Available online at www.sciencedirect.com
SCIENCE DIRECT
ADVANCES IN Mathematics
Advances in Mathematics 204 (2006) 572–618
www.elsevier.com/locate/aim

Tilting theory and cluster combinatorics

Aslak Bakke Buan[a], Bethany Rose Marsh[b,*], Markus Reineke[c], Idun Reiten[a], Gordana Todorov[d]

[a] *Institutt for Matematiske Fag, Norges Teknisk-Naturvitenskapelige Universitet, N-7491 Trondheim, Norway*
[b] *Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, England*
[c] *Mathematisches Institut, Fachbereich Mathematik und Informatik, Universität Münster, Einsteinstrae, D-48149 Münster, Germany*
[d] *Department of Mathematics, Northeastern University, Boston, MA 02115, USA*

Received 15 March 2005; accepted 8 June 2005

Communicated by Michael Hopkins
Available online 19 August 2005

Abstract

We introduce a new category $\mathcal{C}$, which we call the *cluster category*, obtained as a quotient of the bounded derived category $\mathcal{D}$ of the module category of a finite-dimensional hereditary algebra H over a field. We show that, in the simply laced Dynkin case, $\mathcal{C}$ can be regarded as a natural model for the combinatorics of the corresponding Fomin–Zelevinsky cluster algebra. In this model, the tilting objects correspond to the clusters of Fomin–Zelevinsky. Using approximation theory, we investigate the tilting theory of $\mathcal{C}$, showing that it is more regular than that of the module category itself, and demonstrating an interesting link with the classification of self-injective algebras of finite representation type. This investigation also enables us to conjecture a generalisation of APR-tilting.

MSC: primary 16G20; 16G70; secondary 16S99; 17B99

Keywords: APR tilting theory; Tilting module; Hom-configuration; Ext-configuration; Complement; Approximation theory; Self-injective algebra; Cluster algebra

* Corresponding author.
E-mail addresses: aslakb@math.ntnu.no (A.B. Buan), reinekem@math.uni-muenster.de (M. Reineke), idunr@math.ntnu.no (I. Reiten), todorov@neu.edu (G. Todorov).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2005.06.003

0. Introduction

In this paper, we introduce a new category, which we call the *cluster category*, associated with any finite-dimensional hereditary algebra H over a field k. This is defined as the quotient $\mathcal{C}$ of the bounded derived category $\mathcal{D}$ of finitely generated modules over H by the functor $F = \tau^{-1}[1]$, where τ denotes the AR-translation and [1] denotes the shift functor. The category $\mathcal{C}$ is triangulated, by a result of Keller [Kel03], and we show that it is also a Krull–Schmidt category. Our main aims are to show how this category can be used to study the tilting theory of H (and related algebras) and to show that it can be used as a model for the combinatorics of an associated Fomin–Zelevinsky [FZ02a] cluster algebra.

Hom-*configurations* are certain collections of non-isomorphic indecomposable objects in $\mathcal{D}$, and were considered in [Rie80a] in connection with the classification of self-injective algebras of finite representation type. We formulate analogous conditions using Ext^1 instead of Hom, and call the resulting collections Ext-*configurations*. We show that they exhibit a behaviour similar to that of Hom-configurations. In particular, they are invariant under the functor F (compare [BLR81], where it is shown that Hom-configurations exhibit a similar kind of invariance in the Dynkin case). As a consequence we can show that they are in 1–1 correspondence with what we call basic cluster-tilting objects in $\mathcal{C}$ (or tilting objects, for short). By showing that a basic tilting object in $\mathcal{C}$ is induced by a basic tilting module over some hereditary algebra derived equivalent to H, we prove that Ext-configurations, like Hom-configurations in the Dynkin case, are induced by basic tilting modules.

The category $\mathcal{C}$ provides an interesting "extension" of the module category of H. It is known that any almost complete basic tilting module $\overline{T}$ over H can be completed to a basic tilting module in at most two different ways [RS90,Ung90] and in exactly two different ways if and only if $\overline{T}$ is sincere [HU89]. However, in the extended category $\mathcal{C}$, the behaviour is more regular: an almost complete basic tilting object always has exactly two complements. We show further that, given one complement M to an almost complete basic tilting object $\overline{T}$, the other can be constructed using approximation theory from [AS80]. Indeed, we show that there is a triangle

$$M^* \to B \to M \to M^*[1]$$

in $\mathcal{C}$, where $B \to M$ is a minimal right add $\overline{T}$-approximation of M in $\mathcal{C}$ and M^* is the other complement to $\overline{T}$. Dually, there is a triangle

$$M \to B' \to M^* \to M[1]$$

in $\mathcal{C}$. In fact, we are able to show that two indecomposable objects M and M^* form such an exchange pair if and only if

$$\dim_{\mathrm{End}(M)} \mathrm{Ext}^1_{\mathcal{C}}(M, M^*) = 1 = \dim_{\mathrm{End}(M^*)} \mathrm{Ext}^1_{\mathcal{C}}(M^*, M).$$

The above results have some interesting interpretations in the Dynkin case in terms of cluster algebras, which were defined by Fomin and Zelevinsky [FZ02a]. These algebras were defined so that the cluster structure (when quantised) should encode multiplicative properties of the dual canonical basis of the quantised enveloping algebra of a semisimple Lie algebra over $\mathbb{C}$, and that it should model the (classical and quantised) coordinate rings of varieties associated to algebraic groups (now shown in several cases—see [BFZ02,Sco03].), with particular relevance to total positivity properties; there have already been many applications to other areas as well [CFZ02,BFZ02,FZ02c,FZ03,GSV02,MRZ03,P03].

The definition is as follows. Let $\mathbb{F} = \mathbb{Q}(u_1, u_2, \ldots, u_n)$ be the field of rational functions in indeterminates $u_1, u_2, \ldots, u_n$. Let $\mathbf{x} \subseteq \mathbb{F}$ be a transcendence basis over $\mathbb{Q}$, and let $B = (b_{xy})_{x,y\in\mathbf{x}}$ be an $n \times n$ sign-skew-symmetric integer matrix with rows and columns indexed by $\mathbf{x}$. In other words, we suppose that for all $x, y \in \mathbf{x}$, $b_{xy} = 0$ if and only if $b_{yx} = 0$, that $b_{xy} > 0$ if and only if $b_{yx} < 0$, and that $b_{xx} = 0$. Such a pair $(\mathbf{x}, B)$ is called a *seed*. Fomin and Zelevinsky [FZ02a,FZ03a] have defined a certain subring $\mathcal{A}(\mathbf{x}, B)$ of $\mathbb{F}$ associated to the seed $(\mathbf{x}, B)$, known as a *cluster algebra*. Given such a seed, and an element $z \in \mathbf{x}$, define a new element $z' \in \mathbb{F}$ via the *binary exchange relation*

$$zz' = \prod_{x\in\mathbf{x}, b_{xz}>0} x^{b_{xz}} + \prod_{x\in\mathbf{x}, b_{xz}<0} x^{-b_{xz}}. \tag{1}$$

In such circumstances, we say that z, z' form an *exchange pair*. Let $\mathbf{x}' = \mathbf{x} \cup \{z'\} \setminus \{z\}$, a new transcendence basis of $\mathbb{F}$. Let B' be the *mutation* of the matrix B in direction z (as defined in [FZ02a]). Then

$$b'_{xy} = \begin{cases} -b_{xy} & \text{if } x = z \text{ or } y = z, \\ b_{xy} + \frac{1}{2}(|b_{xz}|b_{zy} + b_{xz}|b_{zy}|) & \text{otherwise.} \end{cases}$$

The row and column labelled z in B are relabelled z' in B'. The pair $(\mathbf{x}', B')$ is called the *mutation* of the seed $\mathbf{x}$ in direction z. Let $\mathcal{S}$ be the set of seeds obtained by iterated mutation of $(\mathbf{x}, B)$. Then the set of *cluster variables* is, by definition, the union χ of the transcendence bases appearing in the seeds in $\mathcal{S}$. These bases are known as *clusters*, and the *cluster algebra* $\mathcal{A}(\mathbf{x}, B)$ is the subring of $\mathbb{F}$ generated by χ. Up to isomorphism of cluster algebras, it does not depend on the initial choice $\mathbf{x}$ of transcendence basis, so can be denoted $\mathcal{A}_B$. In general, coefficients appear in the relation (1), but here we take all of these coefficients to be 1 as this is enough to describe the connections with representation theory that we consider.

If its matrix is skew-symmetric, a seed $(\mathbf{x}, B)$ determines a quiver with vertices corresponding to its rows and columns, and b_{ij} arrows from vertex i to j whenever $b_{ij} > 0$. If χ is finite, the cluster algebra $\mathcal{A}_B$ is said to be of finite type. In [FZ03a], it is shown that, up to isomorphism, the cluster algebras of finite type can be classified by the Dynkin diagrams; they are precisely those for which there exists a seed whose corresponding quiver is of Dynkin type. In this case, Fomin and Zelevinsky associate a non-negative integer, known as the *compatibility degree*, to each pair of cluster variables (see Section 4). Two variables are said to be *compatible* provided that their compatibility degree is zero, and clusters are maximal compatible subsets of χ.

Suppose that H is the path algebra of a simply laced Dynkin quiver of type Δ. We show that the indecomposable objects in $\mathcal{C}$ are in 1–1 correspondence with the cluster variables in a cluster algebra $\mathcal{A}$ of type Δ. Using results from [MRZ03] we show that, for the two indecomposable objects X, Y in $\mathcal{C}$, dim $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$ is equal to the compatibility degree of the corresponding cluster variables.

The advantage of our approach here is that it allows us to give a direct interpretation of all clusters in terms of tilting objects: it follows from the above that the clusters of $\mathcal{A}$ are in 1–1 correspondence with the basic tilting objects in $\mathcal{C}$. We develop this relationship further: the existence of exactly two complements for any almost complete basic tilting object in $\mathcal{C}$ then corresponds to the fact that for any almost complete cluster there are exactly two ways to complete it to a cluster (by adding a cluster variable). A consequence of our result above is a new proof of the result [FZ03a, 3.5, 4.4] that two cluster variables form an exchange pair (i.e. appear in an exchange relation—see Eq. (1)) if and only if their compatibility degree is 1. We conjecture that in this case the middle term B in the triangle above is the direct sum of the indecomposable objects corresponding to the cluster variables appearing in one term of the exchange relation [FZ03a, 1.1], with the middle term B' of the dual triangle corresponding to the other term (see Conjecture 9.3), suggesting that it might be possible to construct the cluster algebra directly from $\mathcal{C}$. Finally, we are able to use the new perspective on tilting theory afforded by cluster algebras and the cluster category to conjecture a generalisation of APR-tilting (see [APR79]).

Caldero et al. [CCS04a] have recently associated a category to the cluster algebra of type A_n, giving a definition via the combinatorics of the corresponding cluster algebra. They have shown that this category is equivalent to the cluster category $\mathcal{C}$ we have associated to a Dynkin quiver of type A_n. Their approach enables them to generalise the denominator theorem of Fomin and Zelevinsky [FZ03a, 1.9], to an arbitrary cluster. Instead, in our approach we consider a more general situation (an arbitrary finite-dimensional hereditary algebra), and the connections with tilting theory and configurations of modules in the derived category. We develop links with cluster combinatorics for all simply laced Dynkin cases in a uniform way.

1. Cluster categories

In this section, we introduce what we call the cluster category of a finite-dimensional hereditary algebra, and discuss some of its elementary properties.

Let H be a finite-dimensional hereditary algebra over a field k, and denote by $\mathcal{D} = D^b(H)$ the bounded derived category of finitely generated H-modules with shift functor [1]. For any category $\mathcal{E}$, we will denote by $\operatorname{ind}\mathcal{E}$ the subcategory of isomorphism classes of indecomposable objects in $\mathcal{E}$; depending on the context we shall also use the same notation to denote the set of isomorphism classes of indecomposable objects in $\mathcal{E}$.

Let $G: \mathcal{D} \to \mathcal{D}$ be a triangle functor, which we also assume satisfies the following properties; see [Kel03].

(g1) For each U in $\operatorname{ind} H$, only a finite number of objects $G^n U$, where $n \in \mathbb{Z}$, lie in $\operatorname{ind} H$.
(g2) There is some $N \in \mathbb{N}$, such that $\{U[n] \mid U \in \operatorname{ind} H, n \in [-N, N]\}$ contains a system of representatives of the orbits of G on $\operatorname{ind} \mathcal{D}$.

We denote by $\mathcal{D}/G$ the corresponding factor category (see [BG81]). The objects are by definition the G-orbits of objects in $\mathcal{D}$, and the morphisms are given by

$$\operatorname{Hom}_{\mathcal{D}/G}(\widetilde{X}, \widetilde{Y}) = \coprod_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(G^i X, Y).$$

Here X and Y are objects in $\mathcal{D}$, and $\widetilde{X}$ and $\widetilde{Y}$ are the corresponding objects in $\mathcal{D}/G$ (although we shall often write such objects simply as X and Y). Note that it follows from our assumptions on G that $\operatorname{Hom}_{\mathcal{D}}(G^i X, Y) \neq 0$ for only a finite number of values of i. It is known from [Kel03] that $\mathcal{D}/G$ is a triangulated category and that the natural functor $\pi: \mathcal{D} \to \mathcal{D}/G$ is a triangle functor. The shift in $\mathcal{D}/G$ is induced by the shift in $\mathcal{D}$, and is also denoted by [1]. In both cases we write as usual $\operatorname{Hom}(U, V[1]) = \operatorname{Ext}^1(U, V)$. We then have

$$\operatorname{Ext}^1_{\mathcal{D}/G}(\widetilde{X}, \widetilde{Y}) = \coprod_{i \in \mathbb{Z}} \operatorname{Ext}^1_{\mathcal{D}}(G^i X, Y),$$

where X, Y are objects in $\mathcal{D}$ and $\widetilde{X}, \widetilde{Y}$ are the corresponding objects in $\mathcal{D}/G$. Note that since there are only finitely many values of i, such that $\operatorname{Hom}_{\mathcal{D}}(G^i X, Y)$ is not zero, there are also only finitely many values of i, such that $\operatorname{Ext}^1_{\mathcal{D}}(G^i X, Y)$ is not zero, for X, Y in $\mathcal{D}$. We remark that the quotient $D^b(H)/[2]$ was considered in [Hap85]; however, this quotient has quite different properties and is not closely linked with cluster algebras.

While several properties hold for arbitrary functors G satisfying (g1) and (g2), we shall mainly be concerned with the special choice of functor $F = \tau^{-1}[1]$, where τ is the AR-translation in $\mathcal{D}$ (which is induced by $D\operatorname{Tr}$ on non-projective indecomposable objects in $\operatorname{ind} H$, and where $\tau(P) = I[-1]$ when P is indecomposable projective and I denotes the indecomposable injective with $\operatorname{soc} I \simeq P/\underline{r}P$).

We shall see various reasons why the factor category $\mathcal{D}/F$ is especially nice. Because of the applications to cluster theory we call it the *cluster category* of H, and we denote it by $\mathcal{C}$.

If we are in the setting with H of finite representation type and k an algebraically closed field, then $\mathcal{D}$ (and thus $\mathcal{C}$) only depends on the underlying graph Δ of the quiver of H, and we write $\mathcal{C} = \mathcal{C}(\Delta)$. Then Δ is a simply laced Dynkin diagram. For this case we give a combinatorial construction of $\operatorname{ind} \mathcal{C}$. We recall the theory of translation quivers from [Rie80b]. If $\Gamma = (\Gamma_0, \Gamma_1)$ is any quiver, with vertices Γ_0 and arrows Γ_1, we recall that, if $x \in \Gamma_0$, then x^+ is the set of the end-points of arrows which start at x, while x^- denotes the set of starting points of arrows which end at x. A *stable translation quiver* is a quiver Γ, without any loops or multiple edges, together with a bijection $\tau : \Gamma_0 \to \Gamma_0$ (known as the *translation*) such that, for all $x \in \Gamma_0$, $x^- = \tau(x)^+$. A morphism of stable translation quivers is defined to be a quiver morphism which commutes with translation.

If Γ is a stable translation quiver, and $a : x \to y$ is an arrow of Γ, then there is a unique arrow $\sigma(a) : \tau(y) \to x$. The rule $a \mapsto \sigma(a)$ defines a bijection from Γ_1 to Γ_1, known as the *polarisation*. The *mesh category* associated to Γ has objects indexed by the vertices of Γ, and morphisms generated by the arrows of Γ, subject to the mesh relations (for all vertices y of Γ)

$$\sum_{a:x\to y} \sigma(a)a = 0.$$

If $\mathcal{E}$ is a Krull–Schmidt category with almost split sequences, we shall denote its AR-quiver by $\Gamma(\mathcal{E})$ (see [Rin84]).

Let Q be the quiver of H and let $\mathbb{Z}Q$ be the stable translation quiver associated to Q (see [Rie80b]). The vertices of $\mathbb{Z}Q$ are labelled by pairs (n, i) with n in $\mathbb{Z}$ and i a vertex of Q. Whenever there is an arrow in Q from i to j there is an arrow from (n, i) to (n, j), and an arrow from (n, j) to $(n + 1, i)$, and these are all the arrows in $\mathbb{Z}Q$. A translation τ is defined on $\mathbb{Z}Q$, just taking (n, i) to $(n - 1, i)$. In this way $\mathbb{Z}Q$ is a stable translation quiver. We denote the corresponding mesh category by $k(\mathbb{Z}Q)$. We have the following:

Proposition 1.1 (*Happel [Hap88, 5.6]*). *Let Q be any quiver of Dynkin type. Then the mesh category $k(\mathbb{Z}Q)$ is equivalent to* $\operatorname{ind}\mathcal{D}$.

It follows that (as a stable translation quiver), $\mathbb{Z}Q$ depends only on the underlying Dynkin diagram Δ, and not on Q. We therefore denote it $\mathbb{Z}\Delta$, and denote the corresponding mesh category by $k(\mathbb{Z}\Delta)$. The AR-quiver of $\mathcal{D}$ is $\Gamma(\mathcal{D}) = \mathbb{Z}\Delta$.

We recall that $F = \tau^{-1}[1]$ is an autoequivalence of $\mathcal{D}$, and therefore permutes the indecomposable objects, inducing a graph automorphism φ (via Proposition 1.1) of $\mathbb{Z}\Delta$. We note that the graph automorphisms induced by τ^{-1} and [1] are independent of the orientation Q, so φ is independent of Q. Since F commutes with τ on $\mathcal{D}$, φ is an automorphism of stable translation quivers. It follows that the quotient graph $\mathbb{Z}\Delta/\varphi$ is also a stable translation quiver, and we can form the corresponding mesh category; this is equivalent to the category $\operatorname{ind}\mathcal{C}(\Delta)$ defined above. The natural epimorphism of stable translation quivers $\pi : \mathbb{Z}\Delta \to \mathbb{Z}\Delta/\varphi$, taking the vertex v of $\mathbb{Z}\Delta$ to its φ-orbit $\pi(v)$, induces the functor π above.

Example. In Fig. 1, we show the AR-quiver of $\mathcal{C}$ in type A_3. The objects 1, 2 and 3 are identified with $1'$, $2'$ and $3'$ (so that, in some sense, the quotient is a Möbius strip).

