# Problem Set 1 for EE227C (Spring 2018): Convex Optimization and Approximation


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## Problem 1: Existence of the Subgradients

**(A)** Let $\mathcal{X}$ be a convex set. Prove that that given any convex function $f\colon \mathcal{X} \to \mathbb{R}$ and any $x \in \mathcal{X}$, there exists at least one vector $g$, called a *subgradient* of $f$ at $x$, such that $f(y) \geqslant f(x) + \langle g, y - x\rangle$ for all $y \in \mathcal{X}$.

To establish this claim, you may follow the steps below. We will only prove the existence under slightly restricted assumptions, but you can assume that the vector $g$ above exists in full generality.

**(A.1)** Define the *Epigraph* of $f$, $\mathrm{Epi}(f) := \{(x,t) \in \mathcal{X} \times \mathbb{R} : f(x) \leqslant t\}$. Prove the $\mathrm{Epi}(f)$ is convex.

**(A.2)** Recall the following definitions from real analysis:

**Definition 1** (Boundary and Interior)**.**

Using the separating hyperplane theorem from the notes (the full version, which applies to arbitrary convex sets not just compact ones), prove the supporting hyperplane theorem.

**Theorem 1** (Supporting Hyperplane)**.** Let $\mathcal{C} \subset \mathbb{R}^n$ be a convex set, and let $x \in \mathrm{Bd}(\mathcal{C})$. Then, there exists a nonzero $w \in \mathbb{R}^n$ such that, for all $y \in \mathcal{C}$, $\langle w, y - x\rangle \geqslant 0$.

*Hint: Find two (not-necessarily compact!) convex sets to apply the separating hyperplane theorem. You might want* $\mathrm{Int}(\mathcal{C})$ *to be one of them - and you should check that* $\mathrm{Int}(\mathcal{C})$ *is convex*

**(A.3)** Using part $i$) and $ii$), prove the existence of a subgradient at $x \in \mathcal{X}$. You may assume that $x \in \mathrm{Int}(\mathcal{X})$ to avoid annoying edge cases.

**(B)** Let $\{f_i\}_{i \in I}$ be a (possibly infinite, uncountable) family of convex functions, and suppose that $f_i(x) < \infty$ for all $x \in \mathcal{X}$. Show that $f(x) := \sup_i f_i(x)$ is convex on $\mathcal{X}$ (you may assume $f(x)$ is finite).

**(C)** Using what we've proven about subgradients, prove that a function $f : \mathcal{X} \to \mathbb{R}$ is convex if and only if it can be written as the supremum of affine functions (e.g. supremum of functions of the form $f_i(x) = \langle a_i, x \rangle + b_i$)

## Problem 2: Properties of Subgradients

Let $f$ be a convex function over a domain $\mathcal{X}$. We will assume $x \in \mathrm{Int}(\mathcal{X})$.

**(A)** Show by way of example that the subgradient is not necssarily unique, but *prove* that the set of all subgradients is closed and convex. We will denote this *set* $\partial f(x)$.

**(B)** Show that $f$ has a directional derivative in each direction. Use this to conclude that a convex $f$ is differentiable at $x$ only if $\partial f(x) = \{\nabla f(x)\}$.

**(C)** Show that if $g_1 \in \partial f_1(x)$ and $g_2 \in \partial f_2(x)$, then $g_1 + g_2 \in \partial(f_1 + f_2)(x)$.

**(D)** Let $f(x) = \sup_i g_i(x)$ which $g_i$ convex. Show that $\mathrm{Conv}\{\partial g_i(x) | g_i(x) = f(x)\} \subseteq \partial f$.

**(E)** Here, you will be asked to show a partial converse to the above statement. Suppose that $\mathcal{X}$ is a compact set, with non-empty interior, and let $f(x) = \max_{w \in \mathcal{X}} \langle w, x \rangle$. Prove that $\partial f(x) \subset \mathrm{Conv}\{w : \langle w, x \rangle = f(x)\}$. Hint: A key step is to show that if $v$ satisfies $\max_{w \in \mathcal{X} \cup \{v\}} \langle w, x \rangle = \max_{w \in \mathcal{X}} \langle w, x \rangle$ for all $x \in \mathbb{R}^n$, then the separating hyperplane theorem implies $v \in \mathrm{Conv}(\mathcal{X})$.

