly see that the

6.4.2 The Chebyshev recurrence relation

Due to the recursive definition of the Chebyshev polynomial, we directly get an iterative algorithm out of it. Transferring the recursive definition to our rescaled Chebyshev polynomial, we have:

$$P_{K+1}(a) = (\eta_k a + \gamma_k)P_k(a) + \mu_k P_{k-1}(a).$$

where we can work out the coefficients η_k, γ_k, μ_k from the recurrence definition. Since $P_k(0) = 1$, we must have $\gamma_k + \mu_k = 1$. This leads to a simple update rule for our iterates:

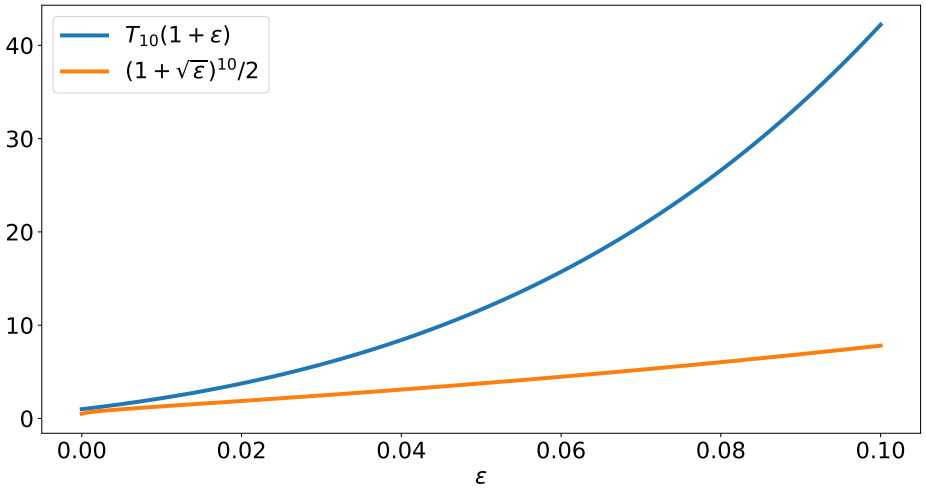

Figure 5: Convergence for naive polynomial and Chebyshev

$$\begin{aligned} X_{k+1} &= (\eta_k A + \gamma_k) X_k + (1 - \gamma_K) X_{k-1} - \eta_k b \\ &= (\eta_K A + (1 - \mu_k)) X_k + \mu_k X_{k-1} - \eta_k b \\ &= X_k - \eta_k (A X_k - b) + \mu_k (X_k - X_{k-1}). \end{aligned}$$

We see that the update rule above is actually very similar to plain gradient descent except for the additional term $\mu_k(x_k - x_{k-1})$. This term can be interpreted as a *momentum* term, pushing the algorithm in the direction of where it was headed before. In the next lecture, we'll dig deeper into momentum and see how to generalize the result for quadratics to general convex functions.

References