We are mostly interested in the factor $\mathcal{C} = \mathcal{D}/F$, where $F = \tau^{-1}[1]$. The next properties, however, we state and prove in a more general setting.

$$
\begin{tikzcd}
 & 3 \arrow[dr] & & \circ \arrow[dr] & & \circ \arrow[dr] & & 1' \\
2 \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & 2' \arrow[ur] \arrow[dr] & \\
 & 1 \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & 3'
\end{tikzcd}
$$

Fig. 1. The AR-quiver of $\mathcal{C}$ in type A_3.

Proposition 1.2. *Let $\mathcal{D} = D^b(H)$ for a finite-dimensional hereditary k-algebra H, and let $G: \mathcal{D} \to \mathcal{D}$ be a triangle functor satisfying* (g1) *and* (g2). *Then the triangulated category $\mathcal{D}/G$ is a Krull–Schmidt category.*

Proof. Let $\widetilde{X}$ be in $\mathcal{D}/G$ induced by X in $\mathcal{D}$. We know that $X = X_1 \coprod \cdots \coprod X_n$ in $\mathcal{D}$, where each X_i is indecomposable, with local endomorphism ring. Since the functor $\pi: \mathcal{D} \to \mathcal{D}/G$ commutes with finite direct sums, we have $\widetilde{X} = \widetilde{X_1} \coprod \cdots \coprod \widetilde{X_n}$.

We then claim that $\mathrm{End}_{\mathcal{D}/G}(\widetilde{X_i})$ is local for each i. So let Y be in $\mathrm{ind}\,\mathcal{D}$. By definition, $\mathrm{Hom}_{\mathcal{D}/G}(\widetilde{Y}, \widetilde{Y}) = \coprod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(G^i Y, Y)$. It is easy to see that

$$
\mathrm{rad}(Y, Y) \coprod \left(\coprod_{i \neq 0} \mathrm{Hom}_{\mathcal{D}}(G^i Y, Y) \right)
$$

is a unique maximal ideal in $\mathrm{Hom}_{\mathcal{D}/G}(\widetilde{Y}, \widetilde{Y})$, which is hence a local ring. Thus, $\mathcal{D}/G$ is a Krull–Schmidt category. $\square$

We remark that triangles in $\mathcal{D}/G$ are not necessarily induced by those in $\mathcal{D}$. However, we have the following:

Proposition 1.3. *Let $\mathcal{D} = D^b(H)$ for a finite-dimensional hereditary k-algebra H, and let $G: \mathcal{D} \to \mathcal{D}$ be a triangle functor satisfying* (g1) *and* (g2). *Then $\mathcal{D}/G$ has almost split triangles induced by those in $\mathcal{D}$, and the AR-quiver is $\Gamma(\mathcal{D})/\varphi(G)$, where φ is the graph automorphism induced by G.*

Proof. Let $\widetilde{X}$ be an indecomposable object in $\mathcal{D}/G$, induced by X in $\mathcal{D}$. Let

$$
\tau X \xrightarrow{f} E \xrightarrow{g} X \xrightarrow{s} \tau X[1]
$$

be an almost split triangle in $\mathcal{D}$. Since $\pi: \mathcal{D} \to \mathcal{D}/G$ is a triangle functor, there is the induced triangle

$$
\widetilde{\tau X} \xrightarrow{\tilde{f}} \widetilde{E} \xrightarrow{\tilde{g}} \widetilde{X} \xrightarrow{\tilde{s}} \widetilde{\tau X}[1] \tag{2}
$$

in $\mathcal{D}/G$. Since $s \neq 0$, we clearly have $\widetilde{s} \neq 0$. Let $\widetilde{Z}$ be in $\operatorname{ind}\mathcal{D}/G$, induced by Z in $\operatorname{ind}\mathcal{D}$, with $\widetilde{Z} \not\simeq \widetilde{X}$, and let $\widetilde{h}: \widetilde{Z} \to \widetilde{X}$ be non-zero. Then $\widetilde{h} = \coprod h_i$, with $h_i \in \operatorname{Hom}_{\mathcal{D}}(G^i Z, X)$. Since $\widetilde{Z} \not\simeq \widetilde{X}$, we have $G^i Z \not\simeq X$ for all i and hence there is some $t_i: G^i Z \to E$, such that $g t_i = h_i$. Let $\widetilde{t} = \coprod t_i$. Then we have $\widetilde{g}\widetilde{t} = \widetilde{h}$, and hence $\widetilde{g}$ is right almost split. Similarly $\widetilde{f}$ is a left almost split map, and hence (2) is an almost split triangle, and the translation $\widetilde{\tau}$ in $\mathcal{D}/G$ is given by $\widetilde{\tau}\widetilde{X} = \widetilde{\tau X}$. Hence it also follows that $\Gamma(\mathcal{D})/\varphi$ is the AR-quiver for $\mathcal{D}/G$. $\square$

Let $D = \operatorname{Hom}_k(\ ,k)$. It is also useful to note that the Serre duality formula $D\operatorname{Ext}^1_{\mathcal{D}}(A, B) \simeq \operatorname{Hom}_{\mathcal{D}}(B, \tau A)$, valid in $D^b(H)$, induces an analogous formula for $\mathcal{D}/G$.

Proposition 1.4. *Let the notation and assumptions be as above. Then for $\widetilde{X}$ and $\widetilde{Y}$ in $\mathcal{D}^b(H)/G$ we have the Serre duality formula*

$$D\operatorname{Ext}^1_{\mathcal{D}/G}(\widetilde{X}, \widetilde{Y}) \simeq \operatorname{Hom}_{\mathcal{D}/G}(\widetilde{Y}, \widetilde{\tau}\widetilde{X})$$

functorial in both $\widetilde{X}$ and $\widetilde{Y}$.

Proof. We have

$$\operatorname{Ext}^1_{\mathcal{D}/G}(\widetilde{X}, \widetilde{Y}) = \coprod_i \operatorname{Hom}_{\mathcal{D}}(G^i X, Y[1]) = \coprod_i \operatorname{Ext}^1_{\mathcal{D}}(G^i X, Y)$$

and $\operatorname{Hom}_{\mathcal{D}/G}(\widetilde{Y}, \widetilde{\tau}\widetilde{X}) = \coprod_i \operatorname{Hom}_{\mathcal{D}}(G^i Y, \tau X)$. We then apply the corresponding formula for $D^b(H)$. $\square$

We end this section with some properties of $\mathcal{C}$. Let $\mathcal{S} = \operatorname{ind}(\operatorname{mod} H \vee H[1])$, i.e. the set consisting of the indecomposable H-modules, together with the objects $P[1]$, where P is an indecomposable projective H-module. Then it can be seen that $\mathcal{S}$ is a fundamental domain for the action of F on $\operatorname{ind}\mathcal{D}$, containing exactly one representative from each F-orbit on $\operatorname{ind}\mathcal{D}$. We recall that there is an oriented graph structure on $\operatorname{ind}\mathcal{D}$, with an arrow from object X to object Y if there is a non-zero map from X to Y.

Proposition 1.5. *Let X and Y be objects in $\mathcal{S}$.*

(a) *We have* $\operatorname{Hom}_{\mathcal{D}}(F^i X, Y) = 0$ *for all* $i \neq -1, 0$.
(b) *If X or Y does not lie on an oriented cycle in $\mathcal{D}$, then* $\operatorname{Hom}_{\mathcal{D}}(F^i X, Y) \neq 0$ *for at most one value of i.*

Proof. (a) We have $\operatorname{Hom}_{\mathcal{D}}(F^i X, Y) = \operatorname{Hom}_{\mathcal{D}}(\tau^{-i} X[i], Y)$. For $i \geqslant 1$, we clearly have $\operatorname{Hom}_{\mathcal{D}}(\tau^{-i} X[i], Y) = 0$. This is obvious for $i > 1$, and for $i = 1$ we only have to consider the case $Y = P[1]$ for P an indecomposable projective H-module. In that case we have $\mathrm{Hom}_{\mathcal{D}}(\tau^{-1}X, P)$, which must be 0. For $i \leqslant -2$ we have that

$$\mathrm{Hom}_{\mathcal{D}}(\tau^{-i}X[i], Y) = \mathrm{Ext}_{\mathcal{D}}^{-i}(\tau^{-i}X, Y) = 0.$$

(b) We have $\mathrm{Hom}_{\mathcal{D}}(F^{-1}X, Y) = \mathrm{Hom}_{\mathcal{D}}(\tau X[-1], Y) = \mathrm{Ext}^1_{\mathcal{D}}(\tau X, Y) \simeq D\,\mathrm{Hom}_{\mathcal{D}}(Y, \tau^2 X) \simeq D\,\mathrm{Hom}_{\mathcal{D}}(\tau^{-2}Y, X)$. If $\mathrm{Hom}_{\mathcal{D}}(X, Y) \neq 0$ and $\mathrm{Hom}_{\mathcal{D}}(\tau^{-2}Y, X) \neq 0$ then it is clear that X and Y lie on a cycle. $\square$

Proposition 1.6. *The indecomposable objects in $\mathcal{C}$ are precisely those of the form $\widetilde{X}$ for X an object in $\mathcal{S}$.*

Proof. It follows from Proposition 1.2 and its proof that the objects $\widetilde{X}$ for X an object in S are indecomposable objects in $\mathcal{C}$. Using the definition of morphisms in $\mathcal{C}$ it is easy to see, using Proposition 1.5(a), that if $X, Y \in S$ are, such that $\widetilde{X} \simeq \widetilde{Y}$ in $\mathcal{C}$ then X and Y are already isomorphic in $\mathcal{D}$. $\square$

Proposition 1.7. (a) *Let X and Y be in $\mathcal{D} = D^b(H)$ for a hereditary k-algebra H. Then we have*

$$\mathrm{Ext}^1_{\mathcal{D}}(Y, X) \simeq D\,\mathrm{Ext}^1_{\mathcal{D}}(FX, Y).$$

(b) *Let $\widetilde{X}$ and $\widetilde{Y}$ be in $\mathcal{C} = \mathcal{D}/F$. Then $\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) \simeq D\,\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{Y}, \widetilde{X})$.*

(c) *Let X, Y be indecomposable kQ-modules. Then*

$$\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) \simeq \mathrm{Ext}^1_{kQ}(X, Y) \coprod D\,\mathrm{Ext}^1_{kQ}(Y, X).$$

(d) *If X, Y are kQ-modules and X is projective then*

$$\mathrm{Hom}_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) \simeq \mathrm{Hom}_{kQ}(X, Y).$$

Proof. (a) We have

$$\mathrm{Ext}^1_{\mathcal{D}}(Y, X) \simeq D\,\mathrm{Hom}_{\mathcal{D}}(\tau^{-1}X, Y) \simeq D\,\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-1}X[1], Y) \simeq D\,\mathrm{Ext}^1_{\mathcal{D}}(FX, Y).$$

(b) follows directly from (a).

(c) We note that, by part (a), $D\,\mathrm{Ext}^1_{\mathcal{D}}(FX, Y) \simeq \mathrm{Ext}^1_{\mathcal{D}}(Y, X) \simeq \mathrm{Ext}^1_{kQ}(Y, X)$. Suppose that $i \neq 0, 1$. Then

$$\mathrm{Ext}^1_{\mathcal{D}}(F^iX, Y) \simeq \mathrm{Hom}_{\mathcal{D}}(F^iX, Y[1]) \simeq \mathrm{Hom}_{\mathcal{D}}(F^iX, \tau FY) \simeq \mathrm{Hom}_{\mathcal{D}}(F^{i-1}\tau^{-1}X, Y).$$

The result then follows from Proposition 1.5(a), noting that $\tau^{-1}X$ is an object in $\mathcal{S}$.

(d) If X is projective, then

$$D\,\mathrm{Hom}_{\mathcal{D}}(F^{-1}X, Y) = D\,\mathrm{Hom}_{D}(\tau X[-1], Y) \simeq D\,\mathrm{Hom}_{\mathcal{D}}(\tau X, Y[1]) \simeq \mathrm{Ext}^1_{\mathcal{D}}(\tau X, Y).$$

But $\tau X \simeq I[-1]$ for some injective module I, so

$$D\,\mathrm{Hom}_{\mathcal{D}}(F^{-1}X, Y) \simeq \mathrm{Ext}^1_{D}(I[-1], Y) \simeq \mathrm{Ext}_{D}(I, Y[1]) \simeq \mathrm{Ext}^2(I, Y) = 0.$$

The claim now follows from Proposition 1.5(a). $\square$

2. Configurations and tilting sets

It has been shown in [MRZ03] that there is an interesting connection between cluster algebras and tilting theory for hereditary algebras. Motivated by this, we start in this section our investigations of tilting theory in cluster categories.

We start by recalling that (combinatorial) Hom-configurations have been investigated for the stable translation quivers $\mathbb{Z}\Delta$ where Δ is a simply laced Dynkin diagram, in connection with the classification of the selfinjective algebras of finite representation type [Rie80a]. Here a subset $\mathcal{T}$ of the vertices in $\mathbb{Z}\Delta$ is a Hom-*configuration* if

(i) $\mathrm{Hom}_{k(\mathbb{Z}\Delta)}(X, Y) = 0$ for all $X \neq Y$ in $\mathcal{T}$, and
(ii) for any vertex Z in $\mathbb{Z}\Delta$ there is some $X \in \mathcal{T}$, such that $\mathrm{Hom}_{k(\mathbb{Z}\Delta)}(Z, X) \neq 0$.

Of course, this can be formulated for the category $D^b(H)$ when Δ is the underlying graph of the quiver of H. Hom-configurations for factors of $\mathbb{Z}\Delta$ are defined in the same way.

We here formulate analogous conditions using Ext^1 instead of Hom, in the more general setting of the categories $\mathcal{D} = D^b(H)$ or $\mathcal{D}/G = D^b(H)/G$ for an arbitrary finite-dimensional hereditary algebra H. We say that a subset $\mathcal{T}$ of non-isomorphic indecomposable objects in $\mathcal{D}$ or $\mathcal{D}/G$ is an Ext-*configuration* if

(E1) $\mathrm{Ext}^1(X, Y) = 0$ for all X and Y in $\mathcal{T}$, and
(E2) for any indecomposable $Z \notin \mathcal{T}$ there is some $X \in \mathcal{T}$, such that $\mathrm{Ext}^1(X, Z) \neq 0$.

Note that in (E2) it is clearly necessary to assume that $Z \notin \mathcal{T}$.

When we have a Hom-configuration $\mathcal{T}$ for $\mathbb{Z}\Delta$, with Δ Dynkin, it is known that $\mathcal{T}$ is stable under the action of τ^{m_Δ}. Here $m = m_\Delta$ is the smallest integer, such that in $k(\mathbb{Z}\Delta)$, the composition of the maps in a path of length greater than or equal to m, is zero. Here $m_{A_n} = n$, $m_{D_n} = 2n-3, m_{E_6} = 11, m_{E_7} = 17$ and $m_{E_8} = 29$ [BLR81]; in each case $m_\Delta = h_\Delta - 1$, where h_Δ is the Coxeter number of Δ. Further, a fundamental domain for the action of τ^{m_Δ} has exactly n objects from $\mathcal{T}$, where n is the number of vertices of Δ, and hence the number of non-isomorphic simple H-modules.

The corresponding role for Ext-configurations is played by the functor $F = \tau^{-1}[1]$ on $D^b(H)$, and this is another reason for the importance of this functor.

Proposition 2.1. *Let $\mathcal{T}$ be an* Ext-*configuration in $\mathcal{D} = D^b(H)$, and let M be in* $\operatorname{ind}\mathcal{D}$. *Then M is in $\mathcal{T}$ if and only if FM is in $\mathcal{T}$.*

Proof. Assume that M is in $\mathcal{T}$. It suffices to show that FM and $F^{-1}M$ are in $\mathcal{T}$. Suppose first that $F^{-1}M \notin \mathcal{T}$. Then by (E2) there is some X in $\mathcal{T}$, such that $\operatorname{Ext}^1_{\mathcal{D}}(X, F^{-1}M) \neq 0$, so $\operatorname{Ext}^1_{\mathcal{D}}(FX, M) \neq 0$. Then $\operatorname{Ext}^1_{\mathcal{D}}(M, X) \neq 0$ by Proposition 1.7, which gives a contradiction to (E1) since M and X are in $\mathcal{T}$. Hence we have $F^{-1}M \in \mathcal{T}$.

Suppose next that $FM \notin \mathcal{T}$. Then by (E2) there is an X in $\mathcal{T}$ such that $\operatorname{Ext}^1_{\mathcal{D}}(X, FM) \neq 0$. Since X is in $\mathcal{T}$ it follows that $F^{-1}X \in \mathcal{T}$ by the first part of the proof. Then $\operatorname{Ext}^1_{\mathcal{D}}(F^{-1}X, M) \simeq \operatorname{Ext}^1_{\mathcal{D}}(X, FM) \neq 0$, which contradicts (E1), since $F^{-1}X$ and M are both in $\mathcal{T}$. $\square$

There is a connection between Ext-configurations in $\mathcal{D}$ and in $\mathcal{D}/G$, with G satisfying (g1) and (g2), which is especially nice for $G = F$.

Proposition 2.2. (a) *Suppose that $\widetilde{\mathcal{T}}$ is an* Ext-*configuration in the factor category $\widetilde{D} = D^b(H)/G$. Then $\mathcal{T} = \{X \in D^b(H) \mid \widetilde{X} \in \widetilde{\mathcal{T}}\}$ is an* Ext-*configuration in $\mathcal{D}$.*

(b) *Let $\mathcal{T}$ be an* Ext-*configuration in $\mathcal{D}$. Then $\widetilde{\mathcal{T}} = \{\widetilde{X} \mid X \in \mathcal{T}\}$ is an* Ext-*configuration in $\mathcal{C} = D^b(H)/F$.*

Proof. (a) Let X and Y be in $\mathcal{T}$. Then $\widetilde{X}$ and $\widetilde{Y}$ are in $\widetilde{\mathcal{T}}$, so $\operatorname{Ext}^1_{\widetilde{D}}(\widetilde{X}, \widetilde{Y}) = 0$. Then $\operatorname{Ext}^1_{\mathcal{D}}(X, Y) = 0$, so (E1) holds.

Let $Z \in \operatorname{ind}\mathcal{D}$, such that Z is not in $\mathcal{T}$. Then $\widetilde{Z}$ is indecomposable in $\widetilde{D}$, with $\widetilde{Z} \notin \widetilde{\mathcal{T}}$. So by (E2) there is an $X \in \operatorname{ind}\mathcal{D}$ with $\widetilde{X} \in \widetilde{\mathcal{T}}$, such that $\operatorname{Ext}^1_{\widetilde{D}}(\widetilde{X}, \widetilde{Z}) \neq 0$ in the factor category. Then $\operatorname{Ext}^1_{\mathcal{D}}(G^n(X), Z) \neq 0$ for some n. But $G^n(X)$ lies in $\mathcal{T}$, since $\widetilde{G^n(X)} = \widetilde{X}$, so $\mathcal{T}$ satisfies (E2). Hence $\mathcal{T}$ is an Ext-configuration in $\mathcal{D}$.