**(F)** Using the previous two subproblems, derive a formula for $\partial \| \cdot \|$, where $\| \cdot \|$ is an arbitrary norm. (Hint: Use 1.C)

## Problem 3: Subgradients of Norms

**(A)** Subgradient of the $\ell_1$ and $\ell_\infty$-norms

**(A.1)** Prove that, for all $x \in \mathbb{R}^n$, $\|x\|_1 = \sup_{y : \|y\|_\infty \leqslant 1} \langle x, y \rangle$, $\|x\|_\infty = \sup_{y : \|y\|_1 \leqslant 1} \langle x, y \rangle$.

**(A.2)** Compute $\partial \|x\|_1$ and $\partial \|x\|_\infty$

**(B)** Subgradient of the $L_1$-norm

**(B.1)** Let $A \in \mathbb{R}^{m\times n}$. Let $\sigma_i(\cdot)$ denote the $i$-th singular value of a matrix. Using the inequality $\sum_{i=1}^{\min(n,m)} \sigma_i(AB) \leqslant \sum_{i=1}^{\min(n,m)} \sigma_i(A)\sigma_i(B)$ for all $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ (this is non-trivial, see <this Stack Exchange>), prove the following: For all $X \in \mathbb{R}^{m\times n}$,

$$\|X\|_{\mathrm{op}} := \max_{Y\in\mathbb{R}^{m\times n}\|Y\|_{\mathrm{nuc}}\leqslant 1} \langle X, Y\rangle \text{ and } \|X\|_{\mathrm{nuc}} = \max_{Y\in\mathbb{R}^{m\times n}:\|Y\|_{\mathrm{op}}\leqslant 1} \langle X, Y\rangle\ , \tag{1}$$

where $\|X\|_{\mathrm{op}} := \sigma_{\max}(X)$, $\|Y\|_{\mathrm{nuc}} := \sum_{i=1}^{\min(n,m)} \sigma_i(Y)$, and $\langle X, Y\rangle := \mathrm{tr}(X^\top Y)$. You may want to refresh yourself on the relationship between traces, eigenvalues and singular values, and some trace tricks. Feel free to use the bound $\sum_i \lambda_i(A) \leqslant \sum_i \sigma_i(A)$ for any squared matrix $A$.

**(B.2)** Compute $\partial\|X\|_{\mathrm{op}}$ and $\partial\|X\|_{\mathrm{nuc}}$. Under what conditions is each subgradient unique? *Hint:* As you prove the result, one direction of the inclusion will be easy but the other may be more difficult to verify. Don't worry about this, but if you would like to be truly rigorous about the other inclusion, consider using the following lemma:

**Lemma 1.** Let $\mathcal{O}(n) := \{O \in \mathbb{R}^{n\times n} : O^\top O = I\}$, and let $X \in \mathbb{R}^{m\times n}$, and let $D \in \mathbb{R}^{m\times n}$ be diagonal (padded with zeros to account for $m \neq n$ mismatch). Then for any maximizers $O_1, O_2$ of $\max_{O_1\in\mathcal{O}(m),O_2\in\mathcal{O}(n)}\langle X, O_1 D O_2^\top\rangle$ is attained with $O_1$ and $O_2$, there exists an an SVD-decomposition of $X$ such that $X = U\Sigma V^\top \in \mathbb{R}^{m\times n}$ must satisfy $U^\top O_1$ and $O_2^\top V$ are diagonal matrices with $\{-1, 1\}$ on the diagonals.

**(C)** Let $\|\cdot\|$ be an arbitary norm (not necessarily Euclidean!) on $\mathbb{R}^n$. Define the dual norm $\|y\|_* := \sup_{x:\|x\|\leqslant 1}\langle x, y\rangle$.

**(C.1)** Show that the dual norm is a norm, and describe its subgradient.

**(C.2)** Show that for all $g, w \in \mathbb{R}^n$, $|\langle g, w\rangle| \leqslant \|g\|_*\|w\|$

**(C.3)** Let $f$ be a convex function on a convex domain $\mathcal{X}$. Show that $f$ is $L$-Lipschitz on $\mathcal{X}$ if an only if, for all $x \in \mathcal{X}$, all $g \in \partial f(x)$, and all $y \in \mathcal{X}$, $\langle g, y - x\rangle \leqslant L\|x - y\|$. Conclude that, if $x \in \mathrm{Int}(\mathcal{X})$, $f$ is $L$-Lipschitz, and $g \in \partial f(x)$ then $\|g\|_* \leqslant L$.

