Problem Set 1 for EE227C (Spring 2018): Convex Optimization and Approximation

Instructor: Moritz Hardt

Email: hardt+ee227c@berkeley.edu

Graduate Instructor: Max Simchowitz

Email: msimchow+ee227c@berkeley.edu

January 20, 2018

Problem 1: Subgradients and Gradients

A) Let $\mathcal{X}$ be a closed set. Prove that that given any convex function $f : \mathcal{X} \to \mathbb{R}$ and any $x \in \mathcal{X}$, there exists a nonempty set $\partial f(x)$ such that, for all $g \in \partial f(x)$, $f(y) \geqslant f(x) + \langle g, y - x \rangle$. Do so following the steps:

i) Define the *Epigraph* of f, $\mathrm{Epi}(f) := \{(x, t) \in \mathcal{X} \times \mathbb{R} : f(x) \leqslant t\}$. Prove the $\mathrm{Epi}(f)$ is convex. Let $\mathrm{Int}(\mathcal{C})$ and $\mathrm{Bd}(\mathcal{C})$ denote the interior and boundary of a set $\mathcal{C}$, respectively. Suppose that $(x, y) \in \mathrm{Bd}(\mathrm{Epi}(f))$. What can you say about the values of $f(x)$ and y?

ii) Using the separating hyperplane theorem from the notes (the full version, which applies to arbitrary convex sets not just compact ones), prove the supporting hyperplane theorem. (see Background)

Theorem 1 (Supporting Hyperplane)**.** Let $\mathcal{C}$ be a convex set, and let $x \in \mathrm{Bd}(\mathcal{C}($. Then, there exists a hyperplane $\mathcal{H}$ passing through x such that $\mathcal{C} \cap \mathcal{H} \subset \mathrm{Bd}(\mathcal{C})$.

Hint: Consider $\mathrm{Int}(\mathcal{C}) := \{x \in \mathcal{C} | x \notin \mathrm{Bd}(\mathcal{C})\}$*. This set is convex (you don't need to prove this). Find an appropriate other convex set to which you can apply the separating hyperplane theorem.*

iii) Using part $i)$ and $ii)$, prove the existence of a subgradient.

iv) Let $\{g_\alpha\}$ be a (possibly infinite, uncountable) family of convex functions, and suppose that $g_\alpha(x) < \infty$ for all $x \in \mathcal{X}$. Show that $g(x) := \sup_\alpha g_\alpha(x)$ is convex on $\mathcal{X}$ (you may assume it's finite).

v) Using what we've proven about subgradients, prove that a function $g : \mathcal{X} \to \mathbb{R}$ is convex if and only if it can be written as the supremum of affine functions (e.g. supremum of $g_\alpha(x) = \langle a_\alpha, x \rangle + b_\alpha$)

B) Some facts about subgradients

i) Show that the subgradient is not unique, but that the set of all subgradients is convex. We will denote this *set* $\partial f(x)$.

ii) Show that if f is differentiable at x, $\partial f(x) = \{\nabla f(x)\}$.

iii) Show that if $g_1 \in \partial f_1(x)$ and $g_2 \in \partial f_2(x)$, then $g_1 + g_2 \in \partial(f_1 + f_2)(x)$.

iv) Danskin's Theorem States

Theorem 2 (Danskin)**.** Let Ω be a convex domain, and let $f(x) = \sup_\alpha g_\alpha(x)$, where g_α are convex functions and finite on Ω. Suppose f is also finite on Ω. Then for all $x \in \Omega$, $\partial f = \mathrm{Conv}\{\partial g_\alpha(x) | \sup_x g_\alpha(x) = f(x)\}$

Prove one direction of the theorem, $\partial f \subseteq \mathrm{Conv}\{\partial g_\alpha(x) | \sup_x g_\alpha(x) = f(x)\}$

v) (Hard) Prove the other direction of Danskin's theorem. Here you may want to work in epigraph space.