(b) Let X, Y be in $\mathcal{T}$, so that $\widetilde{X}, \widetilde{Y}$ are in $\widetilde{\mathcal{T}}$. Suppose for a contradiction that $\operatorname{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) \neq 0$. Then there is some integer n such that $\operatorname{Ext}^1_{\mathcal{D}}(F^n(X), Y)) \neq 0$. Since $F^n(X) \in \mathcal{T}$ by Proposition 2.1, we have a contradiction to (E1) for $\mathcal{T}$. Hence $\widetilde{\mathcal{T}}$ also satisfies (E1).

Now suppose that $Y \in \operatorname{ind}\mathcal{D}$ is such that $\widetilde{Y} \notin \widetilde{\mathcal{T}}$. Then $Y \notin \mathcal{T}$, so there is an $X \in \mathcal{T}$, such that $\operatorname{Ext}^1_{\mathcal{D}}(X, Y) \neq 0$, by (E2) for $\mathcal{T}$. Then $\operatorname{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) \neq 0$. Since $\widetilde{X}$ is in $\widetilde{\mathcal{T}}$, it follows that $\widetilde{\mathcal{T}}$ satisfies (E2). Therefore $\widetilde{\mathcal{T}}$ is an Ext-configuration in $\mathcal{C}$. $\square$

The concept of Ext-configurations is closely related to tilting theory for hereditary algebras. Recall that for a hereditary algebra H, an H-module T is said to be a *tilting module* if

(a) $\operatorname{Ext}^1_H(T, T) = 0$, that is T is *exceptional*, and there is an exact sequence $0 \to H \to T_0 \to T_1 \to 0$ with T_0 and T_1 in $\operatorname{add} T$ (see [HR82]).

There are some useful equivalent characterisations [Bon81]:

(b) T is exceptional and has n non-isomorphic indecomposable direct summands (possibly with multiplicities), where n is the number of non-isomorphic simple modules, or

(c) T is exceptional and has a maximal number of non-isomorphic indecomposable direct summands.

A tilting module is said to be *basic* if all of its direct summands are non-isomorphic.

Motivated by this we say that in the categories $D^b(H)$ or $D^b(H)/G$ a set of non-isomorphic indecomposable objects $\mathcal{T}$ is a *tilting set* if it is an exceptional set, that is $\mathrm{Ext}^1(T, T') = 0$ for all T, T' in $\mathcal{T}$, and it is maximal with respect to this property. For $D^b(H)$ there is already the concept of tilting complexes, which is quite different, since there the vanishing of $\mathrm{Ext}^i_{\mathcal{D}}(T, T')$ for $i \neq 0$ is required. For the case $\mathcal{C} = D^b(H)/F$ we say that T in $\mathcal{C}$ is a *tilting object* if $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$ and T has a maximal number of non-isomorphic direct summands. We note that an object in $\mathcal{C}$ is a basic tilting object if and only if it is the direct sum of all objects in a tilting set $\mathcal{T}$. We shall later see that all tilting sets in $\mathcal{C}$ are finite, so that there will always be a corresponding basic tilting object.

We now discuss the connection between tilting sets, tilting objects and Ext-configurations.

Proposition 2.3. *Let $\widetilde{\mathcal{T}}$ be a set of non-isomorphic objects in* $\operatorname{ind}\mathcal{C}$. *Then $\widetilde{\mathcal{T}}$ is a tilting set if and only if it is an* Ext-*configuration.*

Proof. Suppose that $\widetilde{\mathcal{T}}$ is a tilting set in $\mathcal{C}$. Then $\widetilde{\mathcal{T}}$ satisfies (E1) by definition. Let $M \in \operatorname{ind}\mathcal{C}$ such that $M \notin \widetilde{\mathcal{T}}$. If $\mathrm{Ext}^1_{\mathcal{C}}(X, M) = 0$ for all X in $\widetilde{\mathcal{T}}$, then $\mathrm{Ext}^1_{\mathcal{C}}(M, X) = 0$ for all X in $\widetilde{\mathcal{T}}$ by Proposition 1.7. Hence $\widetilde{\mathcal{T}} \cup \{M\}$ is exceptional, contradicting the maximality of $\widetilde{\mathcal{T}}$. Hence there is some $X \in \widetilde{\mathcal{T}}$ such that $\mathrm{Ext}^1_{\mathcal{C}}(X, M) \neq 0$, so that (E2) holds, so $\widetilde{\mathcal{T}}$ is an Ext-configuration in $\mathcal{C}$. We are done if M is exceptional. By results of Section 3 we can assume that $\widetilde{\mathcal{T}} = \operatorname{add}\mathcal{T}$ for a tilting H-module $\mathcal{T}$. We can also assume that M is an H-module, since otherwise it is clearly exceptional in $\mathcal{C}$. Since $\mathrm{Ext}^1_{\mathcal{C}}(M, T) = 0$ we have $\mathrm{Ext}^1_H(T, M) = \mathrm{Ext}^1_H(M, T) = 0$ by Proposition 1.7, and hence M is in $\operatorname{Fac}\mathcal{T} \cap \operatorname{Sub}\mathcal{T} = \operatorname{add}\mathcal{T}$.

Next suppose $\widetilde{\mathcal{T}}$ is an Ext-configuration in $\mathcal{C}$. Then $\widetilde{\mathcal{T}}$ is exceptional. By (E2), for all $M \notin \widetilde{T}$ there is some $X \in \widetilde{\mathcal{T}}$ such that $\mathrm{Ext}^1_{\mathcal{C}}(X, M) \neq 0$. It follows that $\widetilde{\mathcal{T}}$ is maximal exceptional, and therefore a tilting set. $\square$

Note that in $\mathcal{D} = D^b(H)$ there are tilting sets which are not Ext-configurations. The problem is that $\mathrm{Ext}^1_{\mathcal{D}}(,)$ is not symmetric.

Example. Suppose that H is the path algebra of a quiver of type A_3. See Fig. 2 for the AR-quiver $\Gamma(\mathcal{D})$, indicating vertices which lie in $\mathcal{T}$ by filled-in circles, and those not in $\mathcal{T}$ by empty circles. The arrows are omitted. It is easy to check that $\mathrm{Ext}^1_{\mathcal{D}}(X, Y) = 0$ for all X, Y in $\mathcal{T}$, and that $\mathcal{T}$ is maximal with this property, since for all $M \notin \mathcal{T}$, there is $X \in \mathcal{T}$ such that $\mathrm{Ext}^1_{\mathcal{D}}(X, M) \neq 0$ or $\mathrm{Ext}^1_{\mathcal{D}}(M, X) \neq 0$. In fact for all $M \notin \mathcal{T}$, there is $X \in \mathcal{T}$ for which $\mathrm{Ext}^1_{\mathcal{D}}(X, M) \neq 0$, except for the module N corresponding to the encircled vertex. We note that $\tau N \in \mathcal{T}$ and $\mathrm{Ext}^1_{\mathcal{D}}(N, \tau N) \neq 0$. So $\mathcal{T}$ is a tilting set in $\mathcal{D}$. We note that $\mathcal{T}$ is not an Ext-configuration, since $\mathrm{Ext}^1_{\mathcal{D}}(X, N) = 0$ for all

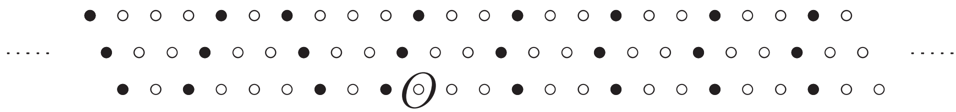


Fig. 2. A tilting set in $\mathcal{D}$ which is not an Ext-configuration.

$X \in \mathcal{T}$, although $N \notin \mathcal{T}$. Note also that this subset is not F-invariant, so could not be an Ext-configuration by Proposition 2.1.

We shall see in Section 3 that any tilting set in $\mathcal{C} = D^b(H)/F$ is induced by a basic tilting module over some hereditary algebra derived equivalent to H. Hence by Proposition 2.3 any Ext-configuration in $\mathcal{C}$ is induced by such a basic tilting module. This gives another analogy with Hom-configurations, since it is known that any Hom-configuration on $\mathbb{Z}\Delta$ for a Dynkin diagram Δ, is induced by a basic tilting H-module for a hereditary algebra H whose quiver has underlying graph Δ.

Let Δ be a simply laced Dynkin diagram, and denote by $\Pi(\Delta)$ the preprojective algebra of type Δ. Then it is known that $\Pi(\Delta)$ has finite representation type if and only if Δ is of type A_1, A_2, A_3 or A_4 (see [DR92]). In type A_1, the stable module category of $\Pi(\Delta)$ has only one indecomposable (simple) object. In types A_2, A_3 and A_4, the stable module category of $\Pi(\Delta)$ can be seen to coincide with the cluster category of type A_1, A_3 and D_6, respectively.

Let n be the number of non-isomorphic simple H-modules, and let t be the number of non-isomorphic indecomposable H-modules for a hereditary algebra of finite representation type. Then we have seen that a fundamental domain for the action of F on $D^b(H)$ has $t+n$ indecomposable objects, and we have mentioned that there are n members of an Ext-configuration. For comparison, a fundamental domain for the action of τ^{m_Δ} is known to have $2t-n$ indecomposable objects, with n members of a Hom-configuration. So we see that in general "more space" is needed to have a Hom-configuration. But in small cases it may be the same, as the following example shows.

Example. Let H be of type A_n. Then $t = \frac{n(n+1)}{2}$, so that we have $\frac{n(n+1)}{2} + n = \frac{n^2+3n}{2}$ members of a fundamental domain for F and $n(n+1) - n = n^2$ for τ^{m_Δ}. We see that for $n = 3$, we get 9 in both cases. In this case the preprojective algebra of H has 9 indecomposable non-projective modules and induces a Hom-configuration on $\mathbb{Z}A_3$.

3. Relationship to tilting modules

In this section we show basic tilting modules in $\operatorname{mod} H$ induce tilting objects in $\mathcal{C} = D^b(H)/F$ for a hereditary algebra H, and that in fact all the basic tilting objects in $\mathcal{C}$ can be obtained from basic tilting modules over hereditary algebras derived equivalent to H. This allows us to deduce additional information on the basic tilting objects in $\mathcal{C}$: A basic exceptional object in $\mathcal{C}$ can be extended to a basic tilting object, and the number of indecomposable direct summands in a basic tilting object is the number n of non-isomorphic simple H-modules. In particular a basic exceptional object in $\mathcal{C}$ with n non-isomorphic indecomposable direct summands is a basic tilting object in $\mathcal{C}$.

We start with the following immediate relationship between exceptional objects in $\operatorname{mod} H$ and in $\mathcal{C}$.

Lemma 3.1. *Let T be an H-module. Then T is exceptional if and only if T is an exceptional object in $\mathcal{C}$.*

Proof. This follows directly from Proposition 1.7(c). $\square$

We use this to show the following.

Proposition 3.2. *Let H be a hereditary algebra with n non-isomorphic simple modules, and let T be a basic exceptional object in $\mathcal{C}$. Then T can be extended to a basic tilting object.*

Proof. We claim that any basic exceptional object T' in $\mathcal{C}$ has at most $2n$ indecomposable summands. Let $T_1, \ldots, T_r, T_{r+1}, \ldots T_t$ be indecomposable objects in $\operatorname{mod} H \vee H[1]$, such that

$$T_1 \coprod \cdots \coprod T_r \coprod \cdots \coprod T_t$$

determines T', where $T_1, \ldots, T_r$ are in $\operatorname{mod} H$ and $T_{r+1}, \ldots, T_t$ are summands of $H[1]$. Then

$$T_1 \coprod \cdots \coprod T_r$$

is a basic exceptional H-module, so that $r \leqslant n$, and hence $t \leqslant 2n$. In particular any basic exceptional object having T as a direct summand has at most $2n$ indecomposable direct summands, and hence T can be extended to a maximal basic exceptional object in $\mathcal{C}$, which is then by definition a basic tilting object. $\square$

By Lemma 3.1, a basic tilting H-module gives rise to a basic exceptional object in $\mathcal{C}$ (as indecomposable kQ-modules are isomorphic as modules if and only if they are isomorphic in $\mathcal{C}$). We shall show that this is in fact a basic tilting object, and that any basic tilting object in $\mathcal{C}$ can be obtained this way.

Theorem 3.3. (a) *Let T be a basic tilting object in $\mathcal{C} = D^b(H)/F$, where H is a hereditary algebra with n simple modules.*

(i) *T is induced by a basic tilting module over a hereditary algebra H', derived equivalent to H.*
(ii) *T has n indecomposable direct summands.*

(b) *Any basic tilting module over a hereditary algebra H induces a basic tilting object for $\mathcal{C} = D^b(H)/F$.*

Proof. (a) (i) Let T be a basic tilting object in $\mathcal{C} = D^b(H)/F$. Let $T_1, \ldots, T_r$ be indecomposable objects in $\operatorname{mod} H \vee H[1]$ inducing T. If no T_i is a summand of $H[1]$, then $T_1 \coprod \cdots \coprod T_r$ is a basic exceptional H-module which we claim is a basic tilting module. If not, we get a basic tilting module by adding a non-zero module as summand. But then this will give rise to a basic exceptional object in $\mathcal{C}$ properly containing T as a direct summand, which is a contradiction to T being a basic tilting object in $\mathcal{C}$.

If no T_i is projective, we have

$$\{T_1, \ldots, T_r\} \subset \tau_{\mathcal{D}}^{-1}(\operatorname{mod} H)$$

and then $T_1 \coprod \cdots \coprod T_r$ is a basic tilting module over a hereditary algebra derived equivalent to H (in fact isomorphic to H, but with a different embedding into $D^b(H)$). Assume now that some T_i is projective. Let first H be of infinite representation type. We assume that there are some T_j which are summands of $H[1]$ (otherwise we are done, by the above argument). If T has no injective direct summands, then $\tau_{\mathcal{C}}^{-1}T$ can be represented by a module in $\operatorname{mod} H$. If T has an injective direct summand (such that $\tau_{\mathcal{C}}^{-1}T$ has a summand in $H[1]$), we can apply $\tau_{\mathcal{C}}^{-1}$ again. It is clear that there is a t, such that $\tau_{\mathcal{C}}^{-t}T$ can be represented by a module in $\operatorname{mod} H$. Hence, T is a module over a hereditary algebra derived equivalent to H, and we proceed as above.

Let now H be of finite representation type, and we use the same notation as above, with $T_1, \ldots, T_r$ in $\operatorname{mod} H \vee H[1]$. We claim that for any simple projective module S not in $\operatorname{add} T$, there is a path to some T_i. Since T is a basic tilting object, we have $\operatorname{Ext}^1_{\mathcal{C}}(T, S) \neq 0$, and hence $\operatorname{Hom}_{\mathcal{C}}(S, \tau_{\mathcal{C}}T) \neq 0$. Since $\operatorname{Hom}_{\mathcal{D}}(S, F(\tau_{\mathcal{D}}T)) = \operatorname{Hom}_{\mathcal{D}}(S, T[1]) = 0$, we must have $\operatorname{Hom}_{\mathcal{D}}(S, \tau_{\mathcal{D}}T) \neq 0$ (using Proposition 1.5(a)), and consequently we have a path of the desired type. Denote by $\alpha(H)$ the sum of the lengths of all paths (where paths through the same sequence of vertices are counted only once) from a simple projective H-module which is not in $\operatorname{add} T$, to some T_i. By possibly replacing H by a derived equivalent hereditary algebra, we can assume that $\alpha(H)$ is smallest possible, when all T_i are in $\operatorname{mod} H \vee H[1]$. If $\alpha(H) > 0$, there is some simple projective H-module S not in $\operatorname{add} T$. By performing an APR-tilt (see [APR79]) using the basic tilting module $M = \tau^{-1}S \coprod P$, where $H = S \coprod P$, to get $H' = \operatorname{End}_H(M)^{\mathrm{op}}$, it is easy to see that $\alpha(H') < \alpha(H)$, and that H' satisfies the desired properties. This contradiction implies that $\alpha(H) = 0$, so that all simple projective H-modules are in $\operatorname{add} T$.

We next want to show that no T_i is a summand of $\tau_{\mathcal{D}}^{-1}H$. Assume to the contrary that there is an indecomposable projective H-module P with $\tau_{\mathcal{D}}^{-1}P$ in $\operatorname{add} T$. There is a simple projective H-module S with $\operatorname{Hom}_H(S, P) \neq 0$, and as we have seen, it is in $\operatorname{add} T$. Since $\operatorname{Ext}^1_{\mathcal{D}}(\tau^{-1}P, S) \simeq \operatorname{Hom}_H(S, P)$, we have a contradiction to T being exceptional. Hence no T_i is a summand of $\tau_{\mathcal{D}}^{-1}H$.

Choose H' derived equivalent to H, such that $\tau_{\mathcal{D}}^{-2}(\operatorname{mod} H \vee H[1]) = \operatorname{mod} H' \vee H'[1]$. Since no T_i is a summand of $\tau_{\mathcal{D}}^{-1}H$, no T_i is a summand of $H'[1]$ (now regarding the

T_i as objects in $\operatorname{mod} H' \vee H'$ [1]; see Proposition 1.6). So T is a basic exceptional H'-module which has to be a basic tilting module.

(a) (ii) This is clearly a consequence of part (i).
(b) Let T be a basic exceptional object in $\mathcal{C}$ induced by a basic tilting H-module.

Then T has n indecomposable direct summands, and can be extended to a basic tilting object by Proposition 3.2. But any basic tilting object has n indecomposable direct summands, and consequently T is a basic tilting object in $\mathcal{C}$. $\square$

We note that the basic tilting modules of kQ are in bijection with the Hom-configurations of $\mathcal{D}$ [BLR81]. The above Theorem indicates a link between tilting sets in $\mathcal{C}$ and basic tilting modules, which, in the light of Propositions 2.2 and 2.3, gives a link between Ext-configurations in $\mathcal{D}$ and basic tilting modules. It would be interesting to find a direct link between the Hom-configurations and the Ext-configurations of $\mathcal{D}$.

The previous investigation holds more generally in the setting of a hereditary abelian category $\mathcal{H}$ with finite-dimensional Hom-spaces and Ext-spaces and with a tilting object T, as introduced and investigated in [HRS96]. We still have Serre duality for $D^b(\mathcal{H})$ and hence almost split triangles, (see [HRS96,RV02]) and Keller's theorem on $\mathcal{C}_{\mathcal{H}} = D^b(\mathcal{H})/F$ being triangulated is proved in this generality [Kel03]. It is also known in this setting that a basic object T in $\mathcal{H}$ is a tilting object if and only if $\operatorname{Ext}^1_{\mathcal{H}}(T,T) = 0$ and the number of indecomposable direct summands of T is equal to the rank of the Grothendieck-group of $\mathcal{H}$. Furthermore, any exceptional object can be extended to a tilting object, see [HU89]. Using this, the previous results carry over to this setting.

When $\mathcal{H}$ is connected and not equivalent to some $\operatorname{mod} H$ for a hereditary algebra H, it is known that $\mathcal{H}$ has no non-zero projective or injective objects, see [HU03].