# Problem 4: Extensions for Gradient Descent

**(A)** In this exercise, you will show some generalizations of the basic grdient descent analysis we saw in class.

**(A.1)** Prove the following statement:

**Proposition 1.** Let $\Omega$ be a convex domain of radius $R$, and let $f$ be a convex function on $\Omega$. Let $x_0 \in \Omega$, and let $x_t = \Pi_\Omega(x_{t-1} - \eta g_t)$, where $\mathbb{E}[g_t|g_1, \ldots, g_{t-1}] \in \partial f(x_{t-1})$, and $\sup_t \mathbb{E}[\|g_t\|^2] \leqslant L^2$ and $\eta = \frac{LR}{\sqrt{T}}$. Prove that

$$\mathrm{Exp}[f(\frac{1}{T}\sum_{t=1}^{T} x_t)] \leqslant \inf_{x\in\Omega} f(x) + \ldots \tag{2}$$

You fill in the ....

**(A.2)** Prove the following statement:

**Proposition 2.** Let $\Omega$ be a convex domain of radius $R$, Let $f_1, f_2, \ldots, f_T$ be $L$-Lipschitz, convex functions on $\Omega$. Given any $x_0 \in \Omega$, let $x_t = \Pi_\Omega(x_{t-1} - \eta g_t))$, where $g_t \in \partial f_{t-1}(x_{t-1})$, and $\eta = \frac{LR}{\sqrt{T}}$. Prove that

$$\frac{1}{T}\sum_{t=1}^{T} f_t(x_t) \leqslant \inf_{x\in\Omega} \frac{1}{T}\sum_{t=1}^{T} f_t(x) + \ldots \tag{3}$$

You fill in the ....

**(B)** In this problem we show that in the stochastic setting, smoothness of the function $f$ does not help. Let $\Omega = [-1, 1]$, let $\sigma$ be a random variable with $\mathbb{P}\mathrm{r}[\sigma = 1] = \mathbb{P}\mathrm{r}[\sigma = -1] = 1/2$, fix an $\epsilon \in (0, 1/4)$. Let $z_1, z_2, \ldots, z_T$ be $T$ i.i.d random variables, such that $z_i|\sigma$ are mutually independent, and

$$\mathbb{P}\mathrm{r}[z_i = 1|\sigma] = 1/2 + \sigma\epsilon \text{ and } \mathbb{P}\mathrm{r}[z_i = -1|\sigma] = 1/2 - \sigma\epsilon \tag{4}$$

You will need the following information

**Lemma 2.** Let $\sigma$ and $z_1, z_2, \ldots, z_T$ be as above. Then there exists a universal constant $C$ such that, if $T \leqslant C\epsilon^{-2}$, any algorithm which returns an estimate $\widehat{\sigma}$ of $\sigma$ from observing $z_1, z_2, \ldots, z_T$ satisfies $\mathbb{P}\mathrm{r}[\widehat{\sigma} \neq \sigma] \geqslant \frac{1}{4}$, where $\mathbb{P}\mathrm{r}$ is taking over the randomness in $\sigma$, $z_1, \ldots, Z_T$, and any randomness in the algorithm.

**(B.1)** Construct a function on $f_\sigma$ such that $\mathbb{E}[z_i|\sigma] = \nabla f_\sigma(x)$ for all $x \in \Omega$. What is the optimum $x^*_\sigma$of $f_\sigma$? What is the "smoothness" of $f_\sigma$?

**(B.2)** Show that there is a universal constant $C'$ such that, for $T \leqslant C'\epsilon^{-2}$, any algorithm which produces iterates $x_1, x_2, \ldots$ ifrom access to the noisy gradients of $f_\sigma$, constructed above, must have $\mathbb{E}[f_\sigma(x_{T+1}) - \min_{x\in[-1,1]} f_\sigma(x)] \geqslant \epsilon$, where $\mathrm{Exp}_\sigma$ is taken over the randomness in $\sigma$, $z_1, \ldots, Z_T$, and any randomness in the algorithm.