Problem 2: Computing Some Subgradients

A) Norms and Matrix Norms

i)) Using the result about the suprema of convex functions, show that $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ are convex functions by expressing each norm as , , ,

Problem 3: Extensions for Gradient Descent

A) Generalizations of SGD

i)) Prove the following statement:

Proposition 1. Let Ω be a convex domain of radius R, and let f be a convex function on Ω. Let $x_0 \in \Omega$, and let $x_t = \Pi_\Omega(x_{t-1} - \eta g_t)$, where $\mathrm{Exp}[g_t | g_1, \dots, g_{t-1}] \in \partial f(x_t)$, and $\sup_t \mathrm{Exp}[\|g_t\|^2] \leqslant L$. Prove that

$$f(\frac{1}{T}\sum_{t=1}^{T} x_t) \leqslant \inf_{x \in \Omega} f(x) + \dots \tag{1}$$

You fill in the $\dots$.

ii) Prove the following statement:

Proposition 2. Let Ω be a convex domain of radius R, Let $f_1, f_2, \dots, f_T$ be L-Lipschitz, convex functions on Ω. Given any $x_0 \in \Omega$, let $x_t = \Pi_\Omega(x_{t-1} - \eta g_t))$, where $g_t \in \partial f_t(x_t)$, and $\eta = \frac{LR}{\sqrt{T}}$. Prove that

$$\frac{1}{T}\sum_{t=1}^{T} f(\frac{1}{T}\sum_{t=1}^{T} x_t) \leqslant \frac{1}{T}\sum_{t=1}^{T} f_t(x_t) \leqslant \inf_{x \in \Omega} \frac{1}{T}\sum_{t=1}^{T} f_t(x_t) + \frac{\dots}{} \tag{2}$$

You fill in the $\dots$.

B) In this problem we show that in the stochastic setting, smoothness of the function f does not help. Let $\Omega = [-1, 1]$, let σ be a random variable with $\mathbb{P}\mathrm{r}[\sigma = 1] = \mathbb{P}\mathrm{r}[\sigma = -1] = 1/2$, fix an $\epsilon \in (0, 1/4)$. Let $z_1, z_2, \dots, z_T$ be T i.i.d random variables, such that $z_i|\sigma$ are mutually independent, and

$$\mathbb{P}\mathrm{r}[z_i = 1|\sigma] = 1/2 + \sigma\epsilon \text{ and } \mathbb{P}\mathrm{r}[z_i = -1|\sigma] = 1/2 - \sigma\epsilon \tag{3}$$

You will need the following information

Lemma 1. Let σ and $z_1, z_2, \dots, z_T$ be as above. Then there exists a universal constant C such that, if $T \leqslant C\epsilon^2$, any algorithm which resturns an estimate $\widehat{\sigma}$ of σ from observing $z_1, z_2, \dots, z_T$ satisfies $\mathbb{P}\mathrm{r}[\widehat{\sigma} \neq \sigma] \geqslant \frac{1}{4}$.

i) Construct a function on f_σ such that $\mathrm{Exp}[z_i|\sigma] = \nabla f_\sigma(x)$ for all $x \in \Omega$. What is the optimum $x^*_\sigma$of f_σ? What is the "smoothness" of f_σ.

ii) Show that there universal constants c such that the following hold: Fix a $T \in \mathbb{N}$, and let $\mathcal{A}$ be an algorithm which is allowed to make T queries $x_t \in [-1, 1]$ and g_t, where $\mathcal{A}$ decides x_t, and recieves responses g_t such that $\mathrm{Exp}[g_t] = \nabla f(x_t)$ and $|g_t| \leqslant 1$ as.s. Then, there is a 0-smooth, 1-Lipschitz function f and a mechanism generating responses g_t such that the iterate x_T satisfies

$$\mathrm{Exp}[f(x_1)] - \inf_{x \in [-1,1]} f(x) \geqslant c/\sqrt{T} \tag{4}$$

iii) Answer to Problem 1(B)(iii) here.

C) Answer to Problem 1(C) here.

D) Answer to Problem 1(D) here.

E) Answer to Problem 1(E) here.

Bacgkround