In this case it is clear that $\operatorname{ind} \mathcal{H}$ is a fundamental domain for $\mathcal{C}$ under the action of F. For if X is in $\operatorname{ind} \mathcal{H}$, then $F^i X$ is in $\operatorname{ind} \mathcal{H}[i]$, so that no other object in the F-orbit of X is in $\operatorname{ind} \mathcal{H}$. And given any Y in $\operatorname{ind} \mathcal{D}$, we have $Y[i] \in \mathcal{H}$ for some i, and so $F^i Y = \tau^{-i} Y[i]$ is in $\mathcal{H}$ since $\mathcal{H}$ is closed under positive and negative powers of τ. We then get the following.

Proposition 3.4. *Let $\mathcal{H}$ be a hereditary abelian k-category over a field k, with finite-dimensional* Hom-*spaces and* Ext^1-*spaces. Assume $\mathcal{H}$ has no non-zero projective or injective objects, and assume that $\mathcal{H}$ has a tilting object. Then there is a natural* 1–1 *correspondence between the exceptional objects in $\mathcal{H}$ and in $\mathcal{C}_{\mathcal{H}} = D^b(\mathcal{H})/F$. The correspondence preserves tilting objects.*

As has previously been done for $\operatorname{mod} H$ and other hereditary categories $\mathcal{H}$ with tilting objects (see [HU03]), one can associate to $\mathcal{C}$ a tilting graph whose vertices are the basic tilting objects, and where there is an edge between two vertices if the corresponding tilting objects have all but one indecomposable summands in common. It is known that for $\operatorname{mod} H$ the graph is not always connected, but this is the case for the hereditary abelian Ext-finite k-categories with tilting objects derived equivalent, but not equivalent, to mod H [HU03], provided k is algebraically closed. Using this last result, we obtain the following.

Proposition 3.5. *Let k be an algebraically closed field. For an indecomposable hereditary k-algebra H, the tilting graph of $\mathcal{C} = \mathcal{C}_H$ defined above is connected.*

Proof. If H is given by a Dynkin diagram, the tilting graph for mod H is connected, as pointed out in [HU03], and hence the same is true for the tilting graph of $\mathcal{C}$.

If H is of infinite representation type, it is known that there is some indecomposable hereditary abelian k-category $\mathcal{H}$ with tilting objects, finite-dimensional Hom-spaces and Ext-spaces and no non-zero projective or injective objects, with $D^b(\text{mod } H)$ equivalent to $D^b(\mathcal{H})$ (see e.g. [HU03]). Consequently $\mathcal{C}_H$ is equivalent to $\mathcal{C}_{\mathcal{H}}$. It follows from Proposition 3.4 that the tilting graph for $\mathcal{H}$ and $\mathcal{C}_{\mathcal{H}}$ are isomorphic. Since it is proved in [HU03] that the tilting graph of $\mathcal{H}$ is connected, our result follows. $\square$

4. Connections with cluster algebras

In this section, we assume that H is the path algebra of a simply laced quiver of Dynkin type, with underlying graph Δ, and that k is algebraically closed. We denote by $\mathcal{A} = \mathcal{A}(\Delta)$ the corresponding cluster algebra [FZ03a]. Let Φ denote the set of roots of the corresponding Lie algebra, and let $\Phi_{\geqslant -1}$ denote the set of *almost positive* roots, i.e. the positive roots together with the negatives of the simple roots. The cluster variables of $\mathcal{A}$ are in 1–1 correspondence with the elements of $\Phi_{\geqslant -1}$. Fomin and Zelevinsky associate a non-negative integer $(\alpha\|\beta)$, known as the *compatibility degree*, to each pair α, β of almost positive roots. This is defined in the following way. Let s_i be the Coxeter generator of the Weyl group of Φ corresponding to i, and let σ_i be the permutation of $\Phi_{\geqslant -1}$ defined as follows:

$$\sigma_i(\alpha) = \begin{cases} \alpha & \alpha = -\alpha_j, \ \ j \neq i, \\ s_i(\alpha) & \text{otherwise.} \end{cases}$$

Let $I = I^+ \sqcup I^-$ be a partition of the set of vertices I of Δ into completely disconnected subsets and define:

$$\tau_\pm = \prod_{i \in I^\pm} \sigma_i.$$

Then $(\ \|\ )$ is defined by setting $(-\alpha_i\|\beta)$ to be the coefficient of α_i in β, and by specifying that it is $\tau_\pm$-invariant. The clusters in $\mathcal{A}$ correspond to the maximal subsets of $\Phi_{\geqslant -1}$ which are pairwise compatible.

In [MRZ03], it was shown that the combinatorics of $\mathcal{A}$ could be obtained from the category of *decorated representations* of a quiver Q with underlying graph Δ. In particular, this allowed the generalised associahedra (Stasheff polytopes) of Chapoton et al. [CFZ02] to be constructed directly from the representation theory of Q, and gave, for the first time, a uniform formula for the number of basic tilting modules over kQ in terms of the degrees of the corresponding Weyl group. The compatibility degree, key to the construction of the associahedron, was interpreted as the dimension of a certain bifunctor from the decorated category to the category of finite-dimensional vector spaces, in the case where the quiver was alternating. This bifunctor can be regarded as a symmetrised version of Ext^1.

In this section, we will show that such a construction can be made in a more symmetric way, via the category $\mathcal{C} = D^b(H)/F$. This approach has the advantage that the category $\mathcal{C}$ is independent of the orientation of the quiver considered. We show that, when the indecomposable objects of $\mathcal{C}$ are labelled appropriately with decorated representations (in a way dependent on the orientation of the quiver) the dimension of an Ext^1-group coincides with the dimension of the symmetrised bifunctor mentioned above. Thus, when $\mathcal{C}$ is labelled in a way corresponding to the alternating quiver, the combinatorics of the corresponding cluster algebra is recovered in terms of Ext^1-groups of $\mathcal{C}$. In particular, we will show that the clusters are in 1–1 correspondence with the basic tilting objects in $\mathcal{C}$.

We first of all show that the Ext^1-groups in $\mathcal{C}$ coincide with the symmetrised Ext^1-groups used for the decorated representations in [MRZ03]. Recall that in [MRZ03] the quiver Q, with vertices Q_0 and arrows Q_1, is replaced by a "decorated" quiver $\widetilde{Q}$, with an extra copy $Q_0^- = \{i_- : i \in Q_0\}$ of the vertices of Q (with no arrows incident with the new copy). A module M over $k\widetilde{Q}$ can be written in the form $M^+ \coprod V$, where $M^+ = \coprod_{i\in Q_0} M_i^+$ is a kQ-module, and $V = \coprod_{i\in Q_0} V_i$ is a Q_0-graded vector space over k. Its *signed dimension vector*, $\mathbf{sdim}(M)$ is the element of the root lattice of the Lie algebra of type Δ given by

$$\mathbf{sdim}(M) = \sum_{i\in Q_0} \dim(M_i^+)\alpha_i - \sum_{i\in Q_0} \dim(V_i)\alpha_i,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the simple roots. By Gabriel's Theorem, the indecomposable objects of $k\widetilde{Q}$-mod are parametrised, via $\mathbf{sdim}$, by the almost positive roots, $\Phi_{\geqslant -1}$, of the corresponding Lie algebra, i.e. the positive roots together with the negative simple roots. The positive roots correspond to the indecomposable kQ-modules, and the negative simple roots correspond to the simple modules associated with the new vertices. We denote the simple module corresponding to the vertex i_- by S_i^-. Let $M = M^+ \coprod V$ and $N = N^+ \coprod W$ be two $k\widetilde{Q}$-modules. The symmetrised Ext^1-group for this pair of modules is defined to be:

$$E_{kQ}(M,N) := \mathrm{Ext}^1_{kQ}(M^+,N^+) \coprod \mathrm{Ext}^1_{kQ}(N^+,M^+) \coprod \mathrm{Hom}^{Q_0}(M^+,W) \coprod \mathrm{Hom}^{Q_0}(V,N^+),$$

where Hom^{Q_0} denotes homomorphisms of Q_0-graded vector spaces.

We define a map ψ_Q from $\operatorname{ind}\mathcal{C}$ to the set of isomorphism classes of indecomposable $k\widetilde{Q}$-modules as follows. Let $\widetilde{X} \in \operatorname{ind}\mathcal{C}$. We can assume that one of the following cases holds:

(1) X is an indecomposable kQ-module M^+.
(2) $X = P_i[1]$ where P_i is the indecomposable projective kQ-module corresponding to vertex $i \in Q_0$.

We define $\psi_Q(\widetilde{X})$ to be M^+ in Case (1), and to be S_i^- in Case (2).

The following is clear:

Proposition 4.1. *The map ψ_Q is a bijection between* $\operatorname{ind} \mathcal{C}$ *and the set of isomorphism classes of indecomposable $k\widetilde{Q}$-modules (i.e. indecomposable decorated representations). It follows that $\gamma_Q := \mathbf{sdim} \circ \psi_Q$ is a bijection between* $\operatorname{ind} \mathcal{C}$ *and* $\Phi_{\geqslant -1}$ *(and thus induces a bijection between* $\operatorname{ind} \mathcal{C}$ *and the set of cluster variables).*

For $\alpha \in \Phi_{\geqslant -1}$ we denote by $M_Q(\alpha)$ the element of $\operatorname{ind} \mathcal{C}$, such that $\gamma_Q(M_Q(\alpha)) = \alpha$.

Proposition 4.2. *Let X, Y be objects of $\mathcal{D}$. Then*

$$E_{kQ}(\psi_Q(\widetilde{X}), \psi_Q(\widetilde{Y})) \simeq \operatorname{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}).$$

Proof. Without loss of generality, we can assume that X and Y are either indecomposable kQ modules or of the form $P_i[1]$ where P_i is an indecomposable projective kQ-module. We first of all consider the case where $X = M^+$ and $Y = N^+$ are both indecomposable kQ-modules. Then $E_{kQ}(\psi_Q(\widetilde{X}), \psi_Q(\widetilde{Y})) = \operatorname{Ext}^1_{kQ}(M^+, N^+) \coprod \operatorname{Ext}^1_{kQ}(N^+, M^+)$ which is isomorphic to $\operatorname{Ext}^1_{\mathcal{C}}(\widetilde{M^+}, \widetilde{N^+})$ by Proposition 1.7. Next, suppose that $X = P_i[1]$ and that $Y = N^+$, where P_i is an indecomposable projective and N^+ is an indecomposable kQ-module. Then

$$E_{kQ}(\psi_Q(\widetilde{X}), \psi_Q(\widetilde{Y})) = \operatorname{Hom}^{Q_0}(S_i^-, N^+)$$

and has dimension given by the multiplicity of α_i in the positive root corresponding to N^+. We also have

$$\begin{aligned}
\operatorname{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) &\simeq \operatorname{Ext}^1_{\mathcal{C}}(\widetilde{P_i[1]}, \widetilde{N^+}) \\
&\simeq \operatorname{Ext}^1_{\mathcal{C}}(\tau^{-1}\widetilde{P_i[1]}, \tau^{-1}\widetilde{N^+}) \\
&\simeq \operatorname{Ext}^1_{\mathcal{C}}(\widetilde{P}_i, \tau^{-1}\widetilde{N^+}) \\
&\simeq \operatorname{Ext}^1_{\mathcal{C}}(\widetilde{\tau^{-1}N^+}, \widetilde{P}_i) \\
&\simeq \operatorname{Hom}_{\mathcal{C}}(\widetilde{P}_i, \widetilde{N^+}) \\
&\simeq \operatorname{Hom}_{kQ}(P_i, N^+),
\end{aligned}$$

the last step by Proposition 1.7. This also has dimension equal to the multiplicity of α_i in the positive root corresponding to N^+.

In this situation, we also have

$$\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{Y}, \widetilde{X}) \simeq \mathrm{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y})$$

of the same dimension, and $E_{kQ}(\psi_Q(\widetilde{Y}), \psi_Q(\widetilde{X})) = \mathrm{Hom}^{Q_0}(N^+, S_i^-)$ with the same dimension, so the only case left to consider is when $X = P_i[1]$ and $Y = P_j[1]$ where P_i and P_j are indecomposable kQ-modules. In this case,

$$E_{kQ}(\psi_Q(\widetilde{X}), \psi_Q(\widetilde{Y})) = 0,$$

and

$$\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) = \mathrm{Ext}^1_{\mathcal{C}}(\widetilde{P_i[1]}, \widetilde{P_j[1]}) \simeq \mathrm{Ext}^1_{\mathcal{C}}(\widetilde{P_i}, \widetilde{P_j}) = 0$$

by Proposition 1.7. □

This proposition shows that $\mathcal{C}$, which is independent of the orientation of its defining quiver, can be regarded as a "symmetrised" (orientation independent) version of the decorated categories $k\widetilde{Q}$-mod, since E_{kQ} can be modelled for all orientations Q of Δ by $\mathcal{C}$, via the labellings ψ_Q.

We therefore have:

Corollary 4.3. *Let $\alpha, \beta \in \Phi_{\geqslant -1}$. Then we have*

$$(\alpha \| \beta)_Q = \dim \mathrm{Ext}^1_{\mathcal{C}}(M_Q(\alpha), M_Q(\beta)),$$

where $(\alpha\|\beta)_Q$ denotes the Q-compatibility degree of α and β (see [MRZ03, Eq. 3.3]).

We also have the following consequences. Let $\Delta(\mathcal{C})$ be the abstract simplicial complex on $\mathcal{C}$ with simplices given by the exceptional sets in $\mathcal{C}$, i.e. the subsets of tilting sets. Thus the maximal simplicies are the tilting sets.

Corollary 4.4. *Let Q be any quiver of type Δ. Then $\Delta(\mathcal{C})$ is isomorphic to the abstract simplicial complex Δ_Q of* [MRZ03, 3.7, 4.11].

Corollary 4.4, together with [MRZ03, 4.11, 4.12], show that the simplicial complex $\Delta(\Phi)$ of [FZ03, p. 6], can be obtained in a natural way from the category $\mathcal{C}$ associated to Φ.

Theorem 4.5. *Let $Q = Q_{\mathrm{alt}}$ be an alternating quiver of type Δ. Then the map $\alpha \mapsto M_{Q_{\mathrm{alt}}}(\alpha)$ between $\Phi_{\geqslant -1}$ and* $\mathrm{ind}\,\mathcal{C}$ *induces a bijection between the following sets*:

(1) *The set of clusters in a cluster algebra of type Δ.*
(2) The set of basic tilting objects in $\mathcal{C}(\Delta)$.

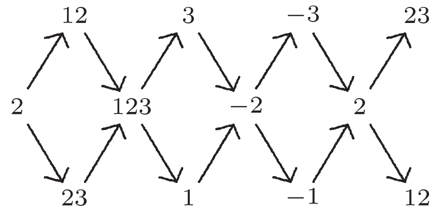


Fig. 3. The labelled AR-quiver of $\mathcal{C}$ in type A_3.

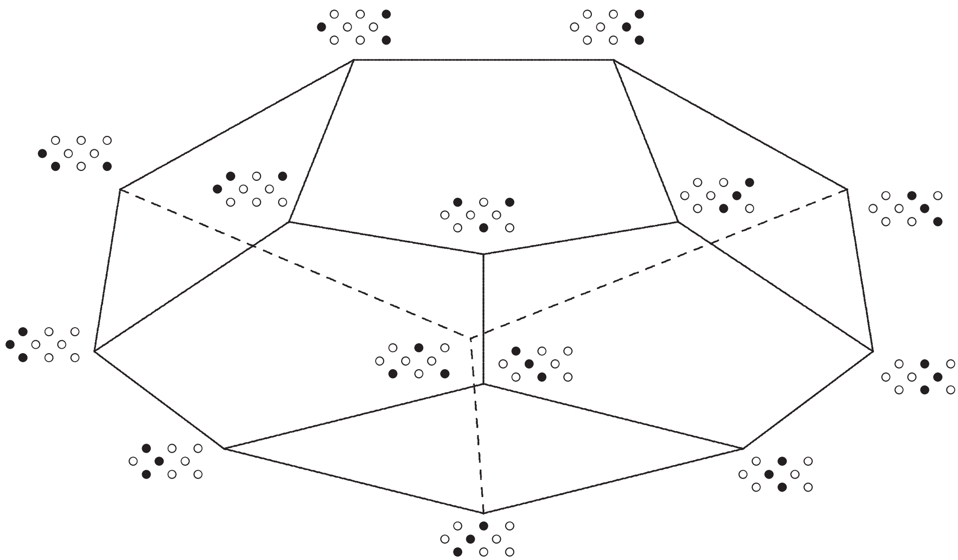


Fig. 4. The 14 tilting sets of $\mathcal{C}$ in type A_3.

Proof. The induced map is given by applying the map $\alpha \mapsto M_{Q_{\mathrm{alt}}}(\alpha)$ pointwise to a cluster regarded as a subset of $\Phi_{\geqslant -1}$. The result follows from Corollary 4.3 and [MRZ03, 4.12]. $\square$

Example. In Fig. 3, we indicate the labelling of $\operatorname{ind}\mathcal{C}$ (via its AR-quiver) from Theorem 4.5 in type A_3. Objects with the same label are identified. A positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ is denoted by $i, i+1, \ldots, j$ and a negative root $-\alpha_i$ is denoted by $-i$.

We recall that it is known that there is a bijection between the clusters in type A_n and the vertices of the n-dimensional associahedron—see [CFZ02, 1.4]. In Fig. 4, we show the 14 tilting sets in type A_3, associated to the vertices of the 3-dimensional associahedron via the bijection in Theorem 4.5. The filled-in circles indicate the elements of the tilting set; note that the duplicated vertices of Fig. 3 do not appear in these diagrams.

Proposition 4.6. *Given a basic* kQ*-tilting module* T, *we can write it as a direct sum* $T = \coprod_{\alpha \in S} X_\alpha$ *where* $S \subset \Phi_+$ *and* X_α *is the indecomposable* kQ*-module corresponding to* $\alpha \in \Phi_+$. *Let* $\varepsilon(T) := \coprod_{\alpha \in S} M_Q(\alpha)$. *Then* $\varepsilon(T)$ *is a basic tilting object of* $\mathcal{C}$, *and* ε *defines an embedding of the set of basic tilting* kQ*-modules into the set of basic tilting objects of* $\mathcal{C}$.

Proof. The result follows immediately from Proposition 1.7. $\square$

Let $\Delta_{\mathrm{mod}}(Q)$ denote the complex of basic exceptional kQ-modules. This is an abstract simplicial complex on the set of isomorphism classes of indecomposable kQ-modules, with the simplices given by the basic exceptional kQ-modules. This complex was studied by Riedtmann and Schofield [RS91], and Unger [Ung96] following a suggestion of C.M. Ringel.

Corollary 4.7. *Let* Q *be any quiver of type* Δ. *The map* ε *induces an embedding of* $\Delta_{\mathrm{mod}}(Q)$ *into* $\Delta(\mathcal{C})$.

Proof. We note that ε actually defines an embedding of the set of basic exceptional kQ-modules into the set of exceptional sets of $\mathcal{C}$. $\square$

For the algebra kQ, where Q is the quiver given by A_n with linear orientation, the tilting graph of the category of finite-dimensional kQ-modules, as discussed in Section 3, can be regarded as the skeleton of a simplicial complex with simplices the faithful basic exceptional modules. This simplicial complex is in fact the Stasheff associahedron of dimension $n-1$, see [BK].