## Problem 5: Generalized Projections

In this problem, we introduce a useful generalization of gradient descent. Let $\mathcal{X} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ be convex sets, and let $\Phi : \mathcal{D} \to \mathbb{R}$ be (a) a strictly convex, (b) continuously differentiable map (c) suppose $\|\nabla\Phi(x)\|$ diverges on $\mathrm{Bd}(\mathcal{D})$, and diverges for any sequence $x_n \in \mathcal{D}$ such that $\lim \|x_n\| = \infty$ and (d) $\nabla\Phi(\mathcal{D}) = \mathbb{R}^n$. We call $\Phi$ a *mirror map*.

**(A)** Define the *Bregman Divergence*

$$D_\Phi(x, y) = \Phi(x) - \Phi(y) - \nabla\Phi(y)^\top(x - y) \tag{5}$$

and the associated $\Phi$ projection

$$\Pi_{\mathcal{X}}^{\Phi}(y) := \arg\min_{x \in \mathcal{X}} D_{\Phi}(x, y) \tag{6}$$

Show that $\Phi(x) = \frac{1}{2}\|x\|_2^2$ is a mirror map for $\mathcal{D} = \mathbb{R}^n$, and compute $D_{\Phi}(x, y)$ and explain what $\Pi_{\mathcal{X}}^{\Phi}(y)$ corresponds to

**(B)** Prove that, for all $x \in \mathcal{X}$ and $y \in \mathcal{D}$,

$$(\nabla\Phi(\Pi_{\mathcal{X}}^{\Phi}(y)) - \nabla\Phi(y))^{\top}(\Pi_{\mathcal{X}}^{\Phi}(y) - x) \leqslant 0 \tag{7}$$

and conclude that

$$D_{\phi}(x, \Phi_{x}(y)) + D_{\phi}(\Phi_{x}(y), y) \leqslant D_{\Phi}(x, y) \tag{8}$$

What does this reduce to when $\Phi(x) = \frac{1}{2}\|x\|_2^2$? For the above, you may use the following lemma:

**Lemma 3.** Let $f$ be convex, and let $\mathcal{X}$ be a closed convex set on which $f$ is differentiable. Then $x^* \in \arg\min_{x \in \mathcal{X}} f(x)$, if and only if, for all $x \in \mathcal{X}$, $\nabla f(x^*)^{\top}(x^* - y) \leqslant 0$ for all $y \in \mathcal{X}$.

**(C)** Consider the following algorithm, known as mirror descent. Let $\mathcal{X} \subset \mathcal{D}$ and $\Phi$ be as above, let $f : \mathcal{X} \to \mathbb{R}$ be convex, let $x_1 \in \mathcal{X}$. Fix an $\eta > 0$. For $t \geqslant 1$, define $y_{t+1}$ such that $\nabla\Phi(y_{t+1}) - \nabla\Phi(x_t) = -\eta g_t$, where $g_t \in \partial f(x_t)$. Prove the following:

**Theorem 2.** Let $\|\cdot\|$ be an *arbitrary* norm on $\mathcal{X}$, and suppose that $\Phi$ is a $\kappa$ strongly-convex mirror map with respect to $\|\cdot\|$ on $\mathcal{X}$. Suppose that $f$ is L-Lipschitz with respect to $\|\cdot\|$. Prove that

$$f\left(\frac{1}{T}\sum_{s=1}^{T} x_s\right) - \min_{x \in \mathcal{X}} f(x) \leqslant \frac{D(x^*, x_1)}{\eta} + \eta\frac{L^2 T}{\kappa} \tag{9}$$

Recall that $\Phi$ is $\kappa$-strongly convex with respect to $\|\cdot\|$ if and only $\Phi(x) - \Phi(y) \leqslant \nabla\Phi(x)^{\top}(x - y) + \frac{\kappa}{2}\|x - y\|^2$.

**(D)** A common setup for mirror descent is on the simplex, where $\mathcal{D} : \{x : x[i] > 0 \forall i \in [n]\}$, $\mathcal{X} := \{x \in \mathcal{D} : \|x\|_1 = 1\}$, and $\Phi(x) = \sum_i x[i] \log x[i]$. Given an iterate $x_t$, compute the updates $y_{t+1}$ and $x_{t+1}$. Here, $x[i]$ is the $i$-th coordinate of $x$.

## Background

**(A)** A ball of radius $\epsilon$ about $x$ is the set $\{y : \|y - x\|_2 \leqslant \epsilon\}$. One can also consider balls with other norms, but they are all qualitatively equivalent to the Euclidean norm.