5. Complements of almost complete basic tilting objects

Let H be a finite-dimensional hereditary algebra with n non-isomorphic simple modules. An H-module $\overline{T}$ is said to be an *almost complete basic tilting module* if it is basic exceptional and has $n-1$ indecomposable direct summands. Then there is automatically an indecomposable module M, such that $\overline{T} \coprod M$ is a basic tilting module. Such an indecomposable module is known as a *complement* to $\overline{T}$. It is known that $\overline{T}$ can be completed to a basic tilting module in at most two different ways [RS90,Ung90] and it can be done in exactly two ways if and only if $\overline{T}$ is sincere [HU89], that is, each simple module occurs as a composition factor of $\overline{T}$. We investigate the analogous concept for the category $\mathcal{C} = D^b(\operatorname{mod} H)/F$, and show that in this context an almost complete basic tilting object has exactly two complements. Hence there is a more regular behaviour in $\mathcal{C}$. Certain classes of hereditary categories exhibit a similar behaviour [HU03]. The analogous question has been investigated for arbitrary artin algebras [CHU94].

We say that a basic exceptional object $\overline{T}$ in $\mathcal{C}$ is an *almost complete basic tilting object* if there is an indecomposable object M in $\mathcal{C}$, such that $\overline{T} \coprod M$ is a basic tilting object. Then we have the following main result of this section.

Theorem 5.1. *Let H be a finite-dimensional hereditary algebra, and $\overline{T}$ an almost complete basic tilting object in $\mathcal{C} = D^b(H)/F$. Then $\overline{T}$ can be completed to a basic tilting object in $\mathcal{C}$ in exactly two different ways.*

Proof. By Theorem 3.3 we can assume that $\overline{T}$ is an H-module. Since $\overline{T}$ is a basic exceptional H-module with $n-1$ non-isomorphic direct summands, where n is the number of non-isomorphic simple H-modules, $\overline{T}$ is an almost complete basic tilting module over H.

Assume $\overline{T}$ is sincere and let M_1 and M_2 be the complements in $\operatorname{mod} H$. Since $\overline{T} \coprod M_1$ and $\overline{T} \coprod M_2$ are basic tilting H-modules, they induce basic tilting objects in $\mathcal{C}$ by Theorem 3.3. Hence M_1 and M_2 are complements to $\overline{T}$ in $\mathcal{C}$. If another complement M_3 comes from an H-module, then $\overline{T} \coprod M_3$ would be a basic exceptional H-module by Lemma 3.1 and hence a basic tilting H-module, which is impossible. Let P be an indecomposable projective H-module. Then $\operatorname{Ext}^1_{\mathcal{C}}(P[1], \overline{T}) \simeq \operatorname{Hom}_{\mathcal{C}}(P, \overline{T}) = \operatorname{Hom}_H(P, \overline{T}) \neq 0$ (using Proposition 1.7(d)), since $\overline{T}$ is sincere. Therefore $P[1]$ cannot be a complement to $\overline{T}$. Hence, we have exactly two complements when $\overline{T}$ is a sincere H-module.

Assume now that $\overline{T}$ is not sincere as an H-module, so that there is exactly one indecomposable H-module which is a complement of $\overline{T}$. It follows as above that there are no more indecomposable H-modules which induce complements of $\overline{T}$ in $\mathcal{C}$. Since $\overline{T}$ is not sincere, there is an indecomposable projective H-module Q, such that $\operatorname{Hom}_H(Q, \overline{T}) = 0$.

Let Γ be the quiver of H, which we can assume to be a basic algebra, and Γ' the subquiver obtained by removing the vertex e of Γ corresponding to Q, and all arrows starting or ending at e. So the corresponding path algebra $k\Gamma'$ is isomorphic to H/HeH. Then $\overline{T}$ is clearly a $k\Gamma'$-module, and we obviously have $\operatorname{Ext}^1_{k\Gamma'}(\overline{T}, \overline{T}) = 0$ since $\operatorname{Ext}^1_H(\overline{T}, \overline{T}) = 0$. Since $k\Gamma'$ has $n-1$ vertices, and $\overline{T}$ has $n-1$ non-isomorphic indecomposable summands, $\overline{T}$ is a basic tilting module over $k\Gamma'$. Therefore $\overline{T}$ is a faithful (and hence sincere) $k\Gamma'$-module. In particular, $\operatorname{Hom}_{k\Gamma'}(P, \overline{T}) \neq 0$ for any indecomposable projective $k\Gamma'$-module P, so that Q is the only indecomposable projective H-module with $\operatorname{Hom}_H(Q, \overline{T}) = 0$.

If $P[1]$, with P an indecomposable projective H-module, is a complement of $\overline{T}$ in $\mathcal{C} = D^b(H)/F$, we must have $\operatorname{Ext}^1_{\mathcal{C}}(P[1], \overline{T}) = 0$, so that $\operatorname{Hom}_{\mathcal{C}}(P, \overline{T}) = 0$, and hence $\operatorname{Hom}_H(P, \overline{T}) = 0$. So we must have $P \simeq Q$; in particular at most one possibility.

Conversely, if $\operatorname{Hom}_{\mathcal{D}}(Q, \overline{T}) = 0$, we have $\operatorname{Ext}^1_{\mathcal{D}}(Q[1], \overline{T}) = 0$ and

$$\operatorname{Ext}^1_{\mathcal{D}}(Q[1], F\overline{T}) = \operatorname{Ext}^1_{\mathcal{D}}(Q, \tau^{-1}\overline{T}) = \operatorname{Ext}^1_{\mathcal{D}}(I[-1], \overline{T}) = \operatorname{Ext}^2_{\mathcal{D}}(I, \overline{T}) = 0,$$

where $I \in \operatorname{mod} H$. We also have

$$\operatorname{Ext}^1_{\mathcal{D}}(Q[1], F^{-1}\overline{T}) = \operatorname{Ext}^1_{\mathcal{D}}(Q[1], \tau\overline{T}[-1]) = 0.$$

Furthermore,

$$\operatorname{Ext}^1_{\mathcal{D}}(\overline{T}, Q[1]) = \operatorname{Ext}^2_{\mathcal{D}}(\overline{T}, Q) = 0,$$

and

$$\mathrm{Ext}^1_{\mathcal{D}}(\overline{T}, F^{-1}(Q)) = \mathrm{Ext}^1_{\mathcal{D}}(\overline{T}, \tau Q[-1]) = \mathrm{Ext}^1_{\mathcal{D}}(\overline{T}, I[-2]) = 0,$$

where $I \in \mathrm{mod}\, H$. Hence we see that $\overline{T} \coprod Q[1]$ is a basic tilting object in $\mathcal{C} = D^b(H)/F$, so that $Q[1]$ is a complement. $\Box$

6. Description of complements via approximations

We shall now see how, starting with a complement of an almost complete basic tilting object, we can construct the other one by using minimal left and right approximations in $\mathcal{C} = D^b(H)/F$. This is possible since $\mathcal{C}$ is a Krull–Schmidt category. We shall also use that $\mathcal{C}$ is in a canonical way a triangulated category, namely the canonical functor $\mathcal{D} \to \mathcal{C}$ is a triangle functor.

We recall the definition of minimal left and right approximations, which come from the theory of covariantly and contravariantly finite subcategories [AS80]. Suppose that $\mathcal{E}$ is an additive category, that χ is an additive subcategory of $\mathcal{E}$, and E is an object of $\mathcal{E}$. A map $Y \to E$ with Y an object of χ is called a *right χ-approximation* if the induced map $\mathrm{Hom}_{\mathcal{E}}(X, Y) \to \mathrm{Hom}_{\mathcal{E}}(X, E)$ is an epimorphism for every object X of χ. There is the dual notion of a *left χ-approximation*. A map $f : E \to F$ in an arbitrary category $\mathcal{E}$ is called *right minimal* if for every $g : E \to E$ such that $fg = f$, the map g is an isomorphism. Then there is the dual notion of *left minimal map*. A right (respectively, left) approximation, that is also right (respectively, left) minimal is called a *minimal right* (respectively, *left*) *approximation*.

So let as before $\overline{T}$ be an almost complete basic tilting object in $\mathcal{C}$, and let M be a complement. Let $f: B \to M$ be a minimal right $\mathrm{add}\,\overline{T}$-approximation of M in $\mathcal{C}$, and complete this map to a triangle

$$M^* \xrightarrow{g} B \xrightarrow{f} M \to M^*[1] \tag{3}$$

in $\mathcal{C}$. We show in this section that M^* is the second complement to $\overline{T}$. This can be seen as a generalisation of a result of Happel and Unger.

Proposition 6.1 (*Happel and Unger [HU89]*). *Let $\overline{T}$ be a sincere almost complete tilting module over a hereditary algebra H. Then there are exactly two non-isomorphic complements M^* and M in* $\mathrm{mod}\, H$, *and an exact sequence*

$$0 \to M^* \to B \to M \to 0, \tag{4}$$

in $\mathrm{mod}\, H$, *where $B \to M$ is a minimal right* $\mathrm{add}\,\overline{T}$*-approximation in* $\mathrm{mod}\, H$.

The exact sequence (4) gives rise to a triangle in $\mathcal{D}$, and thus to a triangle in $\mathcal{C}$.

Lemma 6.2. *Assume $\overline{T}$ is an almost complete tilting object in $\mathcal{C}$ induced by a sincere almost complete tilting module in* $\operatorname{mod} H$. *Then the triangle* (3) *in $\mathcal{C}$ is induced by the exact sequence* (4).

Proof. We need to show that the right $\operatorname{add}\overline{T}$-approximation $B \to M$ in $\operatorname{mod} H$, is also a right $\operatorname{add}\overline{T}$-approximation in $\mathcal{C}$. View (4) as a triangle in $\mathcal{D}$ and apply $\operatorname{Hom}_{\mathcal{D}}(F^{-1}\overline{T},\ )$ to it, to obtain an exact sequence

$$\operatorname{Hom}_{\mathcal{D}}(F^{-1}\overline{T}, B) \to \operatorname{Hom}_{\mathcal{D}}(F^{-1}\overline{T}, M) \to \operatorname{Hom}_{\mathcal{D}}(F^{-1}\overline{T}, M^*[1]),$$

where $\operatorname{Hom}_{\mathcal{D}}(F^{-1}\overline{T}, M^*[1]) = \operatorname{Hom}_{\mathcal{D}}(\tau\overline{T}, M^*[2]) = 0$. Thus, the claim follows by Proposition 1.5. □

To prove that M^* is a second complement to $\overline{T}$ we use the following preliminary results.

Lemma 6.3. *With the above notation, we have* $\operatorname{Ext}^1_{\mathcal{C}}(\overline{T}, M^*) = 0 = \operatorname{Ext}^1_{\mathcal{C}}(M^*, \overline{T})$.

Proof. Applying $\operatorname{Hom}_{\mathcal{C}}(\overline{T},\ )$ to the triangle $M^* \to B \to M \to M^*[1]$ we get the exact sequence

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(\overline{T}, M^*) &\to \operatorname{Hom}_{\mathcal{C}}(\overline{T}, B) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(\overline{T}, f)} \operatorname{Hom}_{\mathcal{C}}(\overline{T}, M) \\ &\to \operatorname{Ext}^1_{\mathcal{C}}(\overline{T}, M^*) \to \operatorname{Ext}^1_{\mathcal{C}}(\overline{T}, B). \end{aligned}$$

Since $\operatorname{Ext}^1_{\mathcal{C}}(\overline{T}, B) = 0$ because B is in $\operatorname{add}\overline{T}$ and $\operatorname{Ext}^1_{\mathcal{C}}(\overline{T}, \overline{T}) = 0$, and $\operatorname{Hom}_{\mathcal{C}}(\overline{T}, f)$ is surjective since $f: B \to M$ is a right $\operatorname{add}\overline{T}$-approximation, we get $\operatorname{Ext}^1_{\mathcal{C}}(\overline{T}, M^*) = 0$. By the symmetry property of $\operatorname{Ext}^1_{\mathcal{C}}(\ )$, we also get $\operatorname{Ext}^1_{\mathcal{C}}(M^*, \overline{T}) = 0$. □

Lemma 6.4. *The map $g: M^* \to B$ is a minimal left* $\operatorname{add}\overline{T}$*-approximation in $\mathcal{C}$.*

Proof. Apply $\operatorname{Hom}_{\mathcal{C}}(\ , \overline{T})$ to the triangle $M^* \to B \to M \to M^*[1]$ to get the exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(B, \overline{T}) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(g, \overline{T})} \operatorname{Hom}_{\mathcal{C}}(M^*, \overline{T}) \to \operatorname{Ext}^1_{\mathcal{C}}(M, \overline{T}).$$

Since $\overline{T} \coprod M$ is a basic tilting object in $\mathcal{C}$, we have $\operatorname{Ext}^1_{\mathcal{C}}(M, \overline{T}) = 0$, and hence

$$\operatorname{Hom}_{\mathcal{C}}(g, \overline{T}): \operatorname{Hom}_{\mathcal{C}}(B, \overline{T}) \to \operatorname{Hom}_{\mathcal{C}}(M^*, \overline{T})$$

is surjective. So $g: M^* \to B$ is a left $\operatorname{add}\overline{T}$-approximation.

We now show that $g: M^* \to B$ is a left minimal map. If it was not, then a summand $0 \to B_1$ would split off, where B_1 is a non-zero summand of B. But then $B_1 \xrightarrow{\simeq} B_1$

would be a direct summand of $f: B \to M$. Since M is indecomposable, we would have $M \simeq B_1$, contradicting that B_1 is in add $\overline{T}$, and that M is a complement of $\overline{T}$. Our claim then follows. $\square$

Lemma 6.5. *M^* is indecomposable.*

Proof. Assume that $M^* = U \coprod V$ with U and V non-zero. Let $f_1: U \to B_1$ and $f_2: U \to B_2$ be minimal left add $\overline{T}$-approximations, and complete the two maps to triangles

$$U \to B_1 \to X \to U[1]$$

and

$$V \to B_2 \to Y \to V[1].$$

The direct sum of the triangles is

$$M^* \to B \to M \to M^*[1]$$

and so $M = X \coprod Y$. Hence $X = 0$ or $Y = 0$. If $X = 0$, then $B_1 \to 0$ is a direct summand of $f: B \to M$, which contradicts f being right minimal. Similarly $Y = 0$ leads to a contradiction. Hence M^* is indecomposable. $\square$

Lemma 6.6. *M^* is not in* add $\overline{T}$.

Proof. If M^* was in add $\overline{T}$, then $g: M^* \to B$ would be an isomorphism, and hence $M = 0$, which is a contradiction. $\square$

To show that $\overline{T} \coprod M^*$ is a basic tilting object in $\mathcal{C}$, it remains to show the following.

Lemma 6.7. $\mathrm{Ext}^1_{\mathcal{C}}(M^*, M^*) = 0$.

Proof. Consider again the triangle

$$M^* \xrightarrow{g} B \xrightarrow{f} M \to M^*[1].$$

Apply $\mathrm{Hom}_{\mathcal{C}}(\ , M)$ to get the exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(B, M) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(g,M)} \mathrm{Hom}_{\mathcal{C}}(M^*, M) \to \mathrm{Ext}^1_{\mathcal{C}}(M, M).$$

$$
\begin{tikzcd}
 & & B \arrow[dl, dashed, "s"'] \arrow[d, "t"] \\
M^* \arrow[r, "g"] \arrow[d, "h"] & B \arrow[r, "f"] \arrow[dl, dashed, "t"] & M \\
M & &
\end{tikzcd}
$$

Fig. 5. Commutative diagram for the proof of Lemma 6.7.

Since $\mathrm{Ext}^1_{\mathcal{C}}(M, M) = 0$, the map $\mathrm{Hom}_{\mathcal{C}}(g, M)$ is surjective. Hence any map $h: M^* \to M$ factors through $g: M^* \to B$. Now apply $\mathrm{Hom}_{\mathcal{C}}(M^*, \ )$ to the triangle to get the exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(M^*, B) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(M^*, f)} \mathrm{Hom}_{\mathcal{C}}(M^*, M) \to \mathrm{Ext}^1_{\mathcal{C}}(M^*, M^*) \to \mathrm{Ext}^1_{\mathcal{C}}(M^*, B),$$

where the last term is zero. To show that $\mathrm{Ext}^1_{\mathcal{C}}(M^*, M^*) = 0$ it is therefore enough to show that $\mathrm{Hom}_{\mathcal{C}}(M^*, f): \mathrm{Hom}_{\mathcal{C}}(M^*, B) \to \mathrm{Hom}_{\mathcal{C}}(M^*, M)$ is surjective, that is, any map $h: M^* \to M$ factors through $f: B \to M$. Then consider the commutative diagram in Fig. 5, where t is obtained from the first lifting, and we get $s: B \to B$ by using that $f: B \to M$ is a right $\mathrm{add}\,\overline{T}$-approximation. So $h = tg = fsg$, and hence $h: M^* \to M$ factors through $f: B \to M$, as desired. This finishes the proof of the lemma. $\square$

We now put the lemmas together to get the following.

Theorem 6.8. *If M is a complement of the almost complete basic tilting object $\overline{T}$ in $\mathcal{C}$, then M^* is another complement, obtained by extending the minimal right* $\mathrm{add}\,\overline{T}$-*approximation to a triangle.*

Proof. We only need to remark that $M \not\simeq M^*$. This follows since $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*) \neq 0$ and $\mathrm{Ext}^1_{\mathcal{C}}(M, M) = 0$. $\square$

It is clear that we can also get dual constructions. That is, start with a complement M, and consider the triangle

$$M \xrightarrow{u} B' \xrightarrow{v} M^{**} \to M[1], \tag{5}$$

where $u: M \to B'$ is a minimal left $\mathrm{add}\,\overline{T}$-approximation. In a dual way we get that $v: B' \to M^{**}$ is a minimal right $\mathrm{add}\,\overline{T}$-approximation, and that M^{**} is a complement of $\overline{T}$ with $M \not\simeq M^{**}$. We then have the following consequence of the previous results.

Proposition 6.9. *Let M be a complement of the almost complete basic tilting object $\overline{T}$ in $\mathcal{C}$. Then $M^* \simeq M^{**}$ is the unique other complement, where M^* is the fibre of the minimal right* add $\overline{T}$*-approximation of M in $\mathcal{C}$, and M^{**} is the cofibre of the minimal left* add $\overline{T}$*-approximation of M in $\mathcal{C}$.*

For an indecomposable exceptional module M, it is well known that the endomorphism ring $\mathrm{End}_H(M)$ is a division ring. However, the endomorphism ring $\mathrm{End}_{\mathcal{C}}(M)$ need not be a division ring, which we later in this section observe in an example. However, if H is of finite representation type, or more generally, if M is (induced by) a preprojective or preinjective module, then $\mathrm{End}_{\mathcal{C}}(M)$ is a division ring. This is a special case of the following.

Lemma 6.10. *Let M be an indecomposable H-module with $\mathrm{Hom}_{\mathcal{D}}(M, \tau^2 M) = 0$. Then $\mathrm{End}_{\mathcal{C}}(M)$ is a division ring.*

Proof. By Proposition 1.5, $\mathrm{End}_{\mathcal{C}}(M) = \mathrm{Hom}_{\mathcal{D}}(M, M) \oplus \mathrm{Hom}_{\mathcal{D}}(M, FM)$. Using the AR-formula and the assumption on M, we obtain

$$\mathrm{Hom}_{\mathcal{D}}(M, FM) = \mathrm{Hom}_{\mathcal{D}}(M, \tau^{-1}M[1]) \simeq D\mathrm{Hom}_{\mathcal{D}}(M, \tau^2 M) = 0$$

and the claim follows. $\square$

We will need to consider the factor rings

$$D_M = \mathrm{End}_{\mathcal{C}}(M)/\mathrm{rad}_{\mathcal{C}}(M, M)$$

and

$$D_{M^*} = \mathrm{End}_{\mathcal{C}}(M^*)/\mathrm{rad}_{\mathcal{C}}(M^*, M^*),$$

which turn out to be isomorphic.

Lemma 6.11. *There is a natural ring-isomorphism $D_M \to D_{M^*}$.*

Proof. Consider the triangle

$$M^* \xrightarrow{g} B \xrightarrow{f} M \to M^*[1].$$

Let α be an element in $\mathrm{End}_{\mathcal{C}}(M)$. Then there is a commutative diagram as in Fig. 6 where γ exists since $B \to M$ is a right add $\overline{T}$-approximation. We claim that the map $\alpha \mapsto \beta$ gives a well-defined ring-homomorphism

$$\mathrm{End}_{\mathcal{C}}(M) \to \mathrm{End}_{\mathcal{C}}(M^*)/\mathrm{rad}_{\mathcal{C}}(M^*, M^*).$$

$$\begin{tikzcd}
M^* \arrow[r, "g"] \arrow[d, dashed, "\beta"] & B \arrow[r, "f"] \arrow[d, dashed, "\gamma"] & M \arrow[r] \arrow[d, "\alpha"] & M[1] \arrow[d, dashed, "\beta[1]"] \\
M^* \arrow[r, "g"] & B \arrow[r, "f"] & M \arrow[r] & M[1]
\end{tikzcd}$$

Fig. 6. Commutative diagram for the proof of Lemma 6.11.

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\gamma^N"] & M \arrow[r] \arrow[d, "0"] & M^*[1] \arrow[d, "\beta^N"] \\
B \arrow[r] & M \arrow[r] & M^*[1]
\end{tikzcd}$$

Fig. 7. Commutative diagram for the proof of Lemma 6.11.

First note that if $g = 0$, then $B = 0$, and the map $M \to M^*[1]$ is an isomorphism. Thus, in this case, the map $\alpha \mapsto \beta$ is well-defined. Assume then that g is non-zero. Let $\alpha \in \mathrm{End}_{\mathcal{C}}(M)$ and fix a map γ, such that $\alpha f = f\gamma$. Then there is some map from M^* to M^* completing the diagram. Assume there are two such maps β_1 and β_2. Then $g(\beta_1 - \beta_2) = 0$, so $\beta_1 - \beta_2$ is not an isomorphism, and thus each choice of γ gives a well-defined element $\overline{\beta} \in \mathrm{End}_{\mathcal{C}}(M^*)/\mathrm{rad}_{\mathcal{C}}(M^*, M^*)$. Let γ_1 and γ_2 be maps from B to B making the diagram commute, and choose corresponding maps $\beta_1, \beta_2 \in \mathrm{End}_{\mathcal{C}}(M^*)$. It then suffices to show that $\overline{\beta_1} - \overline{\beta_2}$ is zero, in other words that $\beta_1 - \beta_2$ is a non-isomorphism. We have $\alpha f = f\gamma_1 = f\gamma_2$. Since $f(\gamma_1 - \gamma_2) = 0$, there is a map $w \colon B \to M^*$, such that $gw = \gamma_1 - \gamma_2$ and thus

$$gwg = (\gamma_1 - \gamma_2)g = g(\beta_1 - \beta_2).$$

Since M^* is not a summand of B, wg is not an isomorphism. If $\beta_1 - \beta_2$ was an isomorphism, then $wg - (\beta_1 - \beta_2)$ would also be an isomorphism. But

$$g(wg - (\beta_1 - \beta_2)) = 0,$$

so then $g = 0$, a contradiction, so the claim is proved.

Using that also $M^* \to B$ is a left add $\overline{T}$-approximation, we obtain that $\alpha \mapsto \overline{\beta}$ is an epimorphism. Assume now α is not an isomorphism, then there is an integer N, such that $\alpha^N = 0$ by Proposition 1.2. Thus, there is a commutative diagram as in Fig. 7 which shows that β^N is not an isomorphism, and thus β is in $\mathrm{rad}_{\mathcal{C}}(M^*, M^*)$. It follows from the minimality of $B \to M$ that if α is an isomorphism, then γ and hence β are isomorphisms. $\square$

We want to show that in $\mathcal{C}$ all non-isomorphisms $M \to M$, actually factor through $B \to M$. The following is useful for this.

Lemma 6.12. *All maps in* $\mathrm{rad}_{\mathcal{C}}(M, M)$ *factor through* $B \to M$ *if and only if all maps in* $\mathrm{rad}_{\mathcal{C}}(M^*, M^*)$ *factor through* $M^* \to B$.

Proof. Apply $\mathrm{Hom}_{\mathcal{C}}(M,\ )$ to the triangle

$$M^* \to B \to M \to M^*[1]$$

to obtain the exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(M, M^*) \to \mathrm{Hom}_{\mathcal{C}}(M, B) \to \mathrm{Hom}_{\mathcal{C}}(M, M) \to \mathrm{Hom}_{\mathcal{C}}(M, M^*[1]) \to 0.$$

Assume any $f \in \mathrm{rad}_{\mathcal{C}}(M, M)$ factors through $B \to M$. This means that

$$\mathrm{Hom}_{\mathcal{C}}(M, M^*[1]) \simeq \mathrm{Hom}_{\mathcal{C}}(M, M)/\mathrm{rad}_{\mathcal{C}}(M, M).$$

Applying $\mathrm{Hom}_{\mathcal{C}}(\ , M^*)$ to the same triangle gives the exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(M, M^*) \to \mathrm{Hom}_{\mathcal{C}}(B, M^*) \xrightarrow{u} \mathrm{Hom}_{\mathcal{C}}(M^*, M^*) \to \mathrm{Hom}_{\mathcal{C}}(M, M^*[1]) \to 0,$$

which means that $\mathrm{Hom}_{\mathcal{C}}(M, M^*[1]) \simeq \mathrm{Hom}_{\mathcal{C}}(M^*, M^*)/I$ where I is the image of the map u. It follows from Lemma 6.11 that I is the radical $\mathrm{rad}_{\mathcal{C}}(M^*, M^*)$. The other implication can be shown similarly. □

We can now prove the promised result about lifting non-isomorphisms in $\mathcal{C}$.

Lemma 6.13. *With the previous notation and assumptions, any non-isomorphism* $M \to M$ *in* $\mathcal{C}$ *factors through* $B \to M$.

Proof. We can assume that $\overline{T} \oplus M$ is induced by an H-module. We first assume that $\overline{T}$ is sincere, so M^* is also induced by a module. By Lemma 6.2, the triangle (3) is induced by an exact sequence of modules. We view this exact sequence as a triangle in $\mathcal{D}$ and apply $\mathrm{Hom}_{\mathcal{D}}(F^{-1}M,\ )$ to it. Since $\mathrm{Hom}_{\mathcal{D}}(F^{-1}M, M^*[1]) = \mathrm{Hom}_{\mathcal{D}}(\tau M, M^*[2]) = 0$, it follows that any non-isomorphism $M \to M$ in $\mathcal{C}$ factors through $B \to M$, using Proposition 1.5.

Now assume that $\overline{T}$ is not sincere. Then M^* is induced by an object $P[1]$ in $\mathcal{D}$, where P is an indecomposable projective H-module, and thus $\mathrm{rad}_{\mathcal{C}}(M^*, M^*) = 0$, using Lemma 6.10. Applying Lemma 6.12, it follows trivially that also in this case, all non-isomorphisms $M \to M$ in $\mathcal{C}$ factor through $B \to M$. □

We can now conclude with the following property of $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*)$.

Proposition 6.14. *Let* M *and* M^* *be the complements of an almost complete tilting object in the cluster category* $\mathcal{C}$. *Then* $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*)$ *has dimension one over each of the division rings* $D_M = \mathrm{End}_{\mathcal{C}}(M)/\mathrm{rad}_{\mathcal{C}}(M, M)$ *and* D_{M^*}.

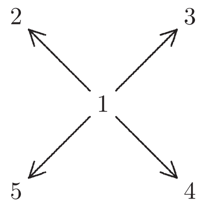


Fig. 8. A quiver of type $\widetilde{D}_4$.

Proof. Apply $\mathrm{Hom}_{\mathcal{C}}(M,\ )$ to the triangle

$$M^* \to B \to M \to M^*[1],$$

to get the exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(M, M^*) \to \mathrm{Hom}_{\mathcal{C}}(M, B) \xrightarrow{u} \mathrm{Hom}_{\mathcal{C}}(M, M) \to \mathrm{Ext}^1_{\mathcal{C}}(M, M^*) \to \mathrm{Ext}^1_{\mathcal{C}}(M, B),$$

where $\mathrm{Ext}^1_{\mathcal{C}}(M, B) = 0$. Isomorphisms $M \to M$ do not lift to B, since M is not a summand of B. Thus, it follows from Lemma 6.13 that $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*) \simeq \mathrm{End}_{\mathcal{C}}(M)/\mathrm{rad}_{\mathcal{C}}(M, M)$. It follows similarly that $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*)$ is one-dimensional over $\mathrm{End}_{\mathcal{C}}(M^*)$. $\square$

Note that for the triangle

$$M^* \to B \to M \to M^*[1]$$

it may happen that B is zero, even though $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*) \neq 0$. This of course means that $M \simeq M^*[1] = \tau M^*$, so in this case the second triangle $\tau M^* = M \to B' \to M^*$ is almost split.

We also notice that Lemma 6.13 has the following interpretation.

Corollary 6.15. *Let T be a tilting object in a cluster category $\mathcal{C}$ for a hereditary algebra over an algebraically closed field. Then the quiver of* $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ *has no loops.*

Example. The following example illustrates Theorem 6.8 in the tame hereditary case. We consider the quiver $\widetilde{D}_4$ with the orientation as in Fig. 8. Let Λ be the path algebra of this quiver over some field k. Let P_i be the indecomposable projective corresponding to vertex i. Then it easy to see that

$$\overline{T} = P_4 \coprod P_5 \coprod \tau^{-1} P_2 \coprod \tau^{-1} P_3$$

is an almost complete basic tilting module, and thus an almost complete basic tilting object in the corresponding category $\mathcal{C}$.

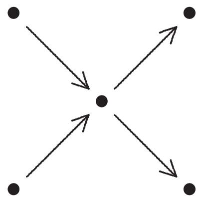


Fig. 9. The quiver of $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$.

It is clear that $\overline{T}$ is sincere, and one complement is easy to find, namely P_1. We use the above approach to find the other complement.

Let R be the cokernel of the embedding $P_1 \to \tau^{-1}P_2 \coprod \tau^{-1}P_3$. Then R is a regular exceptional module with composition factors S_1, S_4, S_5. In the AR-quiver it is at the mouth of a tube of rank two, so $\tau^2 R \simeq R$. Thus, R is an example of an indecomposable exceptional object, with $\mathrm{End}_{\mathcal{C}}(R)$ *not* a division ring.

In $\mathrm{mod}\,\Lambda$ there are exact sequences

$$0 \to P_1 \to \tau^{-1}P_2 \coprod \tau^{-1}P_3 \to R \to 0$$

and

$$0 \to P_4 \coprod P_5 \to P_1 \to \tau R \to 0.$$

Thus, in $\mathcal{D}$ there are triangles

$$P_1 \to \tau^{-1}P_2 \coprod \tau^{-1}P_3 \to R \to P_1[1]$$

and

$$F^{-1}R \to P_4 \coprod P_5 \to P_1 \to F^{-1}R[1].$$

The images of these triangles in $\mathcal{C}$ are exactly the triangles described in Theorem 6.8.

Thus, we obtain that the other complement of $\overline{T}$ is R, and B and B' are given by $P_4 \coprod P_5$ and $\tau^{-1}P_2 \coprod \tau^{-1}P_3$, respectively.

If we let $T = \overline{T} \coprod P_1$ and $T' = \overline{T} \coprod R$, then the endomorphism ring $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ is the path algebra of the quiver in Fig. 9, while $\mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$ is the path algebra of the quiver in Fig. 10, with relations $ac - bf, ec - df, ga - he, gb - hd, cg, ch, fg, fh$.

7. Description of exchange pairs

As usual let H be a hereditary finite-dimensional algebra, and $\mathcal{C}$ the factor category $\mathcal{D}^b(H)/F$, with $F = \tau^{-1}[1]$. We say that two non-isomorphic indecomposable

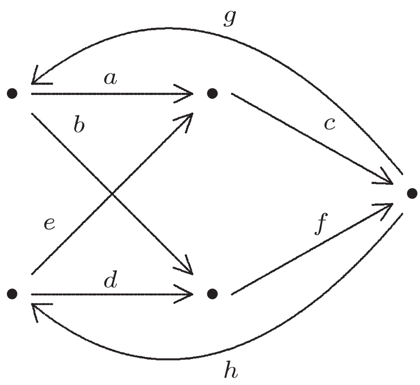


Fig. 10. The quiver of $\mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$.

objects in $\mathcal{C}$ form an *exchange pair* if they are complements of the same almost complete basic tilting object. In this language, we have seen that if M and M^* form an exchange pair, then $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*) \simeq \mathrm{Ext}^1_{\mathcal{C}}(M^*, M)$ is one-dimensional over $D_M = \mathrm{End}_{\mathcal{C}}(M)/\mathrm{rad}_{\mathcal{C}}(M, M)$ and over D_{M^*}. We now want to show that also the converse holds.

Assume that M, M^* are exceptional and that $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*) = \mathrm{Ext}^1_{\mathcal{C}}(M^*, M)$ is one-dimensional over D_M and over D_{M^*}. We can therefore choose non-split triangles

$$M^* \to B \to M \to M^*[1] \tag{6}$$

and

$$M \to B' \to M^* \to M[1] \tag{7}$$

in $\mathcal{C}$, where we use the same notation as before. We want to find an almost complete basic tilting object $\overline{T}$ having M and M^* as complements. We start building up $\overline{T}$ by showing that $B \coprod B' \coprod M$ and $B \coprod B' \coprod M^*$ are exceptional objects in $\mathcal{C}$.

Lemma 7.1. *In the above notation we have*:

$$\mathrm{Ext}^1_{\mathcal{C}}\left(B \coprod B' \coprod M, B \coprod B' \coprod M\right) = 0$$

and

$$\mathrm{Ext}^1_{\mathcal{C}}\left(B \coprod B' \coprod M^*, B \coprod B' \coprod M^*\right) = 0.$$

Proof. Apply $\mathrm{Hom}_{\mathcal{C}}(M,\ )$ to (6) to get the exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(M, M^*) &\to \mathrm{Hom}_{\mathcal{C}}(M, B) \to \mathrm{Hom}_{\mathcal{C}}(M, M)\\ &\xrightarrow{\alpha} \mathrm{Ext}^1_{\mathcal{C}}(M, M^*) \to \mathrm{Ext}^1_{\mathcal{C}}(M, B) \to \mathrm{Ext}^1_{\mathcal{C}}(M, M).\end{aligned}$$

Since $\alpha \neq 0$ and $\dim_{D_M} \mathrm{Ext}^1_{\mathcal{C}}(M, M^*) = 1$, while $\mathrm{Ext}^1_{\mathcal{C}}(M, M) = 0$ by assumption, it follows that $\mathrm{Ext}^1_{\mathcal{C}}(M, B) = 0$. Analogously, we get $\mathrm{Ext}^1_{\mathcal{C}}(M^*, B') = 0$.

Apply $\mathrm{Hom}_{\mathcal{C}}(\ , M^*)$ to (6) to get the exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(M, M^*) &\to \mathrm{Hom}_{\mathcal{C}}(B, M^*) \to \mathrm{Hom}_{\mathcal{C}}(M^*, M^*)\\ &\xrightarrow{\beta} \mathrm{Ext}^1_{\mathcal{C}}(M, M^*) \to \mathrm{Ext}^1_{\mathcal{C}}(B, M^*) \to \mathrm{Ext}^1_{\mathcal{C}}(M^*, M^*).\end{aligned}$$

Since $\beta \neq 0$ and $\dim_{D_{M^*}} \mathrm{Ext}^1_{\mathcal{C}}(M, M^*) = 1$, while $\mathrm{Ext}^1_{\mathcal{C}}(M^*, M^*) = 0$ by assumption, we get $\mathrm{Ext}^1_{\mathcal{C}}(B, M^*) = 0$. Analogously, we get from (7) that $\mathrm{Ext}^1_{\mathcal{C}}(B', M) = 0$.

Apply $\mathrm{Hom}_{\mathcal{C}}(B \coprod B',\ )$ to (6) to get the exact sequence

$$\mathrm{Ext}^1_{\mathcal{C}}\left(B \coprod B', M^*\right) \to \mathrm{Ext}^1_{\mathcal{C}}\left(B \coprod B', B\right) \to \mathrm{Ext}^1_{\mathcal{C}}\left(B \coprod B', M\right)$$

and hence $\mathrm{Ext}^1_{\mathcal{C}}(B \coprod B', B) = 0$. Apply $\mathrm{Hom}_{\mathcal{C}}(B \coprod B',\ )$ to (7) to get the exact sequence

$$\mathrm{Ext}^1_{\mathcal{C}}\left(B \coprod B', M\right) \to \mathrm{Ext}^1_{\mathcal{C}}\left(B \coprod B', B'\right) \to \mathrm{Ext}^1_{\mathcal{C}}\left(B \coprod B', M^*\right),$$

and hence $\mathrm{Ext}^1_{\mathcal{C}}(B \coprod B', B') = 0$. This finishes the proof of the lemma. □

We remark that this implies that M and M^* cannot be direct summands of $B \coprod B'$. We have that $B \coprod B'$ is an exceptional object in $\mathcal{C}$, and hence can be extended to a tilting object by Lemma 3.2. So let T' be a complement in $\mathcal{C}$, that is $T = B \coprod B' \coprod T'$ is a tilting object in $\mathcal{C}$. We want to show that either M or M^* is a direct summand of T and if we remove all copies of this summand, we get a new tilting object by adding the other one.

The proof of this is based upon the following crucial result. Here, for X an object of $\mathcal{C}$, $\mathrm{Supp}_{\mathcal{C}}(\ , X)$ denotes the objects in $\mathcal{C}$ which have a non-zero homomorphism to X.

Lemma 7.2. *With the above notation, we have*

$$\mathrm{Supp}_{\mathcal{C}}(\ , \tau M) \subset \{M^*\} \cup \mathrm{Supp}_{\mathcal{C}}(\ , \tau B) \cup \mathrm{Supp}_{\mathcal{C}}(\ , \tau B').$$

$$\begin{tikzcd}
\tau M^* \arrow[r, "a"] \arrow[d, equal] & A \arrow[r, "r"] \arrow[d, dashed, "b_1"] & M^* \arrow[d, dashed, "b_2"] \\
\tau M^* \arrow[r, "\tau g"] & \tau B \arrow[r, "\tau f"] & \tau M
\end{tikzcd}$$

Fig. 11. Commutative diagram for the proof of Lemma 7.2.