**(B)** For a set $\mathcal{X} \subset \mathbb{R}^n$, its closure $\overline{\mathcal{X}}$ is defined as the set of all $x \in \mathbb{R}^n$ (not necessarily in $\mathcal{X}$) such that, for all $\epsilon > 0$, there exists a $y \in \mathcal{X}$ such that $\|x - y\| \leqslant \epsilon$. In other words, for every $\epsilon > 0$, the ball of radius $\epsilon$ around $x$ intersects $\mathcal{X}$. $\mathrm{Int}(\mathcal{X})$ is defined as the set of all points $x \in \mathcal{X}$ such that there exists an $\epsilon > 0$ for which, for all $y : \|x - y\| \leqslant \epsilon$, $y \in \mathcal{X}$; it other words, for some $\epsilon > 0$, the ball of radius $\epsilon > 0$ around $x$ lies entirely in $\mathcal{X}$. Lastly, we define the boundary $\mathrm{Bd}(\mathcal{X}) := \overline{\mathcal{X}} - \mathrm{Int}(\mathcal{X}) = \{x \in \overline{\mathcal{X}} : x \notin \mathrm{Int}(\mathcal{X})\}$.

**(C)** A set is said to be *open* if $\mathcal{X} = \mathrm{Int}\mathcal{X}$, and closed if $\mathcal{X} \supseteq \mathrm{Bd}(\mathcal{X})$. A set $\mathcal{X} \subset \mathbb{R}^n$ is called compact if and only if it is closed and bounded.

**(D)** Given a set of real numbers $\{a_i\}_{i \in I}$ (here $I$ is an index set), $\sup_{i \in I}\{a_i\}$ is the smallest $a \in \mathbb{R}$ such that $a \geqslant a_i$ for all $i \in I$. If there is no such smallest $a$, $\sup\{a_i\}_{i \in I} = \infty$. Otherwise, $\sup\{a_i\}_{i \in I} = a_* \in \mathbb{R}$, and for every $\epsilon > 0$, there exists some $i = i(\epsilon) \in I$ such that $a_i \geqslant a_* - \epsilon$.

**(E)** When there exists an $i_*$ such that $a_{i_*} = \sup\{a_i\}_{i \in I}$, we say that the supremum is attained, and may replace sup with max for maximum. A finite set always has a maximum. When a maximum exists, we write $\arg\max_{i \in I}\{a_i\} := \{a_i : i \in I, a_i = \{\sup_{i' \in I} a_{i'}\}\}$ to denote the *set* of maximizers.

**(F)** $\inf\{a_i\}_{i \in I}$ is defined as the least $a \in \mathbb{R}$ such that $a_i \geqslant a$ for all $i \in I$, and analogous properties hold.

**(G)** Defining $f(x) = \sup_{i \in I} f_i(x)$, means that for every $x$, compute $\sup_{i \in I}\{f_i(x)\}$.

**(H)** A norm is $\|\cdot\|$ is a function from $\mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ such that $\|\alpha x\| = |\alpha|\|x\|$ for any $\alpha \in \mathbb{R}$, $\|x + y\| \leqslant \|x\| + \|y\|$, and $\|x\| \geqslant 0$, and $\|x\| = 0 \iff x = 0$.

**(I)** A sequence $x_n$ is said to converge to a limit $x_*$ if, for every $\epsilon \geqslant 0$, there is an $N = N(\epsilon)$ sufficiently large that $\|x_n - x_*\| \leqslant \epsilon$ for all $n \geqslant N$. We then write $\lim_{n \to \infty} x_n = x_*$.

**(J)** If $f$ is continuous and $\lim_{n \to \infty} x_n = x_*$, then $\lim_{x_n \to \infty} f(x_n) = f(x_*)$. If $f$ is continuous and $\mathcal{X}$ is compact, then $-\infty < \inf_{x \in \mathcal{X}} f(x) \leqslant \sup_{x \in \mathcal{X}} < \infty$. Moreover, there exist $x_-$ and $x_+ \in \mathcal{X}$ such that $f(x_i) = \inf_{x \in \mathcal{X}} f(x)$ and $x_+ = \sup_{x \in \mathcal{X}} f(x)$; hence, $\arg\min_{x \in \mathcal{X}} f(x)$ and $\arg\max_{x \in \mathcal{X}} f(x)$ are well-defined, and we can replace sup and max with inf and min.