Proof. Consider the triangles

$$M^* \xrightarrow{g} B \xrightarrow{f} M \to M^*[1]$$

and

$$M \to B' \to M^* \to M[1].$$

Rewrite the last triangle as

$$M^* \xrightarrow{h} \tau M \xrightarrow{k} \tau B' \to M^*[1]$$

where we use that $M^*[1] = \tau^{-1} M^*$ and $\tau M^* = M^*[1]$ in $\mathcal{C}$. This gives rise to an exact sequence of functors

$$\mathrm{Hom}_{\mathcal{C}}(\ , M^*) \to \mathrm{Hom}_{\mathcal{C}}(\ , \tau M) \to \mathrm{Hom}_{\mathcal{C}}(\ , \tau B') \to \mathrm{Hom}_{\mathcal{C}}(\ , M^*[1]) \to \cdots .$$

Assume that X is an indecomposable object which is not isomorphic to M^*, and which is in $\mathrm{Supp}_{\mathcal{C}}(\ , \tau M)$, and let $s \in \mathrm{Hom}_{\mathcal{C}}(X, \tau M)$ be a non-zero map.

If $ks: X \to \tau B'$ is not zero, then X is in $\mathrm{Supp}_{\mathcal{C}}(\ , \tau B')$. If $ks = 0$, then there is some $s': X \to M^*$, such that $s = hs'$. Denote by

$$\tau M^* \xrightarrow{a} A \xrightarrow{r} M^* \to \tau M^*[1]$$

the almost split triangle in $\mathcal{C}$ for M^*. Since $X \not\simeq M^*$, there is some map $s'': X \to A$, such that $s' = rs''$. Consider the commutative diagram in Fig. 11, where the map $b_1: A \to \tau B$ exists since the first triangle is almost split and the second one is not split, and b_2 is then the induced map.

We claim that the map $b_1 s'': X \to A \to \tau B$ is non-zero. Note that $(\tau f) b_1 s'' = b_2 r s'' = b_2 s'$. Since $\mathrm{Ext}^1_{\mathcal{C}}(M, M^*)$ has dimension one over D_M, it follows that $\mathrm{Hom}_{\mathcal{C}}$ $(M^*, \tau M)$ also has dimension one. Since b_2 and h are both non-zero elements in $\mathrm{Hom}_{\mathcal{C}}(M^*, \tau M)$, it follows that there is a non-zero map $\phi: \tau M \to \tau M$, necessarily an isomorphism, such that $b_2 = \phi k$. Hence $b_2 s' = \phi h s' = \phi s \neq 0$, and consequently $b_1 s'' \neq 0$. This finishes the proof of the lemma. $\square$

We need some additional preliminary results.

Lemma 7.3. *Let the assumptions and notation be as before.*

(a) $\mathrm{Ext}^1_{\mathcal{C}}(M, T_i) = 0$ *for any indecomposable summand* T_i *of* T *which is not isomorphic to* M^*.
(b) $\mathrm{Ext}^1_{\mathcal{C}}(M^*, T_i) = 0$ *for any indecomposable summand* T_i *of* T *which is not isomorphic to* M.

Proof. (a) Assume to the contrary that $\mathrm{Ext}^1_{\mathcal{C}}(M, T_1) \neq 0$ for some T_1 an indecomposable summand of T, with $T_1 \not\simeq M^*$. We have $\mathrm{Ext}^1_{\mathcal{C}}(M, T_1) \simeq D\,\mathrm{Hom}_{\mathcal{C}}(T_1, \tau M) \neq 0$, and hence by Lemma 7.2, either $\mathrm{Hom}_{\mathcal{C}}(T_1, \tau B) \neq 0$ or $\mathrm{Hom}_{\mathcal{C}}(T_1, \tau B') \neq 0$, so that $\mathrm{Ext}^1_{\mathcal{C}}(B, T_1) \neq 0$ or $\mathrm{Ext}^1_{\mathcal{C}}(B', T_1) \neq 0$. But this contradicts the fact that $B \coprod B' \coprod T'$ is exceptional, and the claim follows.

(b) The proof is dual to the proof of (a). $\square$

We can now get the following.

Lemma 7.4. *If* M^* *is not a direct summand of* T, *then* M *is a direct summand of* T, *and if* $T = M^k \coprod \overline{T}$ *(with* M *not a direct summand of* $\overline{T}$*), then* $M^* \coprod \overline{T}$ *is also a tilting object.*

Proof. Assume that M and M^* are not summands of T. Then by Lemma 7.3, $T \coprod M$ is exceptional, contradicting the fact that T is a tilting object.

Assume still that M^* is not a summand of T, so that $T = M^k \coprod \overline{T}$ where M is not a summand of $\overline{T}$ and $k > 0$. By Lemma 7.3, $M^* \coprod \overline{T}$ is an exceptional object with the "correct" number of indecomposable non-isomorphic direct summands, and is hence a tilting object. $\square$

Summarising, we now have the following.

Theorem 7.5. *Two exceptional indecomposable objects* M *and* M^* *form an exchange pair if and only if* $\dim_{D_M} \mathrm{Ext}^1_{\mathcal{C}}(M, M^*) = 1 = \dim_{D_{M^*}} \mathrm{Ext}^1_{\mathcal{C}}(M^*, M)$.

The following example shows that it is necessary to assume that both Ext^1-spaces are one-dimensional, that is, one is not the consequence of the other.

Example. Consider the ring

$$H = \begin{pmatrix} \mathbb{R} & 0 \\ {}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}} & \mathbb{C} \end{pmatrix}.$$

The AR-quiver of $D^b(H)$ is shown in Fig. 12. We have

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{C}} \begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix} = 1$$

$$\begin{tikzcd}[column sep=small]
 & & \begin{pmatrix}\mathbb{R}\\ \mathbb{C}\end{pmatrix} \arrow[dr] & & \begin{pmatrix}\mathbb{R}\\ 0\end{pmatrix} \arrow[dr] & & \begin{pmatrix}\mathbb{R}\\ \mathbb{C}\end{pmatrix}[1] & \\
\cdots & \begin{pmatrix}0\\ \mathbb{C}\end{pmatrix} \arrow[ur] & & \begin{pmatrix}\mathbb{R}^2\\ \mathbb{C}\end{pmatrix} \arrow[ur] & & \begin{pmatrix}0\\ \mathbb{C}\end{pmatrix}[1] \arrow[ur] & & \cdots
\end{tikzcd}$$

Fig. 12. The AR-quiver of $D^b(H)$.

and

$$\dim_{\mathbb{R}} \operatorname{Ext}^1_{\mathcal{C}} \begin{pmatrix}\mathbb{R}\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \mathbb{C}\end{pmatrix} = 2.$$

Hence $\left\{ \begin{pmatrix}\mathbb{R}\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \mathbb{C}\end{pmatrix} \right\}$ is not an exchange pair.

Finally, suppose that H is the path algebra of a quiver of simply laced Dynkin type Δ with an alternating orientation. Let $\mathcal{A}(\Delta)$ denote the corresponding cluster algebra. By Proposition 4.1, we know that there is a 1–1 correspondence between the cluster variables of $\mathcal{A}(\Delta)$ and $\operatorname{ind}\mathcal{C}$. By Theorem 4.5 we know that this induces a bijection between the basic tilting objects of $\mathcal{C}$ and the clusters of $\mathcal{A}(\Delta)$. We have the following interpretation of Theorem 7.5.

Theorem 7.6 (*Fomin and Zelevinsky [FZ03a, 3.5, 4.4]*). *Suppose $\mathcal{A}(\Delta)$ is the cluster algebra associated to an arbitrary Dynkin diagram of simply laced type. Let x, y be two cluster variables of $\mathcal{A}(\Delta)$. Then x, y form an exchange pair if and only if their compatibility degree is equal to* 1.

8. Graphical calculus

In this section, we assume the quiver Q to be of simply laced Dynkin type. We shall give a graphical calculus for computing the triangles in Section 6 (see Theorem 6.8 and the comment afterwards).

Suppose that M, M^* are indecomposable objects of $\mathcal{C} = D^b(kQ)/F$. We know that $\operatorname{End}_{\mathcal{C}}(M) \simeq \operatorname{End}_{\mathcal{C}}(M') \simeq k$—use Proposition 1.7(c) and the fact that every indecomposable object in $\mathcal{C}$ is in the τ-orbit of an indecomposable projective module. Suppose that $\operatorname{Ext}^1_{\mathcal{C}}(M^*, M)$ is one-dimensional over k. We know by Theorem 7.5 that this is the equivalent to assuming that M, M^* are the two complements of an almost complete basic tilting object $\overline{T}$ of $\mathcal{C}$. We would like to construct triangles

$$M^* \xrightarrow{g} B \xrightarrow{f} M \to M^*[1] \tag{8}$$

and

$$M \xrightarrow{u} B' \xrightarrow{v} M^* \to M[1] \tag{9}$$

where $f : B \to M$ is a minimal right add $\overline{T}$-approximation of M, and $u : M \to B'$ is a minimal left add $\overline{T}$-approximation of M.

Without loss of generality (by applying APR-tilts if necessary), we can assume that M is a simple projective kQ-module P. Suppose first that $M^* = P'[1]$ is the shift of an indecomposable projective kQ-module P'. Then

$$\mathrm{Ext}^1_{\mathcal{C}}(M^*, M) = \mathrm{Hom}_{kQ}(P', P)$$

(see Proposition 1.7(d)), since P' is projective. But, since P is simple projective, this is non-zero (and necessarily one-dimensional) if and only if $P \simeq P'$. By applying the autoequivalence $\tau_{\mathcal{C}}^{-1}$ to P and $P'[1]$ we are reduced to the situation where M and M^* are the start and end terms, respectively, of an almost split sequence of kQ-modules. We are then in the case discussed after the proof of Proposition 6.14, and we see that $B = 0$ and B' is the middle term of the almost split sequence involving M and M^*.

We are now left with the case where M and M^* are both modules over kQ, with M projective. By Proposition 1.7(c), we have

$$\mathrm{Ext}^1_{\mathcal{C}}(M^*, M) \simeq \mathrm{Ext}^1_{kQ}(M^*, M) \coprod \mathrm{Ext}^1_{kQ}(M, M^*).$$

Since M is projective, $\mathrm{Ext}^1_{kQ}(M, M^*) = 0$. Then we have a unique non-trivial extension

$$0 \to M \to E \to M^* \to 0$$

of kQ-modules. There is a corresponding triangle

$$M \to E \to M^* \to M[1]$$

in $\mathcal{D}$ which induces a non-split triangle in $\mathcal{C}$. Since the triangle (9) is (up to isomorphism) the unique non-split triangle in $\mathcal{C}$ with start term M and end term M^*, we have that E is isomorphic to B'. In the case where $\mathrm{Ext}^1_{kQ}(M, M^*) \simeq k$ we obtain a middle term isomorphic to B. Note that by switching the roles of M and M^* (using Proposition 1.7(b)), and applying $\tau_{\mathcal{C}}$ as appropriate, we can compute both middle terms B and B' using the module category alone. We have reduced the problem to the following:

Problem 8.1. *Let Q be a simply laced Dynkin quiver, and let M, M^* be indecomposable kQ-modules satisfying $\mathrm{Ext}^1_{kQ}(M, M^*) \simeq k$ (and therefore $\mathrm{Ext}^1_{kQ}(M^*, M) = 0$). Compute the middle term of the unique non-trivial extension represented by a non-zero element of $\mathrm{Ext}^1_{kQ}(M, M^*)$.*

Let M and M^* be indecomposable kQ-modules, such that $\dim \mathrm{Ext}^1_{kQ}(M, M^*) = 1$, and let

$$\zeta : 0 \to M^* \to X \to M \to 0$$

be the unique non-trivial extension mentioned above. We will now develop a graphical method (in terms of the AR-quiver) for the determination of X. We recall that the *starting function* s_U of an indecomposable kQ-module U is defined as the function $V \mapsto \dim \mathrm{Hom}_{kQ}(U, V)$ on indecomposable kQ-modules. All such starting functions are depicted in [Bon84]. Similarly, the ending function e_U is defined as the function $V \mapsto \dim \mathrm{Hom}_{kQ}(V, U)$.

Lemma 8.2. *Let U and V be indecomposable representations of kQ, such that* $\mathrm{Hom}_{kQ}(U, V) \neq 0$ *and* $\mathrm{Ext}^1_{kQ}(V, U) = 0$. *Then* $\dim \mathrm{Hom}_{kQ}(U, V) = 1$.

Proof. The above condition translates to

$$s_U(V) \neq 0, \quad s_U(\tau V) = 0$$

by the AR-formula. Now direct inspection of the tables in [Bon84] gives the above result. □

This result can also be established in a theoretical way, using the result [vHoh94], Corollary to main theorem. Since the table in [Bon84] will play a central role in the following, the above proof is more adapted to the theme of this section.

Proposition 8.3. *Let M, M^* and X be as above. Then X is the direct sum of one copy of each indecomposable kQ-module V fulfilling*

$$\mathrm{Hom}_{kQ}(M^*, V) \neq 0 \neq \mathrm{Hom}_{kQ}(V, M) \text{ and } \mathrm{Ext}^1_{kQ}(V, M^*) = 0 = \mathrm{Ext}^1_{kQ}(M, V).$$

Proof. Let V be an indecomposable direct summand of X. We first show that the stated homological conditions on V are satisfied. If $\mathrm{Hom}_{kQ}(M^*, V) = 0$, then V has to appear as a direct summand of M. Since M is indecomposable, this implies $M = V$, and the sequence ζ splits, a contradiction. Thus $\mathrm{Hom}_{kQ}(M^*, V) \neq 0$.

In the induced exact sequence

$$\mathrm{Hom}_{kQ}(M^*, M^*) \xrightarrow{d} \mathrm{Ext}^1_{kQ}(M, M^*) \to \mathrm{Ext}^1_{kQ}(X, M^*) \to \mathrm{Ext}^1_{kQ}(M^*, M^*) = 0,$$

the map d is surjective, since $\mathrm{Ext}^1_{kQ}(M, M^*)$ is one-dimensional and the sequence ζ is non-split. Thus $\mathrm{Ext}^1_{kQ}(X, M^*) = 0$, and in particular $\mathrm{Ext}^1_{kQ}(V, M^*) = 0$. We can argue dually to obtain the other two conditions on V.

Enumerate the isomorphism classes of indecomposables with the above properties as $\{V_1, \ldots, V_s\}$; thus we can write $X = \oplus_{i=1}^s V_i^{m_i}$, and we have to prove that $m_i = 1$ for all $i = 1 \dots s$. Consider the induced exact sequence

$$0 \to \mathrm{Hom}_{kQ}(M, X) \to \mathrm{Hom}_{kQ}(X, X) \to \mathrm{Hom}_{kQ}(M^*, X) \to \mathrm{Ext}^1_{kQ}(M, X).$$

From the above, we can conclude that $\mathrm{Ext}^1_{kQ}(M, X) = 0$. Since any V_i maps to M, we also have $\mathrm{Hom}_{kQ}(M, X) = 0$, thus $\mathrm{Hom}_{kQ}(M, V_i) = 0$ since the category $\mathrm{mod}\, kQ$ is representation-directed. We arrive at an isomorphism

$$\mathrm{Hom}_{kQ}(X, X) \simeq \mathrm{Hom}_{kQ}(M^*, X).$$

Since $\mathrm{End}_{kQ}(X)$ contains the semisimple ring $\oplus_{i=1}^s M_{m_i}(\mathrm{End}_{kQ}(V_i))$ as a subring, we can estimate:

$$\begin{aligned}\sum_{i=1}^s m_i^2 \leqslant \sum_{i,j=1}^s m_i m_j \dim \mathrm{Hom}_{kQ}(V_i, V_j) &= \dim \mathrm{Hom}_{kQ}(X, X)\\ = \dim \mathrm{Hom}_{kQ}(M^*, X) = \sum_{i=1}^s m_i \dim \mathrm{Hom}_{kQ}(M^*, V_i) &= \sum_{i=1}^s m_i,\end{aligned}$$

using Lemma 8.2. Thus $m_i \in \{0, 1\}$ for all $i = 1 \dots s$, and $\mathrm{Hom}_{kQ}(V_i, V_j) = 0$ whenever $i \neq j$ and $m_i = 1 = m_j$.

Similarly, we see that for each $i = 1 \dots s$, we have isomorphisms

$$\mathrm{Hom}_{kQ}(X, V_i) \simeq \mathrm{Hom}_{kQ}(M^*, V_i) \text{ and } \mathrm{Hom}_{kQ}(V_i, X) \simeq \mathrm{Hom}_{kQ}(V_i, M).$$

Given a fixed V_i, we choose non-zero maps $f : M^* \to V_i$ and $g : V_i \to M$. The above isomorphisms yield factorisations $f = r\alpha$ and $g = \beta s$, where $\alpha : M^* \to X$ and $\beta : X \to M$ are the maps in the short exact sequence ζ. Since $r \neq 0$ and $s \neq 0$ we can choose summands V_j and V_k of X, such that $r_j\alpha \neq 0$ and $\beta s_k \neq 0$, where r_j is the restriction of r to V_j and s_k is the composition of s with the projection onto V_k.

It is enough to prove that $s_k r_j \neq 0$. Then, since V_j and V_k are direct summands of X, we obtain from the above that $j = k$, and therefore that $i = j = k$ since there are no oriented cycles of homomorphisms in the category of kQ-modules.

We have $\mathrm{Ext}^1_{kQ}(M, V_i) = 0$ by assumption, and $\mathrm{Ext}^1_{kQ}(M^*, V_i) = 0$ since kQ is representation-directed and $\mathrm{Hom}_{kQ}(M^*, V_i) \neq 0$. These two facts together imply $\mathrm{Ext}^1_{kQ}(X, V_i) = 0$, thus in particular $\mathrm{Ext}^1_{kQ}(V_k, V_i) = 0$, since V_k is a direct summand of X. This vanishing condition allows us to apply the Happel–Ringel Lemma [HR82] to conclude that $s_k \neq 0$ must be mono or epi. If s_k is mono, then $s_k r_j \neq 0$ since $r_j \neq 0$, and we are done. So assume that s_k is epi. By possibly applying the AR-translate, we can assume without loss of generality that M^* is projective. This provides us with a surjection

$$(s_k \circ \_) : \mathrm{Hom}_{kQ}(M^*, V_i) \to \mathrm{Hom}_{kQ}(M^*, V_k),$$

thus an isomorphism since both spaces are one-dimensional by Lemma 8.2. But this implies that $s_k r_j \neq 0$. This finishes the proof. □

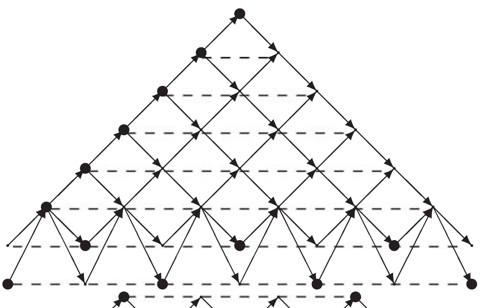
The starting and ending functions of an indecomposable kQ-module U can be computed in terms of the AR-quiver: the function s_U is determined by defining $s_U(V) = 1$ on the slice starting in U, and by additivity $s_U(\tau^{-1}(V)) = \sum_i s_U(C_i) - s_U(V)$ for a mesh $V \to \bigoplus_i C_i \to \tau^{-1}V$. We can now define

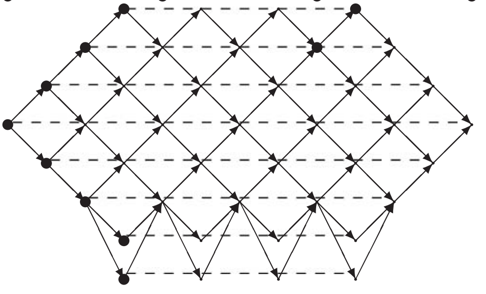
Definition 8.4. The starting frame $F_s(U)$ (resp., the ending frame $F_e(U)$) of an indecomposable kQ-module U consists of all vertices V of the AR-quiver, such that $s_U(V) \neq 0 = s_U(\tau V)$ (resp., $e_U(V) \neq 0 = e_U(\tau^{-1}V)$).

As an immediate corollary to the above proposition, we get

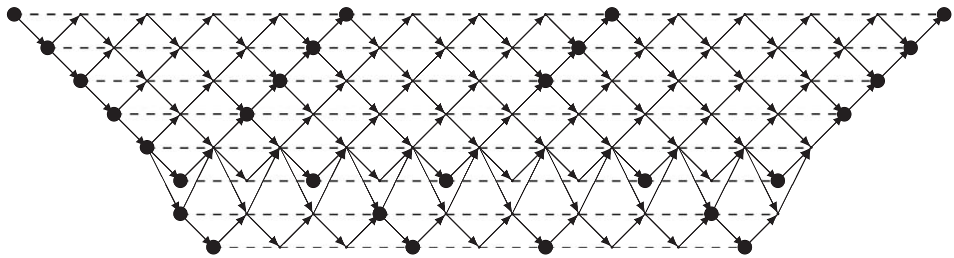
Corollary 8.5. *Given indecomposables M and M^*, such that $\mathrm{Ext}^1_{kQ}(M, M^*)$ is one-dimensional, the unique non-trivial extension X of M by M^* is given as the direct sum of all indecomposables belonging to the intersection $F_s(M^*) \cap F_e(M)$.*

The starting and ending frames can now be worked out using the tables in [Bon84]. In type A, they are easily seen to coincide with the slice starting (resp., ending) in an indecomposable. For type D and E, the frames look in general more complicated. Below, we first show two "typical" examples in type D_8. The starting frame of the respective minimal vertex of the picture is shown (the solid circles), embedded in a portion of the AR-quiver.





Finally, we show the "most complicated" starting frame in type E_8:



9. Interpretation and conjectures

In this section we will consider further links with cluster algebras, including interpretations of some of the preceding results. We will make some conjectures in this direction and also provide some examples giving supporting evidence for the conjectures.

Let H be a finite-dimensional hereditary algebra, with quiver Γ. For vertices i and j of Γ, let n_{ij} denote the number of arrows from i to j in Γ. Let X be the matrix with rows and columns indexed by the vertices of Γ (we choose a total ordering)

$$x_{ij} = \begin{cases} n_{ij}, & n_{ij} \neq 0, \\ -n_{ji}, & n_{ij} = 0. \end{cases}$$

Let $\mathcal{A}(H)$ be the corresponding cluster algebra. Let $\mathcal{C}$ be the cluster category associated to H and denote by $\mathrm{ind}^\circ\,\mathcal{C}$ the set of exceptional indecomposable objects of $\mathcal{C}$.

Conjecture 9.1. *There is a* 1–1 *correspondence between the cluster variables of* $\mathcal{A}(H)$ *and* $\mathrm{ind}^\circ\,\mathcal{C}$ *inducing a* 1–1 *correspondence between the clusters of* $\mathcal{A}(H)$ *and the basic tilting objects in* $\mathcal{C}$.

We have seen (see Section 4) that this conjecture holds in the case where H is the path algebra of a simply laced Dynkin quiver. In this case, we make a further conjecture:

Conjecture 9.2. *Let C be a cluster of the cluster algebra of simply laced Dynkin type, and let T be the corresponding tilting object of the cluster category $\mathcal{C}$ of the same type. Let Q_C denote the algebra associated to C in* [CCS04a, Section 1]. *Then* $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ *is isomorphic to Q_C.*

Suppose that $\overline{T}$ is an almost complete basic tilting object of $\mathcal{C}$. Let M, M^* be the complements of $\overline{T}$, and let

$$M^* \xrightarrow{g} B \xrightarrow{f} M \to M^*[1]$$

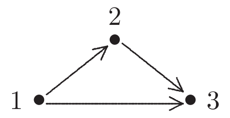


Fig. 13. A quiver of type $\widetilde{A_2}$.

and

$$M \xrightarrow{u} B' \xrightarrow{v} M^{**} \to M[1]$$

be the triangles (3) and (5) from Section 6.

We make the following conjecture:

Conjecture 9.3. *In the above situation, let $B = \coprod_{i \in I} B_i^{d_i}$ (respectively, $B' = \coprod_{j \in J} (B'_j)^{e_j}$) be the direct sum decomposition of B (respectively, B'), where the B_i are all non-isomorphic and the B'_j are all non-isomorphic. Let x, x' be the cluster variables corresponding to M, M^*, and for $i \in I$ (respectively, $j \in J$) let x_i (respectively, x'_j) be the cluster variable corresponding to B_i (respectively, B'_j). Then the exchange relation in the cluster algebra $\mathcal{A}(H)$ (see Eq. (1) in the introduction) takes the form*

$$xx' = \prod_{i \in I} x_i^{d_i} + \prod_{j \in J} (x'_j)^{e_j}.$$

In particular, B and B' should have no common direct summands.

We note that in the simply laced Dynkin case, this conjecture can be reformulated, via the discussion in Section 8, to give a conjecture providing a direct interpretation of the cluster exchange relation in terms of short exact sequences of kQ-modules (see Problem 8.1). We also note that if Conjecture 9.3 holds then it can be seen that the rule for matrix mutation (see the introduction) describes the change in the quiver of the algebra $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ when one indecomposable direct summand of the basic tilting object T is exchanged for another. We give an example of this below. Finally, we remark that if Conjectures 9.1 and 9.3 both hold, then Theorems 1.11 and 1.12 (without coefficients) in [FZ03a] hold for the corresponding cluster algebra.

Example. Let H be the path algebra of the quiver as shown in Fig. 13. Then the corresponding cluster algebra $\mathcal{A}(H)$ has seed given by the transcendence basis $\{u_1, u_2, u_3\}$ of $\mathbb{Q}(u_1, u_2, u_3)$ and matrix

$$X = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

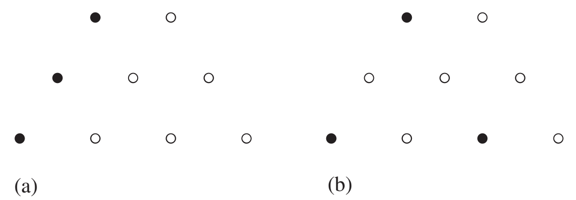


Fig. 14. Two basic tilting objects of $\mathcal{C}$ in type A_3: (a) summands of T and (b) T'.

The corresponding cluster algebra was investigated in [FZ02a, 7.8]—the *brick wall* example. Let P_1, P_2, P_3 denote the indecomposable projective modules corresponding to the vertices of the graph of H. Let R denote the regular indecomposable module with dimension vector $(1,0,1)$. Then $T = \widetilde{P_1} \coprod \widetilde{P_2} \coprod \widetilde{P_3}$ is a basic tilting object of $\mathcal{C}$. Choosing $M = \widetilde{P_2}$ and $\overline{T} = \widetilde{P_1} \coprod \widetilde{P_3}$ we see that $\widetilde{P_2} \to \widetilde{P_1}$ is a minimal left $\mathrm{add}(\overline{T})$-approximation of $\widetilde{P_2}$ and obtain the triangle:

$$\widetilde{P_2} \to \widetilde{P_1} \to \widetilde{R} \to \widetilde{P_2}[1]$$

in $\mathcal{C}$. It follows that $T' = \widetilde{P_1} \coprod \widetilde{R} \coprod \widetilde{P_3}$ is again a basic tilting object of $\mathcal{C}$. The matrix X' of the quiver of $\mathrm{End}(T')^{\mathrm{op}}$ corresponding to T' is:

$$X' = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix},$$

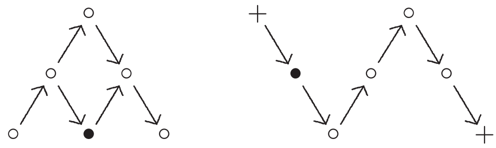
which is easily seen to be the mutation of the matrix X at 2.

Suppose that H is a finite-dimensional hereditary algebra, $\overline{T}$ is an almost complete basic tilting object of $\mathcal{C}$, and M and M^* are the two complements of $\overline{T}$. Let $T = \overline{T} \coprod M$ and $T' = \overline{T} \coprod M^*$ be the two completions of $\overline{T}$ to a basic tilting object. Let $\Gamma = \mathrm{End}_{\mathcal{C}}(\overline{T} \coprod M)^{\mathrm{op}}$ and $\Gamma' = \mathrm{End}_{\mathcal{C}}(\overline{T} \coprod M^*)^{\mathrm{op}}$ be the endomorphism algebras, taken over $\mathcal{C}$. Denote by S_M (respectively, S_{M^*}) the simple top of the Γ-module $\mathrm{Hom}_{\mathcal{C}}(T, M)$ (respectively, the Γ'-module $\mathrm{Hom}_{\mathcal{C}}(T, M^*)$). Then we conjecture that the category of Γ-modules and the category of Γ'-modules are related in the following way:

Conjecture 9.4. *The categories* $\mathrm{mod}\,\Gamma/\mathrm{add}\,S_M$ *and* $\mathrm{mod}\,\Gamma'/\mathrm{add}\,S_{M^*}$ *are equivalent.*

This can be viewed as a generalisation of APR-tilting [APR79].

Example. We give an example illustrating Conjecture 9.4. Take Δ to be the Dynkin diagram of type A_3. Then the AR-quiver of $\mathcal{C}$ is given in Fig. 1. Let T be the direct sum of the indecomposable objects corresponding to the filled-in circles in Fig. 14(a) and let T' be the direct sum of the indecomposable objects corresponding to the filled-in circles in Fig. 14(b). Thus $\overline{T}$ is the almost complete basic tilting object which is the



(a) (b)

Fig. 15. The AR-quivers of (a) Γ and (b) Γ'.

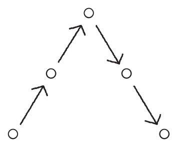


Fig. 16. Thecommon sub-translation quiver of the AR-quivers of Γ and Γ'.

(a) (b)

Fig. 17. The quivers of the algebras (a) Γ and (b) Γ'.

direct sum of the objects corresponding to the two filled-in circles common to T and T'. Here, we display the AR-quiver of $\mathcal{C}$ slightly differently in order to demonstrate this example (noting that it appears on a Möbius band).

The AR-quivers of $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ and $\Gamma' = \mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$ are given in Fig. 15. The two vertices labelled by a "+" are identified, and the simples S_M and S_{M^*} are shown by filled-in circles. We can see that the full sub-translation quiver of the AR-quiver of Γ consisting of all of the vertices except S_M is isomorphic to the full sub-translation quiver of Γ' consisting of all of the vertices except S_{M^*}. See Fig. 16.

This also gives a nice example of mutation. The quivers of Γ and Γ' are shown in Fig. 17; Γ has no relations, but for Γ' the relations are that the product of any pair of composable arrows is zero. The corresponding mutation is the mutation at 2 of the matrix X to X', where

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X' = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Note added in proof: We give the following update on the status of the conjectures in Section 9. Conjecture 9.2 has been solved in [BMR05], and partially in [CCS04b].

A local version of Conjecture 9.3 is proved in [BMR04b]. Note that settling the still unsolved Conjecture 9.1 would give a proof of the conjecture as it is formulated in this paper. In the case of simply laced Dynkin quivers it was solved independently in [CCS04b]. Conjecture 9.4 was solved in [BMR04a]. Cluster categories have also proved useful in the Hall algebra approach to cluster algebras of Caldero and Chapoton [CC].

Acknowledgements

We thank A. Zelevinsky for many helpful discussions. B. Marsh would like to thank Northeastern University, Boston and NTNU, Trondheim, for their kind hospitality. Aslak Bakke Buan was supported by a grant from the Norwegian Research Council. Bethany Marsh was supported by a Leverhulme Fellowship, study leave from the University of Leicester, EPSRC Grant GR/R17547/01, NSF funding through a research grant of A. Zelevinsky and support from the Norwegian Research Council.

References

[APR79] M. Auslander, M.I. Platzeck, I. Reiten, Coxeter functors without diagrams, Trans. Amer. Math. Soc. 250 (1979) 1–46.

[AS80] M. Auslander, S.O. Smalø, Preprojective modules over Artin algebras, J. Algebra 66 (1980) 61–122.

[BFZ02] A. Berenstein, S. Fomin, A. Zelevinsky, Cluster algebras III: upper bounds and double Bruhat cells, Duke Math. J. 126 (1) (2005) 1–52.

[Bon81] K. Bongartz, Tilted algebras, in: M. Auslander, E. Lluis (Eds.), Representations of algebras (Puebla, 1980), Lecture Notes in Mathematics, vol. 903, Springer, Berlin, New York, 1981, pp. 26–38.

[Bon84] K. Bongartz, Critical simply connected algebras, Manuscripta Math. 46 (1–3) (1984) 117–136.

[BG81] K. Bongartz, P. Gabriel, Covering spaces in representation-theory, Invent. Math. 65 (3) (1981/2) 331–378.

[BLR81] O. Bretscher, C. Läser, C. Riedtmann, Selfinjective and simply connected algebras, Manuscipta Math. 36 (1981) 253–307.

[BK] A.B. Buan, H. Krause, Tilting and cotilting for quivers of type $\tilde{A}_n$, J. Pure Appl. Algebra 190 (1–3) (2004) 1–21.

[BMR04a] A.B. Buan, B. Marsh, I. Reiten, Cluster-tilted algebras, Preprint, February 2004, Trans. Amer. Math. Soc., to appear.

[BMR04b] A.B. Buan, B. Marsh, I. Reiten, Cluster mutation via quiver representations, Preprint arXiv:math.RT/0412077, 2004.

[BMR05] A.B. Buan, B. Marsh, I. Reiten, Cluster-tilted algebras of finite representation type, Preprint, 2005.

[CC] P. Caldero, F. Chapoton, Cluster algebras as Hall algebras of quiver representations, Preprint arXiv:math.RT/0410187, 2004.

[CCS04a] P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations arising from clusters (A_n case), Trans. Amer. Math. Soc., to appear, PII: S0002-9947(05)03753-0.

[CCS04b] P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations and cluster tilted algebras, Preprint arXiv:math.RT/0411238, 2004.

[CFZ02] F. Chapoton, S. Fomin, A. Zelevinsky, Polytopal realizations of generalized associahedra, Canad. Math. Bull. 45 (4) (2002) 537–566.

[CHU94] F. Coelho, D. Happel, L. Unger, Complements to partial tilting modules, J. Algebra 170 (1) (1994) 184–205.

[DR92] V. Dlab, C.M. Ringel, The module theoretical approach to quasi-hereditary algebras, in: H. Tachikawa, S. Brenner (Eds.), Representations of algebras and related topics (Kyoto, 1990), London Mathematical Society, Lecture Note Series, vol. 168, Cambridge University Press, Cambridge, 1992, pp. 200–224.

[FZ02a] S. Fomin, A. Zelevinsky, Cluster algebras I: foundations, J. Amer. Math. Soc. 15 (2) (2002) 497–529.

[FZ03a] S. Fomin, A. Zelevinsky, Cluster algebras II: finite type classification, Invent. Math. 154 (1) (2003) 63–121.

[FZ02c] S. Fomin, A. Zelevinsky, The Laurent phenomenon, Adv. Appl. Math. 28 (2) (2002) 119–144.

[FZ03] S. Fomin, A. Zelevinsky, Y-systems and generalized associahedra, Ann. Math. 158 (3) (2003).

[GSV02] M. Gekhtman, M. Shapiro, A. Vainshtein, Cluster algebras and Poisson geometry, Moscow Math. J. 3 (2003) 899–934.

[Hap85] D. Happel, Tilting sets on cylinders, Proc. London Math. Soc. 51 (3) (1985) 21–55.

[Hap88] D. Happel, Triangulated categories in the representation theory of quivers, LMS Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[HRS96] D. Happel, I. Reiten, S. Smalø, Tilting in abelian categories and quasitilted algebras, Mem. Amer. Math. Soc. 120 (575) (1996).

[HR82] D. Happel, C. Ringel, Tilted algebras, Trans. Amer. Math. Soc. 274 (2) (1982) 399–443.

[HU89] D. Happel, L. Unger, Almost complete tilting modules, Proc. Amer. Math. Soc. 107 (3) (1989) 603–610.

[HU03] D. Happel, L. Unger, On the set of tilting objects in hereditary categories, Preprint, 2003.

[Kel03] B. Keller, On Triangulated orbit categories, Preprint arXiv:math.RT /0503240, 2005.

[MRZ03] B. Marsh, M. Reineke, A. Zelevinsky, Generalized associahedra via quiver representations, Trans. Amer. Math. Soc. 355 (10) (2003) 4171–4186.

[P03] D.I. Panyushev, *AD*-nilpotent ideals of a Borel subalgebra: generators and duality, J. Algebra 274 (2) (2004) 822–846.

[RV02] I. Reiten, M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15 (2) (2002) 295–366.

[Rie80a] C. Riedtmann, Representation-finite self-injective algebras of class A_n, in: V. Dlab, P. Gabriel (Eds.), Representation theory, II (Proceedings of the Second International Conference, Carleton University, Ottawa, Ont., 1979), Lecture Notes in Mathematics, vol. 832, Springer, Berlin, 1980, pp. 449–520.

[Rie80b] C. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurück, Comment. Math. Helv. 55 (2) (1980) 199–224.

[RS90] C. Riedtmann, A. Schofield, On open orbits and their complements, J. Algebra 130 (2) (1990) 388–411.

[RS91] C. Riedtmann, A. Schofield, On a simplicial complex associated with tilting modules, Comment. Math. Helv. 66 (1) (1991) 70–78.

[Rin84] C.M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Mathematics, vol. 1099, Springer, Berlin, 1984.

[Sco03] J. Scott, Grassmannians and cluster algebras, Preprint arXiv:math.CO/ 0311148, 2003, Proc. London. Math. Soc., 2003, to appear.

[Ung90] L. Unger, Schur modules over wild, finite-dimensional path algebras with three simple modules, J. Pure Appl. Algebra 64 (2) (1990) 205–222.

[Ung96] L. Unger, The simplical complex of tilting modules over quiver algebras, Proc. London Math. Soc. 73 (3) (1996) 27–46.

[vHoh94] H. von Höhne, On the dimension vectors in preprojective components, Bull. London Math. Soc. 26 (2) (1994) 147–152.